Scale-dependent bias from primordial non-Gaussianity in general relativity

David Wands[1] and Anže Slosar[2, 3]

[1] Institute of Cosmology and Gravitation, University of Portsmouth, Dennis Sciama Building, Portsmouth, PO1 3FX, United Kingdom

[2] Berkeley Center for Cosmological Physics, Physics Department and Lawrence Berkeley National Laboratory, University of California, Berkeley California 94720, USA

[3] Faculty of Mathematics and Physics, University of Ljubljana, Slovenia

(Dated: June 11, 2009)

In this note we examine the derivation of scale-dependent bias due to primordial non-Gaussianity of the local type in the context of general relativity. We justify the use of the Poisson equation in general relativistic perturbation theory and thus the derivation of scale-dependent bias as a test of primordial non-Gaussianity, using the spherical collapse model. The corollary is that the form of scale-dependent bias does not receive general relativistic corrections on scales larger than the Hubble radius. This leads to a formally divergent correlation function for biased tracers of the mass distribution which we discuss.

PACS numbers: 98.80.Jk, 98.80.Cq

I. INTRODUCTION

One of the most powerful discriminants between various models for the origin of structure in the Universe is the non-Gaussianity of the primordial fluctuations. The non-linear evolution of inhomogeneities on super-Hubble scales during or after slow-roll inflation can give rise to the local form of non-Gaussianity where the primordial curvature perturbation can be given as a local function of a Gaussian random field [1, 2, 3, 4]. This occurs, for example in the curvaton scenario, due to the late-decay of a weakly coupled scalar field after inflation [5], or in the new ekpyrotic models of an accelerated collapse with two or more fields [6, 7, 8, 9].

Local non-Gaussianity is commonly characterised by a single parameter, $f_{\rm NL}$, where the local primordial Newtonian potential on large scales in the matter-dominated era is given by [10]

$$\phi_N(x) = \phi_G(x) + f_{\rm NL}\left(\phi_G^2(x) - \langle\phi_G^2\rangle\right) , \qquad (1)$$

and $\phi_G(x)$ is a Gaussian random field[1].

In Newtonian gravity the Poisson equation,

$$-\nabla^2\phi_N = 4\pi G\rho\delta\,, \qquad (2)$$

gives the local density contrast, δ, as a function not of the local Newtonian potential directly, but rather as a function of the spatial divergence of the Newtonian potential. A consequence of combining equation (1) with the Poisson equation (2) is that the non-Gaussian part of the density field is not a local function of the Gaussian part.

If we re-write the physical Laplacian in terms of comoving coordinates in a Friedmann-Robertson-Walker (FRW) universe, $\nabla^2 = \partial^2/a^2$, where a is the scale factor, we can write the Poisson equation as

$$-\partial^2\phi_N = 4\pi G a^2\rho\delta\,. \qquad (3)$$

The Fourier transform of this equation gives

$$k^2\phi_N(\mathbf{k}) = \frac{3}{2}a^2H^2\delta(\mathbf{k}) \qquad (4)$$

where we use the Friedmann equation to give the background density in terms of the Hubble expansion, H. It is the scale-dependence of this relation which is ultimately responsible for the diverging scale-dependent bias due to non-Gaussianity as $k \to 0$.

It is commonly assumed that the formation of astrophysical objects is a local function of the dark matter overdensity (smoothed over some scale R):

$$\delta_g(\mathbf{x}) = f(\delta(\mathbf{x}), (\nabla\delta(\mathbf{x}))^2, \nabla^2\delta(\mathbf{x}), \ldots) + s(\mathbf{x}), \qquad (5)$$

where $s(\mathbf{x})$ is a non-deterministic stochastic component. Iff R is large enough so that δ is small and if Fourier modes are decoupled, then the overdensity of astrophysical objects traces dark matter fluctuations in the Fourier space in the limit of $k \to 0$:

$$\delta_g(\mathbf{k}) = b\delta(\mathbf{k}) + s(\mathbf{k}), \qquad (6)$$

where $b = f'(0)$ is the so-called bias parameter. The corresponding power spectrum is given by

$$P_{gg}(k) = b^2P(k) + \sigma_s^2. \qquad (7)$$

The stochastic white noise tail has never been observed and hence we will neglect it here. The important point is

