Updating Automata Networks

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Mises à jour de réseaux d'automates

THÈSE

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École normale supérieure de Lyon

Laboratoire de l'informatique du parallélisme

Updating Automata Networks

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fulfilment of the requirements for the degree
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François	ROBERT	Invited

À Sylvain

“. . . et les quelques pages de démonstration qui suivent tirent toute leur force du fait que l’histoire est entièrement vraie, puisque je l’ai imaginée d’un bout à l’autre. Sa réalisation matérielle proprement dite consiste essentiellement en une projection de la réalité, en atmosphère biaise et chauffée, sur un plan de référence irrégulièrement ondulé et présentant de la distorsion. ”

BORIS VIAN

Title: Updating Automata Networks

Abstract: This thesis is concerned with the events and the organisation of events that take place within networks of abstract predetermined elements called "automata". In these networks, automata incite one another to switch states in agreement with predefined rules which, precisely, define the net-work. When an automaton effectively conforms to the influences it receives from others, its state is said to be *updated*. The elementary events that are considered here are thus automata state changes. To define an update mode for all the automata of a network allows to select some events among all those that are *a priori* possible. It also allows to organise and order the events relatively so as to impose, for example, that independent events occur simultaneously or so that simply, they happen close enough to disallow the occurrence of any other events in between. Informally, update modes can be interpreted as the expressions of influences incoming from outside the network, forbidding certain changes, or else, as the formalisation of a relaxed and relative version of time flow. This thesis proposes to study their influences on network behaviours. And to distinguish their influences from that of network structures, it starts by highlighting the role of certain structural motives. After that, it explores in particular the information that is "encoded" in a sequence of updates as well as the general impact of synchronism in updates.

Keywords: Automata network, update modes and schedules, time flow relative to automata networks, (a)synchronism, attractor, parallel update schedule, general transition graph, structural cycle.

Titre : Mises à jour de réseaux d'automates

Résumé : Cette thèse s'intéresse aux évènements et aux ordonnancements d'évènements se produisant au sein de réseaux d'éléments conceptuels prédéterminés. Dans ces réseaux, les éléments, appellés plutôt "automates", s'incitent les uns les autres à changer d'état en accord avec des règles prédéfinies qui, précisément, définissent le (fonctionnement du) réseau. Lorsqu'un automate se conforme effectivement aux influences qu'il reçoit de la part des autres, on dit que son état est *mis à jour*. Les évènements élémentaires considérés sont les changements d'états des automates. Définir un mode de mise à jour pour l'ensemble des automates d'un réseau permet de sélectionner certains évènements parmi l'ensemble de ceux qui sont a priori possibles. Cela permet aussi d'organiser et d'ordonner les évènements les uns par rapport aux autres de façon, par exemple, à imposer que des évènements indépendants se produisent simultanément ou simplement, de manière assez rapprochée pour qu'aucun autre événement ne puisse se produire pendant leur occurrence. Informellement, les modes de mise à jour peuvent donc être interprétés comme l'expression d'influences extérieures au réseau interdisant certains changements, ou alors comme la formalisation d'une version relâchée et relative de l'écoulement de temps. Cette thèse propose d'étudier leur influence sur le comportement des réseaux. Et afin de distinguer cette influence de celle de la structure des réseaux, elle commence par mettre en évidence le rôle de certains motifs structurels. Après ça, elle s'intéresse en particulier à l'information “encodée” dans une séquence de mises à jour et à l'impact du synchronisme dans celles-ci.

Mots-clés : Réseau d'automates, mode de mise à jour, écoulement du temps relatif aux réseaux d'automates, (a)synchronisme, attracteur, mode parallèle, graphe de transitions général, circuit structurel.

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Tradition makes provision for a section of acknowledgments in manuscripts of Ph.D. theses. But actually, I really have no need for this pretext to be convinced that my manuscript must indeed include such a section. It even is essential for me and certainly also for the right reading of the work I'm presenting here that my manuscript start with this section. So here it is (in French because my gratitude speaks French more often than it does English).

Mon travail de thèse s'est implanté dans un contexte scientifique que je n'avais pas contribué à bâtir mais dont j'ai évidemment bénéficié de façon fondamentale. Je crois être encore loin d'avoir bien mesuré l'importance du rôle subtil que joue ce contexte dans ma compréhension des choses aujourd'hui. Mais j'en ai tout de même conscience. Ma thèse est avant tout le fruit d'un héritage scientifique qui m'a été transmis par voies directes et indirectes de mes aînés. Je ne veux pas donner l'impression que je fais là une référence civile et abstraite à mes prédécesseurs lointains. Les Leibniz, d'Alembert, Lebesgue et autres illustres ancêtres scientifiques ont certainement fait du très bon travail. Leurs noms m'évoquent l'intérêt, voire la passion que j'ai eus pour mes cours de licence. De ce plaisir là, je me souviens. Du contenu des cours …je n'en dirais pas autant. Donc malgré tout le respect que je dois à leur très grande œuvre qui n'a laissé quasiment aucune trace dans mon heureuse mémoire incompétente, ici, je ne les remercierai pas. Je préférerais citer des personnes que des noms, celles qui ont eu une influence tangible sur moi au travers de leurs articles, de leurs exposés ou des discussions que j'ai eues avec elles ou avec des personnes qui ont été porteuses de leurs idées. La liste de ces personnes sans lesquelles le présent document n'existerait pas commence par François Robert, Eric Goles et Jacques Demongeot. Mon travail de thèse est une conséquence immédiate du leur. De façon consciente ou non, j'ai indéniablement été imprégnée par leurs idées et leurs points de vue. Mais la liste est certainement bien plus longue que la bibliographie de ce document. Et j'espère que tous ceux que je n'aurai pas mentionnés ici et dont le travail n'est pas non plus cité me le pardonneront, tout comme ceux que j'aurai seulement cités sans vraiment rendre honneur à leur contribution. Je m'en remets à ces quelques mots très inspirateurs que François Robert m'a écrits récemment à propos de l'élaboration de son livre *Discrete Iterations* [102]:

> *"...Alors, cette tâche aboutie, le bouquin prend son existence autonome et fait dorénavant partie d'un bien culturel de la collectivité mathématicienne qui en fait exactement ce qu'elle veut, y compris le critiquer et surtout, espérance basique de l'auteur, le prolonger, aller plus loin... Vous l'avez rencontré et j'en suis ravi ! La conséquence c'est que je vous prie de vous sentir tout à fait libre vis-à-vis de lui et de moi* [...] *le fait que vous vous en soyez nourrie est le principal, inutile de vouloir précisément rendre compte de ce que vous y avez utilisé, la référence bibliographique suffit : les enfants et petits enfants vont de l'avant et n'ont pas à se sentir débiteurs vis-à-vis de leurs parents ou grand-parents...qui sont eux surtout très heureux d'avoir de la descendance qui à son tour construit sa vie en s'alimentant à des sources diverses."* F. ROBERT

Il y a tout de même d'autres personnes qu'il m'est facile de mentionner parce qu'elles se sont impliquées dans mon travail de thèse voire y ont contribué très directement.

Pour commencer, je remercie Thomas Ward. Ses articles ainsi que les échanges que j'ai eus le grand plaisir d'avoir avec lui m'ont apporté un éclairage nouveau et une meilleure compréhension du travail que je présente dans le chapitre 2.

Je suis spécialement redevable envers les deux relecteurs des articles *Combinatorics of Boolean automata circuits dynamics* [32] et *Disjunctive networks and update schedules* [48]. Le premier de ces articles a donné lieu à une grosse partie du chapitre 2. Le deuxième est à l'origine des parties sur les réseaux disjonctifs, principalement dans le chapitre 3. Tous deux ont été relus avec une attention évidente et une rigueur appelant des corrections qui m'ont sans doute permis de les améliorer de façon substantielle. Encore une fois, ma compréhension de mon propre travail s'en est trouvée augmentée. Qui qu'ils soient, je les remercie vivement.

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Ceci m'amène aux quatre personnages principaux de ma thèse :
Eric Goles, Éric Rémila, Sylvain Sené et Jacques Demongeot.

Eric (Goles), ensemble nous avons produit les résultats sur les réseaux disjonctifs et sur les modes blocs-séquentiels (présentés dans la section 1 du chapitre 2 et dans le chapitre 3). Ils correspondent à une partie importante de ma thèse. Je te remercie donc chaudement pour ces résultats mais je te remercie aussi pour le rôle plus général que tu as joué dans mon parcours. Certains de mes premiers pas dans la recherche ont été faits sous ton encadrement. Ces choses-là sont difficiles à mesurer, mais je pense que ça n'a pu que beaucoup compter dans ma propre façon de faire de la recherche.

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Maintenant j'en viens à vous, Jacques et Sylvain. Cette thèse est le fruit de notre travail à tous les trois. Sylvain, bien sûr, l'étude des circuits en parallèle, c'est à dire le noyau de ma thèse (*cf.* section 3), c'est toi qui l'a proposée, tout comme l'idée que je trouve si élégante de distinguer la non-monotonie pour étudier son influence en elle-même (*cf.* section 2.4). Et les discussions plus abstraites qui servent d'ossature à ma thèse sont issues de nos très nombreuses conversations. Et tout ce que je peux avoir de courage intellectuel vient de toi … mais ça serait idiot d'essayer d'énumérer tout ce que je te dois, alors je m'arrête là. Jacques, je pense qu'il est évident que le document tout entier est imprégné de votre vision, de votre énergie impressionnante et de votre créativité absolument époustouflante. Je ne pense pas avoir été à la hauteur de la tâche de profiter de toutes les avalanches d'idées que vous avez très généreusement mises à ma disposition (bien que celles qui n'aient pas été exploitées dans ma thèse ne soient pas toutes oubliées pour autant), je m'enorgueillis de ce que j'ai su exploiter. L'honneur le plus grand est d'avoir eu le privilège de travailler avec vous deux et de bénéficier et m'enrichir autant de vos intelligences respectives pour lesquelles j'ai un extraordinaire respect et tellement d'admiration. Si ce n'est rien que de vous avoir connus, je suis fière et reconnaissante.

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OPENING

This thesis is concerned with *interaction networks*, that is, formal objects in which conceptual entities interact together in a simple manner to form a network. In this network, the only events that occur are due to the effective implementation of influences between entities. A singularity of the networks that are considered here is that the behaviours of these entities are assumed to be predetermined. In any particular configuration of the whole network, the way that one element in it reacts to the actual, particular influences of other network elements is supposed to obey some predefined rules (although all elements are not necessarily supposed to conform to the same rules). To this predetermination, network elements owe the name "*automata*".

The main objects of interest of this thesis are thus fundamentally very simple. In spite of this, some of my elder researchers and I still believe that studying these objects and devoting a PhD dissertation to them is worthwhile. In fact, as discussed in greater details in the first section of Chapter 1, I believe that precisely because of their simplicity, networks of interacting automata are especially interesting. Indeed, on the one hand, to build a "complex" system whose behaviour is challenging to understand, there is no need to involve "complex bricks". In reality, the example of interaction networks shows that simple interacting elements assemble to produce global effects that are not so simple to predict or even to understand. On the other hand, generally, the simpler the abstraction the more it has representational capacity and thus applications. Of course, in itself, this is not an argument that supports the study of automata networks (the concept of a point is quite simple, has a great deal of applications but I can't think of one of them that would be enough to motivate several years of dedicated attention to its study). But in addition to their ability to represent many real-life systems of interacting entities, formal interaction networks such as automata networks also have the characteristic property of being able to capture pertinently some of the *basic* but essential intricacies and heterogeneities that underlie some of these systems. And they do so without burdening the formalisation with unnecessary and unmanageable complexities, precisely because their simplicity disallows them to. Section 1 of Chapter 1 aims at expliciting this further. The rest of Chapter 1 describes exhaustively the features of the particular automata networks that are studied in this thesis. The main supplementary particularity of these consists in a further simplification which imposes that automata have only two possible states (which can be understood, for instance, as "active" and "inactive").

On this basis, our central problem is to better understand how the networks work. More precisely, the events that are considered in the present context consist in the local implementation of the influences between automata. Concretely, these are accounted for by the updating of the states of elements, making them adapt to their current incoming influences. The possible sets of these events (or updates), their results, and the ways they can successively occur is what determines the *global behaviours* of networks. Thus, in other terms, our central problem is to derive an understanding of how the different defining properties of networks relate to the way they globally behave. Of course, this raises the question of *what*, rigorously speaking, are the defining properties of networks. This question is settled by an answer *pro tempore* at the very beginning of this document to allow for some primary developments before it is openly addressed (in Section 3, Chap. 4).

Generally, the more specific aim of this thesis, which justifies fundamentally the minimality of the networks considered, is to better understand how precisely does the sequencing of events, *i.e.* the updating mode, impact on the behaviour of a network and also how its influence relates to other features of networks (such as their underlying interaction structures and the *nature* of the interactions that they involve) that also are decisive for their behaviours.

Chapter 2 starts by concentrating on the simplest of all update modes: the parallel update mode (or schedule) which repeatedly updates (the states of) all automata at once. It focuses on simple networks, with simple interaction rules between automata and/or simple underlying interaction structures. This way, it derives some useful primary results and insights for more general cases, that is, for more arbitrary networks with less constraints in their updating. The next chapter, Chapter 3, takes a very similar approach but precisely, relaxes the updating constraints by allowing some *sequentialisation* of local events. While still concentrating on deterministic update schedules, one of its main purposes derives from the comparison that can be made between the results that it presents and those of Chapter 2.

Next, comes Chapter 4. It discusses issues concerning the relation between various network features, how they manifest, how they may be observed and with what consequences to our general understanding of network behaviours. On these grounds it ends by addressing the question of the defining of networks in our context. With regards to its matter, this chapter could have figured right after preliminary Chapter 1. However, I have chosen otherwise to fit it in between Chapters 3 and 5 because it uses some notions that are made clearer and more tangible by the previous developments and because it serves as a motivation for the next chapter.

Driven particularly by the need for means of comparing the impact of update modes on network behaviours, Chapter 5 concentrates on the most general descriptions that can be made of network behaviours, namely, *general transition graphs* or GTGs. Informally, GTGs represent the set of *all* updates that can possibly happen in all situations among the set of situations that can effectively be formalised in our context, that is, in all network configurations. Thus, rather than defining a specific deterministic sequencing of events, they describe the whole set of possible ones. This way, defining a particular update mode for a network amounts to picking some of the updates “listed” in its GTG. Notably, in this framework, Chapter 5 ends by questioning the role of synchronism, *i.e.* how the possibility to update several automata at once (as with the parallel update schedule which severely imposes that *all* automata be updated at once) impacts globally on the network.

Finally, Chapter 6 concludes by confronting the various results and ideas presented throughout the thesis. Examining them *a posteriori* with some added hindsight, it discusses how they can be interpreted. And pursuing further an implicit connection that has already been suggested in this introduction, it highlights how the notion of time flow in systems of interacting automata relates to that of update modes via the idea that these precisely, by definition, determine the organisations of events in these systems. With this baseline idea, Chapter 6 exploits the various theoretical results presented in this thesis to propose some new perspectives involving the abstraction of time in automata networks.

CONTENTS

THE THEORY OF BOOLEAN AUTOMATA NETWORKS

1

1 The fundamental Boolean model

Formal developments and discussions in this thesis are extensively based on *Boolean automata networks*, or BANs, for short. And they are set in the framework that is defined by their definition and features. Informally, general *automata networks* are mathematical objects consisting in networks of interacting elements. An automata network of size n involves a set of n multi-state elements interacting with one another. Elements are supposed to have fixed predetermined behaviours that only possibly depend on the current states of other elements but are otherwise autonomous and predetermined. This is why they are called *automata* [15, 102, 103, 135]. In the general case, their set of possible states is any (finite) discrete set. In the particular case of *Boolean* automata networks considered here, they are supposed to take only two possible states, 0 (*inactive*) and 1 (*active*). Interactions between automata consist in *pre-defined* influences of some automata states on other automata states. For the sake of simplicity, we rather speak of *influences between automata*. In parts of this document, I assume (*locally*) *monotone* interactions, that is, whatever the automata i and j, if j can influence i then, either it always tends to activate i (pushing it towards state 1) or it always tends to inhibit i (pushing it towards state 0). It cannot both have a positive influence on i under certain conditions and a negative one under others. However, the running objective of this thesis is to understand what impacts on the overall behaviour of a network, as defined by the behaviours of each of its automata. The role of non-monotony is therefore a very interesting subject in itself. It is mentioned in Chapters 5 and 6. Concretely, as long as an automaton's state is not *updated*, it remains unchanged, heedless of what happens in the rest of the network. When it *is* updated, if necessary, it adapts to obey the set of all current influences that it receives from other automata in the network. By encompassing the whole set of automata states at once, a more global point of view can be taken. This way, focus can be put on the transitions "performed" by the entire BAN, as a result of one or several local automata state switches. The transitions and series of transitions of a BAN define its evolution or *behaviour*. The work presented here aims at providing some formally supported insights on the ways that BANs behave and on the ways that their behaviours are related to their other features.

1.1 Representational capacity & methodological framework

Without mentioning the Turing universality of BANs as a computation model, there are two main reasons for which I am convinced that the general *Boolean model* defined by BANs is fundamental. First, the intrinsic simplicity of these objects potentially grants them a great many applications. BANs can be used to model any system that satisfies the following three basic properties: (1) it is an *interaction system, i.e.* a system composed of separate interacting entities, (2) each of its entities has two notable/extreme states so that entities may be modelled pertinently by Boolean automata and (3) neither the events that the system is subjected to nor the underlying mechanisms responsible for them can be observed directly, only the outcome of accomplished changes can (and from them, possibly, information on their causes can be *infered*). Thus, as evidenced by the looseness of these conditions, BANs are generic models for a wide variety of systems encountered in nature. And since McCulloch & Pitts [74] and later Kauffman [61] and Thomas [117] first introduced Boolean models of neural and genetic networks, automata networks have been extensively used as formal representations of biological regulation systems [3, 4, 7, 11, 10, 21, 22, 23, 27, 30, 50, 34, 35, 44, 46, 57, 59, 62, 64, 75, 92, 93, 94, 99, 96, 97, 98, 102, 103, 109, 110, 115, 118, 119, 120, 122, 123, 129, 130].

But beyond their ability to provide a basic, formal means of representing real systems and their already proven practicality in terms of modelling, BANs also owe to their simplicity another characteristic advantage. They make it possible to define what I believe is an ideal *framework* in which to start addressing properly some fundamental (modelling) issues relative to interaction systems, and whose well-boundedness is necessary to ascertain the usability of theoretical properties uncovered in it. Instances of these issues are the various theoretical problems addressed in this thesis. And the question on which it is hinged – that of the global impact of the sequencing of events – especially, is an issue that I believe, given the present state of our understanding, cannot yet benefit from being set in an elaborate framework where the attention it can be given is necessarily more divided onto matters that do not directly and *fundamentally* relate to it. For this precise question, it seems pertinent on the contrary to initially set investigations in a framework such as that of BANs which allows not much more than to formalise the ability of network elements *to change or not to change* in each given network configuration. Generally, this second reason in favour of the Boolean model is discussed further in the next two paragraphs.

1.2 Sound navigation between reality and theory

It may be argued that modelling elements of a real network by Boolean automata which only have two possible states is a severe restriction which, in most contexts, results in an excessive oversimplification of reality. If the elements in a network do take more than two states, then it is probable that the whole range of their different states and the subtle nuances between them impact appreciably on the behaviour of other network elements and, *a fortiori*, on the global network behaviour. Consequently, the system may satisfy properties that are likely to elude a "Boolean modelling". By essence, such a modelling can indeed only focus on the roughest and the most obvious events, such as switches between two extreme states. For this reason, it may be argued that in some cases, a discrete modelling by *multi-state* automata [73, 99, 96, 115, 123] rather than Boolean automata, or perhaps even a continuous setting [14, 20, 54, 55, 76, 87, 113, 115, 131] is better suited. Now, however, in itself, determining the representational power of a model is essential. It cannot be bypassed if the predictions of the model are to be used safely and if information derived from its analysis is intended to explain a portion of "reality". With elaborate models that aim at better fitting to observations, or more subtly, various aspects of a real system and possibly of its environment are supposed to be accounted for simultaneously. Usually, this implies to rely on a wider range of parameters. In any case, the resulting inherent complexity of these models tends to obfuscate some essential questions and some unavoidable inaccuracies of the modelling process [85]. In particular, it makes more difficult rigorous, consistent identifications of the *modelling features* of a model (those that effectively represent something) and of the *modelled features* of the corresponding real system (those that are effectively accounted for by a property of the model). With BANs, these problems more obviously call for attention. Especially because of the Boolean nature of automata, information that can possibly be drawn from these models necessarily and evidently is partial, basic and much more qualitative than quantitative. Thus, the elementariness of their definition imposes significant, apparent restrictions to their modelling capacities and this way, favours a retrospective analysis of the modelling exercise itself [85]. It follows that, if given attention, the explanatory scope of BANs and of the very coarse Boolean modelling can be expected to be less ambiguous and easier to exploit than that of finer models.

1.3 Causality & scale changes

Further in these lines, BANs make it easier to manipulate a strict minimal notion of causality when exploring relationships between various system features. Thus, they allow to derive information that is perhaps not so ambitious in the sense that it concerns subtle, local properties but that is for this reason precisely, more reliable. This is especially important in interdisciplinary contexts involving "complex" systems (of which interaction networks are typical instances) whose features can be considered with different levels of abstraction and understood under the light of various possible interpretations. In these systems, although *local* characteristics[1] are very simple to understand and to describe, they combine to produce *global* emergent system behaviours that are more difficult to explain and to predict. Under the "complex systems paradigm", the main concern is the explanation of these global behaviours precisely. Studies endeavour to consider systems with hindsight and advance causal connections between global and local properties using a top-down approach. But their applied incentives risk generating implicit mental associations that tend to fill in for the obligatory approximations issuing from unexplicited scale changes in these connections (*cf.* Section 1, Chap. 6 and [85]). The simplicity of BANs can be exploited to avoid this. Rather than aiming investigations directly towards the explaining of global properties that have immediate applications, the notion of causality can be given sharp, dedicated attention while endeavouring to explicit its pre-requisite conditions and to maintain thorough comprehension of the scale changes that it involves. Methodologically, this imposes to take a constructive, bottom-up approach to any particular problem, define rigorously local properties of interest first, and then examine their relations, possibly with some gradual hindsight.

In this framework, the rest of the present chapter lists and defines formally the main features of BANs considered in the literature, and gives a few preliminary results along the way. This lays the grounds of what I call the "Theory of BANs".

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[1] *e.g. structural properties involving a manipulable set of individual interactions,* or *punctual behavioural properties involving a few specific system configurations and transitions,* as opposed to *more general properties involving entire network architectures, evolving systems,* and perhaps even *classes of systems.*

2 Basic features of Boolean automata networks

2.1 States, configurations and words

First of all, we introduce some conventions and notations. In the sequel, unless specified otherwise, the BANs that are considered are supposed to have size $n \in \mathbb{N}$ and their automata are assumed to be numbered from 0 to $n-1$. The set $\mathbf{V} = \{0, \dots, n-1\}$ refers to the set of network automata. As mentioned above, automata are supposed to have only two possible states. The binary set containing these two states is denoted by $\mathbb{B} = \{0, 1\}$. Global states of networks, called **configurations** in the sequel, are vectors of the set $\mathbb{B}^n$. If $x = (x_0 \dots x_{n-1}) \in \mathbb{B}^n$ (also written $x = x_0 \dots x_{n-1}$ for the sake of clarity in some contexts) is the network configuration, then the i^{th} component $x_i \in \mathbb{B}$ of this vector is the **state** of the i^{th} network automaton. And more generally (*cf.* Example 1.1), for any subset $\mathrm{W} \subseteq \mathbf{V}$ of automata, x_{W} denotes the configuration of the sub-network induced by W (if $\mathrm{W} = \{\sigma(0), \sigma(1), \dots\}$ and $y = x_{\mathrm{W}}$, then $y_k = x_{\sigma(k)}$). In the present context, special attention is paid to switches of automata states starting in a given network configuration. For this reason, the following notations concerning network configurations will be useful[2]:

$$\forall x = (x_0 \dots x_{n-1}) \in \mathbb{B}^n,$$

$$\begin{array}{ll} (1) & \forall i \in \mathbf{V},\ \overline{x}^{i} = (x_0 \dots x_{i-1} \neg x_i\, x_{i+1} \dots x_{n-1}), \\ (2) & \forall \mathrm{W} = \mathrm{W}' \uplus \{i\} \subseteq \mathbf{V},\ \overline{x}^{\mathrm{W}} = \overline{(\overline{x}^{i})}^{\mathrm{W}'} = \overline{(\overline{x}^{\mathrm{W}'})}^{i} \text{ and} \\ (3) & \overline{x} = \overline{x}^{\mathbf{V}} = (\neg x_0 \dots \neg x_{n-1}) \text{ as well as} \\ (4) & \forall b \in \mathbb{B},\ b^n = (b\, b \dots b) \in \mathbb{B}^n. \end{array} \tag{1.1}$$

We will note $\mathrm{d}(x, y)$ the Hamming distance between any two network configurations $x, y \in \mathbb{B}^n$:

$$\mathrm{d}(x, y) = |\mathrm{D}(x, y)| \text{ where } \mathrm{D}(x, y) = \{i \in \mathbf{V} \mid x_i \neq y_i\}. \tag{1.2}$$

And to switch from Boolean values in $\mathbb{B}$ to values in $\{-1, 1\}$ and back we use:

$$\forall b \in \mathbb{B},\ \mathbf{s}(b) = \begin{cases} 1 & \text{if } b = 1 \\ -1 & \text{if } b = 0 \end{cases} = 2b - 1 \in \{-1, 1\}$$

$$\text{and } \forall a \in \{-1, 1\},\ \mathbf{b}(a) = \mathbf{s}^{-1}(a) = \begin{cases} 0 & \text{if } a = -1 \\ 1 & \text{if } a = 1 \end{cases} = \frac{a+1}{2} \in \mathbb{B}.$$

Finally, since many developments concentrate on properties of configurations, it is useful to introduce some terminology and notations concerning (binary)

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[2] $\uplus$ denotes the disjoint union of sets ($A = B \uplus C \Leftrightarrow A = B \cup C \wedge B \cap C = \emptyset$) and $\neg$ denotes the negation of a Boolean value ($\neg 0 = 1$ and $\neg 1 = 0$).

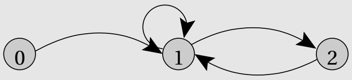
words. Thus, for any word $w \in \mathbb{B}^n$ of size n, and any $i \leq j < n$, we write $w[i,j]$ to denote the word of length $j - i + 1$ equal to $w[i,j] = w_i w_{i+1} \dots w_j$. A word $w = w_0 w_1 \dots w_{n-1} \in \mathbb{B}^n$ is said to be a **circular word** or a **necklace** of length n when $\forall i \in \mathbb{Z}$, the letter w_i is considered to be equal to $w_i = w_{i \bmod n}$.

2.2 Network structures and graph theory terminology

To describe a BAN $\mathcal{N}$, it is often convenient to start by representing its underlying *connectivity graph* [102, 103] or (**interaction**) **structure** (*cf.* Example 1.1). This can be done with a digraph $\mathbf{G} = (\mathbf{V}, \mathbf{A})$ whose set of nodes $\mathbf{V}$ is assimilated to the set of automata in the network and whose set of arcs $\mathbf{A}$ represents the set of interactions that take place in it: an arc $(j,i) \in \mathbf{A}$ of this digraph represents the influence that (the state of) automaton $j \in \mathbf{V}$ may possibly have on (the state of) automaton $i \in \mathbf{V}$. Let us emphasise that for the arc (j,i) to belong to $\mathbf{A}$, node j is not necessarily supposed to have a *constant* effective impact on i. It is merely supposed to have an impact in *some* network configurations and in at least one of them (*cf.* (1.3) below). In some works [93, 94, 99, 98, 109], a digraph $\mathbf{G}(x) = (\mathbf{V}, \mathbf{A}(x))$ is defined for every configuration $x \in \mathbb{B}^n$. It contains arcs $(j,i) \in \mathbf{A}(x)$ if and only if j does indeed have an appreciable influence on i in x (thus, $\mathbf{A} = \bigcup_{x \in \mathbb{B}^n} \mathbf{A}(x)$).

Example 1.1.

The figure below represents the structure $\mathbf{G} = (\mathbf{V}, \mathbf{A})$ *(where* $\mathbf{V} = \{0,1,2\}$ *and* $\mathbf{A} = \{(0,1),(1,1),(1,2),(2,1)\}$*) of a network of three automata. Automaton* 0 *is influenced by no automaton which means that it always tends to take the same state. Automaton* 1 *is influenced by all three automata including itself. And automaton* 2 *is influenced just by automaton* 1.



In network configuration $x = (1\,1\,0)$*, the configuration of set* $\mathrm{W} = \{0,2\} \subseteq \mathbf{V}$ *(or of the sub-graph or sub-network induced by* W*) is* $x_{\mathrm{W}} = (1\,0)$.

Considering an arbitrary digraph $\mathbf{G} = (\mathbf{V}, \mathbf{A})$, the set $\mathbf{V}^-_{\mathbf{G}}(i) = \{j \mid (j,i) \in \mathbf{A}\} \subseteq \mathbf{V}$ (resp. $\mathbf{V}^+_{\mathbf{G}}(i) = \{j \mid (i,j) \in \mathbf{A}\}$) denotes the **in-** (resp. **out-**) **neighbourhood** of an arbitrary node $i \in \mathbf{V}$, that is, its set of **in-** (resp. **out-**) **neighbours** and $\deg^-_{\mathbf{G}}(i) = |\mathbf{V}^-_{\mathbf{G}}(i)|$ (resp. $\deg^+_{\mathbf{G}}(i) = |\mathbf{V}^+_{\mathbf{G}}(i)|$) denotes its **in-** (resp. **out-**) **degree**. In the sequel we use the term **path** (resp. **undirected path**) to refer to an ordered list $\{i_0, i_1, \dots, i_\ell\} \subseteq \mathbf{V}$ of nodes such that $\forall k < \ell$, $(i_k, i_{k+1}) \in \mathbf{A}$ (resp. either $(i_k, i_{k+1}) \in \mathbf{A}$ or $(i_{k+1}, i_k) \in \mathbf{A}$). Closed paths are called **cycles**. Unless specified otherwise, paths and cycles are supposed to be directed and non-necessarily simple (nodes can be repeated). In the sequel, the abbreviation SCC is used for *strongly connected component.* SCCs of a digraph that have no outgoing (resp. incoming) arcs are called **terminal SCCs** (resp. **source SCCs**). SCCs that have at least two nodes are called **non-trivial SCCs**.

Abusing terminology, we also use the term **cycle** to refer to a digraph $\mathbb{C}_n = (\mathbf{V}, \mathbf{A})$ of the form of the one pictured on the left of Fig. 1.1. By default, the set of nodes of such a digraph of size or length n is assimilated to $\mathbb{Z}/n\mathbb{Z}$ so that, considering two nodes $i, j \in \mathbf{V}$, $i + j$ denotes node $i + j \bmod n$. The set of its arcs is then $\mathbf{A} = \{(i, i+1) \mid i \in \mathbb{Z}/n\mathbb{Z}\}$.

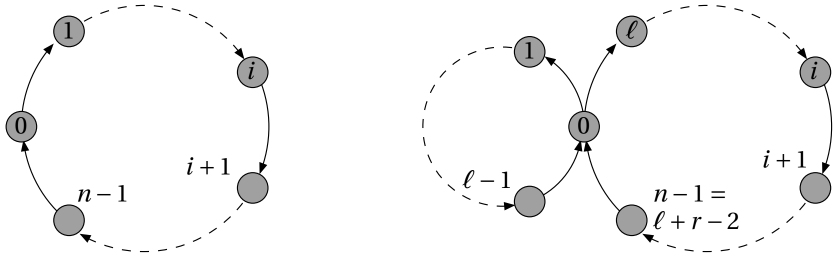


Figure 1.1: *Left: a cycle* $\mathbb{C}_n = (\mathbf{V}, \mathbf{A}) = (\mathbb{Z}/n\mathbb{Z}, \{(i, i+1) \mid i \in \mathbb{Z}/n\mathbb{Z}\})$ *of length* n. *Right: a double-cycle* $\mathbb{D}_{\ell r} = \mathbb{C}_\ell \times \mathbb{C}_r$.

A **double-cycle** $\mathbb{D}_{\ell r} = (\mathbf{V}, \mathbf{A})$, written $\mathbb{D}_{\ell r} = \mathbb{C}_\ell \times \mathbb{C}_r$, is a digraph of size $n = \ell + r - 1$ like the one represented on the right of Fig. 1.1. It consists of two sub-graphs called its **side-cycles** that intersect on node 0. The set of nodes of the **left-cycle** $\mathbb{C}_\ell$ (resp. of the **right-cycle** $\mathbb{C}_r$) is a $\mathbf{V}^L = \mathbb{Z}/\ell\mathbb{Z}$ (resp. $\mathbf{V}^R = \{0\} \cup \{\ell - 1 + i \mid i \neq 0 \in \mathbb{Z}/r\mathbb{Z}\}$). Its size ℓ (resp. r) is called the the **left-** (resp. **right-**) **size** of $\mathbb{D}_{\ell r}$.

2.3 Local transition functions

The structure $\mathbf{G} = (\mathbf{V}, \mathbf{A})$ of a BAN $\mathcal{N}$ gives the existence of the oriented interactions that it involves: an arbitrary automaton of the network depends on each of its in-neighbours at some point, and it depends on no other. But $\mathbf{G}$ neither specifies the nature of these interactions nor the conditions under which they effectively occur. This is done by assigning a **local transition function** $\mathrm{f}_i : \mathbb{B}^n \to \mathbb{B}$ to each automaton $i \in \mathbf{V}$ (*cf.* Example 1.2) so that the following is satisfied:

$$\forall i \in \mathbf{V},\ \exists x \in \mathbb{B}^n,\ \mathrm{f}_i(x) \neq \mathrm{f}_i(\overline{x}^j) \ \Leftrightarrow\ (j, i) \in \mathbf{A}. \tag{1.3}$$

As mentioned in the previous paragraph, this implies the *minimality* of $\mathbf{G}$ in the sense that all of its arcs are supposed to represent *effective* interactions. With this new definition, sets of arcs of digraphs $\mathbf{G}(x)$, $x \in \mathbb{B}^n$ mentioned above equal $\mathbf{A}(x) = \{(j, i) \mid \mathrm{f}_i(x) \neq \mathrm{f}_i(\overline{x}^j)\}$.

Example 1.2.

In agreement with the structure of Example 1.1, automata 0, 1 *and* 2 *could for instance be assigned the following local transition functions, respectively:* $\forall x \in \mathbb{B}^n$, $\mathrm{f}_0(x) = 1$, $\mathrm{f}_1(x) = x_1 \vee (x_0 \wedge \neg x_2)$, $\mathrm{f}_2(x) = \neg x_1$.

By default, for the sake of simple notations, the arity of a local transition function f_i is supposed to equal the network size n. However, for the same reasons, in some cases (*cf.* Examples 1.3 and 1.5), we will rather assume that the arity of f_i equals the in-degree of i in $\mathbf{G}$:

$$\mathrm{f}_i : \mathbb{B}^{\deg^-_{\mathbf{G}}(i)} \to \mathbb{B}. \tag{1.4}$$

Formally, BANs of size $n \in \mathbb{N}$ are defined exactly by a set of local transition functions of cardinal n. This choice of definition is discussed later in Section 3, Chap. 4.

Example 1.3. BACs and BADs

A BAN *of size n whose structure is a cycle (resp. a double-cycle) as in Fig. 1.1 is called a* ***Boolean automata cycle*** *(resp. a* ***Boolean automata double-cycle****) or* ***BAC*** *(resp.* ***BAD****) for short. With the notation of (1.4), the minimality of structures expressed in (1.3) requires that the set of local transition functions* $\mathscr{C}$ *and* $\mathscr{D}$ *that respectively define a* BAC *and a* BAD *satisfy:*

$$\mathscr{C} \subseteq \{\mathrm{id}, \mathrm{neg}\}^n \text{ and } \mathscr{D} \setminus \{\mathrm{f}_0\} \subseteq \{\mathrm{id}, \mathrm{neg}\}^{n-1}$$

where $\mathrm{id} : a \in \mathbb{B} \mapsto a$ *and* $\mathrm{neg} : a \in \mathbb{B} \mapsto \neg a$.

2.4 Local monotony, signed paths and frustrated arcs

The local transition functions f_i of many of the BANS $\mathscr{N} = \{\mathrm{f}_i \mid i \in \mathbf{V}\}$ considered in this document are **locally monotone** (*cf.* Examples 1.4 and 1.5): for any $j \in \mathbf{V}^-_{\mathbf{G}}(i)$, f_i either satisfies

$$\forall x \in \mathbb{B}^n, x_j = 0 \Rightarrow \mathrm{f}_i(x) \leq \mathrm{f}_i(\overline{x}^j),$$

in which case arc (j, i) is said to be **positive** (the state of i tends to imitate that of j) and we write $\mathrm{sign}_{\mathscr{N}}(j, i) = +1$, or it satisfies

$$\forall x \in \mathbb{B}^n, x_j = 0 \Rightarrow \mathrm{f}_i(x) \geq \mathrm{f}_i(\overline{x}^j),$$

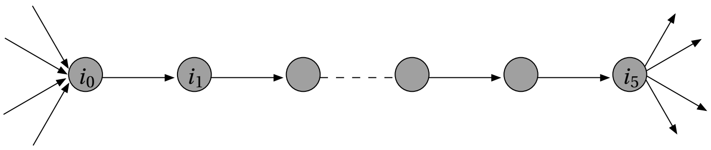
in which case (j,i) is said to be **negative** (the state of i tends to negate that of j) and we write $\mathrm{sign}_{\mathcal{N}}(j,i) = -1$. BANs containing only locally monotone local transition functions (see in particular the BADs of Example 1.5 studied in Chapter 2) are called (**locally**) **monotone** themselves. Assuming the (local) monotony of BANs is a notable, convenient initial simplification. However, the way that non-monotony impacts on network behaviours is an interesting subject *per se* whose study is initiated in Section 2.4, Chap. 5.

Example 1.4. Disjunctive & conjunctive BANs

Notable examples of locally monotone local transition functions $\mathrm{f}_i : \mathbb{B}^n \to \mathbb{B}$ *that can be assigned to an arbitrary automaton* $i \in \mathbf{V}$ *of a* BAN $\mathcal{N}$ *are the Boolean functions that output the disjunction and conjunction of the inputs* $\mathbf{b}^i_j(x) = \mathbf{b}\big(\mathrm{sign}_{\mathcal{N}}(j,i)\cdot\mathbf{s}(x_j)\big)$ *of automaton i (where* $j \in \mathbf{V}^-_{\mathbf{G}}(i)$*):*

$$\mathrm{f}_i : x \mapsto \bigvee_{(j,i)\in\mathbf{A}} \mathbf{b}^i_j(x) \;\; \textit{and} \;\; \mathrm{f}_i : x \mapsto \bigwedge_{(j,i)\in\mathbf{A}} \mathbf{b}^i_j(x). \qquad (1.6)$$

Locally monotone BAN*s only containing functions of the first sort (resp. of the second) are called* ***disjunctive BANs*** *or* ***DANs*** *(resp.* ***conjunctive BANs****). Both sorts of* BAN*s are entirely defined by their signed structures. And when these contain no negative arcs, they are said to be* ***positive*** *and are entirely defined by their (unsigned) structures. Thus, abusing language, in the sequel we will speak of a* BAN $\mathbf{G} = (\mathbf{V},\mathbf{A})$ *to mean a positive disjunctive/conjunctive* BAN *with structure* $\mathbf{G}$. *In particular, in a positive* DAN $\mathbf{G} = (\mathbf{V},\mathbf{A})$, *when the state of any* $i \in \mathbf{V}$ *is updated, i takes state* 1 *if and only if one of its in-neighbours is currently in state* 1.



Since the local transition functions of automata with in-degree 1 in $\mathbf{G}$ must be non-constant, they either equal id or neg (*cf.* Example 1.3). Therefore, they necessarily are locally monotone. Let us call **nude path** (*cf.* figure above) a path $P = \{i_0, i_1, \dots, i_\ell\}$ without any intersections, *i.e.* such that $\forall 0 \le k < \ell$, $\deg^+_{\mathbf{G}}(i_k) = 1$ and $\forall 0 < k \le \ell$, $\deg^-_{\mathbf{G}}(i_k) = 1$. We define the **sign** of such a path P by:

$$\mathrm{sign}_{\mathcal{N}}(P) = \prod_{0<k<\ell} \mathrm{sign}_{\mathcal{N}}(i_k, i_{k+1})$$

and we say that P is **positive** if $\mathrm{sign}_{\mathcal{N}}(P) = +1$, *i.e.* if it contains an even number of negative arcs, and say that P is **negative** if $\mathrm{sign}_{\mathcal{N}}(P) = -1$, *i.e.* if it contains an odd number of negative arcs. Also, for a nude path $P = \{i_0, i_1, \dots, i_\ell\}$ or $P = \{0, 1, \dots, \ell\}$ in a BAN $\mathcal{N} = \{\mathrm{f}_i\}$, we use the following notation (*cf.* Example 1.5):

$$\mathrm{f}[\ell,1] = \mathrm{f}_\ell \circ \mathrm{f}_{\ell-1} \circ \ldots \circ \mathrm{f}_1. \tag{1.7}$$

Example 1.5. Signs of BACs and BADs

A BAC $\mathscr{C} = \{\mathrm{f}_i \mid i \in \mathbb{Z}/n\mathbb{Z}\}$ *whose structure* $\mathbb{C}_n$ *defines a positive (resp. negative) closed nude path of length n is called a* ***positive* BAC** *(resp. a* ***negative* BAC***) and is denoted by* $\mathscr{C} = \mathscr{C}_n^+$ *(resp.* $\mathscr{C}_n^-$ *). It satisfies* $\mathrm{f}[i, i+1] = \mathrm{id}$ *(resp.* neg*),* $\forall i \in \mathbb{Z}/n\mathbb{Z}$*. We write:*

$$\mathrm{sign}(\mathscr{C}) = \mathrm{sign}_{\mathscr{C}}(\mathbb{C}_n) = +1 \textit{ (resp. } -1\textit{)}. \tag{1.8}$$

Requiring that a BAD $\mathscr{D}$ *with structure* $\mathbb{D}_{\ell r} = \mathbb{C}_\ell \times \mathbb{C}_r$*, be locally monotone imposes that arcs* $(\ell-1, 0)$ *and* $(n-1, 0)$ *may be signed as well as all the others. In this case, we define the* ***left-sign*** $s = \mathrm{sign}_{\mathscr{D}}(\mathbb{C}_\ell) \in \{-,+\}$ *and the* ***right-sign*** $s' = \mathrm{sign}_{\mathscr{D}}(\mathbb{C}_r) \in \{-,+\}$ *of* $\mathscr{D}$ *to be, respectively, the signs of its left and right cycles if they were isolated. And we use the notation* $\mathscr{D} = \mathscr{D}_{\ell r}^{ss'}$*. Then,* $\mathscr{D}_{\ell r}^{++}$*,* $\mathscr{D}_{\ell r}^{--}$ *and* $\mathscr{D}_{\ell r}^{-+}$ *are respectively called a* ***positive*** BAD*, a* ***negative*** BAD *and a* ***mixed*** BAD*. Further, there are only two possibilities for the local transition function of intersection automaton* 0*:*

$$\mathrm{f}_0(x_{\ell-1}, x_{n-1}) \; = \; \mathrm{f}_0^L(x_{\ell-1}) \diamond \mathrm{f}_0^R(x_{n-1}) \tag{1.9}$$

where $\diamond \in \{\wedge, \vee\}$ *and where, denoting* $\mathbf{b}_j^i : x_j \mapsto \mathbf{b}(\mathrm{sign}(j,i) \cdot \mathbf{s}(x_j))$*,*

$$\mathrm{f}_0^L = \mathbf{b}_{\ell-1}^0, \;\; \mathrm{f}_0^R = \mathbf{b}_{n-1}^0 \in \{\mathrm{id}, \mathrm{neg}\}.$$

Thus, a locally monotone BAD *is either a conjunctive or a disjunctive* BAN*. Furthermore, although automaton* 0 *of* $\mathscr{D}_{\ell r}^{ss'}$ *does not belong to any nude path, in analogy to (1.7) we define:*

$$\mathrm{f}[i,j]^L = \begin{cases} \mathrm{f}[i,j] & \textit{if } i \geq j \in \mathbf{V}^L, \\ \mathrm{f}_i \circ \mathrm{f}_{i-1} \circ \ldots \circ \mathrm{f}_0^L \circ \mathrm{f}_{\ell-1} \circ \ldots \mathrm{f}_j & \textit{if } i < j \in \mathbf{V}^L \end{cases} \tag{1.10}$$

and similarly we define $\mathrm{f}[i,j]^R$ *(replacing L by R and* $\ell - 1$ *by* $n-1$*). Let us note that* $s = - \;\Leftrightarrow\; \mathrm{f}[i, i+1]^L = \mathrm{neg} \; \forall i \in \mathbf{V}^L$ *and similarly for the right side and* s'*.*

In a configuration $x \in \mathbb{B}^n$, signed arcs $(j,i) \in \mathbf{A}$ such that:

$$\mathbf{s}(x_j) \cdot \mathbf{s}(x_i) \neq \mathrm{sign}_{\mathscr{N}}(j,i)$$

are said to be **frustrated** [17, 44, 128, 133]. The set of frustrated arcs in x is denoted by **FRUS**(x).

2.5 Events and updates

In any network configuration, zero, one or several **punctual events** may take place. Here, the punctual events that are considered consist in the update of one or several automata states. More precisely, supposing that the network is currently in configuration $x \in \mathbb{B}^n$, we say that automaton $i \in \mathbf{V}$ is **updated** if its state switches from x_i, its current state, to $\mathrm{f}_i(x)$, its new state. Possibly, $\mathrm{f}_i(x) = x_i$ holds so the update of i is not necessarily effective in x. In any case, this local event yields a global network configuration change (possibly ineffective) which is described by the i-**update function** $\mathrm{F}_i : \mathbb{B}^n \to \mathbb{B}^n$ (*cf.* Example 1.6)[3]:

$$\forall x \in \mathbb{B}^n,\ \mathrm{F}_i(x) = (x_0 \dots x_{i-1}\, \mathrm{f}_i(x)\, x_{i+1} \dots x_{n-1}). \tag{1.11}$$

This event is said to be **atomic** because it involves only one automaton. A **non-atomic** event corresponds on the contrary to the simultaneous update of several automata. In the general case, the W-**update function**[4] $\mathrm{F}_\mathrm{W} : \mathbb{B}^n \to \mathbb{B}^n$ describes the network configuration change that results from the non-atomic update of all automata in an arbitrary set $\mathrm{W} \subseteq \mathbf{V}$:

$$\forall x \in \mathbb{B}^n,\ \forall i \in \mathbf{V},\ \mathrm{F}_\mathrm{W}(x)_i = \begin{cases} \mathrm{f}_i(x) & \text{if } i \in \mathrm{W}, \\ x_i & \text{otherwise.} \end{cases} \tag{1.12}$$

Let us emphasise that the *punctuality* of events mentioned above refers to their happening in a *unique step* whereas the *atomicity* of events characterises their *nature*. All atomic as well as all non-atomic events are punctual. No other punctual events are considered here but the next paragraph mentions more general events consisting in series of successive punctual ones.

2.6 Transitions and derivations

Network **transitions** are couples $(x, y) \in \mathbb{B}^n \times \mathbb{B}^n$ that represent changes of network configurations (from x to y) due to the occurrence of one or a series of punctual events. Transitions that involve only one punctual event are called **elementary**. They satisfy $y = \mathrm{F}_\mathrm{W}(x)$ for some (possibly empty) set $\mathrm{W} \subseteq \mathbf{V}$ of automata and are denoted as follows:

$$x \longrightarrow y,\ x \xrightarrow{\mathrm{W}} y \text{ or } x \text{ —W→ } y.$$

The set of all elementary transitions of a BAN $\mathcal{N}$ is:

$$\mathrm{T}_\mathcal{N} = \bigcup \{(x, \mathrm{F}_\mathrm{W}(x)) \mid x \in \mathbb{B}^n,\ \mathrm{W} \subseteq \mathbf{V}\}. \tag{1.13}$$

---

[3] Note that usually $\mathrm{F}_i(x)$ is not to be confused with $\mathrm{F}(x)_i$.

[4] $\forall i \in \mathbf{V}$, F_i obviously equals $\mathrm{F}_{\{i\}}$ but we prefer the first notation.

Example 1.6.

The following table defines the update functions F_1 *and* $\mathrm{F}_{\{0,2\}}$ *for the network considered in Examples 1.1 and 1.2:*

$x = (x_0\, x_1\, x_2)$	$\mathrm{f}_0(x)$	$\mathrm{f}_1(x)$	$\mathrm{f}_2(x)$	$\mathrm{F}_1(x)$	$\mathrm{F}_{\{0,2\}}(x)$
(000)	1	0	1	(000)	(101)
(001)	1	0	1	(001)	(101)
(010)	1	1	0	(010)	(110)
(011)	1	1	0	(011)	(110)
(100)	1	1	1	(110)	(101)
(101)	1	0	1	(101)	(101)
(110)	1	1	0	(110)	(110)
(111)	1	1	0	(111)	(110)

There are two main types of elementary transitions. **Asynchronous** or **atomic transitions** correspond to atomic updates. **Synchronous** or **non-atomic transitions** correspond to non-atomic updates. When emphasis needs to be put on the atomicity (resp. non-atomicity) of an elementary transition $x \xrightarrow{\{i\}} y = \mathrm{F}_i(x)$ (resp. $x \xrightarrow{\mathrm{W}} y = \mathrm{F}_{\mathrm{W}}(x)$, $|\mathrm{W}| > 1$) it is rather written:

$$x \longrightarrow\!\!\!\triangleright\ y,\ x \overset{i}{\longrightarrow\!\!\!\triangleright}\ y \text{ or } x \xrightarrow{i}\!\!\!\triangleright\ y \text{ (resp. } x \longrightarrow\!\!\!\blacktriangleright\ y,\ x \overset{\mathrm{W}}{\longrightarrow\!\!\!\blacktriangleright}\ y \text{ or } x \xrightarrow{\mathrm{W}}\!\!\!\blacktriangleright\ y).$$

The reflexive and transitive closures of $\longrightarrow$, $\longrightarrow\!\!\!\triangleright$ and $\longrightarrow\!\!\!\blacktriangleright$ are respectively denoted by $\longrightarrow\!\!\!\!\twoheadrightarrow$, $\longrightarrow\!\!\!\triangleright\!\!\triangleright$ and $\longrightarrow\!\!\!\blacktriangleright\!\!\blacktriangleright$. General network **transitions** $(x, y) \in \mathbb{B}^n \times \mathbb{B}^n$, *i.e.* $x \longrightarrow\!\!\!\!\twoheadrightarrow y$, are sequences of zero, one or several elementary transitions:

$$\begin{aligned} & x \longrightarrow\!\!\!\!\twoheadrightarrow y \Leftrightarrow \\ & \quad \exists \ell \in \mathbb{N},\ \exists x^1, \dots, x^{\ell-1} \in \mathbb{B}^n,\ x \longrightarrow x^1 \longrightarrow \dots \longrightarrow x^{\ell-1} \longrightarrow y. \end{aligned} \tag{1.14}$$

In other terms, any transition $x \longrightarrow\!\!\!\!\twoheadrightarrow y$ corresponds to an ordered list of sets $(\mathrm{W}_t)_{1 \le t \le \ell}$ such that $y = \mathrm{F}_{\mathrm{W}_\ell} \circ \ldots \circ \mathrm{F}_{\mathrm{W}_2} \circ \mathrm{F}_{\mathrm{W}_1}(x)$. When this list is known, we use it to label the transition and specify the sequence of punctual updates it involves:

$$x \xrightarrow{\mathrm{W}_1, \mathrm{W}_2, \ldots, \mathrm{W}_\ell}\!\!\!\!\twoheadrightarrow y.$$

Derivations (or **trajectories**[5]) are ordered lists of elementary or non-elementary transitions (x^0, x^1), (x^1, x^2),..., $(x^{\ell-1}, x^\ell)$, simply written:

---

[5] Unless the network is assumed to be a dynamical system, we prefer the less restrictive term *derivation*. It conveys the notion of *existence* of possible system evolutions without suggesting necessarily that the system is assumed to evolve at all. The term *trajectory* suggests a significantly different paradigm in which the system is placed in a temporalised setting (*cf.* [85] and Section 1, Chap. 6).

$$x^0 \twoheadrightarrow x^1 \twoheadrightarrow x^2 \twoheadrightarrow \ldots \twoheadrightarrow x^{\ell-1} \twoheadrightarrow x^{\ell}.$$

Derivations only involving elementary transitions are said to be **elementary** (*cf.* Example 1.7).

Example 1.7.

The derivation bellow involves three elementary transitions, (x^0, x^1), (x^1, x^2) *and* (x^3, x^4), *as well as one non-elementary transition,* (x^2, x^3), *which could itself be broken into a derivation of several elementary transitions if the updates it involves were known:*

$$x^0 \xrightarrow{\mathrm{W}} x^1 = \mathrm{F}_{\mathrm{W}}(x^0) \overset{i}{\longrightarrow\!\triangleright} x^2 = \mathrm{F}_i(x^1) \twoheadrightarrow x^3 \xrightarrow{\mathrm{W}'} x^4 = \mathrm{F}_{\mathrm{W}'}(x^3).$$

2.7 Automata stability and transition effectiveness

As mentioned above, when an automaton is updated in a given configuration $x \in \mathbb{B}^n$, it does not necessarily change states. We define the set $\mathcal{U}(x)$ of automata that can indeed change states in x and that do so if and only if they are updated (they are refered to as automata that “call for change” or “for an updating” in [92]):

$$\mathcal{U}(x) = \{i \in \mathbf{V} \,|\, \mathrm{f}_i(x) \neq x_i\}.$$

Automata in $\mathcal{U}(x)$ are said to be **unstable** in x and those in $\overline{\mathcal{U}}(x) = \mathbf{V} \setminus \mathcal{U}(x)$ are said to be **stable** in x. Note that frustration relates to instability in that, in a monotone BAN, all unstable automata $i \in \mathcal{U}(x)$ have at least one incoming frustrated arc $(j,i) \in \mathbf{FRUS}(x)$. We denote by $u(x) = |\mathcal{U}(x)|$ **the number of automata that are unstable in** x or **the number of instabilities in** x. It can be understood as the *velocity* or *momentum* of the BAN in x. Configurations x in which all automata are stable ($u(x) = 0, \overline{\mathcal{U}}(x) = \mathbf{V}$) are called **stable configurations**. It is important to note that the only couples $(x,y) \in \mathbb{B}^n \times \mathbb{B}^n$ that indeed are elementary network transitions are the couples that satisfy: $\mathrm{D}(x,y) \subseteq \mathcal{U}(x)$ (*cf.* (1.2) for notations). And for any subset $\mathrm{W} \subseteq \mathbf{V}$, the following holds:

$$\forall x \in \mathbb{B}^n,\ \mathrm{F}_{\mathrm{W}}(x) \ = \ \mathrm{F}_{\mathrm{W} \cap \mathcal{U}(x)}(x) \ = \ \overline{x}^{\mathrm{W} \cap \mathcal{U}(x)}.$$

Let $\mathrm{D} = \mathrm{D}(x,y) = \mathrm{W} \cap \mathcal{U}(x)$ and let us suppose that $\mathrm{D} \neq \emptyset$. By (1.13), $x \xrightarrow{\mathrm{D}} \mathrm{F}_{\mathrm{D}}(x) = \overline{x}^{\mathrm{D}}$ and $x \xrightarrow{\mathrm{W}} \mathrm{F}_{\mathrm{W}}(x) = \overline{x}^{\mathrm{D}}$ rigorously denote the same transition $(x, \overline{x}^{\mathrm{D}})$. However, in the sequel, the convention of (1.13) that defines transitions as unlabelled couples will be abused because $x \xrightarrow{\mathrm{D}} \overline{x}^{\mathrm{D}}$ and $x \xrightarrow{\mathrm{W}} \overline{x}^{\mathrm{D}}$ do not correspond to the same events (*i.e.* updates) although the events that they correspond to respectively produce the same effect. $x \xrightarrow{\mathrm{D}} \overline{x}^{\mathrm{D}}$ and the update of D are said to be **effective**; $x \xrightarrow{\mathrm{W}} \overline{x}^{\mathrm{D}}$ and the update of W are said to be **partially effective** or **partially null**. And further, any transition $(x, x) \in \mathrm{T}_{\mathcal{N}}$ labelled by a subset of $\overline{\mathcal{U}(x)}$ is called a **null transition**. By extension, we also naturally speak of (partially) effective and null non-elementray transitions or derivations.

Elementary transitions are ordered according to the number automata that they effectively update. Thus, a transition $x \xrightarrow{\mathrm{W}} y$ is said to be **smaller** than a transition $x' \xrightarrow{\mathrm{W}'} y'$ if $|\mathrm{W} \cap \mathcal{U}(x)| = \mathrm{d}(x, y) < |\mathrm{W}' \cap \mathcal{U}(x')| = \mathrm{d}(x', y')$.

3 Update modes & schedules

Generally, an **update mode** defines a restriction of the set of updates, and thus of transitions, that can be made by a BAN. An *operating mode* [102, 103] or *update schedule* is a deterministic update mode. More precisely, an **update schedule** δ of a set $\mathbf{V}$ of automata – and by extension, of a BAN whose set of automata is $\mathbf{V}$ – is defined by an ordered (finite or infinite) list $(\mathrm{W}_t)_{t \in S}$ ($S \subseteq \mathbb{N}$) of non-empty sets of automata ($\forall t \in S,\ \emptyset \neq \mathrm{W}_t \subseteq \mathbf{V}$). We write $\delta := (\mathrm{W}_t)_{t \in S}$ or just $\delta := \mathrm{W}_0, \mathrm{W}_1, \ldots, \mathrm{W}_t, \ldots$. Starting in an arbitrary configuration x, a BAN that is updated with $\delta := (\mathrm{W}_t)_{t \in S}$ sequentially takes configurations $\mathrm{F}_{\mathrm{W}_0}(x)$, $\mathrm{F}_{\mathrm{W}_1} \circ \mathrm{F}_{\mathrm{W}_0}(x), \ldots, \mathrm{F}_{\mathrm{W}_t} \circ \ldots \circ \mathrm{F}_{\mathrm{W}_0}(x), \ldots$ and follows the elementary derivation:

$$x \xrightarrow{\mathrm{W}_0} \mathrm{F}_{\mathrm{W}_0}(x) \xrightarrow{\mathrm{W}_1} \cdots \xrightarrow{\mathrm{W}_t} \mathrm{F}_{\mathrm{W}_t} \circ \cdots \circ \mathrm{F}_{\mathrm{W}_0}(x) \xrightarrow{\mathrm{W}_{t+1}} \cdots \tag{1.15}$$

In particular, δ only allows the BAN to perform elementary transitions that update one of the sets W_t, $t \in S$. If $\mathrm{U} \subseteq \mathbf{V}$ is a set of automata that differs from all of these sets W_t, $\forall t \in S$, then $x \xrightarrow{\mathrm{U}} \mathrm{F}_{\mathrm{U}}(x)$ is not an elementary transition that can be done by the BAN under δ. Also, the sets W_t themselves cannot either be updated in any configuration. They can only be in the configurations of the sets $X_t \subseteq \mathbb{B}^n$ defined by induction:

$$\begin{cases} X_0 = \mathbb{B}^n, \\ \forall t \in S,\ X_{t+1} = \mathrm{F}_{\mathrm{W}_t}(X_t) = \{\mathrm{F}_{\mathrm{W}_t}(x) \mid x \in X_t\}. \end{cases} \tag{1.16}$$

Thus, $\forall x \in \mathbb{B}^n$, $\forall t \in S$, δ allows $x \xrightarrow{\mathrm{W}_t} \mathrm{F}_{\mathrm{W}_t}(x)$ if and only if $x \in X_t$.

3.1 Periodic update schedules

Periodic update schedules of arbitrary period $p \in \mathbb{N}$ correspond to infinite periodic lists $\delta := \mathrm{W}_0, \mathrm{W}_1, \ldots, \mathrm{W}_{p-1}, \mathrm{W}_0, \mathrm{W}_1, \ldots, \mathrm{W}_{p-1}, \ldots$ (*e.g.* the update schedule of period 2 in Example 1.8). For the sake of simplicity they are rather defined by finite ordered lists of size p: we write $\delta :\equiv (\mathrm{W}_t)_{t \in \mathbb{N}/p\mathbb{N}}$ or $\delta :\equiv \mathrm{W}_0, \mathrm{W}_1, \ldots, \mathrm{W}_{p-1}$. And they can also be defined as functions (*cf.* Example 1.9) so that $\forall i \in \mathbf{V}$, $\delta(i)$ is

Example 1.8.

Suppose that the BAN *considered in Examples 1.1 and 1.2 is updated by the periodic update schedule* $\delta :\equiv \{1\},\{0,2\},\{1\},\{0,2\},\ldots$ *Since* δ *does not allow the atomic update of automaton* 2*, any labelled elementary transition of the form* $(0\,x_1\,x_2) \xrightarrow{\{2\}} (0\,x_1\,\neg x_1)$ *is impossible. The sets* $X_t,\ t \in \mathbb{N}$ *defined in (1.16) equal (cf. Example 1.6):* $X_0 = \mathbb{B}^3$*,* $X_1 = \mathbb{B}^3 \setminus \{(1\,0\,0)\}$ *and,* $\forall t \geq 2$*,* $X_t = \{(1\,0\,1),(1\,1\,0)\}$*. Thus,* δ *does not allow the set* $\{0,2\}$ *to be updated in configuration* $(1\,0\,0)$ *either.*

the set of steps involving automaton i in the periodic sequence $(\mathrm{W}_t)_{t\in\mathbb{N}/p\mathbb{N}}$:

$$\delta : \mathbf{V} \to \mathscr{P}(\mathbb{N}/p\mathbb{N}) \text{ such that } \forall t \in \mathbb{N}/p\mathbb{N},\ t \in \delta(i) \Leftrightarrow i \in \mathrm{W}_t \qquad (1.17)$$

where $\mathscr{P}(S)$ is the power set of S. And since each subset $\mathrm{W}_t,\ t \in \mathbb{N}/p\mathbb{N}$, must be non-empty in order for the update schedule to effectively have period p, δ must satisfy $\forall t \in \mathbb{N}/p\mathbb{N},\ \exists i \in \mathbf{V},\ t \in \delta(i)$.

