Automates cellulaires : structures

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Cellular Automata: Structures

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A main concern of “Complex Systems”:

a relatively simple microscopic rule

*completely defined local rule (given)*

*may produce*

a very complex macroscopic behavior

*far more complex global rule (induced)*
A main concern of “Complex Systems”:

- a relatively simple microscopic rule
  
  *completely defined local rule (given)*

  *may produce*

- a very complex macroscopic behavior
  
  *far more complex global rule (induced)*

- Cellular Automata provide a simple – not simplistic – and uniform model for studying this problem.
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1. Definitions
2. Classifications
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4. Abstract Bulking
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**Definition.** A \( d\text{-CA} \) \( \mathcal{A} \) is a 4-uple \((\mathbb{Z}^d, S, N, \delta)\) where:

- \( S \) is the finite state set of \( \mathcal{A} \);
- \( N \subset \mathbb{Z}^d \), finite, is the neighborhood of \( \mathcal{A} \);
- \( \delta : S^{\lvert N \rvert} \rightarrow S \) is the local rule of \( \mathcal{A} \).
Definition. A $d$-CA $\mathcal{A}$ is a 4-uple $(\mathbb{Z}^d, S, N, \delta)$ where:

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A configuration $C$ is a mapping from $\mathbb{Z}^d$ to $S$. 
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- $\delta : S^{|N|} \rightarrow S$ is the local rule of $A$.

A configuration $C$ is a mapping from $\mathbb{Z}^d$ to $S$.

The global rule applies $\delta$ uniformly according to $N$:

$$\forall p \in \mathbb{Z}^d, \quad G(C)_p = \delta(C_{p+N_1}, \ldots, C_{p+N_\nu})$$
$\sigma = (\mathbb{Z}, \{■, □\}, \{-1\}, q \mapsto q)$

$\Sigma_2 = (\mathbb{Z}, \{■, □\}, [-1, 0], (q, q') \mapsto q \oplus q')$,
where $(\{■, □\}, \oplus)$ is isomorphic to $(\mathbb{Z}_2, +)$
Examples (2)

\((\mathbb{Z}, \{\bullet, \square\}, [-1, 1], \text{maj})\),
where \(\text{maj}\) is majority between 3

\((\mathbb{Z}, \{\square, \gray{\square}, \yellow{\square}, \brown{\square}, \blue{\square}, \red{\square}\}, [-1, 1], \delta_6)\)
Endow $S$ with the trivial topology.

Endow $S^\mathbb{Z}^d$ with the induced product topology.

The *shift* $\sigma_v : S^\mathbb{Z}^d \rightarrow S^\mathbb{Z}^d$ is defined as

$$\sigma_v(C)_p + v = C_p.$$
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**Theorem**[Hedlund 69]. A map $G : S^\mathbb{Z}^d \to S^\mathbb{Z}^d$ is the global rule of a $d$-CA if and only if it is continuous and commutes with shifts.
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**Theorem** [Hedlund 69]. A map $G : S^\mathbb{Z}^d \to S^\mathbb{Z}^d$ is the global rule of a $d$-CA if and only if it is continue and commutes with shifts.

**Consequences.** We can freely compose CA and invert bijective CA to obtain new CA.
A CA $\mathcal{A}$ is isomorphic to a CA $\mathcal{B}$ ($\mathcal{A} \cong \mathcal{B}$) if there exists a bijective map $\varphi : S_\mathcal{A} \rightarrow S_\mathcal{B}$ such that

$$\overline{\varphi} \circ G_\mathcal{A} = G_\mathcal{B} \circ \overline{\varphi}$$
A CA $\mathcal{A}$ is isomorphic to a CA $\mathcal{B}$ ($\mathcal{A} \cong \mathcal{B}$) if there exists a bijective map $\varphi : S_{\mathcal{A}} \rightarrow S_{\mathcal{B}}$ such that