[1] Note that we adopt the sign convention of Komatsu & Spergel [10] which is widely used to characterise primordial non-Gaussianity; the Newtonian potential is positive for a point mass $\phi_N(r) = +Gm/r$. Many theoretical calculations of primordial non-Gaussianity, starting with that of Maldacena [11], use the opposite sign for the Newtonian potential, as for instance used in the influential review by Mukhanov, Feldman and Brandenberger [12], and this leads to a different sign for $f_{\rm NL}$. Although the choice of sign is conventional, observations are sensitive to the sign of $f_{\rm NL}$.

that bias b should approach a constant as k approaches zero. Deviation from this implies either that the formation process is non-local or that Fourier modes are coupled, i.e., the dark matter density fluctuations non-Gaussian.

In concrete models of formation of astrophysical objects, it is assumed that the function f is such, that it puts objects in the peaks of the local density field [13]. Such density peaks collapse under their own gravity and form virialized objects. The local number density of such objects in Lagrangian space is given by

$$n = n_0(1 + b_L\delta_l), \tag{8}$$

where b_L stands for Lagrangian space bias, δ_l is the contribution from the very long wavelength modes that essentially modulate the mean density of the effective local cosmology. Therefore

$$b_L = n_0^{-1}\frac{\partial n}{\partial \delta_l} \tag{9}$$

and the more usual Eulerian-space bias is given by $b = 1 + b_L$ [14, 15, 16, 17]. This approach towards formation of astrophysical objects is the essence of the so-called peak-background split. If one understands how the local effective cosmology is modulated by the presence of the large-scale modes and how the formation of objects is affected by the change in the local effective cosmology, then one can calculate the bias parameter for that particular tracer.

The bias is therefore given by the interplay between the small-scale collapse dynamics and the large scales, which modulate the local cosmology through fluctuations present in the very large scale modes. The purpose of this paper is to analyse how can this be viewed within the context of general relativity.

The distinctive scale-dependence of large-scale structure has recently been proposed as a powerful test of the local form of primordial non-Gaussianity [18]. For a biased tracer of structure, it can be shown that local non-Gaussianity induces a scale dependent bias $b = b_G + \Delta b$, with [19]:

$$\Delta b = \frac{2 f_{\rm NL}}{\alpha(k)}\frac{\partial \log n}{\partial \log \sigma_8}, \tag{10}$$

where

$$\alpha(k) = \frac{2c^2 T(k) D(z)}{3\Omega_m H_0^2}k^2\,, \tag{11}$$

is the Fourier space conversion factor between the present day density perturbations and the Newtonian potential during matter domination on large scales, including the linear transfer function $T(k)$, which is approaches unity on large scales, and growth factor normalised to be $D(z) = 1/(1+z)$ in the matter domination. $\partial \log n/\partial \log \sigma_8$ describes how the number density of astrophysical objects, n, is affected by the change of the amplitude of small-scale fluctuations σ_8 *in Gaussian cosmologies*. Therefore, although the initial derivation relied on the statistics of Gaussian fields, it is now clear that the effect is present for all local theories for the formation of the astrophysical objects [20, 21, 22, 23]. In particular, for a universal form of the mass function $n(M)$, the above equation simplifies to

$$\Delta b = f_{\rm NL}(b_G - 1)\frac{3\delta_* \Omega_m H_0^2}{c^2 k^2 T(k) D(z)}, \tag{12}$$

where b_G is the Gaussian bias and $\delta_* \sim 1.68$ is the linear over-density at collapse for the spherical collapse model and other symbols have their usual meaning.

A striking feature of the Equation (12) is that the scale-dependent correction diverges as the wave-vector k approaches zero. This is a direct consequence of the Newtonian form of the Poisson equation in the Fourier space. Since the Newtonian approximation is expected to break down at the scales comparable to or larger than the Hubble scale, it is timely to ask whether the divergence of the scale-dependent bias is real or an artefact of the Newtonian approximation. This is the main question that we address in this brief report.

We start by analysing the Poisson equation in the context of general relativistic perturbation theory in Section II. Next we review the well known *exact* general relativistic solution, namely the spherical collapse solution (Section III). By investigating the metric in the limit of small perturbations, we can identify the gauge in which the spherhical collapse calculation is performed. This justifies the form the Poisson equation for linear perturbations on all scales. In Section IV we discuss the formally divergent correlation function of a biased tracer of structure and show how this is absent in observations of real correlation functions. We conclude in Section V.