The **global transition function** of $\delta :\equiv (\mathrm{W}_t)_{t\in\mathbb{N}/p\mathbb{N}}$ is:

$$\mathrm{F}[\delta] = \mathrm{F}_{\mathrm{W}_{p-1}} \circ \ldots \circ \mathrm{F}_{\mathrm{W}_1} \circ \mathrm{F}_{\mathrm{W}_0} : \mathbb{B}^n \to \mathbb{B}^n. \qquad (1.18)$$

The definition of this function allows to focus on series of p elementary transitions rather than on single elementary transitions so that (1.15) can be simplified to the following derivation that is not necessarily elementary (and where $\mathrm{F}[\delta]^k$ denotes the k^{th} iterate of $\mathrm{F}[\delta]$):

$$x \twoheadrightarrow \mathrm{F}[\delta](x) \twoheadrightarrow \mathrm{F}[\delta]^2(x) \twoheadrightarrow \ldots \twoheadrightarrow \mathrm{F}[\delta]^k(x) \twoheadrightarrow \ldots$$

This change of point of view on the network transitions that are considered amounts to a change of the *granularity* of events that are observed. Its impact and meaning are discussed in Section 2, Chap. 4.

3.2 Fair update schedules

A first particular class of periodic update schedules is the class of **fair update schedules** [48, 127]. These are defined as the periodic update schedules that update each automaton at least once (*cf.* Example 1.9). Unlike the simple and block-sequential update schedules defined below, they may update some automata more often than others. A fair update schedule $\delta : \mathbf{V} \to \mathscr{P}(\mathbb{N}/p\mathbb{N}))$ is said to be k-**fair** if for all automata i and j:

$$|\delta(i)| \leq k \cdot |\delta(j)|,$$

i.e. within each period, i is not updated more than k times as often as j is.

3.3 Simple update schedules

Another notable class of periodic update schedules is that of *simple update schedules*. Contrary to a fair update schedule, a **simple update schedule** δ does not update any automaton more than once within each period:

$$\forall i \in \mathbf{V},\ |\delta(i)| \leq 1. \tag{1.19}$$

3.4 Block-sequential update schedules

The intersection of the classes of simple and 1-fair update schedules defines the well-known class of **block-sequential update schedules** [7, 30, 34, 47, 102, 103, 129] introduced as *serial-parallel update schedules* in [102, 103]. Their sequences of updates involves *exactly once* each automaton. Thus, they can be defined either by a finite list $(\mathrm{W}_t)_{t\in\mathbb{N}/p\mathbb{N}}$ such that $\mathbf{V} = \biguplus_{t\in\mathbb{N}/p\mathbb{N}} \mathrm{W}_t$ or, abusing notations introduced above, by a function $\delta : \mathbf{V} \to \mathbb{N}/p\mathbb{N}$ (*cf.* Example 1.9).

The **parallel update schedule** is the unique block-sequential update schedule of period $p = 1$. It updates all automata of the BAN in one step, simultaneously. Since it will be extensively used in the sequel for its simplicity, we denote it by π:

$$\forall i \in \mathbf{V},\ \pi(i) = 0 \text{ and } \pi :\equiv \mathbf{V}.$$

Sequential update schedules are block-sequential update schedules $\sigma :\equiv (\mathrm{W}_t)_{t\in\mathbb{N}/n\mathbb{N}}$ with period equal to the size $n = |\mathbf{V}|$ of the BAN; they update only one node at a time: $\forall i \neq j \in \mathbf{V}, \sigma(i) \neq \sigma(j)$ and $\forall t \in \mathbb{N}/n\mathbb{N}, |\mathrm{W}_t| = 1$.

4 Network behaviours and transition graphs

A **transition graph** of a BAN is any digraph $\mathscr{T} = (X, \mathrm{T})$ whose nodes represent configurations and whose set of arcs $\mathrm{T} \subseteq X \times X$ represents a subset of the BAN's transitions. It is called a ***labelled* transition graph** when its arcs represent *labelled* transitions (*cf.* Section 2.6). For any subset $\mathrm{W} \subseteq \mathbf{V}$ of the BAN's set of automata and any sub-graph $\mathrm{H} = (\mathrm{W}, \mathbf{A}_\mathrm{H})$ of its structure $\mathbf{G} = (\mathbf{V}, \mathbf{A})$, we define the transition graph of W and of H as the digraph $\mathscr{T}' = (X', \mathrm{T}')$ which is the restriction of $\mathscr{T}$ that concerns W only, *i.e.* that only involves configurations $x_\mathrm{W} \in \mathbb{B}^{|\mathrm{W}|}$ (also noted $x_\mathrm{H} \in \mathbb{B}^{|\mathrm{W}|}$) instead of configurations $x \in \mathbb{B}^n$ of the whole BAN. In $\mathscr{T}'$, an effective transition $x_\mathrm{W} \longrightarrow\!\!\!\!\!\rightarrow y_\mathrm{W} \neq x_\mathrm{W}$ is represented if and only if transition $x \longrightarrow\!\!\!\!\!\rightarrow y$ is represented in T. Generally, we define the binary relation $\mathrm{T}^* \subseteq \longrightarrow\!\!\!\!\!\rightarrow$ as the reflexive and transitive closure of the relation T: $(x, y) \in \mathrm{T}^*$ if and only if there exists in $\mathscr{T}$

Example 1.9.

For a BAN *of size* $n = 6$, *the* 3*-fair update schedule* $\delta :\equiv \{2,5\},\{0,1,4\},\{1,2,3\},\{0,1,4,5\}$, *the block-sequential update schedule* $\beta :\equiv \{2\},\{3,4\},\{0,1,5\}$, *the sequential update schedule* $\sigma :\equiv \{5\},\{3\},\{1\},\{0\},\{2\},\{4\}$ *and the parallel update schedule* $\pi :\equiv \{0,1,2,3,4,5\}$ *can be defined as functions:*

$$\delta : \begin{cases} \mathbf{V} & \rightarrow & \mathscr{P}(\mathbb{N}/4\mathbb{N}) \\ 0 & \mapsto & \{1,3\} \\ 1 & \mapsto & \{1,2,3\} \\ 2 & \mapsto & \{0,2\} \\ 3 & \mapsto & \{2\} \\ 4 & \mapsto & \{1,3\} \\ 5 & \mapsto & \{0,3\} \end{cases} \qquad \sigma : \begin{cases} \mathbf{V} & \rightarrow & \mathbb{N}/6\mathbb{N} \\ 0 & \mapsto & 3 \\ 1 & \mapsto & 2 \\ 2 & \mapsto & 4 \\ 3 & \mapsto & 1 \\ 4 & \mapsto & 5 \\ 5 & \mapsto & 0 \end{cases}$$

$$\beta : \begin{cases} \mathbf{V} & \rightarrow & \mathbb{N}/3\mathbb{N} \\ i \in \{0,1,5\} & \mapsto & 2 \\ 2 & \mapsto & 0 \\ i \in \{3,4\} & \mapsto & 1 \end{cases} \qquad \pi : \begin{cases} \mathbf{V} & \rightarrow & \mathbb{N}/1\mathbb{N} \\ \forall i \in \mathbf{V} & \mapsto & 0. \end{cases}$$

a path/derivation from x to y. **Transient**[6] **configurations** are then defined as the configurations $x \in \mathbb{B}^n$ that satisfy:

$$\exists y \in \mathbb{B}^n,\ (x,y) \in \mathrm{T}^* \wedge (y,x) \notin \mathrm{T}^*.$$

Any sub-graph of $\mathscr{T}$ that involves transient configurations is said to be transient itself. Configurations that are not transient are called **recurrent**. Recurrent configurations induce the terminal SCCs of $\mathscr{T}$. Generally, these SCCs are called **limit/asymptotic behaviours** or **attractors** [19, 33, 62]. A configuration x with no outgoing arcs in $\mathscr{T}$ except possibly a loop $(x,x) \in \mathrm{T}$ is called a **fixed configuration** and by extension, so is the attractor that it induces. Attractors that are not fixed configurations are called **oscillating** or **unstable attractors**[7]. The **attraction basin** of an attractor $\mathscr{A}$ is the sub-graph of $\mathscr{T}$ induced by the set of configurations x such that $(x,y) \in \mathrm{T}^*$ for some y in $\mathscr{A}$. Generally, attraction basins of different attractors are not necessarily disjoint.

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[6] Ideally, to be consistent with the remark made in footnote 5, all time-related terminology such as “transient” and “recurrent” should be avoided, for example by replacing it with graph theory terminology.

[7] also *cyclic attractors* in [97] and *sustained oscillations* in [118] and limit cycles for deterministic transition graphs as it will be mentioned later in the document.

In practice, a BAN is often associated to a unique transition graph $\mathscr{T} = (X, \mathrm{T})$ that is meant to describe non-ambiguously the alleged *"normal", "unrestricted" global behaviour* of the BAN: T represents the set of all BAN transitions considered as possible (among those that effectively are possible by the BAN definition, *cf.* Section 2.6) so that any BAN derivation that is not a path of $\mathscr{T}$ is disregarded. In the sequel, we informally use the term **system** to imply "*a* BAN *behaving in agreement with a specific, given transition graph* $\mathscr{T}$". There are several important characteristics that transition graphs and systems may have. They are listed in the next paragraphs.

4.1 Elementary and effective transition graphs

A first notable characteristic of transition graphs concerns the nature of their transitions. If $\mathscr{T} = (X, \mathrm{T})$ contains only elementary transitions, then it is said to be **elementary**; otherwise, if it contains at least one non-elementary transition, it is said to be **non-elementary**. Non-elementary transition graphs are used in particular to describe the behaviour of a BAN that is observed only once in a while, for instance, only once per period of an update schedule (*cf.* transition graph $\mathscr{T}_{[\delta]}$ in Section 4.3). Thus, rather than being a feature of the system itself, elementariness concerns the *observation* of a system's behaviour.

Let us recall that by definition, the elementariness of a transition $x \longrightarrow y$ imposes that $\forall i \in \mathbf{V},\ y_i \in \{x_i, \mathrm{f}_i(x)\}$. As a result, if $(x, y) \in \mathrm{T}$ is an arc of $\mathscr{T}$ satisfying $\exists i \in \mathbf{V},\ x_i = \mathrm{f}_i(x) \neq y_i$, then $\mathscr{T}$ is not elementary. In this case, $\mathscr{T}$ can only still effectively be a transition graph if (x, y) can be split into an elementary derivation. Thus, for the BAN of Examples 1.1, 1.2 and 1.6, transition $(0\,0\,0) \longrightarrow (1\,1\,0)$ is impossible but transition $(0\,0\,0) \twoheadrightarrow (1\,1\,0)$ is not because the sequence $(0,0,0) \longrightarrow (1\,0\,0) \longrightarrow (1\,1\,0)$ is possible. In similar lines, suppose that in configuration x, two allegedly elementary transitions are possible. Because any automaton i that is updated by both transitions necessarily takes state $\mathrm{f}_i(x)$ in both resulting configurations, the following situation is impossible:

$$\begin{tikzcd} & y \\ x \arrow[ur, "\mathrm{W}"] \arrow[dr, "\mathrm{W}'"'] & \\ & y' \end{tikzcd} \qquad \text{where } \exists i \in \mathrm{W} \cap \mathrm{W}',\ y_i \neq y'_i.$$

However, again, a similar situation might be possible if the transitions $x \twoheadrightarrow y$ and $x \twoheadrightarrow y'$ are not supposed to be elementary. This shows that the nature of transitions (elementary or not) figuring in the transition graph $\mathscr{T}$ associated to a BAN is an essential precision to understand properly the observed behaviour of the BAN described by $\mathscr{T}$ (*cf.* Chapter 4).

A notable elementary transition graph of a BAN $\mathcal{N}$ is its **general transition graph** (**GTG**) $\mathscr{T}_{\mathcal{N}} = (\mathbb{B}^n, \mathrm{T}_{\mathcal{N}})$ which contains all elementary transitions of $\mathcal{N}$ (*cf.* (1.13)). In this graph, any configuration $x \in \mathbb{B}^n$ has out-degree satisfying $2^{u(x)} = |\mathscr{P}(\mathscr{U}(x))| \leq \deg^+_{\mathbf{G}}(x) \leq |\mathscr{P}(\mathbf{V})| = 2^n$ although for the sake of clarity loops (*i.e.* null transitions) $(x,x) \in \mathrm{T}_{\mathcal{N}}$ are sometimes ommitted so $2^{u(x)} - 1 \leq \deg^+_{\mathbf{G}}(x) \leq 2^n - 1$.

The **asynchronous transition graph** (**ATG**) of a BAN $\mathcal{N}$ is another elementary transition graph. It is the spanning sub-graph $\mathscr{T}^{\mathrm{a}}_{\mathcal{N}} = (\mathbb{B}^n, \mathrm{T}^{\mathrm{a}}_{\mathcal{N}})$ of the GTG $\mathscr{T}_{\mathcal{N}}$ whose set of transitions equals the set of all asynchronous transitions of $\mathcal{N}$:

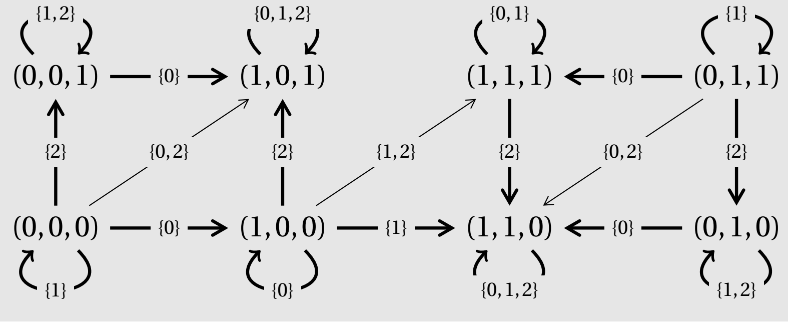
$$\mathrm{T}^{\mathrm{a}}_{\mathcal{N}} = \bigcup \{(x, \mathrm{F}_i(x)) \mid x \in \mathbb{B}^n,\ i \in \mathbf{V}\}.$$

In $\mathscr{T}^{\mathrm{a}}_{\mathcal{N}}$, each node $x \in \mathbb{B}^n$ has out-degree satisfying $u(x) \leq \deg^+(x) \leq n$. Generally, all sub-graphs of the ATG are described as asynchronous [10, 11, 21, 57, 93, 97, 109, 115, 120].

Let us highlight that in agreement with Section 2.6 there are several ways to label transitions of an elementary transition graph: an effective transition (x,y) can be labelled by any set W such that $\mathrm{D}(x,y) \subseteq \mathrm{W}$ and a null transition (x,x) can be labelled by any subset of $\overline{\mathscr{U}(x)}$. In the sequel, the choice will depend on the context and on the information that is intended to be expressed in that particular situation. However, most often, loops (x,x) are labelled by $\overline{\mathscr{U}(x)}$ and other arcs (x,y) are labelled by $\mathrm{D}(x,y)$ (*cf.* Example 1.10).

Example 1.10.

The GTG *of the* BAN *of Examples 1.1, 1.2 and 1.6 is the digraph represented below where the arcs represented with thicker lines are the arcs of the* ATG.



4.2 Deterministic systems and transition graphs

Next, a system and the transition graph $\mathscr{T} = (X, \mathrm{T})$ associated to it may be **deterministic** if in $\mathscr{T}$ the maximal out-degree of any node is 1 (*e.g.* the transition graphs $\mathscr{T}_{[\delta]}$ and $\mathscr{T}_\delta$ introduced in Section 4.3 relative to a periodic update schedule δ). Chapters 2 and 3 focus on these types of systems and additional terminology in relation to their behaviours is introduced on Page 36. In the case where $X = \mathbb{B}^n$, a **global transition function** $\mathrm{F} : X \to X$ may be defined straightforwardly so that $\mathscr{T}$ is the graph of F, *i.e.* $\mathrm{T} = \{(x, \mathrm{F}(x)) \mid x \in X\}$. And when there is no ambiguity concerning the system that is considered, for any of its configuration $x \in \mathbb{B}^n$, we write:

$$x = x(0) \text{ and } \forall\, t \in \mathbb{N}^*,\ x(t) = \mathrm{F}(x(t-1)) = \mathrm{F}^t(x).$$

Then, the **orbit** $\mathscr{O}_x$ of an arbitrary $x \in \mathbb{B}^n$ equals the subset of $\mathbb{B}^n$ that induces the derivation (or trajectory) in $\mathscr{T}$ starting in x:

$$\mathscr{O}_x = \{\mathrm{F}^t(x) \mid t \in \mathbb{N}\} = \{x(t) \mid t \in \mathbb{N}\}.$$

4.3 Transition graphs induced by periodic update schedules

A BAN updated according to a specific periodic update schedule $\delta :\equiv (\mathrm{W}_t)_{t \in \mathbb{N}/p\mathbb{N}}$ defines a deterministic system whose behaviour may be described by two different transition graphs (*cf.* Example 1.11). The first one, $\mathscr{T}_{[\delta]}$, is not necessarily elementary (it is only if $p = 1$, *e.g.* $\delta = \pi$). It equals the graph of the global transition function $\mathrm{F}[\delta]$ (*cf.* (1.18)):

$$\mathscr{T}_{[\delta]} = (\mathbb{B}^n, \mathrm{T}_{[\delta]}) \tag{1.20}$$

where $\mathrm{T}_{[\delta]} = \{(x, \mathrm{F}[\delta](x)) \mid x \in \mathbb{B}^n\} = \{(x(t), x(t+1))\}$.

The second transition graph associated to δ, $\mathscr{T}_\delta = (X, \mathrm{T})$, is used to describe exhaustively the behaviour of the BAN under δ. It is the elementary version of $\mathscr{T}_{[\delta]}$ in which each transition $x \twoheadrightarrow \mathrm{F}[\delta](x)$ is replaced by the series of p elementary transitions that it represents, $x \longrightarrow \mathrm{F}_{W_0}(x) \longrightarrow \mathrm{F}_{W_1} \circ \mathrm{F}_{W_0}(x) \longrightarrow \ldots \longrightarrow \mathrm{F}_{W_{p-1}} \circ \ldots \circ \mathrm{F}_{W_1} \circ \mathrm{F}_{W_0}(x)$. Its set of arcs therefore represents the set of elementary transitions (*cf.* (1.16) for definition of X_t):

$$\mathrm{T}_\delta = \bigcup_{t \in \mathbb{N}/p\mathbb{N}} \{(x, \mathrm{F}_{\mathrm{W}_t}(x)) \in X_t \times X_{t+1}\}$$

although it does not necessarily equal it. Indeed, usually, $\mathrm{T} \neq \mathrm{T}_\delta$ and $X \neq \mathbb{B}^n$ because decomposing transitions of $\mathscr{T}_{[\delta]}$ this way makes the system depend on its history of updates. Whether W_0 can be updated in configuration x, for instance, depends on how far are we along on the periodic sequence of updates imposed by δ when x is reached. And this is not necessarily deterministic as long as only x is considered because it depends on the very first configuration of the derivation that lead to x. Generally, in order to avoid loosing any information and effectively describe the complete *deterministic* behaviour of the system, for any configuration x that belongs to k different sets X_t, $\mathscr{T}_\delta$ needs to contain k copies of x (to see this, consider merging nodes of $\mathscr{T}_\delta$ representing the same configuration in Example 1.11). This way, like $\mathscr{T}_{[\delta]}$, $\mathscr{T}_\delta$ consistently remains deterministic. Section 1.1, Chap. 3 and Chapter 4 discuss further the relations between $\mathscr{T}_{[\delta]}$, $\mathscr{T}_\delta$ and their meaning.

Example 1.11.

The transition graph $\mathscr{T}_{[\delta]}$ of the BAN *of Examples 1.1, 1.2 and 1.6 associated to the update schedule given in Example 1.8 is the following:*

$$\begin{tikzcd}
(0,0,0) \arrow[dr, ->>] & & (0,1,0) \arrow[dr, ->>] & & (1,0,0) \arrow[dl, ->>] \\
& (1,0,1) & & (1,1,0) & \\
(0,0,1) \arrow[ur, ->>] & & (0,1,1) \arrow[ur, ->>] & & (1,1,1) \arrow[ul, ->>]
\end{tikzcd}$$

and its elementary version $\mathscr{T}_\delta$ is:

$$\begin{array}{lllllll}
(0,0,0) \xrightarrow{\{1\}} (0,0,0) \xrightarrow{\{0,2\}} (1,0,1) & \quad & (1,0,0) \xrightarrow{\{1\}} (1,1,0) \xrightarrow{\{0,2\}} (1,1,0) \\
(0,0,1) \longrightarrow (0,0,1) \longrightarrow (1,0,1) & \quad & (1,0,1) \longrightarrow (1,0,1) \longrightarrow (1,0,1) \\
(0,1,0) \longrightarrow (0,1,0) \longrightarrow (1,1,0) & \quad & (1,1,0) \longrightarrow (1,1,0) \longrightarrow (1,1,0) \\
(0,1,1) \longrightarrow (0,1,1) \longrightarrow (1,1,0) & \quad & (1,1,1) \longrightarrow (1,1,1) \longrightarrow (1,1,0)
\end{array}$$

As mentioned above, our choice of definitions imposes that certain BAN behaviours be baned. Adding the supplementary constraint of an update schedule $\delta :\equiv (\mathrm{W}_t)_{t\in\mathbb{N}/p\mathbb{N}}$ restricts further the situations that may be considered possible. In particular, for any subsets $\mathrm{W}_t, \mathrm{W}_{t'} \subseteq \mathbf{V}$ that belong to the defining list of δ and for any configuration $x \in \mathbb{B}^n$, configurations $y = \mathrm{F}_{\mathrm{W}_t}(x)$ and $z = \mathrm{F}_{\mathrm{W}_{t'}}(x)$ must satisfy $y_{\mathrm{W}_t \cap \mathrm{W}_{t'}} = z_{\mathrm{W}_t \cap \mathrm{W}_{t'}}$. If x belongs to both the sets X_3 and X_5, for instance, the situation below is consistent with δ only if $\forall\, i \in \mathrm{W}_3 \cap \mathrm{W}_5$, $y_i = z_i = \mathrm{f}_i(x)$.

$$\begin{tikzcd}
y^0 \arrow[r, "\mathrm{W}_0"] & y^1 \arrow[r, "\mathrm{W}_1"] & y^2 \arrow[dr, "\mathrm{W}_2"] & & z = \mathrm{F}_{\mathrm{W}_5}(x) \\
& & & \begin{array}{l} x = \mathrm{F}_{\mathrm{W}_2}(y^2) \\ \;\; = \mathrm{F}_{\mathrm{W}_4}(z^4) \end{array} \arrow[ur, "\mathrm{W}_5"] \arrow[dr, "\mathrm{W}_3"'] & \\
\cdots\cdots \arrow[r, "\mathrm{W}_2"] & z^3 \arrow[r, "\mathrm{W}_3"] & z^4 \arrow[ur, "\mathrm{W}_4"'] & & y = \mathrm{F}_{\mathrm{W}_3}(x)
\end{tikzcd}$$

4.4 State transition systems and context-dependent systems

The example of graph $\mathscr{T}_\delta$ defined in the previous paragraph shows that the set X of nodes of a transition graph $\mathscr{T} = (X, \mathrm{T})$ is not necessarily a subset of $\mathbb{B}^n$. It can also be a multiset with basis set a subset of $\mathbb{B}^n$. Thus, X may contain several copies of a configuration $x \in \mathbb{B}^n$. When the BAN behaviour differs in each, these copies cannot be merged without loosing information. In this case, the system or BAN behaviour is said to be **context-dependent**. As argued in Section 2.5, Chap. 4, the example of $\mathscr{T}_\delta$ also shows that update schedules are a natural way to insert a context-dependency of BANs.

When the set of nodes of $\mathscr{T}$ is simply a subset of $\mathbb{B}^n$ (*e.g.* transition graphs $\mathscr{T}_{\mathscr{N}}$, $\mathscr{T}_{\mathscr{N}}^{\mathrm{a}}$ and $\mathscr{T}_{[\delta]}$), $\mathscr{T}$ defines exactly a **state transition system** [16, 43, 66]. Its set of nodes $X \subseteq \mathbb{B}^n$ is the set of states of the system and its set of arcs $\mathrm{T} \subseteq X \times X$ (or $\mathrm{T} \subseteq X \times \mathscr{P}^*(\mathbf{V}) \times X$ if $\mathscr{T}$ is labelled) is the set of system transitions. In this case, the BAN behaviour is **context-free** or **memory-less**: the set of its transitions that are possible in any configuration is independent of the derivations that lead to that configuration.

4.5 Dynamical systems

A (discrete-time) **dynamical system**[8] is a triplet $\mathscr{D} = (S, \Theta, \phi)$ where S is the **state space** of the system, $\Theta \subseteq \mathbb{N}$ is its **time domain** and $\phi : S \times \Theta \to S$ is the **evolution rule** describing the system's dynamics. It satisfies:

$$\forall s \in S,\ \forall t, t' \in \Theta,\ \phi(s, 0) = s \text{ and } \phi(\phi(s,t), t') = \phi(s, t + t').$$

$\phi(s,t)$ represents the state of the system at time t so that the *trajectory* of $\mathscr{D}$ initiated in state $s \in S$ is induced by $\{\phi(s,t) \mid t \in \Theta\}$. Let us note that for any initial state $s \in S$, the function $\phi_s : t \mapsto \phi(s,t)$ associates to every time step t, a *unique* image $\phi(s,t)$. In particular, it defines the unique successor $\phi(s,1)$ of every state $s \in S$. This allows for two types of dynamical systems: deterministic and stochastic. Formally, for the latter type, S rather denotes a *measurable space* associated to the state space of the system and to a probability measure μ such that ϕ is measure-preserving.

A notable example of a deterministic dynamical system in the present context is a BAN that is updated with a periodic update schedule $\delta :\equiv (\mathrm{W}_k)_{k \in \mathbb{N}/p\mathbb{N}}$. If its behaviour is described by $\mathscr{T}_{[\delta]}$, then it defines a deterministic dynamical system $\mathscr{D} = (\mathbb{B}^n, \mathbb{N}, \phi)$ where $\forall x \in \mathbb{B}^n$, $\forall t \in \mathbb{N}$, $\phi(x,t) = \mathrm{F}[\delta]^t(x)$. If the behaviour of the BAN is described by $\mathscr{T}_\delta$, then, since a dynamical system is necessarily context-free, the BAN behaviour must be defined as *collection* of dynamical systems, one for each maximal, connected sub-graph (or derivation) of $\mathscr{T}_\delta$. In this case, the maximal derivation starting in $x \in \mathbb{B}^n$ corresponds to a system of the form $(\mathscr{O}_x, \mathbb{N}, \phi)$ where $\mathscr{O}_x \subseteq \mathbb{B}^n$ and $\phi(x,t) = \mathrm{F}_{W_d} \circ \ldots \circ \mathrm{F}_{W_1} \circ \mathrm{F}_{W_0} \circ \mathrm{F}[\delta]^k$, $\forall t = k \cdot p + d \equiv d \bmod p$.

---

[8] Because the state space of systems considered here is discrete, continuous-time dynamical systems will not be mentioned at all so we bypass the difficulty of choosing a time space Θ by assuming it is discrete.

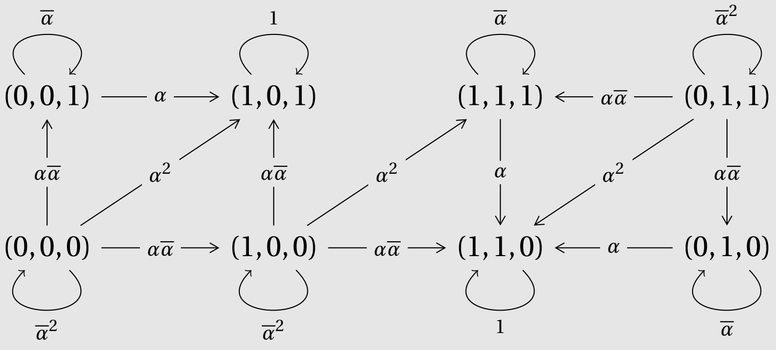
Generally, when the system behaviour described by $\mathscr{T} = (\mathbb{B}^n, \mathrm{T})$ is non-deterministic, there is no immediate way of defining it as a dynamical system. However, with some additional indications or hypotheses, probabilities may be assigned to each transition of T (*cf.* Example 1.12). This way, $\mathscr{T}$ can be seen as the graph of a Markov chain on $\mathbb{B}^n$ and the adjacency matrix of $\mathscr{T}$ can be turned into a *stochastic transition matrix P*, the Markovian matrix of dimension $2^n \times 2^n$ whose component $P_{x,y}$ represents the probability that the BAN performs transition (x, y) when it actually is in configuration x: $P_{x,y} = \mathbb{P}\big(x(t+1) = y \mid x(t) = x\big)$. Then, the BAN behaviour can be defined as a stochastic dynamical system whose evolution rule is given by $\phi(\mu, 0) = \mu$ and $\phi(\mu, t) = \mu \cdot P^t$ where μ is an arbitrary probability law on $\mathbb{B}^n$ ($\mu \in [0,1]^{2^n}$ and $\sum_{x \in \mathbb{B}^n} \mu_x = 1$). This way, if $\mu = \mu(0)$ is the probability law of the initial network configuration $x(0) \in \mathbb{B}^n$, then $\mu(t) = \phi(\mu, t)$ is that of the network configuration $x(t) \in \mathbb{B}^n$ at time step t ($\forall x \in \mathbb{B}^n$, $\mu_x(t) = \mathbb{P}\big(x(t) = x\big)$). In this context, the notion of attractor corresponds to that of stationary law (*cf.* [27, 107]).

Example 1.12.

Consider the BAN *of Examples 1.1, 1.2 and 1.6 whose* GTG $\mathscr{T}_{\mathscr{N}} = (\mathbb{B}^3, \mathrm{T}_{\mathscr{N}})$ *is given in Example 1.10. Let us suppose that at each time step, any automaton* $i \in V$ *is updated with probability* $\alpha = 1 - \overline{\alpha} \in [0,1]$ *[36, 90, 104] and let P be the stochastic transition matrix defined by (cf. Section 2.7 for notations):*

$$\forall (x,y) \in \mathbb{B}^3 \times \mathbb{B}^3, P_{x,y} = \begin{cases} \alpha^{\mathrm{d}(x,y)} \cdot \overline{\alpha}^{|\mathscr{U}(x)| - \mathrm{d}(x,y)} & \textit{if } (x,y) \in \mathrm{T}_{\mathscr{N}}, x \neq y \\ 0 & \textit{otherwise.} \end{cases}$$

This yields the following Markov graph corresponding to $\mathscr{T}_{\mathscr{N}}$ *in which previous notations are momentarily dropped to label transitions by their probabilities:*



5 Simulations & canonical networks

Let $\mathscr{N}$ and $\mathscr{N}'$ be any two BANs of respective sizes n and m. We say that $\mathscr{N}'$ **simulates** $\mathscr{N}$ (*cf.* Examples 1.13 and 1.14) and write $\mathscr{N} \lhd \mathscr{N}'$ if the labelled GTG $\mathscr{T}_{\mathscr{N}}$ of $\mathscr{N}$ is isomorphic to a sub-graph of the labelled GTG $\mathscr{T}_{\mathscr{N}'}$ of $\mathscr{N}'$:

$$\mathscr{N} \lhd \mathscr{N}' \;\Leftrightarrow\; \exists \phi : \mathbb{B}^n \to \mathbb{B}^n,$$
$$\forall \mathrm{W} \subseteq \mathbf{V},\ \forall x \in \mathbb{B}^n,\ x \xrightarrow{\mathrm{W}} y \;\Rightarrow\; \phi(x) \xrightarrow{\mathrm{W}} \phi(y).$$

The bisimulation relation that is the symmetric closure of $\lhd$ is denoted by $\bowtie$ ($\mathscr{N} \bowtie \mathscr{N}'$ is equivalent to the labelled GTGs $\mathscr{T}_{\mathscr{N}}$ and $\mathscr{T}_{\mathscr{N}'}$ being isomorphic). Moreover, let $\mathscr{A}$ (resp. $\mathscr{A}'$) be the sub-graph of $\mathscr{T}_{\mathscr{N}}$ (resp. of $\mathscr{T}_{\mathscr{N}'}$) induced by its recurrent configurations. We say that $\mathscr{N}'$ **simulates** $\mathscr{N}$ **asymptotically** and write $\mathscr{N} \lhd\!| \mathscr{N}'$ if $\mathscr{A}$ is isomorphic to a sub-graph of $\mathscr{A}'$ and we denote by $|\!\bowtie\!|$ the symmetric closure of $\lhd\!|$.

Example 1.13. Isomorphism between disjunctive and conjunctive BANs

Any DAN *(cf. Example 1.4)* $\mathscr{N} = \{\mathrm{f}_i\}$ *is bisimulated by the conjunctive* BAN $\mathscr{N}' = \{\mathrm{f}'_i : x \mapsto \neg\mathrm{f}_i(\overline{x})\}$ *that has the same signed structure:* $\mathscr{N} \bowtie \mathscr{N}'$. *Indeed, the application* $\phi : x \in \mathbb{B}^n \mapsto \overline{x} \in \mathbb{B}^n$ *that maps configurations of one network to configurations of the other is an isomorphism of their* GTG*s.*

Example 1.14. Simulation of a sub-DAN

If $\mathbf{G}$ *is a structure defining a positive* DAN *and if* $\mathbf{H}$ *is a* SCC *of* $\mathbf{G}$*, then:*

$$\mathbf{H} \lhd \mathbf{G}.$$

Indeed, consider the set of SCC*s* $\mathbf{H}' \neq \mathbf{H}$ *of* $\mathbf{G}$ *from which* $\mathbf{H}$ *can be reached. If the states of all these* SCC*s are set to* $0^{|\mathbf{H}'|}$*, then the states of their automata have no influence on that of* $\mathbf{H}$*. So* $\mathbf{H}$ *is free to behave as it would if it were isolated from the rest of* $\mathbf{G}$*.*

Now, in different lines, we let $\Bumpeq$ be the equivalence relation (*cf.* Fig. 1.2) that relates two BANs $\mathscr{N}$ and $\mathscr{N}'$ with the same structures, globally differing only in the localisation and number of negative arcs in their nude paths (*cf.* Section 2.4) without differing in the signs of any maximal nude path (the parities of the numbers of negative arcs in maximal nude paths is the same for $\mathscr{N}$ and $\mathscr{N}'$).

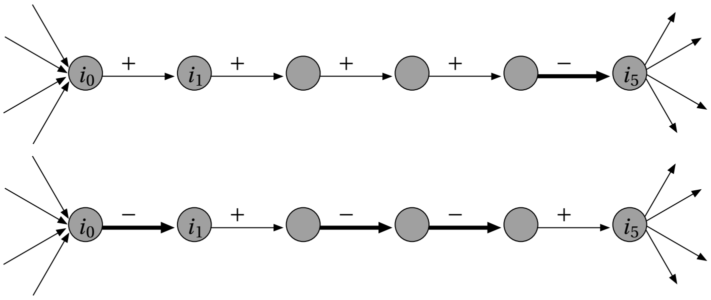


Figure 1.2: *Two negative nude paths of length $\ell = 5$ that are equivalent by $\Bumpeq$. The one above is canonical (cf. Definition 1).*

Lemma 1.1.

For any two BANs $\mathscr{N}$ *and* $\mathscr{N}'$*:* $\mathscr{N} \Bumpeq \mathscr{N}' \;\Rightarrow\; \mathscr{N} \bowtie \mathscr{N}'$.

Proof: Let $\mathbf{G} = (\mathbf{V}, \mathbf{A})$ be the common structure of $\mathscr{N} = \{\mathrm{f}_i\}$ and $\mathscr{N}' = \{\mathrm{h}_i\}$, let $L \subseteq \mathbf{V}$ be the set of automata that belong to a nude path of $\mathbf{G}$ and let $L^* \subsetneq L$ be the set of automata that belong to L without being the start node of a maximal nude path, *i.e.* $i \in L^* \Rightarrow \deg^-_{\mathbf{G}}(i) = 1$. Finally, let $\phi : \mathbb{B}^n \to \mathbb{B}^n$ be the function such that $\forall i \in \mathbf{V}, \forall x \in \mathbb{B}^n, \phi(x)_i = \phi_i(x_i) \in \mathbb{B}$ where:

$$\phi_i = \begin{cases} \mathrm{id} & \text{if } i \notin L^*, \\ \mathrm{f}[i_k, i_1] \circ \mathrm{h}[i_k, i_1] & \text{if } i = i_k \in \{i_0, i_1, \dots, i_k\} \subseteq L,\ i_0 \notin L^*. \end{cases}$$

From the three points below derives that $\forall \mathrm{W} \subseteq \mathbf{V}$, $\forall i \in \mathbf{V}$ and $\forall x \in \mathbb{B}^n$, $\phi(\mathrm{F_W}(x))_i = \mathrm{H_W}(\phi(x))_i$, holds ($\mathrm{H_W}$ is the W-update function of $\mathscr{N}'$), proving Lemma 1.1. Let $\mathrm{W} \subseteq \mathbf{V}$.

- If $i \notin \mathrm{W}$, then $\forall x \in \mathbb{B}^n$, $\mathrm{F_W}(x)_i = \mathrm{H_W}(x)_i = x_i$ so $\phi(\mathrm{F_W}(x))_i = \mathrm{H_W}(\phi(x))_i$.
- If $i = i_k \in L^* \cap \mathrm{W}$ belongs to the maximal nude path $P = \{i_0, i_1, \dots, i_k, \dots, i_\ell\}$ then: $\phi(\mathrm{F_W}(x))_i = \phi_i(\mathrm{F_W}(x)_i) = \phi_i(\mathrm{f}_i(x_{i_{k-1}})) = \mathrm{f}[i_k, i_1] \circ \mathrm{h}[i_k, i_1] \circ \mathrm{f}_{i_k}(x_{i_{k-1}})$,
$$\begin{aligned} \text{so } k > 1 \;\Rightarrow\; \phi(\mathrm{F_W}(x))_i &= \mathrm{f}_{i_k} \circ \mathrm{f}_{i_k} \circ \mathrm{f}[i_{k-1}, i_1] \circ \mathrm{h}_{i_k} \circ \mathrm{h}[i_{k-1}, i_1](x_{i_{k-1}}) \\ &= \mathrm{h}_i(\phi(x)_{i_{k-1}}) = \mathrm{H_W}(\phi(x))_i \\ \text{and } k = 1 \;\Rightarrow\; \phi(\mathrm{F_W}(x))_i &= \mathrm{f}_{i_1} \circ \mathrm{h}_{i_1} \circ \mathrm{f}_{i_1}(x_{i_0}) = \mathrm{h}_{i_1}(x_{i_0}) = \mathrm{h}_{i_1}(\phi(x)_{i_0}) \\ &= \mathrm{H_W}(\phi(x))_i. \end{aligned}$$
- If $i = i_\ell \in L^* \cap \mathrm{W}$ is the end of a nude path P ($\deg^+_{\mathbf{G}}(i) \neq 1$) then, since $\mathrm{sign}_{\mathscr{N}}(P) = \mathrm{sign}_{\mathscr{N}'}(P)$ is true, $\mathrm{f}[i_\ell, i_1] = \mathrm{h}[i_\ell, i_1]$ and $\phi_{i_\ell} = \mathrm{id}$ are also true. As a consequence, if $i \in \mathrm{W} \setminus L^*$ then, $\mathrm{f}_i = \mathrm{h}_i$, $\phi_i = \mathrm{id}$ and $\forall j \in \mathbf{V}^-_{\mathbf{G}}(i)$, $\phi_j = \mathrm{id}$ and therefore, $\phi(\mathrm{F_W}(x))_i = \mathrm{F_W}(x)_i = \mathrm{f}_i(x) = \mathrm{h}_i(x) = \mathrm{h}_i(\phi(x)) = \mathrm{H_W}(\phi(x))_i$. □

The relation $\Bumpeq$ equates BANs that behave identically locally except, possibly, in specific points internal to their nude paths in a way that has no impact at all outside of these paths. In the rest of this document, we concentrate on *canonical*

BANs defined precisely in Definition 1 as members of equivalence classes of $\Bumpeq$ that have the least negative arcs.

Definition 1. Canonical BANs

A ***canonical*** BAN $\mathscr{N}$ *is a* BAN *in which all maximal nude paths* $P = \{i_0, i_1, \ldots, i_k, \ldots, i_\ell\}$ *have at most* 1 *negative arc, the last one:*

$$\forall 0 < k < \ell,\ \mathrm{sign}_{\mathscr{N}}(i_{k-1}, i_k) = +1 \textit{ and } \mathrm{sign}_{\mathscr{N}}(i_{\ell-1}, i_\ell) = \mathrm{sign}_{\mathscr{N}}(P)$$

i.e. $\forall 0 < k < \ell$, $f_{i_k} = \mathrm{id}$ *whatever the sign of* P*, and also, if* P *is positive, then* $f_{i_\ell} = \mathrm{id}$ *and if* P *is negative, then* $f_{i_\ell} = \mathrm{neg}$.

THE PARALLEL UPDATE SCHEDULE

2

In this chapter we concentrate on the parallel update schedule[9] so that all BANs $\mathscr{N} = \{f_i \mid i \in \mathbf{V}\}$ considered are associated to the global transition function:

$$F = F[\pi] : \begin{cases} \mathbb{B}^n & \rightarrow & \mathbb{B}^n \\ x & \mapsto & F(x) = (f_0(x)\, f_1(x) \ldots f_{n-1}(x)) \end{cases}$$

and to the transition graph:

$$\mathscr{T} = \mathscr{T}_{[\pi]} = (\mathbb{B}^n, T) \text{ where } T = \{(x, F(x)) \mid x \in \mathbb{B}^n\}.$$

The justification and meaning of this direct restriction of our study of BANs to the simplest of all periodic update schedules is discussed later, at the end of this chapter and again in Chapter 6. For now, let us just mention that first, implementing π and performing the **V**-updates that it requires amounts to making "optimal use" of the BAN's momentum $u(x) = |\mathscr{U}(x)|$ in any configuration x. And second, importantly, the simplicity of this update schedule precisely, allows for developments whose advantage is to provide preliminary insights on some features of BANs and their properties (especially structural) influencing their behaviours. In particular, as will be highlighted in Section 1.2, Chap. 3, π is the only periodic update schedule for which the structure of a BAN has a direct, straightforward meaning that can be related to its behaviour. Thus, it is natural and it

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[9] The work presented in this chapter mainly regroups results that were presented in [48] (for Section 1), [31, 32, 81] (for Section 3) and [25, 26, 83] (for Sections 4 to 7).

makes sense to start by exploiting the “straightforwardness” of the parallel update schedule before complexifying the study of BANs by adding new influences on their behaviours in the form of various update modes.

For BANs updated with the parallel update schedule, or any other periodic update schedule, as for any deterministic system with state space $\mathbb{B}^n$ (*cf.* Section 4.2, Chap. 1), fixed configurations are rather called **fix points** since they are precisely the fix points of the global transition function F. Other attractors are called **limit cycles**. $\mathscr{T}$ being deterministic, attractors or **p-attractors** are the simple cycles of $\mathscr{T}$ of length p; they have the following form for some configuration $x \in \mathbb{B}^n$ and some $p \in \mathbb{N}$:

$$\begin{tikzcd}
x \arrow[r, two heads] & \mathrm{F}(x) \arrow[r, two heads] & \mathrm{F}^2(x) \arrow[rr, dotted, no head] & & {} \arrow[d, dotted, no head] \\
\mathrm{F}^{p-1}(x) \arrow[u, two heads] & \mathrm{F}^{p-2}(x) \arrow[l, two heads] & {} \arrow[l, two heads] & {} \arrow[r, dotted, no head] & {}
\end{tikzcd}$$

All multiples of p are called **periods** of x and of every other configuration $\mathrm{F}^t(x)$ of this p-attractor; p is the **minimal period** of these configurations and the (minimal) **period** of the attractor. We write:

$$\mathscr{X}(p) \;=\; \{x \in \mathbb{B}^n \,|\, x = \mathrm{F}^p(x)\} \tag{2.1a}$$

to denote the set of configurations of period p and:

$$\mathtt{X}(p) \;=\; |\mathscr{X}(p)| \tag{2.1b}$$

to denote the number of them. Globally, any integer $p \in \mathbb{N}$ such that $\mathtt{X}(p) \geq 1$, is called a **period of the system** and any period of the system that is the minimal period of one of its recurrent configurations is called a **minimal period of the system**. The smallest integer ω such that all recurrent configurations of the system have period ω is called its **order**. $\mathscr{X} = \mathscr{X}(\omega)$ is thus the system's set of recurrent configurations and $\mathtt{X}(\omega)$ is their number. We say that the order ω is **reached** when it is the minimal period of some configuration. In particular, when $\omega = 1$, it is necessarily reached, all attractors are fix points and the system is said to be **fixed**. On the contrary, a system with order $\omega > 1$ has limit cycles and is said to **cycle**.

Let us note that the map $(t, x) \in \mathbb{Z}/\omega\mathbb{Z} \times \mathscr{X} \mapsto \mathrm{F}^t(x) = x(t) = x(t \bmod \omega) \in \mathscr{X}$ defines an action of the additive group $\mathbb{Z}$ on the set of recurrent configurations $\mathscr{X}$. For this action, the orbit $\mathscr{O}_x$ of a configuration $x \in \mathscr{X}$ induces the attractor of (minimal) period $|\mathscr{O}_x|$ containing x in $\mathscr{T}$. The *stabiliser* of x equals the set of its periods: $\mathscr{S}t_x \;=\; \{p \in \mathbb{Z}/\omega\mathbb{Z} \,|\, \mathrm{F}^p(x) = x\} \;=\; \{k \cdot |\mathscr{O}_x| \in \mathbb{Z}/\omega\mathbb{Z}\}$ (it satisfies the orbit-stabiliser theorem: $|\mathscr{O}_x| = \omega / |\mathscr{S}t_x|$); and the set of fix points by any $p \in \mathbb{Z}/\omega\mathbb{Z}$ equals $\mathscr{X}(p)$. Letting $\mathscr{S}t_x^* = \mathscr{S}t_x \setminus \{0\}$, the order ω of the system can be shown to equal:

$$\omega \;=\; \min \bigcap_{x \in \mathscr{X}} \mathscr{S}t_x^* \;=\; \mathrm{lcm}\{|\mathscr{O}_x| \mid x \in \mathscr{X}\}.$$

In the next paragraphs, we focus on BACs, BADs and positive DANs to show how the order of a system relates to its structure and especially to the cycles embedded in it.

1 Positive disjunctive networks

As mentioned in Example 1.4, positive disjunctive BANs (DANs) have the particularity of being completely defined by their structures. Thus, they help to understand the direct impact that structural properties of a network have on its possible behaviours in the absence of other notable influences.

Let $\mathbf{G} = (\mathbf{V}, \mathbf{A})$ be a positive DAN of size n and let M be the adjacency matrix of $\mathbf{G}$ (*i.e.* the $n \times n$ matrix such that $\forall i, j \in \mathbf{V}, \mathrm{M}_{ij} = 1 \;\Leftrightarrow\; (i, j) \in \mathbf{A}$). Then, with the parallel update schedule π, the non-elementary behaviour of $\mathbf{G}$ is given by:

$$\forall x \in \mathbb{B}^n,\ \forall t \in \mathbb{N},\ x(t) \;=\; \mathrm{F}^t(x) \;=\; x \cdot \mathrm{M}^t.$$

The next lemma closely relates to results in [44] and to the formula in [60] giving the number of fixed points of conjunctive and disjunctive BANs. We recall that the set of fix points of a BAN under the parallel update schedule and under all other block-sequential update schedules equals its set of stable configurations [102, 103] because:

$$x = \mathrm{F}(x) \;\Leftrightarrow\; \forall i \in \mathbf{V}, \mathrm{f}_i(x) = x_i \;\Leftrightarrow\; \forall \mathrm{W} \subseteq \mathbf{V},\ \mathrm{F}_{\mathrm{W}}(x)_i = x_i. \tag{2.2}$$

This is why despite the title of the present chapter, Lemma 2.1 is stated more generally in terms of arbitrary block-sequential update schedules.

Lemma 2.1. Fix points of positive DANs

Two positive DAN*s whose structures reduce to the same digraph (the digraph in which nodes represent* SCC*s of each structure) have the same set of fix points under all block-sequential update schedules. In each of these any* SCC *of size m has a uniform state: either* 0^m *or* 1^m*.*

Proof: If one automaton i becomes fixed in state 1, the whole SCC C to which it belongs will eventually become fixed in state $1^{|\mathrm{C}|}$. Indeed, since all automata of the DAN are updated at each step, automata that are on a path that starts on i will progressively all become fixed in state 1 as well. Thus, in a fix point, either all automata of a SCC are fixed to state 0, or they all are fixed to state 1. And in the latter case, all automata that can be reached from that SCC also end up being fixed in state 1.

Conversely, if $\mathbf{G}$ contains several non-trivial SCCs, then it has a non-trivial source SCC C. $\mathbf{G}$ has a fix point x in which the state of C is fixed to $x_{\mathrm{C}} = 0^{|\mathrm{C}|}$ and the states of all other non-trivial SCCs C' are fixed to $x_{\mathrm{C}'} = 1^{|\mathrm{C}'|}$ (they can remain in that state because they are non-trivial). □

Let us note that by definition of a DAN, a source node $i \in \mathbf{V}$ ($\deg^{-}_{\mathbf{G}}(i) = 0$) has constant local transition function $\mathrm{f}_i : x \in \mathbb{B}^n \mapsto 0$. A consequence of Lemma 2.1 is that positive DANs with a unique SCC are characterised by the fact that they only have configuration 0^n as fix point involving an automaton in state 0. And if this SCC is non-trivial then they are characterised by the fact that 0^n and 1^n are their only fix points.

The **index of imprimitivity** $\eta(\mathbf{G})$ of a strongly connected digraph $\mathbf{G}$ denotes the greatest common divisor of all cycles in $\mathbf{G}$ [13] (also called the *loop number* in [60]). Let us recall that the property $\eta(\mathbf{G}) = 1$ is equivalent to the adjacency matrix M being *primitive, i.e.* M is an irreducible square matrix for which there exists a positive integer m such that $\forall k \geq m$, M^k is a strictly positive matrix [13]. Here, this means that for any configuration $x \neq 0^n$, there exists an integer t such that $x(t) = x \cdot \mathrm{M}^t = 1^n$. In other terms, if M is primitive, then all configurations except 0^n evolve towards the fix point 1^n and thus the system is fixed. The first main part of Lemma 2.2 (which was proven in [44] and again in [60]) extends this remark.

Lemma 2.2. Order of a positive DAN

Under the parallel update schedule, the order $\omega_{\mathbf{H}}$ of any strongly connected component $\mathbf{H}$ of a positive DAN $\mathbf{G}$ *equals its index of imprimitivity $\eta(\mathbf{H})$ and is reached. As a consequence, $\mathbf{G}$ cycles if and only if it contains one* SCC *that has index of imprimitivity strictly greater than 1 and its order equals $\omega_{\mathbf{G}} = \mathrm{lcm}\{\eta(\mathbf{H}) \mid \mathbf{H}$ is an* SCC *of* $\mathbf{G}\}$.

Proof: As demonstrated in Example 1.14, $\mathbf{H} \lhd \mathbf{G}$ so we concentrate on the case where $\mathbf{H} = \mathbf{G}$ is strongly connected. By induction on length ℓ_{ij} of a path from $i \in \mathbf{V}$ to $j \in \mathbf{V}$, it can be shown that $\forall x \in \mathbb{B}^n, x_i = 1 \Rightarrow x_j(\ell_{ij}) = 1$ (using the fact that at every time step $t + k \in \mathbb{N}$ all automata, and in particular the k^{th} automata on the path, are updated by π). When $i = j$, this implies that $\forall x \in \mathbb{B}^n, \exists k \geq 1, x_i(k\ell_{ii}) = 1 \Rightarrow \forall k' \geq k, x_i(k'\ell_{ii}) = 1$. If $x \in \mathscr{X}(p) \wedge x_i = 0$, letting $k' = kp \geq k$, this leads to the contradiction $0 = x_i = x_i(kp\ell_{ii}) = 1$, proving that any recurrent configuration satisfies $\forall i \in \mathbf{V}, x_i = 0 \Rightarrow \forall k \geq 1, x_i(k\ell_{ii}) = 0$ and thus $\forall i \in \mathbf{V}, x_i = x_i(\ell_{ii})$. Thus, any (minimal) period must divide all cycle lengths ℓ_{ii}, and it must divide their gcd $\eta(\mathbf{G})$.

If $\eta(\mathbf{G}) = 1$ then all periods equal $\eta(\mathbf{G}) = 1$. If $\eta(\mathbf{G}) > 1$, then $\mathbf{V}$ can be partitioned into $\eta(\mathbf{G})$ non-empty sets $\mathbf{V}_k \neq \emptyset$, $k \in \mathbb{Z}/\eta(\mathbf{G})\mathbb{Z}$ as follows (*cf.* [13]):

$$\mathbf{V} = \biguplus_{k \in \mathbb{Z}/\eta(\mathbf{G})\mathbb{Z}} \mathbf{V}_k \text{ such that } \mathbf{A} \subseteq \bigcup_{k \in \mathbb{Z}/\eta(\mathbf{G})\mathbb{Z}} \mathbf{V}_k \times \mathbf{V}_{k+1}. \tag{2.3}$$

In this case, configuration $x \in \mathbb{B}^n$ defined by $x_{\mathbf{V}_0} = 1^{|\mathbf{V}_0|}$ and $x_{\mathbf{V}\setminus\mathbf{V}_0} = 0^{n-|\mathbf{V}_0|}$, for example, has period $\eta(\mathbf{G})$. In both cases the order of **G** is reached. □

2 Degrees of freedom and cycles

The previous section concretely relates the attractors of positive DANs to their structures, and especially to their underlying structural SCCs and cycles. More precisely, since fix points only depend on the *reduced* versions of the structures of these BANs by Lemma 2.1, cycles are what determine their ability to behave asymptotically in more ways that what is appointed by fix points (through the index of imprimitivity of structures, they actually determine more than that since they are responsible for attractor periods).

Now, to serve as a guideline in our discussion, let us introduce the informal notion of **degrees of freedom** of a network or system (not necessarily a DAN submitted to π). Even though this requires to start by disregarding transient behaviours, for the sake of simplicity, let us initially define this notion formally as equivalent to the total number T of different attractors of a system. It might certainly be beneficial to reconsider later the problem of finding a more accurate, informative measure. Besides taking into account transient behaviours, it seems natural, for instance, that the degree of freedom of a BAN should integrate its ability to cycle, and further perhaps also, its ability to cycle *with large periods*, requiring the propagation of a small number of instabilities through a widespread area of the structure, involving many automata that successively become unstable and pass on their instabilities to a few of their out-neighbours. For now, however, we make do with a basic notion of degrees of freedom that simply counts the various ways in which a BAN can behave asymptotically. Let us remark that this notion echoes the *punctual* notion of momentum $u(x) = |\mathcal{U}(x)|$ relative to a configuration $x \in \mathbb{B}^n$ (*cf.* Section 2.7, Chap. 1).

Since Thomas [118] first emphasised this, closed directed chains of interactions – *i.e.* cycles – are recognised [4, 23, 52, 63, 86, 92, 94, 99, 96, 97, 98, 115, 116, 119, 122] as the most basic structural motifs that allow and are responsible for diversity and complexity in interaction networks behaviours[10]. Generally, this idea has become intuitive since indeed, in a network whose structure is acyclic, the "in-

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[10] Interestingly, in a significantly although not unrelated framework [65], we have found problems involving cycles which on the contrary, are more tractable than the corresponding problems involving trees (but a proper comparative analysis of both frameworks remains to be done).

formation" runs linearly from the source nodes that have constant states towards the sink nodes whose states influence that of no other so the network can only end in a stable configuration. In our context, the example of positive DANs agrees with this and gives some further details on the relation between structural cycles and network behaviours. For these special networks whose behaviours are directly controlled by their structures, the previous section shows that it is indeed the underlying structural cycles that are primarily responsible for the networks' degrees of freedom. This first formal explicit argument on the role of cycles motivates the special attention that is given to them in the next sections (as well as again in Section 2, Chap. 3, Chap. 3 and in Section 1, Chap. 5, Chap. 5).

3 Isolated cycles

Thus, we now concentrate on the particular instance of disjunctive networks that are BACs. In the next series of lemmas, we determine the periods, order and number of recurrent configurations of these networks under the parallel update schedule π, we characterise the configurations $x \in \mathscr{X}(p)$ of minimal period p and in Lemma 2.5 we show how some BACs can simulate one another.

One of the main ideas that is exploited in the proofs of these lemmas is the following. Since every automaton is updated at each step, the state $x_i(t)$ of an automaton i at step t is responsible for the state $x_{i+1}(t+1) = \mathrm{f}_{i+1}(x_i(t))$ of its out-neighbour at the next step. And it also is responsible for the state $x_{i+k}(t+k)$ of a automaton that is further away and at a subsequent step. So all in all, the states of automata can be seen as simply progressing along the cycles.

Following Section 5, Chap. 1, without loss of generality, only one BAC of each size n and each sign $s \in \{-,+\}$ can be studied as a representative of all other BACs $\mathscr{C}_n^s$ of same signs and sizes. In agreement with Definition 1, we focus on canonical BACs which have the least negative arcs: the **canonical positive** BAC $\mathscr{C}_n^+$ of size n has no negative arcs, the **canonical negative** BAC $\mathscr{C}_n^-$ has arc $(n-1,0)$ as unique negative arc (*cf.* Fig. 2.1).

Considering configurations of BACs as binary necklaces, the global transition function $\mathrm{F} = \mathrm{F}[\pi]$ of a positive BAC acts as a rotation on configurations: $\forall x \in \mathbb{B}^n, \mathrm{F}(x) = x_{n-1}x[0, n-2]$ (*cf.* Section 2.1, Chap. 1 for notations). The global transition function of a negative BAC satisfies: $\mathrm{F}(x) = \neg x_{n-1}x[0, n-2]$. From these remarks follow the immediate relations [81, 111] that exist between the behaviours of BACs under π which are the object of this section, the shift register machines studied in [51, 112], the problem studied in [134] concerning cycles in a digraph under certain quadratic maps as well as binary (labelled or unlabelled) necklaces and Lyndon words [12, 53, 68, 69, 70, 71, 72, 105, 106]. As a consequence of the lat-

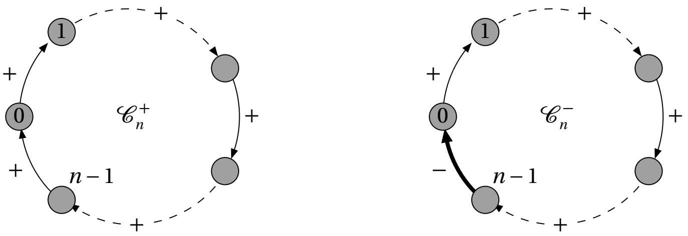

Figure 2.1: *Signed structures of canonical positive (left) and negative (right)* BACs *of size n.*

ter relation in particular, the results of this section can mostly be derived directly from results concerning binary necklaces and Lyndon words. The proofs still appear here in the formalism of BANs because their point of view differs slightly. In addition, they provide intuitions for the more complex cases of BADs studied in the next section. These cases yield problems related to necklaces with forbidden sequences. They involve longer proofs that have been moved to the appendix to preserve the fluidity of the global development of this chapter but their baseline ideas are similar to those used in this section on BACs.

The first lemma below extends a result proven in [40].

Lemma 2.3. Periods of BACs

Under the parallel update schedule, all configurations of a BAC $\mathscr{C}$ *are recurrent, with period n if* $\mathscr{C} = \mathscr{C}_n^+$ *is positive* ($\mathscr{X}_n^+(n) = \mathbb{B}^n$)*, and with period* $2n$ *if* $\mathscr{C} = \mathscr{C}_n^-$ *is negative* ($\mathscr{X}_n^-(2n) = \mathbb{B}^n$)*. Moreover, no period of* $\mathscr{C}_n^-$ *divides n, i.e. any period* $p \in \mathbb{N}$ *of* $\mathscr{C}_n^-$ *is an even divisor of* $2n$ *satisfying* $n = q \cdot p/2$ *for some odd* $q \in \mathbb{N}$.

