Large‑scale perturbations on the brane and the isotropy of the cosmological singularity

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We present the complete set of propagation and constraint equations for the kinematic and non‑local first order quantities which describe general linear inhomogeneous and anisotropic perturbations of a flat FRW braneworld with vanishing cosmological constant and decompose them in the standard way into their scalar, vector and tensor contributions. A detailed analysis of the perturbation dynamics is performed using dimensionless variables that are specially tailored for the different regimes of interest; namely, the *low energy GR regime*, the *high energy regime* and the *dark energy regime*. Tables are presented for the evolution of all the physical quantities, making it easy to do a detailed comparison of the past asymptotic behaviour of the perturbations of these models. We find results that exactly match those obtained in the analysis of the spatially inhomogeneous G_2 braneworld cosmologies presented recently; i.e., that isotropization towards the $\mathcal{F}_b$ model occurs for $\gamma > 4/3$.

I. INTRODUCTION

A well known problem of cosmology is to explain the very high degree of isotropy observed in the Cosmic Microwave Background (CMB). In a theory such as general relativity, where isotropy is a special rather than generic feature of cosmological models, we need a dynamical mechanism able to produce isotropy. Inflation was proposed, among other reasons, as a way to isotropize the universe. Inflation is effective in this sense, but it needs homogeneous enough initial data in order for inflation to begin [1]. Although one could perhaps adopt the view that one smooth enough patch in an otherwise non‑smooth initial universe is all that is needed to explain observations, this may not be satisfactory [2]: the isotropy problem remains open in standard cosmology.

Recently, a number of authors [3, 4, 5, 6, 7, 8, 9, 10, 11, 12] have addressed the issue of isotropization in the context of braneworld cosmology based on a generalization of the Randall and Sundrum model [13, 14]. Here the bulk is 5‑dimensional and contains only a cosmological constant, assumed to be negative (see [15] for a comprehensive review).

In all cases considered, an interesting result was found: unlike general relativity, where in general the cosmological singularity is anisotropic, the past attractor for spatially homogeneous anisotropic models in the brane is a simple Robertson‑Walker (RW) model $\mathcal{F}_b$ [7, 16]. Since this result was also found to hold for Bianchi IX models [7, 8] as well as for some inhomogeneous models, the author suggested that the isotropic singularity could be a generic feature of brane cosmological models.

In a recent paper [17], this conjecture was supported by studying the dynamics of a class of *spatially inhomo‑geneous G_2* cosmological models in the braneworld scenario. A *numerical* analysis of the governing system of evolution equations led to the result that for $\gamma > 4/3$ isotropization towards a simple RW model $\mathcal{F}_b$ occurs as $\tau \to -\infty$ for *all* initial conditions. In the case of radiation ($\gamma = 4/3$), the models were still found to isotropize as $\tau \to -\infty$, albeit slowly. It can therefore be concluded that an initial isotropic singularity occurs in all of these G_2 spatially inhomogeneous brane cosmologies for a range of parameter values which include the physically important cases of radiation and a scalar field source. The numerical results were confirmed by a qualitative dynamical analysis and a detailed calculation of the past asymptotic decay rates [17].

A similar result is also obtained in a related perturbative study where a careful analysis of generic linear inhomogeneous and anisotropic perturbations of the $\mathcal{F}_b$ model [18] was conducted. Solutions were obtained for the large‑scale evolution of *scalar*, *vector* and *tensor* perturbations showing that the $\mathcal{F}_b$ model is stable in the past (as $\tau \to -\infty$) with respect to generic inhomogeneous and anisotropic perturbations provided the matter is described by a non‑inflationary perfect fluid with γ‑law equation of state parameter satisfying $\gamma > 1$. In particular, it was shown that the expansion normalised shear vanishes as $\tau \to -\infty$, signalling isotropization.

Brane cosmology thus has the very attractive feature of having isotropy built in, and although inflation in this context would still be the most likely way of producing the fluctuations seen in the CMB, there would be no need for special initial conditions for it to start. Also, the Penrose conjecture [19] on gravitational entropy and an initially vanishing measure of the Weyl tensor might be satisfied, c.f. [20].

The aim of this paper is to give a more compressive large‑scale perturbative analysis of flat Friedmann‑Robertson‑Walker (FRW) brane models with vanishing cosmological constant by combining the *high energy* results in [18] with an analysis of the other important stages in the braneworld evolution, namely the *low energy GR* and *dark energy* regimes. To make this precise we define dimensionless variables that are specially tailored for each regime of interest. In this way we are able to clarify further the past asymptotic behaviour of these models and obtain results which match the analysis of the spatially inhomogeneous G_2 cosmologies presented in [17]; i.e., that isotropization towards the $\mathcal{F}_b$ model occurs for $\gamma > 4/3$.

The paper is organised as follows. In section II we will give a brief summary of the braneworld scenario and the induced field equations on the brane. In section III we introduce dimensionless expansion normalised variables and derive the complete set of propagation and constraint equations for the *kinematic*, *inhomogeneity* and *non‑local* quantities. In section IV we split these equations into *scalar*, *vector* and *tensor* parts, which we then analyse and discuss in sections V‑VII for the *low energy*, *high‑energy* and *dark radiation* dominated regimes respectively. Finally, in section VIII we present our conclusions. For the most part we follow the notation and convention of [15, 18].

II. BRANE DYNAMICS

A. Geometric Formulation

The implementation of the braneworld scenario considered in [14] assumes that the whole spacetime is 5‑D and governed by the 5‑D field equations ($A, B = 0, ..., 4$):

$$G^{(5)}_{AB} = -\Lambda_{(5)} g^{(5)}_{AB} + \kappa^2_{(5)} \delta(\chi)[-\lambda g_{AB} + T_{AB}] . \quad (1)$$

These represent a 4‑D brane at $\chi = 0$ embedded in a vacuum bulk with metric $g^{(5)}_{AB}$ and cosmological constant $\Lambda^{(5)}$; $\kappa^2_{(5)}$ is the 5‑D gravitational constant, λ is the brane tension, g_{AB} and T_{AB} are respectively the metric and the energy-momentum on the brane. The 4‑D field equations induced on the brane are derived geometrically from (1) assuming a Z_2 symmetry with the brane at the fixed point, leading to modified Einstein equations with new terms representing bulk effects:

$$G_{ab} = -\Lambda g_{ab} + \kappa^2 T^{\text{tot}}_{ab} , \quad (2)$$

where

$$T^{\text{tot}}_{ab} = T_{ab} + \frac{6}{\lambda} S_{ab} - \frac{1}{\kappa^2} E^{(5)}_{ab} . \quad (3)$$

As usual $\kappa^2 = 8\pi/M^2_{\rm p}$, and ($a, b = 0, ..., 3$). The various physical constants and parameters appearing in the

equations above are not independent, but related to each other by

$$\lambda = 6 \frac{\kappa^2}{\kappa^4_{(5)}} , \quad \Lambda = \tfrac{1}{2}[\Lambda_{(5)} + \kappa^2 \lambda] . \quad (4)$$

The tensor S_{ab} represents non‑linear matter corrections given by

$$S_{ab} = \tfrac{1}{12} T^c{}_c T_{ab} - \tfrac{1}{4} T_{ac} T^c{}_b + \tfrac{1}{24} g_{ab} \left[3 T_{cd} T^{cd} - (T^c{}_c)^2 \right] . \quad (5)$$

$E^{(5)}_{ab}$ is the projection of the 5‑D Weyl tensor $C^{(5)}_{ABCD}$ on to the brane: $E^{(5)}_{ab} = C^{(5)}_{ABCD} n^C n^D g^A_a g^B_b$, where n_A is the normal to the hypersurface $\chi = 0$ ($n_A n^A = -1$).

Although the whole dynamics are 5‑D and given by (1), from the 4‑D point of view $E^{(5)}_{ab}$ is a non‑local source term that carries bulk effects onto the brane.

The energy‑momentum tensor T_{ab} is assumed to be conserved on the brane:

$$\nabla^b T_{ab} = 0 , \quad (6)$$

and on using the 4‑D contracted Bianchi identities $\nabla^b G_{ab} = 0$ an additional constraint is obtained:

$$\nabla^a E^{(5)}_{ab} = \frac{6\kappa^2}{\lambda} \nabla^b S_{ab} , \quad (7)$$

which shows how the non‑local bulk effects are sourced by the evolution and spatial inhomogeneity of the brane matter content.

B. Cosmological Dynamics on the Brane

In the following we describe the matter on the brane by a perfect fluid with barotropic equation of state $p = (\gamma - 1)\rho$. As usual, we require $\gamma \geq 0$ to satisfy the dominant energy condition and $\gamma \leq 2$ to preserve causality and therefore we restrict our analysis to values of $0 < \gamma \leq 2$. The case $\gamma = 0$ can be treated similarly to our analysis below, but using different variables. We do not study this special case here, but refer the reader to [23] for details on how to treat this case.

If u^a is the matter 4‑velocity and $h_{ab} = g_{ab} + u_a u_b$ projects into the comoving rest space of a fundamental observer, the brane energy momentum tensor is given by

$$T_{ab} = \rho \, u_a u_b + p h_{ab} . \quad (8)$$

One can also decompose $E^{(5)}_{ab}$ in such a way that it is equivalent to a trace-less energy momentum tensor with energy density ρ^*, energy flux q^*_a and anisotropic pressure π^*_{ab} (see [15] for details):

$$-\tfrac{1}{\kappa^2} E^{(5)}_{ab} = \rho^*(u_a u_b + \tfrac{1}{3} h_{ab}) + q^*_a u_b + q^*_b u_a + \pi^*_{ab} . \quad (9)$$

Since there is no evolution equation for the non‑local anisotropic pressure π^*_{ab} we restrict our analysis to the case $\pi^*_{ab} = 0$. This is dynamically justified in early time

regimes [15, 17, 21]. Note in particular, that using (3) the total energy density is given by

$$\rho^{\text{tot}} = \rho + \frac{1}{2\lambda}\rho^2 + \rho^* \,. \tag{10}$$

In this way we can see that there are three essentially different energy regimes: when $\rho \gg \frac{\rho^2}{\lambda}, \rho^*$ we recover general relativity (GR). When $\frac{\rho^2}{\lambda} \gg \rho, \rho^*$ we obtain the high energy limit, and when $\rho^* \gg \rho, \frac{\rho^2}{\lambda}$ we obtain the dark radiation dominated regime.