II. GENERAL RELATIVISTIC POISSON EQUATION

One might naively expect corrections to the Newtonian Poisson equation (3) of order $(aH/k)^2$ which would become large close to the Hubble scale and change the scale-dependence of $\alpha(k)$ on large scales. However, the form of the general relativistic constraint equations is gauge dependent.

In a completely homogeneous universe, the FRW metric $ds^2 = dt^2 - a^2\delta_{ij}dx^i dx^j = a^2\left(d\eta^2 - \delta_{ij}dx^i dx^j\right)$ has no gauge freedom. At the first order of perturbations, on the other hand, the coordinate freedom gives rise to ambiguities in the definition of the density perturbation in an inhomogeneous spacetime (see, for instance, Ref. [24]). In particular the freedom to make an inhomogeneous redefinition of the time coordinate $\eta \rightarrow \eta + \delta\eta(x, \eta)$ leads to a redefinition of the density perturbation $\delta\rho \rightarrow \delta\rho - \rho'\delta\eta$, where a prime denotes derivatives with respect to conformal time, η.

The most general scalar perturbation[2] of the spatially flat Friedman-Robertson-Walker (FRW) metric can be written as [12, 24]

$$ds^2 = a^2\left\{(1+2A)d\eta^2 - 2(\partial_i B)dx^i d\eta - [(1-2\psi)\delta_{ij} + 2(\partial_i\partial_j E)]dx^i dx^j\right\}. \quad (13)$$

but the metric perturbations A, B, ψ and E are all, like the density perturbation, $\delta\rho$, gauge-dependent.

In an arbitrary gauge there is a first-order energy constraint

$$-3\mathcal{H}\left(\psi' + \mathcal{H}A\right) + \partial^2\left[\psi + \mathcal{H}(E'-B)\right] = 4\pi G a^2 \delta\rho\,. \quad (14)$$

where $\mathcal{H} = aH$ is the conformal Hubble rate, and a first-order momentum constraint

$$\partial_i\left(\psi' + \mathcal{H}A\right) = -4\pi G a^2 \partial_i(\delta q) \quad (15)$$

where $\partial_i(\delta q)$ is the 3-momentum. We can combine these two equations to obtain a general relativistic Poisson equation

$$\partial^2\left[\psi + \mathcal{H}(E'-B)\right] = 4\pi G a^2\left[\delta\rho - 3\mathcal{H}\delta q\right]\,. \quad (16)$$

This is exactly the same as the Newtonian Poisson equation (3) as long as we identify $-\phi_N$ with the metric potential in the Newtonian or longitudinal gauge

$$-\phi_N = \psi_\ell \equiv \psi + \mathcal{H}(E'-B)\,, \quad (17)$$

and $\rho\delta$ with the density perturbation in the comoving-orthogonal gauge

$$\rho\delta = \delta\rho_c \equiv \delta\rho - 3\mathcal{H}\delta q\,. \quad (18)$$

i.e., the density perturbation in a gauge in which $B = 0$ and $\delta q = 0$.

The general relativistic Poisson equation (16) relates gauge-invariant combinations of metric and matter perturbations, which nonetheless have the interpretation as potential or density perturbations in two different gauges [25, 26].

If the primordial Newtonian potential, ϕ_N, has a local non-Gaussianity as given in Equation (1) then the comoving-orthogonal density contrast, δ, will indeed have a non-local non-Gaussianity.

Note however, that in some gauges there are general relativistic corrections to the Poisson equation. For example, in the longitudinal gauge the shear potential, $E' - B$, vanishes and on large scales the regular solution, $-\phi_N = A_\ell = \psi_\ell = \text{const}$, gives a relativistic correction to the Poisson equation

$$3\mathcal{H}^2\phi_N - \partial^2\phi_N = 4\pi G a^2 \delta\rho_\ell\,. \quad (19)$$

[2] We neglect vector and tensor perturbations which are decoupled from scalar perturbations at first order.

Thus on super-Hubble scales ($|\partial^2\phi_N| \ll \mathcal{H}^2|\phi_N|$) the density contrast in the longitudinal gauge, $\delta_\ell = \delta\rho_\ell/\rho$, is proportional to the Newtonian potential

$$\delta_\ell(x) \simeq -2\phi_N(x) = -2\phi_G(x) - 2f_{\rm NL}\left(\phi_G^2(x) - \langle\phi_G^2\rangle\right)\,. \quad (20)$$

This density contrast, δ_ℓ, *does* have a local non-Gaussianity in the large scale limit.