Proof: Any configuration $x = x(0) \in \mathbb{B}^n$ of $\mathscr{C} = \mathscr{C}_n^s$ satisfies:

$$\begin{aligned}\forall i \in \mathbf{V} = \mathbb{Z}/n\mathbb{Z},\ x_i(2n) &= \mathrm{f}_i(x_{i-1}(2n-1)) \\ &= \mathrm{f}[i, i-t+1](x_{i-t}(2n-t)) \quad (\forall 1 \leq t \leq n) \\ &= \mathrm{f}[i, i+1](x_i(n)) \\ &= \mathrm{f}[i, i+1] \circ \mathrm{f}[i, i+1](x_i(0)) \ = \ x_i(0).\end{aligned}$$

If $s = +$, then $\mathrm{f}[i, i+1] = \mathrm{id}$ so $x_i(2n) = x_i(n) = x_i(0)$: x has period n. If $s = -$, then $\mathrm{f}[i, i+1] = \mathrm{neg}$ so $x_i(2n) = \neg x_i(n) = \neg\neg x_i(0) = x_i(0)$: x has period $2n$ and its minimal period p divides $2n$ without dividing n, *i.e.* $\exists q \in \mathbb{N},\ n = q \cdot \frac{p}{2},\ \frac{q}{2} \notin \mathbb{N}$. □

Lemma 2.4. Recurrent configurations of BACs

Let $p \in \mathbb{N}$ be a divisor of $n = qp \in \mathbb{N}$, let $x = x(0) \in \mathbb{B}^n$ be an arbitrary configuration of the BAC $\mathscr{C}_n^s$, *and let $w \in \mathbb{B}^p$ be the binary necklace of length p defined by $w_i = w_{i \bmod p} = x_{i \bmod p}$, $\forall i \in \mathbb{Z}$. Then:*

$$\begin{aligned} x \in \mathscr{X}_n^+(p) \quad &\Leftrightarrow \quad x = w^q \\ &\Leftrightarrow \quad \forall t \in \mathbb{Z},\ x_0(t) = w_{-t} \qquad\qquad \textit{if } s = + \end{aligned}$$

and

$$\begin{aligned} x \in \mathscr{X}_n^-(2p) \quad &\Leftrightarrow \quad x = z^{(q-1)/2} w,\ z = w\overline{w} \in \mathbb{B}^{2p} \\ &\Leftrightarrow \quad \forall t \in \mathbb{Z},\ x_0(t) = z_{-t} \qquad\qquad \textit{if } s = -. \end{aligned}$$

Proof: Let $s = +$ so that $p = n/q$ is a period of $\mathscr{C}_n^+$. The following holds:

$$\begin{aligned} x \in \mathscr{X}_n^+(p) \quad &\Rightarrow \quad \forall i \in \mathbf{V}, i = kp + j \equiv j \bmod p, \\ &\qquad x_i = x_i(kp) = \mathrm{f}[i, i-kp+1](x_{i-kp}(0)) = x_{i-kp}(0) = x_j = w_j \\ &\Rightarrow \quad x = w^q \\ &\Rightarrow \quad \forall t' \in \mathbb{Z},\ t' \equiv t \bmod n, \\ &\qquad x_0(t') = x_0(t) = \mathrm{f}[0, n-t+1](x_{n-t}(0)) = x_{n-t} = w_{n-t} = w_{-t'} \end{aligned}$$

and by induction on k, the last property can be shown to imply $\forall i = kp + j \in \mathbf{V}$, $i \equiv j \bmod p, x_i(p) = x_0(p-j) = w_j = x_0(-j) = x_i(0)$ and thus $x \in \mathscr{X}_n^+(p)$. Now let $s = -$ so that $2p = 2n/q$ is a period of $\mathscr{C}_n^-$. Because $q = n/p$ is odd and thus $n + p \equiv 0 \bmod 2p$:

$$\begin{aligned} x \in \mathscr{X}_n^-(2p) \quad &\Rightarrow \quad \forall i \in \mathbf{V},\ i = 2kp + j \equiv j \bmod 2p, \\ &\qquad x_i = x_i(2kp) = \mathrm{f}[i, i-2kp+1](x_j(0)) = x_j \\ &\qquad\quad = \begin{cases} w_j & \text{if } j < p \\ x_j(n+p) = \neg x_j(p) = \neg x_{j-p} = \neg w_{j-p} & \text{if } j \geq p \end{cases} \\ &\Rightarrow \quad x = z^{(q-1)/2} w,\ z = w\overline{w} \in \mathbb{B}^{2p} \\ &\Rightarrow \quad \forall t' \in \mathbb{Z},\ t' \equiv t \bmod 2n, \\ &\qquad x_0(t') = x_0(t) = \begin{cases} \neg x_{n-t}(0) = \neg z_{p-t} = z_{-t} = z_{-t'} & \text{if } t < n \\ \neg x_0(t-n) = \neg\neg x_{2n-t}(0) = z_{-t} = z_{-t'} & \text{if } t \geq n \end{cases} \end{aligned}$$

where the last property can be proven to imply $x \in \mathscr{X}_n^-(2p)$ by showing $\forall i = kp + j \in \mathbf{V}$, $x_i(p) = x_j(p) = x_0(p-j) = x_j = x_i$ with an induction on k. □

Corollary 2.1. Order of BACs

The order of a BAC $\mathscr{C}_n^s$, *which equals n if $s = +$ and $2n$ if $s = -$ is reached.*

Proof: Let $x = 10^{n-1} \in \mathbb{B}^n = \mathscr{X}_n^+(n)$ and $y = 1^n \in \mathbb{B}^n = \mathscr{X}_n^-(2n)$. Because x can clearly not be written $x = w^q$ for some $w \in \mathbb{B}^{n/q}$, $q > 1$, Lemma 2.4 implies that $x \notin \mathscr{X}_n^+(n/q)$, $\forall q > 1$. Similarly, we show that $y \notin \mathscr{X}_n^-(2n/q)$, $\forall q > 1$. □

Let $\mathcal{N}$ and $\mathcal{N}'$ be two arbitrary BANs of respective sizes n and m and with transition graphs under the parallel update schedule π respectively equal to $\mathcal{T}$ and $\mathcal{T}'$. Restricting to π the simulation $\lhd$ introduced in Section 5, Chap. 1, we write $\mathcal{N} \blacktriangleleft \mathcal{N}'$ if and only if $\mathcal{T}$ is isomorphic to a sub-graph of $\mathcal{T}'$ and we denote by $\blacktriangleright\!\!\blacktriangleleft$ the symmetric closure of $\blacktriangleleft$. Also, when the sub-graph of $\mathcal{T}$ induced by the set $\mathcal{X}(p)$ of configurations of period p of $\mathcal{N}$ is isomorphic to a sub-graph of $\mathcal{T}'$, we write $\mathcal{N} \blacktriangleleft^p \mathcal{N}'$. And we denote by $\blacktriangleright\!\!\blacktriangleleft^p$ the symmetric closure of $\blacktriangleleft^p$. Note that $\mathcal{N} \blacktriangleleft^p \mathcal{N}' \Rightarrow \mathtt{X}(p) \leq \mathtt{X}(p)'$ and $\mathcal{N} \blacktriangleright\!\!\blacktriangleleft^p \mathcal{N}' \Rightarrow \mathtt{X}(p) = \mathtt{X}(p)'$. For the order $p = \omega$ of $\mathcal{N}$, we use: $\blacktriangleleft^\omega = \blacktriangleleft\!|$ and $\blacktriangleright\!\!\blacktriangleleft^\omega = |\!\blacktriangleright\!\!\blacktriangleleft\!|$.

Lemma 2.5. Simulation between BACs

$\forall n, m \in \mathbb{N}, \forall s \in \{-,+\}, \forall p | \gcd(n,m)$, $\mathscr{C}_n^s \blacktriangleright\!\!\blacktriangleleft^p \mathscr{C}_m^s$ *and* $\mathscr{C}_n^- \blacktriangleleft\!| \mathscr{C}_{2n}^+$.

Proof: Let F and F′ be respectively be the global transition functions of $\mathscr{C} = \mathscr{C}_n^s$ and $\mathscr{C}' = \mathscr{C}_m^s$ under π. And for any divisor k of $\gcd(n,m)$, let:

$$\phi_k : \begin{cases} \mathbb{B}^n & \to & \mathbb{B}^m \\ x & \mapsto & x[0,k-1]^{m/k} \end{cases} . \tag{2.4}$$

By Lemma 2.4, ϕ_k maps configurations of period p of $\mathscr{C}$ to configurations of period p of $\mathscr{C}'$, where $p = k$ if $s = +$, and $p = 2k$ if $s = -$ (so that in both cases, p is a period of both BACs). Then for any such k, p and $x = w^{n/k} \in \mathcal{X}_n^s(p)$, letting $u = w_{k-1} w[0,k-2] \in \mathbb{B}^k$ if $s = +$, and $u = \neg w_{k-1} w[0,k-2]$ if $s = -$, the following holds:

$$\begin{tikzcd} x = w^{n/k} \arrow[r, mapsto, "\mathrm{F}"] \arrow[d, mapsto, "\phi_k"] & \mathrm{F}(x) = u^{n/k} \arrow[d, mapsto, "\phi_k"] \\ \phi_k(x) = w^{m/k} \arrow[r, mapsto, "\mathrm{F}'"] & \mathrm{F}'(\phi_k(x)) = u^{m/k} = \phi_k(\mathrm{F}(x)) \end{tikzcd}$$

The second part of the lemma can be derived from Lemma 2.4 by considering a canonical negative BAC $\mathscr{C}_n^-$ and a non-canonical positive BAC $\mathscr{C}_{2n}^+$ with exactly two negative arcs, $(n-1,n)$ and $(2n-1,0)$. □

Corollary 2.2. Number of recurrent configurations of each period

The number of configurations of period any divisor $p \in \mathbb{N}$ of the order of $\mathscr{C}_n^s$ equals[10] *(cf. Fig. 2.2 for an example):*

$$|\mathcal{X}_n^s(p)| = \begin{cases} |\mathcal{X}_p^+(p)| = \mathtt{X}^+(p) = 2^p & \textit{if } s = + \\ \mathtt{X}_n^-(p) = \neg(p|n) \cdot 2^{\frac{p}{2}} & \textit{if } s = - \end{cases}$$

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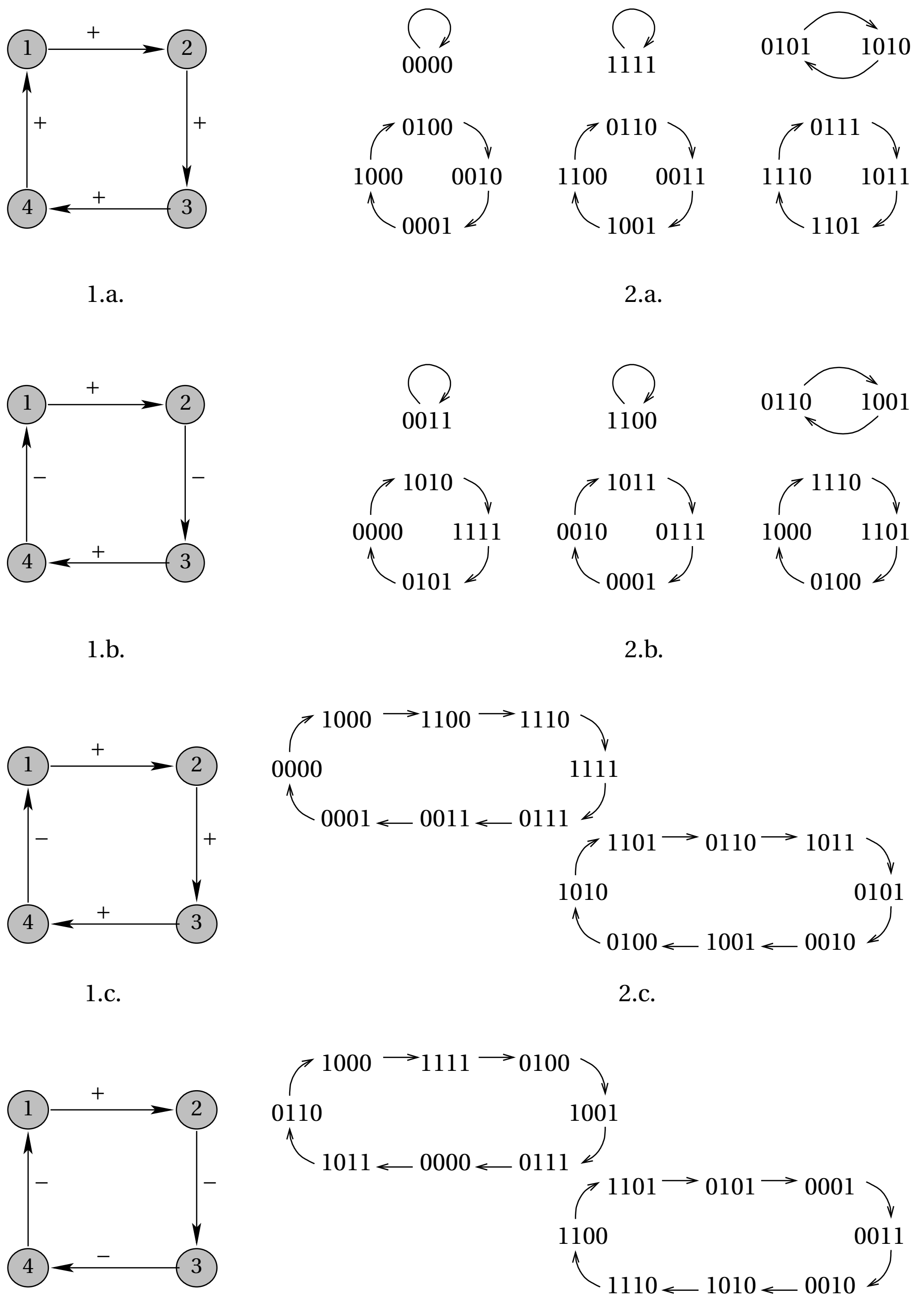
[10] $\neg(p|n) = 0$ if p divides n and 1 otherwise.



1.a. 2.a.

1.b. 2.b.

1.c. 2.c.

1.d. 2.d.

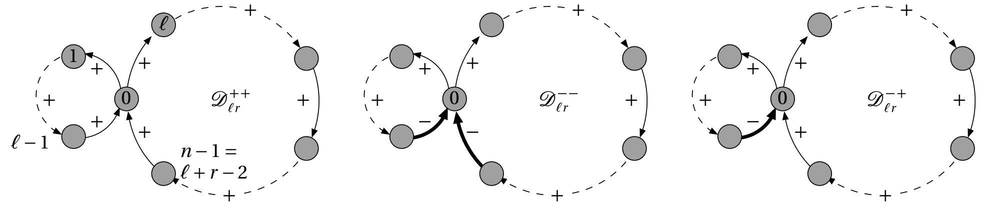
Figure 2.2: *Figures 1.a, b, and c represent four different signed cycles of size $n = 4$. Those of figures 1.a and b are positive while those of 1.c and d are negative. Figures 2.a, b, c and d respectively picture their transition graphs under the parallel update schedule π. In all four cases, all configurations are recurrent.*

The exponential number of recurrent configurations (which yields an exponential number of attractors in Theorem 2.1 below) of BACs found in this section appears to contradict the experimental and theoretical results in [4, 35, 61] suggesting that the interaction networks considered in these studies have only very little different attractors ($\mathcal{O}(\sqrt{n})$ in the case of connectivity 2 BANs considered in [61] and [4], 1 or 2 in the case of the small threshold BANs studied in [35]). A first explanation of this assumes in line with [24], that real interaction networks as random networks, only involve very little cycles. So the total number of attractors that these are responsible for remains small compared to the sizes of the networks. But another argument is provided by the next sections in agreement with the idea, supported in [8] by statistical physics methods, that regulation networks could mostly be small-world networks with a high clustering coefficient, implying the existence of many cycle intersections acting as "gears" on cycles and on their propensity to oscillate.

4 Intersected cycles

Of course, understanding how isolated cycles behave is not enough to understand their role when they are embedded in larger BAN structures. To gain some further intuition beyond that provided by Section 1, we now focus on the behaviours of BADs in parallel to understand formally how cycles interact via structural intersections of the simplest type, involving just one automaton (automaton $0 \in \mathbf{V}$). Later, Section 7 explores the generalisation of this study to other, wider spanning types of intersections.

Following Example 1.13 and Definition 1 in Chapter 1, without loss of generality, we concentrate on **canonical locally monotone BADs** that we define as the disjunctive BADs $\mathscr{D}_{\ell r}^{ss'}$ that have at most two negative arcs, the arcs $(\ell-1, 0)$ whose sign equals that of the left-cycle and the arc $(n-1, 0)$ whose sign equals that of the right-cycle (with the notation of (1.5) in Example 1.5: $\diamond = \vee$ and $\mathrm{f}_0 : (x_{\ell-1}, x_{n-1}) \mapsto \mathbf{b}(s \cdot \mathbf{s}(x_{\ell-1})) \vee \mathbf{b}(s' \cdot \mathbf{s}(x_{n-1}))$). Thus, **canonical positive**, **negative** and **mixed BADs** respectively have the following signed structures (without loss of generality, we do not consider mixed BADs $\mathscr{D}_{\ell r}^{+-}$):



where $\bigcirc \overset{+}{\dashrightarrow} \bigcirc$ denotes a path involving only positive arcs. In addition to the notations and preliminary remarks given in Chapter 1, we introduce the following:

$$\forall i \in \mathbf{V}, \quad i^{\Upsilon} = \begin{cases} 0 & \text{if } i \in \mathbf{V}^L \\ \ell - 1 & \text{if } i \in \mathbf{V}^R \end{cases} \quad \text{and} \quad i^{\dagger} = i - i^{\Upsilon}.$$

Concentrating on $i^{\dagger}$ rather than $i = i^{\Upsilon} + i^{\dagger}$ allows to deal with all automata independently of whether they belong to the left-side ($i = i^{\dagger} \in \mathbf{V}^L$) or to the right-side ($i = (\ell - 1) + i^{\dagger} \in \mathbf{V}^R$), when possible. Similarly to the case of BACs (*cf.* beginning of previous section), the baseline idea of the proofs of results concerning BADs is that the state $x_i(t)$ of one automaton $i \in \mathbf{V}$ at a certain step t can be retraced to the states of other automata at previous steps, these latter states having circulated along the cycles step by step until they reached i at step t. The additional difficulty with respect to the case of BACs derives from what happens at the junction, where automaton 0 is. In particular, formally, for any configuration $x \in \mathbb{B}^n$ of $\mathscr{D}_{\ell r}^{ss'}$ and any automaton $i \in \mathbf{V}$, if $x(-i^{\dagger})$ exists (*i.e.* if $\exists y \in \mathbb{B}^n, \mathrm{F}^{i^{\dagger}}(y) = x$), then the state of i can be expressed simply using a previous state of automaton 0:

$$x_i \ = x_i(0) = \mathrm{f}[i,1](x_0(-i^{\dagger})) = \ x_0(-i^{\dagger}). \tag{2.5}$$

And for configurations x such that $x(-\max\{\ell, r\})$ exists, the state $x_0 = x_0(0)$ of automaton 0 can be related as follows to two of its own previous states, $x_0(-\ell)$ and $x_0(-r)$:

$$\begin{aligned} x_0 = \mathrm{f}_0^L(x_{\ell-1}(-1)) \vee \mathrm{f}_0^R(x_{n-1}(-1)) = \mathrm{f}[0,1]^L(x_0(-\ell)) \vee \mathrm{f}[n-1,0]^R(x_0(-r)) \\ = \mathrm{f}_0^L(x_0(-\ell)) \vee \mathrm{f}_0^R(x_{n-1}(x_0(-r)). \end{aligned} \tag{2.6}$$

Both (2.5) and (2.6) hold for any recurrent configuration (considering that $x(-t) = x(\omega - t)$, ω being the BAN order) and for any configuration $x \in \mathbb{B}^n$, they also obviously do for $x(t)$, $\forall t \geq \max\{\ell, r\}$.

Lemma 2.6. Periods of BADs

Periods $p > 1$ of a BAD $\mathscr{D}_{\ell r}^{ss'}$ *divide the lengths of its positive cycles without dividing the lengths of its negative cycles (if there are any).*

Proof: To prove Lemma 2.6, we first need to relate the behaviour of the intersection automaton 0 of a BAD to the behaviour of the rest of the network in the sense that almost as soon as automaton 0 has started cycling then, the entire BAD starts cycling:

$$\begin{aligned} &\exists d \in \mathbb{N}, \forall t \in \mathbb{N}, x_0(t) = x_0(t + k \cdot d), \forall k \in \mathbb{N} \\ &\qquad \Rightarrow \ \forall t \geq max\{\ell, r\}, x(t) = x(t + k \cdot d), \forall k \in \mathbb{N}. \end{aligned} \tag{2.7}$$

And this comes from: $\forall i \in \mathbf{V}$, $\forall t \geq i^{\dagger}$, $x_i(t + k \cdot d) = x_0(t + k \cdot d - i^{\dagger}) = x_0(t - i^{\dagger}) = x_i(t)$. Now, let $x = x(0) \in \mathbb{B}^n$ be an arbitrary configuration of $\mathscr{D}_{\ell r}^{ss'}$. It satisfies:

$$x_0(\ell) \ = \ \mathrm{f}_0^L(x_{\ell-1}(\ell - 1)) \vee \mathrm{f}_0^R(x_{n-1}(\ell - 1)) \ = \ \mathrm{f}_0^L(x_0(0)) \vee \mathrm{f}_0^R(x_{n-1}(\ell - 1)). \tag{2.8}$$

Let us first consider the case where $s = +$ so that $\mathrm{f}_0^L = \mathrm{id}$. In this case, (2.8) implies: $\forall x \in \mathbb{B}^n,\ x_0 = 1 \Rightarrow \forall k \geq 0,\ x_0(k\ell) = 1$. Moreover, if x is recurrent with period p and $x_0 = 0 \neq x_0(\ell) = 1$, then, letting $k = p \geq 1$, this implies that $0 = x_0 = x_0(\ell \cdot p) = x_0(\ell) = 1$. This contradiction proves that if $s = +$ then $x_0 = x_0(\ell)$ for any recurrent configuration x. Using (2.7), we derive that periods of a BAD $\mathscr{D}_{\ell r}^{ss'}$ divide the lengths of its positive side-cycles and if $s = s' = +$, then they divide $\gcd(\ell, r)$ as well as the length $\ell{+}r$ of the larger encompassing positive cycle.

Now let $s = -$ so that $\mathrm{f}_0^L = \mathrm{neg}$. If x is recurrent with period $p > 1$, then so is $x(t), \forall t \in \mathbb{N}$ and $\exists t \in \mathbb{N}, x_0(t) = 0$. For this t, by (2.8), $x_0(t + \ell) = 1$ holds so that the period of $x(t)$ and x cannot divide ℓ. Thus, periods of BADs do not divide the lengths of their negative side-cycles. And if $s = - \neq s' = +$, a period $p > 1$ divides r without dividing ℓ, so in this case, p does not divide the length $\ell{+}r$ of the larger encompassing negative cycle.

If $s = s' = -$ and $r \geq \ell$, then $\forall t \geq r - \ell$:

$$\begin{aligned} x_0(t + \ell{+}r) &= \neg x_0(t + r) \vee \neg x_0(t + \ell) \\ &= \neg\big(\neg x_0(t + r - \ell) \vee \neg x_0(t)\big) \vee \neg\big(\neg x_0(t) \vee \neg x_0(t - r + \ell)\big) \\ &= x_0(t) \wedge \big(x_0(t + r - \ell) \vee x_0(t - r + \ell)\big) \end{aligned}$$

so either $x_0(t) = 0$ implying that $x_0(t + \ell{+}r) = 0$, or, in the case where $x(t)$ is recurrent, $x_0(t) = 1$ and for the same reason as before, $x_0(t + \ell{+}r) = 1$. Thus, if $x(t)$ is recurrent, then $x_0(t + \ell{+}r) = x(t)$. With (2.7), this proves that periods of $\mathscr{D}_{\ell r}^{--}$ divide $\ell{+}r$, the length of the larger encompassing positive cycle. □

Lemma 2.7. Periods of BADs are smaller

Periods of a BAD $\mathscr{D}_{\ell r}^{ss'}$ *either are no greater than* $\max\{\ell, r\}$ *or equal* $\ell{+}r$*, which only is possible if* $(s, s') = (-,-)$ *or if* $(s, s') = (+,+) \wedge \ell = r$.

Proof: If $s = s' = +$ or if $s \neq s'$, Lemma 2.7 follows directly from Lemma 2.6. Let $p > \ell \geq r$ be a period of $\mathscr{D}_{\ell r}^{--}$. By Lemma 2.6, $\exists k \in \mathbb{N}^*,\ 2\ell \geq \ell{+}r = kp > k\ell$. This implies that $k = 1$ and $p = \ell{+}r$. □

Let us note that Lemma 2.7 implies that the order of a BAD is never greater than the orders of its largest positive cycle and of its largest side-cycle. Now, relative to the side-signs $s, s' \in \{-,+\}$ of a BAD $\mathscr{D}_{\ell r}^{ss'}$ and to an integer $d \in \mathbb{N}$, we define the set $\mathscr{W}^d(p)$ of necklaces of size $p \geq d$ as follows:

$$\begin{aligned} \mathscr{W}^d(p) = \{w \in \mathbb{B}^p \quad | \quad & \forall u, u' \in \mathbb{B}^{d-1}, \\ & (s, s') \neq (+,+) \Rightarrow 0u0 \text{ is not a factor of } w \text{ and} \\ & (s, s') = (-,-) \Rightarrow 1u1u'1 \text{ is not a factor of } w \quad \}. \end{aligned} \tag{2.9}$$

In analogy to Lemma 2.4 concerning BACs, Lemma 2.8 below characterises the recurrent configurations of period p of BADs using necklaces of length p.

Lemma 2.8. Recurrent configurations of BADs

Let $x \in \mathbb{B}^n$ *be a configuration of the* BAD $\mathscr{D} = \mathscr{D}_{\ell r}^{ss'}$. *For any divisor* p *of the order of* $\mathscr{D}$, *the circular word* $w \in \mathbb{B}^p$ *defined by:*

$$\forall j < p,\ w_j = x_0(p-j)$$

$$\textit{satisfies:}\quad x \in \mathscr{X}(p) \quad \Leftrightarrow \quad w \in \mathscr{W}^d(p) \textit{ and } \begin{cases} x^L = w^q\, w[0, d-1] \\ x^R = w^{q'}\, w[0, d'-1] \end{cases}$$

where $\ell = qp + d \equiv d \bmod p$, $r = q'p + d' \equiv d' \bmod p$.

The complete proof of Lemma 2.8 figures on Page 141 of the appendix. Informally, the idea is that the global transition function of a canonical BAD just shifts automata states except possibly around the intersection automaton 0 (*cf.* (2.5)). This yields the expression of an arbitrary configuration x as in Lemma 2.8 using x^L and x^R and a word $w \in \mathbb{B}^p$, $p < n$. Then, accounting in (2.6) for the periods of BADs (*cf.* Lemma 2.6) allows to specify the conditions that must be satisfied by configurations of period p. And Lemma 2.8 follows from the direct relation existing between these conditions and the definition of $\mathscr{W}^d(p)$ in (2.9). The proof of the next result, Lemma 2.9 also figures in the appendix, on Page 143. It consists in exhibiting aperiodic words $w \in \mathscr{W}^d(p)$ to characterise configurations of minimal period p in each case $(s, s' \in \{-,+\})$ and in agreement with Lemma 2.8.

Lemma 2.9. Periods and order of BADs

The order of a BAD $\mathscr{D}_{\ell r}^{ss'}$ *(where* $\Delta = \gcd(\ell, r)$ *and* $\ell{+}r = K\Delta$*) equals:*

$$\omega \;=\; \begin{cases} \Delta & \textit{if } (s,s') = (+,+) \\ r & \textit{if } (s,s') = (-,+) \\ \frac{\ell+r}{2} = 2\Delta & \textit{if } (s,s') = (-,-) \textit{ and } K = 4 \\ \ell{+}r & \textit{if } (s,s') = (-,-) \textit{ and } K \neq 4. \end{cases}$$

Further, any divisor p *of* ω *is a minimal period of* $\mathscr{D}_{\ell r}^{ss'}$ *except if* $(s,s') = (-,-)$ *and either* $p = 6\Delta_p = 6$ *or* $p = 4\Delta_p$ *(where* $\Delta_p = \gcd(\Delta, p)$*). Thus, the order* ω *of* $\mathscr{D}_{\ell r}^{ss'}$ *is reached unless* $(s,s') = (-,-)$ *and* $\omega = \ell + r = 6$.

To illustrate Lemma 2.9, consider the BAD $\mathscr{D}_{3,3}^{--}$. Its order is $\omega = 3+3 = 6$ and it has configurations of minimal period $p = 6 = 2\Delta$ (6 of them form an attractor of period 6) since $\Delta_6 = \Delta = 3 \neq 1$. On the contrary, the BAD $\mathscr{D}_{11,1}^{--}$ has none because $\Delta_6 = \gcd(6, 11, 1) = 1$.

Corollary 2.3. Asymptotic simulations between BADs and BACs

$$\mathscr{D}_{\ell r}^{++} \mathrel{|\!\blacktriangleright\!\!\blacktriangleleft\!|} \mathscr{C}_{\gcd(\ell,r)}^{+},\quad \mathscr{D}_{\ell r}^{-+} \mathrel{\blacktriangleleft\!|} \mathscr{C}_{r}^{+},\quad \mathscr{D}_{\ell r}^{--} \mathrel{\blacktriangleleft\!|} \mathscr{C}_{\ell+r}^{+} \textit{ and } \mathscr{D}_{\ell\ell}^{ss} \mathrel{|\!\blacktriangleright\!\!\blacktriangleleft\!|} \mathscr{C}_{\ell}^{s}.$$

Proof: Using the notations of Lemma 2.8. $\forall w \in \mathbb{B}^p$, we define the configuration $x = w^{(n)} \in \mathbb{B}^n$ of a BAD $\mathcal{D}^{ss'}_{\ell r}$ by:

$$x = w^{(n)} \;\Rightarrow\; \left(x^L = w^q w[0, d-1] \;\wedge\; x^R = w^{q'} w[0, d'-1]\right) \tag{2.11}$$

and we extend this definition to BACs so that when x is meant to be a configuration of a BAC of size $n = qp + d \equiv d \bmod p$ rather than of a BAD, it satisfies $x = w^{(n)} = w^q w[0, d-1] \in \mathbb{B}^n$. This way, by Lemmas 2.4 and 2.8, all configurations $x \in \mathscr{X}(p)$ of (minimal) period p can be written $x = w^{(n)}$ for some (aperiodic) $w \in \mathbb{B}^p$ (belonging to $\mathscr{W}^d(p)$ in the case of BADs). For two BANs $\mathcal{N}$ and $\mathcal{N}'$ of respective sizes n and m, let $\phi_p : x \in \mathbb{B}^n \mapsto x[0, p-1]^{(m)} \in \mathbb{B}^m$. This application that maps a configuration $w^{(n)} \in \mathbb{B}^n$ of $\mathcal{N}$ to the configuration $w^{(m)} \in \mathbb{B}^m$ of $\mathcal{N}'$ for any $w \in \mathbb{B}^p$. Corollary 2.3 can be proven similarly to Lemma 2.5 by using ϕ_p to show that $\mathcal{N} \blacktriangleleft\!| \mathcal{N}'$. □

The **Lucas sequence** $(\mathtt{L}(n))_{n\in\mathbb{N}}$ [88, 95] (sequence `A204` of the OEIS [111]) is defined by:

$$\begin{cases} \mathtt{L}(1) = 1, \\ \mathtt{L}(2) = 3, \\ \mathtt{L}(n) = \mathtt{L}(n-1) + \mathtt{L}(n-2),\ \forall n > 2. \end{cases} \tag{2.12}$$

It counts the number of circular binary words without the factor 00, *i.e.*:

$$\mathtt{L}(n) = |\mathscr{W}^1(n)| \text{ in the case where } (s, s') = (-, +) \ (cf.\ (2.9)). \tag{2.13}$$

The **Perrin sequence** $(\mathtt{P}(n))_{n\in\mathbb{N}}$ [1] (sequence `A1608` of the OEIS [111]) is defined by:

$$\begin{cases} \mathtt{P}(0) = 3, \\ \mathtt{P}(1) = 0, \\ \mathtt{P}(2) = 2, \\ \mathtt{P}(n) = \mathtt{P}(n-2) + \mathtt{P}(n-3),\ \forall n > 2. \end{cases} \tag{2.14}$$

It satisfies the following result, proven on Page 145 of the appendix.

Lemma 2.10. The Perrin sequence

For $n > 0$, $\mathtt{P}(n)$ counts the number of circular words of size n without the subsequences 00 and 111, i.e. when $(s, s') = (-, -)$ in (2.9), $\mathtt{P}(n) = |\mathscr{W}^1(n)|$.

Lemma 2.11. Number of recurrent configurations of a BAD

For any divisor p of the order ω of a BAD $\mathscr{D} = \mathscr{D}_{\ell r}^{ss'}$, *the number of configurations of period p of $\mathscr{D}$ is:*

$$|\mathscr{X}_\omega(p)| = \begin{cases} \mathtt{X}^+(p) & = 2^p & \textit{if } (s,s') = (+,+) \\ \mathtt{X}_\ell^{-+}(p) & = \mathtt{L}(\frac{p}{\Delta_p})^{\Delta_p} & \textit{if } (s,s') = (-,+) \\ \mathtt{X}_\Delta^{-+}(p) & = \mathtt{P}(\frac{p}{\Delta_p})^{\Delta_p} & \textit{if } (s,s') = (-,-) \end{cases}$$

where $\Delta = \mathrm{gcd}(\ell, r)$ *and* $\Delta_p = \mathrm{gcd}(p, \ell) = \mathrm{gcd}(p, \Delta)$. *In particular, $\mathscr{D}$ has as many fix points as it has positive side-cycles.*

Proof: By Lemma 2.8, the number of configurations of period p equals $|\mathscr{W}^d(p)|$, the number of fix points equals $|\mathscr{W}^d(1)|$. If $(s,s') = (+,+)$, then $\mathscr{W}^d(p) = \mathbb{B}^p$ and $\mathscr{W}^d(1) = \{0^{(n)}, 1^{(n)}\}$. Otherwise, if $(s,s') = (-,+)$, then $\mathscr{W}^d(1) = \{1^{(n)}\}$ and if $(s,s') = (-,-)$, then $\mathscr{W}^d(1) = \emptyset$.

Let $p > 1$. Any word $w \in \mathbb{B}^p$ can be written as an interleaving of a certain number a of circular sub-words $w(1), w(2), \ldots, w(a)$ of size $m = \frac{p}{a}$ such that (*cf.* Fig. 2.3) $\forall j \leq a$, $w(j)_0 = w_j$ and $\forall i < m$, $w(j)_i = w_{i'+d}$ if $w(j)_{i-1} = w_{i'}$. Then, $m \times d = k \times p$ holds for a certain *minimal* integer k, *i.e.* $md = \mathrm{lcm}(d,p) = \frac{dp}{\mathrm{gcd}(d,p)}$. Consequently, any word $w \in \mathbb{B}^p$ can be written as an interleaving of $a = \Delta_p$ sub-words of length $m = \frac{p}{\Delta_p}$.

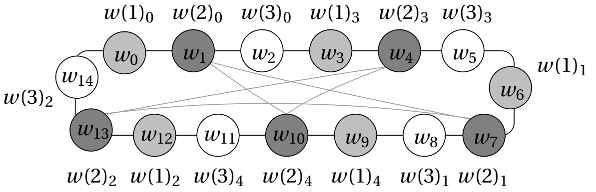


Figure 2.3: *A circular word* $w \in \mathbb{B}^p$, $p = 15$, *represented as an interleaving of* $\Delta_p = \mathrm{gcd}(6, p) = 3$ *words* $w(1)$, $w(2)$ *and* $w(3)$ *of size* $\frac{p}{\Delta_p} = 5$, *corresponding respectively to nodes in light grey, dark grey and white.*

If $w \in \mathscr{W}^d(p)$, then each of the sub-words $w(j)$, $j < \Delta_p$ in this writing belongs to $\mathscr{W}^1(m)$. If $(s,s') = (-,+)$ (resp. $(s,s') = (-,-)$), by (2.13) (resp. by Lemma 2.10), $|\mathscr{W}^1(m)| = \mathtt{L}(m)$ (resp. $|\mathscr{W}^1(m)| = \mathtt{P}(m)$) and as a result $|\mathscr{W}^d(p)| = |\mathscr{W}^1(m)|^{\Delta_p}$. □

5 Combinatorial characterisation of limit behaviours

The two previous sections determine the periods and orders of BACs and BADs and they also characterise and count the recurrent configurations of each period.

Using Dirichlet convolutions whose properties we now recall, the present section draws from these results a more complete combinatorial description of the asymptotic behaviours of these two types of networks.

Let $\mathbb{1}$ be the function $\mathbb{1} : n \in \mathbb{N} \mapsto 1$ and let $\star$ denote the *Dirichlet convolution* [2], that is, the binary operator such that for any two arithmetic functions f and g:

$$f \star g : n \in \mathbb{N} \mapsto \sum_{p|n} f(p) \cdot g\big(n/p\big).$$

We recall that the set of arithmetic functions with point-wise addition and Dirichlet convolution is a commutative ring. The multiplicative identity of this ring is the function $\delta : \mathbb{N} \to \mathbb{N}$ defined by $\delta(1) = 1$ and $\forall n > 1, \delta(n) = 0$. Let us also recall that the inverse of function $\mathbb{1}$ for the Dirichlet convolution is the Möbius function (see [2], for instance):

$$\mu : n \in \mathbb{N} \mapsto \begin{cases} 0 & \text{if } n \text{ is not square-free} \\ 1 & \text{if if } n > 0 \text{ has an even number of prime factors} \\ -1 & \text{if } n > 0 \text{ has an odd number of prime factors.} \end{cases}$$

If $n = \prod_{i=0}^{k} p_i$ where the p_i's are distinct positive primes, then $\mu(n) = (-1)^k$. The importance of this function in the present context lies in the *Möbius inversion formula* which is derived from $\mathbb{1} \star \mu = \delta$ and which is satisfied by all arithmetic functions f and g:

$$g = f \star 1 \;\Rightarrow\; f = g \star \mu,$$

$$\textit{i.e.}\ \forall n \in \mathbb{N},\ g(n) = \sum_{p|n} f(p) \;\Rightarrow\; f(n) = \sum_{p|n} g(p) \cdot \mu\big(n/p\big).$$

Another notable property is the resulting relation between the Möbius function and the Euler totient φ: since φ satisfies $\forall n \in \mathbb{N},\ n = \varphi \star \mathbb{1}(n)$, it holds that $\varphi = \mu \star id$, where $id : n \in \mathbb{N} \mapsto n$.

In combinatorial terms, the limit behaviour of a deterministic system such as a BAN $\mathscr{N}$ updated in parallel can be described by the four quantities [89] that are listed below together with the equalities that relate them. Let p be a divisor of the order ω of the system and let $inv : n \in \mathbb{N} \mapsto 1/n$.

- The **number** $\mathtt{X}(p)$ **of configurations of period** p:

$$\mathtt{X} = \widetilde{\mathtt{X}} \star \mathbb{1}; \tag{2.15a}$$

- The **number** $\widetilde{\mathtt{X}}(p)$ **of configurations of minimal period** p:

$$\widetilde{\mathtt{X}} = \mathtt{X} \star \mu; \tag{2.15b}$$

- The **number** $\mathtt{A}(p) = \widetilde{\mathtt{X}}(p)/p$ **of** p**-attractors**:

$$\mathtt{A} = inv \cdot (\mathtt{X} \star \mu); \tag{2.15c}$$

- The **number** $\mathtt{T}(p)$ **of attractors of period a divisor of** p:

$$\mathtt{T} = \mathtt{A} \star \mathbb{1} = inv \cdot (\mathtt{X} \star \varphi) \tag{2.15d}$$

where the last equality above, is a well known formula in the context of necklaces (just like (2.15c) which corresponds to the *Witt formula* counting the number of Lyndon words) [12, 53, 69, 71, 72, 105] and corresponds to the Burnside's orbit-counting Lemma. It holds because completely multiplicative functions such as inv distribute over $\star$.

As a result of the existence of the relations (2.15a) to (2.15d), to determine any of the three quantities $\widetilde{\mathtt{X}}$, $\mathtt{A}$ and $\mathtt{T}$ relative to a given network $\mathscr{N}$, it suffices to determine the quantity $\mathtt{X}$. This yields the following theorem which sums up in the table on Page 54 the combinatorial results concerning BACs and BADs derived in this chapter.

Theorem 2.1. Combinatorics of BACs and BADs

For any network $\mathscr{N}$ which is either a BAC *or a* BAD*, Table 5 gives the order ω of $\mathscr{N}$, its numbers $\mathtt{X}(p)$ and $\widetilde{\mathtt{X}}(p)$ of configurations of (minimal) period p ($\mathtt{X}(\omega)$ being its total number of recurrent configurations), its number of p-attractors and its total number of attractors.*

Proof: Corollary 2.1 and Lemma 2.9 determine ω. Corollary 2.2 and Lemma 2.11 give $\mathtt{X}(p)$. The rest of Theorem 2.1 follows from (2.15), noting for the case of negative BACs $\mathscr{C}_n^-$, that $p|2n = pq$ implies for any $d \in \mathbb{N}$, $\big(d|p \wedge \neg d|n\big) \Leftrightarrow \big(d|p \wedge \frac{2n}{d} \text{ is odd}\big) \Leftrightarrow \big(d' = \frac{p}{d}|p \wedge d' \text{ is odd}\big)$. □

Thus, for instance, letting $\mathtt{Q} \in \{\mathtt{X}, \widetilde{\mathtt{X}}, \mathtt{A}, \mathtt{T}\}$ (*cf.* (2.15)), a mixed BAD $\mathscr{D}_{3,6}^{-+}$ has $1 = \mathtt{Q}_3^{-+}(1)$ fix point. Also, it has $\mathtt{X}_3^{-+}(2) = 3$ (resp. $\widetilde{\mathtt{X}}_3^{-+}(2) = 2$) configurations of (resp. minimal) period 2, $\mathtt{X}_3^{-+}(3) = 1$ (resp. $\widetilde{\mathtt{X}}_3^{-+}(3) = 0$) of (resp. minimal) period 3 and $\mathtt{X}_3^{-+}(6) = 27$ (resp. $\widetilde{\mathtt{X}}_3^{-+}(6) = 24$) of (resp. minimal) period 6. Therefore, it has $\mathtt{A}_3^{-+}(2) = 1$ attractor of period 2, $\mathtt{A}_3^{-+}(6) = 4$ of period 6 and $\mathtt{T}_3^{-+}(6) = 6$ attractors in total.

Computer simulations of the behaviours of BACs and BADs under π produced Tables 2.1, 2.2, 2.4 and 2.5, illustrating Theorem 2.1. In Tables 2.1 and 2.2 for BACs (Tables 2.4 and 2.5 for BADs are mentioned again later), as predicted by Lemmas 2.3 and 2.5 (among others), all numbers appearing on one line are identical. In particular, the first line of Table 2.1 confirms that all positive cycles have two fixed points, whereas Table 2.2 confirms that negative cycles have none. This recalls the results of [4] that characterised positive cycles by this property and from

n p	1	2	3	4	5	6	7	8	9	10	11	12	21	22
1	2	2	2	2	2	2	2	2	2	2	2	2	2	2
2	–	1	–	1	–	1	–	1	–	1	–	1	–	1
3	–	–	2	–	–	2	–	–	2	–	–	2	2	–
4	–	–	–	3	–	–	–	3	–	–	–	3	–	–
5	–	–	–	–	6	–	–	–	–	6	–	–	–	–
6	–	–	–	–	–	9	–	–	–	–	–	9	–	–
7	–	–	–	–	–	–	18	–	–	–	–	–	18	–
8	–	–	–	–	–	–	–	30	–	–	–	–	–	–
9	–	–	–	–	–	–	–	–	56	–	–	–	–	–
10	–	–	–	–	–	–	–	–	–	99	–	–	–	–
11	–	–	–	–	–	–	–	–	–	–	186	–	–	186
12	–	–	–	–	–	–	–	–	–	–	–	335	–	–
21	–	–	–	–	–	–	–	–	–	–	–	–	99858	–
22	–	–	–	–	–	–	–	–	–	–	–	–	–	190557
$\mathtt{T}^+$	2	3	4	6	8	14	20	36	60	108	188	352	99880	190746

Table 2.1: *Behaviours of positive* BACs*. Cell (p,n) gives the number $\mathtt{A}^+(p)$ of p-attractors of $\mathscr{C}_n^+$ (– stands for 0) and the last line gives its total number of attractors $\mathtt{T}^+(n)$.*

n p	1	2	3	4	5	6	7	8	15	16	17	18	21	22
2	1	–	1	–	1	–	1	–	1	–	1	–	1	–
4	–	1	–	–	–	1	–	–	–	–	–	1	–	1
6	–	–	1	–	–	–	–	–	1	–	–	–	1	–
8	–	–	–	2	–	–	–	–	–	–	–	–	–	–
10	–	–	–	–	3	–	–	–	3	–	–	–	–	–
12	–	–	–	–	–	5	–	–	–	–	–	5	–	–
14	–	–	–	–	–	–	9	–	–	–	–	–	9	–
16	–	–	–	–	–	–	–	16	–	–	–	–	–	–
30	–	–	–	–	–	–	–	–	1091	–	–	–	–	–
32	–	–	–	–	–	–	–	–	–	2048	–	–	–	–
34	–	–	–	–	–	–	–	–	–	–	3855	–	–	–
36	–	–	–	–	–	–	–	–	–	–	–	7280	–	–
42	–	–	–	–	–	–	–	–	–	–	–	–	49929	–
44	–	–	–	–	–	–	–	–	–	–	–	–	–	95325
$\mathtt{T}^-_n$	1	1	2	2	4	6	10	16	1096	2048	3856	7286	49940	95326

Table 2.2: *Behaviours of negative* BACs*. Cell (p,n) gives the number $\mathtt{A}^-_n(p)$ of p-attractors of $\mathscr{C}_n^-$ (– stands for 0) and the last line gives its total number of attractors $\mathtt{T}^-_n(2n)$.*

	Positive BACs $\mathscr{N} = \mathscr{C}_n^+$	Negative BACs $\mathscr{N} = \mathscr{C}_n^-$	Positive BADs $\mathscr{N} = \mathscr{D}_{\ell r}^{++}$	Mixed BADs $\mathscr{N} = \mathscr{D}_{\ell r}^{-+}$	Negative BADs $\mathscr{N} = \mathscr{D}_{\ell r}^{--}$
Order ω of $\mathscr{N}$	n	$2n$	Δ	r	$\begin{cases} \frac{\ell+r}{2} & \text{if } \frac{\ell+r}{\Delta} = 4 \\ \ell+r & \text{otherwise} \end{cases}$
Number of configurations of period p, $p \vert \omega$	$\mathtt{X}^+(p) \quad = \quad 2^p$	$\mathtt{X}_n^-(p) \quad = \quad \neg(p \vert n) \cdot 2^{\frac{p}{2}}$	$\mathtt{X}^+(p)$	$\mathtt{X}_\ell^{-+}(p) \quad = \quad \neg(p \vert \ell) \cdot \mathtt{L}\left(\frac{p}{\Delta_p}\right)^{\Delta_p}$	$\mathtt{X}_\Delta^{--}(p) \quad = \quad \neg(p \vert \Delta) \cdot \mathtt{P}\left(\frac{p}{\Delta_p}\right)^{\Delta_p}$
Number of configurations of minimal period p, $p \vert \omega$	$\widetilde{\mathtt{X}}^+(p) \quad = \quad \sum_{d \vert p} \mu\left(\frac{p}{d}\right) \cdot 2^d$ $= \quad \mathtt{A27375}(p)$	$\widetilde{\mathtt{X}}_n^-(p) \quad = \quad \sum_{\text{odd } k \vert p} \mu(k) \cdot 2^{\frac{p}{2k}}$	$\widetilde{\mathtt{X}}^+(p)$	$\widetilde{\mathtt{X}}_\ell^{-+}(p) \quad = \quad \sum_{\substack{d \vert p, \\ \neg(d \vert \ell)}} \mu\left(\frac{p}{d}\right) \cdot \mathtt{L}\left(\frac{d}{\Delta_d}\right)^{\Delta_d}$	$\widetilde{\mathtt{X}}_\Delta^{--}(p) \quad = \quad \sum_{\substack{d \vert p, \\ \neg(d \vert \Delta)}} \mu\left(\frac{p}{d}\right) \cdot \mathtt{P}\left(\frac{d}{\Delta_d}\right)^{\Delta_d}$
Number of attractors of period p, $p \vert \omega$	$\mathtt{A}^+(p) \quad = \quad \frac{\widetilde{\mathtt{X}}^+(p)}{p}$ $= \quad \mathtt{A1037}(p)$	$\mathtt{A}_n^-(p) \quad = \quad \frac{\widetilde{\mathtt{X}}_n^-(p)}{p}$ $= \quad \mathtt{A48}(\frac{p}{2})$	$\mathtt{A}^+(p)$	$\mathtt{A}_\ell^{-+}(p) \quad = \quad \frac{\widetilde{\mathtt{X}}_\ell^{-+}(p)}{p}$	$\mathtt{A}_\Delta^{=}(p) \quad = \quad \frac{\widetilde{\mathtt{X}}_\Delta^{--}(p)}{p}$
Total number of attractors	$\mathtt{T}^+(n) \quad = \quad \frac{1}{n} \sum_{d \vert n} \varphi\left(\frac{n}{d}\right) \cdot 2^d$ $= \quad \mathtt{A31}(n)$	$\mathtt{T}_n^-(2n) \quad = \quad \frac{1}{2n} \sum_{\text{odd } k \vert 2n} \varphi(k) \cdot 2^{\frac{n}{2k}}$ $= \quad \mathtt{A16}(n)$	$\mathtt{T}^+(\Delta)$	$\mathtt{T}_\ell^{-+}(r) \quad = \frac{1}{r} \sum_{\substack{d \vert r, \\ \neg(d \vert \ell)}} \varphi\left(\frac{r}{d}\right) \cdot \mathtt{L}\left(\frac{d}{\Delta_d}\right)^{\Delta_d}$	$\mathtt{T}_\Delta^{--}(\ell+r) \quad = \frac{1}{N} \sum_{\substack{d \vert N, \\ \neg(d \vert \Delta)}} \varphi\left(\frac{N}{d}\right) \cdot \mathtt{P}\left(\frac{d}{\Delta_d}\right)^{\Delta_d}$

Table 2.3: *Combinatorial description of the (asymptotic) behaviours of all* BACs *and* BANs*, where* $\Delta = \gcd(\ell, r)$ *and* $\Delta_p = \gcd(p, \ell) = \gcd(p, \Delta)$ *and* $\neg(p \vert m)$ *equals* 0 *if* p *divides* m *and* 1 *otherwise.*

which the authors derived that threshold networks with *arbitrary* structures containing only negative cycles have no fixed points. Other particular cases of couples (p, n) may be pointed out. When $n = 2^k$, for instance, since 1 is the only odd divisor of n, the only period of the negative BAC $\mathscr{C}_n^-$ is its order $\omega = 2n$, so $\mathtt{A}_{2n}^- = \mathtt{T}_n^- = 2^{n-k-1}$ (*cf.* cells (16,8) and (32,16) of Table 2.2). Also, if n is prime then, because $\mu(n) = -1$, the positive BAC $\mathscr{C}_n^+$ has only two types of attractors, that is, fix points and ω-attractors of maximal period $\omega = n$, so $\mathtt{T}_n^+ = 2 + \mathtt{A}_n^+$ where $\mathtt{A}_n^+ = (2\mu(n) + 2^n\mu(1))/n = (2^n - 2)/n$.

6 Comparisons and bounds

The previous sections provide explicit formulae (*cf.* Theorem 2.1) to characterise the combinatorics of all types of BACs $\mathscr{C}_n^s$ as well as of all types of BADs $\mathscr{D}_{\ell r}^{ss'}$ ($s, s' \in \{-,+\}$, $n, \ell, r \in \mathbb{N}$). However, in themselves, these formulae and the general understanding of the behaviours of BACs and BADs that follows from Sections 3 and 4 have a limited scope when they are placed in the wider framework of arbitrary BANs. To exploit them effectively and draw an understanding of how BAN behaviours relate to their fundamental structural motifs requires to compare the behaviours of BACs and BADs. Thus, Lemma 2.5 and Corollary 2.3 above have shown how different BACs and BADs can simulate one another. But further than that, the remarks made at the end of Section 3 suggest to explore how interactions between cycles via simple intersections of the types involved in double-cycles cause significant losses in the degrees of freedom of the overall network, that is (*cf.* Section 2), in its total number $\mathtt{T}(\omega)$ of attractors. In agreement with this suggestion, the present section compares the degrees of freedom yielded by isolated cycles $\mathbb{C}_n$ ($\mathtt{T}^+(n)$ and $\mathtt{T}_n^-(2n)$) with those yielded by intersected cycles $\mathbb{D}_{\ell r} = \mathbb{C}_\ell \times \mathbb{C}_r$ ($\mathtt{T}^{++}(\Delta)$, $\mathtt{T}_\ell^{-+}(r)$, and $\mathtt{T}_\Delta^{--}(\ell + r)$).

Let $\mathscr{N}$ be either a BAC $\mathscr{C}_n^s$ or a BAD $\mathscr{D}_{\ell r}^{ss'}$ with order ω, let p be a divisor of ω and let $\mathtt{Q} \in \{\mathtt{X}, \widetilde{\mathtt{X}}, \mathtt{A}, \mathtt{T}\}$ be any of the four quantities introduced in (2.15) and mentioned in Theorem 2.1 for the special cases of BACs and BADs. Corollary 2.3 implies[11]:

$$\mathtt{Q}_{\ell,r}^{++}(p) = \mathtt{Q}^+(p), \quad \mathtt{X}_{\ell,r}^{-+}(p) \le \mathtt{X}^+(p), \quad \mathtt{X}_{\ell,r}^{--}(p) \le \mathtt{X}^+(p) \text{ and } \mathtt{Q}_{\ell,\ell}^{ss}(p) = \mathtt{Q}_\ell^s(p).$$

But $\mathtt{X}_{\ell,r}^{ss'}(p)$ can also be bounded more precisely as stated by the next lemma which is to be compared with the expressions of $\mathtt{X}(p)$ and $\mathtt{X}_n^-(p)$ given in Corollary 2.2. The proof of this lemma is given in the appendix, on Page 145. It

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[11] For the sake of clarity, we slightly abuse notations of Theorem 2.1 and confuse $\mathtt{Q}_k^s$ and $\mathtt{Q}_k^{ss'}$ respectively denoting quantities relative to $\mathscr{C}_n^s$ and $\mathscr{D}_{\ell r}^{ss'}$ dependent on $k \in \mathbb{N}$ (and also $\mathtt{Q}^+$ and $\mathtt{Q}^{++}$ for positive BACs and BADs, independent of any $k \in \mathbb{N}$) with $\mathtt{Q}_n^s(p)$ and $\mathtt{Q}_{\ell,r}^{ss'}(p)$.

relies on the last result (Lemma 2.11) presented in the previous section, giving literal expressions of $\mathtt{X}^{ss'}_{\ell,r}(p)$ for BADs. And it also relies on properties satisfied by the Lucas [95] and Perrin [1] sequences in relation to the **golden ratio** $\gamma = (1+\sqrt{5})/2 \approx 1.61803399$ (root of $x^2 - x - 1 = 0$) and to the **plastic number** [132] $\xi \approx 1.32471796 \in \mathbb{R}$ (real root of $x^3 - x - 1 = 0$) respectively.

Lemma 2.12. Upper bound on $\mathtt{X}(p)$

For a divisor $p = K_p\Delta_p$ ($\Delta_p = \gcd(\ell, r, p)$) of the order of a BAD $\mathscr{D}^{ss'}_{\ell r}$, *the number of configurations of period p is bounded as follows*[11]*:*

$$\begin{array}{lcccll} \gamma^p & \sim & \mathtt{X}^{-+}_{\ell,r}(p) & \leq & \sqrt{3}^p & \textit{if}\,(s,s') = (-,+) \\ \xi^p & \sim & \mathtt{X}^{--}_{\ell,r}(p) & \leq & \begin{cases} 3^{\frac{p}{3}} & \textit{if}\, K_p = 3 \\ \sqrt{2}^p & \textit{if}\, K_p \neq 3 \end{cases} & \textit{if}\,(s,s') = (-,-). \end{array}$$

Bypassing some rare exceptions (*cf.* Lemma 2.9) let us concentrate on BADs $\mathscr{D}^{ss'}_{\ell r}$ whose orders are reached and equal to $\omega = r$ if $(s,s') = (-,+)$ and to $\omega = \ell + r$ if $(s,s') = (-,-)$. The patterns that can be observed in Tables 2.4 and 2.5 may now be explained *informally* as follows, by extending expressions concerning $\mathtt{X}$ in Lemma 2.12 and its proof to $\mathtt{T}$. The pertinence of this extension is supported later by Theorem 2.2 which puts forward that the great majority of all recurrent configurations of BADs have maximal period $p = \omega$ so that $\mathtt{T}(\omega) = \Theta(\mathtt{X}(\omega)/\omega)$. For the case of mixed BADs, by examining expressions figuring in the proof of Lemma 2.12 on Page 146, it can be shown that if p is odd, then $\mathtt{X}^{-+}_{\ell}(p)$ is maximal when Δ_p is minimal, *i.e.* when $\Delta_p = 1$; and if p is even then, on the contrary, $\mathtt{X}^{-+}_{\ell}(p)$ is maximal when Δ_p is maximal, *i.e.* when $\Delta_p = p/2$. With $p = \omega$ this supports the patterns of Table 2.4. For the case of negative BADs, let us consider a given $\ell \in \mathbb{N}$ and suppose that $\ell \geq r$. Lemma 2.12 suggests that $\mathtt{X}^{--}_{\Delta}(\omega)$ is maximal when ω is, but also that it is even greater if $K_\omega = K = 3$. Thus, (informally, still) if ℓ is even, then $\mathtt{X}^{--}_{\ell,r}(\omega)$ is maximal when $\omega = \ell + r = 3\Delta$, that is, when $\Delta = r$ and $\ell = 2r$. When ℓ is even, $\mathtt{X}^{--}_{\ell,r}(\omega)$ simply is maximal when ω is maximal, *i.e.* when $r = \ell$ (in which case $\mathtt{X}^{--}_{\Delta}(\omega) = \mathtt{X}^{-}_{n}(\omega)$).

Let $\mathtt{T}(\omega)$ be the total number of attractors of a BAN $\mathscr{N}$ which is either a BAC or a BAD with order ω. Theorem 2.1 and Lemma 2.12 directly imply the following, where $K = (\ell + r) / \gcd(\ell, r)$:

$$\mathtt{T}(\omega) \;\leq\; \frac{(\varphi \star \mathtt{Y})(\omega)}{\omega} \tag{2.16}$$

$$\text{where } \forall p \in \mathbb{N},\ \mathtt{Y}(p) = a^p, \quad a = \begin{cases} 2 & \text{if } \mathscr{N} = \mathscr{C}_n^+ \text{ or } \mathscr{N} = \mathscr{D}_{\ell r}^{++} \\ \sqrt{2} & \text{if } \mathscr{N} = \mathscr{C}_n^- \text{ or } \mathscr{N} = \mathscr{D}_{\ell r}^{--} \wedge \neg\,(3|K) \\ \sqrt{3} & \text{if } \mathscr{N} = \mathscr{D}_{\ell r}^{-+} \\ 3^{1/3} & \text{if } \mathscr{N} = \mathscr{D}_{\ell r}^{--} \wedge (3|K). \end{cases}$$

To go further, let us note that obviously T is necessarily greater than what it would be if all recurrent configurations of $\mathscr{N}$ had the greatest possible minimal period, that is, ω. Conversely, T is necessarily smaller than what it would be if all recurrent configurations had the smallest possible minimal period, that is, if they were all fix points. From this remark derives:

$$\frac{\mathtt{X}(\omega)}{\omega} \;\leq\; \mathtt{T}(\omega) \;\leq\; \mathtt{X}(\omega).$$

For BACs $\mathscr{N} = \mathscr{C}$, it is known (especially in the context of binary necklaces [100]) that the upper bound can actually be made much smaller because there exists an injective map that associates aperiodic necklaces of size a divisor $p < \omega$ of ω to aperiodic necklaces of size ω. The next result, Theorem 2.2 generalises this to BADs. The baseline idea of its proof (given on Page 147 of the appendix) is to define an injective application $\Gamma : \mathbb{B}^p \to \mathbb{B}^\omega$ mapping aperiodic words of $\mathscr{W}(p)$ characterising configurations of minimal period p (*cf.* Lemma 2.8) onto aperiodic words of $\mathscr{W}(\omega)$ characterising configurations of minimal period the BAD's order ω.

Theorem 2.2.

The total number $\mathtt{T}(\omega)$ *of attractors of a* BAN $\mathscr{N} \notin \{\mathscr{D}_{5,1}^{--}, \mathscr{D}_{1,5}^{--}\}$ *which is either a* BAC *or a* BAD *and has order* ω, *is related to its total number* $\mathtt{X}(\omega)$ *of recurrent configurations by:*

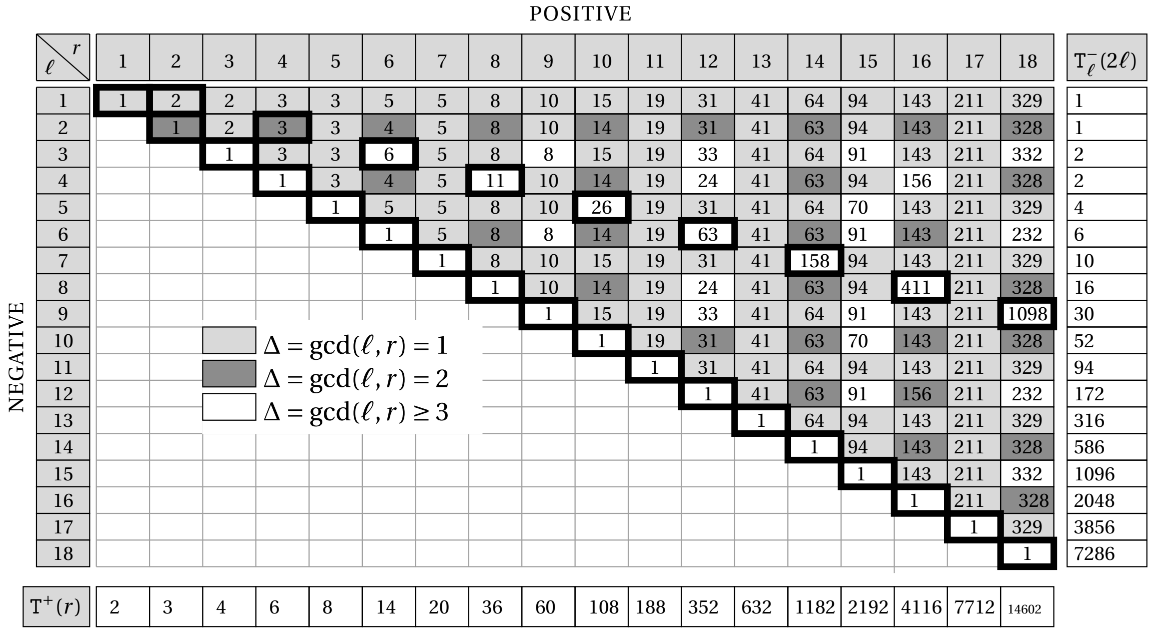
$$\frac{\mathtt{X}(\omega)}{\omega} \;\leq\; \mathtt{T}(\omega) \;\leq\; 2 \cdot \frac{\mathtt{X}(\omega)}{\omega},$$

i.e. the expected value of attractor periods of $\mathscr{N}$ *is very high:*

$$\sum_{p|\omega} p \cdot \frac{\mathtt{A}(p)}{\mathtt{T}(\omega)} = \frac{\mathtt{X}(\omega)}{\mathtt{T}(\omega)} \;\geq\; \frac{\omega}{2}.$$

And if $\mathscr{N} \in \{\mathscr{D}_{5,1}^{--}, \mathscr{D}_{1,5}^{--}\}$, *then* $\omega = 6$, $\mathtt{X}(6) = 5$ *and* $\mathtt{T}(6) = 2$.