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**Definition.** $\mathcal{A} \subseteq \mathcal{B}$ if there exists an injective map $\varphi : S_{\mathcal{A}} \rightarrow S_{\mathcal{B}}$ such that this diagram commutes:

\[
\begin{array}{ccc}
C & \xrightarrow{\varphi} & \overline{\varphi}(C) \\
\downarrow G_{\mathcal{A}} & & \downarrow G_{\mathcal{B}} \\
G_{\mathcal{A}}(C) & \xrightarrow{\varphi} & \overline{\varphi}(G_{\mathcal{A}}(C))
\end{array}
\]
Closure (1)

- An *autarkic* CA $\overline{\psi}$ is a CA with neighborhood $\{0\}$ and local rule $\psi : S \rightarrow S$. (notice that $\overline{\psi}$ is ultimately periodic)

- An *elementary shift* is a shift $\sigma_\nu$ such that $\|\nu\|_1 = 1$. 
An autarkic CA $\overline{\psi}$ is a CA with neighborhood $\{0\}$ and local rule $\psi : S \rightarrow S$. (notice that $\overline{\psi}$ is ultimately periodic)

An elementary shift is a shift $\sigma_v$ such that $\|v\|_1 = 1$.

The composition $A \circ B$ of two CA $A$ and $B$ satisfies

$$G_{A \circ B} = G_A \circ G_B.$$ 

The Cartesian product $A \times B$ of two CA satisfies

$$G_{A \times B} = G_A \times G_B.$$
A new characterization of CA

**Theorem.** The set of CA is the closure of the set of autarkic CA and elementary shifts by the operations of composition, Cartesian product and subautomaton.
A new characterization of CA

**Theorem.** The set of CA is the closure of the set of autarkic CA and elementary shifts by the operations of composition, Cartesian product and subautomaton.

**Theorem.** The set of *reversible* (bijective) CA is the closure of the set of *bijective* autarkic CA and elementary shifts by the operations of composition, Cartesian product and subautomaton.
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“[...] In class 1, the behavior is very simple, and almost all initial conditions lead to exactly the same uniform final state.

In class 2, there are many different possible final states, but all of them consist just of a certain set of simple structures that either remain the same forever or repeat every few steps.

In class 3, the behavior is more complicated, and seems in many respects random, although triangles and other small-scale structures are essentially always at some level seen.

And finally [...] class 4 involves a mixture of order and randomness: localized structures are produced which on their own are fairly simple, but these structures move around and interact with each other in very complicated ways. [...]”

S. Wolfram [ANKOS, chapter 6, pp. 231–235]

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Only one proposition of classification (to our knowledge)
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(to our knowledge)


● **Grouping** relies on an algebraic approach
Only one proposition of classification (to our knowledge)


**Grouping** relies on an algebraic approach

**Idea.** Define a quasi-order on CA using the subautomaton relation, up to some geometrical transformation of these CA.
How to eliminate the periodic background pattern? You can zoom out and use shades of grey...

\[ C'_p = \frac{1}{9} \sum_{\nu \in [0, 2]^2} C_{3p + \nu} \]
How to eliminate the periodic background pattern?
...but also make blocks of bottom cells of the squares

$$C'_p = \left( C_{3p+(0,0)}, C_{3p+(1,0)}, C_{3p+(2,0)} \right)$$
We consider 1D CA with neighborhood $[-1, 1]$.

- Define the $k$th power $A^k$ of a CA $A$.

**Definition.** A CA $B$ simulates a CA $A$, $A \preceq B$, if there exists $m$ and $n$ such that $A^m \subseteq B^n$. 

**Theorem.** The relation $\preceq$ is a quasiorder. It admits a global minimum, some equivalence classes at the bottom of the order correspond to simple known CA families. It admits no global maximum.
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Claim. The grouping operation doesn’t take into account some classical geometrical transformations of the literature, natural in the context of:

- Transformation from CA to OCA,
- Nilpotency,
- Intrinsic Universality.
Classical transformations are usually of the type:

\[
\begin{array}{cccccc}
q_{n,1+k} & \cdots & q_{n,m+k} & q_{n,m+1+k} & \cdots & q_{n,2m+k} \\
q_{1,1} & \cdots & q_{1,m} & q_{1,m+1} & \cdots & q_{1,2m} \\
\end{array}
\]

\[
G^{\langle m,n,k \rangle}_A = o_m^{-1} \circ \sigma_k \circ G^n_A \circ o_m
\]
• A *geometrical transformation* on space-time diagrams *transforms a cellular automaton into a new one by combining cells* of a space-time diagram of the first one to construct a space-time diagram of the second one.
• A geometrical transformation on space-time diagrams transforms a cellular automaton into a new one by combining cells of a space-time diagram of the first one to construct a space-time diagram of the second one.