Both the density perturbation in the comoving-orthogonal and longitudinal gauge can be given as gauge-invariant combinations of gauge-dependent variables, and on sub-Hubble scales ($|\partial^2\phi_N| \gg \mathcal{H}^2|\phi_N|$) we find from Eqs. (19) and (16) that $\delta\rho_\ell \simeq \delta\rho_c$ and these two possible definitions of the density perturbation coincide. The question then arises as to which is the relevant density perturbation from which to calculate the bias of galaxies and large-scale structure.

III. SPHERICAL COLLAPSE

The usual Newtonian calculation of the abundance of collapsed objects such as galaxies or quasars assumes that the collapse is a function of the local density field smoothed on some scale R. But which is the appropriate density variable as we approach the Hubble scale? Although initial conditions are set at early times where the linear perturbation theory is valid, collapse requires a non-linear calculation which is generally only possible in general relativity if we make some assumption of simplifying symmetry. The most simple example is that of spherical collapse. In the absence of any pressure perturbation (as for pressureless matter, with or without a cosmological constant) the model allows us to compute the non-linear collapse of a top-hat overdensity described by a closed FRW model embedded within a flat FRW exterior.

The Friedman equation for the unperturbed outer universe is given by:

$$H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G\rho_0}{3a^3}, \quad (21)$$

while the inner region has a perturbed scale factor, $\tilde a$, and small positive curvature, K, and hence

$$\tilde H^2 = \left(\frac{\dot{\tilde a}}{\tilde a}\right)^2 = \frac{8\pi G\rho_0}{3\tilde a^3} - \frac{K}{\tilde a^2}. \quad (22)$$

Without the loss of generality, we can set $8\pi G\rho_0/3 = 1$. The Friedman equation for the closed inner universe can then be expressed in a parametric form

$$\tilde a = \frac{1}{2K}(1-\cos\theta) \quad (23)$$

$$t = \frac{1}{2K^{3/2}}(\theta - \sin\theta). \quad (24)$$

Expanding $\tilde{a}$ and t to fifth order in θ, one can show that in the limit of $\theta \ll 1$,

$$\tilde{a} = \left(\frac{3t}{2}\right)^{2/3}\left(1 - \frac{1}{20}\theta^2\right) \tag{25}$$

We see that to leading order the scale factors match as $\tilde{a}$ approaches zero, if we have $a = (3t/2)^{2/3}$ as the solution for the exterior flat universe.

The linear over-densities are thus given by

$$\delta = \frac{a^3}{\tilde{a}^3} - 1 \simeq -3\frac{\delta a}{a} \simeq \frac{3}{20}\theta^2 \simeq \frac{3}{5}Ka, \tag{26}$$

where $\delta a = \tilde{a} - a$. We have thus recovered the expected result that linear perturbations grow with a in the matter era and that this spherically symmetrical system has a one parameter solution. When the inner perturbation collapses, $\theta_* = 2\pi$, $t_* = \pi K^{-3/2}$ and the linear density at that moment is

$$\delta_* = \frac{3}{5}\left(\frac{3\pi}{2}\right)^{2/3} \sim 1.68, \tag{27}$$

which is the standard result.

But to decide which linear density perturbation this is, we need to identify the coordinate choice implicit in the spherical collapse model. Let us look at the same process from the metric perturbation perspective. Inside the overdense region the perturbed, non-linear metric is given by a closed FRW metric

$$ds^2 = dt^2 - \tilde{a}^2\left[\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2\right], \tag{28}$$

where $K > 0$. For $Kr^2 \ll 1$, this can be recast in Cartesian form

$$ds^2 = dt^2 - \tilde{a}^2\left[\delta_{ij} + Kr^2(\partial_i r)(\partial_j r)\right] dx^i dx^j\,. \tag{29}$$

Comparing this with the linearly perturbed FRW metric of equation (13) we can identify the scalar metric perturbations:

$$A = 0 \tag{30}$$

$$B = 0 \tag{31}$$

$$\psi = \frac{Kr^2}{4} - \frac{\delta a}{a} \tag{32}$$

$$E = \frac{Kr^4}{16} \tag{33}$$

In addition, the spherical collapse picture is implicitly set up using comoving spatial coordinates, x^i, as the expansion or collapse is determined solely by the local scale factor, $\tilde{a}$, and there are no peculiar velocities. Since $B = 0$ we are working in a comoving-orthogonal system with vanishing 3-momentum, $\delta q = 0$.