Thus, almost all periodic configurations have the greatest possible minimal period. In the context of automata networks, this can perhaps be related to the



POSITIVE

NEGATIVE

$\ell \backslash r$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	$\mathrm{T}^-_\ell(2\ell)$
1	1	2	2	3	3	5	5	8	10	15	19	31	41	64	94	143	211	329	1
2		1	2	3	3	4	5	8	10	14	19	31	41	63	94	143	211	328	1
3			1	3	3	6	5	8	8	15	19	33	41	64	91	143	211	332	2
4				1	3	4	5	11	10	14	19	24	41	63	94	156	211	328	2
5					1	5	5	8	10	26	19	31	41	64	70	143	211	329	4
6						1	5	8	8	14	19	63	41	63	91	143	211	232	6
7							1	8	10	15	19	31	41	158	94	143	211	329	10
8								1	10	14	19	24	41	63	94	411	211	328	16
9									1	15	19	33	41	64	91	143	211	1098	30
10										1	19	31	41	63	70	143	211	328	52
11											1	31	41	64	94	143	211	329	94
12												1	41	63	91	156	211	232	172
13													1	64	94	143	211	329	316
14														1	94	143	211	328	586
15															1	143	211	332	1096
16																1	211	328	2048
17																	1	329	3856
18																		1	7286
$\mathrm{T}^+(r)$	2	3	4	6	8	14	20	36	60	108	188	352	632	1182	2192	4116	7712	14602	

░ $\Delta = \gcd(\ell, r) = 1$
▓ $\Delta = \gcd(\ell, r) = 2$
□ $\Delta = \gcd(\ell, r) \geq 3$

Table 2.4: *Total number* T^{-+}_ℓ *of attractors*[11] *of a mixed* BAD $\mathscr{D}^{-+}_{\ell r}$ *in cell* (ℓ, r) *if* $\ell \leq r$ *and otherwise, if* $\ell > r$, $\mathrm{T}^{-+}_\ell = \mathrm{T}^{-+}_{\Delta,r}$ *according to Section 4, where* $\Delta = \gcd(\ell, r)$*). The last column and line respectively recall the total number of attractors of the* BACs $\mathscr{C}^-_\ell$ *and* $\mathscr{C}^+_r$.

instability of automata in a periodic configuration: having very large attractor periods allows for attractors with very little momentum, that is, in which very little automata are unstable but where the rare instabilities need a lot of time to gradually be propagated all around the (double-) cycle and come back to its initial location.

Besides its meaning in terms of the expected values of periods, the importance of Theorem 2.2 lies in that, combined with Lemma 2.12 and Theorem 2.1, it allows to derive more precise comparisons between the total number of attractors of positive BACs and that of the four other types BACs and BADs (*cf.* Tables 2.1 to 2.5):

Corollary 2.4.

Let $\mathrm{T}^+_\omega = \mathrm{T}^+_\omega(\omega)$ *be the total number of attractors of* $\mathscr{C}^+_\omega$.

$$\begin{cases} \mathrm{T}^-_\omega(2\omega) & \leq \frac{1}{2}\mathrm{T}^+_\omega & \textit{if } \mathscr{N} = \mathscr{C}^-_\omega \\ \mathrm{T}^{++}_{\ell,r}(\omega) & = \mathrm{T}^+_\omega & \textit{if } \mathscr{N} = \mathscr{D}^{++}_{\ell r} \quad \textit{s.t. } \Delta = \omega \\ \mathrm{T}^{-+}_{\ell,r}(\omega) & \leq 2\left(\frac{\sqrt{3}}{2}\right)^\omega \mathrm{T}^+_\omega & \textit{if } \mathscr{N} = \mathscr{D}^{-+}_{\ell\omega} \\ \mathrm{T}^{--}_{\ell,r}(\omega) & \leq 2\left(\frac{3^{\frac{1}{3}}}{2}\right)^\omega \mathrm{T}^+_\omega & \textit{if } \mathscr{N} = \mathscr{D}^{--}_{\ell r} \quad \textit{s.t. } \ell + r = \omega \wedge K = 3 \\ \mathrm{T}^{--}_{\ell,r}(\omega) & \leq 2\left(\frac{\sqrt{2}}{2}\right)^\omega \mathrm{T}^+_\omega & \textit{if } \mathscr{N} = \mathscr{D}^{--}_{\ell r} \quad \textit{s.t. } \ell + r = \omega \wedge K \neq 3 \end{cases}$$

where $\Delta = \gcd(\ell, r)$ *and* $\ell + r = K\Delta$.

NEGATIVE

ℓ \ r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\mathtt{T}^-_\ell(2\ell)$
1	1																1
2	1	1															1
3	1	1	2														2
4	1	2	1	2													2
5	2	1	2	2	4												4
6	1	1	3	3	2	6											6
7	2	2	3	2	4	3	10										10
8	2	3	2	8	3	4	6	16									16
9	3	2	2	3	5	9	7	7	30								30
10	2	4	3	4	17	7	7	10	11	52							52
11	4	3	5	6	7	7	11	11	16	19	94						94
12	3	4	9	2	7	42	11	33	17	23	28	172					172
13	5	6	7	7	11	11	16	19	24	28	39	46	316				316
14	6	7	7	10	11	17	105	23	28	38	46	60	75	586			586
15	7	7	10	11	4	17	24	28	44	125	60	66	97				1096
16	7	10	11	33	19	23	28	278	46	60	75	88					2048
17	11	11	16	19	24	28	39	46	60	75	97						3856
18	11	17	17	23	28	6	46	60	729	96							7286
19	16	19	24	28	39	46	60	75	97								13798
20	19	23	28	32	125	60	75	88									26216
21	24	28	44	46	60	66	10										49940
22	28	38	46	60	75	96											95326
23	39	46	60	75	97												182362
24	46	60	66	88													349536
25	60	75	97														671092
26	75	96															1290556
27	97																2485534

■ $\omega = \ell + r$ prime
□ ω even
▒ ω odd

$\ell = r$, $\Delta = \frac{\omega}{2}$
$\ell = \frac{3}{2}r$, $\Delta = \frac{\omega}{5}$
$\ell = 2r$, $\Delta = \frac{\omega}{3}$
$\ell = 4r$, $\Delta = \frac{\omega}{5}$

Table 2.5: *Total number* $\mathtt{T}^{--}_{\Delta}$ *of attractors*[11] *of a negative* BAD $\mathscr{D}^{--}_{\ell r}$. *The last column recall the total number of attractors of the negative* BAC $\mathscr{C}^{-}_{\ell}$.

7 Other cycle intersections

Finally, to extend the scope of results presented above and in particular that of Theorem 2.2, let us consider some other ways that two cycles may intersect and interact. Let $\mathcal{N} = \{\mathrm{f}_i\}$ be a BAN with a structure **G** that is the digraph on the left of Fig. 2.4 (resp. on the right and s.t. $\mathrm{f}_b : (x_{c_L}, x_{c_R}) \in \mathbb{B}^2 \mapsto \mathrm{f}^L_b(x_{c_L}) \diamond \mathrm{f}^R_b(x_{c_R})$, $\diamond \in \{\vee, \wedge\}$).

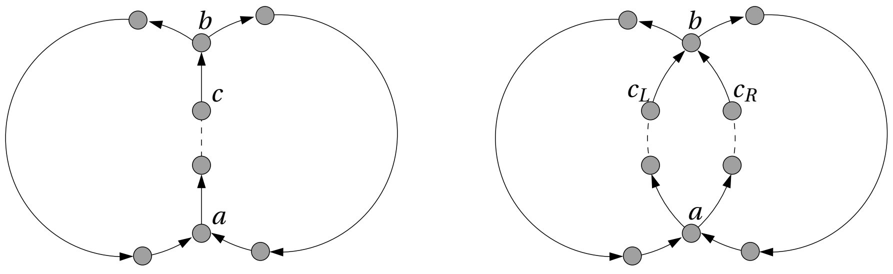


Figure 2.4: *Structure of* $\mathcal{N}$.

Let $\mathscr{N}' = \{\mathrm{h}_i\}$ be the BAN whose structure $\mathbf{G}'$ is pictured on the left (resp. on the right) of Fig. 2.5, and that is defined so that all of its signed paths represent a path of the same sign in $\mathscr{N}$. Its local transition functions satisfy:

$$\forall i \in \mathbf{V},\ i \neq b_L, b_R,\ \mathrm{h}_i = \mathrm{f}_i \text{ and } \mathrm{h}_{b_L} = \mathrm{h}_{b_R} = \mathrm{f}_b \text{ (resp. } \mathrm{h}_{b_L} = \mathrm{f}_b^L \text{ and } \mathrm{h}_{b_R} = \mathrm{f}_b^R\text{)}.$$

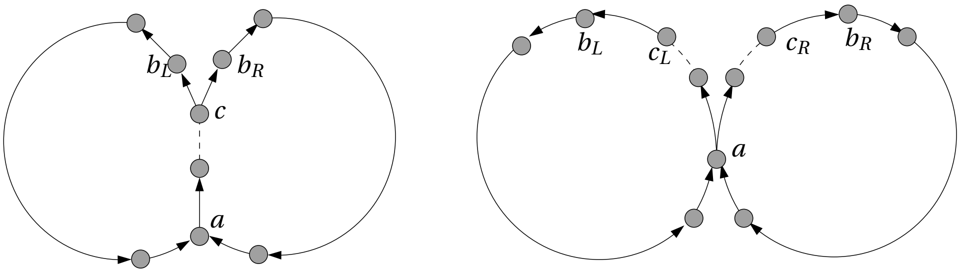


Figure 2.5: *Structure of* $\mathscr{N}'$.

Now, let us define the function ϕ that maps configurations of $\mathscr{N}$ to configurations of $\mathscr{N}'$ so that $\forall x, \forall i \neq b_L, b_R,\ \phi(x)_i = x_i$ and $\phi(x)_{b_L} = \phi(x)_{b_R} = x_b$. Then, it is easy to see that in both cases: $\mathrm{H}(\phi(x)) = \phi(\mathrm{F}(x))$ and thus: $\mathscr{N} \blacktriangleleft \mathscr{N}'$ (in particular because $\forall x, \phi(x)_{b_L} = \phi(x)_{b_R}$). This and an induction on the length of the path from a to b in the first case yield:

$$\exists \mathscr{D} = \mathscr{D}^{ss'}_{\ell r},\ \mathscr{N} \blacktriangleleft \mathscr{D}. \tag{2.17}$$

And since it necessarily holds that $\phi(\mathscr{X}_{\mathscr{N}}(p)) \subseteq \mathscr{X}_{\mathscr{N}'}(p)$, the upper bounds of Theorem 2.2 remain valid for these two other basic types of "double-cycles", in the sense that cycles that intersect on wider parts of their structures have less attractors than comparable intersected or isolated cycles[12]. Further, with an induction on the number and sizes of the intersection paths, it can also be extended to more general cases of networks whose structure is as illustrated in Fig. 2.6.

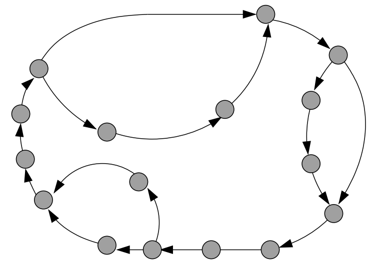


Figure 2.6: *Arbitrary cycle intersections.*

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[12] We deliberately remain imprecise here because the objective is of course not to make an exhaustive and detailed analysis of each type of BAN involving cycle intersections. On the contrary, as mentioned above, the idea is to understand from a more general but informal point of view what are the effects of forcing cycles to interact via structural intersections.

8 Perspectives and scope under the parallel update schedule

A. Drastic decrease in the degrees of freedom. In summary and conclusion of the previous sections, we have given explicit formulae that describe exhaustively the behaviours of BACs and BADs under the parallel updating. As a result we have observed that cycles that intersect tend to severely hinder their respective degrees of freedom (*cf.* Section 2). Their number of possible limit behaviours falls drastically (by an exponential factor) when they are made to interact. The previous section suggests that the more cycles intersect and, generally, the more they communicate via paths of interactions in the structure, the more there are constraints limiting even further their degrees of freedom. These results go against the idea of estimating the number of attractors of an arbitrary BAN by multiplying the degrees of freedom of the cycles embedded in its structure (as it has been done to count fix points in [5], where indeed the network for which the upper bound is reached involves non-interacting cycles). On the contrary, it seems that the more there are interconnected cycles (and also, the more there are negative cycles among them), the less freely can the whole BAN behave. This suggests that a BAN with a strongly connected structure has no more attractors (and perhaps even, it has a lot less) than what it would have if its structure only contained its largest positive cycle. This is confirmed by the study of the positive DANs in Section 1. By Lemma 2.2, the order of $\omega = \eta$ of such a BAN equals the index of imprimitivity of its structure (*i.e.* the gcd of all cycle lengths). And its structure can be arranged into the shape of a "macro-cycle" of length η (whose macro-nodes are the $\mathbf{V}_k \subseteq \mathbf{V}$ of (2.3)) [13]. So the BAN's behaviour in this case does indeed evoke that of an isolated positive cycle with length and order equal to η.

Another, more precise way to formulate this idea is to say that the behaviour induced by intersecting cycles may be *simulated* using less cycles and no negative ones (this extrapolates the simulation relations that have been uncovered in Lemmas 2.5 and 2.3 and in Equation (2.17)). In brief, this means that an arbitrary BAN can be simulated by a BAN that has a simpler structure and therefore more degrees of freedom.

B. Elementary bricks and grammar of BANs. From a constructive angle, this suggests a *structural grammar of* BAN*s* based on (1) a small small set of *elementary brick networks* or *modules* [21, 22, 41, 77, 101] with describable behaviours and simple scalable structures (*e.g.* directed paths, cycles and perhaps simple functional motifs that are frequently encountered in the structures of real biological networks) as well as on (2) a collection of *elementary connections* between them (*e.g.* different types of bottlenecks and bifurcations like those studied in the previous section, linking cycles). Studying the combinatorics (especially, in terms of the degrees of freedom) of different bricks in a similar fashion to the analysis of BACs and BADs presented above, comparing the behaviours of similar bricks with different scales and describing the way different bricks interact through the different sorts of elementary connections, could certainly help develop further a concrete modular and scalable understanding of relations between BAN behaviours and their underlying structural motifs.

C. Hierarchical classification & simulations. Further, this suggests a hierarchical classification of BAN structures that regroups on a same level structures that have comparable *degrees of simplicity/complexity*, *i.e.* those that have the same size and involve modules of similar scales and *intricateness*. The results of this chapter then support the additional idea that structures on the same level would yield similar degrees of freedom. Thus, in relation to the simulations of Lemmas 2.5 and 2.3 and of equation (2.17), under some conditions (to be specified), the behaviours and functions of a BAN at a given level could be simulated or "approximated" with some ambiguities by a BAN figuring at a higher level on the same branch of the hierarchy, with a simpler structure involving larger, more independent and less numerous modules. And at the top of the hierarchy would figure the most "universal" BANs, those with the most degrees of freedom, the simplest structures, capable of simulating the behaviour of a larger number of more complex and constrained BANs.

D. Purpose. Replacing sub-networks by networks that are higher in the hierarchy would allow to choose and adjust the degree of precision (or of approximation) with which parts of a network are described, according to the problems at hand. This way, unnecessary details could be ignored and the complexity that they are responsible for could be bypassed. Thus, one of the practical purposes of these researches is to introduce new, more manipulable descriptions of network features (both their structures and the behaviours that are induced by their structures). Another is to define ways to decompose networks that instructively (w.r.t. modelling issues, for instance) suggest different, *ad hoc* perspectives on their features.

E. Information processing, weak-points & elementary connections. Beyond the results of this chapter and the work proposed above to better understand how bricks interact, if special attention is put on the inter-brick connections (rather than on the bricks themselves), on their precise nature and on their impact on the global network behaviours, then researches can be oriented towards the question of how information circulates and is shared between different communicating, functional mechanisms underlying a system. Here, the notion of information can for instance be quantified in terms of punctual and local instabilities running through the systems structures, either spreading at bifurcations or condensing at bottlenecks. Incidentally, this could help refine the idea expressed in [26] that cycle intersections represent notable fragilities in the underlying interaction architectures of systems. Special effort could be put into generalising this to all elementary connections mentioned above and highlighting the type of interaction motifs that are especially sensitive to mistakes in the sense that if any entity (*e.g.* the intersection automaton 0 in a BAD) or interaction that it involves is modified, then the information transmission is severely disrupted and the impact on the global system behaviour is considerable, causing a drastic decrease of the degrees of freedom/ambiguity and perhaps causing what may be represented by a change of level in the hierarchy and/or in the scale of the behaviour's scope.

F. Emergence. The hierarchy of BANs mentioned above also provides a promising framework with tangible means to study the notion emergence in interaction systems. Focusing on different levels of abstraction – corresponding to different hierarchy levels – in the descriptions of network structures and behaviours, the way system behaviours emerge from one level to the next could be studied *per se*. Here again, I believe special attention could advantageously be brought to the precise way separate structural modules communicate so as to relate behaviours emerging at a given level with the interaction motifs that define the difference between that specific level and the one underneath.

G. Network phylogeny. The evolution of genetic regulation systems (or, possibly, other systems that also involve information processing) could be compared to the hierarchy and decompositions mentioned above by examining several known genetic networks of living organisms (if possible, on different levels of the same branch in the evolutionary tree). Insights on the way to relate BAN structures and the natural complexification and refinement of genetic networks over time follow from results of this chapter and from discussions in [26]. Elementary interactions between elementary brick networks can be considered as additional constraints that have been imposed on a network which would otherwise have had a simpler structure. For instance, instead of considering two intersecting cycles straightforwardly as a whole structure, they can either be considered (1) as two separate cycles that have been imposed to interact, or (2) as one large encompassing cycle that has been imposed a new constraint. Conjectures in [26] formulate the idea that these theoretical constraints can be modelled by those that are added naturally to genetic networks during evolution with the two following consequences: (1) to force different, initially independent parts of a system to communicate and work together in phase and (2) to add constraints that act as "gears", filtering possible network behaviours and limiting their degrees of freedom so as to make their final behaviour less ambiguous and thus also more robust.

H. In defence of the parallel update schedule. To end this chapter, let us recall its initial strong assumption imposing on BANs the parallel update schedule π.

In terms of modelling, this appears to be a severe restriction. However, Chapter 6 discusses the way that update schedules can be interpreted with respect to time and notions of duration, simultaneity, precedence, and causality. From this, follow some arguments in favour of synchronism and thus of π. Further than that, the present chapter evidences the methodological pertinence of π. Owing to its simplicity, non-trivial developments can be made. The pertinence of these is dual. First, the next chapter (Section 1.2, Chap. 3) shows how block-sequential update schedules all relate in a very strong way to the parallel update schedule. This allows to transpose the study of a BAN under any block-sequential update schedule to a study of a BAN under π. But importantly, in addition to this, results derived under the assumption of π provide useful ideas and intuitions as suggested above in this conclusion. Finally, let us also note that the results of Section 6 may be taken as a proof of the existence of some behavioural properties of BANs that are related more intimately to their structures than to their update modes or schedules, although these properties may require a certain special updating to be *revealed*. Indeed, it appears that in the case of BACs and BADs studied here, the restriction in the degrees of freedom of cycles is due more to the structural constraint (embodied by the intersection) that imposes two cycles to “work in phase” than to the choice of π *per se* (this is discussed later in Section 3.C, Chap. 6).

3

DETERMINISTIC UPDATE SCHEDULES

The focus of this chapter[13] is put on the periodic update schedules defined in Section 3, Chap. 1. The *meaning* of these objects is discussed further in the next chapter as well as in Chapter 6. For now, we take them mainly as "prototypes" from which we endeavour to draw some primary understanding of how imposing updating constraints impacts on a network's behaviour. Our stance here consists notably in taking as reference the parallel update schedule π (which is the matter of concern of the previous chapter).

To start, we first describe a very natural way to confuse updatings that globally produce the same effect without necessarily involving the same elementary derivations. A second straightforward relation between update schedules is introduced later in Section 1, Chap. 4. It can pertinently be compared with the one we are about to present because contrary to it, the similarity relation of Section 1, Chap. 4 identifies updatings that involve very similar elementary sequences of events but appear to induce very dissimilar behaviours from an exterior point of view observing the BAN configuration only once per update period.

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[13] This chapter regroups results from [6], [47], [48] and [80].

1 Simple and block-sequential update schedules

1.1 Basic, global similarity

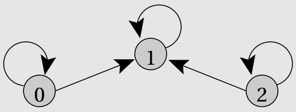
Let us consider an arbitrary BAN $\mathscr{N} = \{\mathrm{f}_i\}$, with structure $\mathbf{G} = (\mathbf{V}, \mathbf{A})$, whose automata are updated according to the simple update schedule δ. Obviously, if $j, i \in \mathbf{V}$ are two independent automata of $\mathscr{N}$ ($(j,i) \notin \mathbf{A}$) then, whether δ updates i after j or not has no incidence on the BAN configuration at the end of any periodic sequence of updates (*cf.* [6, 7, 80]). Indeed, if $(j,i) \notin \mathbf{A}$ and $x = x(t)$ is the BAN configuration at the beginning of an update sequence, then the next update of i makes its state turn to $\mathrm{f}_i(x) = x_i(t+1)$ whatever the current state of j, that is, independently of what happens to j between the instant at which $\mathscr{N}$ is in configuration $x(t)$ and the instant at which i is updated (before $\mathscr{N}$ has finished turning into configuration $x(t+1)$). Thus, as formalised further in the rest of this section, we take two simple update schedule descriptions δ and δ' to define the same update schedule as soon as the following holds:

$$\forall (j,i) \in \mathbf{G},\ \delta(j) < \delta(i) \quad \Leftrightarrow \quad \delta'(j) < \delta'(i). \tag{3.1}$$

We define the equivalence relation $\asymp$ to equate any two such **basically equivalent** update schedules δ and δ' (*cf.* Example 3.1).

Example 3.1. Basic similarity between simple update schedules

Quotienting simple update schedules by $\asymp$, for a BAN *whose structure is the digraph represented on the left below, the block-sequential update schedules listed on the right are all equal since $\delta(1) \leq \delta(0)$ and $\delta(1) \leq \delta(2)$ in all of them.*



- $\{0,1,2\}$,
- $\{0,1\},\{2\}$,
- $\{1,2\},\{0\}$,
- $\{1\},\{0,2\}$,
- $\{1\},\{0\},\{2\}$,
- $\{1\},\{2\},\{0\}$.

Following [102, 103], we introduce the **local transition functions** $\mathrm{f}[\delta]_i$ **relative to** δ so that:

$$\mathrm{F}[\delta](x) = (\mathrm{f}[\delta]_0(x), \ldots, \mathrm{f}[\delta]_{n-1}(x)). \tag{3.2}$$

In particular, for $\delta = \pi$, since $\mathrm{F}[\pi](x) = \mathrm{F}_{\mathbf{V}}(x)$, it holds that: $\mathrm{f}[\pi]_i = \mathrm{f}_i$, $\forall i \in \mathbf{V}$. To pin-point the difference between arbitrary simple update schedules and π, we call **inversion** any arc of the set:

$$\mathrm{I}[\delta] \ = \ \{(j,i) \mid \delta(i) > \delta(j)\} \subseteq \mathbf{A}.$$

π is characterised by $\mathrm{I}[\pi] = \emptyset$. In a cycle, for instance, if the arc $(i, i+1)$ of δ is an inversion, then, under δ, $x_{i+1}(t+1)$ depends on $x_i(t+1)$ instead of $x_i(t)$ as it would if $\delta(i+1) \leq \delta(i)$ were true and if δ were equal to π. Generally, it is easy to see that the functions $\mathrm{f}[\delta]_i$ are explicitly defined as follows:

$$\forall i \in \mathbf{V}, \delta(i) \neq \emptyset, \quad \mathrm{f}[\delta]_i : x \in \mathbb{B}^n \mapsto \mathrm{f}_i(x_0^*, \ldots, x_{n-1}^*)$$
$$\text{where } x_j^* = \begin{cases} x_j & \text{if } (j,i) \notin \mathrm{I}[\delta], \\ \mathrm{f}[\delta]_j(x) & \text{if } (j,i) \in \mathrm{I}[\delta] \end{cases}$$
$$\text{and } \forall i \in \mathbf{V}, \delta(i) = \emptyset, \mathrm{f}[\delta]_i : x \mapsto x_i. \tag{3.3}$$

This confirms the remark made above (3.1) concerning the identical global behaviours induced by two equivalent update schedules δ and δ':

$$\delta \bowtie \delta' \quad \Leftrightarrow \quad \mathrm{I}[\delta] = \mathrm{I}[\delta'] \quad \Rightarrow \quad \forall i \in \mathbf{V}, \mathrm{f}[\delta]_i = \mathrm{f}[\delta']_i \quad \Leftrightarrow \quad \mathrm{F}[\delta] = \mathrm{F}[\delta']. \tag{3.4}$$

Lemma 3.1.

Let $\mathbf{G} = (\mathbf{V}, \mathbf{A})$ *be the structure of an arbitrary* BAN. *If a set* I *of inversions is given, then a block-sequential update schedule* β *such that* $\mathrm{I} = \mathrm{I}[\beta]$ *can be computed in time* $\mathcal{O}(|\mathbf{V}| + |\mathbf{A}|)$ *[80, 81]. In particular, if* $\mathbf{G} = \mathbb{C}_n$ *is a cycle, either* $\mathrm{I} = \emptyset = \mathrm{I}[\pi]$, *or a sequential update schedule* σ *such that* $\mathrm{I} = \mathrm{I}[\sigma]$ *can be computed in* $\mathcal{O}(n)$ *steps.*

Proof: Let $\mathbf{G}' = (\mathbf{V}, \mathbf{A}')$ be the digraph obtained from $\mathbf{G}$ by inverting all arcs $(j,i) \in \mathrm{I}$. $\mathbf{G}'$, involves two types of arcs: (1) arcs $(j,i) \in \mathbf{A} \setminus \mathrm{I} \subseteq \mathbf{A}'$ that satisfy $\delta(i) \leq \delta(j)$ and (2) arcs $(j,i) \in \mathbf{A}'$ such that $(i,j) \in \mathrm{I}$ which satisfy $\delta(i) < \delta(j)$. $\mathbf{G}'$ does not contain any directed cycles involving an arc of the second sort (the contrary would imply that $\exists i \in \mathbf{V}, \delta(i) < \delta(i)$). Let $\mathbf{G}'' = (\mathbf{V}'', \mathbf{A}'')$ be the acyclic digraph obtained by reducing all directed cycles of $\mathbf{G}'$ to a unique node. Then, in linear time $\mathcal{O}(n + |\mathbf{A}''|)$, a topological ordering of the nodes of $\mathbf{G}''$ can be computed. By construction, this ordering defines a block-sequential update schedule with inversion set exactly equal to I. The special case of cycles follows directly. □

In other terms, if $\mathcal{N}$ is updated with an unknown block-sequential update schedule δ and both $\mathbf{G}$ and $\mathrm{I}[\delta]$ are known, then an update schedule β such that $\beta \bowtie \delta$ may be determined in linear time. Let us note that with an arbitrary simple update schedule an automaton i in a cycle may never be updated and then everything is as if i had a constant state and the cycle was really a path starting in i and ending in $i-1$. This is why Lemma 3.1 focuses on block-sequential update schedules. Also for a cycle, since there are $\binom{n}{k}$ ways to choose a set $\mathrm{I} \subseteq \mathbf{A} = \{(i, i+1) \mid i \in \mathbb{Z}/n\mathbb{Z}\}$ of $0 \leq k < n$ inversions, the block-sequential update schedules of $\mathbf{V} = \mathbb{Z}/n\mathbb{Z}$ are partitioned into $\sum_{k=0}^{n-1} \binom{n}{k} = 2^n - 1$ different equivalence classes for $\bowtie$. Each of these has between $0 = |\mathrm{I}[\pi]|$ and $n-1$ inversions.

The relation $\asymp$ can be generalised to all periodic update schedules, in particular those that update some automata several times within an update period. The idea is the same as before: what matters is that for every $(j,i) \in \mathbf{A}$, when i is updated for the k^{th} time by δ at step $\delta_k(i) \in \delta(i) = \{\delta_1(i), \delta_2(i), \ldots\}$ of the update period of δ, and when i is updated for the k^{th} time by δ' at step $\delta'_k(i) \in \delta'(i)$ of the update period of δ', then j has already been updated the same number of times in both cases:

$$\forall (j,i) \in \mathbf{A},\ \forall k, m \leq \max\{|\delta(i)|, |\delta'(i)|\},\quad \delta_m(j) \leq \delta_k(i) \ \Leftrightarrow\ \delta'_m(j) \leq \delta'_k(i). \tag{3.5}$$

Extending $\asymp$ so that it relates any two update schedules satisfying (3.5) yields Lemma 3.2's generalisation of (3.4). We omit its proof which can be done by induction on the steps of the update sequences (m and k).

Lemma 3.2.

For any BAN $\mathcal{N}$ *and any update schedules* δ *and* δ'*:* $\delta \asymp \delta' \ \Rightarrow\ \mathrm{F}[\delta] = \mathrm{F}[\delta']$.

By definition, $\asymp$ relates update schedules inducing identical global effects (*i.e. non-elementary* behaviours): $\delta \asymp \delta' \ \Rightarrow\ \mathscr{T}_{[\delta]} = \mathscr{T}_{[\delta']}$ (*cf.* Section 4, Chap. 1). But it is important to note that generally the update schedules involve different sequences of elementary transitions, *i.e.* $\delta \asymp \delta'$ does not imply that $\mathscr{T}_{\delta} = \mathscr{T}_{\delta'}$. As a result, when $\delta \asymp \delta'$, a BAN submitted to δ does not necessarily transit through the same configurations and in the same order as it does under δ'. In the absence of any exterior influence, this is insignificant, because the differences in the non-elementary steps affect independent parts of the BAN in a way that has no global impact at all on its behaviour, provided that it is isolated. However, as discussed later in this thesis, if the BAN can be perturbed, then the precise sequence of elementary steps it takes may make a difference. For instance, consider the two equivalent simple update schedules $\delta :\equiv \{0\},\{1\} \asymp \delta' :\equiv \{0,1\}$ of a BAN with arc set not including $(1,0)$. Globally, both yield the same behaviour. However, during the transition $x \twoheadrightarrow y = \mathrm{F}[\delta](x) = \mathrm{F}[\delta'](x)$ induced by both update schedules, the BAN may undergo an exterior influence in configuration $\mathrm{F}_0(x)$ which might distinguish the outcome of the sequence $x \xrightarrow{0} \mathrm{F}_0(x) \xrightarrow{1} y$ (induced by δ) from that of the sequence $x \xrightarrow{\{0,1\}} y$ (induced by δ').

The next section is substantially based on Robert's [102, 103] work. It also revisits several results established in Goles' thesis [44]. It explores further how simple and block-sequential update schedules relate to the parallel update schedule and especially how they impact on BAN behaviours through the architecturing of interactions on which they force a "transitivity" (*cf.* Lemma 3.3 below) as a consequence of the changes that they impose on local transition functions (*cf.* (3.2) and (3.3) above).

1.2 Induced transitivity

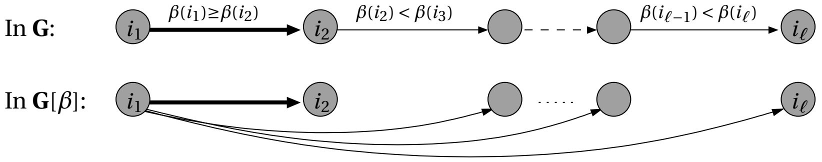
For an arbitrary BAN $\mathscr{N} = \{\mathrm{f}_i\}$ updated with a simple update schedule β, we define a new BAN $\mathscr{N}[\beta] = \{\mathrm{h}_i = \mathrm{f}[\delta]_i\}$ defined by the Boolean functions $\mathrm{f}[\delta]_i$ introduced in (3.3) (*cf.* Examples 3.2, 3.3 and 3.4). Letting $\mathrm{H}[\pi]$ be its global transition function under π, the definition of the $\mathrm{f}[\delta]_i$s in (3.2) implies that $\mathrm{F}[\beta] = \mathrm{H}[\pi]$, *i.e.* the behaviour of $\mathscr{N}$ under β is identical to that of $\mathscr{N}[\beta]$ under the parallel update schedule π: both systems have $\mathscr{T}_{[\beta]}$ as non-elementary transition graph. The same holds for all other simple update schedules $\beta' \asymp \beta$ with the same set of inversions (*cf.* Section 1.1) since they are associated to the same local transition functions. Following this, in the sequel, the set of simple update schedules is taken to be quotiented by $\asymp$ (if $\beta' \asymp \beta$, then β' is considered as a another writing of β) and we speak indifferently of the system defined by $\mathscr{N}$ submitted to β and of the system defined by $\mathscr{N}[\beta]$ submitted to π.

When $\beta = \pi$, the dependencies recorded in the structure $\mathbf{G}$ of $\mathscr{N}$ convey some effective meaning with respect to the behaviour of the system (in the sense that by (1.3), if $(j, i) \in \mathbf{A}$, then there effectively is a configuration x in which j influences i). When, $\beta \neq \pi$ is a different simple update schedule, this no longer is true. By examining the structure $\mathbf{G}[\beta]$ of $\mathscr{N}[\beta]$, this section aims at deriving a first "visual" intuition concerning the impact of β on the behaviour of $\mathscr{N}$ through the "change of dependencies" that it causes. This is especially useful for positive DANs whose behaviours are completely defined by their update schedules/modes and by their structures (*cf.* Example 3.2 and Section 3). First, we specify the set of arcs of $\mathbf{G}[\beta]$.

Lemma 3.3. Arcs in the structure $\mathbf{G}[\beta]$ of $\mathscr{N}[\beta]$

Let $\mathbf{G} = (\mathbf{V}, \mathbf{A})$ be an arbitrary digraph and β a simple update schedule of its nodes. Then, arc set $\mathbf{A}[\beta]$ of $\mathbf{G}[\beta]$ is characterised by:

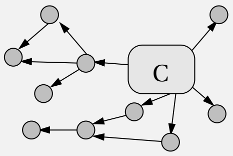
$$(j, i) \in \mathbf{A}[\beta] \quad \Leftrightarrow \quad \textit{in } \mathbf{G}\textit{, there exists a path } P_{ji} = \{j = i_1, i_2, \dots i_\ell = i\}$$
$$\textit{s.t. } \beta(i_1) \geq \beta(i_2) \textit{ and } \forall\ 1 < k < \ell,\ \beta(i_k) < \beta(i_{k+1}).$$



Proof: First, let us suppose that there exists such a path P_{ji} in $\mathbf{G}$. Let $x \in \mathbb{B}^n$ be an arbitrary configuration. By definition of the functions $\mathrm{h}_i = \mathrm{f}[\delta]_i$ and of P_{ji}, configuration $x(1) = \mathrm{F}[\beta](x) = \mathrm{H}[\pi](x)$ is such that: (*i*) $x_{i_{k+1}}(1) = \mathrm{h}_{i_{k+1}}(x)$ depends on $x_{i_k}(1)$, $\forall\, 1 < k < \ell$ and (*ii*) $x_{i_2}(1) = \mathrm{h}_{i_2}(x)$ depends on x_j. With an induction on k this leads to $\forall\, 1 < k < \ell$, $x_{i_{k+1}}(1)$ depends on $x_{i_2}(1)$ and thus on x_j. In particular, $x_i(1)$ depends on x_j so $(j,i) \in \mathbf{A}[\beta]$. Conversely, let us suppose that $(j,i) \in \mathbf{A}[\beta]$ so that $x_i(1)$ depends on x_j. The existence of the path P_{ji} in $\mathbf{G}$ is proven by induction on $\beta(i)$. First suppose that $\beta(i) = 0$. It can only be that $\beta(i) \leq \beta(j)$ and, by definition of $\mathrm{h}_i = \mathrm{f}[\delta]_i$, $(j,i) \in \mathbf{A}$. Next, suppose that $\beta(i) > 0$ then, either again $\beta(i) \leq \beta(j)$ and $(j,i) \in \mathbf{A}$ or, there is a $k \in \mathbf{V}^-_{\mathbf{G}}(i)$ such that $\beta(k) < \beta(i)$ and $x_k(1)$ depends on x_j. In the latter case, by induction hypothesis there exists a path P_{jk} with the desired properties from j to k. Using arc (k,i), P_{jk} can be extended to a path with the desired properties from j to i. □

Lemma 3.4.

Let $\mathbf{G} = (\mathbf{V},\mathbf{A})$ *be a strongly connected digraph such that* $\deg^-_{\mathbf{G}}(i) > 0, \forall i \in \mathbf{V}$, *and let* β *be a simple update schedule of its nodes. Then,* $\mathbf{G}[\beta]$ *is comprised of one unique non-trivial* SCC C *and possibly some outgoing acyclic sub-digraphs (cf. figure on the right).*



Proof: Because, $\deg^-_{\mathbf{G}}(i) > 0, \forall i \in \mathbf{V}$, it also holds that $\deg^-_{\mathbf{G}[\beta]}(i) > 0, \forall i \in \mathbf{V}$, *i.e.* with both update schedules π and β, the state of any node/automata depends on the state of at least one other node/automata. Thus, there exists a non-trivial SCC in $\mathbf{G}[\beta]$. Let us suppose that there exists two distinct non-trivial SCCs C_1 and C_2 in $\mathbf{G}[\beta]$ and let $j \in \mathrm{V}_{C_1}$ and $i \in \mathrm{V}_{C_2}$ be two nodes in each of them. Also, let $j' \in \mathrm{V}_{C_1}$ be another (or the same) node of C_1 such that $(j',j) \in \mathbf{A}[\beta]$ (this node exists because C_1 is non-trivial). Because $\mathbf{G}$ is strongly connected, it contains a path $\mathscr{P}_{ji} = \{j = i_0, i_1, \ldots, i_\ell = i\}$ from j to i. Let $r < \ell$ be the smallest integer such that $\beta(i_r) \geq \beta(i_{r+1})$, if it exists, and let $r = \ell$, otherwise. It can be shown that the sub-path of $\mathscr{P}_{ji}$ that starts in i_r and ends in i is necessarily a series of zero (if $i_r = i$), one or several paths of the form described in Lemma 3.3. By Lemma 3.3, each of these paths are turned into single arcs in $\mathbf{G}[\beta]$ so that, in this digraph, there exists a path from i_r to i. Using (3.3) (the definition of $\mathrm{f}[\beta]_i$) and that $\forall 0 < k \leq r$, $\beta(i_{k-1}) < \beta(i_k)$, it can also be shown that $\forall x \in \mathbb{B}^n$, $\forall 0 < k \leq r$, $\mathrm{f}_{i_k}(x)$ and in particular $\mathrm{f}[\beta]_{i_r}(x)$ depends on $\mathrm{f}[\beta]_j(x)$ which in turn depends on $x_{j'}$. Thus, in $\mathbf{G}[\beta]$, there exists a path from j' to i_r, and consequently, a path from $j' \in \mathrm{V}_{C_1}$ to $i \in \mathrm{V}_{C_2}$. For similar reasons, there also exists a path from a node of C_2 to one of C_1 so that both components can not be distinct. From this contradiction we derive that there only is one non-trivial SCC C in $\mathbf{G}[\beta]$. All nodes $i \notin \mathrm{V}_C$ can be reached from C but belong to no non-trivial SCC. Thus, they necessarily constitute acyclic sub-digraphs outgoing C. □

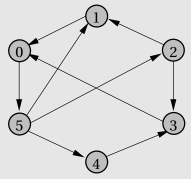
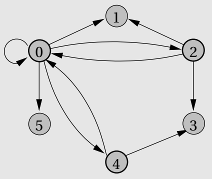
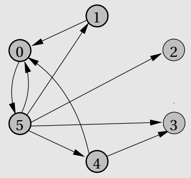
Lemma 3.5.

Let β *be a simple update schedule of the nodes of the digraph* $\mathbf{G} = (\mathbf{V},\mathbf{A})$. *Then,* $\mathbf{G}[\beta]$ *is strongly connected if and only if* $\mathbf{G}$ *is strongly connected and* $\forall i \in \mathbf{V}$, $\exists j \in \mathbf{V}^+_{\mathbf{G}}(i)$, $\beta(i) \geq \beta(j)$.

Proof: First, suppose that $\mathbf{G}$ is strongly connected and $\forall i \in \mathbf{V}$, $\exists j \in \mathbf{V}^+_{\mathbf{G}}(i)$, $\beta(i) \geq \beta(j)$. By Lemma 3.4, $\mathbf{G}[\beta]$ contains a unique non-trivial SCC C. By Lemma 3.3, $\forall i \in V$, $\exists j \in \mathbf{V}^+_{\mathbf{G}}(i)$, $(i, j) \in A[\beta]$ so $\forall i \in \mathbf{V}$, $\deg^+_{\mathbf{G}[\beta]}(i) > 0$. As a result, C contains all nodes of $\mathbf{V}$. Conversely, if $\mathbf{G}$ is not strongly connected then, there exists $i, j \in \mathbf{V}$ with no path in $\mathbf{G}$ from i to j. By Lemma 3.3, paths in $\mathbf{G}[\beta]$ are induced by paths in $\mathbf{G}$: there cannot be a path from i to j in $\mathbf{G}[\beta]$ unless there is one in $\mathbf{G}$. Thus, $\mathbf{G}[\beta]$ is not strongly connected. If $\mathbf{G}$ is strongly connected but there exists $i \in \mathbf{V}$ such that $\forall j \in \mathbf{V}^+_{\mathbf{G}}(i)$, $\beta(j) > \beta(i)$ then, by Lemma 3.3 again, $\deg^+_{\mathbf{G}[\beta]}(i) = 0$ and $\mathbf{G}[\beta]$ is not strongly connected. □

Example 3.2.

Consider a BAN $\mathcal{N} = \{\mathrm{f}_i\}$ *of size* 6 *with structure* $\mathbf{G}$ *and three block-sequential update schedules: the parallel update schedule* $\pi :\equiv \mathbf{V}$*, a sequential update schedule* $\sigma :\equiv \{5\}\{3\}\{1\}\{0\}\{2\}\{4\}$ *and another block-sequential update schedule* $\beta :\equiv \{2\}, \{3, 4\}, \{0, 1, 5\}$*. The table below pictures the structures* $\mathbf{G} = \mathbf{G}[\pi]$, $\mathbf{G}[\sigma]$ *and* $\mathbf{G}[\beta]$ *of the* BAN*s* $\mathcal{N} = \mathcal{N}[\pi]$, $\mathcal{N}[\sigma]$ *and* $\mathcal{N}[\beta]$*. If* $\mathcal{N}$ *is a positive* DAN*, as mentioned in Example 1.4, it is completely defined by its structure* $\mathbf{G}$*. By definition of* $\mathrm{f}[\delta]$ *(for an arbitrary* δ*),* $\mathbf{G}[\beta]$ *is also a positive* DAN*. The table below also gives the dependencies between states of automata according to the update schedule, supposing that* $\mathcal{N}$ *is a positive* DAN.

	$\pi :\equiv \{0,1,2,3,4,5\}$	$\sigma :\equiv \{5\}\{3\}\{1\}\{0\}\{2\}\{4\}$	$\beta :\equiv \{2\}, \{3,4\}, \{0,1,5\}$
	 $\mathbf{G} = \mathbf{G}[\pi]$	 $\mathbf{G}[\sigma]$	 $\mathbf{G}[\beta]$
$i \in \mathbf{V}$	$x_i(t+1)$		
0	$x_1(t) \vee x_3(t)$	$x_1(t+1) \vee x_3(t+1)$ $= x_0(t) \vee x_2(t) \vee x_4(t)$	$x_1(t) \vee x_3(t+1)$ $= x_1(t) \vee x_4(t) \vee x_5(t)$
1	$x_2(t) \vee x_5(t)$	$x_2(t) \vee x_5(t+1)$ $= x_2(t) \vee x_0(t)$	$x_2(t+1) \vee x_5(t)$ $= x_5(t)$
2	$x_5(t)$	$x_5(t+1) = x_0(t)$	$x_5(t)$
3	$x_2(t) \vee x_4(t)$	$x_2(t) \vee x_4(t)$	$x_2(t+1) \vee x_4(t)$ $= x_5(t) \vee x_4(t)$
4	$x_5(t)$	$x_5(t+1) = x_0(t)$	$x_5(t)$
5	$x_0(t)$	$x_0(t)$	$x_0(t)$

Informally, the previous lemmas (especially Lemma 3.4) suggest that forcing a block-sequential update schedule that is not π, *i.e.* forcing some sequentialisation in the updates, "compresses" the information held in $\{\mathrm{f}_i\}$ and in **G**. This way, a "network operating core" is defined that endorses the "function" of the original network, in a more efficacious and less costly (in the spacial terms of the number of automata involved to produced the same effect) manner.

Again, in the next section, for the same reasons as those pointed out in the previous chapter, we give special attention to cycles and show how results of the present section, especially Lemma 3.3, apply to them simply. The importance of this is that together with the results presented before on the behaviours of BACs in parallel, it yields a combinatorial characterisation of the behaviours of BACs under *any* block-sequential update schedule. The section that follows after that focuses on positive DANs and exploits once more their privileged relation to their structures.

2 Cycles & update schedules

Below, we call **pseudo-cycle** of length m and sign s (*cf.* Example 3.3) a BAN whose structure embeds a cycle $\mathbb{C}_m$ of length m and sign s, as well as, possibly, some other arcs outgoing $\mathbb{C}_m$.

Corollary 3.1. BACs and update schedules

1. *Let β be a simple update schedule of a* BAC $\mathscr{C} = \mathscr{C}_n^s$ *with* $k = |\mathrm{I}[\beta]|$ *inversions. Then $\mathscr{C}[\beta]$ is a pseudo-cycle of length $n-k$ and sign s and as a result it bisimulates asymptotically the* BAC $\mathscr{C}^s_{n-k}[\pi] = \mathscr{C}^s_{n-k}$ *under π:*

$$\mathscr{C}[\beta] \mid\!\blacktriangleright\!\!\blacktriangleleft\!\mid \mathscr{C}^s_{n-k}.$$

2. *For two block-sequential update schedules β and β' of $\mathscr{C}$, if* $\mathrm{I}[\beta] \neq \mathrm{I}[\beta']$ *then β and β' induce no limit cycle in common.*
3. *Thus* $\mathrm{I}[\beta] = \mathrm{I}[\beta'] \Leftrightarrow \mathscr{C}[\beta] \mid\!\blacktriangleright\!\!\blacktriangleleft\!\mid \mathscr{C}[\beta'] \Leftrightarrow \mathscr{C}[\beta] \blacktriangleright\!\!\blacktriangleleft \mathscr{C}[\beta']$ *and the set of block-sequential update schedules of $\mathscr{C}$ induces* 2^n *different possible (asymptotic) behaviours* $\mathscr{T}_{[\beta]}$.
4. *Let $\mathscr{A} \subseteq \mathbb{B}^n$ be a set of p configurations of $\mathscr{C}$. Either under no block-sequential update schedule does $\mathscr{A}$ induce an attractor of $\mathscr{C}$, or there exists a sequential update schedule σ under which it does and σ can be computed in $\mathscr{O}(pn)$ steps together with* $\mathrm{I}[\sigma]$.

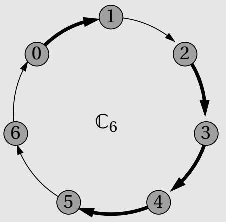
Proof: 1. $\forall i \in \mathbf{V} = \mathbb{Z}/n\mathbb{Z}$ let:

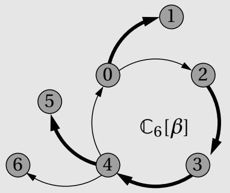
$$i^{\preccurlyeq} = \max\{k < i \mid \beta(k) \geq \beta(k+1)\} \tag{3.6}$$

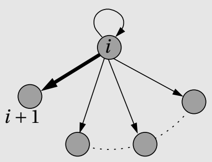
where the maximum is taken cyclically so that the number of arcs on a path from $i^{\preccurlyeq}$ to i is minimal. Then, $\mathbf{A}[\beta] = \{(i^{\preccurlyeq}, i)\}$ so $\mathbb{C}_n[\beta]$ contains a cycle $\mathbb{C}_{n-k}$ of length $n-k$ induced by the set of nodes $\{i \in \mathbf{V} \mid \exists j \in \mathbf{V}, i = j^{\preccurlyeq}\}$. All k other nodes j not belonging $\mathbb{C}_{n-k}$ depend on a unique node that does belong to $\mathbb{C}_{n-k}$. Moreover, the following holds: $\forall i \in \mathbf{V}, \mathrm{f}_i[\beta] = \mathrm{f}[i, i^{\preccurlyeq}]$. As a consequence, with respect to $\mathscr{C}[\beta]$, $\mathbb{C}_{n-k}$ has sign $\mathrm{sign}(\mathbb{C}_{n-k}) = s = \mathrm{sign}(\mathbb{C}_n)$. Automata in $\mathbb{C}_n[\beta]$ not belonging to $\mathbb{C}_{n-k}$ have no influence on $\mathbb{C}_{n-k}$. They only "obey" to the automata in it. Thus, $\mathscr{C}[\beta]$ and $\mathscr{C}^s_{n-k}$ behave similarly asymptotically.

Example 3.3. Pseudo-cycles

For any simple update schedule β *such that* $\mathrm{I}[\beta] = \{(1,2),(5,6),(6,0)\}$ *(i.e. for any* $\beta \propto \{5\},\{4,6\},\{0,1\},\{2,3\}$*),* $\mathbb{C}_6[\beta]$ *is the pseudo-cycle pictured below in the middle, containing a cycle of length* 4*, where* $0 = 1^{\preccurlyeq} = 2^{\preccurlyeq}$*,* $2 = 3^{\preccurlyeq}$*,* $3 = 4^{\preccurlyeq}$ *and* $4 = 0^{\preccurlyeq} = 5^{\preccurlyeq} = 6^{\preccurlyeq}$ *(cf. (3.6)).*







Generally, each of the n equivalence classes of $\propto$ *that have* $|\mathrm{I}| = n-1$ *inversions is characterised by the unique node i* $\in \mathbf{V}$ *such that* $(i, i+1) \notin \mathrm{I}$*. It has cardinal* 1 *since it only contains the sequential update schedule* $\sigma_i :\equiv \{i+1\},\{i+2\},\ldots,\{i-1\}\{i\}$*. And by Corollary 3.1, because of the loop on node i in its structure* $\mathbb{C}_n[\sigma_i]$ *(pictured on the right above), the limit behaviour of* $\mathscr{C}[\sigma_i]$ *only consists of fixed points if* $\mathscr{C}$ *is positive and of limit cycles of period* 2 *if* $\mathscr{C}$ *is negative.*

2. For any $x \in \mathbb{B}^n$, we write $x(t) = \mathrm{F}[\beta](x)$ and $x'(t) = \mathrm{F}[\beta'](x)$. Suppose that $(i, i+1) \in \mathrm{I}[\beta] \setminus \mathrm{I}[\beta']$ (*i.e.* $\beta(i+1) > \beta(i) \wedge \beta'(i+1) \leq \beta'(i)$) and that $x \in \mathbb{B}^n$ is such that $\forall t \in \mathbb{N}, x(t) = x'(t)$. Then $\forall t \in \mathbb{N}$, it holds that: $x(t)_{i+1} = \mathrm{f}_{i+1}(x(t)_i) = \mathrm{f}_{i+1}(x'(t)_i) = x'(t+1)_{i+1} = x(t+1)_{i+1}$, *i.e.* automaton $i+1$ is stable in the orbit of x. By induction this implies that all automata eventually become stable in the orbit of x which ends on a fix point.

3. The number 2^n of different possible behaviours comes from the number of possible sets of inversions $\mathrm{I}[\beta]$.

4. All update schedules β such that $\mathscr{A} = \{x(t) \mid t \in \mathbb{Z}/p\mathbb{Z}\}$, $p > 1$ induces a limit cycle of $\mathscr{C}[\beta]$ have the same set of inversion $\mathrm{I} = \mathrm{I}[\beta]$ by point 2. Thus, with $\mathscr{C}$ and $\mathscr{A}$

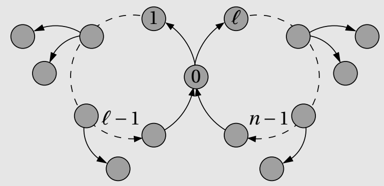
given as input, the following algorithm can output I and σ if they exist. Its step 3 can be done in time $\mathcal{O}(n)$ (*cf.* Lemma 3.1) and all other steps can be done in time $\mathcal{O}(pn)$.

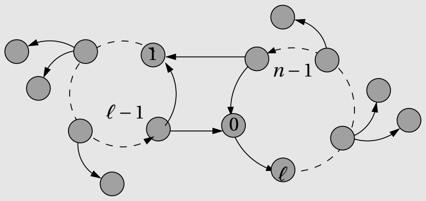
```
1.  Compute $\mathscr{A}^\pi = \{y(t) = \mathrm{F}[\pi](x(t-1)) \mid t \in \mathbb{Z}/p\mathbb{Z}\}$
2.  Compute $\mathrm{I} = \{(i-1, i) \mid \exists t \in \mathbb{Z}/p\mathbb{Z},\ x_i(t) \neq y_i(t)\}$
3.  Compute sequential $\sigma$ such that $\mathrm{I}[\sigma] = \mathrm{I}$
4.  Compute the set $\mathscr{A}^\sigma = \{\mathrm{F}[\sigma]^t(x(0)) \mid t \in \mathbb{Z}/p\mathbb{Z}\}$
5.  If $\mathscr{A}^\sigma = \mathscr{A}$, then output $\sigma$.
```

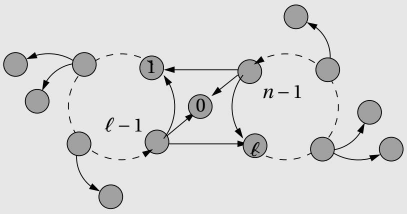
□

Example 3.4. Pseudo-double-cycles

Let β be a simple update schedule of a BAD $\mathscr{D} = \mathscr{D}^{ss'}_{\ell r}$. *The structure of* $\mathscr{D}[\beta]$ *has the form on the right if* $\beta(1), \beta(\ell) \leq \beta(0)$. *It has the form on the left (resp. right) below if* $\beta(1) > 0$ *and* $\beta(\ell) \leq \beta(0)$ *(resp. if* $\beta(1), \beta(\ell) > 0$*).*







3 Positive disjunctive networks & update schedules

Section 1, Chap. 2 focused on positive DANs under the parallel update schedule π and proved how the behaviours of these systems relate to their underlying structural SCCs and cycles. The present section is set in the continuity of this study. Before taking fair update schedules into account in Section 3.2, it first proposes to exploit the results of [44, 102, 103] and of Section 1 and combine them to those of Section 1, Chap. 2 in order to derive some properties concerning the behaviours of positive DANs under block-sequential update schedules. Let us point out here that the properties of positive DANs highlighted in the sequel can be compared to the known intractability of related problems concerning the description of BAN behaviours and in particular to the NP-completeness of the fix-point-existence problem (FPE) in various frameworks[14] [9, 37, 38, 39, 124, 125, 126].

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[14] The NP-completeness of FPE is proven on Page 153 of the appendix for locally monotone BANs.

In the rest of this section, without loss of generality, all digraphs considered are supposed to be connected and to have non-null minimal in-degrees ($\forall i \in \mathbf{V}, \deg^-_{\mathbf{G}}(i) > 0$). The reason why the second restriction can be done safely here has already been noted in Section 1, Chap. 2: in a DAN, source automata necessarily have constant state 0 so they have no impact on the states of other automata; they can be removed from the **G** without any consequences on the states of the remaining automata and this "pruning" of the original digraph can be carried out in polynomial time.

Before we give the first preliminary lemma of this section, we recall that in Section 5, Chap. 1, Example 1.14 shows how a positive DAN can simulate the behaviour of any of its own SCCs. This remark is important here because it allows to extend properties of strongly connected positive DANs to arbitrary positive DANs.

Lemma 3.6.

Let $\mathbf{G} = (\mathbf{V}, \mathbf{A})$ *be a positive* DAN *updated with update schedule* δ.

(i) If δ *is fair and if* $i \in \mathbf{V}$ *becomes fixed in state* 1, *then all nodes on a path that starts in* i *also become fixed in state* 1.

(ii) If δ *is block-sequential and if* $i \in \mathbf{V}$ *is fixed in state* 0, *then all nodes on a path that ends in* i *are also fixed in state* 0.

Proof: Let P_{ij} be a path of length ℓ_{ij} from $i \in \mathbf{V}$ (with state fixed to 1) to an arbitrary $j \in \mathbf{V}$. (*i*) relies on the two following facts: (1) in a positive DAN, a node takes state 1 if it is updated while one of its in-neighbours is currently in state 1; (2) a fair update schedule updates all nodes eventually. An induction on ℓ_{ij} then proves that all nodes on P_{ij} gradually become fixed to state 1. If $j \in \mathbf{V}^-_{\mathbf{G}}(i)$, $x \in \mathbb{B}^n$ and $t \in \mathbb{N}$ are such that $x(t)_j = 1$, then under a block-sequential β, right after its change of states, j is not updated again before i is. Thus, either $x(t)_i = 1$ (if $\beta(i) > \beta(j)$) or $x(t)_i = 1$ (if $\beta(i) \leq \beta(j)$). In both cases, i takes state 1 as a consequence of j having taken state 1. This proves (*ii*). □

A useful, immediate consequence of this lemma is the following.

Corollary 3.2.

Let **G** *be a* SCC *of an arbitrary positive* DAN. *If* **G** *contains a loop-node, then* **G** *can cycle under no block-sequential update schedule and it can cycle under fair update schedules only if its loop-node never takes state* 1 *as a result of an update.*

3.1 Block-sequential update schedules

As mentioned earlier, a positive DAN **G** updated with a block-sequential update schedule β is completely defined by $\mathbf{G}[\beta]$. So following Lemma 2.2 in Section 1, Chap. 2, we know that **G** cycles under β if and only if[15] $\eta(\mathbf{G}[\beta]) > 1$. And further, because **G** can simulate the behaviour of any of its SCCs, **G** cycles under β if and only if **G** has an SCC H such that $\eta(\mathrm{H}[\beta]) > 1$. A natural question is "If **G** is fixed under the parallel update schedule (*i.e.* $\eta(\mathbf{G}) = 1$), can we determine whether **G** cycles under some block-sequential update schedule β (*i.e.* whether there exists β, such that $\eta(\mathbf{G}[\beta]) > 1$)?" Because the number of block-sequential update schedules of n nodes is exponential (*cf.* Section E, Chap. 6), here, we endeavour to answer this question by relying on the structural properties of **G**. The two following propositions answer this question for some particular classes of positive DANs. In the first of these, a **symmetric digraph** refers to a digraph $\mathbf{G} = (\mathbf{V}, \mathbf{A})$ such that $\forall i, j \in V,\ (i, j) \in \mathbf{A} \Rightarrow (j, i) \in \mathbf{A}$. It recalls a result proven in [45, 49].

Proposition 3.1.

A symmetric digraph **G** *cycles under no other block-sequential update schedule* β *than the parallel update schedule* $\beta = \pi$ *and it cycles under* π *if and only if it contains no cycle of odd length, in which case its order is* 2.

Proof: For any nodes i and j such that $(i, j), (j, i) \in \mathbf{A}$, if $\beta(i) > \beta(j)$ then i is a loop-node in $\mathbf{G}[\beta]$ by Lemma 3.3 so, its order[15] being $\eta(\mathbf{G}[\beta]) = 1$, $\mathbf{G}[\beta]$ does not cycle. Thus, the only update schedule that can possibly induce limit cycles is π. If **G** contains a cycle of odd length ℓ, then, since it also contains cycles of length 2, its order is $\eta(\mathbf{G}) = \gcd(\ell, 2) = 1$. Otherwise, it is $\eta(\mathbf{G}) = 2$. □

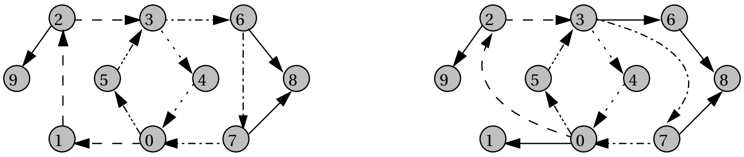


Figure 3.1: *Illustration of Proposition 3.2. Left: a digraph* **G** *such that* $\eta(\mathbf{G}) = 1$. *Right: the digraph* $\mathbf{G}[\beta]$ *satisfying* $\eta(\mathbf{G}[\beta]) = 4$, *where* $\beta :\equiv \{0,1,3,4,5,6,8,9\},\{2,7\}$. *Arcs with different dotted or dashed lines belong to different nude paths in a non-trivial* SCC.