• Formally, it is a pair \((k, \Lambda)\) where

\[
\Lambda : \mathbb{N} \times \mathbb{Z}^d \rightarrow (\mathbb{N} \times \mathbb{Z}^d)^k
\]
To apply a transformation \((k, \Lambda)\) to a space-time diagram \(\Delta\) over \(S\), we define \(\Lambda_S : S^{\mathbb{N} \times \mathbb{Z}^d} \to (S^k)^{\mathbb{N} \times \mathbb{Z}^d}\) by

\[
\Lambda_S(\Delta)(t, p) = (\Delta(\Lambda(t, p)_1), \ldots, \Delta(\Lambda(t, p)_k))
\]
To apply a transformation \((k, \Lambda)\) to a space-time diagram \(\Delta\) over \(S\), we define \(\Lambda_S : S^{N \times Z^d} \rightarrow (S^k)^{N \times Z^d}\) by

\[
\Lambda_S(\Delta)(t, p) = (\Delta(\Lambda(t, p)_1), \ldots, \Delta(\Lambda(t, p)_k))
\]

We define an operation rather similar to composition:

\[(k', \Lambda') \odot (k, \Lambda) = (kk', \Lambda' \circ \Lambda)\]

where

\[(\Lambda' \circ \Lambda)(t, p) = (\Lambda(\Lambda'(t, p)_1), \ldots, \Lambda(\Lambda'(t, p)_k'))_k)\]
We also introduce $\tilde{\Lambda}$ as

$$\tilde{\Lambda} : 2^{\mathbb{N} \times \mathbb{Z}^d} \longrightarrow 2^{\mathbb{N} \times \mathbb{Z}^d}$$

$$X \longmapsto \bigcup_{(t,p) \in X} \{ \Lambda(t, p)_1, \ldots, \Lambda(t, p)_k \}$$
We also introduce $\tilde{\Lambda}$ as

$$
\tilde{\Lambda} : 2^{\mathbb{N} \times \mathbb{Z}^d} \rightarrow 2^{\mathbb{N} \times \mathbb{Z}^d}
$$

$$
X \mapsto \bigcup_{(t,p) \in X} \{ \Lambda(t, p)_1, \ldots, \Lambda(t, p)_k \}
$$

A good geometrical transformation satisfies

1. $\forall A, \exists B, \{ \overline{\Lambda}_{S,A}(\Delta) \}_{\Delta \in \text{Diag}(A)} = \text{Diag}(B)$;
We also introduce $\tilde{\Lambda}$ as

$$
\tilde{\Lambda} : 2^{\mathbb{N} \times \mathbb{Z}^d} \rightarrow 2^{\mathbb{N} \times \mathbb{Z}^d}
$$

$$
X \mapsto \bigcup_{(t,p) \in X} \{ \Lambda(t,p)_1, \ldots, \Lambda(t,p)_k \}
$$

A *good* geometrical transformation satisfies

1. \( \forall A, \exists B, \{ \bar{\Lambda}_{A}(\Delta) \}_{\Delta \in \text{Diag}(A)} = \text{Diag}(B) \);

2. \( \forall t \in \mathbb{N}, \tilde{\Lambda}(\{ t + 1 \} \times \mathbb{Z}^d) \not\subseteq \tilde{\Lambda}(\{ t \} \times \mathbb{Z}^d) \).
\[ P_{F,v}(t,p) = t \odot (F \oplus (p \odot v)) \]