Equation (6) gives the usual energy and momentum conservation equations:

$$\dot{\rho} + 3\gamma H \rho = 0 \,, \tag{11}$$
$$(\gamma - 1)\mathrm{D}_a\rho + \gamma\rho A_a = 0 \,, \tag{12}$$

where a dot denotes $u^b\nabla_b$, $H = 3\mathrm{D}^a u_a$ is the Hubble parameter of the background, $A_b = \dot{u}_b$ is the 4-acceleration, and D_a denotes the spatially projected covariant derivative.

Using equations (6) and (7) we can obtain conservation equations for the non-local quantities ρ^* and q_a^*. Restricting to linear perturbations of Robertson-Walker models [22, 23, 24, 25], we obtain [35]:

$$\dot{\rho}^* + 4H\rho^* + \mathrm{D}^a q_a^* = 0 \,, \tag{13}$$
$$\dot{q}_a^* + 4Hq_a^* + \tfrac{1}{3}\mathrm{D}_a\rho^* + \tfrac{4}{3}\rho^* A_a = -\tfrac{\rho}{\lambda}\gamma\mathrm{D}_a\rho \,. \tag{14}$$

Finally we note that the generalised Friedmann equation on the brane for a flat, homogeneous isotropic background with vanishing 4-D cosmological constant Λ is

$$H^2 = \frac{\kappa^2\rho}{3} + \frac{\kappa^2\rho^2}{6\lambda} + \frac{\kappa^2\rho^*}{3} \,. \tag{15}$$

III.   COSMOLOGICAL PERTURBATIONS

In the following sections we will present a complete description of general large-scale inhomogeneous perturbations for several different flat ($K = 0$) homogeneous isotropic background models with vanishing cosmological constant on the brane ($\Lambda = 0$).

Following the dynamical systems approach developed in [31, 32, 33] and extended to the braneworld scenario in [4, 6] we define the dimensionless density parameter

$$\Omega_\rho \equiv \frac{\kappa^2\rho}{3H^2} \tag{16}$$

as in general relativity, and

$$\Omega_\lambda \equiv \frac{\kappa^2\rho^2}{6\lambda H^2}, \quad \Omega_{\rho^*} \equiv \frac{\kappa^2\rho^*}{3H^2} \tag{17}$$

corresponding to the non-GR contributions to the Friedmann equation. In this way we can classify the various background solutions by their coordinates $(\Omega_\rho\,\Omega_\lambda, \Omega_{\rho^*})$ in the phase space of Friedmann-Robertson-Walker (FRW) models.

The point $(1, 0, 0)$ corresponds to the flat GR Friedmann model with $\lambda^{-1} = \rho^* = 0$ and $a(t) \sim t^{2/3\gamma}$. The point $(0, 1, 0)$ corresponds to the high energy model $\mathcal{F}_b$ with $\rho^* = 0$ and $\rho \ll \rho^2/\lambda$; the scale factor is given by $a(t) \sim t^{1/3\gamma}$, which can be found by a limiting process [7, 16]. Finally the point $(0, 0, 1)$ corresponds to a model (R) with scale factor evolution $a(t) \sim t^{1/2}$, satisfying $\rho = \lambda^{-1} = 0$.

In what follows, we develop the perturbation equations for a general background model $(\Omega_\rho\,\Omega_\lambda, \Omega_{\rho^*})$ and decompose the propagation and constraint equations into their respective scalar, vector and tensor contributions in the usual way. We then evaluate these equations for the three backgrounds described above and interpret the results.

A.   Dimensionless Variables

The projected 4-D field equations (2) can be covariantly split using the Ricci identities and the Bianchi identities [15]. In the previous section we have already given the conservation equations for energy and momentum (11), (12) and for the non-local energy density ρ^* and flux q_a^* (13), (14). The remaining equations correspond to propagation and constraint equations for the kinematic quantities i.e., the acceleration A_b, the vorticity ω_b and the shear σ_{ab}, together with the electric and magnetic parts of the Weyl tensor E_{ab}, H_{ab} corresponding to the non-local gravitational field on the brane.

Instead of using the standard quantities we define dimensionless expansion normalised variables by

$$W_a \equiv \frac{\omega_a}{H} \,, \ \Sigma_{ab} \equiv \frac{\sigma_{ab}}{H} \,, \ \mathcal{E}_{ab} \equiv \frac{E_{ab}}{H^2} \,, \ \mathcal{H}_{ab} \equiv \frac{H_{ab}}{H^2} \,. \tag{18}$$

We emphasise that E_{ab} and the expansion normalised quantity $\mathcal{E}_{ab}$ must not be confused with the 5-D Weyl tensor $E_{ab}^{(5)}$. It turns out that using the dimensionless vorticity variable

$$W_a^* \equiv aHW_a \tag{19}$$

simplifies the calculations below; however, it should be noted that W_a and not W_a^* is the physically relevant quantity. We also use the dimensionless logarithmic time derivative τ, defined by

$$\{\}' = \frac{d}{d\tau} = \frac{d}{d\,ln(a)} = \frac{1}{H}\frac{d}{dt} \,. \tag{20}$$

Density perturbations are physically characterised by the comoving fractional density gradient defined by [22]:

$$\Delta_a \equiv \frac{a}{\rho}\mathrm{D}_a\rho \,. \tag{21}$$

In addition, it is convenient to define the following dimensionless gradients describing inhomogeneity in the expansion rate H and in the non-local energy density ρ^* and flux q_a^*

$$Z_a^* \equiv \frac{3a}{H}\mathrm{D}_a H \,, \ U_a^* \equiv \frac{\kappa^2 a}{H^2}\mathrm{D}_a\rho^* \,, Q_a^* \equiv \frac{\kappa^2 a}{H}q_a^* \,. \tag{22}$$

Note that although Δ_a is not defined for the exact dark radiation background (R) where $\rho = 0$, it is is well-defined in a neighbourhood of (R) and since (R) is a saddle point in the phase space of the background homogeneous models [4], this solution can never be exactly attained. It is therefore sufficient to study arbitrarily small but non-zero inhomogeneous perturbations of this model. In other words we may evaluate the perturbation equations *arbitrarily close* to the background (R), but not *on* the exact background itself.

The above discussion suggests that it makes more sense to define specially tailored inhomogeneity variables by normalising them with respect to the dominant energy density term in the Friedmann equation.

For the low-energy limit we use the usual dimensionless perturbation quantities :

$$\Delta_a^{(\mathrm{LE})} \equiv \frac{a}{\rho}\mathrm{D}_a\rho,\ U_a^{(\mathrm{LE})} \equiv \frac{a}{\rho}\mathrm{D}_a\rho^*,\quad Q_a^{(\mathrm{LE})} \equiv \frac{1}{\rho}q_a^*,\quad (23)$$

however since ρ_* is the dominant term in the *dark radiation* dominated regime, the appropriate inhomogeneity variables are

$$\Delta_a^{(\mathrm{DE})} \equiv \frac{a}{\rho^*}\mathrm{D}_a\rho,\ U_a^{(\mathrm{DE})} \equiv \frac{a}{\rho^*}\mathrm{D}_a\rho^*,\quad Q_a^{(\mathrm{DE})} \equiv \frac{1}{\rho^*}q_a^*.$$
$$(24)$$

Finally in the high-energy limit we define

$$\Delta_a^{(\mathrm{HE})} \equiv \frac{\lambda a}{\rho^2}\mathrm{D}_a\rho\,,\ U_a^{(\mathrm{HE})} \equiv \frac{\lambda a}{\rho^2}\mathrm{D}_a\rho^*\,,\quad Q_a^{(\mathrm{HE})} \equiv \frac{\lambda}{\rho^2}q_a^*\,.$$
$$(25)$$

since the leading energy density term is proportional to ρ^2/λ, which becomes our normalisation factor in this case.

When decomposing the equations into harmonics we will have to deal with the curls of some of the variables. One approach would be to eliminate the curls by deriving second and higher order equations, or alternatively introduce new spatial harmonics corresponding to the curls of the original harmonics. Instead, we find it more convenient to define new variables corresponding to the curls of the original quantities and derive propagation and constraint equations for them. In this way we obtain a complete closed set of linear differential equations which can be easily solved. Note that *all* of the additional propagation and constraint equations have to be satisfied, since these equations are necessary to close the system.

We denote the curl of a quantity with an overbar and the key variables of this type are:

$$\bar{W}_a^* \equiv \frac{1}{H}\mathrm{curl}\,W_a^*\,,\ \bar{\Sigma}_{ab} \equiv \frac{1}{H}\mathrm{curl}\,\Sigma_{ab}\,,$$

$$\bar{\mathcal{E}}_{ab} \equiv \frac{1}{H}\mathrm{curl}\,\mathcal{E}_{ab}\,,\bar{\mathcal{H}}_{ab} \equiv \frac{1}{H}\mathrm{curl}\,\mathcal{H}_{ab} \quad (26)$$

and

$$\bar{Q}_a^* \equiv \frac{1}{H}\mathrm{curl}\,Q_a^*\,. \quad (27)$$

B. Dimensionless Linearized Propagation and Constraint Equations

The complete set of propagation and constraint equations for the kinematic and non-local quantities on the brane were developed in [15]. Here we extend this work by presenting the complete set of evolution and constraint equations for the dimensionless variables defined by (18-19) and (26).

We begin with the generalised Raychaudhuri equation

$$-\frac{H'}{H} = (1+q) + \frac{\gamma-1}{\gamma}\frac{1}{3aH^2}\mathrm{D}^a\Delta_a, \quad (28)$$

where

$$q = \tfrac{3}{2}\gamma\Omega_\rho + 3\gamma\Omega_\lambda + 2\Omega_{\rho^*} - 1$$

is the usual deceleration parameter.