Since $A = 0$, we are also working in a synchronous coordinate system [which was implicitly assumed in the use of the same time derivative in Eqs.(21) and (22)]. In a comoving-orthogonal coordinate system the conservation of momentum requires

$$A_c = -\frac{\delta P_c}{\rho + P}\,. \tag{34}$$

For pressureless matter momentum conservation thus requires $A = 0$ in a comoving-orthogonal gauge, i.e., the gauge is also synchronous [27], so that the proper cosmic time in the overdense interior and unperturbed exterior are the same.

Thus we see that the initial data for the spherical collapse problem which is set up at early times in the matter era where we may use relativistic perturbation theory is set up in a comoving-orthogonal gauge. In particular the relevant density perturbation which is used to determine whether a given region will collapse by the present time is the comoving-orthogonal density perturbation defined in Equation (18) which is related to the Newtonian potential by the general relativistic Poisson equation (16).

Note that the general relativistic Poisson equation (16) shows that for the growing mode solution $\phi_N =$ const. we have $\delta\rho_c \propto a^{-2}$, as the physical spatial gradients decay in an expanding universe. Thus the density contrast, $\delta = \delta\rho_c/\rho \propto a$ grows on all scales in the matter era, as in Newtonian gravity and in the linearised spherical collapse solution, Equation (26).

Since the Poisson equation, (16), relates the Newtonian potential (17) to the density perturbation in the comoving-orthogonal gauge (18), and the spherical collapse model uses the density perturbation in the comoving-orthogonal gauge to determine the collapse of overdensities, we conclude that the Newtonian form of the scale-dependent bias does not receive any corrections arising from general relativity on the super-Hubble scales in the spherical collapse model.

A. Spherical collapse in longitudinal gauge

It is of course always possible to consider spherical collapse in different coordinates. If we redefine the conformal time and radial coordinate

$$\begin{aligned} \eta &= \left(1 - \frac{1}{20}K(r_0^2 - r_\ell^2)\right)\eta_\ell\,, \\ r &= \left(1 + \frac{1}{20}K\eta_\ell^2 + K(r_0^2 - r_\ell^2)\right) r_\ell\,, \end{aligned} \tag{35}$$

then the line element (29) can be written (to first order in K) in a longitudinal gauge

$$ds^2 = a_\ell^2\left[(1 - 2\phi_N)d\eta_\ell^2 - (1 + 2\phi_N)\left(dr_\ell^2 + r_\ell^2 d\Omega^2\right)\right], \tag{36}$$

where $a_\ell = a(\eta_\ell)$ and the Newtonian potential for $r_\ell < r_0$ is

$$\phi_N = \frac{3}{20}K(r_0^2 - r_\ell^2)\,. \tag{37}$$

Note that we need to specify the size of the overdensity, r_0, so that the coordinate time coincides with the time in the unperturbed background, $r \geq r_0$.

Note that the density contrast in the longitudinal gauge, given by Eq. (19), is no longer spatially homogeneous

$$\delta_\ell = \frac{3}{5} K a_\ell \left[1 + \frac{1}{2} \left(\frac{r_0^2 - r_\ell^2}{r_H^2} \right) \right] , \tag{38}$$

where $r_H = \mathcal{H}^{-1}$ is the comoving Hubble length. The linear central density contrast at the collapse time is given by

$$\delta_{\ell *}|_{r=0} = \frac{3}{5} \left(\frac{3\pi}{2} \right)^{2/3} \left[1 + \frac{1}{2} \left(\frac{r_0^2}{r_H^2} \right) \right] . \tag{39}$$

In contrast to the result in the comoving-orthogonal gauge, Eq. (27), the criterion for collapse in the longitudinal gauge is not simply a function of the local density, but it also depends upon the scale of the overdensity for $r_0 \sim r_H$. Thus, although the GR correction to the Poisson equation in the longitudinal gauge, Eq. (19), implies that the density contrast on super-Hubble scales ($r_0 \gg r_H$) becomes a local function of the Newtonian potential, the collapse condition, Eq. (39), becomes non-local and this is the origin of the scale-dependent bias as seen in the longitudinal gauge.