Transformed CA global rule:

\[ o_{F,v}^{-1} \circ G \circ o_{F,v} \]
Cutting

Transformed CA global rule:

\[ C_T(t, p) = (tT, p) \]

\[ G^T \]
Transformed CA global rule:

\[ \sigma_s \circ G \]

\[ S_s(t, p) = (t, p \oplus ts) \]
We define **PCS** transformations as

\[ \text{PCS}_{F,v,T,s} = P_{F,v} \circ S_s \circ C_T \]

\[ \text{PCS}_{F,v,T,s}(t, p) = tT \odot (F \oplus (p \odot v \oplus ts)) \]

Transformed CA global rule:

\[ o_{F,v}^{-1} \circ \sigma_s \circ G_T \circ o_{F,v} \]
We define PCS transformations as

\[ \text{PCS}_{F,v,T,s} = P_{F,v} \circ S_s \circ C_T \]

\[ \text{PCS}_{F,v,T,s}(t, p) = tT \odot (F \oplus (p \odot v \oplus ts)) \]

Transformed CA global rule:

\[ o_{F,v}^{-1} \circ \sigma_s \circ G^T \circ o_{F,v} \]

PCS transformations are closed under composition.
Theorem. A geometrical transformation is a good geometrical transformation if and only if it can be expressed as a PCS transformation.

The proof highly relies on the uniformity of cellular automata and the construction of counter-examples.
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● We don’t want to reproof that we have a quasi-order for each kind of grouping we introduce.

● Some properties are generic and do not rely on painful computation at the level of geometrical transformations but come from more abstract properties.

● We introduce a logical theory to uniformize the work with grouping.
Definition. An *abstract bulking* $\mathcal{A}$ is a logical theory on the signature

\[(\text{Obj}, \text{Trans}; \text{apply} : \text{Obj} \times \text{Trans} \to \text{Obj}, \text{divide} \subseteq \text{Obj} \times \text{Obj}, \text{combine} : \text{Trans} \times \text{Trans} \to \text{Trans}).\]

Notation. An object $y$ simulates an object $x$ if they satisfy the formula

\[x \preceq y \equiv \exists \alpha \exists \beta (x^\alpha \mid y^\beta)\]
Axioms (1)

**Combination.** \((\text{Trans}, \cdot)\) is a monoid.

\[ A \vdash \exists \forall \alpha (\alpha \cdot 1 = \alpha \land 1 \cdot \alpha = \alpha) \land \forall \alpha \forall \beta \forall \gamma ((\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)) \]

**Compatibility.** \((\text{Trans}, \cdot)\) acts on \(\text{Obj}\) through \text{apply}.

\[ A \vdash \forall x (x^1 = x) \land \forall x \forall \alpha \forall \beta \left( (x^\alpha)^\beta = x^{\alpha \cdot \beta} \right) \]
Divisibility. \textit{divide} is a quasi-order on \textit{Obj}.

\[ \mathcal{A} \vdash \forall x \; (x \mid x) \land \forall x \forall y \forall z \; ((x \mid y \land y \mid z) \rightarrow x \mid z) \]

Transitivity. \textit{apply} is compatible with \textit{divide}.

\[ \mathcal{A} \vdash \forall x \forall y \forall \alpha \; (x \mid y \rightarrow x^\alpha \mid y^\alpha) \]
Surjectivity. apply preserve the richness of objects.

\[ \forall \alpha \forall x \exists y \left( x \mid y^\alpha \right) \]
**Proximity.** apply keeps objects nearby. There exists two functions $\zeta$ and $\xi$, such that

\[
A \models \forall x \forall \alpha \forall \beta \left( (\chi^\alpha) \zeta(x, \beta) \mid (\chi^\beta) \xi(x, \alpha, \beta) \right)
\]
Theorem. “≼ is a quasi-order” is a bulking property.

\[ \forall x (x \preceq x) \land \forall x \forall y \forall z ((x \preceq y \land y \preceq z) \rightarrow x \preceq z) \]
Theorem. “≼ is a quasi-order” is a bulking property.

\[ \forall x \, (x \preceq x) \land \forall x \forall y \forall z \, ((x \preceq y \land y \preceq z) \to x \preceq z) \]

- \( u \) is universal if \( \forall x \, (x \preceq u) \).
- \( u \) is strongly universal if \( \forall x \exists \alpha \, (x \mid u^\alpha) \).

