Codes de Gray généralisés à l'énumération des objets d'une structure combinatoire sous contrainte

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"Por mi raza hablará el espíritu."

Introduction version française

Les problèmes étudiés tout au long de cette thèse concernent le cube de Fibonacci et le cube de Lucas.

Le cube de Fibonacci Γ_n est un sous-graphe isométrique de l'hypercube, mais avec un nombre de Fibonacci de sommets. Il a été initialement introduit par W-J. Hsu dans [Hsu93] comme réseau d'interconnexion.

Un *réseau d'interconnexion* peut être représenté par un graphe $G = (V, E)$, où V désigne les processeurs et E les liens de communication pour l'échange de données entre les processeurs.

Selon Bertsekas et Tsitsiklis (cf.[BT97]), les réseaux d'interconnexion sont généralement évalués en fonction de leur aptitude à certaines tâches de communication standard. Certains critères typiques incluent le diamètre du réseau qui est la distance maximale entre chaque paire de sommets ; la connectivité du réseau qui fournit une mesure du nombre de chemins indépendants entre toute paire de sommets et la flexibilité donnée en exécutant efficacement une large variété d'algorithmes. Cela se traduit par l'étude du problème du plongement qui demande si un graphe invité est un sous-graphe d'un graphe hôte. Si un plongement graphique existe, nous pouvons appliquer des algorithmes conçus pour les graphes hôtes pour travailler efficacement sur les graphes invités.

L'hypercube est un réseau d'interconnexion populaire avec 2^n sommets qui représentent des processeurs autonomes connectés avec n voisins. L'hypercube représente une classe importante dans la théorie des graphes principalement en raison de ses propriétés structurales.

Comme réseau d'interconnexion, l'hypercube a des propriétés très intéressantes telles que la récurrence, la symétrie et la connectivité. En outre, de nombreuses structures topologiques qui apparaissent lors de la parallélisation d'un algorithme peuvent être plongées dans l'hypercube. Voir [HHW88] pour une étude de certaines propriétés structurales de l'hypercube, dont les cycles hamiltoninens et les plongements. Nous faisons référence aussi à [SS88], pour un examen large des propriétés de l'hypercube qui rendent ce graphe si attrayant.

En revanche, lors de la mise en œuvre d'un système informatique parallèle, plus de processeurs et de mémoires peuvent être ajoutés, comme le budget le permet. Dans le cas de l'hypercube, le nombre de processeurs doit être une puissance de 2. Lorsque le réseau est mis en œuvre avec un plus petit nombre de processeurs, de nombreux liens de communication peuvent être inutilisés. Ainsi, d'autres structures topologiques sont nécessaires afin de permettre l'ajout d'un petit nombre de nœuds tout en minimisant le gaspillage des ressources. (Voir [DYN03].)

Parmi d'autres modèles alternatifs, le cube de Fibonacci, qui est inspiré par les nombres de Fibonacci, s'avère être un réseau d'interconnexion attrayant en raison de sa structure topologique et de sa croissance plus modérée.

Dans le chapitre 2, nous introduisons le cube de Fibonacci.

Dans la section 2.2, nous présentons quelques préliminaires relatifs aux nombres de Fibonacci. Ces nombres doivent leur nom à Léonard de Pise, plus connu sous le nom de Fibonacci et sont liés à de nombreux problèmes d'énumération.

The $n^{\text{ième}}$ *nombre de Fibonacci*, $n \geq 2$, est déterminé par la relation de récurrence suivante :

$$F_n = F_{n-1} + F_{n-2} \text{ avec les valeurs initiales } F_0 = 0,\ F_1 = 1.$$

En d'autres termes, chaque nombre dans la suite est la somme des deux nombres précédents. Soit $g(x)$ la fonction génératrice de la suite de Fibonacci, alors

$$g(x) = \sum_{n\geq 0} F_n\, x^n = \frac{x}{1-x-x^2}.$$

De nombreuses propriétés ont été trouvés dans la suite de Fibonacci et comme Zeckendorf l'a démontré, voir [GKP94], chaque nombre entier positif a une représentation unique comme somme de nombres de Fibonacci non consécutifs. On en déduit une représentation binaire du nombre entier, sans 1 consécutifs. En outre, un nombre compris entre 0 et F_{n-1} exige $(n-2)$ positions pour être représenté. Une chaîne $(b_1\, b_2 \cdots b_n)$ où $b_j \in \{0,1\}$ et $b_j \cdot b_{j+1} = 0$ pour tout i dans $\{0, 1, \ldots n-1\}$ sera définie comme une *chaîne de Fibonacci de longueur* n.

Le *cube de Fibonacci* Γ_n est défini comme le sous-graphe de l'hypercube Q_n, induit par les chaînes de Fibonacci de longueur n où les sommets adjacents de Γ_n diffèrent une position et le nombre de sommets de Γ_n est F_{n+2}. Voir la figure 1.1. Nous notons $f(x)$ la fonction génératrice du nombre de sommets de Γ_n. Alors

$$f(x) = \sum_{n\geq 0} |V(\Gamma_n)| x^n = \frac{1+x}{1-x-x^2}.$$

Dans la section 2.3, nous abordons différentes décompositions récursives du cube de Fibonacci. Principalement la *décomposition fondamentale de* Γ_n, qui indique que le cube de de Fibonacci Γ_n se decompose en deux sous-graphes