The remaining propagation equations are given by

$$W_a^{*\prime} = (3\gamma - 4)W_a^*\,,$$
$$\Sigma_{ab}' = (q-1)\Sigma_{ab} - \mathcal{E}_{ab} - \tfrac{\gamma-1}{\gamma}(aH)^{-2}\mathrm{D}_{<a}\Delta_{b>}\,,$$
$$\mathcal{E}_{ab}' = (2q-1)\mathcal{E}_{ab} - (q+1)\Sigma_{ab}$$
$$\qquad - \tfrac{1}{2aH^2}\mathrm{D}_{<a}Q^*_{\,b>} + \bar{\mathcal{H}}_{ab}\,,$$
$$\mathcal{H}_{ab}' = (2q-1)\mathcal{H}_{ab} - \bar{\mathcal{E}}_{ab}\,, \quad (29)$$

which are subject to following dimensionless constraints:

$$a\mathrm{D}^b W_b^* = 0\,,$$
$$a\mathrm{D}^b\Sigma_{ab} = \bar{W}_a^* + \tfrac{2}{3}Z_a^* - Q_a^*\,,$$
$$H\bar{\Sigma}_{ab} = -\tfrac{1}{aH}\mathrm{D}_{<a}W_{b>}^* + H\mathcal{H}_{ab}\,,$$
$$a\mathrm{D}^b\mathcal{E}_{ab} = (\Omega_\rho + 2\Omega_\lambda)\Delta_a + \tfrac{1}{3}U_a^* - Q_a^*\,,$$
$$a\mathrm{D}^b\mathcal{H}_{ab} = 2(q+1)W_a^* - \tfrac{1}{2}\bar{Q}_a^*\,, \quad (30)$$

where the angle bracket is defined by

$$D_{<a}W_{b>}^* \equiv D_{(a}W_{b)}^* - \tfrac{1}{3}D^c W_c^* h_{ab}\,. \quad (31)$$

Equations for the inhomogeneity variables Δ_a, Z_a^*, Q_a^* and Q_a^* are given by

$$\Delta_a{}' = (3\gamma - 3)\Delta_a - \gamma Z_a^*\,,$$
$$Z_a^{*\prime} = (q-1)Z_a^* - \tfrac{3}{2}[\,\Omega_\rho + (6\gamma+2)\Omega_\lambda - \tfrac{4\gamma-4}{\gamma}\Omega_{\rho^*}\,]\,\Delta_a$$
$$\qquad - U_a^* - 6(\gamma-1)\bar{W}_a - \tfrac{\gamma-1}{\gamma}\tfrac{1}{H^2}\mathrm{D}^2\Delta_a\,,$$
$$Q_a^{*\prime} = (q-2)Q_a^* - \tfrac{1}{3}U_a^* + (4\tfrac{\gamma-1}{\gamma}\Omega_{\rho^*} - 6\gamma\Omega_\lambda)\Delta_a\,,$$
$$U_a^{*\prime} = (2q-2)U_a^* - 4\Omega_{\rho^*}Z_a^*$$
$$\qquad + \tfrac{12\gamma-12}{\gamma}\Omega_{\rho^*}\Delta_a - \tfrac{1}{H^2}\mathrm{D}_a(\mathrm{D}^b Q_b^*)\,. \quad (32)$$

Finally, using the definitions (26) and (27), we obtain equations for the curls of the original quantities:

$$\bar{W}^{*\prime}_a = (q + 3\gamma - 4)\bar{W}_a^*\,,$$
$$\bar{\Sigma}_{ab}' = (2q-1)\bar{\Sigma}_{ab} - \bar{\mathcal{E}}_{ab}\,,$$
$$\bar{\mathcal{E}}_{ab}' = (3q-1)\bar{\mathcal{E}}_{ab} - (q+1)\bar{\Sigma}_{ab} + \tfrac{3}{2H^2}\mathrm{D}_{<a}\mathrm{D}^c\mathcal{H}_{b>c}$$
$$\qquad - \tfrac{1}{4aH^2}\mathrm{D}_{<a}\bar{Q}_{b>}^* - \tfrac{1}{H^2}\mathrm{D}^2\mathcal{H}_{ab}\,,$$
$$\bar{\mathcal{H}}_{ab}' = (3q-1)\bar{\mathcal{H}}_{ab} - \tfrac{3}{2H^2}\mathrm{D}_{<a}\mathrm{D}^c\mathcal{E}_{b>c} + \tfrac{1}{H^2}\mathrm{D}^2\mathcal{E}_{ab}\,,$$
$$\bar{Q}^{*\prime}_a = (2q-2)\bar{Q}_a^* + 4[(6\gamma-8)\Omega_{\rho^*} - 9\gamma^2\Omega_\lambda]W_a^* \quad (33)$$

which are subject to the following constraints (obtained by taking the curls of (30)):

$$a\mathrm{D}^b\bar{W}_b^* = 0\,,$$
$$a\mathrm{D}^b\bar{\Sigma}_{ab} = 2(q+1)W_a^* - \tfrac{1}{2}\bar{Q}_a^* - \tfrac{1}{2H^2}\mathrm{D}^2 W_a^*\,,$$
$$\bar{\mathcal{H}}_{ab} = \tfrac{1}{2aH}\mathrm{D}_{<a}\bar{W}_{b>}^* - \tfrac{1}{H^2}\mathrm{D}^2\Sigma_{ab} + \tfrac{3}{2H^2}\mathrm{D}_{<a}\mathrm{D}^c\Sigma_{b>c}\,,$$
$$a\mathrm{D}^b\bar{\mathcal{E}}_{ab} = 2(q+1)W_a^* - \tfrac{1}{2}\bar{Q}_a^*\,,$$
$$a\mathrm{D}^b\bar{\mathcal{H}}_{ab} = (q+1)\bar{W}_a^* + \tfrac{1}{4H^2}(\mathrm{D}^2 Q_a^* - \mathrm{D}_a(\mathrm{D}^b Q_b^*))\,,$$
$$a\mathrm{D}^a\bar{Q}_a^* = 0\,. \tag{34}$$

IV. HARMONIC DECOMPOSITION

In order to solve these equation we employ the standard approach of expanding the variables in these equations in terms of scalar (S), vector (V) and tensor (T) harmonics Q [36]. These harmonics are eigenfunctions of the covariantly defined Laplace-Beltrami operator [24]:

$$\mathrm{D}^2 Q \equiv \mathrm{D}_a\mathrm{D}^a Q = -\tfrac{k^2}{a^2}Q\,, \tag{35}$$

where k is the wave number corresponding to a comoving scale $\lambda \equiv 2\pi a/k$. This yields a covariant and gauge invariant splitting into three sets of evolution and constraint equations for scalar, vector and tensor modes.

Thus a scalar X, vector X^a (orthogonal to u^a) and tensor X^{ab} (orthogonal to u^a) can be expanded as follows

$$X = X^S Q^S\,,$$
$$X_a = k^{-1}X^S Q_a^S + X^V Q_a^V\,,$$
$$X_{ab} = k^{-2}X^S Q_{ab}^S + k^{-1}X^V Q_{ab}^V + X^T Q_{ab}^T\,. \tag{36}$$

In what follows we drop the subscripts S, V, T and also restrict our analysis to the long wavelength limit defined by $\frac{k^2}{a^2 H^2} << 1$.

A. Scalar Perturbations

In the long wavelength limit the scalar evolution equations for the kinematics (which follow after expanding (29) in terms of scalar harmonics) are given by

$$\Sigma' = (q-1)\Sigma - \mathcal{E}\,,$$
$$\mathcal{E}' = (2q-1)\mathcal{E} - (q+1)\Sigma\,, \tag{37}$$

and these are subject to the following constraints (which follow from (30)):

$$W = \bar{W} = \mathcal{H} = \bar{\mathcal{H}} = \bar{\Sigma} = \bar{\mathcal{E}} = \bar{Q} = 0\,,$$
$$2Z_* = 2\Sigma + 3Q_*\,,$$
$$3(\Omega_\rho + 2\Omega_\lambda)\Delta = 2\mathcal{E} + 3Q_* - U_*\,. \tag{38}$$

In addition, we have scalar evolution equations for the inhomogeneity variables (which follow from (32) after harmonic analysis):

$$\Delta' = (3\gamma - 3)\Delta - \gamma Z_*\,,$$

$$Z_*' = (q-1)Z_* - [\tfrac{3}{2}\Omega_\rho + (9\gamma+3)\Omega_\lambda \tag{39}$$
$$- 6\tfrac{\gamma-1}{\gamma}\Omega_{\rho^*}]\Delta - U_*\,,$$
$$Q_*' = (q-2)Q_* - \tfrac{1}{3}U_* + (4\tfrac{\gamma-1}{\gamma}\Omega_{\rho^*} - 6\gamma\Omega_\lambda)\Delta\,,$$
$$U_*' = (2q-2)U_* - 4\Omega_{\rho^*}Z_* + 12\tfrac{\gamma-1}{\gamma}\Omega_{\rho^*}\Delta\,. \tag{40}$$

B. Vector Perturbations

Expanding equations (29) and (33) in terms of vector harmonics, we obtain the following evolution equations for the kinematic and non-local quantities together with their curls:

$$W_*' = (3\gamma - 4)W_*\,,$$
$$\Sigma' = (q-1)\Sigma - \mathcal{E} - 2(\gamma-1)\bar{W}_*\,,$$
$$\mathcal{E}' = (2q-1)\mathcal{E} - (q+1)\Sigma + \bar{\mathcal{H}}\,,$$
$$\mathcal{H}' = (2q-1)\mathcal{H} - \bar{\mathcal{E}}\,,$$
$$\bar{W}_*' = (3\gamma - 4 + q)\bar{W}_*\,,$$
$$\bar{\Sigma}' = (2q-1)\bar{\Sigma} - \bar{\mathcal{E}}\,,$$
$$\bar{\mathcal{E}}' = (3q-1)\bar{\mathcal{E}} - (q+1)\bar{\Sigma}\,,$$
$$\bar{\mathcal{H}}' = (3q-1)\bar{\mathcal{H}}\,,$$
$$\bar{Q}_*' = (2q-2)\bar{Q}_* + 4[(6\gamma-8)\Omega_{\rho^*} - 9\gamma^2\Omega_\lambda]W_*\,.$$