Proposition 3.2.

Let **G** *be a non-trivial strongly connected component of* H. *If* **G** *contains no loops nor any nude path of length* 1, *then* **G** *and* H *can cycle for a certain block-sequential update schedule.*

---

[15] $\eta(\mathbf{G})$ is the index of imprimitivity of **G** and equals its order under π according to Section 1, Chap. 2.

Proof: Let β be a block-sequential update schedule such that for any nude path $\{i_0,\dots,i_k\}$ of odd length $k \geq 3$, $\beta(i_1) < \beta(i_2)$ and $\forall r \leq k,\ r \neq 1,\ \beta(i_r) \geq \beta(i_{r+1})$. Then, for any such nude path, by Lemma 3.3, the following holds. The path $\{i_2,\dots,i_k\}$ remains unchanged in $\mathbf{G}[\beta]$. The path $\{i_0,i_2\}$ is replaced by the arc $(i_0,i_2) \in A[\beta]$. The arc $(i_1,i_2) \in A \setminus A[\beta]$ disappears. Thus, globally, the nude path $\{i_0,i_1,\dots,i_k\}$ of odd length k in $\mathbf{G}$ becomes a nude path $\{i_0,i_2,\dots,i_k\}$ of even length $k-1$ in $G[\beta]$. All nude paths and consequently all cycles of $G[\beta]$ have an even length. As a result, $\eta(G[\beta]) > 1$ is even (*cf.* Fig. 3.1). $\square$

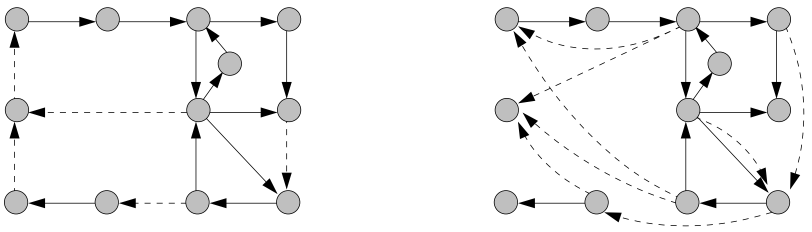


Figure 3.2: *Illustration of Proposition 3.3. Left: a digraph* $\mathbf{G} = (\mathbf{V},\mathbf{A})$. *Arcs in full lines define a* 3*-cycle-cover* W *of* $\mathbf{G}$. *Right:* $\mathbf{G}[\beta]$ *where* β *is an update schedule with inversion set* $\mathrm{I}[\beta] = \mathbf{A}\setminus\mathrm{W}$, *i.e. the set of arcs in dashed lines on the left, in* $\mathbf{G}$. *Having order* $\eta(\mathbf{G}[\beta]) = 3 > 1$, *the positive* DAN $\mathbf{G}[\beta]$ *cycles.*

The next proposition characterises in terms of their structures the strongly connected positive DANs that cycle for a certain block-sequential update schedule. It uses *k-cycle-covers.* Informally, the idea is to pick a multiple of k arcs in each cycle of a positive DAN $\mathbf{G}$ and define an update schedule β that precisely ousts all arcs that were not chosen so that those remaining in $\mathbf{G}[\beta]$ form cycles of lengths a multiple of k. Formally, k-cycle-covers are defined as follows. Let $\mathbf{G} = (\mathbf{V},\mathbf{A})$ be a strongly connected digraph. A cycle $\mathrm{C} = (\mathbf{V}_\mathrm{C},\mathbf{A}_\mathrm{C})$ of $\mathbf{G}$ is said to be **k-covered by** $\mathrm{W} \subseteq \mathbf{A}$ if $\exists q \geq 1$ such that $|\mathbf{A}_\mathrm{C} \cap \mathrm{W}| = kq$. The set W is said to be a **k-cycle-cover** of $\mathbf{G}$ if and only if:

(i) all cycles of $\mathbf{G}$ are k-covered by W and

(ii) there are no undirected cycles $\{i_0,i_1,\dots,i_m = i_0\}$ containing at least one arc of $\mathbf{A}\setminus\mathrm{W}$ and satisfying the following: $\forall r < m$, either $(i_r,i_{r+1}) \in \mathbf{A}\cap\mathrm{W}$ or $(i_{r+1},i_r) \in \mathbf{A}\setminus\mathrm{W}$.

The purpose of the second condition is to ensure that a block-sequential update schedule β can indeed be defined as intended, on the basis of the k-cycle-cover W. More precisely, *(ii)* ensures the existence of a block-sequential β satisfying $\mathrm{I}[\beta] = \mathbf{A}\setminus\mathrm{W}$ without $\exists i \in \mathbf{V},\ \beta(i) > \beta(i)$ (*cf.* Fig. 3.2). The formal proof of Proposition 3.3 can be found on Page 150 of the appendix.

Proposition 3.3.

If **G** *is strongly connected, it has a k-cycle-cover* ($k > 1$) *if and only if it has a block-sequential update schedule* β *such that k divides* $\eta(\mathbf{G}[\beta])$. *Thus, a positive* DAN H *cycles for some block-sequential update schedule if and only if it contains a non-trivial* SCC **G** *that has a k-cycle-cover for some* $k > 1$.

To complete this section, we give one last result concerning the behaviours of positive DANs under block-sequential update schedules.

Lemma 3.7. All positive DANs can be fixed

For any positive DAN **G** *there exists a block-sequential update schedule* β *such that* $\mathbf{G}[\beta]$ *only has fix points.*

Proof: For every non-trivial SCC $C = (\mathbf{V}_C, \mathbf{A}_C)$ of **G**, let $i_C \in \mathbf{V}_C$ be a certain node of C and let $P_C = \{i_C = i_1, \dots, i_\ell = i_C\}$ be a closed path starting and ending in i_C. We can define β as follows. For every non-trivial SCC C associated to the couple (i_C, P_C), β satisfies $\beta(i_{r+1}) > \beta(i_r)$, $\forall 1 < r < \ell$ and all other arcs belong to $\mathbf{A} \setminus \mathrm{I}[\beta]$. By Lemma 3.3, $\forall$C, i_C is a loop-node in $\mathbf{G}[\beta]$. By Lemma 2.2, $\mathbf{G}[\beta]$ does not cycle. □

3.2 Fair update schedules and classification

Compared to asynchronous update modes and to the parallel update schedule π, block-sequential update schedules have the notable advantage of allowing both a certain degree of synchronism and of asynchronism. Indeed, in modelling real systems (especially biological), it often seems quite unrealistic and unsafe to discard either one altogether (*cf.* Example 5.1 and Chapter 6). But it also seems unrealistic to impose automata to be updated exactly as often. This introduces the advantage of fair update schedules that allow some automata to be updated more often than others.

This section is devoted to the classification of positive DANs according to their behaviours with respect to all fair update schedules. In analogy to the notation introduced above for simple update schedules, we use $\mathcal{N}[\delta]$ to refer to a BAN $\mathcal{N}$ updated with δ. However, it is important to note that unlike with block-sequential update schedules, in the case of general update schedules (including fair ones), $\mathcal{N}[\delta]$ *cannot* simply be seen as a BAN under π and if **G** is the structure of $\mathcal{N}$ then $\mathbf{G}[\delta]$ does *not* denote a BAN structure. Indeed, since automata are possibly updated several times within an update period, there is no straightforward way to define a digraph that conveys the direct dependencies between automata as they are appointed by δ (contrary to the case where $\delta = \beta$ is block-sequential, as demonstrated by the definition of $\mathbf{G}[\beta]$ in Section 1.2). Nonetheless, if $\mathcal{N}$ is a positive DAN, since **G** is equivalently used to denote $\mathcal{N}$, we also use $\mathbf{G}[\delta]$ equivalently to denote $\underline{\text{BAN}}$ $\mathcal{N}[\delta]$.

The basis of our classification is the four classes of networks B_1, B_2, B_3, B_4 introduced in [27] and studied by Elena in his PhD Thesis [35] under the names **Fi**, **Cy**, **Mi** and **Ev**. Because Elena focused on a larger set of BANs[16], we adapt the definition of these classes to fit our study of positive DANs with positive in-degree. More precisely, since each of these BANs obviously has at least two fix points, 0^n and 1^n (*cf.* Lemma 2.1 on Page 37), there is no point in distinguishing them on the criteria that they have fix points or not. We chose to replace this criteria by that of having fix points different from 0^n and 1^n. This way, we end up with the six classes defined in Table 3.1. We recall again that the set of fix points of a network is independent of the update schedule. Thus, every positive DAN belongs to one of the six classes of Table 3.1.

	Positive DANs with	
	fix points other than $\neq 0^n$, 1^n	limit cycles
Fi	✘	✘
Fi′	✔	✘
Cy	✘	✔
Mi	✔	✔
Ev	✘	for some US ✔, for others ✘
Ev′	✔	for some US ✔, for others ✘

Table 3.1: *Classification of positive* DANs *with positive in-degree.* US *stands for update schedule. Column 1 lists the names of the classes that are defined on the corresponding lines according to the properties given in Line 1. The two bottom right cells define* $Ev \cup Ev'$ *as the set of positive* DANs *having limit cycles for some* USs *and none for others. In the remaining cells of Column 2 (resp. 3),* ✔ *indicates that the* DANs *have other fix points than* 0^n *and* 1^n *(resp. they have limit cycles) for all fair* USs*;* ✘ *means that they have for none.*

Theorem 3.1 below details the content of the six classes of Table 3.1. It mentions **weakly-loop-free** components which are defined as non-trivial sub-digraphs $\mathrm{H} = (\mathbf{V}_\mathrm{H}, \mathbf{A}_\mathrm{H})$ of a digraph $\mathbf{G} = (\mathbf{V}, \mathbf{A})$ that satisfy the following three conditions:

(i) H is strongly connected,

(ii) $\mathbf{V}_\mathrm{H}$ contains no loop-node,

(iii) There is no node $i \in \mathbf{V}$ such that $\mathbf{V}_\mathrm{H} \subseteq \mathbf{V}^-_\mathbf{G}(i)$ and there is a path from i to $\mathbf{V}_\mathrm{H}$.

To contain no weakly-loop-free component, an arbitrary digraph must either be cycle-free or all of its non-trivial SCCs must contain loop-nodes (*cf.* Fig. 3.3, right).

---

[16] He studied threshold BANs with local transition functions of the form: $f_i(x) = \sum_j H(w_{ij} \cdot x_j - \theta_i)$ where H is the Heaviside function ($H(a) = 0$ if $a < 0$ and $H(a) = 1$ otherwise), w_{ij} is the weight attributed to the arc (j, i) and θ_i is the activation threshold of i.

Indeed, any non-trivial SCC with no loop-nodes is a weakly-loop-free component itself. But this condition is not sufficient because non-trivial SCCs containing loop-nodes can also contain weakly-loop-free components (*cf.* Fig. 3.3, left). It can be determined in polynomial time whether or not a strongly connected digraph, and as a consequence, any digraph, contains a weakly-loop-free component[17].

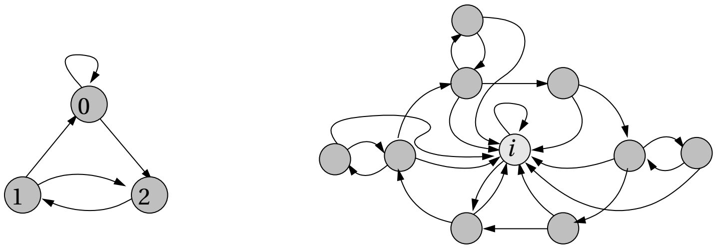


Figure 3.3: *Left: a strongly connected digraph in which nodes* 1 *and* 2 *induce a weakly-loop-free component. As a positive* DAN*, this digraph has a limit cycle induced by* $\{(0\,1\,0),(0\,0\,1)\} \subseteq \mathbb{B}^3$ *under the fair update schedule* $\{1,2\},\{0,1,2\}$. *Right: a strongly connected digraph* $\mathbf{G} = (\mathbf{V},\mathbf{A})$ *with no weakly-loop-free component. In any configuration* $x \neq 0^{|\mathbf{V}|}$, $\exists j \in \mathbf{V}^-_{\mathbf{G}}(i) = \mathbf{V}$, $x_j = 1$ *so as soon as i is updated it takes and remains in state* 1 *and by Lemma 3.6, the network cannot cycle.*

Theorem 3.1.

Classes of positive DAN*s with no source nodes defined in Table 3.1 satisfy:*

- *(i)* $\mathbf{Mi} = \mathbf{Cy} = \emptyset$.
- *(ii)* $\mathbf{Fi} \cup \mathbf{Ev}$ *is the set of positive* DAN*s that contain a unique non-trivial* SCC. $\mathbf{Fi}' \cup \mathbf{Ev}'$ *is the set of positive* DAN*s that contain several.*
- *(iii)* $\mathbf{Ev} \cup \mathbf{Ev}'$ *is the set of positive* DAN*s that contain at least one weakly-loop-free component.* $\mathbf{Fi} \cup \mathbf{Fi}'$ *is the set of positive* DAN*s that contain none.*

Proof: (*i*) and (*ii*) derive directly from Lemmas 3.7 and 2.1, respectively. (*iii*) characterises the set of positive DANs that can cycle and relies on the following claim which is proven in the appendix, on Page 151:

> *A positive* DAN **G** *can cycle under some fair update schedule if and only if it contains a non-trivial* SCC *in which there is a weakly-loop-free component. And if* **G** *contains a weakly-loop-free simple cycle, then it cycles under a* 2*-fair update schedule.*

□

---

[17] Construct the sub-digraph $\mathbf{G}'$ of the strongly connected digraph $\mathbf{G} = (\mathbf{V},\mathbf{A})$ where all loop-nodes and their incident arcs have been removed; find the non-trivial SCCs of $\mathbf{G}'$; and look for one of them, $\mathrm{C} = (\mathbf{V}_\mathrm{C},\mathbf{A}_\mathrm{C})$, satisfying $\forall i \in \mathbf{V}, \mathbf{V}_\mathrm{C} \not\subseteq \mathbf{V}^-_{\mathbf{G}}(i)$.

Let us again restrict our attention to block-sequential update schedules. For any of the six classes $\mathbf{C}$ defined in Table 3.1, let $\mathbf{C}_{\mathrm{B}}$ be the class of positive DANs that is defined similarly to $\mathbf{C}$ except that only block-sequential update schedules are considered. Then, by the results of Section 3.1, replacing the classes $\mathbf{C}$ by $\mathbf{C}_{\mathrm{B}}$ in Theorem 3.1, points (*i*) and (*ii*) remain true. As for point (*iii*), it can be replaced, for instance, by the following claim that derives from Proposition 3.3: $\mathbf{Ev} \cup \mathbf{Ev}'$ is the set of positive DANs in which there is a non-trivial strongly connected component with a k-cycle-cover for some $k > 1$. Fig. 3.4 gives two examples ($\mathbf{G}_3$ and $\mathbf{G}_4$) of digraphs that do not cycle for any block-sequential update schedule but that do for some other fair update schedules. Thus, the positive DAN classes satisfy the following strict inclusions:

$$\mathbf{Ev}_{\mathrm{B}} \subsetneq \mathbf{Ev},\ \mathbf{Ev}'_{\mathrm{B}} \subsetneq \mathbf{Ev}',\ \mathbf{Fi} \subsetneq \mathbf{Fi}_{\mathrm{B}} \text{ and } \mathbf{Fi}' \subsetneq \mathbf{Fi}'_{\mathrm{B}}.$$

Fig. 3.4 sums up the results concerning all classes introduced in Table 3.1 for fair and block-sequential update schedules.

3.3 Conclusion and perspectives

Despite the apparent simplicity of positive DANs and although their underlying structures are very tightly related to their behaviours, this section which only aims at distinguishing DANs that cycle and DANs that don't, shows that allowing a certain liberty in the update schedule yields a new level of difficulty in the analysis of the behaviours of BANs. Generally, this study calls for a more precise description of the limit behaviours of positive DANs under fair and block-sequential update schedules and of the way their recurrent configurations are distributed in their attractors.

In this section, we have classified networks into four classes in both the cases of fair and block-sequential update schedules. We have showed that all networks can loose their limit cycles with a certain update schedule. We have also characterised those networks that may cycle for either a fair update schedule or a block-sequential one. In the more general case of fair update schedules, we can determine in polynomial time whether a positive DAN whose structure is known is able to cycle or not. When we restrict our attention to just block-sequential update schedules, Proposition 3.3 gives a characterisation of the networks that cycle. It does so by means of the network structure only, allowing us to bring our study of behaviours under different block-sequential update schedules back to a planer study of a digraph property (that of having a k-cycle-cover for some $k > 1$) that is not without similarities with the *Feedback-Arc-Set* problem [42]. One question remains unanswered, however: can we determine in polynomial time if a positive DAN is able to cycle for a certain block-sequential update schedule?

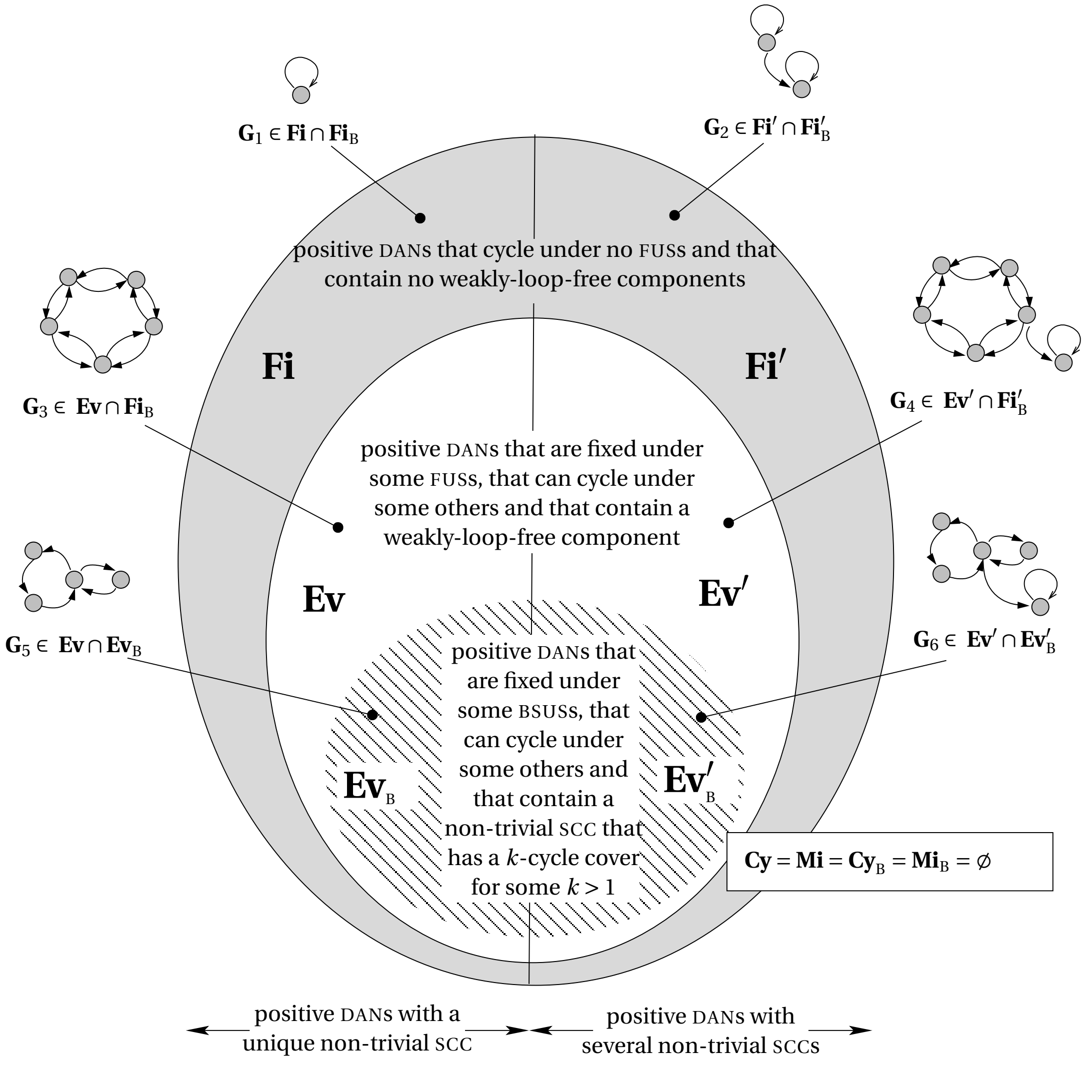


Figure 3.4: *Classes of positive* DAN*s.* US *stands for update schedule;* FUS *and* BSUS *respectively stand for fair and block-sequential* US*. Positive* DAN*s that do not belong to* $\mathbf{Ev}_{\mathrm{B}}$ *(resp.* $\mathbf{Ev}'_{\mathrm{B}}$*) belong to* $\mathbf{Fi}_{\mathrm{B}}$ *(resp.* $\mathbf{Fi}'_{\mathrm{B}}$*). Digraphs* $\mathbf{G}_1$ *to* $\mathbf{G}_6$ *are examples of networks corresponding to each section of the diagram. In these examples, all non-trivial* SCC*s containing a loop-node are unable to cycle whatever the* US *(by Theorem 3.1, (iii)).* $\mathbf{G}_3 \in \mathbf{Fi}_{\mathrm{B}}$ *and* $\mathbf{G}_4 \in \mathbf{Fi}'_{\mathrm{B}}$ *follow from Proposition 3.1,* $\mathbf{G}_3 \in \mathbf{Ev}$ *and* $\mathbf{G}_4 \in \mathbf{Ev}'$ *from Theorem 3.1, (iii). By Proposition 3.2,* $\mathbf{G}_5$ *satisfies* $\eta(\mathbf{G}_5) = 1$ *but can still cycle under a* BSUS *by Proposition 3.2. The same is true for the top component of* $\mathbf{G}_6$ *so* $\mathbf{G}_5 \in \mathbf{Ev}_{\mathrm{B}}$ *and* $\mathbf{G}_6 \in \mathbf{Ev}'_{\mathrm{B}}$.

Also, we can mention that in [91], Reidys focuses on asynchronous fair update schedules and proves that the property of keeping the same set of recurrent configurations is independent of whether or not automata can be updated several times in a sequence. It would be interesting to determine if Reidys's result extends to the conditions we have set here allowing synchronous updatings, *i.e.* whether in our setting, updating automata more often than others significantly impacts (*cf.* Section 3.C, Chap. 6 on Page 117) or not on the asymptotics of a BAN behaviour.

Finally, Lemma 3.7 states that all positive DANs can be fixed with a block-sequential update schedule. Theorem 3.1 states that almost all positive DANs can be made to cycle with a fair update schedule. This suggests that BANs can be made to do almost anything by playing on their update schedules and puts forward the idea that “information” can be coded into the updating of a network (*cf.* Section 3.A, Chap. 6). In these lines, it remains to address the more precise question of whether update schedules can be chosen to force cycles of a certain period. For block-sequential update schedules this amounts to considering whether, given $p \in \mathbb{N}$, an update schedule δ exists such that $\mathbf{G}[\delta]$ has a p-cycle cover, or just, more simply, whether or not having a k-cycle-cover induces having a 2-cycle-cover, *i.e.* can a cycling BAN cycle with period $\eta(\mathbf{G}[\delta]) = 2$.

OBSERVING NETWORKS

4

This intermediary, less formal chapter aims at highlighting and discussing some of the ins and outs of discrete formal interaction networks – such as BANs – and of their update modes. It proposes to explore the notion of causality with respect to these networks and discuss how it can be used and perceived. Also, its last concluding section on the defining of networks serves as a motivation to the next chapter that concentrates on GTGS (*i.e.* transition graphs that require no prior specification of an update mode).

The focus is put on transition graphs. More precisely, the running general point of view of the present chapter is that of an exterior *observer* of an arbitrary system which can possibly be modelled by a BAN. The observer may have full, partial or no knowledge of the system characteristics (mainly, those that can be represented by the structure, set of interactions and behaviour of a BAN). But *a priori*, the system's underlying mechanisms producing the events observed are not known. And dynamical changes underwent by the system are not supposed to be observed directly either. Instead, it is some of the successive states taken by the system which are observed, modelled as transitions and recorded in a transition graph $\mathscr{T} = \mathscr{T}_{\text{obs}}$. Metaphorically, one can imagine that a computer program simulates the behaviour of a BAN $\mathscr{N}$ and prints out on a monitor the configurations that it takes. On the one hand, the program may be designed to hide some parts of the information when it is executed. On the other, the observer's diligence in watching the screen may not be perfect. They may, for instance, only look at it periodically, or from time to time at random, or they may perhaps forget to look at it altogether. In any case, the lists of successive configurations that are observed yield the transition graph $\mathscr{T}$. And from $\mathscr{T}$, sets of automata, of interactions and of local transition functions can then be *inferred* to define a new BAN $\mathscr{N}'$ intended to model the original, simulated and observed BAN $\mathscr{N}$. Thus, initially, the only information at hand on the system is $\mathscr{T}$, the formalisation of the observer's experience of the system's behaviour.

The first section of this chapter defines a new equivalence relation on update schedules which formalises the importance of the "observation protocol". The next section examines how the different features of BAN inter-relate and with what consequences regarding our understanding of them and of their role in the behaviours of BANs, and also for our possible utilisations of them. In the continuity of this, Section 3 ends the chapter on the subject of defining networks.

Let us highlight that *periodic* update *schedules* are put forward again here. But they are not regarded for themselves as in the previous chapter. They rather serve as manipulable representatives of general update modes, *i.e.* of any possible restrictions that can be made of the set of transitions and derivations of a BAN to those that are considered as effectively representing the system's behaviour – as opposed to those that are possible by definition but do not appear in $\mathscr{T}$ (*e.g.* the restriction consisting in disregarding elementary transition $x \xrightarrow{\{0,1\}} \overline{x}^{\{0,1\}}$ when $\delta :\equiv \{1\},\{0,2\}$). In particular, any periodic update schedule $\delta :\equiv (\mathrm{W}_t)_{t\in\mathbb{Z}/p\mathbb{Z}}$ of period p can be taken to formally represent the *non-periodic* update schedule $\delta' := (\mathrm{W}_t)_{t<p}$ defined by the *finite* sequence of p updates $\mathrm{W}_0,\dots,\mathrm{W}_{p-1}$.

1 Elementary similarity

Contrary to relation $\asymp$ introduced in Section 1.1, Chap. 3, we now relate update schedules that may appear to be very different from a distant point of view (that of an observer only observing the system's configurations once in a while, *e.g.* once per update period), but fundamentally involve the same elementary events.

Let $\delta :\equiv \mathrm{W}_0,\mathrm{W}_1,\dots,\mathrm{W}_{p-1}$ and $\delta' :\equiv \mathrm{W}_1,\dots,\mathrm{W}_{p-1},\mathrm{W}_0$, be two periodic update schedules of an arbitrary BAN $\mathscr{N}$. Then, for any elementary derivation:

$$\begin{tikzcd}
x(0) \arrow[r, "\mathrm{W}_0"] & x(1) \arrow[r, "\mathrm{W}_1"] & x(2) \arrow[r, "\mathrm{W}_2"] & \cdots\cdots\cdots \arrow[r, "\mathrm{W}_{p-1}"] & x(p) = \mathrm{F}[\delta](x(0)) \arrow[d, "\mathrm{W}_0"] \\
\cdots\cdots\cdots\cdots\cdots & x(2p) = \mathrm{F}[\delta](x(1)) \arrow[l, "\mathrm{W}_0"'] & & \cdots\cdots\cdots \arrow[ll, "\mathrm{W}_{p-1}"'] & x(p+1) \arrow[l, "\mathrm{W}_1"']
\end{tikzcd}$$

of $\mathscr{N}$ under δ, $\mathscr{N}$ under δ' can perform the following elementary derivation:

$$\begin{tikzcd}
 & y(0) = x(1) \arrow[r, "\mathrm{W}_1"] & y(1) = x(2) \arrow[r, "\mathrm{W}_2"] & \cdots\cdots \arrow[r, "\mathrm{W}_{p-1}"] & y(p-1) = x(p) \arrow[d, "\mathrm{W}_0"] \\
\cdots\cdots\cdots & y(2p-1)) = x(2p) \arrow[l, "\mathrm{W}_0"'] & \cdots\cdots \arrow[l, "\mathrm{W}_{p-1}"'] & & \begin{array}{l} y(p) = \mathrm{F}[\delta'](y(0)) \\ = x(p+1) \end{array} \arrow[ll, "\mathrm{W}_1"']
\end{tikzcd}$$

We define the equivalence relation $\asymp$ that relates two update schedules that differ only by a circular permutation of their sequence of updates, *e.g.* : $\{1\},\{0,2\},\{1,2\} \asymp \{0,2\},\{1,2\},\{1\} \asymp \{1,2\},\{1\},\{0,2\}$. And generally, with $\delta :\equiv (\mathrm{W}_t)_{t\in\mathbb{N}/p\mathbb{N}}$ and $\delta' :\equiv (\mathrm{W}'_t)_{t\in\mathbb{N}/p\mathbb{N}}$:

$$\delta \ \asymp \ \delta' \quad \Leftrightarrow \quad \exists k \in \mathbb{Z}/p\mathbb{Z}, \forall t \in \mathbb{Z}/p\mathbb{Z}, \mathrm{W}'_t = \mathrm{W}_{t+k}. \tag{4.1}$$

On Page 155 of the appendix, the number of block-sequential update schedules is compared to the number of different classes of these update schedules for the relation $\asymp$. Generally, with the notation of (4.1) and letting $\mathrm{F}' = \mathrm{F}_{\mathrm{W}_{k-1}} \circ \ldots \circ \mathrm{F}_{\mathrm{W}_1} \circ \mathrm{F}_{\mathrm{W}_0}$ and $\mathrm{F}'' = \mathrm{F}_{\mathrm{W}_{p-1}} \circ \ldots \circ \mathrm{F}_{\mathrm{W}_{k+1}} \circ \mathrm{F}_{\mathrm{W}_k}$, it holds that $\mathrm{F}[\delta] = \mathrm{F}'' \circ \mathrm{F}'$ and $\mathrm{F}[\delta'] = \mathrm{F}' \circ \mathrm{F}''$. Thus, an elementary derivation starting in $x \in \mathbb{B}^n$ under δ (resp. δ') becomes identical, at its k^{th} (resp. $(p-k)^{\text{th}}$) step to the elementary derivation that starts in $\mathrm{F}'(x)$ (resp. $\mathrm{F}''(x)$) under δ' (resp. δ).

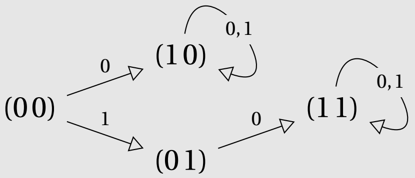
According to Example 4.1, for short derivations (of length smaller than k and $p-k$), the first steps can have a decisive impact on the evolution and future of a BAN so that equating two equivalent update schedules $\delta \asymp \delta'$ is not pertinent. Nonetheless, it *is* when transient derivations are long enough, for instance when δ and δ' are periodic and more than one of their respective periods are considered (unlike in Example 4.1). Then, except for their first few steps, δ and δ' yield the exact same elementary behaviours. However, between two observations of the global BAN configuration, δ and δ' can possibly perform any sequence of updates (depending on their primary definitions). So, as a result of some (partial) non-elementary observations of a derivation that corresponds to:

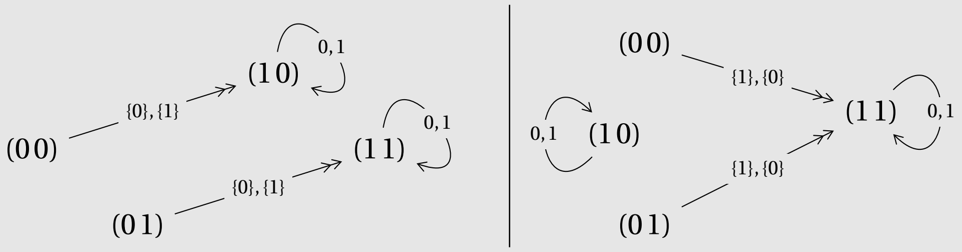
$$\begin{tikzcd}
 & & & & \mathrm{F}[\delta'](y) \arrow[d, phantom, "\parallel"] & & \mathrm{F}[\delta']^2(y) \arrow[d, phantom, "\parallel"] & \\
\cdots \arrow[r, "\mathrm{F}''"] & x \arrow[r, "\mathrm{F}'"] & y \arrow[r, "\mathrm{F}''"] & z \arrow[r, "\mathrm{F}'"] \arrow[d, phantom, "\parallel"] & u \arrow[r, "\mathrm{F}''"] & v \arrow[r, "\mathrm{F}'"] \arrow[d, phantom, "\parallel"] & w \arrow[r, "\mathrm{F}''"] & \cdots, \\
 & & & \mathrm{F}[\delta](x) & & \mathrm{F}[\delta]^2(x) & &
\end{tikzcd}$$

there is no prior reason for the sequences of successive configurations x, z, v and y, u, w that are respectively recorded under δ and δ', to look anything alike and be relatable (not, without any additional specifications, at least). In other terms, while $\mathscr{T}_\delta$ and $\mathscr{T}_{\delta'}$ may be almost identical, $\mathscr{T}_{[\delta]}$ and $\mathscr{T}_{[\delta']}$, on the contrary, may differ significantly. This shows that differences in transition graphs may only be due to the "instant" at which the system configuration is observed. This remark can be compared with the discussion at the end of Section 1.1, Chap. 3 concerning the impact of the differences between two update schedules that are equivalent by the basic, global equivalence relation $\propto$, in a context where exterior perturbations are assumed to be possible. In such a context, differences that are undistinguished by $\propto$ (in the sequencing of elementary events) may be appreciably more consequential than the differences undistinguished by $\asymp$ (which only lie in the "observation protocol").

Example 4.1.

According to its ATG *(cf. digraph on the right), starting in configuration* $(0\,0)$*, the* BAN $\mathcal{N} = \{\mathrm{f}_0 : x \mapsto 1, \mathrm{f}_1 : x \mapsto \neg x_0 \vee x_1\}$ *ends up looping on configuration* $(1\,0)$ *if it is updated with* $\delta :\equiv \{0\},\{1\}$ *and it ends up looping on configuration* $(1\,1)$ *if it is updated with* $\delta' :\equiv \{1\},\{0\}$*, although* $\delta \asymp \delta'$*. Transition graphs* $\mathcal{T}_{[\delta]}$ *and* $\mathcal{T}_{[\delta']}$ *remain very different:*





2 Observing & inferring

The previous section raises the question of how systems are observed and with what consequences to our understanding of their underlying mechanisms (as represented by their structures $\mathbf{G} = (\mathbf{V}, \mathbf{A})$ and, more precisely, by their sets of local transition functions $\mathscr{F} = \{\mathrm{f}_i\}$) and of the causes that are responsible for the events observed and formalised via a transition graph $\mathcal{T}_{\mathrm{obs}} = \mathcal{T}$. To address this question, the present section proposes to analyse some of the relations that exist between the different features of BANs, and how an observer holding one type of information on a network may complete it by inferring some of another type.

2.1 Structure & local transition functions

To start, let us point out that the structure $\mathbf{G} = (\mathbf{V}, \mathbf{A})$ of a BAN can obviously be derived from its set of local transition functions $\mathscr{F} = \{\mathrm{f}_i \mid i \in \mathbf{V}\}$, without any ambiguity thanks to the minimality of structures imposed by (1.3). But it takes exponential time with respect to the size n of the BAN to draw **G** from $\mathscr{F}$, even if the f_is are given in conjunctive normal form (CNF)[18].

Throughout the rest of this section, we consider a transition graph $\mathscr{T} = (\mathbb{B}^n, \mathrm{T})$ induced by a certain update mode δ (not necessarily periodic nor deterministic). Abusing notations, whatever δ, we write $\mathscr{T} = \mathscr{T}_{[\delta]}$ to emphasise that the transitions recorded in $\mathscr{T}$ and the *granularity* of events that they represent are subjected to the definition of δ. As mentioned at the beginning of the chapter, $\mathscr{T}$ represents the behaviour of a system whose underlying mechanisms are unknown *a priori*[19].

2.2 Observing fixity & movement

Let $\deg^{\ddagger}_{\mathscr{T}}(x) = |\{x \neq y \,|\, (x,y) \in \mathrm{T}\}|$ denote the **effective out-degree** of a node x in $\mathscr{T}$. Fixed configurations x of $\mathscr{T}$ are characterised by $\deg^{\ddagger}_{\mathscr{T}}(x) = 0$ (*cf.* Section 4, Chap. 1). But notably, they are not necessarily *stable* configurations: $\deg^{\ddagger}_{\mathscr{T}}(x) = 0 \wedge \mathscr{U}(x) \neq \emptyset$ may hold if all unstable automata of $\mathscr{U}(x)$ are updated an even number of times between two observations of the BAN (represented by the two extremities of a transition in T).

Thus, for instance, the updating δ may allow several updates of the same automaton between two observations (this is possible if δ is a fair update schedule, for example, but not if it is a simple update schedule). This way, in fixed configuration $x \in \mathbb{B}^n$ of $\mathscr{T}$, automata in $\mathscr{U}(x)$ may *appear* to be fixed in state x_i although they actually are updated several times during the sequence of (effective) elementary transitions corresponding to $(x,x) \in \mathrm{T}$. Then, they can switch states an even, non null number of times between the beginning and the end of this sequence in a way that allows them to come back into their initial states x_i before the BAN is observed again. The possibility of such reversible changes between two observable configurations x and $\mathrm{F}[\delta](x)$ of a BAN highlights the difficulty in interpreting the absence of movement, when $\mathscr{T}$ is the only information at hand.

Another possibility is that automata in $\mathscr{U}(x)$ are not updated at all in x. For instance, if δ is a simple update schedule, then the sequence of periodic updates defined by δ may *omit* altogether the update of $\mathscr{U}(x)$ (*i.e.* $\forall i \in \mathscr{U}(x), \delta(i) = \emptyset$),

---

[18] Indeed, given $j \in \mathbf{V}$ and the CNF definition of f_i, the problem of determining whether there exists $x \in \mathbb{B}^n$ such that $\mathrm{f}_i(x) \neq \mathrm{f}_i(\overline{x}^j)$ is NP-complete. It is easy to show that Problem SAT [18, 42] can be reduced to it by associating the local transition function $\mathrm{f}_i : x \in \mathbb{B}^{n+1} \mapsto \phi(y) \vee x_j \in \mathbb{B}$ (where $y = (x_0 \ldots x_{j-1}\, x_{j+1} \ldots x_n) \in \mathbb{B}^n$) for any instance $\phi(y)$ of SAT, with n literals.

[19] Let us note that knowing $\mathscr{T}$ implies knowing the size $n = |\mathbf{V}|$ of the system. In a modelling context, this often implies knowing the set of interacting elements that are modelled by the n automata of **V**, meaning in particular, that the *interior* and the *exterior* of the system have been identified so as to assume that all events observed have interior causes.

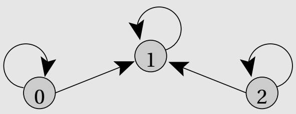
making ineffective the transition $x \twoheadrightarrow x = \mathrm{F}[\delta](x)$ although $u(x) > 1$.

Let us note however that with simple update schedules, contrary to fair update schedules, movement and fixity can be identified non-ambiguously. If transition $(x, y) \in \mathrm{T}$ is observed as a result of a whole period of updates by simple update schedule δ, then all automata $i \in \mathbf{V}$ that have not *apparently* changed states during this transition ($y_i = x_i$) have *effectively* not done so. In spite of this, simple update schedules $\delta :\equiv (\mathrm{W}_t)_{t \in \mathbb{Z}/p\mathbb{Z}}$ with periods $p > 1$ (*e.g.* block-sequential update schedules $\delta \neq \pi$) do not allow an *immediate* identification of the *elementary* periods of limit cycles. To loop on a configuration $x(0)$ belonging to a limit cycle that has "apparent" period k (*i.e.* a limit cycle of $\mathscr{T}$ induced by a set $\{x(t) \mid t \in \mathbb{Z}/k\mathbb{Z}\} \subseteq \mathbb{B}^n$ of k configurations) might require *any* number of effective elementary changes between k and $p \times k$, not necessarily $p \times k$ because the elementary version of $x(t) \twoheadrightarrow x(t+1) = \mathrm{F}[\delta](x(t))$ might involve null transitions.

Thus, according to the remarks above, generally, given $\mathscr{T}$, stable automata (and configurations) cannot be distinguished from unstable ones. With block-sequential update schedules (which are instances of both fair and simple update schedules), it is however possible to identify stable configurations since these are the fixed configurations of $\mathscr{T}$ (*cf.* (2.2) on Page 37). Notably, if δ imposes that transitions observed and recorded in $\mathscr{T} = \mathscr{T}_{[\delta]}$ involve exactly one update of each automaton, then instabilities are either "put to use" or "cancelled".

Example 4.2.

Consider the BAN $\mathcal{N}$ *with the structure and set of local transition functions:*



$$\begin{cases} f_0 : x \mapsto \neg x_0 \\ f_1 : x \mapsto x_0 \vee x_1 \vee x_2 \\ f_2 : x \mapsto \neg x_2 \end{cases}$$

and let $\delta \equiv \{1\}, \{0\}, \{1\}, \{0\}$ *and* $\delta' \equiv \{1\}, \{2\}, \{1\}, \{2\}$ *be two different periodic update schedules of* $\mathcal{N}$*. They induce the same global transition functions* $\mathrm{F}[\delta] = \mathrm{F}[\delta'] : x \mapsto (x_0\, 1\, x_2)$ *so the* BAN*'s behaviour under* δ *globally appears to be identical to its behaviour under* δ' *(*$\mathscr{T}_{[\delta]} = \mathscr{T}_{[\delta']}$*) although at the level of elementary events it is not (*$\mathscr{T}_{\delta} \neq \mathscr{T}_{\delta'}$*). In a configuration such as* $(0\,0\,0)$*, automaton* 1 *is activated for different reasons in each case. In the first, its activation is due to the momentary activation of* 0 *and the influence* $(2, 1)$ *is not implemented. In the second, it is due to the momentary activation of* 2 *and the influence* $(0, 1)$ *is not implemented.*

2.3 Witnessing causes

Under simple update schedules (and block-sequential update schedules in particular), *all* events that effectively occur are observed and recorded in $\mathscr{T}$. Each change underwent by the BAN results directly from 0, 1 or several other observed changes, although determining which exactly is not necessarily possible, except for asynchronous transitions (*cf.* Example 4.3). A consequence is that there is a "proof" of any underlying mechanism – *i.e.* of any application of a f_i resulting in the change $x_i \rightsquigarrow \mathrm{f}_i(x)$ – that is effectively "used" in a transition of $\mathscr{T}$. This property is exactly what allows Algorithm 1 below to compute (in time $\mathscr{O}(n^2 \cdot 2^n)$) the set of local transition functions of a BAN, given as input a simple δ and $\mathscr{T}_{[\delta]}$.

```
Algorithm 1: $\langle \mathscr{T}, \delta \rangle \rightsquigarrow \mathscr{F}$
  INPUT: the (deterministic) digraph $\mathscr{T} = (\mathbb{B}^n, \mathrm{T})$ of a function $\mathrm{F} : \mathbb{B}^n \to \mathbb{B}^n$ and
  a simple update schedule $\delta :\equiv (\mathrm{W}_t)_{t \in \mathbb{N}/p\mathbb{N}}$.
  OUTPUT: A set of local transition functions $\mathscr{F} = \{\mathrm{f}_i : \mathbb{B}^n \to \mathbb{B} \mid i < n\}$ such that
  $\mathscr{T} = \mathscr{T}_{[\delta]}$ and $\mathrm{F} = \mathrm{F}[\delta]$.
  forall $x \in \mathbb{B}^n$ do
      $y \longleftarrow \mathrm{F}(x)$;
      forall $i \in \mathrm{W}_0$ do
          $\mathrm{f}_i(x) \longleftarrow y_i$;
      forall $t < p$ do
          $x \longleftarrow \mathrm{F}_{W_t}(x)$;
          forall $i \in \mathrm{W}_{t+1}$ do
              $\mathrm{f}_i(x) \longleftarrow y_i$.
```

In the special case where $\delta = \pi$, this algorithm amounts to drawing the set of local transition functions directly from the transitions by setting [67, 129, 130]:

$$\forall (x, y) \in \mathrm{T}, \quad \mathrm{f}_i : x \mapsto y_i. \tag{4.2}$$

Similarly and more generally, when $\mathscr{T}$ is elementary, a set of n local transition functions that induce $\mathscr{T}$ can be computed in linear time ($\mathscr{O}(2^n n^2)$) with respect to the size of $\mathscr{T}$ by exploiting:

$$\forall i \in \mathbf{V}, \mathrm{f}_i : x \in \mathbb{B}^n \mapsto \begin{cases} \neg x_i & \text{if } \exists (x, y) \in \mathrm{T},\, y_i \neq x_i, \\ x_i & \text{otherwise,} \end{cases} \tag{4.3}$$

which simplifies to the following if $\mathscr{T}$ is asynchronous:

$$\mathrm{f}_i : x \in \mathbb{B}^n \mapsto \begin{cases} \neg x_i & \text{if } (x, \overline{x}^i) \in \mathrm{T}, \\ x_i & \text{otherwise.} \end{cases} \tag{4.4}$$

In the most general case, however, because of the possible non-elementariness of the transitions in T, $\mathscr{T}$ might not record all the relevant information concerning the effective BAN behaviour. As mentioned above, events may be missed due to reversible changes occuring between two observations of the BAN's configuration. As a result, the causes of some of the events that *are* observed may be hidden. And as Example 4.2 shows, there might be different implicit causes for the same explicit event.

Example 4.3.

Consider a BAN $\mathscr{N}$ *with automata set* $\mathbf{V} = \{0,1,2\}$ *which is known to obey to a specific periodic update schedule* δ. *Let us suppose that as a result of a whole period of updates of* δ*, it is performs transition:*

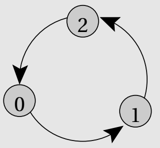
$$x = (0\,0\,0) \twoheadrightarrow (0\,0\,1) = \overline{x}^2.$$

Under these conditions, if in addition, δ *is known to be simple, then, since no other automata change states during this transition, the change of states of automaton* 2 *necessarily results from its immediate update in configuration* x. *Because* δ *is simple, there are no hidden causes for this event. And if* f_2 *is unknown, then it can be derived from the sole observation of transition* $(x, \overline{x}^2)$ *that* $\mathrm{f}_2(x) = 1$. *In other terms, it can be derived that something in* x *– the inactivity of any* $i \in \mathbf{V}$ *or any combination of these inactivities – causes automaton* 2 *to activate. So* $\mathscr{N}$ *must involve at least one negative influence of an automaton* $i \in \mathbf{V}$ *on* 2.

On the contrary, if δ *had been an arbitrary periodic update schedule, for instance the* 2*-fair update schedule* $\delta \equiv \{0\},\{1\},\{2\},\{0\},\{1\}$, *then transition* $(x, \overline{x}^2)$ *could have summed up the following series of elementary transitions:*

$$x \longrightarrow (1\,0\,0) \longrightarrow (1\,1\,0) \longrightarrow (1\,1\,1) \longrightarrow (0\,1\,1) \longrightarrow \overline{x}^2$$

in consistency with the structure and local transition functions on the right (and in contradiction with the existence of a negative arc $(i,2) \in \mathbf{A}$, $i \in \mathbf{V}$ *implied by the simplicity of* δ *above). In this case, automaton* 2 *would have changed states because of a more complex set of causes consisting in a series of consecutive events, some of which being reversed between x and* $\overline{x}^2$ *and as a consequence, unobserved.*



$$\begin{cases} f_0 : x \mapsto \neg x_2 \\ f_1 : x \mapsto x_0 \\ f_2 : x \mapsto x_1 \end{cases}$$

2.4 On the observer's information

Thus, the "observation protocol" might be responsible for a lack of information which in turn prevents the deriving of supplementary information without ambiguities. This *incompletion issue* is doubled with the issue of what the observer is aware of. If the updating δ of the observed BAN is known to be simple, then Algorithm 1 is assured to produce a correct set of local transition functions $\mathscr{F}$ inducing $\mathscr{T}$ and so are Equations (4.2), (4.3) and (4.4) in the case where transitions of $\mathscr{T}$ are known to be elementary. On the contrary, if δ is known to be fair, then the utilisation of Algorithm 1 is known to be unsafe. And if $\mathscr{T}$ is provided without any supplementary indication concerning the granularity, nature and order of events represented by its transitions and derivations, then no information (such as $\mathscr{F}$ and $\mathbf{G}$) can safely be drawn from $\mathscr{T}$. To compensate for this incompletion and to go any further, $\mathscr{T}$ needs to be backed up with some *assumptions*.

In a modelling context, this is non-negligible because the representational capacity of information drawn on the basis of these assumptions will be bounded by them in a way that can be significant (*e.g.* if a non-elementary transition is taken for an elementary one and if an atomic event is mistaken for a series of non-atomic ones and *vice versa*) [85]. The pertinence of causal relationships put forward this way will be strongly dependent on their validity. As an immediate example, consider using (4.2) to infer the set of local transition functions of a BAN that is actually updated with a block-sequential update schedule δ. This amounts to mistaking the BAN $\mathscr{N}[\delta] = \{\mathrm{f}[\delta]_i\}$ with structure $\mathbf{G}[\delta]$ updated in parallel for the BAN $\mathscr{N} = \{\mathrm{f}_i\}$ with structure $\mathbf{G}$. This raises the question addressed in Section 3 of how a BAN is defined: by its underlying mechanisms or by the way that it is observed to behave[20].

2.5 Mechanisms implementation & updating related contexts

Even when the observation protocol is known and allows to witness every change that occurs, there remains a third possible source of incompletion. Rather than containing a proof of *every* underlying, defining mechanism of the BAN, $\mathscr{T}$ just contains a proof of those that are implemented by δ. This might make a difference if δ is an arbitrary simple update schedule since in some configurations, some instabilities might then be ignored and the local transition functions pro-

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[20] Deriving information from $\mathscr{T}$ also raises the question of the realism/pertinence of the utilised information at hand. For instance, consider adapting Algorithm 1 so that instead of the whole definition of δ, it only needs to be given its set of inversions $\mathrm{I}[\delta]$ (*cf.* Section 1.1, Chap. 3). This amounts to inputting the characterisation of an equivalence class of $\asymp$ and seems natural because the definition of $\asymp$ is natural. But then, in addition to $\mathbf{V}$, $\mathbf{A}$ needs to be known. In other terms, the structure $\mathbf{G}$ of $\mathscr{N}$ needs to be known. This seems to be a lot of information for a network whose mechanisms are unknown and intended to be revealed by an analysis of $\mathscr{T}$.

duced by Algorithm 1 might only be partially defined. Also, $\mathscr{T}$ might be elementary but the updating δ responsible for it, might not make use of *every* existing mechanism of the BAN (*cf.* Example 4.2). In this case, $\mathscr{T}$ is not the GTG and the set $\mathscr{F}$ of local transition functions that might be drawn using (4.3) will incompletely describe the set of *all* possible mechanisms of the BAN (*cf.* examples in [85]).

On the contrary, if δ is block-sequential, then this is not the case because δ is precisely an instance of both fair and simple update schedules. Indeed, during any transition $(x, y) \in \mathrm{T}$, at some point, δ "puts into action" *every* existing underlying mechanism of the BAN, if it is actually possible. And it does so in a directly observable manner (*i.e.* only once so as to ensure that changes are reversed before the end of the transition). The combination of these two properties results in *every* mechanism of the system leaving a trace in $\mathscr{T}$ whenever it potentially can cause an effect.

Importantly in these lines let us note that the updating can be taken to play the role of a "**context**", forbidding the occurrence of some events while permitting that of others[21]. Notably, the principal and most direct way to *formalise* a notion of "exterior influence" or "context-dependency" *in* the theory of BANs is through update modes. Indeed they are the only formal objects that can impose constraints on the behaviours of BANs in a manner that can be considered as "exterior" because *their definitions are independent of that of* BANs (*cf.* Example 4.4). This is supported precisely by the *context-dependent* transition graph $\mathscr{T}_\delta$ associated to them (*cf.* Section 4.3, Chap. 1).

2.6 Different causes, same effects

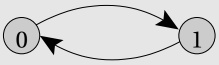
The previous paragraphs evidence the importance of the information held by the definition of the update mode δ. To end this section and go further in these lines, let us now assume that both $\mathscr{T}$ and the set of local transition functions $\mathscr{F}$ of the BAN behaving according to $\mathscr{T}$ are given (rather than $\mathscr{T}$ and δ as in Algorithm 1). Then, the update mode δ that induces $\mathscr{T}$ cannot usually be inferred. Indeed, with the knowledge of both $\mathscr{T}$ and $\mathscr{F}$ the update schedules of Example 4.2 cannot be distinguished and obviously, nor can two basically equivalent simple update schedules $\delta \propto \delta'$ (*cf.* Section 1.1, Chap. 3). Moreover, in [7], an example is given of a strongly connected BAN $\mathscr{N}$ whose global behaviour is identical under two different non-basically equivalent block-sequential update schedules, *i.e.* $\mathscr{T}_{[\delta]} = \mathscr{T}_{[\delta']}$ but $\neg(\delta \propto \delta')$. This implies that $\mathscr{F}$ and $\mathscr{T}_{[\delta]}$ are not even enough to derive the set of inversions $\mathrm{I}[\delta]$ when the BAN is known to be updated with a simple update schedule (but by Corollary 3.1, it can in the special case of BACs).

---

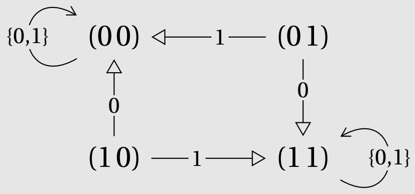
[21] Note, however, that the restrictions that can be represented through an update mode δ cannot just forbid some influences to act on a particular automaton in a particular situation (*cf.* Section 3.1).

Example 4.4.

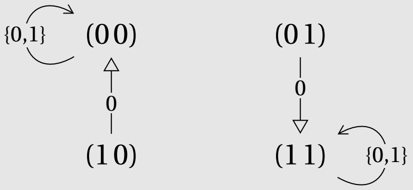
Let $\mathcal{N} = \{\mathrm{f}_0 : x \mapsto x_1, \mathrm{f}_0 : x \mapsto x_0\}$ *be a positive* BAC *of length* 2 *with the following structure* $\mathbf{G} = (\mathbf{V}, \mathbf{A})$*:*



Assume that for some reason, $\mathcal{N}$ *can only perform asynchronous transitions and that it can potentially perform them all, the behaviour of* $\mathcal{N}$*, in the absence of any exterior perturbations, is best described by its* ATG*:*



But, if $\mathcal{N}$ *is subjected to some exterior forces that incite active automata to be updated or to change states faster than inactive ones, then transitions* $(0\,1) \xrightarrow{0} (1\,1)$ *and* $(1\,0) \xrightarrow{1} (1\,1)$ *in the* ATG *may be considered as unlikely. Also the behaviour of* $\mathcal{N}$ *might be restricted in the sense that automaton* 0 *tends to be much faster in switching states than automaton* 1*. Then,* $\mathcal{N}$ *behaves as if it were obeying to the sequential update schedule* $\sigma := \{0\}, \{1\}$ *and its behaviour is:*



In the transition graph $\mathcal{T}_{[\sigma]}$ *above, the influence that automaton* 0 *has on automaton* 1 *is not revealed at all because of the assumed precedence of* 0*-updates over* 1*-updates. Mistaking this graph for the* ATG *would thus yield an incomplete description of the structure of* $\mathcal{N}$*, lacking influence* $(0,1) \in \mathbf{A}$.

3 Defining networks

In the previous section, the question of how systems must be defined arises several times. As a conclusion to this chapter, and on the basis of the series of remarks made above, we now propose to consider the problem of formally defining networks and the implications of the different ways that this can be done. We recall that at the beginning of this document, in Section 2.3, Chap. 1, we defined BANs as follows:

DEF 0 : definition of BANs

A BAN $\mathcal{N}$ *of size n is a set of n Boolean functions :* $\mathcal{N} = \{\mathrm{f}_i : \mathbb{B}^n \to \mathbb{B} \mid i < n\}$.

This choice of definition is disputable for several reasons.

3.1 Structural defining

First, let us confront it to a less restrictive defining of BANs via their structures $\mathbf{G} = (\mathbf{V}, \mathbf{A})$. To do this highlights how DEF 0 implicitly assumes that, under all circumstances, *an automaton either obeys all of its influences at once, or it obeys none at all.* In other words, in each configuration, an automaton $i \in \mathbf{V}$ is either not updated or it is in which case none of its influences $(j, i) \in \mathbf{A}$ can be ignored. This restriction cannot be bypassed by playing on the definition of the update mode[21]. However, it can be partially resolved by a judicious design of the local transition functions f_i of the BAN. For instance, if originally $\mathrm{f}_i : x \mapsto x_j$ makes i depend on j in all circumstances, then i can be made independent of j when j is inactive by replacing f_i by $\mathrm{f}'_i : x \mapsto x_j \vee x_i$. But this new independence of i regarding j can nevertheless only be added relatively to a given (set of) configuration(s)[22].

This leads to noting that DEF 0 also implicitly assumes the non-trivial hypothesis imposing that *the result of the interactions taking place in a given configuration depends* only *on that configuration.* They are otherwise predetermined. For instance, consider a BAN $\mathcal{N}$ in which automaton i positively and effectively influences automaton j in configuration $x \in \mathbb{B}^n$: $\mathrm{f}_j(x) = x_i = 1 \neq x_j = 0 = \mathrm{f}_j(\overline{x}^i)$. Then, when $\mathcal{N}$ is in configuration x, whatever its current environment, if the influence (i, j) is implemented (*i.e.* if j is updated in x), then i causes j to activate. DEF 0 imposes that the nature of the influence represented by arc $(i, j) \in \mathbf{A}$ depends only on x; it forbids any situation consistent with x in which the active automaton i deactivates j: in x, i has only one, predetermined way of influencing j.

Defining networks by means of their structures $\mathbf{G} = (\mathbf{V}, \mathbf{A})$ specifies the *existence* of inter-automata connections rather than also specifying the *natures* of the influences that they represent as DEF 0 does. A consequence is that the two limitations highlighted above can be avoided. This less restrictive structural defining may turn out to be more appropriate (or equivalent, *e.g.* positive DANs) in some contexts (*e.g.* modelling social networks [58]) where different questions are addressed that focus more on events concerning influences (mainly, whether they are effective/*functional* or not in a given situation) than on events concerning automata as we do here (by concentrating on whether they are updated or not in a given configuration). Thus, with regards to a structural defining, the pertinence of DEF 0 remains disputable and subjected to the precise problems that are of interest.

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[22] More generally, automaton i can be made independent of j's influence in configuration $y \in \mathbb{B}^n$ by simply replacing f_i by $\mathrm{f}'_i : x \mapsto \big(\mathrm{f}_i(x) \wedge (x \neq y)\big) \vee \big(\mathrm{f}(y) \wedge (x = y)\big)$ where $\mathrm{f}(x) = \mathrm{f}_i(x)[{}^0/_{x_j}, {}^0/_{\neg x_j}]$ is obtained by replacing every occurrences of both the literal x_j and its negation $\neg x_j$ by 0 in the expression of $\mathrm{f}_i(x)$ (*e.g.* if $\mathrm{f}_i(x) = \big(x_1 \vee \neg x_2 \vee \neg x_4\big) \wedge \big(x_2 \vee x_3\big)$, then $\mathrm{f}(x) = \big(x_1 \vee \neg x_4\big) \wedge x_3$).

3.2 Behavioural defining

DEF 0 can also pertinently be confronted to a defining of BANs via their behaviours. Indeed, some of the issues emphasised in Section 2 may be addressed by simply deciding that "a system is what it does". But depending on how we come to be acquainted with the behaviour of a network, on whether the "behaviour of a network" is taken to mean "the set of *all* of its potential behaviours" or not and whether it implies "in all possible circumstances" or not, our resulting understanding of the system may substantially differ. Section 2 endeavoured to show that the causality relations uncovered from a given transition graph $\mathscr{T}$ (which is supposed to represent the way a system works, precisely) intricately depends on how these questions are answered. So to settle on defining a system by the way it works leaves the important question of how to define the way a system works.

In the continuity of the discussion of Section 2, such an *outcome-oriented perspective* straightforwardly calls for a defining of BANs by their behaviours. For the sake of the clarity of the rest of this discussion, let us use the term "**DEF$_\mathrm{b}$0**" to refer to this behavioural definition by which a BAN $\mathscr{N}$ is the transition graph $\mathscr{T}$ describing its behaviour. Under the general point of view assuming that a network is what it outputs, DEF$_\mathrm{b}$0 can be seen as an *extensional* definition of systems, *i.e.* of BAN behaviours, while *a priori* DEF 0 is *intensional* in the sense that rather than listing the set of possible transitions and behaviours of a BAN $\mathscr{N}$, it comprehensively defines the mechanisms that produce them. However, choosing DEF$_\mathrm{b}$0 instead of DEF 0, requires a different understanding of the latter. In the absence of any additional specification (such as an updating requirement), the information held by the set $\mathscr{F}$ of local transition functions of a BAN $\mathscr{N}$ is equivalent to the information held by its GTG[23]. As a consequence, under DEF$_\mathrm{b}$0, DEF 0 defines the subset of systems whose behaviours are described by their GTGs. Let us make a series of remarks to emphasise how DEF$_\mathrm{b}$0 settles some of the problems mentioned above deriving from the choice of an outcome-oriented view on systems.

First, to define a BAN extensively by its behaviour $\mathscr{T}$ requires to make implicit the part of $\mathscr{T}$ that is due only to the observation protocol. As a result, such a defining fails to equate two different observations of the same network behaving identically (*e.g.* if $\delta \asymp \delta'$, $\mathscr{T}_{[\delta]}$ and $\mathscr{T}[\delta']$ will be interpreted as very different systems).