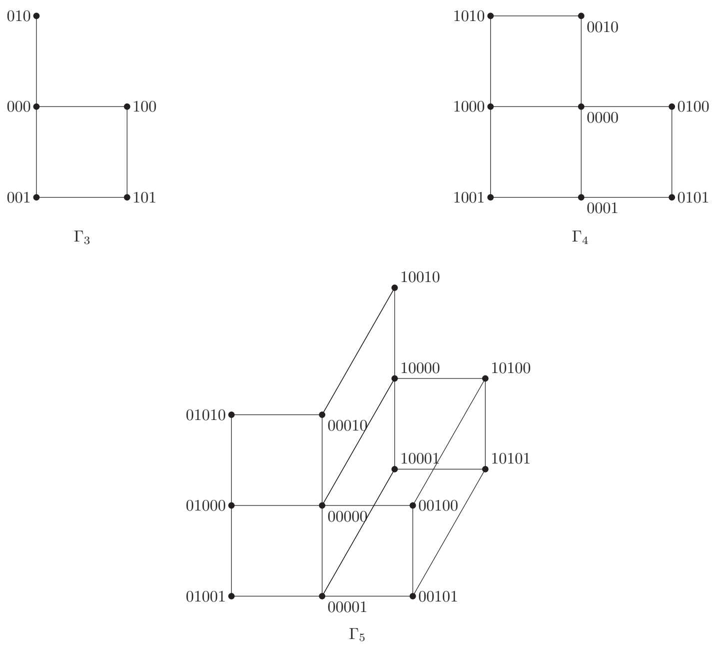


FIGURE 1: Γ_3, Γ_4 and Γ_5

disjoints qui sont isomorphes à Γ_{n-1} et à Γ_{n-2}. En outre, chaque sommet dans le graphe isomorphe à Γ_{n-2} a exactement un voisin dans le sous-graphe isomorphe à Γ_{n-1}.

La décomposition fondamentale de Γ_n peut être appliquée de manière récursive à ses sous-graphes Γ_{n-1} et/ou Γ_{n-2}. Beaucoup de propriétés importantes seront déduites de cette décomposition, comme le fait de pouvoir plonger le cube de Fibonacci dans un autre cube de Fibonacci de plus petite taille. L'idée principale consiste à décomposer Γ_n en Γ_{n-1} et Γ_{n-2}. Ensuite, nous fusionnons les arêtes qui relient ces deux sous-graphes en une unique arête. Le graphe qui en résulte est isomorphe à Γ_1. En appliquant cette méthode itérative à chaque sous-graphe, on obtient à chaque fois, un sous-graphe isomorphe à Γ_k. Nous appelons chacun de ces graphes le *Cube de Fibonacci Quotient*.

Dans la section 2.4, nous énonçons d'autres résultats sur la structure du cube de Fibonacci. Parmi ceux-ci, nous mentionnons le résultat dû à Klavžar [Kla05], qui dit que le cube de Fibonacci est un graphe médian. D'autres résultats tirés de [MCS01], concernant le nombre d'arêtes de Γ_n sont aussi enoncés. Dans [KMP11], nous trouvons certains résultats énumératifs concernant le nombre de sommets de Γ_n d'un degré donné, et le nombre de sommets d'un degré et d'un poids donnés. Le nombre de sous-graphes induits isomorphes à Q_n dans le cube de Fibonacci est étudié dans [KM12a] et le nombre de sous-graphes maximaux induits isomorphes à Q_n dans le cube de Fibonacci est étudié dans [Mol11]. Enfin, nous citons une étude de [KM12b] sur *l'indice de Wiener* des cubes de Fibonacci qui est une application en chimie associé à ces cubes .

La dernière section présente une variante et deux généralisations du cube de Fibonacci. Le premier, nommé le cube de Lucas est obtenu par l'élimination de toutes les chaînes qui commencent et finissent avec 1 dans le cube de Fibonacci.

Le cube de Lucas Λ_n, introduit par E. Munarini, C.Cippo et N. Zagaglia Salvi dans [MCS01], est un sous-graphe du n-cube qui possède également des propriétés intéressantes comme ses décompositions récursives en deux sous-graphes disjoints isomorphes à Γ_{n-1} et Γ_{n-3}.

Le nombre de sommets du cube de Lucas, $|V(\Lambda_n)|$ pour $n \geq 1$ est L_n, le $n^{ième}$ nombre de Lucas, où L_n est défini comme $L_{n-1} + L_{n-2}$ pour $n \geq 2$ avec $L_0 = 2$, $L_1 = 1$.

La fonction génératrice de $|V(\Lambda_n)|$ est

$$l(x) = \sum_{n\geq 0} |V(\Lambda_n)| \; x^n = \frac{1+x^2}{1-x-x^2}.$$

Nous allons également étudier ces graphes dans tous les chapitres qui vont suivre.

Selon Xu, (cf. [Xu01]), puisque certaines applications parallèles telles que celles du traitement de l'image et du signal sont à l'origine conçues sur une architecture de cycle, il est important d'avoir un plongement efficace du cycle dans un réseau.