These equations are subject to the following constraints, which are obtained from (30) and (34):

$$4Z_* = 3\Sigma + 6Q_* - 6\bar{W}_*\,,$$
$$6(\Omega_\rho + 2\Omega_\lambda)\Delta = 3\mathcal{E} + 6Q_* - 2U_*\,,$$
$$\bar{\Sigma} = \mathcal{H}\,,$$
$$\mathcal{H} = 4(q+1)W_* - \bar{Q}_*$$
$$\bar{\mathcal{H}} = 2(q+1)\bar{W}_* = 0\,,$$
$$\bar{\Sigma} = 4(q+1)W_* - \bar{Q}_*\,,$$
$$\bar{\mathcal{E}} = 4(q+1)W_* - \bar{Q}_*\,. \tag{41}$$

Equations for the vector parts of the inhomogeneity variables follow from (32) and are given by

$$\Delta' = (3\gamma - 3)\Delta - \gamma Z_*\,,$$
$$Z_*' = (q-1)Z_* - [\tfrac{3}{2}\Omega_\rho + 3(3\gamma+1)\Omega_\lambda - 6\tfrac{\gamma-1}{\gamma}\Omega_{\rho^*}]\Delta$$
$$- U_* - 6(\gamma-1)\bar{W}_*\,,$$
$$Q_*' = (q-2)Q_* - \tfrac{1}{3}U_* + (4\tfrac{\gamma-1}{\gamma}\Omega_{\rho^*} - 6\gamma\Omega_\lambda)\Delta\,,$$
$$U_*' = (2q-2)U_* - 4\Omega_{\rho^*}Z_* + 12\tfrac{\gamma-1}{\gamma}\Omega_{\rho^*}\Delta\,. \tag{42}$$

C. Tensor Perturbations

Finally the long wavelength behaviour of tensor perturbations are obtained by expanding (29) and (33) in terms of tensor harmonics:

$$\Sigma' = (q-1)\Sigma - \mathcal{E}\,,$$
$$\mathcal{E}' = (2q-1)\mathcal{E} - (q+1)\Sigma\,,$$

$$\mathcal{H}' = (2q-1)\mathcal{H} - \bar{\mathcal{E}},$$
$$\bar{\Sigma}' = (2q-1)\bar{\Sigma} - \bar{\mathcal{E}},$$
$$\bar{\mathcal{E}}' = (3q-1)\bar{\mathcal{E}} - (q+1)\bar{\Sigma}, \qquad (43)$$

subject to the following constraints

$$\bar{\mathcal{H}} = 0, \quad \bar{\Sigma} = \mathcal{H}. \qquad (44)$$

V. LOW ENERGY LIMIT: THE GR BACKGROUND

We begin with perturbations in the low‑energy limit, defined by $\rho \gg \rho^2/\lambda$ and $\rho \gg \rho^*$ or $\Omega_\rho \gg \Omega_\lambda$ and $\Omega_\rho \gg \Omega_{\rho^*}$. We therefore evaluate the perturbation equations (38-43) in the limit $(\Omega_\rho, \Omega_\lambda, \Omega_{\rho^*}) \to (1, 0, 0)$. Using the energy conservation equation (11), the Friedmann equation (15) can be solved to give the background scale factor a and the Hubble parameter H:

$$a(t) = (t/t_0)^{2/3\gamma}, \quad H = H_0 a^{-3\gamma/2}, \qquad (45)$$

where we fix an arbitrary initial condition by choosing $a_0 = a(t_0) = 1$. The deceleration parameter is given by

$$q = \tfrac{3}{2}\gamma - 1, \qquad (46)$$

and, as usual,

$$\rho = \rho_0 a^{-3\gamma}, \qquad (47)$$

where $H_0^2 = \frac{\kappa^2}{3}\rho_0$.

A. Scalar Perturbations

In this case the propagation equations for the kinematic quantities are

$$\Sigma' = (\tfrac{3}{2}\gamma - 2)\Sigma - \mathcal{E},$$
$$\mathcal{E}' = (3\gamma - 3)\mathcal{E} - \tfrac{3}{2}\gamma\Sigma, \qquad (48)$$

while the constraints are given by

$$W = \bar{W} = \mathcal{H} = \bar{\mathcal{H}} = \bar{\Sigma} = \bar{\mathcal{E}} = \bar{Q} = 0,$$
$$2Z_* = 2\Sigma + 3Q_*,$$
$$3\Delta = 2\mathcal{E} + 3Q_* - U_*, \qquad (49)$$

and the equations for the inhomogeneity variables become

$$\Delta' = (3\gamma - 3)\Delta - \gamma Z_*,$$
$$Z_*' = (\tfrac{3}{2}\gamma - 2)Z_* - \tfrac{3}{2}\Delta - U_*,$$
$$Q_*' = (\tfrac{3}{2}\gamma - 3)Q_* - \tfrac{1}{3}U_*,$$
$$U_*' = (3\gamma - 4)U_*. \qquad (50)$$

The above equations can be easily solved to give

$$\Sigma = \Sigma_0 a^{3\gamma-2} + \Sigma_1 a^{\frac{3}{2}\gamma-3},$$
$$\mathcal{E} = -\tfrac{3}{2}\gamma\Sigma_0 a^{3\gamma-2} + \Sigma_1 a^{\frac{3}{2}\gamma-3}, \qquad (51)$$

and

$$\Delta = -\gamma\Sigma_0 a^{3\gamma-2} + (\tfrac{2}{3}\Sigma_1 + Q_0^*)a^{\frac{3}{2}\gamma-3} - \tfrac{\gamma}{3\gamma-2}U_0^* a^{3\gamma-4},$$
$$Z_* = \Sigma_0 a^{3\gamma-2} + (\Sigma_1 + \tfrac{3}{2}Q_0^*)a^{\frac{3}{2}\gamma-3} - \tfrac{1}{3\gamma-2}U_0^* a^{3\gamma-4},$$
$$Q_* = Q_0^* a^{\frac{3}{2}\gamma-3} - \tfrac{2}{3}\tfrac{1}{3\gamma-2}U_0^* a^{3\gamma-4},$$
$$U_* = U_0^* a^{3\gamma-4} \qquad (52)$$

for $\gamma \neq \tfrac{2}{3}$, and

$$\Delta = -\tfrac{2}{3}\Sigma_0 + (\tfrac{2}{3}\Sigma_1 + Q_0^* - \tfrac{1}{3}U_0^*)a^{-2} - \tfrac{1}{3}U_0^* \ln a \; a^{-2},$$
$$Z_* = \Sigma_0 + (\Sigma_1 + \tfrac{3}{2}Q_0^*)a^{-2} - \tfrac{1}{2}U_0^* \ln a \; a^{-2},$$
$$Q_* = Q_0^* a^{-2} - \tfrac{1}{3}U_0^* \ln a \; a^{-2},$$
$$U_* = U_0^* a^{-2} \qquad (53)$$

if $\gamma = \tfrac{2}{3}$.

Here Σ_0, Σ_1, Q_0^*, U_0^* are arbitrary constants of integration, corresponding to the four independent modes.

Finally the solutions for Q_*, U_* can be converted into the the physical quantities $Q^{(\mathrm{LE})}$, $U^{(\mathrm{LE})}$ (which correspond to the scalar modes of Q_a, U_a defined in (23)) using the background solutions for H and ρ:

$$Q^{(\mathrm{LE})} = \tfrac{1}{3H_0}Q_0^* a^{3\gamma-4} - \tfrac{2}{9H_0}\tfrac{1}{3\gamma-2}U_0^* a^{\frac{9}{2}\gamma-5},$$
$$U^{(\mathrm{LE})} = \tfrac{1}{3}U_0^* a^{3\gamma-4} \qquad (54)$$

for $\gamma \neq \tfrac{2}{3}$ and

$$Q^{(\mathrm{LE})} = \tfrac{1}{3H_0}Q_0^* a^{-2} - \tfrac{1}{9H_0}U_0^* \ln a \; a^{-2},$$
$$U^{(\mathrm{LE})} = \tfrac{1}{3}U_0^* a^{-2}. \qquad (55)$$

if $\gamma = \tfrac{2}{3}$.

B. Vector Perturbations

For vector perturbations the complete set of propagation equations for the kinematic and non‑local quantities are given by the ten‑dimensional system

$$W_*' = (3\gamma - 4)W_*,$$
$$\Sigma' = (\tfrac{3}{2}\gamma - 2)\Sigma - \mathcal{E},$$
$$\mathcal{E}' = (3\gamma - 3)\mathcal{E} - \tfrac{3}{2}\gamma\Sigma,$$
$$\mathcal{H}' = (3\gamma - 3)\mathcal{H} - \bar{\mathcal{E}},$$
$$\bar{\Sigma}' = (3\gamma - 3)\bar{\Sigma} - \bar{\mathcal{E}},$$
$$\bar{\mathcal{E}}' = (\tfrac{9}{2}\gamma - 4)\bar{\mathcal{E}} - \tfrac{3}{2}\gamma\bar{\Sigma},$$
$$\bar{Q}_*' = (3\gamma - 4)\bar{Q}_*. \qquad (56)$$

subject to the following constraints

$$4Z_* = 3\Sigma + 6Q_*,$$
$$6\Delta = 3\mathcal{E} + 6Q_* - 2U_*,$$
$$\mathcal{H} = \bar{\Sigma} = \bar{\mathcal{E}} = 6\gamma W_* - \bar{Q}_*,$$
$$\bar{\mathcal{H}} = \bar{W}_* = 0, \qquad (57)$$

TABLE I: Large scale contributions of the different modes to the geometric and kinematic quantities for the low energy background. We assume $0 < \gamma \le 2$, $\gamma \ne \frac{2}{3}$, and we omit non-zero constant coefficients. The first line in this table should be read as $\Sigma = \alpha \Sigma_0 a^{3\gamma-2} + \beta \Sigma_1 a^{\frac{3}{2}\gamma-3}$, where α, β are some non-zero constants, for all scalar, vector and tensor modes. General Relativity is recovered when $Q_0^* = U_0^* = 0$.