More generally, dropping the condition of spherical symmetry, we can combine the two different expressions of the GR Poisson equations (16) and (19) to obtain a manifestly non-local expression for the density contrast in the longitudinal gauge in terms of the comoving-orthogonal density contrast

$$\delta_\ell = \left(1 - 3\mathcal{H}^2 \partial^{-2} \right) \delta \, . \tag{40}$$

IV. CORRELATION FUNCTION

The necessary consequence of the fact that there are no general relativistic corrections to the form of the scale dependent bias on scales larger than the horizon is that the correlation function formally diverges as we show next. The correlation function of objects is given by the Fourier transform of the power spectrum

$$\xi(r) = \frac{1}{(2\pi)^3} \int d^3k P(k) e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{41}$$

In the homogeneous and isotropic universe, this can be integrated to

$$\xi(r) = \frac{1}{2\pi^2} \int P(k) \frac{\sin(kr)}{kr} k^2 dk. \tag{42}$$

The dark matter power spectrum $P_{\rm dm}(k) = k^{n_s}$ for small k, where n_s is the spectral index of primordial fluctuations. Therefore, for $P(k) = (b + \Delta b)^2 P_{\rm dm}(k)$, the $P(k)$ is proportional to $k^{n_s - 4}$ (for objects with $b \neq 1$) as k approaches zero and hence the integrand is proportional to k^{n_s-2} in this limit. The result is that $\xi(r)$ diverges for *every* r.

How does one then relate the measurements of the large scale structure to the observations in non-Gaussian universes? The hint is given by the fact, that root mean square of fluctuation in the field of tracers, i.e. $\sigma = \sqrt{\xi(0)}$ is infinite. However, when we measure the fluctuations in a given survey, we measure the mean density from the same survey, effectively forcing the fluctuations to average to zero. Therefore, the measured density fluctuations $\tilde{\delta}$ are given by

$$\tilde{\delta}(\mathbf{x}) = \delta(\mathbf{x}) - \bar{\delta}, \tag{43}$$

where

$$\bar{\delta} = \int d^3\mathbf{x} W(\mathbf{x}) \delta(\mathbf{x}) \tag{44}$$

with $W(\mathbf{x})$ being the normalised survey window.

This means that the measured correlation function is given by

$$\begin{aligned} \tilde{\xi}(r) &= \int d^3\mathbf{x} W(\mathbf{x}) \tilde{\delta}(\mathbf{x}) \tilde{\delta}(\mathbf{x} + \mathbf{r}) \\ &= \int d^3\mathbf{x} W(\mathbf{x}) \delta(\mathbf{x}) \delta(\mathbf{x} + \mathbf{r}) - \bar{\delta}^2. \end{aligned} \tag{45}$$

The expectation value for $\tilde{\xi}(r)$, averaged over all possible realisations of the underlying density fields is thus given by

$$\begin{aligned} \left\langle \tilde{\xi}(r) \right\rangle &= \int d^3\mathbf{x} W(\mathbf{x}) \left\langle \delta(\mathbf{x}) \delta(\mathbf{x} + \mathbf{r}) \right\rangle - \left\langle \bar{\delta}^2 \right\rangle \\ &= \xi(r) - \left\langle \bar{\delta}^2 \right\rangle . \end{aligned} \tag{46}$$

The second term of this equation is the variance of $\bar{\delta}$ over an ensemble over possible realisations of the underlying field. As the universe is ergodic, we can replace many realisations of the underlying field with many copies of the window function over a single realisation and hence the second term can be written as

$$\begin{aligned} \left\langle \bar{\delta}^2 \right\rangle &= \left\langle (\delta * W)^2(\mathbf{x}) \right\rangle \\ &= \frac{1}{(2\pi)^3} \int d^3k W_k(\mathbf{k})^2 P(k) = \sigma_W^2, \end{aligned} \tag{47}$$

where $*$ denotes convolution and W_k is the Fourier transform of the window function. The last term is therefore nothing more than the variance of the field smoothed over the window function of the survey. We can therefore write

$$\left\langle \tilde{\xi}(r) \right\rangle = \xi(r) - \sigma_W^2 . \tag{48}$$

This is an expected result. We simply cannot measure variance coming from scales larger than the survey.