Dans le chapitre 3, nous discutons l'Hamiltonicité dans le cube de Fibonacci et dans le cube de Lucas. Comme Γ_n est bipart, si il admet un cycle Hamiltonien alors nécessairement le nombre de sommets est pair. Cette condition est aussi suffisante. En d'autre termes, Γ_n peut avoir un cycle Hamiltonien si et seulement si $n \equiv 1(mod\, 3)$.
À partir de la décomposition fondamentale, dans [LHC94], il est prouvé que le cube Fibonacci contient un chemin Hamiltonien $\mathcal{P}_n = (0\,\mathcal{P}^R_{n-1},\, 10\,\mathcal{P}^R_{n-2})$ avec $\mathcal{P}_0 =$ la chaîne vide, $\mathcal{P}_1 = (0,1)$ et $\mathcal{P}_2 = (01, 00, 10)$. En outre, Liu, Hsu et Chung ont construit $\mathcal{H}_n$, un sous-graphe isomorphe à Γ_n induit par les sommets de $\mathcal{P}_n$. Ensuite, les auteurs ont construit dans $\mathcal{H}_n$, des cycles de chaque longueur paire entre 4 et $|V(\Gamma_n)|$ pour $n \equiv 1(mod\, 3)$. Sinon, les cycles construits ont toutes les longueurs paires entre 4 et $|V(\Gamma_n)| - 1$ sommets.

Dans l'article [Kla], Klavžar a proposé le problème de la caractérisation des sommets v sommets de $V(\Gamma_n)$ pour lesquels le graphe $V(\Gamma_n) - v$ contient un cycle Hamiltonien, étant donné $n \not\equiv 1 (mod\, 3)$. Inspirés par ce dernier problème et les résultats précédents, nous continuons à étudier l'Hamiltonicité dans les cubes de Fibonacci.

Considérons la bipartition $V(\Gamma_n) = (V^{od}(\Gamma_n), V^{ev}(\Gamma_n))$ avec
$V^{od}(\Gamma_n) = \{u \in \Gamma_n \mid u \text{ contient un nombre impair de } 1\}$ et
$V^{ev}(\Gamma_n) = \{v \in \Gamma_n \mid v \text{ contient un nombre pair de } 1\}$.
Ensuite, nous prouvons que pour $n \geq 3$, $|V^{ev}(\Gamma_n)| - |V^{od}(\Gamma_n)| = |V^{od}(\Gamma_{n-3})| - |V^{ev}(\Gamma_{n-3})|$. Cette proposition est utilisée pour montrer que $|V^{ev}(\Gamma_n)| - |V^{od}(\Gamma_n)| = (-1)^{\lfloor \frac{n+2}{3} \rfloor}$, et par conséquent l'un des ensembles $V^{ev}(\Gamma_n)$ ou $V^{od}(\Gamma_n)$ a un élement de plus que l'autre ensemble. Soit $V^P(\Gamma_n)$ cet ensemble on a

$$V^P(\Gamma_n) = \begin{cases} V^{ev}(\Gamma_n) & \text{si } \lfloor \frac{n+2}{3} \rfloor \text{ est pair,} \\ V^{od}(\Gamma_n) & \text{si } \lfloor \frac{n+2}{3} \rfloor \text{ est impair.} \end{cases}$$

Nous démontrons alors

Théorème 0.0.1. *Pour $n \not\equiv 1\,(mod\, 3)$, $n \geq 5$; soit $v \in V^P(\Gamma_n)$. Alors $\Gamma_n - v$ contient un cycle Hamiltonien. De plus, $\Gamma_3 - (010)$ contient un cycle Hamiltonien.*
En outre, si $v \notin V^P(\Gamma_n)$, alors $\Gamma_n - v$ ne contient pas de cycle Hamiltonien.

Pour démontrer cela, nous utilisons à nouveau $\mathcal{H}_n$, le graphe isomorphe à Γ_n induit par les sommets du chemin Hamiltonien $\mathcal{P}_n$ de Γ_n. La figure 2 représente $\mathcal{H}_n$ et les sommets de $V^P(\Gamma_n)$ en gris.
Nous construisons ensuite les cycles de $V(\Gamma_n) - v$, en distinguant deux cas

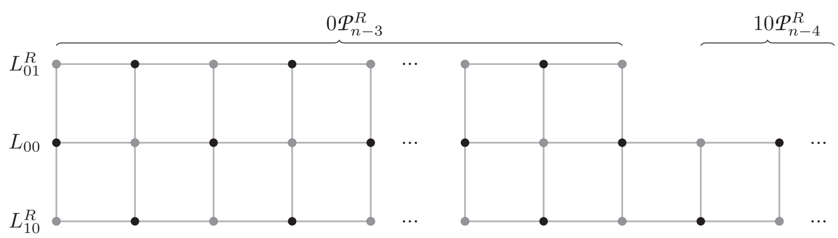


FIGURE 2: Sommets de $V^P(\Gamma_n)$

principaux. Dans le premier cas, le sommet v de $V^P(\Gamma_n)$ est dans le "côté gauche" du graphe. Dans le deuxième cas, v est dans le "côté droit" du graphe. Dans notre graphe de la figure 2, le "côté gauche" du graphe est composé de tous les sommets de $0\mathcal{P}^R_{n-3}$ et le "côté droit" du graphe est composé de tous les sommets de $10\mathcal{P}^R_{n-4}$. Dans le premier cas, nous construisons un cycle