harmonic	scalar				vector				tensor			
mode	$a^{3\gamma-2}$	$a^{\frac{3}{2}\gamma-3}$	$a^{3\gamma-4}$	$a^{\frac{9}{2}\gamma-5}$	$a^{3\gamma-2}$	$a^{\frac{3}{2}\gamma-3}$	$a^{3\gamma-4}$	$a^{\frac{9}{2}\gamma-5}$	$a^{3\gamma-2}$	$a^{\frac{3}{2}\gamma-3}$	$a^{3\gamma-4}$	$a^{\frac{9}{2}\gamma-3}$
Σ	Σ_0	Σ_1	-	-	Σ_0	Σ_1	-	-	Σ_0	Σ_1	-	-
$\mathcal{E}$	Σ_0	Σ_1	-	-	Σ_0	Σ_1	-	-	Σ_0	Σ_1	-	-
$\mathcal{H}$	-	-	-	-	-	-	$\mathcal{H}_0$	-	-	-	$\mathcal{H}_0$	$\mathcal{H}_1$
W	-	-	-	-	-	-	-	W_0^*	-	-	-	-
Δ	Σ_0	$2\Sigma_1 + 3Q_0^*$	U_0^*	-	Σ_0	$\Sigma_1 + 2Q_0^*$	U_0^*	-	-	-	-	-
Z^*	Σ_0	$2\Sigma_1 + 3Q_0^*$	U_0^*	-	Σ_0	$\Sigma_1 + 2Q_0^*$	U_0^*	-	-	-	-	-
$Q^{(\mathrm{LE})}$	-	-	Q_0^*	U_0^*	-	-	Q_0^*	U_0^*	-	-	-	-
$U^{(\mathrm{LE})}$	-	-	U_0^*	-	-	-	U_0^*	-	-	-	-	-

while the propagation equations for the inhomogeneity variables are

$$\Delta' = (3\gamma - 3)\Delta - \gamma Z_* \,,$$
$$Z_*{}' = (\tfrac{3}{2}\gamma - 2)Z_* - \tfrac{3}{2}\Delta - U_* \,,$$
$$Q_*{}' = (\tfrac{3}{2}\gamma - 3)Q_* - \tfrac{1}{3}U_* \,,$$
$$U_*{}' = (3\gamma - 4)U_* \,. \tag{58}$$

Solutions can again be easily obtained and are given by

$$\Sigma = \Sigma_0 a^{3\gamma-2} + \Sigma_1 a^{\frac{3}{2}\gamma-3} \,,$$
$$\mathcal{E} = -\tfrac{3}{2}\gamma \Sigma_0 a^{3\gamma-2} + \Sigma_1 a^{\frac{3}{2}\gamma-3} \,,$$
$$\mathcal{H} = \bar{\Sigma} = \bar{\mathcal{E}} = \mathcal{H}_0 a^{3\gamma-4} \,,$$
$$W = (aH)^{-1}W_* = \tfrac{3}{2}\gamma W_0^* a^{\frac{9}{2}\gamma-5} \,,$$
$$\bar{Q}_* = (6\gamma W_0^* - \mathcal{H}_0)a^{3\gamma-4} \,, \tag{59}$$

and

$$\Delta = -\tfrac{3}{4}\gamma \Sigma_0 a^{3\gamma-2} + (\tfrac{1}{2}\Sigma_1 + Q_0^*)a^{\frac{3}{2}\gamma-3} - \tfrac{\gamma}{3\gamma-2}U_0^* a^{3\gamma-4} \,,$$
$$Z_* = \tfrac{3}{4}\Sigma_0 a^{3\gamma-2} + \tfrac{3}{4}(\Sigma_1 + 2Q_0^*)a^{\frac{3}{2}\gamma-3} - \tfrac{1}{3\gamma-2}U_0^* a^{3\gamma-4} \,,$$
$$Q^{(\mathrm{LE})} = \tfrac{1}{3H_0}Q_0^* a^{3\gamma-4} - \tfrac{2}{9H_0}\tfrac{1}{3\gamma-2}U_0^* a^{\frac{9}{2}\gamma-5} \,,$$
$$U^{(\mathrm{LE})} = \tfrac{1}{3}U_0^* a^{3\gamma-4} \tag{60}$$

for $\gamma \ne \frac{2}{3}$, and

$$\Delta = -\tfrac{1}{2}\Sigma_0 + (\tfrac{1}{2}\Sigma_1 + Q_0^* - \tfrac{1}{3}U_0^*)a^{-2} - \tfrac{1}{3}U_0^* \ln a \; a^{-2} \,,$$
$$Z_* = \tfrac{3}{4}\Sigma_0 + (\tfrac{3}{4}\Sigma_1 + \tfrac{3}{2}Q_0^*)a^{-2} - \tfrac{1}{2}U_0^* \ln a \; a^{-2} \,,$$
$$Q^{(\mathrm{LE})} = \tfrac{1}{3H_0}Q_0^* a^{-2} - \tfrac{1}{9H_0}U_0^* \ln a \; a^{-2} \,,$$
$$U^{(\mathrm{LE})} = \tfrac{1}{3}U_0^* a^{-2} \tag{61}$$

if $\gamma = \frac{2}{3}$. Again, we have converted the solutions for Q_*, U_* into the physical quantities $Q^{(\mathrm{LE})}$, $U^{(\mathrm{LE})}$.

There are six independent modes corresponding to the constants of integration Σ_0, Σ_1, W_0, $\mathcal{H}_0$, Q_0^* and U_0^*.

C.   Tensor Perturbations

For tensor perturbations the propagation equations in the long wavelength limit are

$$\Sigma' = (\tfrac{3}{2}\gamma - 2)\Sigma - \mathcal{E} \,,$$
$$\mathcal{E}' = (3\gamma - 3)\mathcal{E} - \tfrac{3}{2}\gamma \Sigma \,,$$
$$\mathcal{H}' = (3\gamma - 3)\mathcal{H} - \bar{\mathcal{E}} \,,$$
$$\bar{\Sigma}' = (3\gamma - 3)\bar{\Sigma} - \bar{\mathcal{E}} \,,$$
$$\bar{\mathcal{E}}' = (\tfrac{9}{2}\gamma - 4)\bar{\mathcal{E}} - \tfrac{3}{2}\gamma \bar{\Sigma} \,, \tag{62}$$

subject to the following constraints:

$$\bar{\mathcal{H}} = 0 \,, \quad \bar{\Sigma} = \mathcal{H} \,. \tag{63}$$

The solutions are

$$\Sigma = \Sigma_0 a^{3\gamma-2} + \Sigma_1 a^{\frac{3}{2}\gamma-3} \,,$$
$$\mathcal{E} = -\tfrac{3}{2}\gamma \Sigma_0 a^{3\gamma-2} + \Sigma_1 a^{\frac{3}{2}\gamma-3} \,,$$
$$\mathcal{H} = \bar{\Sigma} = \mathcal{H}_0 a^{3\gamma-4} + \mathcal{H}_1 a^{\frac{9}{2}\gamma-3} \,,$$
$$\bar{\mathcal{E}} = \mathcal{H}_0 a^{3\gamma-4} - \tfrac{3}{2}\gamma \mathcal{H}_1 a^{\frac{9}{2}\gamma-3} \,, \tag{64}$$

where Σ_0, Σ_1, $\mathcal{H}_0$, $\mathcal{H}_1$ are four independent constants of integration.

VI.   THE DARK ENERGY ERA

The dark energy dominated regime is characterised by $\rho^* \gg \rho$ and $\rho^* \gg \rho^2/\lambda$ or $\Omega_{\rho^*} \gg \Omega_\rho$ and $\Omega_{\rho^*} \gg \Omega_\lambda$, so we now evaluate the perturbation equations in the limit $(\Omega_\rho, \Omega_\lambda, \Omega_{\rho^*}) \to (0, 0, 1)$.

The background solution (R), has the same metric as a flat radiation FRW model, with $\rho = 0$ and $\rho^* = \rho_0^* a^{-4}$.

The scale factor a, Hubble parameter H and deceleration parameter q are given by

$$a(t) = (t/t_0)^{1/2}, \quad H = H_0 a^{-2}, \quad q = 1. \qquad (65)$$

As explained in section III the perturbation equations are only defined for small but non-zero energy density ρ, or equivalently arbitrarily close but not *on* the exact background (R). We therefore use $\rho = \rho_0 a^{-3\gamma}$ for small but non-zero values of ρ_0.

A. Scalar Perturbations

In the case of scalar perturbations the propagation equations reduce to:

$$\Sigma' = -\mathcal{E}, \qquad (66)$$
$$\mathcal{E}' = \mathcal{E} - 2\Sigma, \qquad (67)$$

subject to the following constraints:

$$W = \bar{W} = \mathcal{H} = \bar{\mathcal{H}} = \bar{\Sigma} = \bar{\mathcal{E}} = \bar{Q} = 0,$$
$$2\Sigma + 3Q_* = 2Z_*,$$
$$2\mathcal{E} + 3Q_* - U_* = 3(\Omega_\rho + 2\Omega_\lambda)\Delta, \qquad (68)$$

while the equations for the inhomogeneity variables are

$$\Delta' = (3\gamma - 3)\Delta - \gamma Z_*,$$
$$Z_*' = 6\frac{\gamma-1}{\gamma}\Delta - U_*,$$
$$Q_*' = -Q_* - \frac{1}{3}U_* + 4\frac{\gamma-1}{\gamma}\Delta,$$
$$U_*' = -4Z_* + 12\frac{\gamma-1}{\gamma}\Delta. \qquad (69)$$

The following solutions can then be obtained:

$$\Sigma = \Sigma_0 a^2 + \Sigma_1 a^{-1},$$
$$\mathcal{E} = -2\Sigma_0 a^2 + \Sigma_1 a^{-1}, \qquad (70)$$

and

$$\Delta = \Delta_0 + \Delta_1 a^{3\gamma-5} + \frac{3\gamma}{3\gamma-7}\Sigma_0 a^2,$$
$$Z_* = \frac{3\gamma-3}{\gamma}\Delta_0 + \frac{2}{\gamma}\Delta_1 a^{3\gamma-5} + 3\frac{3\gamma-5}{3\gamma-7}\Sigma_0 a^2,$$
$$Q_* = \frac{2\gamma-2}{\gamma}\Delta_0 + \frac{4}{3\gamma}\Delta_1 a^{3\gamma-5} + \frac{4}{3}\frac{3\gamma-4}{3\gamma-7}\Sigma_0 a^2 - \frac{2}{3}\Sigma_1 a^{-1},$$
$$U_* = \frac{6\gamma-6}{\gamma}\Delta_0 + \frac{4}{\gamma}\Delta_1 a^{3\gamma-5} + \frac{12}{3\gamma-7}\Sigma_0 a^2 \qquad (71)$$

for $\gamma \neq \frac{7}{3}$. We do not give the solutions for $\gamma = \frac{7}{3}$, since all values of $\gamma > 2$ are outside the region of interest.