Properties
Theorem. “≼” is a quasi-order is a bulking property.

\[ \forall x (x \preceq x) \land \forall x \forall y \forall z ((x \preceq y \land y \preceq z) \rightarrow x \preceq z) \]

- \( u \) is universal if \( \forall x (x \preceq u) \).
- \( u \) is strongly universal if \( \forall x \exists \alpha (x \mid u^\alpha) \).

Theorem. “If there exists a strongly universal objet then each universal object is strongly universal” is a bulking property.
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Idea. Use abstract bulking theory with:

- **Obj** the set of d-CA,
- **Trans** the set of **PCS** transformations,
- **apply** the transformation operator,
- **divide** the subautomaton relation,
- **combine** the composition of transformations.
Idea. Use abstract bulking theory with:

- **Obj**: the set of d-CA,
- **Trans**: the set of PCS transformations,
- **apply** the transformation operator,
- **divide** the subautomaton relation,
- **combine** the composition of transformations.

**Argh!** The **Proximity** axiom is **not** satisfied.
\( \tilde{P} : \text{restriction on } P \text{ transformations.} \)

\[
\tilde{P}(m_1, \ldots, m_d), \tau = P \prod_{i=1}^{d} [0, m_i - 1], (\sigma_{\tau(1)}, \ldots, \sigma_{\tau(d)}) \otimes m
\]
Idea. Use abstract bulking theory with:

- **Obj** the set of \( d \)-CA,
- **Trans** the set of \( \tilde{PCS}' \) transformations,
- **apply** the transformation operator,
- **divide** the subautomaton relation,
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It works: All the axioms are satisfied.
Idea. Use abstract bulking theory with:

- **Obj**: the set of $d$-CA,
- **Trans**: the set of $\tilde{PCS}'$ transformations,
- **apply**: the transformation operator,
- **divide**: the subautomaton relation,
- **combine**: the composition of transformations.

($\tilde{PCS}'$ and $\tilde{PCS}$ define the same relation of simulation)
Idea. Use abstract bulking theory with:

- **Obj** the set of \( d \)-CA,
- **Trans** the set of \( \tilde{\text{PCS}}' \) transformations,
- **apply** the transformation operator,
- **divide** the subautomaton relation,
- **combine** the composition of transformations.

(\( \tilde{\text{PCS}}' \) and \( \tilde{\text{PCS}} \) define the same relation of simulation)

It works: All the axioms are satisfied.
Applying a \( \tilde{\text{PCS}} \) transformation \( \langle m_\tau, n, k \rangle \) to a CA \( A \):

\[
G_A^{\langle m_\tau, n, k \rangle} = o_{m_\tau}^{-1} \circ \sigma_k \circ G_A^n \circ o_{m_\tau}
\]
Applying a \( \tilde{\text{PCS}} \) transformation \( \langle m_\tau, n, k \rangle \) to a CA \( A \):

\[
G_{A}^{\langle m_\tau, n, k \rangle} = o_{m_\tau}^{-1} \circ \sigma_{k} \circ G_{A}^{n} \circ o_{m_\tau}
\]

**Definition.** A CA \( A \) is *simulated* by a CA \( B \), \( A \leq B \), if there exists two \( \tilde{\text{PCS}} \) transformations \( \langle m_\tau, n, k \rangle \) and \( \langle m'_\tau, n', k' \rangle \) such that:

\[
A^{\langle m_\tau, n, k \rangle} \subseteq B^{\langle m'_\tau, n', k' \rangle}
\]
Applying a $\tilde{\text{PCS}}$ transformation $\langle m_\tau, n, k \rangle$ to a CA $A$:

$$G^\langle m_\tau, n, k \rangle_A = o_{m_\tau}^{-1} \circ \sigma_k \circ G^n_A \circ o_{m_\tau}$$