Hamiltonien dans $\Gamma_n - v$ en suivant le modèle utilisé par Liu et al pour construire le cycle Hamiltonien dans Γ_n pour $n \equiv 1 (mod\, 3)$. Dans le dernier cas, nous utilisons la récursivité du chemin Hamiltonien $\mathcal{P}_{n-2} = 0\mathcal{P}^R_{n-3}, 10\,\mathcal{P}^R_{n-4} = 010\mathcal{P}_{n-5}, 00\mathcal{P}_{n-4}, 10\mathcal{P}^R_{n-4}$. Ainsi, un sommet v dans le "côté droit" de $\mathcal{H}_n$ sera toujours adjacent à un sommet dans le "côté gauche". Nous utilisons cette arête pour former le cycle désiré. Nous concluons cette section avec un corollaire qui indique qu'il existe des cycles de chaque longueur pair entre 4 et $|V(\Gamma_n) - v|$ quand n est impair. La preuve de ce corollaire commence avec les cycles Hamiltoniens que nous avons précédemment décrits. Nous retirons alors une paire de sommets pour former un nouveau cycle à chaque fois.
Dans la section 3.2, nous construisons des cycles presque Hamiltoniens pour les cubes de Lucas de la même manière que nous l'avons fait dans la section précédente.
Nous remarquons ensuite le fait que, pour $n \geq 1$, alors

$$|V^{ev}(\Lambda_n)| - |V^{od}(\Lambda_n)| = \begin{cases} -1\,(-1)^{\lfloor \frac{n+2}{3} \rfloor} & \text{if } n \equiv 1\,(mod\,3), \\ 1\,(-1)^{\lfloor \frac{n+2}{3} \rfloor} & \text{if } n \equiv 2\,(mod\,3), \\ 2\,(-1)^{\lfloor \frac{n+2}{3} \rfloor} & \text{if } n \equiv 0\,(mod\,3), \end{cases}$$

pour mentionner que si $n \not\equiv 0\,(mod\,3)$, l'un des ensembles de la partition $V(\Lambda_n) = (V^{ev}(\Lambda_n), V^{od}(\Lambda_n))$ a une chaîne de plus que l'autre ensemble ; si $n \equiv 0\,(mod\,3)$, alors un des ensembles de la partition $(V^{ev}(\Lambda_n), V^{od}(\Lambda_n))$ a deux chaînes plus que l'autre ensemble. Nous appelons cet ensemble $V^P(\Lambda_n)$, $n \geq 1$, où :

$$V^P(\Lambda_n) = \begin{cases} V^{ev}(\Lambda_n) \begin{cases} \text{si } n \equiv 1\,(mod\,3) \text{ et } & \lfloor \frac{n+2}{3} \rfloor \text{ est impair,} \\ \text{si } n \equiv 2\,(mod\,3) \text{ et } & \lfloor \frac{n+2}{3} \rfloor \text{ est pair,} \\ \text{si } n \equiv 0\,(mod\,3) \text{ et } & \lfloor \frac{n+2}{3} \rfloor \text{ est pair.} \end{cases} \\ V^{od}(\Lambda_n) \begin{cases} \text{si } n \equiv 1\,(mod\,3) \text{ et } & \lfloor \frac{n+2}{3} \rfloor \text{ est pair,} \\ \text{si } n \equiv 2\,(mod\,3) \text{ et } & \lfloor \frac{n+2}{3} \rfloor \text{ est impair,} \\ \text{si } n \equiv 0\,(mod\,3) \text{ et } & \lfloor \frac{n+2}{3} \rfloor \text{ est impair.} \end{cases} \end{cases}$$

Ensuite, nous construisons un sous-graphe $\mathcal{H}'_n$, isomorphe à Λ_n, induit par les sommets $0\,\mathcal{P}_{n-1}$ et $10\,\mathcal{P}_{n-2} \setminus \{$chaînes qui se terminent par 1$\}$ représentés dans les figures 3, 4 et 5 pour $n \equiv 1(mod\,3)$, $n \equiv 2(mod\,3)$ et $n \equiv 0(mod\,3)$ respectivement, où les sommets de $V^P(\Lambda_n)$ sont indiqués en gris.

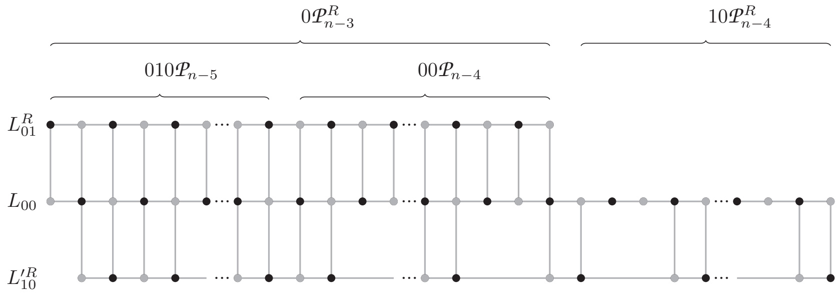


FIGURE 3: Sommets de $V^P(\Lambda_n)$ pour $n \equiv 1 (mod\,3)$

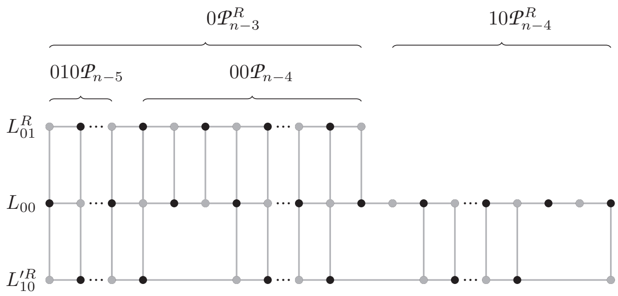


FIGURE 4: Sommets de $V^P(\Lambda_n)$ pour $n \equiv 2 (mod\,3)$

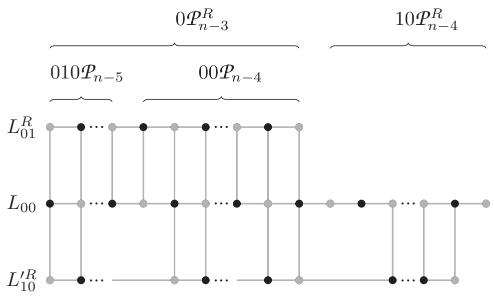


FIGURE 5: Sommets de $V^P(\Lambda_n)$ pour $n \equiv 0 (mod\,3)$

Nous démontrons alors le résultat suivant :