There are four constants of integration Σ_0, Σ_1, Δ_0, Δ_1 corresponding to the four independent modes.

The scalar contributions to the physical quantities defined in (24) can then easily be obtained:

$$Q^{(\mathrm{DE})} = \frac{2\gamma-2}{3H_0\gamma}\Delta_0 a + \frac{4}{9H_0\gamma}\Delta_1 a^{3\gamma-4} + \frac{4}{9H_0}\frac{3\gamma-4}{3\gamma-7}\Sigma_0 a^3$$
$$- \frac{2}{9H_0}\Sigma_1,$$
$$U^{(\mathrm{DE})} = U_*. \qquad (72)$$

In particular, the density perturbation $\Delta^{(\mathrm{DE})}$ can be written as $\Omega_{\rho^*}\Delta^{(\mathrm{DE})} = \Omega_\rho\Delta$, and using the fact that $\Omega_\rho, \Omega_\lambda \geq 0$, we find that $\Omega_\rho\Delta = \Omega_\lambda\Delta \to 0$ as $\rho \to 0$. Hence the density perturbations are given by

$$\Delta^{(\mathrm{DE})} = \Omega_\rho\Delta_0 + \Omega_\rho\Delta_1 a^{3\gamma-5} + \frac{3\gamma}{3\gamma-7}\Omega_\rho\Sigma_0 a^2 \qquad (73)$$

and are suppressed as one approaches the dark energy solution.

B. Vector Perturbations

In the case of vector perturbations the propagation equations are

$$W_*' = (3\gamma - 4)W_*, \qquad (74)$$
$$\Sigma' = -\mathcal{E}, \qquad (75)$$
$$\mathcal{E}' = \mathcal{E} - 2\Sigma, \qquad (76)$$
$$\mathcal{H}' = \bar{\Sigma}' = \bar{\mathcal{E}}' = 0, \qquad (77)$$
$$\bar{Q}_*' = 8(3\gamma - 4)W_*, \qquad (78)$$

which are subject to the following constraints

$$3\Sigma + 6Q_* = 4Z_*,$$
$$3\mathcal{E} + 6Q_* = 2U_* + 6(\Omega_\rho + 2\Omega_\lambda)\Delta,$$
$$\bar{\Sigma} = \mathcal{H},$$
$$\mathcal{H} = 8W_* - \bar{Q}_* = 0,$$
$$\bar{\mathcal{H}} = \bar{W}_* = 0,$$
$$\bar{\Sigma} = \bar{\mathcal{E}} = 8W_* - \bar{Q}_*. \qquad (79)$$

The inhomogeneity variables evolve according to

$$\Delta' = (3\gamma - 3)\Delta - \gamma Z_*,$$
$$Z_*' = 6\frac{\gamma-1}{\gamma}\Delta - U_*,$$
$$Q_*' = -Q_* - \frac{1}{3}U_* + 4\frac{\gamma-1}{\gamma}\Delta,$$
$$U_*' = -4Z_* + 12\frac{\gamma-1}{\gamma}\Delta. \qquad (80)$$

The solutions to this system are

$$\Sigma = \Sigma_0 a^2 + \Sigma_1 a^{-1},$$
$$\mathcal{E} = -2\Sigma_0 a^2 + \Sigma_1 a^{-1},$$
$$\mathcal{H} = \bar{\Sigma} = \bar{\mathcal{E}} = \mathcal{H}_0,$$
$$W = 2W_0^* a^{3\gamma-3},$$
$$\bar{Q}_* = 8W_0^* a^{3\gamma-4} + \mathcal{H}_0, \qquad (81)$$

and

$$\Delta = \Delta_0 + \Delta_1 a^{3\gamma-5} + \frac{3}{4}\frac{3\gamma}{3\gamma-7}\Sigma_0 a^2,$$
$$Z_* = \frac{3\gamma-3}{\gamma}\Delta_0 + \frac{2}{\gamma}\Delta_1 a^{3\gamma-5} + \frac{9}{4}\frac{3\gamma-5}{3\gamma-7}\Sigma_0 a^2,$$
$$Q_* = \frac{2\gamma-2}{\gamma}\Delta_0 + \frac{4}{3\gamma}\Delta_1 a^{3\gamma-5} + \frac{3\gamma-4}{3\gamma-7}\Sigma_0 a^2 - \frac{1}{2}\Sigma_1 a^{-1},$$
$$U_* = \frac{6\gamma-6}{\gamma}\Delta_0 + \frac{4}{\gamma}\Delta_1 a^{3\gamma-5} + \frac{9}{3\gamma-7}\Sigma_0 a^2, \qquad (82)$$

where again $\gamma \neq \frac{7}{3}$.

This time there are six constants of integration: Σ_0, Σ_1, $\mathcal{H}_0$, W_0^*, Q_0^*, Δ_1.

The vector modes of the physical quantities defined in (24) can then be found:

$$\Delta^{(DE)} = \frac{\kappa^2 \rho_0}{3H_0^2}\Delta_0 a^{4-3\gamma} + \frac{\kappa^2 \rho_0}{3H_0^2}\Delta_1 a^{-1} + \frac{\kappa^2 \rho_0}{4H_0^2}\frac{3\gamma}{3\gamma-7}\Sigma_0 a^{6-3\gamma} ,$$

$$Q^{(DE)} = \frac{2\gamma-2}{3H_0\gamma}\Delta_0 a + \frac{4}{9H_0\gamma}\Delta_1 a^{3\gamma-4} + \frac{4}{9H_0}\frac{3\gamma-4}{3\gamma-7}\Sigma_0 a^3$$
$$- \frac{2}{9H_0}\Sigma_1 ,$$

$$U^{(DE)} = U_* . \tag{83}$$

C. Tensor Perturbations

The tensor parts of the propagation equations in the long wavelength limit are:

$$\Sigma' = -\mathcal{E} ,$$
$$\mathcal{E}' = \mathcal{E} - 2\Sigma ,$$
$$\mathcal{H}' = \mathcal{H} - \bar{\mathcal{E}} ,$$
$$\bar{\Sigma}' = \bar{\Sigma} - \bar{\mathcal{E}} ,$$
$$\bar{\mathcal{E}}' = 2\bar{\mathcal{E}} - 2\bar{\Sigma} , \tag{84}$$

subject to the following constraints:

$$\bar{\mathcal{H}} = 0 , \quad \bar{\Sigma} = \mathcal{H} . \tag{85}$$

The solutions are

$$\Sigma = \Sigma_0 a^2 + \Sigma_1 a^{-1} ,$$
$$\mathcal{E} = -2\Sigma_0 a^2 + \Sigma_1 a^{-1} ,$$
$$\mathcal{H} = \bar{\Sigma} = \mathcal{H}_0 + \mathcal{H}_1 a^3 ,$$
$$\bar{\mathcal{E}} = \mathcal{H}_0 - 2\mathcal{H}_1 a^3 , \tag{86}$$

with a constant of integration for each of the independent modes: Σ_0, Σ_1, $\mathcal{H}_0$, $\mathcal{H}_1$.

VII. HIGH ENERGY LIMIT: THE $\mathcal{F}_b$ BACKGROUND

The high energy limit is characterised by $\frac{\rho^2}{\lambda} \gg \rho$ and $\frac{\rho^2}{\lambda} \gg \rho^*$ or $\Omega_\lambda \gg \Omega_\rho$ and $\Omega_\lambda \gg \Omega_{\rho^*}$ so this time we evaluate the perturbation equations in the limit $(\Omega_\rho, \Omega_\lambda, \Omega_{\rho^*}) \to (0, 1, 0)$.

This model corresponds to a stationary (equilibrium) point $\mathcal{F}_b$ in the phase space of homogeneous Bianchi models [7, 8], as well as in the phase space of the special class of inhomogeneous G_2 cosmological models. In both cases $\mathcal{F}_b$ is found to be the source, or past attractor, for the generic dynamics for $\gamma > 1$ ($\gamma = 1$ is also included in the homogeneous case), consistent with [3, 4, 5, 6]. The stability of this result is now examined through an analysis of the perturbation equations for this case.

The background scale factor a, Hubble function H and deceleration parameter q of these models are given by

$$a(t) = (t/t_0)^{1/3\gamma} , \quad H = H_0 a^{-3\gamma}, \quad q = 3\gamma - 1 , \tag{87}$$

where again we fix an arbitrary initial condition by choosing $a_0 = a(t_0) = 1$. The energy density behaves in the usual way:

$$\rho = \rho_0 a^{-3\gamma} . \tag{88}$$

From the Friedmann equation (15) we find that $H_0^2 = \frac{\kappa^2}{6\lambda}\rho_0^2$.