DEF$_\mathrm{b}$0 also disregards the opposite possibility that different causes may have the same effect. In particular, it requires to identify any two networks that produce the same effect *under the present conditions* (*cf.* Section 2.4). And it also imposes to confuse different situations in which the same network outputs the same be-

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[23] Given a set of local transition functions $\mathscr{F} = \{\mathrm{f}_i \mid i < n\}$, the corresponding GTG can be built in linear time $\mathscr{O}(2^{2n} n)$ (*cf.* (4.3) for the converse).

haviour for different reasons (*cf.* Section 2.6 and Example 4.2). One reason why this might be significant is mentioned at the end of Section 1.1, Chap. 3: in a general context where the possibility of exterior influences is not disregarded, two systems that do not involve the same mechanisms (*e.g.* $\mathscr{N}[\delta]$ under π and $\mathscr{N}$ under δ) or more simply that just do not transit through the same configurations, may be perturbed differently and thus eventually experience very different evolutions.

Even without considering the possibility of exterior influences not accountable in the theory of BANS, $\text{DEF}_\text{b}0$ still imposes a similar restriction through the notion of updating (*cf.* Section 2.5). Indeed, it implies that a system can only behave according to the same transition graph $\mathscr{T}$. Unless $\mathscr{T}$ is the BAN's GTG and the outcome-oriented perspective is dropped (which amounts to adopting DEF 0^{23}), $\mathscr{T}$ describes the behaviour of a BAN *submitted* to a certain constraint which takes the formal form of the update mode which is associated to $\mathscr{T}$. So by equating $\mathscr{N}$ and $\mathscr{T}$, $\text{DEF}_\text{b}0$ presumes that this constraint is unavoidable, *i.e.* that there are no different conditions in which $\mathscr{N}$ can be set and observed, in which it may behave differently than what is prescribed by $\mathscr{T}$. These two ways that $\text{DEF}_\text{b}0$ has of implicitly denying the possibility of a significant environment, limits the potential expressiveness and scope of BANS.

Now, on the one hand, when the systems that are considered and represented by BANS have not been engineered by ourselves and are thus very poorly understood, an intensional defining usefully agrees with the need for some reverse-engineering. On the other hand, the extensional, outcome-oriented point of view imposed by $\text{DEF}_\text{b}0$ is natural in many frameworks, especially when systems are *used* rather than analysed or when they are embedded in larger systems and represent modules of these that are not intended to be decomposed. It also is natural in a modelling context where the problems of interest first require the formalisation of an observed system behaviour in terms of a transition graph $\mathscr{T} = \mathscr{T}_\text{obs}$. Notably, as mentioned in the introduction of the present chapter, in this case, what is modelled is some observations of a system's transitions and not its effective dynamical changes nor its underlying mechanisms. A modelling of the latter must be inferred from $\mathscr{T}$. And as Section 2 demonstrates, this inference can only be made with some complementary information or, for lack of which, with some unproven[24] assumptions. These assumptions embody the fact that the modelling beyond $\mathscr{T}$ is contingent on the way the system is observed (to produce $\mathscr{T}$), on the circumstances in which it is observed, and on the way the ambiguity due to the various sources of incompletion (of the sort highlighted in Section 2) in the

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[24] At most, the validity or plausibility of these assumptions can only be *supported* by some informal arguments. And these can neither issue only from the reality of the observed system nor be drawn from its theoretic model alone. They must however be consistent with both (*cf.* Section 1, Chap. 6). [85] discusses the difficulty of pertinently putting forward, justifying and using hypotheses that are indispensable but also essential for the modelling process.

information at hand is chosen to be compensated. In this context, an extensional defining of networks tends to “hardwire” these contingencies within the very definition of the system. In other terms it incites the tacit transferring of these unavoidable – although essentially useful – assumptions into a pre-accepted BAN definition. A consequence of this is that instead of being put forward and exploited *perse*, the assumptions turn into unquestioned, implicit approximations. On the contrary, with an intensional defining, the expliciting of assumptions of the sort uncovered in Section 2 cannot be bypassed. Our choice of DEF 0 is precisely motivated by this remark and by the fact that in a theoretical framework where supplementary assumptions are not always needed, DEF 0 allows for the most general point of view, potentially considering *all* possible transitions of a BAN, *i.e.* all those figuring in its GTG.

Let us highlight however that our choice, as any choice, remains essentially disputable and dependent on our subjective perspective on the problems we approach and on the intention we lend them. Indeed, to motivate this choice we have put forward application-oriented arguments. Absolutely speaking, it remains that our choice, as any choice, imposes a specific point of view which, by definition, is not self-sufficient. If only to highlight its own implicit implications, it needs to be confronted with different points of view. But much more than that, from a purely theoretic angle, it can undoubtedly benefit positively from being compared to and completed by other approaches, including those that adopt a stance closer to DEF$_{b}$0, some of which have already undeniably proven their pertinence and provided important results for our context (in a very non-exhaustive fashion, we can cite some of the works that can perhaps most immediately be related with the work presented in this thesis: [92, 93, 94, 96, 97, 98, 99, 102, 103, 108, 109]).

5

GENERAL TRANSITION GRAPHS

As mentioned in Section 2.5, Chap. 4, update modes have the notable advantage of being the only objects *in* the theory of BANs that are *not* involved in the definitions of BANs and that can account for *environmental influences*. They also are especially pertinent because the behaviours that they induce give indications on the behavioural properties of BANs (*cf.* Section 8, Chap. 2). On the one hand, many studies [6, 7, 34, 47, 50, 82] as well as the previous chapters have emphasised how substantially can updatings impact on BAN behaviours. Below, Section 2 and Example 5.3 go further in these lines by showing how the slightest non-atomicity of an update can make an appreciable difference in terms of limit behaviours[25]. The choice of one specific updating method is therefore undoubtedly consequential. On the other hand however, in practice, justifications in favour of one updating rather than another may be hard to provide – unless they are chosen on the grounds of the convenience that they allow in formal developments and the insights that they provide on specific properties for more general settings (as I claim is the case with π in Chapter 2). In cases where an update schedule *can* indeed convey some of the reality of an observed system behaviour, Example 5.1 considers the result of assuming (reasonably) that errors can occur in the implementation of the update sequence (*e.g.* the omission of the update of one or several automata from time to time), even if very rare. Provided this possibility of occasional updating disruptions, Example 5.1 reveals – using GTGs – that some behaviours induced by deterministic update schedules may in

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[25] Results of this chapter were previously presented in [79] (for Section 1), [82, 84] (for Section 2).

fact be less likely and thus, less meaningful from a modelling point of view. It suggests that the apparent stability of some behaviours may only be an *artifact* of the update mode.

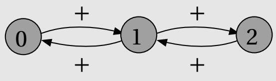
Thus, two notable features of GTGs may be put forward to support their pertinence. The first is that in a framework where the aim is to represent some real

Example 5.1. The floral morphogenesis of *Arabidopsis thaliana*

In [75], Mendoza and Alvarez-Buylla introduced a BAN *to model the genetic regulation of the floral morphogenesis of plant Arabidopsis thaliana. We denote by $\mathscr{M}$ the simplified version of this* BAN *that was given in [29, 28].*

Under all block-sequential update schedules, and in particular under the one studied in [75], $\mathscr{M}$ has six fix points which model effective or hypothetical cellular types of the plant [75]. Under the parallel update schedule π, it has seven additional limit cycles of period 2 [107] which have no known biological meaning.

To decrease further the number of configurations involved, let us focus only on the two distinct sub-networks, $\mathscr{M}_1$ and $\mathscr{M}_2$ of $\mathscr{M}$ whose structures are respectively[a]*:*



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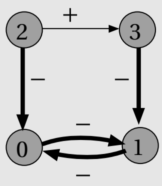
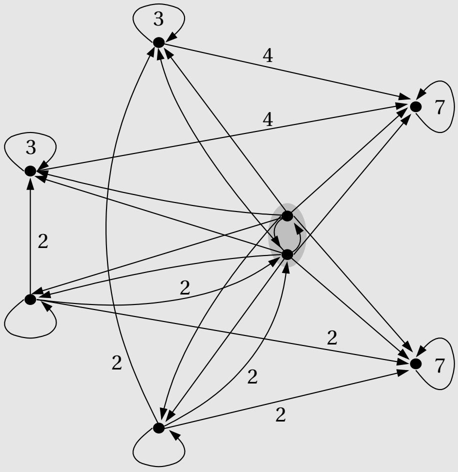


and whose GTG*s, $\mathscr{T}_1$ and $\mathscr{T}_2$ are respectively:*



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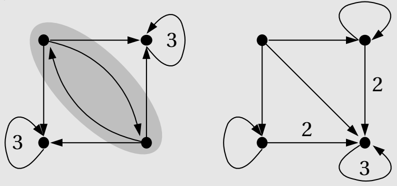

$x_2 = x_3 = 0$ $x_2 = x_3 = 1$

where synchronous transitions (x, y) are labelled by $|\mathscr{P}(\overline{\mathscr{U}}(x) \setminus \mathrm{D}(x, y))|$, the number of different updates that can be made to perform that particular transtion and where, because node 3 takes the same state as node 2 as soon as it is updated, $\mathscr{T}_2$ is decomposed into two parts: the one on the left (resp. right) involving configurations $x \in \mathbb{B}^4$, $x_2 = x_3 = 0$ (resp. $x_2 = x_3 = 1$).

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[a] Where nodes $0, 1, 2$ of $\mathscr{M}_1$ and nodes $0, 1, 2, 3$ of $\mathscr{M}_2$ respectively correspond to AP3, BFU, PI and AP1, AG, EMF1, TFL1 in [75, 107].

Example 5.1. (continued) The floral morphogenesis of *Arabidopsis thaliana*

Nodes of $\mathscr{M}$ outside $\mathscr{M}_1$ and $\mathscr{M}_2$ can be ignored: because they take state 0 after at most a couple of updates and because of their local transition functions (threshold functions[16]*) they end up having no definite influence on other nodes. $\mathscr{M}_1$ and the* SCC *in $\mathscr{M}_2$ are the only two non-trivial* SCC*s of $\mathscr{M}$, acting as the "motors" of its dynamics. Under π, in any attractor $\mathscr{A}$ of $\mathscr{M}$, both $\mathscr{M}_1$ and $\mathscr{M}_2$ are in one of their own attractors (i.e. they behave as they would do if they were isolated from the rest of $\mathscr{M}$). In particular, the seven limit cycles of $\mathscr{M} = \mathscr{M}[\pi]$ are induced by the limit cycles of $\mathscr{M}_1 = \mathscr{M}_1[\pi]$ and $\mathscr{M}_2 = \mathscr{M}_2[\pi]$ (shadowed in grey in their* GTG*s).*

For the sake of intuition, let[b] $\mathrm{R}(\mathrm{X}) = 1/\deg^+(\mathrm{X})$ *and* $\mathrm{L}(\mathrm{X}) = \deg^-(\mathrm{X})/|\mathrm{T}|$ *respectively denote the "robustness" and "likeliness" of $\mathrm{X} \subseteq \mathbb{B}^n$ in the* GTG $\mathscr{T} = (\mathbb{B}^n, \mathrm{T})$. *With these criteria, in $\mathscr{T}_1$ and $\mathscr{T}_2$, sets of fixed points have significantly greater robustness and likeliness than the sets of shadowed configurations. This implies that on the one hand, the fix points of $\mathscr{M}$ are "robust" and "likely" in the sense that many elementary transitions lead to them and none leave them. And on the other hand, not only very few transitions lead to the limit cycles of $\mathscr{M}[\pi]$, but also, there exist some leading away from them. Starting in an arbitrary configuration, $\mathscr{M}[\pi]$ has therefore very little chances to end in a limit cycle, and if ever it does however, it is very likely to leave it eventually.*

Let us assume that π does indeed best represent the updating in the real system modelled by $\mathscr{M}$. But let us also assume the possibility of updating errors. Thus, a transition $x \xrightarrow{\mathrm{W} \subsetneq \mathbf{V}} F_{\mathrm{W}}(x)$ may be performed instead of $x \xrightarrow{\mathbf{V}} F_{\mathbf{V}}(x)$ because of the omission of the update of automata in $\mathbf{V} \setminus \mathrm{W}$. *With this assumption, the discussion above suggests that, contrary to its fix points, the limit cycles of $\mathscr{M}$ under the parallel update schedule are highly improbable to be reached and maintained over time. So they can only model behaviours of the real system that also are highly improbable to be observed effectively.*

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[b] Where $\forall \mathrm{X} \subseteq \mathbb{B}^n$, $\overline{\mathrm{X}} = \mathbb{B}^n \setminus \mathrm{X}$, $\deg^+(\mathrm{X}) = |\mathrm{T} \cap (\mathrm{X} \times \overline{\mathrm{X}})|$ and $\deg^-(\mathrm{X}) = |\mathrm{T} \cap (\overline{\mathrm{X}} \times \mathrm{X})|$.

systems that are not known well enough to determine precisely their contexts of evolution, it may be more appropriate or safer to avoid *any* restriction on the BAN behaviours considered, rather than to assume that the same pre-defined updating rules are infallibly and exactly respected. GTGs embody the possibility to do so in the theory of BANs. In line with the discussion of the previous chapter, they ensure the maximal generality in this sense (as far as the theory allows), maintaining the eventuality that the system behave differently under different conditions that can or not be formalised in terms of updating modes or more generally in the terms of the theory of BANs. In that, GTGs can be considered as conveying more complete and realistic information on network behaviours than do other transition graphs (*e.g.* $\mathscr{T}_{[\delta]}$ or the ATG) whose restrictions to the set of possible transitions deprive them from self-sufficiency.

Of course, GTGs have one major drawback: for BANs of size n they have 2^n nodes and up to $2^n \times (2^n - 1)$ arcs, that is, $2^n - 1$ arcs more than the transition graphs $\mathscr{T}_{[\delta]}$ induced by deterministic update schedules. So GTGs cannot reasonably be computed for BANs of arbitrary sizes and thus, they cannot be used directly to model the behaviour of a specific real system, unless its size is small[26].

In the framework of this thesis, GTGs have a second notable property which prevails their representational capacity and (non-)practicability. By their exhaustiveness they are a means of studying further and more generally how updatings impact on BAN behaviours. Using them seems a proper starting point to explore with the largest possible encompassing view, what properties depend intrinsically on update modes (*e.g.* the attractors induced by π in Example 5.1) and what are rather dependent on other BAN features such as their structures (*e.g.* the effect of cycle intersections under π, *cf.* Section 8, Chap. 2). And again, the possibility provided by update modes to represent a *context* for BAN evolutions as well as the privileged relation that they have with a formal notion of time flow (*cf.* Section 1.5, Chap. 6) ascertain the pertinence of these questions and of the study of GTGs.

Before we move on, let us give an example of how GTGs can shed light on updating related problems. Consider the classes of (threshold) BANs defined in [35] (and copied to define the classes of positive DANs introduced and studied in Section 3.2, Chap. 3). **Fi** (resp. **Cy**) groups together the BANs that have no limit cycles (resp. no fix points) under all block-sequential update schedules[27]. **Mi** groups together those that have both fix points and limit cycles under all block-sequential update schedules. And **Ev** contains the remaining BANs which have fix points and, under some update schedules have limit cycles and under others have none. Elena studied in particular the sizes of these classes depending on the sizes n of BANs (with $n \leq 7$). An analysis of the resulting statistics reveals a strong correlation between the evolutions of the proportions of classes **Fi** and **Ev** when n grows. Confronting behaviours induced by block-sequential update schedules to the behaviours exhaustively described by GTGs, this correlation immediately becomes very straightforward: in GTGs, classes **Fi** and **Ev** cannot be distinguished. Indeed,

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[26] But in the absence of evidence in favour of one particular updating, nor can any other transition graph; the sake of exhaustiveness would require to consider them all which would end up being more costly than to concentrate on the GTG (*cf.* the counting of block-sequential update schedules on Page 155 of the appendix).

[27] We recall again that the set of fix points of a BAN under a block-sequential update schedule equals its set of stable configurations and is the same under all such update schedules.

let $\mathcal{N} \in \mathbf{Ev}$ and let $\mathcal{A}$ be a limit cycle of $\mathcal{N}$ updated with δ (*i.e.* a cycle in $\mathcal{T}_{[\delta]}$). If under $\delta' \neq \delta$, $\mathcal{N}$ has no limit cycles, then $\mathcal{A}$ is transient in $\mathcal{T}_{[\delta']}$ and for all configurations $x \in \mathcal{A}$, there is a derivation $x \longrightarrow\!\!\!\!\!\!\rightarrow y$ in $\mathcal{T}_{[\delta']}$ leading from x to a stable configuration y. This derivation exists in the GTG. As a consequence, no configurations of $\mathcal{N}$ are recurrent except the fix points/stable configurations that can be observed under all block-sequential update schedules. With respect to GTGs, the class of BANs that have no stable configurations equals **Cy**. The class of BANs that only have stable configurations as attractors contains both the classes **Fi** and **Ev**. And the class of BANs that have both is included in **Mi**.

1 General transition graphs of cycles

Thomas [118] conjectured that negative cycles are necessary to have oscillating attractors and this was confirmed in several different contexts [52, 63, 86, 94, 99, 114]. But in the context of BANs submitted to periodic update schedules, the results of Chapters 2 and 3 contradict this conjecture. Positive cycles can also induce limit cycles. And what is more, they can induce much more than do negative cycles of the same size. In the terms of Section 2, Chap. 2, they are responsible for greater degrees of freedom. The present section characterises the GTGs of BACs. This way, it shows how considering all elementary transitions lifts this contradiction with Thomas' conjecture. Doing so, it highlights the importance of the notion of instabilities.

Let us recall again that by Lemma 1.1 in Section 5, Chap. 1, we can concentrate on canonical BACs. What is more, by the definitions of the relation $\bowtie$ and of GTGs, there is an isomorphism between the GTGs of any two BACs of same signs and sizes preserving the number $u(x) = |\mathcal{U}(x)|$ of unstable nodes in x. Also, the following lemma holds (it can be proven using function $x \in \mathbb{B}^n \mapsto xx \in \mathbb{B}^{2n}$):

Lemma 5.1. Simulation of a negative BAC by a positive BAC

There is an isomorphism between the GTG *of a negative* BAC $\mathcal{C}_n^-$ *of size* n, $\forall n$, *and a sub-graph of the* GTG *of any positive* BAC $\mathcal{C}_{2n}^+$ *of size* $2n$:

$$\mathcal{C}_n^- \lhd \mathcal{C}_{2n}^+$$

which maps configurations $x \in \mathbb{B}^n$ *of* $\mathcal{C}_n^-$ *with* $u(x)$ *unstable nodes to configurations* $y \in \mathbb{B}^{2n}$ *of* $\mathcal{C}_{2n}^+$ *with* $u(y) = 2u(x)$ *unstable nodes.*

The results in the following Lemma appear in [92] (a simple proof is also given on Page 153 of the appendix). They give some useful properties of $u(\cdot)$ using the following notation:

$$u_{min} = \min\{u(x) \mid x \in \mathbb{B}^n\} \quad \text{and} \quad u_{max} = \max\{u(x) \mid x \in \mathbb{B}^n\}.$$

Lemma 5.2. Number of instabilities of BACs

A BAC $\mathscr{C}_n^s$ *of size n and sign* $s \in \{-,+\}$ *satisfies:*

	$s = +$	$s = -$
$u(x)$ *is:*	*even*	*odd*
$u_{min} =$	0	1
$u_{max} =$	$\begin{cases} n & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$	$\begin{cases} n-1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$

Let $\mathscr{C} = \mathscr{C}_n^+$. Any configuration $x \in \mathbb{B}^n$ of $\mathscr{C}$ can be written as a sequence of m alternating "blocks" of consecutive 0s or consecutive 1s:

$$x = 1^{i_0} 0^{n_0} 1^{n_1} \ldots 0^{n_{m-2}} 1^{n_{m-1}-i_0} \tag{5.1}$$

where $\sum_{0 \le k < m} n_k = n$ and where i_0 denotes the first automaton of the first block of 0s. Obviously, $u(x)$ equals the number m of blocks and $\mathscr{U}(x) = \{i \mid x_i \neq x_{i-1}\} = \{i_k \mid k < m\}$ is the set of first automata $i_k = i_0 + \sum_{z<k} n_z$ of each block $B_k = \{i_k, i_{k+1}, \ldots, i_{k+n_k-1}\}$, $k < m$. And as a consequence, $\forall m \in \mathbb{N}$, the set $\mathrm{X}_m = \{x \mid u(x) = m\} \subseteq \mathbb{B}^n$ has cardinal:

$$|\mathrm{X}_m| = 2\binom{n}{m} \tag{5.2}$$

(once $\mathscr{U}(x)$ and $\{i_k\}$ have been chosen, it remains to choose $x_{i_0} \in \mathbb{B}$).

During any elementary transition $x \xrightarrow{\mathrm{W}} y$, $\mathrm{W} \subseteq \mathbf{V}$, blocks can only change at their frontiers. No new block can therefore be created inside a pre-existing block and it cannot either be at the frontier of two consecutive blocks B_{k-1}, B_k: the only possible change that can happen there requires $i_k \in \mathrm{W}$ to be updated so that, as a consequence, i_k is lost by B_k and gained by B_{k-1}. This implies that y has no more blocks than x, *i.e.* $u(y) \le u(x)$ and more generally:

$$\forall x, y \in \mathbb{B}^n,\ x \twoheadrightarrow y \Rightarrow u(y) \le u(x) \tag{5.3}$$

which suggests that $u(\cdot)$ can be seen as a potential energy for the whole system.

Now, let us show that in the GTG of $\mathscr{C}$, any $x \in \mathbb{B}^n$ written as in (5.1) (with $u(x) = m$ blocks) is strongly connected with configuration $\mathbf{x}^{(m)}$ in which all blocks of x have been reduced to size 1 except the first which has been augmented to size $r = n - (m-1)$:

$$\mathbf{x}^{(m)} = 0^r (10)^{\frac{m}{2}-1} 1 \in \mathbb{B}^n. \tag{5.4}$$

To do so, first let $\mathrm{F} = \mathrm{F}_{\mathbf{V}} = \mathrm{F}[\pi]$ and:

$$\widetilde{x} = \mathrm{F}^{n-i_0}(x) = 0^{n_0} 1^{n_1} \dots 0^{n_{m-2}} 1^{n_{m-1}} \in \mathbb{B}^n.$$

By definition of transitions and because $\mathrm{F}^{n-i_0} = (\mathrm{F}^{i_0})^{-1}$ is invertible, x and $\widetilde{x}$ are strongly connected in the GTG $\mathscr{C}$:

$$\forall x \in \mathbb{B}^n, \quad x \twoheadleftarrow\!\!\!\twoheadrightarrow \widetilde{x}. \tag{5.5}$$

Further, $\forall k < m, n_k \geq 2$, let $R_k = \mathrm{F}_{n_k - 2} \circ \dots \mathrm{F}_{n_{k-1}+1} \circ \mathrm{F}_{n_{k-1}}$ be the function that, applied to $\widetilde{x}$, reduces its k^{th} block to size 1 while augmenting the previous block to size $n_{k-1} + n_k - 1$, *i.e.*, letting $a = k \bmod 2$:

$$R_k(\widetilde{x}) \;=\; 0^{n_0} 1^{n_1} \dots \underbrace{a^{n_{k-1}+n_k-1}}_{\text{new } B_{k-1}} \underbrace{\neg a}_{\text{new } B_k} \underbrace{a^{n_{k+1}}}_{B_{k+1}} \dots 0^{n_{m-2}} 1^{n_{m-1}}.$$

Finally, let $\overline{R}_k = R_{k-1} \circ \dots R_1 \circ R_0 \circ R_{m-1} \circ \dots \circ R_{k+1}$ so that $R_k = (\overline{R}_k R_k \overline{R}_k)^{-1}$ is also invertible and:

$$\forall x \in \mathbb{B}^n, \; \widetilde{x} \twoheadleftarrow\!\!\!\twoheadrightarrow R_k(\widetilde{x}). \tag{5.6}$$

Then alternating derivations of both the forms of (5.5) and (5.6) yields (*cf.* (5.4)):

$$x \twoheadleftarrow\!\!\!\twoheadrightarrow \mathbf{x}^{(m)}. \tag{5.7}$$

Let $\mathrm{W}_d = \{r + 2k \mid 0 \leq k < d\} \subseteq \mathbf{V}$, be the set of the first d automata that are in state 1 in $\mathbf{x}^{(m)}$, $\forall 0 \leq d < m/2$. Then:

$$\begin{cases} d < \frac{m}{2} & \Rightarrow \;\; \mathbf{x}^{(m)} \xrightarrow{\mathrm{W}_d} \mathrm{F}_{\mathrm{W}_d}(\mathbf{x}^{(m)}) = 0^{r+d}\,(10)^{\frac{m}{2}-1-d} 1 = \mathbf{x}^{(m-2d)} \\ d = \frac{m}{2} & \Rightarrow \;\; \mathbf{x}^{(m)} \xrightarrow{\mathrm{W}_d} \mathrm{F}_{\mathrm{W}_d}(\mathbf{x}^{(m)}) = 0^n = \mathbf{x}^{(0)}. \end{cases} \tag{5.8}$$

Summing up this development (in particular (5.3), (5.7) and (5.8)) and extending it to negative BACs using Lemma 5.1, we obtain Proposition 5.1 below which prolongs [92] by giving a description of the GTGs of BACs (*cf.* examples in Fig. 5.1 and Fig. 5.2). Let us highlight that in this proposition, for the negative BAC $\mathscr{C}_n^-$, layer $\mathscr{L}_1$ is the *stable cyclic attractor* of [94] and the limit cycle of maximal period the order $\omega = 2n$ of the BAC under the parallel update schedule.

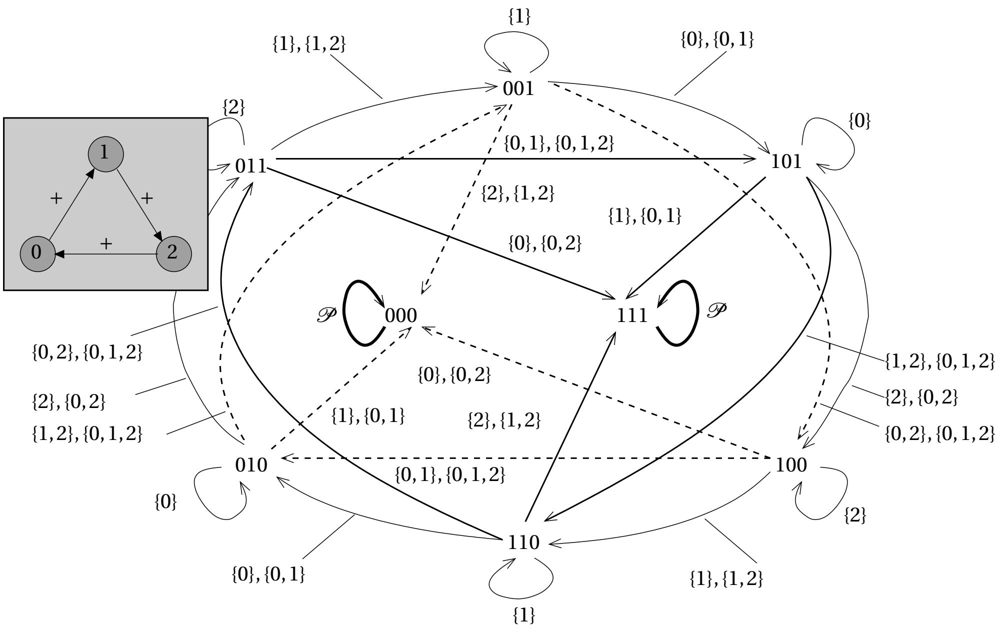

Figure 5.1: GTG *of a positive* BAC *of size* 3 *whose structure is pictured in the frame. The different fonts of arcs have been chosen for the sake of clarity and not of meaning. Abusing conventions, they are labelled by the set of all updates that can be made to perform the corresponding transition, i.e. every arc* (x, y) *is labelled by the set of all* $\mathrm{D}(x, y) \cup \mathrm{W}$ *where* $\mathrm{W} \subseteq \overline{\mathscr{U}}(x) \setminus \mathrm{D}(x, y)$. *The label* $\mathscr{P}$ *is short for* $\mathscr{P}(\mathbf{V}) \setminus \{\emptyset\}$.

Proposition 5.1. GTGs of BACs

For any configurations $x, x', y, y' \in \mathbb{B}^n$ *of a* BAC $\mathscr{C}_n^s$ *satisfying* $u_{min} \leq u(y) = u(y') < u(x) = u(x') \leq u_{max}$:

$$x \twoheadleftarrow\!\!\twoheadrightarrow x' \longrightarrow\!\!\!\!\twoheadrightarrow y \twoheadleftarrow\!\!\twoheadrightarrow y'$$

holds but for no $z \in \mathrm{X}_{u(x)}$ *does* $y \longrightarrow\!\!\!\!\twoheadrightarrow z$ *hold. So the* GTG *of* $\mathscr{C}_n^s$ *is a layered digraph with* $\lceil \frac{n}{2} \rceil$ *layers. For* $0 \leq k < \lceil \frac{n}{2} \rceil$ *and for* $u = 2k$ *if* $s = +$ *and* $u = 2k + 1$ *if* $s = -$, *layer* $\mathscr{L}_k$ *is an* SCC *that is induced by the set* X_u *of* $2\binom{n}{u}$ *configurations* x *in which exactly* $u(x) = u$ *automata are unstable. And for all* $k < k'$, *there is an elementary transition from* $\mathscr{L}_{k'}$ *to* $\mathscr{L}_k$ *but no path from* $\mathscr{L}_k$ *to* $\mathscr{L}_{k'}$.

In conclusion of this section, for BACs, the *momentum* or number of local instabilities $u(x)$ defines the potential energy of configurations and determines the layered organisation of their GTGs: the less there are unstable automata in a configuration, the closer this configuration is to the global asymptotic network behaviour. This means that problems concerning the descriptions of certain properties of BAN behaviours that are generally difficult [9, 37, 38, 39, 124, 125, 126] can easily be bypassed in the case of BACs. Given an arbitrary configuration of these special BANs, it suffices to count its local instabilities to know the number of events (updates) that are required to reach the greatest overall asymptotic net-

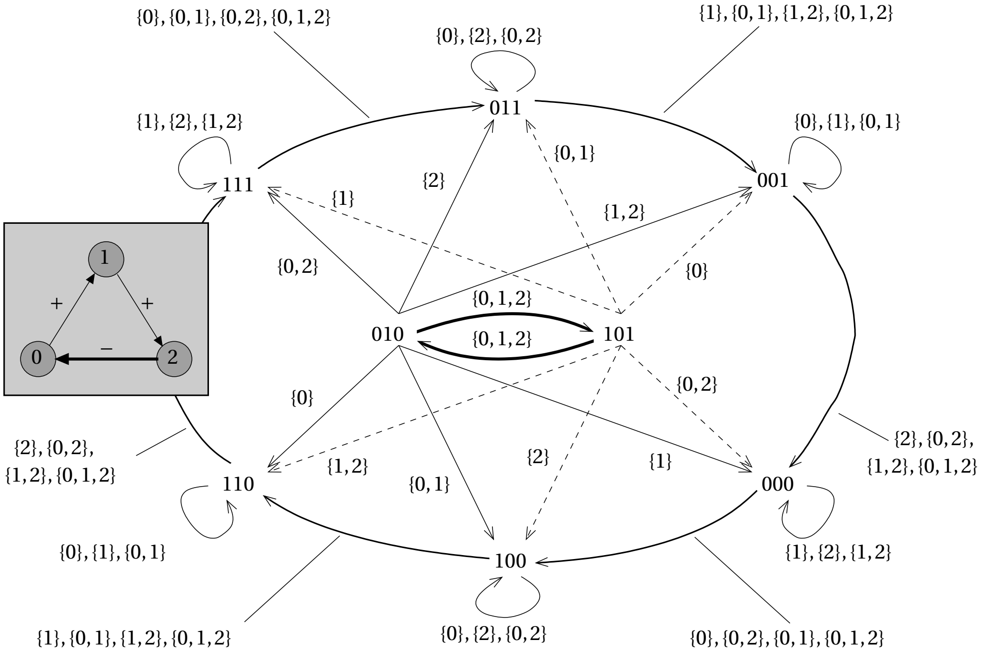

Figure 5.2: GTG *of a negative* BAC *of size* 3 *whose structure is pictured in the frame. See caption of Fig. 5.1.*

work stability and the least local instabilities. Thus, the number of local instabilities $u(x)$ carries a lot of information; in the case of "cyclic BANs", it holds almost *all* the information.

Reformulating this in the terms of Example 5.1, suggests that the most *stable* or *robust* and *likely* behaviours of a cycle are those of lesser $u(x)$. In particular, configurations of BACs that are involved in the same attractor under the parallel update schedule, are strongly connected in the GTG but are not recurrent unless their common potential energy $u(x)$ is minimal. This agrees with the possible artefactual nature of attractors induced by the parallel update schedule pointed out in Example 5.1. In the case of positive cycles, the only durable outcomes of the evolution of the BAC are its stable configurations (contrary to negative cycles which have no stable configurations). Now, in the previous chapters we have seen that with deterministic update schedules involving synchronism, especially the parallel update schedule π, cycles can cycle or induce non-stable attractors. In the present case, the degrees of freedom (*cf.* Section 2, Chap. 2) that allow cycles to do that seem to be "pondered" by a propensity to decrease the momentum $u(x)$. With this analysis of the GTGs of BACs, we find that as long as a parallel updating is maintained, so are future possibilities because this potential energy $u(x)$ cannot fall.

2 Synchronism *vs* asynchronism

In this section, we compare GTGs and ATGs to study the impact of synchronism and identify the cases where an addition of synchronism changes substantially the possible (limit) behaviours of a BAN. Thus, we are looking for BANs for which synchronism does not just add shortcuts to asynchronous derivations but rather also adds derivations that can be *mimicked* by no asynchronous derivations. This study leads to a classification of BANs sensitivity to synchronism proposed below in Section 3.C, Chap. 6.

By default in this section, $\mathcal{N} = \{\mathrm{f}_i \mid i \in \mathbf{V}\}$ is a locally monotone BAN of size n and $\mathbf{G} = (\mathbf{V}, \mathbf{A})$ is its structure. Elementary transitions $x \xrightarrow{\mathrm{W}} y$ are assumed to be effective. All transition labels W denoting sets of updated automata are "minimal": $\mathrm{W} = \mathrm{D}(x, y) = \{i \mid x_i \neq y_i\}$. Also, if a configuration $x \in \mathbb{B}^n$ of $\mathcal{N}$ is such that $\mathbf{A} = \mathbf{FRUS}(x)$, then $\mathcal{N}$ and **G** are said to be **frustrated in** x or x**-frustrated**. If in addition all automata are unstable in x, *i.e.* $\mathcal{U}(x) = \mathbf{V}$, then they are said to be **critically** x**-frustrated**. And they are said to be (**critically**) **frustrable** if there effectively exists an x in which they are (critically) x-frustrated. The same terminology is used for *sub*-networks of $\mathcal{N}$ and *sub*-graphs of **G**. We focus especially on **frustrable cycles**. By definition, these can be totally frustrated if they are isolated. And in an isolated cycle, frustrated arcs are arcs incoming unstable nodes $((i, i+1) \in \mathbf{FRUS}(x) \;\Leftrightarrow\; i+1 \in \mathcal{U}(x))$. So a frustrated isolated cycle is necessarily critically frustrated. Thus by Lemma 5.2, frustrable cycles either are *positive cycles with an even length or are negative cycles with an odd length.* When embedded in larger structures, they may loose the property of being critically frustrable (*cf.* Example 5.2).

2.1 Frustrations & instabilities

Let us start with some preliminary remarks concerning frustrations and instabilities. First, we relate instabilities to arcs of $\mathbf{G} = (\mathbf{V}, \mathbf{A})$ and in particular to positive and negative loops[28].

Note 1. Arcs & instabilities

For any automata $i, j \in \mathbf{V}$*:*

$$\begin{aligned}
(j, i) \in \mathbf{A} \quad &\Leftrightarrow \quad \exists x \in \mathbb{B}^n, \quad i \in \mathcal{U}(x) \cap \overline{\mathcal{U}}(\overline{x}^j) \\
(i, i) \in \mathbf{A} \wedge \mathrm{sign}_{\mathcal{N}}(i, i) = + \quad &\Leftrightarrow \quad \exists x \in \mathbb{B}^n, \quad i \in \overline{\mathcal{U}}(x) \cap \overline{\mathcal{U}}(\overline{x}^i) \\
(i, i) \in \mathbf{A} \wedge \mathrm{sign}_{\mathcal{N}}(i, i) = - \quad &\Leftrightarrow \quad \exists x \in \mathbb{B}^n, \quad i \in \mathcal{U}(x) \cap \mathcal{U}(\overline{x}^i).
\end{aligned}$$

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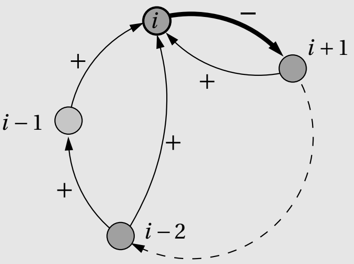
[28] We recall the notations: $\overline{\mathcal{U}}(x) = \mathbf{V} \setminus \mathcal{U}(x)$, $\overline{\mathbf{FRUS}}(x) = \mathbf{A} \setminus \mathbf{FRUS}(x)$ and $\forall \mathrm{W} \subseteq \mathbf{V}, \overline{\mathrm{W}} = \mathbf{V} \setminus \mathrm{W}$.

Example 5.2. Frustrable but not effectively



Consider the signed structure $\mathbf{G} = (\mathbf{V}, \mathbf{A})$ *on the right where* $\mathbf{V}^-_{\mathbf{G}}(i) = \{i-2, i-1, i+1\}$ *(the* ATG *and* GTG *of a* BAN *of size* 4 *with a signed structure of this form are given in Fig. 2.5). The Hamiltonian cycle* $\mathbb{C}_n = (\mathbf{V}_{\mathbb{C}}, \mathbf{A}_{\mathbb{C}})$ *of* $\mathbf{G}$ *may be frustrable but not critically because* $\mathbf{V}_{\mathbb{C}} \subseteq \mathscr{U}(x)$ *and* $\mathbf{A}_{\mathbb{C}} \subseteq \textbf{FRUS}(x)$ *cannot be satisfied at once. Indeed, let us suppose otherwise that* $x \in \mathbb{B}^n$ *satisfies both these conditions. By definition, since* $\mathbb{C}_n$ *is (totally) frustrated in* x*,* $(x_{i-2}x_{i-1}x_ix_{i+1}) \in \mathbb{B}^4$ *must equal either* (1011) *or* (0100)*. In the first case,* $i \in \mathscr{U}(x)$ *imposes that* $\mathrm{f}_i(x) = \neg x_i = 0$*; in the second, it imposes that* $\mathrm{f}_i(x) = 1$*. Confusing notations* $\mathrm{f}_i(x)$ *and* $\mathrm{f}_i(x_{\mathbf{V}^-_{\mathbf{G}}(i)})$*, both cases require that* $\mathrm{f}_i(\mathrm{bbb}) = \neg\mathrm{b}$ *for a certain* $\mathrm{b} \in \mathbb{B}$*. According to the signs of arcs, this is not possible: i depends positively on all 3 of its in-neighbours.*

Proof: The first equivalence amounts to the definition of arcs in a BAN structure. By the minimality of BAN structures (*cf.* (1.3)) and the definition of signed arcs (on Page 17), (i,i) is a positive (resp. negative) loop if and only if there exists $\exists x \in \mathbb{B}^n,\ x_i = 1$ such that $1 = x_i = f_i(x) > f_i(\overline{x}^i) = \overline{x}^i_i = 0$ (resp. $0 = \overline{x}^i_i = f_i(x) < f_i(\overline{x}^i) = x_i = 1$). □

Then, we add some straightforward relations between frustrations and instabilities:

Note 2. Adding/removing frustrations

Adding (resp. removing) frustrated arcs incoming an unstable (resp. stable) automaton cannot stabilise (resp. destabilise) it:

$$\begin{array}{rlcl} \forall x,y \in \mathbb{B}^n, & i \in \mathscr{U}(x) \ \wedge\ (\mathbf{V}^-_{\mathbf{G}}(i) \cap \textbf{FRUS}(x) \subseteq \mathbf{V}^-_{\mathbf{G}}(i) \cap \textbf{FRUS}(y)) & \Rightarrow & i \in \mathscr{U}(y) \\ \textit{and} & i \in \overline{\mathscr{U}}(x) \ \wedge\ (\mathbf{V}^-_{\mathbf{G}}(i) \cap \overline{\textbf{FRUS}}(x) \subseteq \mathbf{V}^-_{\mathbf{G}}(i) \cap \overline{\textbf{FRUS}}(y)) & \Rightarrow & i \in \overline{\mathscr{U}}(y). \end{array}$$

Proof: By hypothesis of local monotony and because the input provided by j to i is:

$$\mathbf{b}^i_j(x) = \mathbf{b}\big(\mathrm{sign}_{\mathcal{N}}(j,i) \cdot \mathbf{s}(x_j)\big) = \begin{cases} x_i & \text{if } (j,i) \notin \textbf{FRUS}(x) \\ \neg x_i & \text{if } (j,i) \in \textbf{FRUS}(x), \end{cases}$$

any local transition function f_i can be written in conjunctive normal form as follows:

$$\mathrm{f}_i : x \in \mathbb{B}^n \mapsto \bigwedge_{k<m} c_k(x)$$

where each of the m disjunctive clauses $c_k(x)$ involve a certain subset $\mathbf{V}^i_k \subseteq \mathbf{V}^-_{\mathbf{G}}(i)$ of the in-neighbours of i so that $\mathbf{V}^-_{\mathbf{G}}(i) = \bigcup_{k<m} \mathbf{V}^i_k$ and:

$$c_k(x) = \bigvee_{j \in \mathbf{V}^i_k} \mathbf{b}^i_j(x) = \bigvee_{j \in \mathbf{V}^i_k,\, (j,i) \in \textbf{FRUS}(x)} \neg x_i \quad \vee \bigvee_{j \in \mathbf{V}^i_k,\, (j,i) \notin \textbf{FRUS}(x)} x_i.$$

□

In similar lines, the next series of equations consist in comparing two different configurations and relating differences in their instabilities and in their frustrations. Equations (5.9a) derive directly from the definitions of $\mathbf{FRUS}$ and $\mathcal{U}$. As for Equations (5.9b), given two configurations x and y and an automaton i, they enumerate different cases defined by whether or not $x_i = y_i$ and whether or not the (in)stability of i is different in x and y. And Equations (5.9c) are the restrictions of (5.9b) to the case where $x \longrightarrow y$ is an elementary transition, *i.e.* $\mathrm{D}(x, y) \subseteq \mathcal{U}(x)$.

Note 3. Frustrations & instabilities

For any automata i and $j \neq i$ in $\mathbf{V}$:

$$\forall x \in \mathbb{B}^n,\ i \in \mathcal{U}(x) \cap \overline{\mathcal{U}}(\overline{x}^j) \;\Rightarrow\; (j,i) \in \mathbf{FRUS}(x) \cap \overline{\mathbf{FRUS}}(\overline{x}^j)$$

$$\forall x \in \mathbb{B}^n,\ (j,i) \in \mathbf{FRUS}(x) \;\Leftrightarrow\; (j,i) \in \overline{\mathbf{FRUS}}(\overline{x}^i) \cap \overline{\mathbf{FRUS}}(\overline{x}^j). \tag{5.9a}$$

For any automaton $i \in \mathbf{V}$ and $\forall x, y = \overline{x}^{\mathrm{D}} \in \mathbb{B}^n$ where $\mathrm{D} = \mathrm{D}(x,y) \subseteq \mathbf{V}$:

$$\begin{aligned}
&i \in \overline{\mathcal{U}}(x) \cap \mathcal{U}(y) \cap \overline{\mathrm{D}} \;\Rightarrow\; \exists j \in \mathrm{D} \cap \mathbf{V}^-_{\mathbf{G}}(i), (j,i) \in \overline{\mathbf{FRUS}}(x) \cap \mathbf{FRUS}(y)\\
&i \in \mathcal{U}(x) \cap \overline{\mathcal{U}}(y) \cap \overline{\mathrm{D}} \;\Rightarrow\; \exists j \in \mathrm{D} \cap \mathbf{V}^-_{\mathbf{G}}(i), (j,i) \in \mathbf{FRUS}(x) \cap \overline{\mathbf{FRUS}}(y)\\
&i \in \mathcal{U}(y) \cap \mathcal{U}(x) \cap \mathrm{D} \;\Rightarrow\; \exists j, j' \in \mathrm{D} \cap \mathbf{V}^-_{\mathbf{G}}(i),\\
&\qquad (j,i) \in \mathbf{FRUS}(x) \cap \overline{\mathbf{FRUS}}(y) \;\wedge\; (j',i) \in \overline{\mathbf{FRUS}}(x) \cap \mathbf{FRUS}(y)
\end{aligned} \tag{5.9b}$$

in particular, if $x \xrightarrow{\mathrm{D}} y$ so that $\mathrm{D} \subseteq \mathcal{U}(x)$:

$$\begin{aligned}
&i \in \overline{\mathcal{U}}(x) \cap \mathcal{U}(y) \;\Rightarrow\; \exists j \in \mathrm{D} \cap \mathbf{V}^-_{\mathbf{G}}(i), (j,i) \in \overline{\mathbf{FRUS}}(x) \cap \mathbf{FRUS}(y)\\
&i \in \mathcal{U}(y) \cap \mathrm{D} \;\Rightarrow\; \exists j, j' \in \mathrm{D} \cap \mathbf{V}^-_{\mathbf{G}}(i),\\
&\qquad (j,i) \in \mathbf{FRUS}(x) \cap \overline{\mathbf{FRUS}}(y) \;\wedge\; (j',i) \in \overline{\mathbf{FRUS}}(x) \cap \mathbf{FRUS}(y)
\end{aligned} \tag{5.9c}$$

implying that if $\mathrm{D} \subseteq \mathcal{U}(y)$, then all $i \in \mathrm{D}$ belong to both a x-frustrated cycle and a y-frustrated cycle (which are not necessarily equal).

In Section 1, it was highlighted that $u(x) = |\mathcal{U}(x)|$ can serve as a potential energy for configurations of isolated BACs. More precisely, the study that it presents supports the idea that, in the absence of structural bottlenecks and bifurcations, updates that are not the parallel update tend to reduce the number $u(x)$ of instabilities. For a BAN with an arbitrary structure, this suggests that $u(x)$ may serve *locally* as a potential energy for configurations. Ultimately, this (certainly, joined with the notion of frustrated arcs) may help to characterise points of no-return, critical configurations, or the *distance* to an attractor. For now, in a preliminary attempt to detail and formalise this idea, we state Lemma 5.3 below. It puts forward conditions that must be satisfied for $u(x)$ to increase to a maximum value $u_{max} = n$ and puts a first emphasis on frustrated cycles. It is is a direct consequence of (5.9c).

Lemma 5.3. Maximal instability and frustrated cycles

Let $\mathcal{N}$ be a BAN *whose automata can all be unstable at once. Let $x \in \mathbb{B}^n$ be a configuration of $\mathcal{N}$ such that $\mathcal{U}(x) = \mathbf{V}$. If x has an incoming transition $y \longrightarrow x$, then all automata of* $\mathrm{D}(y,x)$ *belong to a x-frustrated cycle and to a y-frustrated cycle (which are not necessarily equal).*

2.2 Non sequentialisable synchronous transitions

Now, we say that a transition $x \longrightarrow y$ is **sequentialisable** if there exists an asynchronous derivation from x to y, *i.e.* if $x \relbar\!\!\rhd\!\!\rhd y$. Otherwise, if there exists a derivation from x to y that involves non sequentialisable transitions that are smaller[29] than $x \relbar\!\!\!\blacktriangleright y$, then transition $x \relbar\!\!\!\blacktriangleright y$ is said to be ***partially*** **sequentialisable**. Otherwise, $x \relbar\!\!\!\blacktriangleright y$ is called a **normal synchronous transition** and rather written $x \pmb{\relbar\!\!\!\blacktriangleright} y$ or $x \pmb{-}\mathrm{D}\pmb{\relbar\!\!\!\blacktriangleright} y$ (where $\mathrm{D} = \mathrm{D}(x,y)$).

Proposition 5.2. Sequentialisable transitions and frustrated cycles

Any synchronous transition $x \relbar\!\!\!\blacktriangleright y$ such that automata in $\mathrm{D}(x,y)$ *do not all belong to the same x-frustrated cycle is partially sequentialisable.*

Proof: Let $\mathbf{G}_\mathrm{D} = (\mathrm{D},\mathbf{A}_\mathrm{D})$ be the sub-graph of $\mathbf{G}$ induced by D. And let m be the size of the digraph obtained by merging all nodes of $\mathbf{G}_\mathrm{D}$ that are on a same x-frustrated cycle. We may consider a simple update schedule/ordering $\delta := (\mathrm{W}_t)_{t<m}$ of $\mathrm{D} = \mathrm{D}(x,y)$ so that any $i,j \in \mathrm{D}$, $(j,i) \in \textbf{FRUS}(x)$, $\delta(i) \leq \delta(j)$ is satisfied, and more precisely, if i,j belong to the same frustrated cycle, then $\delta(i) = \delta(j)$ and otherwise, $\delta(i) \neq \delta(j)$.

Let $z = \mathrm{F}_{\mathrm{W}_{t-1}} \circ \ldots \mathrm{F}_{\mathrm{W}_1} \circ \mathrm{F}_{\mathrm{W}_1}(x)$ and let $i \in \mathrm{W}_t$. Consider an arbitrary incoming frustrated arc $(j,i) \in \textbf{FRUS}(x)$. Either $j \notin \mathrm{D}$ or, by definition of δ, $\delta(j) \geq \delta(i)$. In both cases, in configuration z, neither j nor i have yet been updated so $z_i = x_i$ and $z_j = x_j$ and thus $(j,i) \in \textbf{FRUS}(x) \cap \textbf{FRUS}(z)$. By Note 2, i is still unstable in z: $i \in \mathcal{U}(x) \cap \mathcal{U}(z)$ and as a consequence, transition $z \xrightarrow{\mathrm{W}_t} \mathrm{F}_{\mathrm{W}_t}(z) = \overline{z}^{\mathrm{W}_t}$ is possible.

The series of all such transitions $(0 \leq t < m)$ induced by δ leads from x to $\mathrm{F}[\delta](x) = \overline{x}^{\uplus_{t<m}\mathrm{W}_t} = \overline{x}^{\mathrm{D}} = y$. And, unless all nodes in D belong to the same frustrated cycle, it has size $m > 1$. Therefore, it is a partial sequentialisation of $x \relbar\!\!\!\blacktriangleright y$. □

Corollary 5.1. Sequentialisable transitions and frustrable cycles

If $\mathcal{N}$ has no critically frustrable cycles of size m, then all of its effective synchronous that change the states of less than m automata are sequentialisable. If it has no critically frustrable cycles, then all of its transitions are sequentialisable.

---

[29] The minimality refers to the ordering of transitions introduced on Page 22 by which $z \longrightarrow w$ is smaller than $z' \longrightarrow w'$ if $\mathrm{d}(z,w) \leq \mathrm{d}(z',w')$.

Now, let $x \xrightarrow{\mathrm{D}} y = \overline{x}^{\mathrm{D}}$ be a minimal[29], normal synchronous transition of $\mathscr{N}$. By its definition, any derivation $x \longrightarrow\!\!\!\!\twoheadrightarrow y$ (if there exist some) involves a normal synchronous transition (otherwise (x, y) would be sequentialisable) that is greater than (x, y) (otherwise (x, y) would not be minimal):

$$x \longrightarrow\!\!\!\!\twoheadrightarrow x' \pmb{\longrightarrow} y' \longrightarrow\!\!\!\!\twoheadrightarrow y \quad \Rightarrow \quad \mathrm{d}(x', y') \geq \mathrm{d}(x, y) = |\mathrm{D}(x, y)| = |\mathrm{D}|.$$

Thus, by the minimality of $x \pmb{\longrightarrow} y$, by $\mathrm{D} = \mathrm{D}(x, y) \subseteq \mathscr{U}(x)$ and by the non-existence of $x \longrightarrow\!\triangleright\!\triangleright\, y$, the transitions represented below in Fig. 5.3 a exist for all automata i in D and all subset E of D.

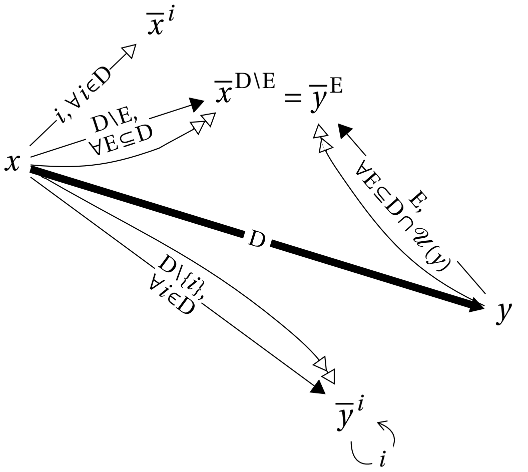


Figure 5.3: *a.*

And further, according to Note 1, they can be completed as suggested in Fig. 5.3 b:

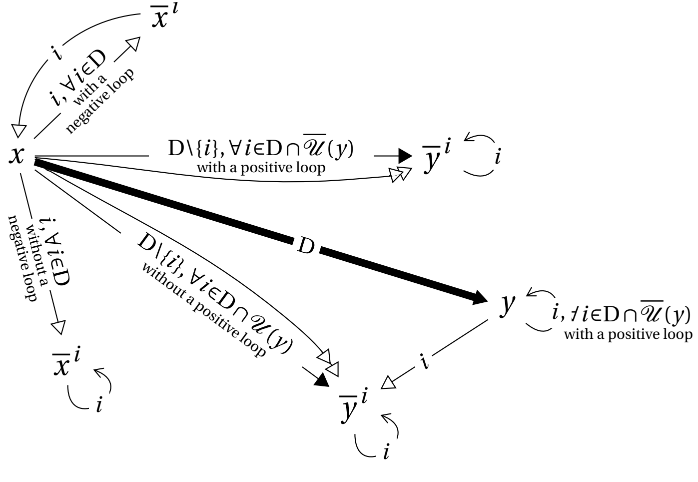


Figure 5.3: *b.*

Note 4. Frustrable cycles are not enough

If $\mathscr{U}(y) \setminus \mathrm{D} = \emptyset$*, then the addition of* $x \longrightarrow\!\!\!\blacktriangleright y$ *to the* ATG *has no impact besides its own existence: if* $x' \longrightarrow\!\!\triangleright\!\triangleright y' \neq y$ *is impossible then* $x' \longrightarrow\!\!\triangleright\!\triangleright x \longrightarrow\!\!\!\blacktriangleright y \longrightarrow\!\!\triangleright\!\triangleright y'$ *also is impossible.*

Proof: Suppose that it is and let $j \in \mathscr{U}(y)$ be such that $y \longrightarrow\!\!\triangleright \overline{y}^{j} \longrightarrow\!\!\triangleright\!\triangleright y'$. If $j \in \mathrm{D}$, then $x' \longrightarrow\!\!\triangleright\!\triangleright x \longrightarrow\!\!\triangleright\!\triangleright \overline{x}^{\mathrm{D}\setminus\{j\}} = \overline{y}^{j} \longrightarrow\!\!\triangleright\!\triangleright y'$ is possible (*cf.* Fig. 5.3). Since this contradicts the impossibility of $x' \longrightarrow\!\!\triangleright\!\triangleright y'$, $j \notin \mathrm{D}$. ☐

Thus, if the addition of $x \longrightarrow\!\!\!\blacktriangleright y$ is to cause any significant change, either y must be a stable configuration ($\mathscr{U}(y) = \mathscr{U}(y) \setminus \mathrm{D} = \emptyset$) so that the addition of $x \longrightarrow\!\!\!\blacktriangleright y$ may cause x to gain the possibility of reaching y (if it couldn't already), or else, $\mathscr{N}$ must have automata that are not on the frustrable cycle involved by $x \longrightarrow\!\!\!\blacktriangleright y$.

Note 5. On the recurrence of y and x

1. *If* y *is recurrent in the* ATG*, then all automata of* D *have a positive loop and are stable in* y*.*
2. *If* x *is recurrent in the* ATG*, then* **G** *contains a negative cycle and if* $\mathbf{V} = \mathrm{D}$*, then a node of* D *has a negative loop.*

Proof: 1. If $i \in \mathrm{D}$ does not have a positive loop, then $i \in \mathscr{U}(y)$. If $i \in \mathscr{U}(y)$, the recurrence of y implies $y \longrightarrow\!\!\triangleright \overline{y}^{i} \longrightarrow\!\!\triangleright\!\triangleright y$. And then (*cf.* Fig. 5.3) $x \longrightarrow\!\!\triangleright\!\triangleright \overline{y}^{i} \longrightarrow\!\!\triangleright\!\triangleright y$ holds, contradicting that $x \longrightarrow\!\!\!\blacktriangleright y$ is non sequentialisable. 2. x being unstable, if it is recurrent in the ATG, it belongs to an unstable attractor of the ATG. By [97], **G** has a negative cycle[30]. In addition, the recurrence of x implies: $\forall \mathrm{E} \subsetneq \mathrm{D},\ x \longrightarrow\!\!\triangleright\!\triangleright \overline{x}^{\mathrm{E}} \longrightarrow\!\!\triangleright\!\triangleright \overline{x}^{j} \longrightarrow\!\!\triangleright x$ for a certain $j \in \mathbf{V}$ (*cf.* Fig. 5.3). If $\mathrm{D} = \mathbf{V}$ this j belongs to $\mathscr{U}(\overline{x}^{j}) \cap \mathscr{U}(x)$. By Note 1, it bears a negative loop in **G**. ☐

2.3 Sensitivity of BANs to the addition of synchronism

Let us consider how the addition of $x \longrightarrow\!\!\!\blacktriangleright y$ to the ATG impacts on possible derivations. By Proposition 5.2, $\mathrm{D} = \mathrm{D}(x, y)$ induces a x-frustrated cycle $\mathbb{C}$. Let us emphasise that generally, any derivation that exists in the ATG also exists in the GTG. For any $z \in \mathbb{B}^n$, we let $\mathscr{A}_z$ (resp. $\mathscr{A}_z^{\star}$) be the set of attractors to which z leads or belongs in the ATG (resp. in the GTG). And we let $\mathscr{L} = \bigcup_z \mathscr{A}_z$ (resp. $\mathscr{L}^{\star} = \bigcup_z \mathscr{A}_z^{\star}$) be the set of all attractors in the ATG (resp. in the GTG).

---

[30] Exploiting further the tools and results of [97, 98] should certainly help to describe more precisely the structure of the BAN when x is recurrent.

With these notations, because of the existence of transition (x,y) in the GTG, any attractor that can be reached by y can also be by x so $\mathscr{A}^{\star}_y \subseteq \mathscr{A}^{\star}_x$. On the contrary, in the ATG, because there are no derivations from x to y ((x,y) is non sequentialisable), $\mathscr{A}_y \subsetneq \mathscr{A}_x$ is impossible. Indeed, either (i) y is transient and the only attractors that it can reach are those of $\mathscr{A}_y = \mathscr{A}_x$ that can be reached from x, either (ii) y is transient and it can reach attractors in $\mathscr{A}_y \setminus \mathscr{A}_x \neq \emptyset$ that cannot be reached from x, or (iii) y is recurrent and since $x \longrightarrow\!\!\rhd\!\!\rhd y$ is impossssible, $y \longrightarrow\!\!\rhd\!\!\rhd x$ also is so $\mathscr{A}_x \cap \mathscr{A}_y = \emptyset$ (in the two latter case, $\mathscr{A}_y \nsubseteq \mathscr{A}_x$). (i), (ii) and (iii) respectively yield cases 2,3 and 4 listed below in Note 6. And it is easy to check that Note 6 encompasses all possible situations.

Note 6. Adding $x \boldsymbol{\longrightarrow} y$ to the ATG

When the normal synchronous transition $x \boldsymbol{\longrightarrow} y$ is added to the ATG *of $\mathscr{N}$, one of the four following cases holds.*

CASE *1: x is transient in the* ATG*. Consequently, the set $\mathscr{L} = \mathscr{L}^{\star}$ of all attractors is unchanged. All configurations $z \in \mathbb{B}^n$ that can reach x in the* ATG*, including x, remain transient but gain the possibility to reach attractors in $\mathscr{A}_y \setminus \mathscr{A}_z$ (i.e. $\mathscr{A}_y = \mathscr{A}^{\star}_y$ and $\mathscr{A}_x \subseteq \mathscr{A}_z \Rightarrow \mathscr{A}^{\star}_z = \mathscr{A}_z \cup \mathscr{A}_y$).*

CASE *2: x is recurrent, y is transient and $\mathscr{A}_y = \mathscr{A}_x$. Consequently, all $z \in \mathbb{B}^n$ on a derivation from y to $\mathscr{A}_x$, including y, become recurrent and are included in $\mathscr{A}^{\star}_x$, causing $\mathscr{A}_x$ to grow (to become $\mathscr{A}^{\star}_x$).*

CASE *3: x is recurrent, y is transient and $\mathscr{A}_y \setminus \mathscr{A}_x \neq \emptyset$. Then x becomes transient causing $\mathscr{L}$ to loose attractor $\mathscr{A}_x$ ($\mathscr{A}^{\star}_x = \mathscr{A}_y = \mathscr{A}^{\star}_y$ and $\mathscr{L}^{\star} = \mathscr{L} \setminus \mathscr{A}_x$).*

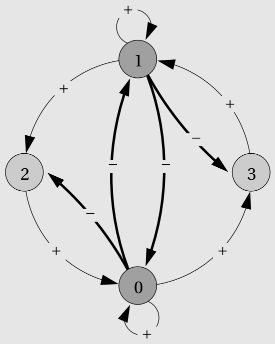
CASE *4: both x and y are recurrent in the* ATG*. Attractor $\mathscr{A}_x$ "empties itself" in $\mathscr{A}_y$ ($\forall z \in \mathscr{A}_x$, z becomes transient and such that $\mathscr{A}_z = \mathscr{A}_x \nsubseteq \mathscr{A}^{\star}_z = \mathscr{A}^{\star}_y$) also causing $\mathscr{L}$ to loose attractor $\mathscr{A}_x$ (to become $\mathscr{L}^{\star}$) as in Example 5.3.*

In particular, case 4 implies that *it is impossible to merge two attractors* by adding one synchronous transition. Generally, Note 6 suggests between three and four different levels of sensitivity that a BAN $\mathscr{N}$ can have to the addition of synchronism (the relative importance of levels $1^{\boxast}$ and $1^{\boxbox}$ being disputable, they are deliberately not ordered). Cases 1 and 2 respectively yield levels $1^{\boxast}$ and $1^{\boxbox}$ below and cases 3 and 4 both yield level 2.