Théorème 0.0.2. *Soit $v \in V^P(\Lambda_n)$ avec $n \not\equiv 0\,(mod\,3)$ et $n \geq 5$. Alors $\Lambda_n - v$ contient un cycle Hamiltonien.*
De plus, si $v \notin V^P(\Lambda_n)$, alors $\Lambda_n - v$ ne contient pas de cycle Hamiltonien.

Dans la preuve, nous considèrons deux cas principaux avec des sous-cas spéciaux pour $n \equiv 1(mod\,3)$ et $n \equiv 2(mod\,3)$ comme nous l'avons fait pour le cube de Fibonacci. Dans le premier cas, le sommet v de $V^P(\Lambda_n)$ est dans le "côté gauche" du graphe. Dans le deuxième cas, v est dans le "côté droit" du graphe. Dans les figures 3 et 4, le "côté gauche" du graphe est constitué de tous les sommets de $0\mathcal{P}^R_{n-3}$ et le "côté droit" du graphe se compose de tous les sommets de $10\mathcal{P}^R_{n-4}$. Nous construisons les cycles Hamiltoniens dans $\Lambda_n - v$ avec des dessins semblables à ceux de $\Gamma_n - v$. Pour conclure ce chapitre, nous prouvons que pour $n \geq 0,\ n \equiv 0\,(mod\,3)$, Λ_n a un chemin de longueur $|\Lambda_n| - 1$ et :

Théorème 0.0.3. *Pour $n \equiv 0\,(mod\,3)$, $n \geq 6$ et l pair avec $4 \leq l \leq |V(\Lambda_n)| - 2$, un cycle de longueur l peut être plongé dans Λ_n.*

La preuve exhibe un cycle Hamiltonien pour $\Lambda_n \setminus \{v_1, v_2\}$ avec des sommets spécifiques v_1 et v_2. Puis en enlevant deux sommets fixés du cycle à chaque étape, nous obtenons tous les cycles désirés.

Nous présentons dans le chapitre 4, les suites d'excentricité des cubes de Fibonacci et Lucas, qui sont les fruits d'un travail effectué avec Michel Mollard.
L'excentricité $e(u)$ d'un sommet u, est la plus grande distance entre u et tous les autres sommets v dans le graphe. Notons que tous les sommets de Γ_n ou Λ_n n'ont pas la même excentricité comme c'est le cas dans Q_n où il n'y a aucune restriction de 1 consécutifs.
Nous définissons la *suite d'excentricité* de G comme la suite $\{a_k\}_{k=0}^{diam(G)}$ d'entiers positifs, où a_k est le nombre de sommets d'excentricité k dans G. Par exemple, dans le tableau suivant, nous donnons le nombre de sommets d'excentricité k dans Γ_n et dans Λ_n pour $n = 0$ jusqu'à 5 qui peut être facilement calculé à la main avec l'aide de la figure 1.1.

n	0	1		2			3				4					5					
k	0	0	1	0	1	2	0	1	2	3	0	1	2	3	4	0	1	2	3	4	5
Γ :	1	0	2	0	1	2	0	0	3	2	0	0	1	5	2	0	0	0	4	7	2
Λ :	1	1	0	0	1	2	0	1	3	0	0	0	1	4	2	0	0	1	5	5	0

Table 1: Nombre de sommets d'excentricité k dans Γ_n et Λ_n.

Dans la section 4.2, nous montrons qu'une chaîne x de Γ_n peut être écrite d'une manière unique comme $x = 0^{l_0}10^{l_1}10^{l_2}\cdots10^{l_p}$ avec $p \geq 0,\ l_0, l_p \geq 0$ et $l_1, \cdots, l_{p-1} \geq 1$. Puis nous associons à chaque chaîne 0^l, un ensemble de chaînes $W(0^l)$ de la façon suivante :

$$W(0^l) = \begin{cases} \{1(01)^{\lfloor \frac{l-1}{2} \rfloor}\} \text{ si } l \text{ est impair,} \\ \{(10)^a(01)^b / 2a + 2b = l, a, b \geq 0\} \text{ si } l \text{ est pair.} \end{cases}$$

Nous calculons ensuite l'excentricité d'une chaîne $x \in \Gamma_n$ comme suit

Théorème 0.0.4. *Pour chaque* $x = 0^{l_0}10^{l_1}10^{l_2}\cdots 10^{l_p}$ *dans* $\mathcal{F}_n$, *avec* p, l_0, $l_p \geq 0$; $l_1, \cdots, l_{p-1} \geq 1$,

$$e(x) = p + \sum_{i=0}^{p} \lfloor \frac{l_i+1}{2} \rfloor$$

De plus, les chaînes qui vérifient l'excentricité de x *sont les chaînes :*

$$y = w_0 0 w_1 0 \cdots w_{p-1} 0 w_p \text{ où } w_i \in W(0^{l_i}) \text{ pour } i = 0, 1, \cdots, p.$$