Let us take a concrete example. For the spherical top-hat window function of size R, the integral (47) becomes

$$\sigma^2(R) = \frac{1}{2\pi^2}\int d^3k P(k) k^2 T^2(kR), \qquad (49)$$

with $T(x) = 3(\sin x - x\cos x)/x^3$. In this case, the Equation (48) can be rewritten as

$$\tilde{\xi}(r) = \frac{1}{2\pi^2}\int P(k)\left(\frac{\sin(kr)}{kr} - T^2(kR)\right)k^2 dk. \qquad (50)$$

This is now a well behaved transform. In the limit of $k \to 0$, the term in brackets becomes $k^2(R^2/10 - r^2/6)$ and hence the integrand is proportional to k^{n_s} and thus convergent.

Note that this cancellation occurs for any finite window function W. If a window function has a largest typical linear scale R, then for $k \ll R^{-1}$, $W_k \sim 1$ as the $\exp(ikR)$ is varying slowly across the window. One these scales $\sin(kr)/kr - W^2(kR)$ is therefore quadratic in k to the leading order.

Finally, there are two other worries associated with a diverging field. First, one might worry that the are formal problems with perturbing a homogeneous universe with a solution that has infra-red divergence. However, the underlying physical field, the dark-matter density, is a completely regular and does not diverge on the large scales. Only some non-linear transformation of this regular field, namely a biased tracer of this field, or a field of peak positions (after smoothing on some scale) has a large-scale divergence. Therefore, this should not be a problem. A more serious issue is that the field of tracers, although it can be arbitrarily biased with respect to the underlying dark matter density field, must still remain positive, i.e. $\delta(\mathbf{x}) > -1$ since the local density of astrophysical objects cannot be negative. To estimate when this becomes important, we demand that the dimensionless power per unit log k, $\Delta^2 = P(k)k^3/4\pi^2$ is much less than unity on large scales. For biased tracer, this reduces a requirement that $f_{\rm NL}(b-1) \ll 10^4$, which is always true for observationally interesting values of $f_{\rm NL}$ and b.

V. CONCLUSIONS

In this paper we have discussed the derivation of scale-dependent bias from primordial non-Gaussianity in the context of general relativity. Our results are somewhat unexciting - we find no corrections arising from general relativistic effects on scales larger than the Hubble length. We have justified this by noting that the spherical collapse and thus peak-background split formalisms that are performed in the Newtonian context can be carried over into general relativity if one works in terms of the density perturbation in the comoving-orthogonal gauge. This justifies the use of the Newtonian form of the Poisson equation (3) on all scales (in contrast to what one might naively think [28]).

This means that the power spectrum of biased tracers of the density field diverges as the scales of interest become larger and larger and leads to the infinite variance in the local density of the biased objects. The formal correlation function is divergent on all scales, but any observed correlation function will be finite, because variance on scales larger than the size of the survey will not be observed.

The reason that general relativity reduces to apparently Newtonian equations is because we are only considering scalar perturbations which obey energy and momentum constraint equations relating the metric to the matter perturbations. Considering first-order perturbations about an FRW background, the constraint equations enforce a Poisson-type relation between the local density perturbation and the metric perturbations on a given constant-time hypersurface, coinciding with a given time in the background spacetime. Coordinate freedom in general relativity allows different choices of time slicing in the perturbed spacetime, but it is the density perturbation in the comoving-orthogonal gauge that follows the same behaviour as the Newtonian density perturbation on all scales.

Assuming spherical symmetry we are able to follow the non-linear collapse in a comoving-orthogonal coordinate system. Spherical symmetry eliminates the vector and tensor parts of the metric perturbation which might have introduced non-Newtonian behaviour. For example, there are no gravitational waves if we impose spherical symmetry. Deviations from spherical symmetry could thus introduce additional general relativistic terms, but the scalar, vector and tensor perturbations are decoupled from scalar density perturbations at first-order, so such effects would only be expected to arise at second- or higher-order [29]. This might ultimately be important when we wish to constrain non-linearity parameters, $f_{\rm NL}$, of order unity but it suggests that general relativistic corrections to the Newtonian results currently being used are negligible for current data.