**Definition.** A CA $A$ is *simulated* by a CA $B$, $A \leq B$, if there exists two $\tilde{\text{PCS}}$ transformations $\langle m_\tau, n, k \rangle$ and $\langle m'_\tau, n', k' \rangle$ such that:

$$A^\langle m_\tau, n, k \rangle \subseteq B^\langle m'_\tau, n', k' \rangle$$

**Theorem.** The relation $\leq$ is induced by an abstract bulking model.
Corollary. The relation $\leq$ is a quasi-order.
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- In dimension 1, the relation $\leq$ refines $\leq$. 
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- In dimension 1, the relation $\leq \square$ refines $\leq$.
- $\times$ corresponds to a local maximum.
Corollary. The relation $\leq$ is a quasi-order.

- In dimension 1, the relation $\leq$ refines $\leq$.
- $\times$ corresponds to a local maximum.
- We still have infinite chains.
Bottom of the order

$\sigma$-Per

$(\mathbb{Z}_p, \oplus)$

level 1

level 0
There is no quasi-universal CA.
**Theorem.** Given a CA, deciding whether it is intrinsically universal is undecidable.
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**Theorem.** There exists no real-time intrinsically universal CA ($\forall A, \exists n, A \subseteq U^{(n,n,0)}$).
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Theorem. There exists no real-time intrinsically universal CA ($\forall A, \exists n, A \subseteq U^{(n,n,0)}$).

- We can construct very small intrinsically universal CA (ex. 1D, von Neumann neighborhood, 6 states)
The structure of products of shifts, $\prod_{i=1}^{k} \sigma_{v_i}$, and CA they simulate can be completely described.
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The relation $\leq$ induces no semi-lattice structure.
The structure of products of shifts, $\prod_{i=1}^{k} \sigma_{v_i}$, and CA they simulate can be completely described.

The relation $\leq$ induces no semi-lattice structure.

**Idea.** Modify bulking so that $\times$ defines a supremum.
New transformations: \((k, l, \Lambda)\) where

\[
\Lambda : \ N \times \mathbb{Z}^d \rightarrow (\llbracket 1, l \rrbracket \times \mathbb{N} \times \mathbb{Z}^d)^k
\]
New transformations: \((k, l, \Lambda)\) where

\[ \Lambda : \mathbb{N} \times \mathbb{Z}^d \rightarrow (\mathbb{N} \times \mathbb{Z}^d)^k \]

PCST \( (F_i, v_i, T_i, s_i)_{i \in [1, l]} \) transforms \( A \) into

\[ \left( o_{F_1, v_1}^{-1} \circ \sigma_{s_1} \circ G_{A}^{T_1} \circ o_{F_1, v_1} \right) \times \cdots \times \left( o_{F_l, v_l}^{-1} \circ \sigma_{s_l} \circ G_{A}^{T_l} \circ o_{F_l, v_l} \right) \]
Idea. Use abstract bulking theory with:

- **Obj**: the set of $d$-CA,
- **Trans**: the set of $\tilde{\text{PCST}}'$ transformations,
- **apply**: the transformation operator,
- **divide**: the subautomaton relation,
- **combine**: the composition of transformations.
A new bulking (2)

Idea. Use abstract bulking theory with:

- **Obj**: the set of d-CA,
- **Trans**: the set of $\tilde{\text{PCST}}'$ transformations,
- **apply**: the transformation operator,
- **divide**: the subautomaton relation,
- **combine**: the composition of transformations.

- $\tilde{\text{PCST}}$ transformations are defined like $\tilde{\text{PCS}}$ ones.
- All the axioms are satisfied.
- The relation of simulation induces a sup-semi-lattice with $\times$ as a supremum operator.
An ideal is a set of equivalence classes stable by $\times$ and lower element by $\leq$.
• CA at the bottom and the top of the order seem to correspond to CA which are easy to describe. What about CA in “the middle”?

• Links between structural properties of bulking and decidability questions have been presented. What about topological properties?

• Study abstract bulking in the case of a different kind of dynamical system, refine the choice of axioms, general properties.