Dans la section 4.3, nous considérons les sous-ensembles $\mathcal{F}_{n,k}^{\cdot\, od}$ et $\mathcal{F}_{n,k}^{\cdot\, ev}$, qui représentent l'ensemble des chaînes de Γ_n avec excentricité k qui se terminent par un nombre impair de 0 pour le premier sous-ensemble et se terminent par un nombre pair (éventuellement nul) de 0 pour le second sous-ensemble. Puis nous calculons la fonction génératrice de la suite d'excentricité de chaque sous-ensemble :

Théorème 0.0.5.

$$f^{\cdot\, ev}(x,y) = f^{ev\, \cdot}(x,y) = \frac{1}{1 - x(x+1)y},$$

$$f^{\cdot\, od}(x,y) = f^{od\, \cdot}(x,y) = \frac{xy}{1 - x(x+1)y},$$

ainsi la fonction génératrice de la suite d'excentricité est

$$\sum_{n,k \geq 0} f_{n,k}\; x^n y^k = \frac{1+xy}{1 - x(x+1)y}.$$

Nous concluons avec un corollaire qui détermine la valeur de $f_{n,k}$:

Corollaire 0.0.1. *Pour chaque* n, k *tel que* $n \geq k \geq 1$,

$$f_{n,k} = \binom{k}{n-k} + \binom{k-1}{n-k}$$

De plus, $f_{0,0} = 1$ *et* $f_{n,0} = 0$ *pour* $n > 0$.

Dans la section suivante, nous calculons la fonction génératrice de la suite d'excentricité des chaînes du cube de Lucas, $\ell(x,y)$.

Jusqu'à présent, nous avons utilisé la lettre $\mathcal{F}$ pour les ensembles de Fibonacci. Nous désignons les ensembles de Lucas par $\mathcal{L}$.

Considérons le sous-ensemble $\mathcal{F}_{n,k}^{\cdot\, od}(\mathcal{F}_{n,k}^{od\, \cdot})$ qui représente l'ensemble des chaînes de Γ_n avec excentricité k qui finissent (commencent) par un nombre impair de 0. De même, $\mathcal{F}_{n,k}^{\cdot\, ev^*}(\mathcal{F}_{n,k}^{ev^*\,\cdot})$ est l'ensemble des chaînes de Γ_n avec excentricité k qui finissent (commencent) par un nombre pair, non nul de 0, et $\mathcal{F}_{n,k}^{\cdot\,\varnothing}(\mathcal{F}_{n,k}^{\varnothing\,\cdot})$ est l'ensemble des chaînes de Γ_n avec excentricité k qui ne finissent (commencent) pas par 0.

Nous montrons alors que les ensembles $\mathcal{L}_{n,k}^{a\,b}$ et $\mathcal{F}_{n,k}^{a\,b}$ sont les mêmes pour tout (a, b) en excluant deux ensembles, $\mathcal{L}^{od\,od}$ et $\mathcal{L}^{\varnothing\,\varnothing}$. Nous calculons les valeurs de $\ell_{n,k}^{od\,od}$ et $\ell_{n,k}^{\varnothing\,\varnothing}$ ainsi que les valeurs de $f_{n,k}^{od\,od}$ et $f_{n,k}^{\varnothing\,\varnothing}$. Ces résultats et le fait que les $\ell_{n,k}$ peuvent être décomposés en

$$\ell_{n,k} = \ell_{n,k}^{od\,od} + \ell_{n,k}^{od\,ev^*} + \ell_{n,k}^{od\,\varnothing} + \ell_{n,k}^{ev^*od} + \ell_{n,k}^{ev^*ev^*} + \ell_{n,k}^{ev^*\varnothing} + \ell_{n,k}^{\varnothing\,od} + \ell_{n,k}^{\varnothing\,ev^*} + \ell_{n,k}^{\varnothing\,\varnothing}, \quad (0.0.1)$$

nous donnent l'équation suivante :

$$\ell_{n,k} = f_{n,k} - f_{n,k}^{od\,od} - f_{n,k}^{\varnothing\,\varnothing} + \ell_{n,k}^{od\,od} + \ell_{n,k}^{\varnothing\,\varnothing}.$$

Puis en appliquant le résultat du théorème précédent, la suite d'excentricité est déterminée comme suit

Théorème 0.0.6. *La fonction génératrice de la suite d'excentricité du cube de Lucas est*

$$\ell(x, y) = \sum_{n,k\geq 0} \ell_{n,k}x^n y^k = \frac{1 + x^2y}{1 - xy - x^2y} + \frac{1}{1 + xy} - \frac{1 - x}{1 - x^2y}.$$

Comme corollaire, on obtient la valeur de $\ell_{n,k}$:

Corollaire 0.0.2. *Pour tout n, k avec $n > k \geq 1$,*

$$\ell_{n,k} = \binom{k}{n-k} + \binom{k-1}{n-k-1} + \varepsilon_{n,k}$$

où

$$\varepsilon_{n,k} = \begin{cases} -1 & si\ n = 2k, \\ \ \ 1 & si\ n = 2k + 1, \\ \ \ 0 & sinon. \end{cases}$$

De plus, $\ell_{0,0} = \ell_{1,0} = 1$, $\ell_{n,0} = 0$ pour $n > 1$ et

$$\ell_{n,n} = \begin{cases} 2 & si\ n\ est\ pair\ (n \geq 2), \\ 0 & si\ n\ is\ impair. \end{cases}$$

La dernière section de ce chapitre donne une autre preuve du théorème 0.0.6 à partir de l'obtention de la fonction génératrice de la suite d'excentricité des chaînes du cube de Lucas avec une approche directe. C'est-à-dire que nous calculons toutes les fonctions génératrices de l'équation 0.0.1.