Example 5.3. Example of a small synchronism-sensitive BAN

Let $\mathscr{N} = \{f_i \mid i < 4\}$ be the BAN *of size 4 whose local transition functions and signed structure given below:*

$$\forall x \in \mathbb{B}^n, \begin{cases} \mathrm{f}_0(x) &=& x_2 \vee (x_0 \wedge \neg x_1) \\ \mathrm{f}_1(x) &=& x_3 \vee (x_1 \wedge \neg x_0) \\ \mathrm{f}_2(x) &=& \neg x_0 \wedge x_1 \\ \mathrm{f}_3(x) &=& x_0 \wedge \neg x_1 \end{cases}$$



According to the transition graphs of Fig. 5.4 and Fig. 5.5, when $x_0 = x_1 = 1$ and $x_2 = x_3 = 0$, the simultaneous update of automata 0 and 1 has an effect that cannot be mimicked by a series of atomic updates. Transition $x = (0011) \longrightarrow \overline{x}$ (in green in Fig. 5.4) is non sequentialisable. If it were, the non-trivial SCC *in Fig. 5.5 b and c would not be terminal in the* ATG *of Fig. 5.5 a. The present section proves that it is especially the frustrable cycle of size 2 involving automata 0 and 1 which is responsible for the impact of this synchronous update.*

LEVEL 0:	*$\mathscr{N}$ is not sensitive at all: all its synchronous transitions either act as shortcuts for asynchronous derivations or, on the contrary, add local, confluent deviations which increase the number of possible steps in a derivation without changing its outcome;*
LEVEL $1^{\boxast}$:	*$\mathscr{N}$ is sensitive in the sense that the addition of synchronism grants additional liberty in the evolutions of some transient configurations that are made to reach a greater number of different attractors;*
LEVEL $1^{\boxdot}$:	*$\mathscr{N}$ is sensitive in the sense that the addition of synchronism causes some transient configurations to become recurrent and thus some (necessarily unstable) attractors to grow;*
LEVEL 2:	*$\mathscr{N}$ is sensitive in the sense that the addition of synchronism destroys some (necessarily unstable) attractors as in Example 5.3.*

For BANs that have level 0 sensitivity to synchronism, adding a synchronous transition does not change the result of any network evolution. In other terms, it does not change the set of asymptotic behaviours that can be reached from an arbitrary configuration. For BANs of levels 0 and $1^{\boxast}$, the set of recurrent configurations of the ATG equals that of the GTG ($\mathscr{X} = \mathscr{X}^\star$). Generally, a configuration can be transient in the ATG and recurrent in the GTG (as for BANs of level $1^{\boxdot}$) and *vice versa* (as for BANs of level 2).

Proposition 5.3. Relation between synchronism sensitivity & structure

0) All synchronism-sensitive BAN*s (i.e. all* BAN*s that do not belong to level* 0*) have a critically frustrable cycle* $\mathbb{C}$ *in their structure* $\mathbf{G}$.

1) A synchronism-sensitive BAN *with only one critically frustrable cycle* $\mathbb{C}$ *induced by automata set* D *and no automata outside of* $\mathbf{V} = \mathrm{D}$ *belongs to level* $1^{\boxast}$. *All automata on* $\mathbb{C}$ *have a positive loop* ($\forall i \in \mathrm{D}, (i, i) \in \mathbf{A}$). *The* BAN *sensitivity to synchronism is due to a non sequentialisable transition* $x \xrightarrow{\mathrm{D}} \overline{x}$ *where* $\overline{x}$ *is a stable configuration* ($\mathcal{U}(\overline{x}) = \emptyset$).

2) All synchronism-sensitive BAN*s of levels* $1^{\boxbox}$ *and* 2 *have a critically frustrable cycle* $\mathbb{C}$, *some automata outside of it and a negative cycle.*

Proof: 0) By Corollary 5.1, without critically frustrable cycles, synchronism only acts as shortcuts and the BAN belongs to level 0. 1) By Note 4 (and the remark below its proof on Page 115), without any automata outside of the $\mathbb{C}$, non sequentialisable transitions add no significant supplementary possibilities, unless they end on a stable configuration. In this case Note 5 implies that all automata on $\mathbb{C}$ bear a positive loop. 2) Levels $1^{\boxbox}$ and 2 come from cases 2 to 4 in Note 6 in which x is recurrent. The rest follows from Note 5. □

2.4 Sensitivity to synchronism & non-monotony

Obviously, to be sensitive to synchronism, a BAN must involve at least two automata. Example 5.4 shows that there are no monotone BANs of size 2 that are very sensitive to synchronism (in the sense of level 2), but there are some non-monotone ones (*cf.* also [84]).

Further, notably, the monotone BAN of Example 5.3 actually also involves non-monotone actions. Indeed, it only involves a few monotone individual interactions between four automata but these are architectured into a *widget* that can globally *mimic* a punctual non-monotone action in the right configuration and with the right synchronous update. More precisely but informally, in this widget, a non-monotone action is *structurally* split into two parts. These two parts consist in the two halves of a XOR: $(x_0\, x_1) \mapsto x_0 \wedge \neg x_1$ and $(x_0\, x_1) \mapsto \neg x_0 \wedge x_1$. They are encoded separately in the local transition functions f_0 and f_1 of two different automata. When the controls on these two parts are lifted (*i.e.* when $x_2 = x_3 = 0$ so that we do indeed have $\mathrm{f}_0(x) = x_0 \wedge \neg x_1$ and $\mathrm{f}_1(x) = \neg x_0 \wedge x_1$), the synchronous update of automata 0 and 1 simultaneously applies f_0 and f_1. Instantly, this amounts to combining influences underwent by 0 and 1 by "simulating" a OR connector between their local transition functions, thereby outputting the global action

$f_0(x) \vee f_1(x)$. Precisely, this puts together the two halves of a XOR with a $\vee$ and produces a global non-monotone action.

Examining the widget of Example 5.3, one can notice that the automata that it involves have different roles. Roughly, automata 0 and 1 encode the non-monotone action mentioned above. The role of automata 2 and 3 is to make "use" of it and ensure the necessary oscillating attractor. This attractor is made dependent on automata 0 and 1 by requiring that $x_0 \vee x_1$ be satisfied. More precisely, the widget is designed so that the oscillating attractor is characterised by this condition. In the ATG, if the condition becomes true, it remains true. Every configuration x such that $x_0 = x_1 = 0$ reaches the stable configuration.

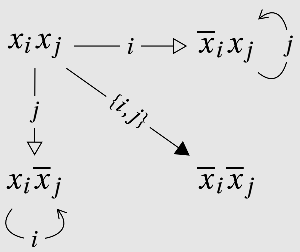
Three automata are enough to monotonously encode a XOR. However, the construction of the widget of Example 5.3 suggests that to make a non-monotone action effectively impact somehow, it must do so through an oscillating attractor. This attractor needs a cycle to induce it. To reduce the number of automata in the widget from 4 to 3 the oscillating attractor could be generated by a unique automaton $k = 2$ with a necessary negative loop, outside of the critically frustrable cycle $\mathbb{C}$ formed by automata 0 and 1. Otherwise, the attractor would have to be generated by a negative cycle involving $k = 2$ and at least one of the automata 0 and 1 on $\mathbb{C}$. Now, let us consider building on the second GTG of Example 5.4 to derive a locally monotone BAN of size 3 with level 2 sensitivity. Here, it seems that one automaton k outside of the set $\{i, j\} \subseteq \mathbf{V}$ inducing the frustrable cycle $\mathbb{C}$ of length 2 is insufficient to ensure monotonously the two conditions that are needed to ensure that both x and $\overline{x}$ be recurrent: (1) the respective instabilities of i and j in $\overline{x}^i$ and in $\overline{x}^j$ as well as (2) the stability of both automata in $\overline{x}$. These considerations suggest that there might be no monotone BANs of size 3 on the level 2 of synchronism-sensitivity. If this intuition turned out true (which should be easy to check), the following would hold: *the smallest* BAN*s that are sensitive to synchronism are non-monotone and have size* 2 *and the smallest monotone* BAN*s that are sensitive to synchronism have size* 4.

2.5 Conclusion and perspectives

This section has put forward (especially with Example 5.3) the existence of a "criticality" that involves updates whose effect is to decrease suddenly and non-reversibly the number of local instabilities $u(x)$. In the direct continuation of the previous section, this raises the question of *how, generally, do local instabilities relate to global (asymptotic) behaviours and, further, how do they relate to the degrees of freedom of a* BAN (which, as noted at the end of Section 1 seem to be constrained by a propensity to reduce local instabilities). For BACs, it was shown that the number of instabilities serves as a potential energy. And notably, for BACs, instabilities are directly related to frustrations $((i-1, i) \in \mathbf{FRUS}(x) \Leftrightarrow i \in \mathcal{U}(x))$.

Example 5.4. Synchronism sensitive BANs of minimal size 2

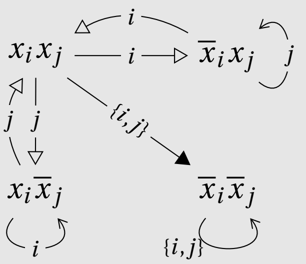
Any synchronism-sensitive BAN *of size* 2 *must have a* GTG *with at least all of the following transitions, for some* $x = x_ix_j \in \mathbb{B}^2$*:*



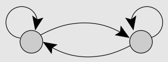
and neither $\overline{x}_ix_j \longrightarrow y = \overline{x} = \overline{x}_i\overline{x}_j$ *nor* $x_i\overline{x}_j \longrightarrow y$. *This implies that there must at least be a cycle of length* 2 *connecting i and j in the structure.*

Also, if x is recurrent, then the GTG *contains* $y \longrightarrow x_i\overline{x}_j$ *or* $y \longrightarrow \overline{x}_ix_j$ *and both automata i and j have a negative loop. If y is recurrent in the* ATG*, it must be a stable configuration and both automata i and j must have a positive loop.*

The two cases 3 and 4 in Note 6 which yield level 2 *sensitivity to synchronism require that x be recurrent and that it become transient with the addition of* (x, y)*. The only way this may be is if y is stable and the* GTG *is isomorphic to:*



In this case, the structure of the BAN *is:*



and it may be checked that both local transition functions f_i, f_j *must either equal* $x \mapsto x_i \oplus x_j$*, or* $x \mapsto \neg(x_i \oplus x_j)$ *(where* $\oplus$ *denotes the* XOR *connector) implying that no arc may be signed and the* BAN *is non-monotone.*

The numbers of each are always equal. The results of this section suggest that the information that is grouped in the equivalent notions of frustrations and instabilities in BACs is divided in more general contexts into the separate notions of frustrations and instabilities. Dually, these evoke a weaker, more local notion of potential that might account for possible irreversible changes in arbitrary BANs.

Also the relation pointed above between this criticality and non-monotony raises the question of *how interactions between unstable automata must be organised if their updates are to be consequential.* This together with the first question calls for further investigations to address the following ones: *How do instabilities evolve in portions of a* BAN*'s structure and especially through the elementary connections proposed in Section 8, Chap. 2?, How do they account for the complexity of a given (asymptotic) behaviour, especially in the sense of the number of different configurations it involves, and how do they relate to the periods or sizes of attractors?* and finally, *How do they relate to the general diversity of a* BAN*'s set of (asymptotic) behaviours* (which is intended to be captured by the notion of *degrees of freedom* introduced in Section 2, Chap. 2)*?*

To close this chapter and section, let us recall that in [102, 103], Robert raises very similar question to those considered here. In particular, in Chapter 5 of [103], he highlights three “frequent (but not systematic) phenomena” that can be observed when comparing the transition graph $\mathscr{T}_{[\pi]}$ induced by the parallel update schedule and the transition graph $\mathscr{T}_{[\sigma]}$ induced by a sequential update schedule σ[31]. These are: the “*bursting*”, the “*aggregation*” and the “*implosion*” of attraction basins. And in the lines of Section 2.4, Robert gives an example in which $\mathscr{T}_{[\sigma]}$ has a fix point and a limit cycle of period 3 while in $\mathscr{T}_{[\pi]}$, the limit cycle is lost. It “*empties itself*” into the fix point which is the only remaining attractor in $\mathscr{T}_{[\pi]}$. This fits into level 2 of the synchronism-sensitivity classification proposed above and it can be shown that the local transition functions of this example involve non-monotony, in agreement with Section 2.4. Interestingly (*cf.* Section 2.C, Chap. 6), Robert concludes on the advantage in terms of algorithmics of $F[\pi]$ over $F[\sigma]$ in this particular case and with respect to the problem of searching for fix points.

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[31] $F[\sigma]$ is called the *Gauss-Seidel* operator associated to $F[\pi]$ in [102, 103].

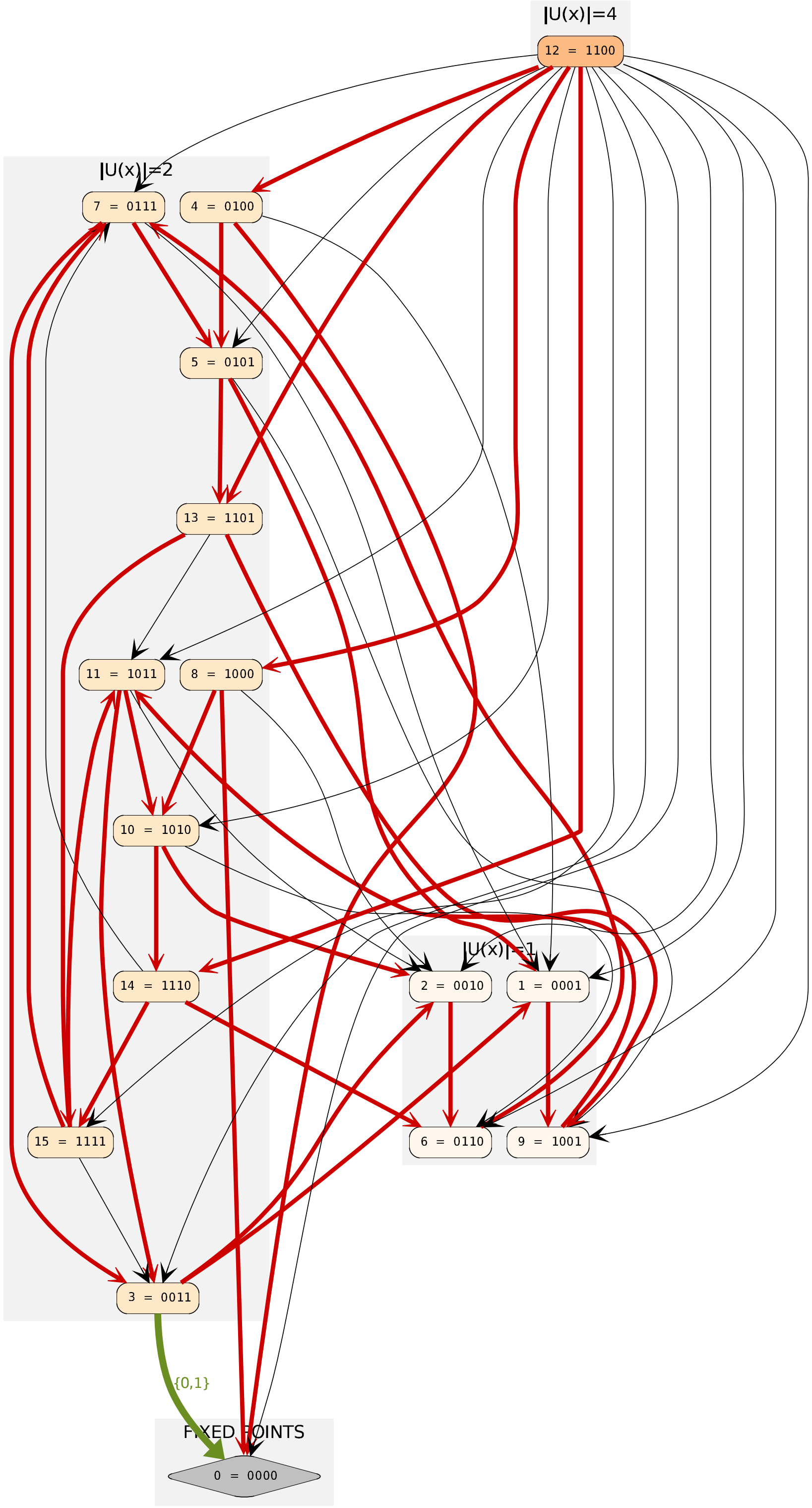
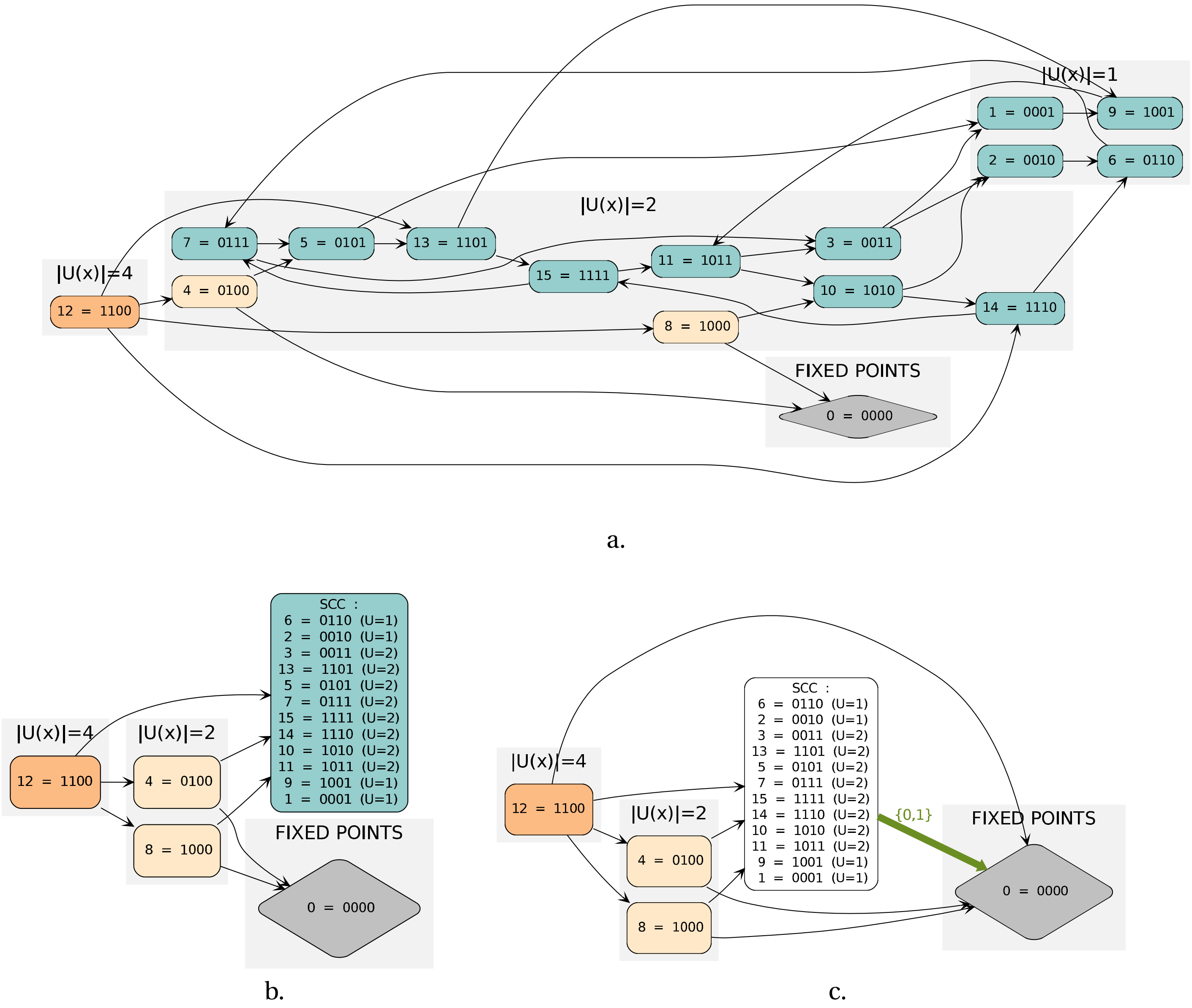


Figure 5.4: GTG *of the* BAN *of Example 5.3. The green arc represents the simultaneous update of automata* 0 *and* 1*. Transitions pictured in red are asynchronous transitions. See caption of Fig. 5.5 for other conventions.*



a.

b.

c.

Figure 5.5: ATG *(a), reduced* ATG *(b) and reduced* GTG *(c) of the* BAN *of Example 5.3 (the reduced version of a digraph is the digraph obtained by reducing each* SCC *of the original digraph to a node). In this figure, as in all other figures of (reduced) transition graphs obtained by computer simulations, configurations* $x = (x_0, x_1, \dots, x_{n-1}) \in \mathbb{B}^n$ *are written in the reversed form* $\sum_{i<n} x_i 2^i = x_{n-1} \dots x_1 x_0$*; configurations with the same number* $u(x) = |\mathscr{U}(x)|$ *of instabilities appear as nodes of the same colour and are grouped together; a non-trivial* SCC *appears in the reduced digraphs as a node labelled by "*SCC*:" followed by a list of all the configurations x that it contains, as well as their* $u(x)$ *value; stable configurations appear as diamond shaped grey nodes; terminal* SCC*s appear as blue nodes.*

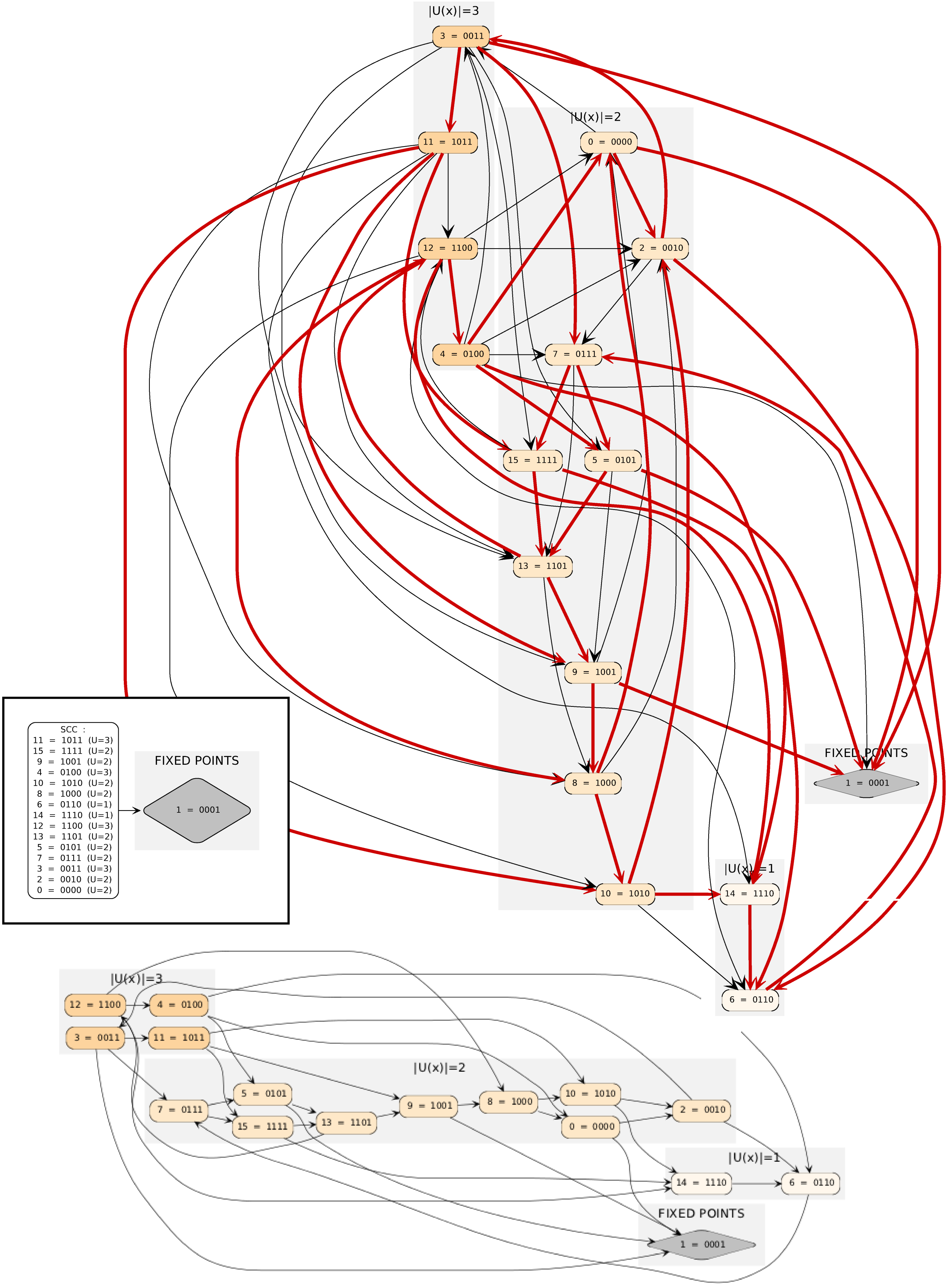


Figure 5.6: *Behaviour of a size* 4 BAN *of the form described in Example 5.2, where* $f_i : x \mapsto x_{i+1} \wedge (x_{i-1} \vee x_{i-2})$ *(*$i = 0$*),* $f_{i+1} = f_{i-2} = \text{neg}$ *and* $f_{i-1} = \text{id}$ *(with the notation of Example 5.2:* $(i-2, i-1,, i, i+1) = (2,3,4,1)$*). The top transition graph is the* BAN*'s* GTG*, the bottom one is its* ATG *and in the frame figures the reduced graph of th* GTG*. See captions of Fig. 5.4 and Fig. 5.5 for conventions used.*

CLOSING DISCUSSION ON TIME & NETWORKS

6

The very concept of *transition* from one network configuration x to another y suggests a notion of time that positions x *before* y. The term *trajectory* that is commonly used instead of *derivation* re-enforces the natural association that can be made between an intuitive idea of time flow and a mathematical concept embodied by an arc in a digraph. The *length* of a series of transitions from x to y evokes the *time* that the system spends to go from one point x to another y in its state space. And, especially in a modelling context where the aim precisely is to relate experiences of reality and abstract concepts, the formal language used tends to adapt to associations that are made to better understand theoretical objects and their properties. In themselves, these associations do not just serve as comprehension aids. More importantly I believe, they can shed some different, possibly enriching light on the theory in a way that is not possible from a clinical, stand-alone theoretic angle. However, the pertinence of these associations is not always obvious because some parts of the theory are only *artifacts* of the formalisation of our observations and thus hold no representational capacity. The importance of this follows from the possible *implicit*. It lies in the risk of becoming misled by intuitive, unquestioned, automatic interpretations.

In practice, the primary, *theorisation* step of any modelling process requires to establish a coherent and well-bounded correspondence between the portion of reality that is intended to be modelled and the theory that is designed to model it [85]. The initial definition of this reality–theory correspondence is a first difficulty because it must insure a rigorous consistency between the various associations that it involves [85]. But once this sound basis has been set, it still remains to understand both the ways in which the modelling is incomplete and the ways in which the theory is meaningless, and as well, of course, the meanings, if any, of the theoretical objects and properties that are handled. In these lines, the first section of this chapter proposes to explore how the notion of time fits into the theory of BANs. It starts by listing different ways that time can be taken into account and how these give rise to different problems and questions. Having argued and settled for a point of view, the next sections build on this and on the results presented earlier in this document to motivate further the development of our understanding of questions that relate to both time and updatings within the theory of BANs.

1 Theorisation of time

1.1 Modelling durations

When BANs are augmented with a dynamical systems view (*i.e.* they are associated to a transition graph that either is deterministic, or is completed with a measure of probability that allows to ponder each of its transitions, *cf.* Section 4.5, Chap. 1) involving a time domain Θ, the implicit abstraction of time is obviously especially present. A first immediate reading of it consists in letting it be a literal match of the real time. This implies in particular that all time steps in Θ and all transitions in the transition graphs are taken to represent the same amount of time, whatever the number of elementary steps they involve and whatever the number of changes they require. Assuming this way that Θ is a discretised version of a real time flow may often be unrealistic, so more loosely, transitions may be associated to different durations. This means that each of them can be labelled by the amount of time that it is supposed to last, or more precisely, by the assumed duration of the event(s) it is supposed to model [11, 64, 110, 117, 119, 121]. In addition to the questions that are mentioned in the next paragraphs which are also natural and pertinent with more general approaches, this point of view (like any point of view) on time yields a set of theoretical questions that are specific to it, that is, questions such as the following which intimately rely on the strong hypothesis that the mathematical concept of time can account for real time flow: *How long does the network take (or is it expected to take) to reach a certain configuration /to start displaying a certain behaviour? What is the (most likely) network configuration that is reached in time $t \in \Theta$? When or how long will the network display this behaviour (or is it expected to so)? How many times is the network expected to reach this configuration during this lapse of time?...*

However, it is important to note that it remains the problem of coherently integrating this essentially continuous notion of real time in a theory that is fundamentally discrete, especially in the sense that all the events that it considers are so, even when they are labelled by real numbers representing their alleged durations. In [85], it is demonstrated how this can naturally (if not necessarily) lead to the abnegation of the discrete modelling paradigm in favour of a continuous one. But beyond the subject of the intrinsic advantages of a discrete modelling (which are discussed in Section 1, Chap. 1 for the case of Boolean modelling), this might be unnecessary as the next paragraphs show. Indeed, again[32], the less hypotheses are made, the more modelling power can the theory be affected.

1.2 Modelling precedence

A second reading of the mathematical concept of time inherent to dynamical systems consists in understanding it as a simple *evolution parameter* defining no more than a relation of precedence between network configurations and without implying any notion of duration. If the two transitions $x \longrightarrow\!\!\!\!\rightarrow y$ and $x' \longrightarrow\!\!\!\!\rightarrow y'$ are both possible, then, with this point of view, it becomes coherent to accept that $x \longrightarrow\!\!\!\!\rightarrow y$ may take much longer to happen than $x' \longrightarrow\!\!\!\!\rightarrow y'$, under certain circumstances, while, perhaps, under different circumstances, the opposite is true (*i.e.* $x' \longrightarrow\!\!\!\!\rightarrow y'$ takes longer than $x \longrightarrow\!\!\!\!\rightarrow y$). Thus, different behaviours of the BAN can take place at different time scales although no additional precisions are provided to distinguish them nor the different possibilities that they yield. The mathematical concept of time regarded here is a *logical* version of time. It requires less information on the nature of transitions and on how they happen. Derivations or trajectories simply are sequences of successive events. The time they take cannot be measured. But the number of (elementary) events they involve can be counted[33]. Consequently, the questions that characterise this point of view on time in BANs and dynamical systems are of the following form: *How many steps does the network take (or is it expected to take) to reach a certain configuration/to start displaying a certain behaviour? What is (the most likely) network configuration that is reached after k steps? Can a given behaviour be observed after a certain other? What trajectories are more likely? What behaviours are more frequent?...*

1.3 Modelling causality

The last stance only requires to consider BAN behaviours as "**causal systems**", that is, state transition systems. It consists in ignoring altogether any associations that can be made between an intuitive idea of time – precedence as well as duration –

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[32] The same remark can be made on the subject of updatings (*cf.* Chapter 4 in particular).

[33] And perhaps so can the numbers of atomic and non-atomic events.

and the theoretical features that follow from formalisation. This way, contrary to the case where BANs are assimilated to dynamical systems, no more information than the transition graph is required. Any arbitrary BAN can be regarded as a causal system stripped from any notion of time. Obviously, there is no notion of duration associated to state transition systems. But neither is there any meaningful notion of precedence that follows naturally and non ambiguously from their definition. Indeed, when several transitions[34] $x \longrightarrow\!\!\!\!\!\rightarrow y^{(k)},\ k \in \mathbb{N}$, are possible in the same configuration $x \in \mathbb{B}^n$, then none of the configurations $y^{(k)}$ is <u>*the*</u> configuration that is reached after x. Each configuration $y^{(k)}$ is only the result of one among several equally possible events which may occur or not with an unknown probability and within an unknown lapse of time. The notion of *moment* is therefore replaced by the notion of *possibility* and duration is replaced by a logical relation between causes and consequences. Two transitions $x \longrightarrow\!\!\!\!\!\rightarrow y$ and $x \longrightarrow\!\!\!\!\!\rightarrow z$ being possible means that x has at least two different "continuations", y and z. State transition systems are essentially non-temporalised systems in which the focus is placed on the *punctual* momentum that the BAN and its automata have in each configuration (*i.e.* on the set of possible state switches determined by $\mathscr{U}(x)$ and on the out-degrees of configurations in the system). Time-related questions such as those that have been mentioned in the previous paragraphs loose their immediate meaning. Further, although the problem of how to prune a transition graph in order to make all trajectories deterministic can obviously be pertinent in the case of a dynamical system[35], it is not in the more general context of state transition systems. Indeed, in this context, transitions are associated to no more information than that of there own existence. Thus, no transition of the system that is *a priori* possible can be disregarded *a posteriori* (even if, absolutely speaking, it can be interpreted as the representation of a highly improbable event). The only pertinent questions in the context of state transition systems are "existence questions" such as: *Can a configuration $y \in \mathbb{B}^n$ that satisfies a certain set of properties be reached from a given configuration $x \in \mathbb{B}^n$? Is a given behaviour possible? Can this transition be made? What new behaviours can be reached or become possible if some new transitions are added?...*

To go further while remaining very close to the remarks of this paragraph, we now give special attention to the notion of synchronism which was also introduced in the first chapter as being a synonym of the non-temporal notion of "non-atomicity".

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[34] $y^{(k)} \in \mathbb{B}^n$ (with $0 \leq k < \deg^+_{\mathscr{T}}(x)$) denotes one of the out-neighbours of x in the digraph $\mathscr{T} = (\mathbb{B}^n, \mathrm{T})$ that defines the state transition system.

[35] But importantly, for this question to mean anything, some indispensable, additional information must be provided, *e.g.* a stochastic transition matrix that specifies what transitions can indeed be ruled out despite that they are predicted as being possible by the basic theoretical model.

1.4 Interpreting the restriction of asynchronism

In many contexts, synchronism is discarded on the grounds of the great unlikeliness of simultaneity in nature. To explore rigorously the meaning of this, let us refer to the hypothesis of asynchronism – by which all non-atomic transitions of a BAN must be ignored – as HA. Also let $x \in \mathbb{B}^n$ be a configuration of an arbitrary BAN $\mathscr{N}$ in which both automata i and j are unstable ($i, j \in \mathscr{U}(x)$). Assuming HA, transitions

$$x \xrightarrow{i} \overline{x}^i \quad \text{and} \quad x \xrightarrow{j} \overline{x}^j$$

are both possible but transition

$$x \xrightarrow{\{i,j\}} \overline{x}^{\{i,j\}}$$

is not. Explaining this by stating that "i and j cannot *both* change states in x" is not enough because it contradicts the very definition of $\mathscr{U}(x)$ that states that on the contrary automata i and j both can change states in x, are "on the verge of" doing so in x, and have no reason not to. Thus, rather, it must be explained by putting forward that "i and j cannot both change states *at once*"[36]. Obviously, to support this interpretation of HA requires that the notion of simultaneity be accounted for. This, in turn, requires to account for a notion of time so the most simple, causal point of view of Section 1.3 is inadequate. Furthermore, a framework that accounts for precedence (*cf.* Section 1.2) does not necessarily disregard simultaneity. However, I claim that it definitely lacks the substance to go all the way up to affecting a meaning to the notion of "unlikeliness" which is put forward to support HA and which involves events that are essentially durable. One way to see this is to suppose that simultaneity is indeed accounted for without the notion of duration. Then, the events that are modelled are *punctual*. They can either be the beginning or the ending of a change. In this context, the modelling is based on a theory that fundamentally allows simultaneity (since in any configuration x, i and j may simultaneously become unstable, so several *beginnings* of changes can indeed occur at once) while HA denies it (or at least denies the simultaneity of *endings*). Thus, the only possible framework in which HA may possibly be justified consistently with an argument that involves a notion of simultaneity is the one of Section 1.1. In other terms, time flow must be assumed to be modelled in a way that the meaning of each transition $x \twoheadrightarrow y$ is augmented with a label that represents its duration.

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[36] *i.e.* informally, a justification of HA must contain a justification of why a system is unable to follow the steepest descent and convert its potential energy into kinetic energy (*cf.* Section 1, Chap. 5). Interestingly, the only update modes that can bypass the need for such a justification are those containing all transitions of $\mathscr{T}_{[\pi]}$.

In this framework, it becomes significant and coherent to consider what happens *during* a transition $x \relbar\joinrel\relbar\joinrel\rhd y$, between the moment where the network is in configuration x and the moment it reaches configuration y. Abusing notations we can write $x \;{-}{\scriptstyle d_k}{\rhd}\; \overline{x}^k$ do denote the asynchronous transition that updates automaton k and lasts d_k units of time. Then, more precisely still, HA must be supported by the idea that "no two automata can *finish* changing states at once".

In our running example, this simplifies to $d_i \neq d_j$. Without loss of generality, let us assume that $d_i < d_j$. Thus, in configuration x, say at time 0, both automata start changing states since, according to the theory, they can. However at time d_i, automaton i has effectively changed states whereas automaton j has not yet had the chance to finish doing so. At this moment, only two situations are *a priori* coherent with both the theory of BANs and our interpretation of HA:

1. Either j has become stable ($j \in \overline{\mathscr{U}}(\overline{x}^i)$) in which case it must be that j is influenced (directly or indirectly) by i, *i.e.* there is an arc or a path from i to j in the structure **G** of $\mathscr{N}$;
2. Or j is not influenced (neither indirectly nor directly) by i (and nor is its instability) and thus, the change of states of i (effective at the current time d_i) has not affected j whatsoever and $j \in \mathscr{U}(\overline{x}^i)$ can still change states in $\overline{x}^i$.

In case 2, we could make the duration of transition $\overline{x}^i \relbar\joinrel\relbar\joinrel\rhd \overline{x}^{\{i,j\}}$ account for the amount of time d_i that has already passed since the moment j started its switch of states, *i.e.* since the network was in configuration x. This way, we would have derivation

$$x \overset{d_i}{\relbar\joinrel\relbar\joinrel\rhd} \overline{x}^i \overset{d_j - d_i}{\relbar\joinrel\relbar\joinrel\relbar\joinrel\relbar\joinrel\rhd} \overline{x}^{\{i,j\}}.$$

However, first, this still does not explain why $x \relbar\joinrel\relbar\joinrel\blacktriangleright \overline{x}^{\{i,j\}}$ is disregarded (nor does it support the simultaneity that HA allows in beginnings of changes). But what is worse perhaps is the underlying risk of sliding towards the modelling of a model – which would be of a continuous nature to have the capacity of supporting consistently the fundamentally continuous notion of time. Indeed, this point of view definitely shifts the attention away from the essential nature of the Boolean (and more generally, discrete) modelling onto the notion of time, which is made here to depend on the history of the overall system, and which intrinsically requires dedicated attention if any interpretation is effectively to be drawn from the model [85].

This since stance obviously wholly disavows the very nature of the theory used for modelling, in case 2, HA imposes to assume that the situation

$$x \overset{d_i}{\relbar\joinrel\relbar\joinrel\relbar\joinrel\relbar\joinrel\rhd} \overline{x}^i \overset{d_j}{\relbar\joinrel\relbar\joinrel\relbar\joinrel\relbar\joinrel\rhd} \overline{x}^{\{i,j\}}$$

is possible while

$$\begin{tikzcd} x \arrow[r, "d_i"] \arrow[rr, bend right, "d_j"'] & \overline{x}^{i} & \overline{x}^{\{i,j\}} \end{tikzcd}$$

is not. This means that even though j starts changing states in x, it must stop and start all over again in $\overline{x}^{i}$, making the whole process last $d_i + d_j$ time units rather than $\max\{d_i, d_j\} = d_j$. Consequently, the change underwent by i in transition $x \xrightarrow{i} \overline{x}^{i}$ which *a priori* involves i alone also has an impact on the possibilities of j. And this impact is non-negligible since it concerns the duration of an event in a context where durations, precisely, have been given significance.

Thus, in the two sole possible cases 1 and 2 listed above, j is influenced by i (this leads to a contradiction in the case 2). This means that HA must either be taken to imply that *two automata cannot be simultaneously unstable unless one has an influence on the other, i.e.* every time $u(x) > 1$, all automata in $\mathcal{U}(x)$ are structurally connected. Clearly, this immediately implies an excessive restriction in terms of modelling. So the only solution to justifying HA with an argument involving simultaneity, consists in interpreting it otherwise, with the much stronger assumption that *no two distinct events can start, finish or occur synchronously, i.e. they cannot even overlap* in the time setting of the real modelled system[37].

I believe that this indispensable implicit assumption following from what seems at first sight to be a natural justification of HA, in most contexts, surpasses the original intentions of the modelling. This snag associated to HA highlights the general difficulty in modelling of justifying restrictions or refinements brought to the original theory. In particular, it shows more generally that choosing to disregard some elementary transitions of a BAN is non-trivial (unless the justification is simply theoretical convenience): no justification can obviously be drawn exclusively from the modelling theory, but nor can any be drawn exclusively from the modelled reality. Necessarily, justifications must be shaped in rigorous consistency with the pre-defined reality–theory correspondence mentioned at the beginning of the present chapter. [85] explores this subject further.

Importantly, let us highlight that this discussion on the justification of HA does not take anything away from the pertinence of this restriction, absolutely speaking. In particular, its utility in a purely theoretical setting remains unquestioned. When looking for stable configurations, for instance, as Robert [102, 103] noticed,

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[37] One consequence of this is to limit the meaning of the digraph $\mathbf{G}(x)$ relative to a BAN structure $\mathbf{G}$ and to a BAN configuration x (*cf.* Page 14) since this digraph fundamentally amounts to a snapshot of the functional interactions of the network in x: without the possibility of simultaneity, the interactions represented in this snapshot are essentially independent. Thus $\mathbf{G}(x)$ is a purely theoretical object, just like the momentum $u(x)$ and its study only is relevant with the causal viewpoint of Section 1.3 which is simply concerned with sets of existing possibilities.

asynchronous derivations often lead faster to the objective. In similar, more general lines, since the demonstration developed in Section 2, Chap. 5, we know that to be significantly sensitive to synchronism (depending on what "significantly" is taken to mean, *cf.* the levels listed on Page 117) requires the network to satisfy some strong conditions. This suggests that in some contexts it might be reasonable to discard synchronism after all (not on the grounds of its unlikeliness with respect to what it represents but rather on that of its little influence), providing additional support for developments under HA.

1.5 Interpreting synchronism & time flow

The previous section demonstrates that synchronism cannot reasonably be definitely discarded so its ins and outs must be questioned to be better understood. As suggested earlier, I claim that one of the most basic, straightforward ways to do this is to consider a more relaxed version of *time flow*. This is very natural because whatever the context, any possible definition of time flow is necessarily entirely relative to the precise sequences of experienced events that are considered in that context. And in the context of BANs, the events considered are the (sequences of) updates materialised by transitions.

This emphasises again the role of update modes. By definition, they filter some events and organise others. Thus, they are responsible for what I believe to be the only natural notion of time flow that can be associated to BANs. As an important result, many problems relative to the updatings (*e.g.* the sensitivity of network behaviours to their updating) can be understood in terms of time flow (although this requires some care). The next sections (*cf.* titles of their subsections) tacitly rely on this idea. Also, this tight relation between an abstraction of time flow and updatings adds some strong incentives to study further the latter in BANs.

To be more precise, in this setting, derivations in a transition graph convey a notion of precedence in agreement with the baseline idea of Paragraph 1.2. But the emphasis is rather put on the intermediary configurations through which the networks transit, or not, thereby selecting a future over a set other possible ones. In this framework, theoretical synchronism – *i.e.* non-atomicity of updates – no longer needs to be interpreted as simultaneity. On the contrary, since the theory has been shown to be able only to account for a *relative* notion of duration, it must rather be understood less restrictively but more safely as the possibility that several events may happen "fast enough" to forbid the occurrence of any other event while they do. Thus, the fact that the network performs transition

$$x \xrightarrow{\{i,j\}} \overline{x}^{\{i,j\}}$$

rather than performing any one of the following derivations

$$x \xrightarrow{i} \overline{x}^i \xrightarrow{j} \overline{x}^{\{i,j\}} \quad \text{and} \quad x \xrightarrow{j} \overline{x}^j \xrightarrow{i} \overline{x}^{\{i,j\}},$$

means that whatever could happen in configurations $\overline{x}^i$ and $\overline{x}^j$ that could short-circuit the derivation of the BAN "intending" to land on $\overline{x}^{\{i,j\}}$, won't happen, or at least won't have the chance to effectively perturb the BAN and make it deviate from its initial "intent".

The study presented in Section 2, Chap. 5 on synchronism-sensitivity can be re-examined further in the light of this new interpretation. Let us concentrate on level 0 of the classification proposed on Page 117. It groups BANs for which synchronism only allows to "go faster" in the sense that adding synchronism only adds shortcuts for asynchronous derivations. In Section 2, Chap. 5, these BANs are presented as "non-sensitive to synchronism". However, it can now be put forward that adding shortcuts to derivations is not in itself insignificant for networks that are seen as dynamical systems. Indeed, the effect of a shortcut may be to select the continuation of a derivation – possibly in a non-reversible fashion – by missing some intermediary steps in which a choice is otherwise possible. And these choices become especially important when the network is not deliberately supposed to be isolated from any possible environmental influences (*cf.* discussion at the end of Section 1.1, Chap. 3). Notably in relation to this, we recall that Section 2.5, Chap. 4 argues that the only way to take these into account *within* the theory of BANs but *independently* of them is through the updating mode, that is, precisely, through the definition of the set of "acceptable" transitions and derivations. In other terms, skipping intermediary configurations as a result of an addition of synchronism or due to constraints of a pre-supposed update mode may possibly reduce the set of global events that are experienced by the system and are otherwise possible. For instance, missing a critical configuration in which several transitions are possible – some of which corresponding to non reversible changes as in Example 5.3 – may cause to implicitly make a definite choice of trajectory and of attractor. Absolutely speaking, the set of possibilities remains the same. But for an evolving dynamical system, the addition of synchronism impacts on the trajectories that are effectively selected and occur.

Less formally, in a wider, modelling context (consistent with the "dynamical systems view"), rather than just representing the instants that frame system hitches (modelled by network transitions), configurations are conceptual breakpoints in the representation of a system's trajectory. These breakpoints (whose precise meanings modelling-wise tightly depend on the modelling intents and are determined by the modelling framework definition [85]) allow a punctual hold on the modelled trajectory, corresponding to the momentary possibility of intercepting the flow of events that are represented. If the network models a system set in a larger environment, then this means that exterior influences can only be taken into account at these pin-points. Thus, forbidding the transit through a given configuration amounts to ignoring the influences that can potentially impact on the system in the modelled situation. And although most often, as Section 2, Chap. 5 explicits, synchronism cannot fundamentally change the set of possible behaviours of a BAN, potentially, it can significantly change their effective behaviours in a given situation.

2 Update modes

Pushing the ideas of the previous paragraph a step forward suggests that to design an update method corresponds to architecturing the time flow experienced by the BANs in question. The ambition in this outlines the considerable limit in terms of representational capacity that unavoidably accompanies any choice of an updating mode. And indeed, we have argued that despite traditional choices made in the literature, assuming asynchronous updatings [10, 11, 57, 64, 93, 99, 96, 97, 98, 110, 109, 115, 118, 119, 122, 120, 123, 130] and disregarding synchronicity altogether is not always reasonable in terms of modelling (*cf.* Section 1.4) and the same goes for assuming deterministic updatings [7, 30, 6, 27, 32, 34, 35, 44, 50, 47, 48, 56, 60, 62, 78, 91, 102, 103, 129, 126] that impose the system to infallibly follow the same predefined schedule. Generally, *any* choice of an updating is hard to justify unless it fits into a methodology such as the one described theoretically in Section 1, Chap. 1 whose aim is to undercover precise *local* or *punctual* causality relations with well identified preconditions and a well-bounded (which often means not so ambitious) representational scope. For this reason, under conditions where a Boolean modelling *is* pertinent, I claim that in the formalism we have considered until now, *all* updatings that define restrictions on the set of possible transitions – *i.e.* all updatings that do not correspond to the GTG – are unrealistic *per se*. The next paragraphs endeavour to evidence how update modes remain especially interesting and useful.

A. Complexity and scales. To effectively exploit the potential representational capacity of update modes requires (1) to evaluate the way that time flow is experienced by the real systems that are modelled and then (2) to formalise it into the theoretical definition of an update mode. The difficulty in this task can (partially) be put on a problem of scales (of time and of structures with respect to the prior knowledge at hand concerning the systems considered). Indeed, as long as a system is in a fundamentally unstable state, the minute sequencings of transient, elementary events matters a lot, although they are hard to determine (otherwise would mean that the system's underlying, defining mechanisms are known very well, in which case, there would be no modelling issue). In other terms, the series of differentiations occuring at a “microscopic” time scale which eventually lead a system to produce a final output behaviour or *function* are decisive (the theoretical cases studied in this thesis in particular, evidence this). And thus the influence of time flow is an additional source of complexity[38] which combines to the “fundamental” complexity of networks issuing precisely from the “the way the net works”, *i.e.* from the interactions between elements/automata and their organisation.

In practice, dealing with both sources of complexity simultaneously is often intractable. This problem can be bypassed if, somehow, one of the two sources of complexity is already, or can be, fathomed and bounded in a way that allows to safely concentrate on the other source of complexity, without risking to mistake the causes of the phenomena that are studied. This motivates further studies in the continuity of those presented in this document aiming at understanding the impact of updatings. Indeed, as argued in Section 8, Chap. 2 as well as in the introduction of Chapter 5 and in the next section, one expected benefit that might be drawn from the possible results of studies on update schedules and modes is to better understand when the influence of these does not matter, or at least when does it not matter “as much” as other influences (*e.g.* influences due to structural properties such as cycle intersections which I conjecture to “overrule” updating constraints imposed by the parallel update schedule under which they were observed).

Of course, the double complexity problem can also be dealt with by concentrating on small networks (favouring the study of GTGS, *cf.* introduction of Chapter 5). To reasonably consider modelling larger networks requires a change of perspective. To compensate for the large BAN sizes, less precisions must be required in the descriptions of their features. This requires the definition of a new framework to which, obviously, the problems and questions considered must be adapted. In these lines fall the considerations on the structure-oriented hierarchy proposed in Section 8, Chap. 2.

Also in these lines, the multi-scale framework informally proposed on Page 156 of the appendix (see also the multi-layered update schedule introduced in [26] to account for chromatin dynamics in a context that models genetic networks) is based on the idea that once a system is globally stabilised – either because it has reached a behaviour of least potential, or because something exterior is maintaining a certain degree of instability in it – its sensitivity to time is lesser (*cf.* Section 3). And then, in some settings, it might make sense to take a step back and observe the system from a more distant point of view and make it interact with other systems, as *modules* of a larger encompassing *macro-system*. In this

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[38] See the conclusion of Chapter 3 which highlights how a BAN can be made to behave almost any way by playing on the definition of its update schedule. See also the next section which precisely focuses on the influence of time flow on interaction networks.

framework, module interactions take place on a "macro time scale" while the automata interactions within each module take place on "micro time scale". This way, the influence of the minute sequencings of events that take place on the micro time scale, within the modules, is decreased in comparison to the wider spanning events that are now also considered. The attention to time is shifted away from unobservable levels of details and towards a greater abstraction which allows to focus on update schedules with more fathomable meanings because of their intent to account for more "macroscopic" phenomena. For this reason, the modelling might gain in pertinence although, again, the problems considered cannot be the same as before.

In particular, with such hindsight on BAN behaviours, I believe that the notions of global instability and of the matching notion of degrees of freedom gain in substance and practicability. Section 8, Chap. 2 proposes to study emergence in the theory of BANs with respect to levels or scales defined by BAN structures. Similarly, the discussion of the present paragraph suggests that the same can perhaps be done with respect to levels of a "macro-update schedule" supposed to represent different time scales. In both contexts ("structural" and "temporal"), I expect that theoretical results such as those presented in this document may be exploited in an exploration of how system behaviours emerge from one level to the next and more generally, how different levels relate by using a progressive list of questions such as: *How can an effect generated by or at a certain level affect a higher one? What perturbations occurring at one level impact on higher ones and how?* conversely, *How can a system behaviour observed at one level have a retroactive impact at a lower one?* and *How can drastic behavioural changes be explained by system weak-points embodied by properties that define frontiers between levels?* As concrete preliminary support for such studies, let us note that the "non-monotone widget" of Example 5.3 (*cf.* Section 2, Chap. 5) gives a tangible example of how a global effect (a non-monotone action) can emerge when local interactions (which individually are monotone) combine properly and automata are updated in the right order.

B. Hardwiring time. The design of a multi-scale framework raises the new question of *how can updating constraints be "hardwired" or "protected" structurally?* Indeed, the informal discussion on Page 156 of the appendix mentions modules of a macro-BAN that would be able to maintain each other stably in transient states (or behaviours) involving many instabilities. For instance, a module whose structure is a cycle could be forced "artificially" to behave according to an SCC in one of the higher layers of its GTG (*cf.* Section 1, Chap. 5), that is, according to one of the unstable attractors induced by the parallel update schedule π. Addressing the question of how to hardwire influences such as those imposed by update modes would help highlight the greater robustness of some update modes owing to certain external structural constraints that make their updatings more likely.

In modelling, it would help make better choices of update modes by ruling out those that are very unlikely to be maintained over time because they have no "structural support". More generally, it would also help identify the networks that are less sensitive to time influences (*cf.* next section).

C. Designing an intelligent update schedule. Further than helping to rule out the update modes that are unlikely and unrobust, future researches on update modes can also aim at helping bypass the choice of an update mode altogether. Indeed, they can be oriented towards the design of an *ad hoc* "intelligent update mode" – or ideally, update schedule – that is characterised by its simplicity and by its ability to capture all main/desired behavioural properties of a network without implicitly suggesting unlikely behaviours (*cf.* above and Example 5.1). To ignore time flow when time flow does not matter, this update mode might first be designed so that it induces a "pruned version" of the GTG in which critical points are kept but deterministic sequences of consecutive transitions are replaced by just one transition. But, generally, I expect this definition to benefit significantly from the improved knowledge on the influence of time flow on the evolutions of interactions systems that is supposed to follow from the various studies suggested in this chapter.

In particular, taking "intelligent" to mean more economic in terms of the size of the associated transition graph $\mathscr{T}_{[\delta]}$, an update schedule may be defined in a modular way, on the basis of the structural hierarchy described in Section 8, Chap. 2. This supports studies in the continuity of Section 2, Chap. 5. Indeed, to explore the sensitivity of BANs to synchronism, Section 2, Chap. 5 looks for ways to sequentialise synchronous transitions. Taking the opposite point of view, one can look for ways to use synchronism safely in order to "shortcut" asynchronous derivations. In [102, 103], Robert addresses a very similar question concerning the "best update schedule", *i.e.* the one leading to fix points in the least steps. Exploiting this, the "intelligent update schedule" could be designed so that in a large network, modules (*cf.* elementary bricks mentioned in Section 8, Chap. 2) that are known to be non-sensitive to synchronism are updated in an unrefined, inexpensive way with the parallel update schedule π (or at least with a maximum of synchronism and exploiting the transitivity induced by update schedules mentioned in Section 1.2, Chap. 3) while sensitive modules are imposed less updating constraints. And globally, the updatings of different modules are assembled in a more refined way. In some other informal terms, the behaviour of the whole network is described with a GTG in which some parts, relative to non-sensitive modules, are simplified. This way, with the possible additional support provided by the "neutrality of π" discussed in Section 3.D, I expect that this hypothetical "intelligent update schedule" can eventually be improved further and made deterministic, modular and scalable. This would help bypass some complexity issues as well as take various instructive viewpoints on network behaviours, according to the problems at hand; and it would also be useful for some of the other problems evoked below and for the study of time scales proposed in Section 2.A above.

3 Impact of time

In this final, closing section, we list some questions and remarks that follow from the work presented in this thesis. Let us emphasise that although these are presented in a sequential manner, they are so very intimately connected that it is more appropriate to consider them as various viewpoints on the same global problem concerning the way that time flow represented by update modes impacts on the behaviours of interaction networks.

A. Encoding information into time. The structural "non-monotone widget" of Example 5.3 (*cf.* Section 2.4, Chap. 5) only involves a few non-monotone individual interactions but these are architectured in a way that they globally "mimic" a punctual non-monotone action if and only if, in the right configuration, the right set of automata are updated at once. Thus, in this example, what is usually hardwired in the structure of a system is simulated by a correct sequencing of events. Conversely to the idea presented in Section 2.B, structural properties are encoded in the updating (this motivates the study of the notion of "elementary/punctual functionality" through the study of $\mathbf{G}(x)$ in a context that allows for non-atomicity). Also, in Chapter 3, it is proven that except for some special networks characterised by very restricted structural properties, all positive disjunctive networks can be made to oscillate asymptotically with an appropriate choice of update schedule. This suggests that if the order of events (automata state changes) in a network can be chosen, then so can its general behaviour. Thus generally, these two examples raise the questions of *what information precisely is held or processed by the definitions of updatings/time flow?* and *how does the "functionality" of a network relate to these?*

B. Filtering behaviours and lifting ambiguities. The influence of the sequencing of events on the behaviours of BANs is undoubtful. The example of simulated non-monotony implies that structural motifs can even define different functions according to the updating that is chosen. More conceptually, this suggests that similarly to the structural constraints highlighted in Section 8, Chap. 2, time flow may also act as a "filter" capable of "lifting ambiguities" and imposing/selecting one behaviour/function of a system when *a priori* its underlying structural engineering defines and allows for several.

C. Updating artefacts & revelations. Some BANs manifest asymptotic behaviours that have very little if not no chances to be sustained unless a precise schedule of updates is respected exactly (*cf.* Example 5.1): the slightest error in this schedule causes the network to move definitely onto a different behaviour. Thus, if ever these behaviours model behaviours of real systems, then these are extremely unlikely to be observed. On the contrary, the decrease in the degrees of freedom of BANs noticed in Chapter 2 seems rather caused by cycle intersections than by the imposed parallel updating π. This suggests that some network properties are pure *artefacts* of update modes (*cf.* the non-monotonic action mentioned above) while others are just *revealed* by them. I believe that this calls for further researches oriented towards distinguishing these two cases and specifying what structural motifs are more sensitive to update errors.

D. The evolution of time sensitivity & the increased neutrality of π. The idea of the previous paragraph concerning cycle intersections whose action is revealed by π can also be stated the following way around: *the more there are cycle intersections in a network structure, the less the parallel update schedule seems to "fake" the asymptotic stability of network behaviours.* Informally, when the number of underlying structural bifurcations increases, the "correctness" of this very special (and perhaps otherwise unlikely) update mode also increases while its impact on network behaviours decreases. Besides its possible utility in the definition of the "intelligent update schedule" of Section 2.C, this observation pertinently relates to the subject of network phylogeny (*cf.* Section 8, Chap. 2). If nature is assumed to complexify the structures of genetic networks by adding what serves as "gears" and "filters" to them [26], then it implies that networks evolve towards a lesser sensitivity to time. As suggested in Section 8.G, Chap. 2, Chap. 6, it would be interesting here also to confront theoretical developments with effective biological data, compare the different phylogenetic stages of real systems with the different levels of granularity in the scalable behaviour descriptions induced by the "intelligent update schedule" and, conversely, endeavour to make this update schedule reflect the evolution of systems in some way.

E. Criticality. Example 5.3 in Section 2, Chap. 5 highlights the existence of network behaviours that are stable as long as one specific (normal) synchronous transition is not made. The criticality of this transition can be taken as due to a sensitivity of the BAN to *simultaneity.* But, it can also be interpreted in terms of a more relaxed notion of relative time flow according to Section 1.5 This subject can be explored further with the aim of characterising more generally the irreversible changes that a BAN can possibly undergo through questions such as: *What specific (sequences of) updates* (*e.g.* defined precisely by the update schedule or, on the contrary, corresponding to mistakes in its implementation) *can suddenly provoke one behaviour to be selected and all others to be definitely ruled out? Under what circumstances do these drastic events happen?* (*i.e. what are the behaviours or critical configurations in which such decisive transitions are possible?*) and *What structural properties allow/cause them?* (*e.g.* non-monotonic interactions in Example 5.3 [84]).

F. Synchronism *vs* asynchronism (again). Paragraph 3.B raises the question of *how to select/eliminate attractors or attraction basins using update schedules, and also perhaps, conversely, how to create some new ones by stabilising some behaviours?* In the direct continuity Section 2, Chap. 5 and of [102, 103] in which Robert proposed to compare the transition graphs associated to π and to sequential update schedules, one can explore further the relative roles of synchronism and asynchronism. The various results concerning cycles derived in this thesis suggest that as long as a certain a parallel updating is maintained, local instabilities which represent the network's momentum are also maintained, and as a result, the number of future evolution possibilities is kept maximal. From the converse point of view, this idea translates into the following. As noted by Robert, it seems that although it does not always do so, sequentialisation often *easily* eliminates unstable oscillating attractors that rely on a strong constraint of non-atomicity (in the sense that they require the on-going simultaneous update of many automata, *cf.* the GTGs of BACs described in Section 1, Chap. 5). On the other side, Section 2, Chap. 5 shows that adding synchronism can also eliminate attractors. But it does so *much less easily* by "emptying" one attractor into another. So synchronism can be understood as being able to select attractors that are *globally more stable.*

APPENDIX

A Complementary proofs for Chapter 2

Proof of Lemma 2.8

This lemma figures on Page 48, in Section 4, Chap. 2.