A. Scalar Perturbations

The scalar propagation equations for this case reduce to

$$\Sigma' = (3\gamma - 2)\Sigma - \mathcal{E} ,$$
$$\mathcal{E}' = (6\gamma - 3)\mathcal{E} - 3\gamma\Sigma , \tag{89}$$

subject to the constraints

$$W = \bar{W} = \mathcal{H} = \bar{\mathcal{H}} = \bar{\Sigma} = \bar{\mathcal{E}} = \bar{Q} = 0 ,$$
$$2Z_* = 2\Sigma + 3Q_* ,$$
$$6\Delta = 2\mathcal{E} + 3Q_* - U_* . \tag{90}$$

The scalar evolution equations for the inhomogeneity variables are

$$\Delta' = (3\gamma - 3)\Delta - \gamma Z_* ,$$
$$Z_*' = (3\gamma - 2)Z_* - 3(3\gamma + 1)\Delta - U_* ,$$
$$Q_*' = (3\gamma - 3)Q_* - \tfrac{1}{3}U_* - 6\gamma\Delta ,$$
$$U_*' = (6\gamma - 4)U_* . \tag{91}$$

Solutions can again be easily obtained by solving the above system of linear equations. They are

$$\Sigma = \Sigma_0 a^{6\gamma-2} + \Sigma_1 a^{3\gamma-3} ,$$
$$\mathcal{E} = -3\gamma\Sigma_0 a^{6\gamma-2} + \Sigma_1 a^{3\gamma-3} , \tag{92}$$

and

$$\Delta = \tfrac{1}{2}Q_0^* a^{-3} - \frac{\gamma(3\gamma+1)}{6\gamma+1}\Sigma_0 a^{6\gamma-2} - \frac{\gamma}{2(6\gamma-1)}U_0^* a^{6\gamma-4} ,$$

$$Z_* = \tfrac{3}{2}Q_0^* a^{-3} + \frac{(3\gamma+1)^2}{6\gamma+1}\Sigma_0 a^{6\gamma-2} + \frac{3\gamma-1}{2(6\gamma-1)}U_0^* a^{6\gamma-4} ,$$

$$Q_* = Q_0^* a^{-3} + \frac{6\gamma^2}{6\gamma+1}\Sigma_0 a^{6\gamma-2} - \tfrac{2}{3}\Sigma_1 a^{3\gamma-3}$$
$$+ \frac{3\gamma-1}{3(6\gamma-1)}U_0^* a^{6\gamma-4} ,$$

$$U_* = U_0^* a^{6\gamma-4} \tag{93}$$

for $\gamma \neq \frac{1}{6}$, and

$$\Delta = (\tfrac{1}{2}Q_0^* - \tfrac{1}{6}U_0^*)a^{-3} - \tfrac{1}{8}\Sigma_0 a^{-1} - \tfrac{1}{12}U_0^* \ln a \, a^{-3} \tag{94}$$

$$Z_* = \tfrac{3}{2}Q_0^* a^{-3} + \tfrac{9}{8}\Sigma_0 a^{-1} - \tfrac{1}{4}U_0^* \ln a \, a^{-3} , \tag{95}$$

$$Q_* = Q_0^* a^{-3} + \tfrac{1}{12}\Sigma_0 a^{-1} - \tfrac{2}{3}\Sigma_1 a^{-\frac{5}{2}} - \tfrac{1}{6}U_0^* \ln a \, a^{-3} \tag{96}$$

$$U_* = U_0^* a^{-3} \tag{97}$$

for $\gamma = \frac{1}{6}$. The scalar parts of the physical quantities $\Delta^{(\mathrm{HE})}$, $Q^{(\mathrm{HE})}$, $U^{(\mathrm{HE})}$ defined by (25) are given by

$$\Delta^{(\mathrm{HE})} = \frac{\kappa^2 \rho_0}{12 H_0^2} Q_0^* a^{3\gamma-3} - \frac{\gamma(3\gamma+1)}{6\gamma+1} \frac{\kappa^2 \rho_0}{6 H_0^2} \Sigma_0 a^{9\gamma-2}$$
$$- \frac{\gamma}{6\gamma-1} \frac{\kappa^2 \rho_0}{12 H_0^2} U_0^* a^{9\gamma-4} ,$$
$$Q^{(\mathrm{HE})} = \frac{1}{6 H_0} Q_0^* a^{3\gamma-4} + \frac{1}{H_0} \frac{\gamma^2}{6\gamma+1} \Sigma_0 a^{9\gamma-3} - \frac{1}{9 H_0} \Sigma_1 a^{6\gamma-4}$$
$$+ \frac{1}{18 H_0} \frac{3\gamma-1}{6\gamma-1} U_0^* a^{9\gamma-5} ,$$
$$U^{(\mathrm{HE})} = \frac{1}{6} U_0^* a^{6\gamma-4} \tag{98}$$

for $\gamma \neq \frac{1}{6}$, and

$$\Delta^{(\mathrm{HE})} = \frac{\kappa^2 \rho_0}{12 H_0^2} (Q_0^* - \frac{1}{3} U_0^*) a^{-\frac{5}{2}} - \frac{\kappa^2 \rho_0}{48 H_0^2} \Sigma_0 a^{-\frac{1}{2}}$$
$$- \frac{\kappa^2 \rho_0}{72 H_0^2} U_0^* \ln a \; a^{-\frac{5}{2}} ,$$
$$Q^{(\mathrm{HE})} = \frac{1}{6 H_0} Q_0^* a^{-\frac{7}{2}} + \frac{1}{6 H_0} \frac{1}{12} \Sigma_0 a^{-\frac{3}{2}} - \frac{1}{9 H_0} \Sigma_1 a^{-3} ,$$
$$- \frac{1}{36 H_0} U_0^* \ln a \; a^{-\frac{7}{2}}$$
$$U^{(\mathrm{HE})} = \frac{1}{6} U_0^* a^{-3} \tag{99}$$

for $\gamma = \frac{1}{6}$.

Σ_0, Σ_1, Q_0^*, U_0^* are arbitrary constants of integration corresponding to the four independent modes.

B. Vector Perturbations

For vector perturbations the propagation equations are

$$W_*{}' = (3\gamma - 4) W_* ,$$
$$\Sigma' = (3\gamma - 2)\Sigma - \mathcal{E} ,$$
$$\mathcal{E}' = (6\gamma - 3)\mathcal{E} - 3\gamma\Sigma ,$$
$$\mathcal{H}' = (6\gamma - 3)\mathcal{H} - \bar{\mathcal{E}} ,$$
$$\bar{\Sigma}' = (6\gamma - 3)\bar{\Sigma} - \bar{\mathcal{E}} ,$$
$$\bar{\mathcal{E}}' = (9\gamma - 4)\bar{\mathcal{E}} - 3\gamma\bar{\Sigma} ,$$
$$\bar{Q}_*' = (6\gamma - 4)\bar{Q}_* - 36\gamma^2 W_* , \tag{100}$$

subject to the following constraints

$$4 Z_* = 3\Sigma + 6 Q_* ,$$
$$12\Delta = 3\mathcal{E} + 6 Q_* - 2 U_* ,$$
$$\mathcal{H} = \bar{\Sigma} = \bar{\mathcal{E}} = 12\gamma W_* - \bar{Q}_* ,$$
$$\bar{\mathcal{H}} = \bar{W}_* = 0 . \tag{101}$$

The propagation equations characterising the inhomogeneities are

$$\Delta' = (3\gamma - 3)\Delta - \gamma Z_* ,$$
$$Z_*{}' = (3\gamma - 2) Z_* - 3(3\gamma + 1)\Delta - U_* ,$$
$$Q_*{}' = (3\gamma - 3) Q_* - \frac{1}{3} U_* - 6\gamma\Delta ,$$
$$U_*{}' = (6\gamma - 4) U_* . \tag{102}$$

Solutions are easily obtained and are given by

$$\Sigma = \Sigma_0 a^{6\gamma-2} + \Sigma_1 a^{3\gamma-3} ,$$
$$\mathcal{E} = -3\gamma\Sigma_0 a^{6\gamma-2} + \Sigma_1 a^{3\gamma-3} ,$$
$$\mathcal{H} = \bar{\Sigma} = \bar{\mathcal{E}} = \mathcal{H}_0 a^{6\gamma-4} ,$$
$$W = 3\gamma W_0^* a^{6\gamma-5} ,$$
$$\bar{Q}_* = 12\gamma W_0^* a^{3\gamma-4} - \mathcal{H}_0 a^{6\gamma-4} , \tag{103}$$

and

$$\Delta^{(\mathrm{HE})} = \frac{\kappa^2 \rho_0}{12 H_0^2} Q_0^* a^{3\gamma-3} - \frac{\gamma(3\gamma+1)}{6\gamma+1} \frac{\kappa^2 \rho_0}{9 H_0^2} \Sigma_0 a^{9\gamma-2}$$
$$- \frac{\gamma}{6\gamma-1} \frac{\kappa^2 \rho_0}{12 H_0^2} U_0^* a^{9\gamma-4} ,$$
$$Z_* = \frac{3}{2} Q_0^* a^{-3} + \frac{3}{4} \frac{(3\gamma+1)^2}{6\gamma+1} \Sigma_0 a^{6\gamma-2} + \frac{3\gamma-1}{2(6\gamma-1)} U_0^* a^{6\gamma-4} ,$$
$$Q^{(\mathrm{HE})} = \frac{1}{6 H_0} Q_0^* a^{3\gamma-4} + \frac{1}{4 H_0} \frac{3\gamma^2}{6\gamma+1} \Sigma_0 a^{9\gamma-3}$$
$$- \frac{1}{12 H_0} \Sigma_1 a^{6\gamma-4} + \frac{1}{18 H_0} \frac{3\gamma-1}{6\gamma-1} U_0^* a^{9\gamma-5} ,$$
$$U^{(\mathrm{HE})} = \frac{1}{6} U_0^* a^{6\gamma-4} \tag{104}$$

for $\gamma \neq \frac{1}{6}$, and

$$\Delta^{(\mathrm{HE})} = \frac{\kappa^2 \rho_0}{12 H_0^2} (Q_0^* - \frac{1}{3} U_0^*) a^{-\frac{5}{2}} - \frac{\kappa^2 \rho_0}{72 H_0^2} \Sigma_0 a^{-\frac{1}{2}}$$
$$- \frac{\kappa^2 \rho_0}{72 H_0^2} U_0^* \ln a \; a^{-\frac{5}{2}} ,$$
$$Z_* = \frac{3}{2} Q_0^* a^{-3} + \frac{27}{32} \Sigma_0 a^{-1} - \frac{1}{4} U_0^* \ln a \; a^{-3} ,$$
$$Q^{(\mathrm{HE})} = \frac{1}{6 H_0} Q_0^* a^{-\frac{7}{2}} + \frac{1}{6 H_0} \frac{1}{16} \Sigma_0 a^{-\frac{3}{2}} - \frac{1}{12 H_0} \Sigma_1 a^{-3}$$
$$- \frac{1}{36 H_0} U_0^* \ln a \; a^{-\frac{7}{2}} , \tag{105}$$
$$U^{(\mathrm{HE})} = \frac{1}{6} U_0^* a^{-3} \tag{106}$$

for $\gamma = \frac{1}{6}$. There are six constants of integration: Σ_0, Σ_1, $\mathcal{H}_0$, W_0^*, Q_0^*, U_0^*.