Enfin, le chapitre 5 présente une étude du problème de la domination et 2-packing pour les cubes de Fibonacci et Lucas. Les résultats présentés ici sont le produit du travail effectué avec Sandi Klavžar, Michel Mollard et Yoomi Rho. Dans la section 5.1, on montre les théorèmes suivants, qui donnent les groupes d'automorphismes des cubes de Fibonacci et Lucas.

Théorème 0.0.7. *Pour tout* $n \geq 1$, $\mathrm{Aut}(\Gamma_n) \simeq \mathbb{Z}_2$.

Théorème 0.0.8. *Pour tout* $n \geq 3$, $\mathrm{Aut}(\Lambda_n) \simeq D_{2n}$.

Dans la section 5.2, on considère le nombre de domination des cubes de Fibonacci et Lucas. Une relation entre les nombres de domination des deux cubes est montrée :

Proposition 0.0.3. *Soit* $n \geq 4$, *alors*

(i) $\gamma(\Lambda_n) \leq \gamma(\Gamma_{n-1}) + \gamma(\Gamma_{n-3})$,

(ii) $\gamma(\Lambda_n) \leq \gamma(\Gamma_n) \leq \gamma(\Lambda_n) + \gamma(\Gamma_{n-4})$.

Ensuite, nous discutons des nombres de domination exacts pour les petites dimensions d'après [PZ12] où les ensembles minimaux dominants de Γ_8 sont determinées, nous calculons les nombres de domination pour Γ_9 et Λ_9.

Alors que nous conjecturons que $\gamma(\Gamma_9) = 17$ et $\gamma(\Lambda_9) = 16$, Ilić et Milošević l'ont confirmé plus tard dans [IM].

La section se termine avec une borne inférieure pour le nombre de domination des cubes de Lucas :

Théorème 0.0.9. *Pour tout* $n \geq 7$, $\gamma(\Lambda_n) \geq \left\lceil \dfrac{L_n - 2n}{n-3} \right\rceil$.

La dernière section est dédiée au 2-packing. On prouve la borne inférieure suivante :

Théorème 0.0.10. *Pour tout* $n \geq 8$, $\rho(\Gamma_n) \geq \rho(\Lambda_n) \geq 2^{2^{\frac{\lfloor \lg n \rfloor}{2}-1}}$.

Nous présentons ensuite les nombres de 2-packing de Γ_n et de Λ_n pour $n \leq 10$ qui ont été trouvés en utilisant l'ordinateur.

Nous concluons le chapitre avec quelques conjectures qui mettent en corrélation les nombres de domination et de 2-packing des deux cubes.

n	0	1	2	3	4	5	6	7	8	9	10
$\gamma(\Gamma_n)$	1	1	1	2	3	4	5	8	12	$\leq$17	-
$\rho(\Gamma_n)$	1	1	1	2	2	3	5	6	9	14	20
$\gamma(\Lambda_n)$	1	1	1	1	3	4	5	7	11	$\leq$16	-
$\rho(\Lambda_n)$	1	1	1	1	2	3	5	6	8	13	18

Table 2: Nombres de domination et du 2-packing pour des cubes de petite taille

Contents

Chapter 1
Introduction

The problems studied throughout this thesis deal with the Fibonacci cube and one of its generalizations: the Lucas cube. The Fibonacci cube Γ_n is an isometric subgraph of the hypercube, but with a Fibonacci number of vertices. It was originally proposed by W-J. Hsu in [Hsu93] as an interconnection network.

An *Interconnection network* can be represented by a graph $G = (V, E)$, where V denotes processors and E the communication links for data exchange among the processors.

According to Bertsekas and Tsitsiklis cf.[BT97], the interconnection networks are usually evaluated in terms of their suitability for some standard communication tasks. Some typical criteria include the diameter of the network which is the maximum distance between any pair of vertices; the connectivity of the network which provides a measure of the number of independent paths between any pair of vertices and the flexibility given by running efficiently a wide variety of algorithms. This translates in the study of the embedding problem which asks if a guest graph is a subgraph of a host graph. Therefore if a graph embedding exists, we can apply algorithms designed to work efficiently for guest graphs to host graphs.

The hypercube is a popular interconnection network of 2^n vertices which represents autonomous processors connected with n neighbors. The hypercube represent an important class in graph theory mainly because of its structural properties.

As an interconnection network, the hypercube has very attractive properties such as recurrency, symmetry and connectivity. Also, many topological structures that arise while parallelizing an algorithm can be embedded into the hypercube. On the contrary, when a parallel computer system is being implemented under this architecture, the number of processors is restricted to be a power of two which becomes impractical as the network's size grows. This can be saved using a network with less vertices. Therefore, among other alternative models, the Fibonacci cube, which is inspired in the Fibonacci numbers, arise as an appealing interconnection network due to its topological structure and its more moderated growth.

In chapter 2, we present some preliminaries related to the Fibonacci numbers. As Zeckendorf proved cf. [Zec72], each positive integer has a unique representation as a sum of nonconsecutive Fibonacci numbers. The result is a binary representation of the integer with no consecutive 1's. Notice that a number between 0 and $F_n - 1$ requires $(n-2)$ positions to be represented.