Acknowledgements

Authors thank the Galileo Galilei Institute for Theoretical Physics for their hospitality and the INFN for partial support during the completion of this work. DW is grateful to Rob Crittenden, Kazuya Koyama, David Lyth and Will Percival for helpful discussions. AS acknowledges useful discussions with Martin White. DW is supported by the STFC, while AS is supported by Berkeley Center for Cosmological Physics.

[1] D. S. Salopek and J. R. Bond, Phys. Rev. D **42**, 3936 (1990), ADS.

[2] A. Gangui, F. Lucchin, S. Matarrese, and S. Mollerach, Astrophys. J. **430**, 447 (1994), arXiv:astro-ph/9312033.

[3] L. Verde, L. Wang, A. F. Heavens, and M. Kamionkowski, Mon. Not. R. Astron. Soc. **313**, 141 (2000), arXiv:astro-ph/9906301.

[4] D. H. Lyth and Y. Rodríguez, Phys. Rev. Lett. **95**, 121302 (2005), arXiv:astro-ph/0504045.

[5] D. H. Lyth, C. Ungarelli, and D. Wands, Phys. Rev. D **67**, 023503 (2003), arXiv:astro-ph/0208055.

[6] P. Creminelli and L. Senatore, Journal of Cosmology and Astro-Particle Physics **11**, 10 (2007), arXiv:hep-th/0702165.

[7] K. Koyama, S. Mizuno, F. Vernizzi, and D. Wands, Journal of Cosmology and Astro-Particle Physics **11**, 24 (2007), arXiv:0708.4321.

[8] E. I. Buchbinder, J. Khoury, and B. A. Ovrut, Phys. Rev. Lett. **100**, 171302 (2008), arXiv:0710.5172.

[9] J.-L. Lehners and P. J. Steinhardt, Phys. Rev. D **77**, 063533 (2008), arXiv:0712.3779.

[10] E. Komatsu and D. N. Spergel, Phys. Rev. D **63**, 063002 (2001), arXiv:astro-ph/0005036.

[11] J. Maldacena, Journal of High Energy Physics **5**, 13 (2003), arXiv:astro-ph/0210603.

[12] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992), ADS.

[13] W. H. Press and P. Schechter, Astrophys. J. **187**, 425 (1974), ADS.

[14] G. Efstathiou, C. S. Frenk, S. D. M. White, and M. Davis, Mon. Not. R. Astron. Soc. **235**, 715 (1988), ADS.

[15] S. Cole and N. Kaiser, Mon. Not. R. Astron. Soc. **237**, 1127 (1989), ADS.

[16] H. J. Mo and S. D. M. White, Mon. Not. R. Astron. Soc. **282**, 347 (1996), ADS.

[17] P. Catelan, F. Lucchin, S. Matarrese, and C. Porciani, Mon. Not. R. Astron. Soc. **297**, 692 (1998), arXiv:astro-ph/9708067.

[18] N. Dalal, O. Doré, D. Huterer, and A. Shirokov, Phys. Rev. D **77**, 123514 (2008), arXiv:0710.4560.

[19] A. Slosar, C. Hirata, U. Seljak, S. Ho, and N. Padmanabhan, Journal of Cosmology and Astro-Particle Physics **8**, 31 (2008), arXiv:0805.3580.

[20] S. Matarrese and L. Verde, Astrophys. J. Lett. **677**, L77 (2008), arXiv:0801.4826.

[21] A. Slosar, C. Hirata, U. Seljak, S. Ho, and N. Padmanabhan, ArXiv e-prints **805** (2008), arXiv:0805.3580.

[22] N. Afshordi and A. J. Tolley, ArXiv e-prints **806** (2008), arXiv:0806.1046.

[23] P. McDonald, ArXiv e-prints **806** (2008), arXiv:0806.1061.

[24] K. A. Malik and D. Wands, ArXiv e-prints (2008), arXiv:0809.4944.

[25] W. Hu, U. Seljak, M. White, and M. Zaldarriaga, Phys. Rev. D **57**, 3290 (1998), ADS.

[26] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980), ADS.

[27] C. Ma and E. Bertschinger, Astrophys. J. **455**, 7 (1995), ADS.

[28] J. B. Dent and S. Dutta, ArXiv e-prints (2008), arXiv:0808.2689.

[29] J.-C. Hwang and H. Noh, Phys. Rev. D **76**, 103527 (2007), arXiv:0704.1927.