C. Tensor Perturbations

The tensor parts of the propagation equations in the long wavelength limit are:

$$\Sigma' = (3\gamma - 2)\Sigma - \mathcal{E} ,$$
$$\mathcal{E}' = (6\gamma - 3)\mathcal{E} - 3\gamma\Sigma ,$$
$$\mathcal{H}' = (6\gamma - 3)\mathcal{H} - \bar{\mathcal{E}} ,$$
$$\bar{\Sigma}' = (6\gamma - 3)\bar{\Sigma} - \bar{\mathcal{E}} ,$$
$$\bar{\mathcal{E}}' = (9\gamma - 4)\bar{\mathcal{E}} - 3\gamma\bar{\Sigma} , \tag{107}$$

subject to the constraints

$$\bar{\mathcal{H}} = 0 , \quad \bar{\Sigma} = \mathcal{H} . \tag{108}$$

The solutions are given by

$$\Sigma = \Sigma_0 a^{6\gamma-2} + \Sigma_1 a^{3\gamma-3} ,$$
$$\mathcal{E} = -3\gamma\Sigma_0 a^{6\gamma-2} + \Sigma_1 a^{3\gamma-3} ,$$
$$\mathcal{H} = \bar{\Sigma} = \mathcal{H}_0 a^{6\gamma-4} + \mathcal{H}_1 a^{9\gamma-3} ,$$
$$\bar{\mathcal{E}} = \mathcal{H}_0 a^{6\gamma-4} - 3\gamma\mathcal{H}_1 a^{9\gamma-3} . \tag{109}$$

There are four constants of integration: Σ_0, Σ_1, $\mathcal{H}_0$, $\mathcal{H}_1$.

TABLE II: Large-scale behaviour of the non-zero physically relevant geometric and kinematic quantities for the different backgrounds in the limit $a \to 0$. The values S, V, T in brackets denote scalar, vector and tensor contributions. One can easily see how the appropriately normalised perturbation quantities defined in (23)-(24) converge for wider ranges of γ as $a \to 0$.

Quantity	mode	$0 < \gamma < \frac{2}{9}$	$< \gamma < \frac{1}{3}$	$< \gamma < \frac{4}{9}$	$< \gamma < \frac{5}{9}$	$< \gamma < \frac{2}{3}$	$< \gamma < \frac{5}{6}$	$< \gamma < 1$	$\gamma < \frac{10}{9}$	$< \gamma < \frac{4}{3}$	$< \gamma < \frac{5}{3}$	$< \gamma < 2$
Low energy limit												
$\Sigma,\ \mathcal{E}$	$\Sigma_0(S,V,T)$	∞	∞	∞	∞	∞	0	0	0	0	0	0
	$\Sigma_1(S,V,T)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
$\mathcal{H}$	$\mathcal{H}_0(V,T)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	0	0
	$\mathcal{H}_1(T)$	∞	∞	∞	∞	∞	0	0	0	0	0	0
W	$W_0^*(V)$	∞	∞	∞	∞	∞	∞	∞	∞	0	0	0
$\Delta = \Delta^{(\mathrm{LE})}$	$\Sigma_0(S,V)$	∞	∞	∞	∞	∞	0	0	0	0	0	0
	$\Sigma_1, Q_0^*(S,V)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
	$U_0^*(S,V)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	0	0
$Q^{(\mathrm{LE})}$	$Q_0^*(S,V)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	0	0
	$U_0^*(S,V)$	∞	∞	∞	∞	∞	∞	∞	∞	0	0	0
$U^{(\mathrm{LE})}$	$U_0^*(S,V)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	0	0
Dark energy limit												
$\Sigma,\ \mathcal{E}$	$\Sigma_0(S,V,T)$	0	0	0	0	0	0	0	0	0	0	0
	$\Sigma_1(S,V,T)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
$\mathcal{H}$	$\mathcal{H}_0(V,T)$	const	const	const	const	const	const	const	const	const	const	const
	$\mathcal{H}_1(T)$	0	0	0	0	0	0	0	0	0	0	0
W	$W_0^*(V)$	∞	∞	∞	∞	∞	∞	∞	0	0	0	0
$\Delta^{(\mathrm{DE})\ a}$	$\rho_0\Sigma_0(S,V)$	0	0	0	0	0	0	0	0	0	0	0
	$\rho_0\Delta_0(S,V)$	0	0	0	0	0	0	0	0	0	∞	∞
	$\rho_0\Delta_1(S,V)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
$Q^{(\mathrm{DE})}$	$\Sigma_0(S,V)$	0	0	0	0	0	0	0	0	0	0	0
	$\Sigma_1(S,V)$	const	const	const	const	const	const	const	const	const	const	const
	$\Delta_0(S,V)$	0	0	0	0	0	0	0	0	0	0	0
	$\Delta_1(S,V)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	0	0
$U^{(\mathrm{DE})}$	$\Sigma_0(S,V)$	0	0	0	0	0	0	0	0	0	0	0
	$\Delta_0(S,V)$	const	const	const	const	const	const	const	const	const	const	const
	$\Delta_1(S,V)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	0
High energy limit												
$\Sigma,\ \mathcal{E}$	$\Sigma_0(S,V,T)$	∞	∞	0	0	0	0	0	0	0	0	0
	$\Sigma_1(S,V,T)$	∞	∞	∞	∞	∞	∞	∞	0	0	0	0
$\mathcal{H}$	$\mathcal{H}_0(V,T)$	∞	∞	∞	∞	∞	0	0	0	0	0	0
	$\mathcal{H}_1(T)$	∞	∞	0	0	0	0	0	0	0	0	0
W	$W_0^*(V)$	∞	∞	∞	∞	∞	∞	0	0	0	0	0
$\Delta^{(\mathrm{HE})}$	$\Sigma_0(S,V)$	∞	0	0	0	0	0	0	0	0	0	0
	$Q_0^*(S,V)$	∞	∞	∞	∞	∞	∞	∞	0	0	0	0
	$U_0^*(S,V)$	∞	∞	∞	0	0	0	0	0	0	0	0
$Q^{(\mathrm{HE})}$	$\Sigma_0(S,V)$	∞	∞	0	0	0	0	0	0	0	0	0
	$\Sigma_1(S,V)$	∞	∞	∞	∞	∞	0	0	0	0	0	0
	$Q_0^*(S,V)$	∞	∞	∞	∞	∞	∞	∞	∞	∞	0	0
	$U_0^*(S,V)$	∞	∞	∞	∞	0	0	0	0	0	0	0
$U^{(\mathrm{HE})}$	$U_0^*(S,V)$	∞	∞	∞	∞	∞	0	0	0	0	0	0

[a]This quantity is suppressed by a factor of ρ_0, hence not significant when approaching the vacuum model (R).

VIII.   RESULTS AND DISCUSSION

In the previous three sections we have developed and solved the perturbation equations for the *low energy, dark energy* and *high energy* backgrounds, respectively.

The main results of our analysis is summarised in Table II, in which we present the early time asymptotics $a \to 0$ of the physically relevant quantities for the different energy regimes. These physically relevant quantities are the harmonically decomposed components of the expansion normalised vorticity, shear, and the electric and magnetic parts of the Weyl tensor (18,19), as well as the appropriate gradients of the energy density ρ, the non-local energy density ρ^* and the non-local flux q_a^* defined in (23)-(25). The remaining quantities appearing in the previous sections are required to close the system of equations, but are otherwise of no particular physical importance.

We can see from Table II that in the low-energy regime the results from general relativity are recovered. In particular, we find the same decaying mode Σ_1 in both the shear Σ and and density gradient Δ, implying that in general relativity the flat RW models are *unstable* with respect to generic linear homogeneous and anisotropic perturbations into the past which was the problem outlined in the introduction and partly the motivation for inflation.

In the dark energy limit, we find that for any value of γ there is a quantity that diverges as $a \to 0$. The dark energy background is, however, an unstable equilibrium point in the state space of flat FRW models [4],

and can therefore only be attained for very special initial conditions.

The main result of this analysis relates to the evolution of the perturbation quantities in the high energy background $\mathcal{F}_b$. We find that, unlike in GR, both shear and density gradient tend to zero at early times if $\gamma > 4/3$. Thus the high-energy models isotropize into the past for realistic equations of state when we include generic linear inhomogeneous and anisotropic perturbations.

IX.   CONCLUSION

In this paper we have given a comprehensive large-scale perturbative analysis of flat FRW braneworld models with vanishing cosmological constant, extending the work presented in [18] by providing a complete analysis of *scalar*, *vector* and *tensor* perturbations for all the important stages of the braneworld evolution, namely the *low energy GR*, *high energy* and *dark energy* regimes. To make this precise we defined dimensionless variables that were specially tailored for each regime of interest. In this way we were able to clarify further the past asymptotic behaviour of these models and obtain results which exactly match the recent work of Coley *et. al.* [17] on the spatially inhomogeneous G_2 braneworld models: isotropization towards the $\mathcal{F}_b$ model occurs for an equation of state parameter $\gamma > 4/3$.

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[35] Strictly speaking, the variables defined in [22] and those defined in the same way in the brane context [15, 26, 27] are 4-D, however they can easily be generalised to 5 - D. Indeed Bardeen like variables [28, 29] have been defined in 5 - D in order to carry out a brane - bulk analysis (e.g. see [30]), however their relation to the covariant quantities used here has not yet been established (see [24, 25] for this relation in 4 - D general relativity).

[36] The properties of these harmonics are discussed extensively in the appendix of [24].