Lemma 2.8. Recurrent configurations of BADs

Let $x \in \mathbb{B}^n$ *be a configuration of the* BAD $\mathscr{D} = \mathscr{D}_{\ell r}^{ss'}$. *For any divisor p of the order of* $\mathscr{D}$, *the circular word* $w \in \mathbb{B}^p$ *defined by:*

$$\forall j < p,\ w_j = x_0(p-j)$$

$$\text{satisfies:} \quad x \in \mathscr{X}(p) \quad \Leftrightarrow \quad w \in \mathscr{W}^d(p) \text{ and } \begin{cases} x^L = w^q\, w[0, d-1] \\ x^R = w^{q'}\, w[0, d'-1] \end{cases}$$

where $\ell = qp + d \equiv d \bmod p$, $r = q'p + d' \equiv d' \bmod p$.

Proof: For an integer $p \in \mathbb{N}$ (intended to be a period of $\mathscr{D}_{\ell r}^{ss'}$), we write:

$$(i)_p \;=\; i_p \;=\; i^{\dagger} \bmod p,\ \forall i \in \mathbf{V} \tag{6.2}$$

and $\forall v \in \mathbb{B}^p$, we let $v^{(n)}$ denote the configuration of $\mathscr{D}_{\ell r}^{ss'}$ satisfying:

$$x = v^{(n)} \;\Rightarrow\; \big(x^L = v^q v[0,d-1] \;\wedge\; x^R = v^{q'} w[0,d'-1]\big).$$

It satisfies $i^\dagger > p \;\Rightarrow\; x_i(p) = x_{i-p}(0) = v_{(i-p)^\dagger} = v_{i^\dagger}$ and thus:

$$\forall v \in \mathbb{B}^p,\; x = v^{(n)} \;\Rightarrow\; \forall i \in \mathbf{V},\; i^\dagger > p,\; x_i(p) = v_{i^\dagger} = x_i. \tag{6.3}$$

Conversely, supposing that x has period p and letting $v \in \mathbb{B}^p$ be a word such that:

$$\begin{cases} v = x[0,p-1] & \text{if } p < \ell{+}r \\ v[0,\ell-1] = x^L[0,\ell-1] & \text{if } p = \ell{+}r,\, \ell \geq r \\ v[0,r-1] = x^R[0,r-1] & \text{otherwise,} \end{cases}$$

configuration x satisfies (*cf.* (2.5)) $\forall i = i^{\curlyvee} + i^\dagger \in \mathbf{V},\; x_i = x_{i^\dagger} = x_{i_p}$ and therefore:

$$x \in \mathscr{X}(p) \;\Rightarrow\; x = v^{(n)}. \tag{6.4}$$

Now, let us suppose that $x \in \mathscr{X}(p)$ and let $k \in \mathbb{N}$ be such that $kp \geq p + \max\{\ell, r\}$. By its definition in Lemma 2.8, $\forall j < p$, the word w satisfies:

$$\begin{aligned} w_j &= x_0(p-j) = x_0(kp-j) \\ &= \mathrm{f}_0^L(x_0((kp-j-\ell) \bmod p)) \vee \mathrm{f}_0^R(x_0((kp-j-r) \bmod p)) \\ &= \mathrm{f}_0^L(x_0((p-j-d) \bmod p)) \vee \mathrm{f}_0^R(x_0((p-j-d') \bmod p)) \\ &= \mathrm{f}_0^L(w_{j+d}) \vee \mathrm{f}_0^R(w_{j+d'}). \end{aligned} \tag{6.5}$$

If $(s,s') = (-,+)$, this implies that $\forall j < p, w_j = \neg w_{j+d} \vee w_j$ (because $d' = 0$ by Lemma 2.6) so $w_j w_{j+d} \neq 00$. If $s = s' = -$, then (6.5) implies that $\forall j < p, w_j = \neg w_{j+d} \vee \neg w_{j-d}$ (because $p | \ell{+}r$ and thus $d' = p - d$ by Lemma 2.6) so $w_{j-d} w_j w_{j+d} \notin \{000, 001, 100, 111\}$. As a result, $w \in \mathscr{W}^d(p)$ holds in all cases. Noting that $v = w$ (because $\forall i < p, v_i = x_i = x_0(-i) = w_i$), this, together with (6.4) proves the first implication ($\Rightarrow$) of Lemma 2.8.

Conversely, let us suppose that $w \in \mathscr{W}^d(p)$ and $x = w^{(n)}$. First, let us also suppose that $p < \max\{\ell, r\} = \ell$ and let $t = kr + t' \equiv t' \bmod r,\; t < \ell$. If $k = 0$, then, because of $x = w^{(n)}$:

$$x_0(t) \;=\; \mathrm{f}_0^L(x_{\ell-t}(0)) \vee \mathrm{f}_0^R(x_{n-t}(0)) \;=\; \mathrm{f}_0^L(w_{d-t}) \vee \mathrm{f}_0^L(w_{d'-t}).$$

If $p > r$ and if $k = 1$, then $x_0(t) \;=\; \mathrm{f}_0^L(x_{\ell-t}(0)) \vee \mathrm{f}_0^R(x_0(t')) = \mathrm{f}_0^L(w_{d-t}) \vee \mathrm{f}_0^L(w_{p-t'}) = \mathrm{f}_0^L(w_{d-t}) \vee \mathrm{f}_0^L(w_{r-t}) = \mathrm{f}_0^L(w_{d-t}) \vee \mathrm{f}_0^L(w_{d'-t})$ by the definition of w in Lemma 2.8.

And if $k > 1$, then, by induction on k, $x_0(t) \;=\; \mathrm{f}_0^L(x_{\ell-t}(0)) \vee \mathrm{f}_0^R(x_0((k-1)r + t')) = \mathrm{f}_0^L(w_{d-t}) \vee \big(\mathrm{f}_0^L(w_{d-t}) \vee \mathrm{f}_0^L(w_{d'-t})\big) \;= \mathrm{f}_0^L(w_{d-t}) \vee \mathrm{f}_0^L(w_{d'-t})$.

If $p < \max\{\ell, r\}$, this yields:

$$
\begin{array}{rcl}
\forall t < \max\{\ell, r\},\ x_0(t) & = & \mathrm{f}_0^L(w_{d-t}) \vee \mathrm{f}_0^L(w_{d'-t}) \\
& = & \begin{cases} w_{-t} \vee w_{-t} & \text{if } s = s' = + \\ w_{d-t} \vee w_{-t} & \text{if } s = - \neq s' = + \\ w_{d-t} \vee w_{-d-t} & \text{if } s = s' = - \end{cases} \\
& = & w_{-t}
\end{array}
$$

where the last but one equality holds because of $w \in \mathscr{W}^d(p)$. Thus, because $x = w^{(n)}$, $\forall i \in \mathbf{V}, i^\dagger = i_p$ we have $x_i(p) = x_0(p - i_p) = w_{i_p} = x_i$ which, with (6.3), proves the second implication ($\Leftarrow$) of Lemma 2.8 in the case where $p < \max\{\ell, r\}$.

To deal with the remaining cases (*cf.* Lemma 2.7), let $(s, s') = (-,-)$ and $p = \ell{+}r$. It holds that $\forall i \in \mathbf{V}$:

$$
\begin{array}{rcl}
x_i(p) & = & x_0(\ell{+}r - i_p) \\
& = & \begin{cases} \neg x_0(r-i) \vee \neg x_0(\ell - i) & \text{if } i < r \leq \ell \\ \neg x_{i-r}(0) \vee \neg x_{i-\ell}(0) & \text{if } r \leq i < \ell \end{cases} \\
& = & \neg w_{i-r} \vee \neg w_{i-\ell} = \neg w_{i+d} \vee \neg w_{i-d}
\end{array}
$$

where the last but least equality holds by definition of w if $i < r \leq \ell$, and by $x = w^{(n)}$ if $r \leq i < \ell$. Then $w \in \mathscr{W}^d(p)$ allows to conclude that $x_i(p) = w_i = x_i$. □

Proof of Lemma 2.9

This lemma figures on Page 48, in Section 4, Chap. 2.

Lemma 2.9. Periods and order of BADs

The order of a BAD $\mathscr{D} = \mathscr{D}_{\ell r}^{ss'}$ *(where* $\Delta = \gcd(\ell, r)$ *and* $\ell{+}r = K\Delta$*) equals:*

$$
\omega = \begin{cases} \Delta & \textit{if } (s,s') = (+,+) \\ r & \textit{if } (s,s') = (-,+) \\ \frac{\ell+r}{2} = 2\Delta & \textit{if } (s,s') = (-,-) \textit{ and } K = 4 \\ \ell{+}r & \textit{if } (s,s') = (-,-) \textit{ and } K \neq 4. \end{cases}
$$

Further, any divisor p *of* ω *is a minimal period of* $\mathscr{D}$ *except if* $(s,s') = (-,-)$ *and either* $p = 6\Delta_p = 6$ *or* $p = 4\Delta_p$ *(where* $\Delta_p = \gcd(\Delta, p)$*). Thus, the order* ω *of* $\mathscr{D}_{\ell r}^{ss'}$ *is reached unless* $(s,s') = (-,-)$ *and* $\omega = \ell + r = 6$.

Proof: We use the notations of Lemma 2.8 and its proof. $\forall s, s' \in \{-,+\}$, we let $\mathscr{A}^d(p) = \{w \in \mathscr{W}^d(p) \mid \forall j < p,\ w \neq w[j, p-1]\,w[0, j-1]\}$ be the set of aperiodic words $w \in \mathscr{W}^d(p)$. Then clearly, if there exists $w \in \mathscr{A}^d(p)$, configuration $x = w^{(n)}$ has minimal period p.

Let us show that there exists a **canonical aperiodic word** $w = \mathrm{w}_{(p,d)} \in \mathscr{A}^d(p)$ in each case, for any period $p > 1$:

- let $\mathrm{w}_{(p,d)} = 01^{p-1} \in \mathbb{B}^p$ if $(s, s') = (+,+)$ or $(s, s') = (-,+)$.

In the latter case, $\mathrm{w}_{(p,d)} \in \mathscr{W}^d(p)$ is true because, by Lemma 2.6, p does not divide ℓ so $d \neq 0$. If $(s, s') = (-,-)$, without loss of generality, we can suppose that $d' = p - d \geq d$. Let $a, b \in \mathbb{N}$ be such that $d' = (a-1)d + b \equiv b \bmod d$ so that $p = d + d' = ad + b$ and $\Delta_p = \gcd(p, \ell, r) = \gcd(p, d, d') = \gcd(b, d)$ equals d if $b = 0$. Then, it can be checked that in all the cases listed below, $\mathrm{w}_{(p,d)} \in \mathscr{A}^d(p)$ is true:

- If $a = 2a' + 1$ is odd, let $\mathrm{w}_{(p,d)} = (1^d 0^d)^{a'} 1^d 0^b$.
- If $a = 2a'$ is even and $b > 0$, let $\mathrm{w}_{(p,d)} = (1^d 0^d)^{a'} 1^b$.
- If $a = 2a'$ is even, $b = 0$, and $a > 6$ let $\mathrm{w}_{(p,d)} = (1^d 0^d)^{a'-3} (1^d 1^d 0^d)^2$.
- If $a = 2$, let $w = (1^d 0^d)$.

The remaining cases correspond to $p = 6d$ and $p = 4d$, where $d = \Delta_p$.

Let $p = 4d$. In this case, $w \in \mathscr{W}^d(p)$ imposes that $\forall k \in \mathbb{Z}/p\mathbb{Z}$, $w_k w_{k+d} w_{k+2d} w_{k+3d}$ belongs to $\{0101, 1010\}$. Therefore, any $w \in \mathscr{W}^d(p)$ can be written $w = uu$ for some aperiodic $u \in \mathscr{W}^d(\frac{p}{2})$ and the configuration $x = w^{(n)}$ satisfies $x^L = u^{2q} u[0, d-1]$ and $x^R = u^{2q'+1} u[0, d-1]$ (*i.e.* $x = u^{(n)}$) and has minimal period $|u| = \frac{p}{2} = 2d < p$. Thus, there are no configurations of minimal period $p = 4d$.

If $K = 4$ and $\Delta = 2^e \Delta'$, $\gcd(\Delta', 2) = 1$, then the set of minimal periods of $\mathscr{D}$ equals $\{2^{e+1} d \mid d | \Delta'\}$ so the order of $\mathscr{D}$ is $\omega = 2^{e+1}\Delta' = 2\Delta$. It is reached since $\mathscr{D}$ has configurations of minimal period $p = 2(\ell \bmod 2\Delta) = 2\Delta$.

Let $p = 6d$. The condition $w \in \mathscr{W}^d(p)$ imposes that:

$$\forall k \in \mathbb{Z}/p\mathbb{Z}, w_k w_{k+d} w_{k+2d} w_{k+3d} w_{k+4d} w_{k+5d} \in \{010101, 101010,\ 011011, 110110, 101101\}. \tag{6.6}$$

- If $\Delta_p = d \geq 2$, let $\mathrm{w}_{(p,d)} = 0^d 1^d 0^{d-1} 1^d 0^d 1^{d+1}$.

Writing $\mathrm{w}_{(p,d)} = 0^{d-1} 0\, 1^{d-1} 1\, 0^{d-1} 1\, 1^{d-1} 0\, 0^{d-1} 1\, 1^{d-1} 1$, it can be checked that $w \in \mathscr{A}^d(p)$. If $d = 1$, by (6.6), any $w \in \mathscr{W}^1(p)$ can either be written $w = uuu$ for some $u \in \mathbb{B}^2$, or it can be written $w = uu$ for some $u \in \mathbb{B}^3$. In both cases, u is aperiodic and $u \in \mathscr{W}^1(|u|)$. In the first (resp. second) case, the configuration $x = w^{(n)}$ satisfies $x^L = u^{3q} u_0$ and $x^R = u^{3q'+2} u_0$ (resp. $x^L = u^{2q} u_0[0, d-1]$

and $x^R = u^{2q'+1}u_0u_1$), *i.e.* $x = u^{(n)}$, and it has minimal period $|u| = 2 < p$ (resp. $|u| = 3 < p$) by Lemma 2.8. Consequently, there are no configurations of minimal period $p = 6$ if $\gcd(\Delta, p) = 1$.

Thus, if $K = 6$ and $\Delta = 1$, then 2 and 3 are minimal periods of $\mathscr{D}$ but 6 is not. So $\omega = \text{lcm}(2,3) = 6\ell{+}r$ is not reached. If $K = 6$ and $\Delta = 2^e3^f\Delta' > 1$ (for some $e, f \in \mathbb{N}$), $\gcd(\Delta',2) = \gcd(\Delta',3) = 1$, the set of minimal periods of $\mathscr{D}$ equals (*cf.* Lemma 2.6):

$$\{2^{e+1}3^{f'}d \mid f' \le f \wedge d|\Delta'\} \cup \{2^{e'}3^{f+1}d \mid e' \le e \wedge d|\Delta'\}.$$

The order of $\mathscr{D}$ which equals the least common multiple of all integers in this set equals $\omega = 2^{e+1}3^{f+1}\Delta' = K\Delta$. It is reached because $\Delta > 1$. □

Proof of Lemma 2.10

This lemma figures on Page 49, in Section 4, Chap. 2.

Lemma 2.10. The Perrin sequence

For $n > 0$, $\mathtt{P}(n)$ counts the number of circular words of size n without the subsequences 00 *and* 111, *i.e. :*

$$\mathtt{P}(n) = |\mathscr{W}^1(n)| \textit{ in the case where } (s,s') = (-,+).$$

Proof: Let $E_n = \mathscr{W}^1(n)$ in the case where $(s,s') = (-,+)$ (E_n is the set of circular words of length n without factors 00 and 111). The statement of Lemma 2.10 is true for $1 \le n \le 3$. If $n \ge 3$, then any word $w \in E_n$ can be written $w = u0v$ where u is a word with no 0, possibly the empty word ε. Further, for $n > 3$, only two disjoint cases are possible: either $v = 10v'$ or $v = 110v'$, for some binary word v'. From this follows that $E_n = \{u010v' \mid u \in \{\varepsilon,1,11\},\ u0v' \in E_{n-2}\} \uplus \{u0110v' \mid u \in \{\varepsilon,1,11\},\ u0v' \in E_{n-3}\}$ and then $|E_n| = |E_{n-2}| + |E_{n-3}|$. □

Proof of Lemma 2.12

Lemma 2.12 which figures in Section 6, Chap. 2, on Page 56 and its proof involve the two roots of $x^2 - x - 1 = 0$, *i.e.* the *golden ratio* $\gamma = \frac{1+\sqrt{5}}{2} \approx 1.61803399$ and $\overline{\gamma} = 1 - \gamma = \frac{1-\sqrt{5}}{2} \approx -0.61803399$. It also involves the three roots of $x^3 - x - 1 = 0$ which are the *plastic number* [132] $\xi \approx 1.32471796 \in \mathbb{R}$ and $\kappa = \frac{1}{2}(-\xi + i \cdot \sqrt{\frac{3}{\xi} - 1})$ and its complex conjugate $\overline{\kappa}$.

Lemma 2.12. Upper bound on $\mathtt{X}(p)$

For a divisor $p = K_p\Delta_p$ ($\Delta_p = \gcd(\ell, r, p)$) of the order of a BAD $\mathcal{D}^{ss'}_{\ell r}$, *the number of configurations of period p is bounded as follows:*

$$\begin{array}{lclcl}
\gamma^p & \sim & \mathtt{X}^{-+}_{\ell,r}(p) & \leq & \sqrt{3}^p \qquad\qquad\qquad\qquad \textit{if}\,(s,s') = (-,+) \\
\xi^p & \sim & \mathtt{X}^{--}_{\ell,r}(p) & \leq & \begin{cases} 3^{\frac{p}{3}} & \textit{if}\, K_p = 3 \\ \sqrt{2}^p & \textit{if}\, K_p \neq 3 \end{cases} \qquad \textit{if}\,(s,s') = (-,-).
\end{array}$$

Proof: The Lucas sequence satisfies [95]:

$$\forall n \in \mathbb{N},\ \mathtt{L}(n) = \gamma^n + \overline{\gamma}^n = \gamma^n + (-\frac{1}{\gamma})^n.$$

Consequences of this and of Lemma 2.11 are:

$$\mathtt{X}^{-+}_{\ell,r}(p) \ = \ \mathtt{L}(K_p)^{\Delta_p} \ = \ (\gamma^{K_p} + \overline{\gamma}^{K_p})^{\Delta_p} \xrightarrow[K_p \to \infty]{} \gamma^p$$

proving $\mathtt{X}^{-+}_{\ell,r}(p) \sim \gamma^p$, and also (using $\gamma^2 = 1 + \gamma$ and $\overline{\gamma} = -\frac{1}{\gamma}$):

$$\begin{aligned}
\mathtt{X}^{-+}_{\ell,r}(p) \ &= \ \sum_{k \leq \Delta_p} \binom{\Delta_p}{k} (-\gamma^2)^{K_p k} \cdot \overline{\gamma}^p \ = \ \overline{\gamma}^p \cdot ((-\gamma^2)^{K_p} + 1)^{\Delta_p} \\
&= \ (-1)^p \cdot |\overline{\gamma}|^p \cdot ((-1)^{K_p} \gamma^{2K_p} + 1)^{\Delta_p} \\
&= \begin{cases} |\overline{\gamma}|^p \cdot (\gamma^{2K_p} - 1)^{\Delta_p} & \text{if } K_p \text{ is odd} \\ |\overline{\gamma}|^p (\gamma^{2K_p} + 1)^{\Delta_p} & \text{if } K_p \text{ is even.} \end{cases}
\end{aligned}$$

Let us note that if p is odd (and necessarily so are K_p and Δ_p), then $\mathtt{X}^{-+}_{\ell,r}(p)$ is maximal when Δ_p is minimal, *i.e.* when $\Delta_p = 1$. And if p is even then, on the contrary, $\mathtt{X}^{-+}_{\ell,r}(p)$ is maximal when Δ_p is maximal, *i.e.* when $\Delta_p = p/2$ (*cf.* Table 2.4 on Page 58). In both cases:

$$\mathtt{X}^{-+}_{\ell,r}(p) \ \leq \ |\overline{\gamma}|^p (\gamma^{2K_p} + 1)^{\Delta_p} \ \leq \ |\overline{\gamma}|^p (\gamma^4 + 1)^{\frac{p}{2}} \ = \ \frac{(3 + 3\gamma)^{\frac{p}{2}}}{\gamma^p} \ = \ 3^{\frac{p}{2}},$$

which proves the first inequality of Lemma 2.12. The rest of Lemma 2.12 concerning $\mathcal{D}^{--}_{\ell r}$ (*cf.* Table 2.5 on Page 59) derives from the following relation that is satisfied by the Perrin sequence [1]:

$$\forall n \geq 2,\ \mathtt{P}(n) = \xi^n + \kappa^n + \overline{\kappa}^n,$$

and from Lemma 2.11 which yields $\mathtt{X}^{--}_{\ell,r}(p) = (\xi^{K_p} + 2\cos(\arg(\kappa^{K_p})) \cdot |\kappa|^{K_p})^{\Delta_p}$, where $|\kappa| = 1/\sqrt{\xi} < 1$, and thus:

$$(\xi^{K_p} - 2|\kappa|^{K_p})^{\Delta_p} \;\le\; \mathtt{X}^{--}_{\ell,r}(p) \;\le\; (\xi^{K_p} + 2|\kappa|^{K_p})^{\Delta_p}.$$

Since $\big(\xi^{K_p} \pm 2|\kappa|^{K_p})^{\Delta_p}\big)/\,\xi^p = \Big(1 \pm 2/\,\xi^{\frac{3}{2}K_p}\Big)^{\Delta_p} \xrightarrow[K_p\to\infty]{} 1$, we have:

$$\left\| \frac{\mathtt{X}^{--}_{\ell,r}(p)}{\xi^p} - 1 \right\| \xrightarrow[p\to\infty]{} 0.$$

Now, if $K_p = 3$, then $\mathtt{X}^{--}_{\ell,r}(p) = \mathtt{P}(3)^{\Delta_p} = 3^{\frac{p}{3}}$. Generally, by the definition of ξ, $\forall a \ge \xi$, it holds that $a+1 \le a^3$. As a consequence, if, for some $b \in \mathbb{R}$, $\mathtt{P}(n) \le ba^n$, $\forall n \le m+1$, then: $\mathtt{P}(m+3) = \mathtt{P}(m+1) + \mathtt{P}(m) \le ba^m(a+1) \le ba^{m+3}$ and by induction on m, $\forall n$, $\mathtt{P}(n) \le ba^n$. Therefore, to prove the last inequality of Lemma 2.12, it suffices to check that it is satisfied for the base cases of the corresponding induction of this form, where $b = 1$ and $a = \sqrt{2}$. □

Proof of Theorem 2.2

Theorem 2.2 appears on 57, in Section 6, Chap. 2.

Theorem 2.2

The total number $\mathtt{T}(\omega)$ *of attractors of a* BAN $\mathscr{N}$ *of order* ω *which is either a* BAC *or a* BAD *is related to its total number* $\mathtt{X}(\omega)$ *of recurrent configurations as follows if* $\mathscr{N} \notin \{\mathscr{D}^{--}_{5,1}, \mathscr{D}^{--}_{1,5}\}$*:*

$$\frac{\mathtt{X}(\omega)}{\omega} \;\le\; \mathtt{T}(\omega) \;\le\; 2\cdot\frac{\mathtt{X}(\omega)}{\omega},$$

i.e. the expected value of attractor periods of $\mathscr{N}$ *is very high:*

$$\sum_{p|\omega} p\cdot\frac{\mathtt{A}(p)}{\mathtt{T}(\omega)} = \frac{\mathtt{X}(\omega)}{\mathtt{T}(\omega)} \;\ge\; \frac{\omega}{2}.$$

If $\mathscr{N} \in \{\mathscr{D}^{--}_{5,1}, \mathscr{D}^{--}_{1,5}\}$*, then* $\omega = 6$, $\mathtt{X}(6) = 5$ *and* $\mathtt{T}(6) = 2$.

Proof: We use the notations of Lemmas 2.9 and 2.11 and Corollary 2.3 and their proofs to prove the upper bound. In particular, $\mathscr{W}(p) \subseteq \mathbb{B}^p$ (resp. $\mathscr{A}(p) \subseteq \mathscr{W}(p)$) is the set of (resp. aperiodic) words satisfying the conditions of Lemmas 2.4 and 2.8: each word of this set characterises a configuration of $\mathscr{N}$ that has (resp. minimal) period p. And we let $\mathscr{A}_\circ(p)$ denote the quotient of $\mathscr{A}(p)$ by the rotation: any word $w \in \mathscr{A}_\circ(p)$ can be written p ways: $w[k, p-1]\,w[0, k-1]$, $k < p$, the canonical one being the least lexicographically. Each word $w \in \mathscr{A}^d_\circ(p)$ characterises the orbit of a configuration of $\mathscr{N}$ that has minimal period p, *i.e.* an attractor of period p of

$\mathcal{N}$, so $|\mathscr{A}_\circ(p)| = \mathtt{A}(p)$. To prove the upper bound of Theorem 2.2, we show that for each type of $\mathcal{N}$, there exists an injective map: $\Gamma : \bigcup_{p|\omega,\, p<\omega} \mathscr{A}_\circ(p) \to \mathscr{A}_\circ(\omega)$.

- If $\mathcal{N} = \mathscr{C}^+_\omega$, we let:

  $$\Gamma^+ : w \in \mathbb{B}^P \mapsto w\,(\neg w_0)^{\omega-p} \in \mathbb{B}^\omega.$$

  For any $w \in \mathbb{B}^p$, $\Gamma^+(w) = w\,1^{\omega-p}$, unless $w = 1 \in \mathbb{B}^1$ is the word characterising fix point $x = 1^\omega$ of $\mathscr{C}^+_\omega$, in which case $\Gamma^+(w) = 10^{\omega-1} \equiv 0^{\omega-1}\,1$. Examining the canonical writing of these, we can derive that Γ^+ is injective and that $\forall\, w \in \mathscr{A}_\circ(p)$, $\Gamma^+(w)$ is aperiodic so $\Gamma^+(w) \in \mathscr{A}_\circ(\omega)$.

- If $\mathcal{N} = \mathscr{C}^-_{\omega/2}$, we let:

  $$\Gamma^- : w = z\overline{z} \in \mathbb{B}^P \mapsto \Gamma^+(z)\Gamma^+(\overline{z}) \equiv 0^{\frac{1}{2}(\omega-p)}\,z\,1^{\frac{1}{2}(\omega-p)}\,\overline{z} \in \mathbb{B}^\omega.$$

  Again, Γ^- is clearly injective and $\forall\, w \in \mathscr{A}_\circ(p)$, $\Gamma^-(w)$ is aperiodic and belongs to $\mathscr{W}(\omega)$ so $\Gamma^-(w) \in \mathscr{A}_\circ(\omega)$.

The set $\mathscr{W}(\omega)$ for $\mathscr{D}^{-+}_{\ell r}$ is included in the set $\mathscr{W}(\omega)$ and $\mathscr{D}^{--}_{\ell r}$ so we concentrate on the case $\mathcal{N} = \mathscr{D}^{--}_{\ell r}$ and the case $\mathcal{N} = \mathscr{D}^{-+}_{\ell r}$ can be deduced directly from it. Also, following Lemma 2.9, the case where $\mathcal{N} = \mathscr{D}^{--}_{\ell r} \wedge \omega = 4\Delta$ does not occur (when $\mathcal{N} = \mathscr{D}^{--}_{\ell r}$, the case where $\ell + r = 4\Delta$ is taken into account by the case where $\omega = (\ell{+}r)/2 = 2\Delta$).

According to the proof of Lemma 2.11, each word of $\mathscr{W}(p) = \mathscr{W}^d(p)$ where $d = \ell \bmod p$ is characterised by an interleaving of $\Delta_p = \gcd(\ell, r, p)$ words of size K_p that belong to $\mathscr{W}^1(K_p)$. We write $w \overset{d}{=} [w(1), w(2), \ldots, w(\Delta_p)]$ for a word $w \in \mathscr{W}^d(p)$ that is characterised this way by the words $w(j)$, $0 < j \le \Delta_p \in \mathscr{W}^1(K_p)$.

If a and b are the Bezout integers such that $a\ell + b\omega = \Delta = \gcd(\ell, r) = \gcd(\ell, \omega)$, then any $i \in \mathbb{Z}/\omega\mathbb{Z}$, $i = m\Delta + j \equiv j \bmod \Delta$, satisfies $i = m(a\ell + b\omega) + j \equiv ma\ell + j \bmod \omega$ so $w_i = w_{mA\ell+j} = w(j)_{ma}$.

Let us note that if $w \overset{\ell}{=} [w(1), w(2), \ldots, w(\Delta)] \in \mathscr{W}^\ell(\omega)$ is periodic, then so are the $w(j)$s. Indeed, let $u \in \mathbb{B}^p, p = K_p\Delta_p < \omega = K\Delta$ such that $w = u^{\omega/p} \in \mathbb{B}^n$. Then, $\forall k \in \mathbb{Z}$, $w_i = w_{i+kp}$ and in particular for $k = \ell/\Delta_p$: $w_i = w_{i+K_p\ell} = w(j)_{ma+K_p}$. This proves that $W(j)$ has period K_p.

Bellow, $\mathrm{w}(k), \mathrm{w}(k)'$ are aperiodic words of $\mathscr{W}^1(k)$ that equal:

$$\begin{cases} \mathrm{w}(k) = (10)^{\frac{k-1}{2}}1, & \mathrm{w}(k)' = (01)^{\frac{k-1}{2}}1 \quad \text{if } k \text{ is odd,} \\ \mathrm{w}(k) = (10)^{\frac{k}{2}-3}(110)^2, & \mathrm{w}(k)' = (01)^{\frac{k}{2}-3}(011)^2 \quad \text{if } k > 6 \text{ is even,} \\ \mathrm{w}(k) = 10, & \mathrm{w}(k)' = 01 \quad \text{if } k = 2, \\ \mathrm{w}(k) = 1110, & \mathrm{w}(k)' = 0111 \quad \text{if } k = 4, \\ \mathrm{w}(k) = 111110, & \mathrm{w}(k)' = 011111 \quad \text{if } k = 6. \end{cases}$$

- If $\mathscr{N} = \mathscr{D}_{\ell r}^{--} \wedge K \neq 6$ or $\mathscr{N} = \mathscr{D}_{\ell r}^{-+}$, we let:

$$\begin{array}{l} \Gamma \ : \ w \overset{d}{=} [w(1), w(2), \dots, w(\Delta_p)] \in \mathbb{B}^p \quad \longmapsto \\ \qquad\qquad u \overset{\ell}{=} [\underbrace{w'(1), w'(2), \dots, w'(\Delta_p)}_{\Delta_p \text{ words in } \mathbb{B}^K}, \underbrace{\mathrm{w}, \mathrm{w}, \dots, \mathrm{w}, \mathrm{w}'}_{\Delta-\Delta_p \text{ words in } \mathbb{B}^K}] \in \mathbb{B}^\omega, \end{array}$$

where:

$$\left\{ \begin{array}{rcl} \forall 1 \leq j \leq \Delta_p, \ u(j) & = & w'(j) \ = \ w(j)^{\frac{K}{K_p}}, \\ u(\Delta) & = & \begin{cases} \mathrm{w}(K_p)' & \text{if } w'(1) = \mathrm{w}(K_p) \\ \mathrm{w}(K_p) & \text{otherwise,} \end{cases} \\ \text{and } \forall 1 \leq j < \Delta - \Delta_p, & & \\ u(\Delta_p + j) & = & \begin{cases} \mathrm{w}(K_p)' & \text{if } w'(\Delta_p) = \mathrm{w}(K_p) \\ \mathrm{w}(K_p) & \text{otherwise.} \end{cases} \end{array} \right.$$

By construction, $\forall w \in \mathscr{W}^d(p)$, $\Gamma(w) \in \mathscr{W}^\ell(\omega)$. And because $\mathrm{w} \in \{\mathrm{w}(K_p), \mathrm{w}(K_p)'\} \subseteq \mathscr{A}(K)$ is aperiodic, by the remark made above, so is $\Gamma(w)$. Thus: $w \in \mathscr{A}_\circ(p) \Rightarrow \Gamma(w) \in \mathscr{A}_\circ(\omega)$. To prove that Γ is injective, let us assume a certain canonical writing of the lists $[w(1), \dots, w(m)], \forall m$, and let $u \overset{\ell}{=} [u(1), \dots u(\Delta)] \in \Gamma(\bigcup_{p|\omega, p<\omega} \mathscr{A}_\circ(p))$.

If $K_p \neq K$, the whole range of consecutive words $u(j) = w'(i)$ can be distinguished from the others because the former are periodic contrary to the latter which belong to $\{\mathrm{w}(K_p), \mathrm{w}(K_p)'\}$. And since u is supposed to be the image of an *aperiodic* word of $\mathscr{A}_\circ(p)$, letting k be the minimal period of the $w'(i)$s, the word w of length kD – where D is the number $w'(i)$s – defined by the list of words $w'(i)[0, k-1]$ is the unique word of $\mathscr{A}_\circ(p)$ such that $u = \Gamma(w)$.

If $K = K_p$, then we can identify in $[u(1), \dots u(\Delta)]$ the largest series $[u(i+1), u(i+2) \dots, u(i+m)]$ of consecutive words belonging to $\{\mathrm{w}(K_p), \mathrm{w}(K_p)'\}$ such that $u(i) \neq u(i+1)$ and $u(i+m) \neq u(i+m+1)$. If $m > \Delta/2$ or if the remaining $u(j)$s do not form a series of the form $[\mathrm{w}, \mathrm{w}, \dots, \mathrm{w}, \mathrm{w}']$ where $\mathrm{w}, \mathrm{w}' \in \{\mathrm{w}(K_p), \mathrm{w}(K_p)'\}$, then they necessarily equal a word of the form $w'(j)$. So again, they define a unique word $w \in \mathbb{B}^p$, $p < \omega$ (where $\Delta_p = \Delta - m$) such that $u = \Gamma(w)$.

Otherwise, $K = K_p$, $m = \Delta/2 = \Delta_p$ ($p = \omega/2$, $d = \ell \bmod p$) and there are words $\mathrm{w}, \mathrm{w}', \mathrm{v}\mathrm{v}' \in \{\mathrm{w}(K_p), \mathrm{w}(K_p)'\}$ necessarily satisfying $\mathrm{w} \neq \mathrm{v}'$ and $\mathrm{w}' \neq \mathrm{v}$, such that the word u can be written: $u \overset{\ell}{=} [\underbrace{\mathrm{w}, \mathrm{w}, \dots, \mathrm{w}, \mathrm{w}'}_{\Delta/2 \text{ words in } \mathbb{B}^K}, \underbrace{\mathrm{v}, \mathrm{v}, \dots, \mathrm{v}, \mathrm{v}'}_{\Delta/2 \text{ words in } \mathbb{B}^K}]$.

If $\mathrm{w} = \mathrm{v} \wedge \mathrm{w}' = \mathrm{v}'$, then the series of $w'(j)$s can only equal $w \overset{d}{=} [\mathrm{w}, \mathrm{w}, \dots, \mathrm{w}, \mathrm{w}']$ where $w \in \mathbb{B}^{\frac{\omega}{2}}$ is the only word of $\mathscr{A}_\circ(\omega/2)$ such that $u = \Gamma(w)$.

Otherwise, the word u can be written:

$$u \overset{\ell}{=} [\ \underbrace{\mathrm{w},\mathrm{w},\ldots,\mathrm{w}}_{\Delta/2 \text{ words in } \mathbb{B}^K}\ ,\ \ \underbrace{\mathrm{v},\mathrm{v},\ldots,\mathrm{v}}_{\Delta/2 \text{ words in } \mathbb{B}^K}\ ],\quad \mathrm{w} \neq \mathrm{v},$$

so that the antecedent $w \in \mathbb{B}^{\frac{\omega}{2}}$ of u by Γ equals either $w \overset{d}{=} [\mathrm{w},\ldots,\mathrm{w}]$ or $w \overset{d}{=} [\mathrm{v},\ldots,\mathrm{v}]$. Explicitly (by definition of the expression of any word $w \in \mathbb{B}^p$ by a list $w \overset{d}{=} [w(1),\ldots,w(m)],\ w(j) \in \mathbb{B}^{\frac{p}{m}}$), in both cases, w can be written $w = z(1)z(2)\ldots z(K)$ for some K words $z(j) = a^{\frac{\omega}{2}} \in \mathbb{B}^{\frac{\omega}{2}}$ $(\forall 1 \leq j \leq K)$ where a is a letter of the word w or of the word v – *i.e.* a letter of $\mathrm{w}_{(K_p)}$ or of $\mathrm{w}_{(K_p)}'$ – depending on the case. But since $\mathrm{w}_{(K_p)}$ is a rotation of $\mathrm{w}_{(K_p)}'$, the two cases yield ws that are rotations of one another and thus equal in $\mathscr{A}_{\circ}^{d}(p)$.

- In the remaining case where $\mathscr{N} = \mathscr{D}_{\ell r}^{--} \wedge K = 6 \wedge \Delta > 1$, we use the same definition of Γ than in the previous case, except that we replace $\mathrm{w}_{(6)}$ and $\mathrm{w}_{(6)}'$ by $\mathrm{w}_{(6)} = 110110$ and $\mathrm{w}_{(6)}' = 101010$. As mentioned above, if $u \overset{\ell}{=} [u(1),\ldots,u(\Delta)]$ has period $p|\omega$, then $\forall 1 \leq j \leq \Delta$, $u(j)$ has period $K_p|K = 6$. But there is no divisor K_p of $K = 6$ that is a period of both $\mathrm{w}_{(6)}$ and $\mathrm{w}_{(6)}'$ so $u = \Gamma(w)$ is aperiodic $\forall w$. The rest of this case is similar to the previous.

□

B Complementary proofs for Chapter 3

Proof of Proposition 3.3

This proposition figures in Section 3.1, Chap. 3, on Page 78.

Proposition 3.3.

If **G** *is strongly connected, it has a k-cycle-cover ($k > 1$) if and only if it has a block-sequential update schedule β such that k divides $\eta(\mathbf{G}[\beta])$. Thus, a positive* DAN H *cycles for some block-sequential update schedule if and only if it contains a non-trivial* SCC **G** *that has a k-cycle-cover for some $k > 1$.*

Proof: Suppose W is a k-cycle-cover of **G**. Let β be a block sequential update schedule such that $\mathrm{I}[\beta] = \mathbf{A} \setminus \mathrm{W}$. The only reason for which β might not be well defined is if its definition induces that $\exists i \in V,\ \beta(i) > \beta(i)$. As proven in [6], this can only happen if there exists an undirected cycle $\{i = i_0, i_1, \ldots, i_m = i\}$ containing at least one arc (i_{r+1}, i_r) (such that $\beta(i_{r+1}) < \beta(i_r)$) and satisfying the following $\forall r < m$:

- if $(i_r, i_{r+1}) \in A$ then $\beta(i_{r+1}) \leq \beta(i_r)$, *i.e.* $(i_r, i_{r+1}) \in \mathrm{W}$, and
- if $(i_{r+1}, i_r) \in A$ then $\beta(i_{r+1}) < \beta(i_r)$, *i.e.* $(i_{r+1}, i_r) \notin \mathrm{W}$.

Point *(ii)* in the definition of a k-cycle-cover excludes this. Therefore, β is well defined. Now, by Lemma 3.3, a cycle of length ℓ in $\mathbf{G}[\beta]$ is necessarily induced by a cycle of $\mathbf{G}$ that contains ℓ arcs of $\mathbf{A} \setminus \mathrm{I}[\beta] = \mathrm{W}$. And since that cycle must be k-covered by W, k must divide ℓ. Thus, k divides the lengths of all cycles in $\mathbf{G}[\beta]$ and it divides $\eta(\mathbf{G}[\beta])$.

Conversely, suppose that β is such that $k > 1$ divides $\eta(\mathbf{G}[\beta])$. We claim that each cycle $\mathrm{C} = (\mathbf{V}_\mathrm{C}, \mathbf{A}_\mathrm{C})$ of $\mathbf{G}$ induces a cycle C' of $\mathbf{G}[\beta]$ of length $|\mathbf{A}_\mathrm{C} \setminus \mathrm{I}[\beta]|$. Indeed, consider the maximal sub-paths of C that have the form described in Lemma 3.3. There necessarily exist some because β would not be well defined otherwise. The extremities of these maximal sub-paths are the nodes $i_r \in \mathbf{V}_\mathrm{C}$ such that $(i_r, i_{r+1}) \in \mathbf{A}_\mathrm{C} \setminus \mathrm{I}[\beta]$. So there are as many of these sub-paths as there are arcs in $\mathbf{A}_\mathrm{C} \setminus \mathrm{I}[\beta]$. By Lemma 3.3, these maximal sub-paths of C are turned into arcs in $\mathbf{G}[\beta]$ that connect their extremities in a way that all these arcs form a cycle C' of $\mathbf{G}[\beta]$. So C' has length $|\mathbf{A}_\mathrm{C} \setminus \mathrm{I}[\beta]|$ (counting each arc as many times as it is used). Since, by definition, $\eta(\mathbf{G}[\beta])$ divides all cycle sizes in $\mathbf{G}[\beta]$, it divides $|\mathbf{V}_{\mathrm{C}'}| = |\mathbf{A}_\mathrm{C} \setminus \mathrm{I}[\beta]|$. And as a result, $\mathrm{W} = \bigcup_\mathrm{C} \mathbf{A}_\mathrm{C} \setminus \mathrm{I}[\beta]$ is a k-cycle-cover of $\mathbf{G}$ (condition *(ii)* in the definition of a k-cycle-cover is satisfied because β is well defined).

The last part of Proposition 3.3 follows from the first and from the simulation highlighted in Example 1.14. □

Proof of Theorem 3.1, (*iii*)

Theorem 3.1 classifies positive DANs according to their behaviours under fair update schedules, into the six classes defined on Page 80, in Section 3.2, Chap. 3. Its first two points are already proven so it remains to prove its third which characterises the set of positive DANs that can cycle. To do so, we prove the following lemma (*weakly-loop-free components* are defined on Page 79).

Lemma.

A positive DAN $\mathbf{G}$ *can cycle under some fair update schedule if and only if it contains a non-trivial* SCC *in which there is a weakly-loop-free component. In addition, if* $\mathbf{G}$ *contains a weakly-loop-free simple cycle, then it cycles under a* 2*-fair update schedule.*

Proof: In this proof, given an update schedule $\delta :\equiv (\mathrm{W}_t)_{t<p}$ and a configuration $x \in \mathbb{B}^n$, we change our notations and write, $\forall\, t' = kp + t \equiv t \bmod p$: $x(t') = \mathrm{F}_{\mathrm{W}_t} \circ \mathrm{F}_{\mathrm{W}_{t-1}} \circ \ldots \circ \mathrm{F}_{\mathrm{W}_0}(\mathrm{F}[\delta]^k(x))$ (rather than $x(t') = \mathrm{F}[\delta]^{t'}(x)$ as before). Also, $x = \overline{0}^i \in \mathbb{B}^n$

denotes the configuration such that $\forall j \neq i$, $x_j = 0$ and $x_i = 1$. We prove the claim in the case where **G** is a non-trivial strongly connected digraph (*cf.* Example 1.14 for the generalisation).

Let H be a maximal weakly-loop-free component of **G**. Let $\mathbb{C} = \{i_0 = i_\ell, i_1, \dots, i_{\ell-1}\}$ be a cycle of **G** that runs through each arc of $\mathbf{A}_\mathrm{H}$ at least once and through no other arcs. And let $\delta :\equiv (\mathrm{W}_t)_{t<p}$ be the update schedule defined as follows (*cf.* (1.17) on Page 22): $\forall k \in \mathbb{Z}/\ell\mathbb{Z}, \delta(i_k) = \{k-1, k\}$ and $\forall j \notin \mathbf{V}_\mathrm{H}, \delta(j) = \{k\}$ where $k \in \mathbb{Z}/\ell\mathbb{Z}$ is such that $i_k \in \mathbf{V}_\mathrm{H} \setminus \mathbf{V}^-_\mathbf{G}(j)$ (k exists by the maximality of H and condition (*iii*) in the definition of a weakly-loop-free component). It can be checked that δ is well defined and fair. Let $x = \overline{0}^{i_0}$. We claim that $\forall k \in \mathbb{Z}/\ell\mathbb{Z}, x(k) = \overline{0}^{i_k}$ and as a result, $\{x(k) = \overline{0}^{i_k} \mid k \in \mathbb{Z}/\ell\mathbb{Z}\}$ is a limit cycle of $\mathbf{G}[\delta]$. Indeed, at step $k \in \mathbb{Z}/\ell\mathbb{Z}$ of the update sequence, the set of nodes that are updated is $\mathrm{W}_k = \{i_k, i_{k+1}\} \cup \{j \notin \mathbf{V}_\mathrm{H} \mid \delta(j) = k\}$. If $x(k) = \overline{0}^{i_k}$, then the following holds: i_k (the unique node of **G** in state 1) has no in-neighbours in state 1, i_{k+1} has one in-neighbour in state 1, all other nodes in W_k do not belong to $\mathbf{V}_\mathrm{H}$ and have no in-neighbours in state 1. As a result, after the update of W_k, $x(k+1)_{i_k} = 0$, $x(k+1)_{i_{k+1}} = 1$ and any $i \in \mathrm{W}_k\mathbf{V}_\mathrm{H}$ remains in state 0 as do all other nodes $i \notin \mathrm{W}_k$ that are not updated. Thus, $x(k+1) = \overline{0}^{i_{k+1}}$ and an induction on k allows to conclude.

If $\mathbb{C}$ is a simple cycle, then, each of its nodes $i_k, k \in \mathbf{V}_\mathbb{C} = \mathbb{Z}/\ell\mathbb{Z}$ is updated exactly twice: once at step k of the update sequence (when it takes state 1) and once at step $k+1$ (when it goes back to state 0). All other nodes are updated once. In this case, δ is 2-fair.

Now, for the converse, suppose that **G** contains no weakly-loop-free component and $\mathscr{A}$ is a limit cycle of $\mathbf{G}[\delta]$, for a fair δ. Let H be the sub-digraph of **G** induced by the nodes whose states are not fixed in $\mathscr{A}$. Necessarily, H is not cycle-free (its minimal in-degree must be at least one for all of its nodes to be able to cycle). Let $\mathrm{C} = (\mathbf{V}_\mathrm{C}, \mathbf{A}_\mathrm{C})$ be a non-trivial strongly-connected component of H with no in-coming arcs in H. By definition of H, all $i \in \mathbf{V}_\mathrm{C}$ take state 1 at some point in $\mathscr{A}$. Because C is not weakly-loop-free, there either is a loop-node $i \in \mathbf{V}_\mathrm{C}$ that takes – and necessarily becomes fixed in – state 1 in $\mathscr{A}$; or there is a $i \in \mathbf{V}$ such that $\mathbf{V}_\mathrm{C} \subseteq \mathbf{V}^-_\mathbf{G}(i)$. In this case, i is also fixed to 1 because $\forall x(t) \in \mathscr{A}, \exists j \in \mathbf{V}_\mathrm{C}, x(t)_j = 1$ (all nodes of C cannot simultaneously be in state 0 without remaining in state 0, which is impossible) and δ being fair, i is necessarily updated during $\mathscr{A}$. In both cases, Lemma 3.6 implies that in $\mathscr{A}$, all nodes of **G** are fixed to state 1 because i is. So $\mathscr{A} = \{1^n\}$ is not a limit cycle. □

C Complementary proofs for Chapter 5

Lemma 5.2. Number of instabilities of BACs

A BAC $\mathscr{C}_n^s$ *of size n and sign* $s \in \{-,+\}$ *satisfies:*

	$s=+$	$s=-$
$u(x)$ *is:*	*even*	*odd*
$u_{min} =$	0	1
$u_{max} =$	$\begin{cases} n & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$	$\begin{cases} n-1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$

Proof: In any configuration $x \in \mathbb{B}^n$ of a BAC $\mathscr{C}_n^s$, the sets of unstable automata $\mathscr{U}(x) \subseteq \mathbf{V}$ and of frustrated arcs $\textbf{FRUS}(x) \subseteq \mathbf{A}$ (*cf.* Page 18) are related as follows:

$$\mathscr{U}(x) \quad = \quad \{i \in \mathbf{V} \mid (i-1,i) \in \textbf{FRUS}(x)\}.$$

The number of unstable automata and the sign s of $\mathscr{C}_n^s$ satisfy:

$$s \;=\; \prod_{i\in\mathbb{Z}/n\mathbb{Z}} \text{sign}(i,i+1) \;=\; \prod_{i\in\mathbb{Z}/n\mathbb{Z}} \text{sign}(i,i+1)\,\Big(\prod_{i\in\mathbb{Z}/n\mathbb{Z}} \mathbf{s}(x_i)\cdot\mathbf{s}(x_{i+1})\Big)$$

$$=\; \prod_{i\in\mathbb{Z}/n\mathbb{Z}} \text{sign}(i,i+1)\cdot\mathbf{s}(x_i)\cdot\mathbf{s}(x_{i+1}) \;=\; \prod_{i\in\overline{\mathscr{U}}(x)} 1 \prod_{i\in\mathscr{U}(x)} -1 \;=\; (-1)^{u(x)}.$$

Because $\mathscr{C}_n^+$ has stable configurations (the fix points of block-sequential update schedules), $u_{min} = 0$. Because $\mathscr{C}_n^-$ has no stable configurations and a^n ($a \in \mathbb{B}$) is a configuration of (canonical) $\mathscr{C}_n^-$ that has one unstable automaton ($i = 0$), $u_{min} = u(a^n) = 1$. Whatever the sign of $\mathscr{C}_n^s$, if n is even, then $u_{max} = u((01)^{\frac{n}{2}})$; if n is odd, then $u_{max} = u(1(01)^{\frac{n-1}{2}})$. □

D The fix-point-existence problem

Proposition. NP-completeness of FPE

The following problem named FPE *is NP-complete.*

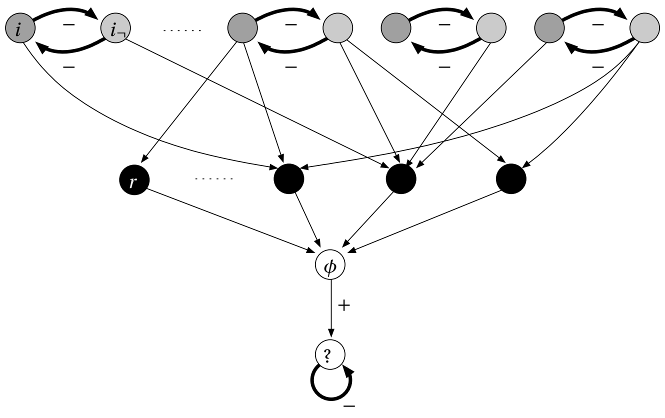
$$\text{FPE:}\begin{cases} \text{INPUT:} & \textit{A locally monotone } \text{BAN } \mathscr{N} = \{\mathrm{f}_i \mid i \in \mathbf{V}\}; \\ \text{QUESTION:} & \textit{Does } \mathscr{N} \textit{ have a stable configuration?} \end{cases}$$

Proof: FPE is obviously NP. Let us show that it is NP-hard by reducing Problem SAT [18, 42] to it. Let $\phi(x) = \bigwedge_r c_r(x)$ be an arbitrary CNF Boolean formula, instance of SAT, with n literals $x_0, \dots x_{n-1} \in \mathbb{B}$ and m disjunctive clauses of the form $c_r(x) = \bigvee_i x_i^{(r)}$ where $x_i^{(r)} \in \{\neg x_i, x_i\}$.

We define the BAN structure $\mathbf{G}_\phi = (\mathbf{V}, \mathbf{A})$ as follows (*cf.* the figure below). Its set of nodes $\mathbf{V}$ is the union of: (1) a set W of nodes (in dark grey) representing (*i.e.* whose states are intended to correspond to the value of) the non-negated literals $x_0, \dots, x_{n-1}$, (2) a set $\mathrm{W}_\neg$ of nodes (in light grey) representing their negated versions $\neg x_0, \dots, \neg x_{n-1}$, (3) a set C of nodes (in black) representing the m clauses $c_r(x)$ of $\phi(x)$, (4) a node ϕ representing $\phi(x)$ itself and (5) a node ? meant to cycle if and only if the previous node is in state 0.

Nodes $i \in \mathrm{W}$ (resp. $i_\neg \in \mathrm{W}_\neg$) have a unique in-neighbour $i_\neg$ (resp. i) which belongs to $\mathrm{W}_\neg$ (resp. W). Nodes $j \in \mathrm{W} \cup \mathrm{W}_\neg$ have an out-going arc $(j, r) \in \mathbf{A}$ ending on a node r of C if and only if the (negated) literal that they represent appears in clause $c_r(x)$ of $\phi(x)$. All incoming arcs of C initiate in $\mathrm{W} \cup \mathrm{W}_\neg$ and all nodes $r \in \mathrm{C}$ have a unique outgoing arc that ends on node ϕ.

To define a locally monotone BAN $\mathcal{N}_\phi = \{\mathrm{f}_i \mid i \in \mathbf{V}\}$ of size $|\mathbf{V}| = 2n + m + 2$, with structure $\mathbf{G}_\phi$, we define the local transition functions so that nodes in $\mathrm{W} \cup \mathrm{W}_\neg$ negate their input, nodes in C perform a disjunction of their inputs, node ϕ performs a conjunction of all its inputs and node ? has $\mathrm{f}_? : x \mapsto x_\phi \vee \neg x_?$ as local transition function. This yields a signed structure of the following form:



The description of the BAN $\mathcal{N}_\phi$ can be computed in polynomial with respect to the size of ϕ.

If $\phi(x)$ is satisfiable, let $a \in \mathbb{B}^n$ be a Boolean vector such that $\phi(a) = 1$. Let $y \in \mathbb{B}^{|\mathbf{V}|}$ be a configuration of $\mathcal{N}_\phi$ such that

$$\begin{cases} \forall i \in \mathrm{W},\ y_i = a_i, \\ \forall i \in \mathrm{W}_\neg,\ y_i = \neg a_i, \\ \forall i \in \mathscr{C} \cup \{\phi, ?\},\ y_i = 1. \end{cases}$$

Then, obviously, all nodes of $\mathrm{W} \cup \mathrm{W}_\neg$ are fixed: $\forall i \in \mathrm{W}$, $\mathrm{f}_i(y) = \neg y_{i_\neg} = \neg\neg a_i = y_i$ and $\forall i_\neg \in \mathrm{W}_\neg$, $\mathrm{f}_{i_\neg}(y) = \neg y_i = \neg a_i = y_{i_\neg}$. Also, by definition, $\forall r \in \mathrm{C}$, $\mathrm{f}_r(y) = \bigvee_{j \in \mathbf{V}^-_{\mathbf{G}_\phi}(i)} y_j = c_r(y) = c_r(a) = 1 = y_r$ and similarly, $\mathrm{f}_\phi(y) = \bigwedge_{j \in \mathbf{V}^-_{\mathbf{G}}(\phi)} y_j = \bigwedge_r c_r(a) = \phi(a) = 1 = y_\phi$. As for node ?, its state satisfies $\mathrm{f}_?(y) = y_\phi \vee \neg y_? = 1 \vee \neg 1 = 1 = y_?$ so all nodes of $\mathscr{N}_\phi$ are stable in y. As a result $\mathscr{N}_\phi$ has a stable configuration if ϕ is satisfiable.

Conversely, if y is a stable configuration of $\mathscr{N}_\phi$, necessarily, for every couple of nodes $(i, i_\neg) \in \mathrm{W} \times \mathrm{W}_\neg$, y must satisfy $y_i \neq y_{i_\neg}$ so that all nodes in $\mathrm{W} \cup \mathrm{W}_\neg$ be stable. And y_ϕ must equal 1 otherwise node ? cannot be stable. Thus, $a = y_{\mathrm{W}} \in \mathbb{B}^n$ does indeed define an assignment of the literals of formula $\phi(x)$ that satisfies it. As a consequence, the instance $\phi(x)$ of SAT is satisfiable if and only if $\mathscr{N}_\phi$ has a stable configuration.

□

E Counting block-sequential update schedules

Block-sequential update schedules of a set **V** define total partial orders on this set (and their inversions[39] define strict weak orders). They are thus counted by sequence `A670`(n) of the OEIS [112]; letting $S(n,q)$ count the surjective applications from a set of n elements to a set of $q \leq n$ and setting $\mathtt{B}_0 = 1$, the number of block-sequential update schedules of a set of n elements therefore equals the following:

$$\mathtt{B}_n = \sum_{0 \leq q < n} \binom{n}{q} \cdot \mathtt{B}_q \;=\; \sum_{1 \leq q \leq n} S(n,q)$$

which can be approximated asymptotically by $\frac{1}{2} \cdot \frac{n!}{(\ln 2)^{n+1}}$ [136].

Quotienting these update schedules by the equivalence relation $\asymp$ introduced in Section 1, Chap. 4 reduces their number by a factor tending towards $n/\ln 2$. Indeed, the number of equivalence classes for $\asymp$ equals:

$$\widehat{\mathtt{B}}_n = \sum_{1 \leq q \leq n} \frac{S(n,q)}{q},$$

and from $S(n+1,q) = q \cdot S(n,q) + q \cdot S(n,q-1)$ results:

$$\begin{aligned} \widehat{\mathtt{B}}_{n+1} &= \sum_{1 \leq q \leq n+1} S(n,q) + \sum_{1 \leq q \leq n+1} S(n,q-1) \;=\; S(n,n+1) + S(n,0) + 2\mathtt{B}_n \\ &= 2\mathtt{B}_n \;\sim\; \frac{n!}{(\ln 2)^{n+1}}, \end{aligned}$$

---

[39] *cf.* Section 1.1, Chap. 3

from which derives:

$$\mathtt{B}_n \quad \sim \quad \frac{n}{\ln 2} \times \widehat{\mathtt{B}}_n.$$

F Time scales & update schedules

Section 2.A, Chap. 6 mentions the possible integration of different time scales in the theory of BANs using multi-scale update schedules (see also [26]). Here, we propose to detail this idea a bit further and *informally*.

Let us consider a collection of separate BANs, called *modules* which independently evolve on a "*microscopic time scale*" denoted by ✺. This time scale is the one from which automata interactions inside modules may be observed. Let us suppose that the modules are globally in non-transient but perturbable states most of the time and that they are made to interact. Here, "network state" must be understood in general terms as *network behaviour* rather than *network configuration*, although this notion clearly calls for much more precisions. Thus, they assemble to form a larger encompassing system which evolves in a "*macroscopic time scale*" denoted by ✳. In this time scale the events that can be observed consist in module interactions and module state changes.

More precisely, a module $\mathscr{M}$ can receive a stimulus from another module $\mathscr{M}'$. As a result, it undergoes some internal modifications occuring within ✺ so that macroscopically, these modifications can be represented by just a switch of global states of the whole module $\mathscr{M}$. Now, there are several ways to understand what these internal modifications correspond to concretely (in addition to the problem of specifying formally what the state of a module is).

First, for example, we might consider that $\mathscr{M}$ is in one of its attractors before it is perturbed. Then, the effect of the stimulus might just be to make it switch from one attractor directly into another. If this is the case, the problem of choosing what update mode is used inside $\mathscr{M}$ is irrelevant. More generally, it might often suffice to consider only a reasonably small part of the GTG of $\mathscr{M}$ surrounding the set of its attractors. Indeed, when $\mathscr{M}$ is not pulled *too* far away by the stimulus, it might often evolve back into an attractor "quick" enough for an updating sequence to make no difference. This is a subject that obviously calls for many investigations in the lines of [28] in order to formalise, bound and check this statement which in other terms suggests that for many BANs, a configuration x that is "close" [40] to one attractor $\mathscr{A}$ is not far from any other.

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[40] *e.g.* in the sense that in the GTG a short derivation leads from x to $\mathscr{A}$, or as in [28], in the sense that the Hamming distance $\mathrm{D}(x, \mathscr{A})$ is small.

We might also consider that before the stimulus emitted by $\mathscr{M}'$, $\mathscr{M}$ is submitted to the influence of another module $\mathscr{M}''$ whose effect is to maintain $\mathscr{M}$ in a transient state which in itself, involves many instabilities. For instance, if $\mathscr{M}$ is simply a cycle this may correspond to maintaining it within a layer of its GTG, in one of the unstable attractors induced by the parallel update schedule π. Let us note that in itself, this gives some supplementary incentives to study update schedules. In addition, it raises a question in close relation to the one addressed in Section 2.B, Chap. 6 on the ways to input a structural influence whose impact on a BAN is to mimic the effect of an update schedule.

In this situation, $\mathscr{M}$ might again evolve as if it would if it were isolated from the rest of the macro-system and launched in a specific state that corresponds to or accounts for its inputs by both $\mathscr{M}$ and $\mathscr{M}'$ (provided these have lasted). Here, unless the GTG of $\mathscr{M}$ is considered, it therefore remains to choose the best, or perhaps fastest way to update $\mathscr{M}$ until either it reaches an attractor that is coherent with its present inputs or it is intercepted and stabilised by another incoming signal. This problem justifies Section 2.C, Chap. 6.

This setting can then be effectively drawn towards a higher level of abstraction by organising the interactions between modules in $\circledast$ through the definition of a "*macro-update mode*" which updates a module at a certain time to make it either react to incoming signals and evolve accordingly or output the result of its evolution. In the second case, the update mode determines when a module is *active* or *functional* in the sense that it can have a global impact on the macro-network. Generally, among others, this wider encompassing perspective on BANs seems to call for some investigations in the lead of [28] on the sensitivity of networks to state perturbations.

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