We will define a string $(b_1\, b_2 \cdots b_n)$ where $b_j \in \{0,1\}$ and $b_j \cdot b_{j+1} = 0$ as a *Fibonacci string of length n*.

The *Fibonacci cube*, Γ_n is the subgraph of Q_n induced by the Fibonacci strings of length n where adjacent vertices of Γ_n differ in one position and the number of vertices in Γ_n is F_{n+2}. See figure 1.1.

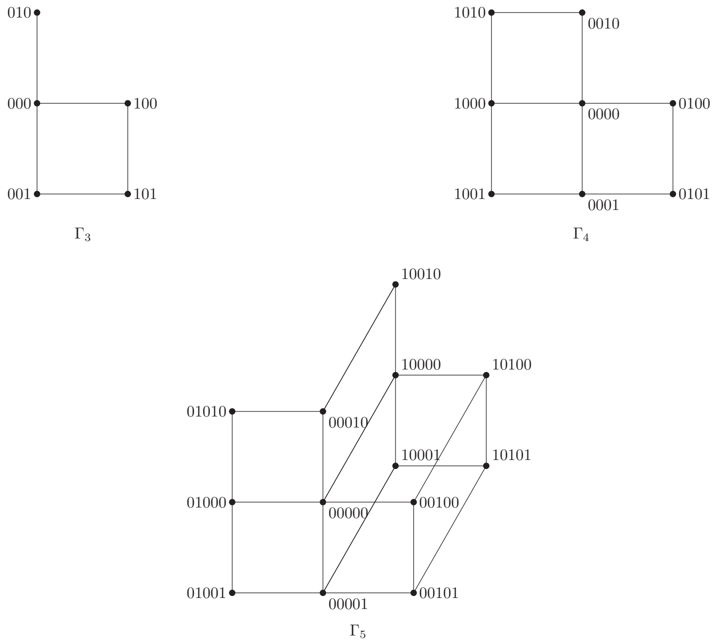


Figure 1.1: Γ_3, Γ_4 and Γ_5

We discuss next, different recursive decompositions of the Fibonacci cube. Mainly, the *fundamental decomposition of* Γ_n which states that the Fibonacci cube Γ_n contains two disjoint subgraphs that are isomorphic to Γ_{n-1} and Γ_{n-2}. Furthermore, each vertex in the subgraph isomorphic to Γ_{n-2} has exactly one neighbor in the subgraph isomorphic to Γ_{n-1}.

The fundamental decomposition of Γ_n can be recursively applied to its subgraphs Γ_{n-1} and/or Γ_{n-2}. Throughout this document, many important properties will be deduced from this decomposition.

We will also mention in this chapter other structural results of the Fibonacci cube. Among these, the result due to Klavžar, cf. [Kla05], that states that the

Fibonacci cube is a median graph. Other results concerning the number of edges of Γ_n are presented as well as other enumerative results. Finally, section 2.5 presents a variant and two generalizations of the Fibonacci cube. The former, named the Lucas cube is obtained by removing all the strings that begin and end with 1 from the Fibonacci cube.

The Lucas cube Λ_n was introduced by E. Munarini, C.Cippo and N. Zagaglia Salvi in [MCS01] as a subgraph of the n-cube that has also attractive properties as its recursive decompositions into two disjoint subgraphs which are isomorphic to Γ_{n-1} and Γ_{n-3}. We will also study these graphs in all the chapters that will follow.

Since some parallel applications such as those in image and signal processing are originally designated on a cycle architecture, it is important to have an effective cycle embedding in a network. (See [Xu01]).

In chapter 3, we discuss the Hamiltonicity in the Fibonacci and the Lucas cubes. Because Γ_n is bipartite, it can have a Hamiltonian cycle only if it has an even number of vertices. In other words, Γ_n can have a Hamiltonian cycle if and only if $n \equiv 1 (mod\, 3)$.
Based on the fundamental decomposition, J. Liu, W. J. Hsu and M. J. Chung proved in [LHC94] that the Fibonaci cube contains a Hamiltonian path $\mathcal{P}_n$. Furthermore, the authors constructed cycles of every even length from 4 to $|V(\Gamma_n)|$ if and only if $n \equiv 1(mod\, 3)$. Otherwise the constructed cycles have every even length from 4 to$|V(\Gamma_n)| - 1$ vertices.

In the survey paper [Kla], Klavžar proposed the problem of characterizing the vertices v of $V(\Gamma_n)$ for which the graph $\Gamma_n - v$ contains a Hamiltonian cycle given that $n \not\equiv 1(mod\, 3)$.

Inspired by this last problem and the previous results, we characterize these vertices which we denote by $V^P(\Gamma_n)$ and continue to study the Hamiltonicity in the Fibonacci cubes. Specifically, we prove the following

Theorem 1.0.4. *For $n \not\equiv 1\, (mod\, 3)$, $n \geq 5$; let $v \in V^P(\Gamma_n)$. Then $\Gamma_n - v$ contains a Hamiltonian cycle. Moreover, $\Gamma_3 - (010)$ contains a Hamiltonian cycle.*
Furthermore, if $v \notin V^P(\Gamma_n)$, then $\Gamma_n - v$ does not contain a Hamiltonian cycle.

The second part of this chapter is consacrated to construct almost Hamiltonian cycles for the Lucas cubes in the same way as we did for the Fibonacci cubes.

We give in chapter 4, the eccentricity sequences of the Fibonacci and the Lucas cubes which are the fruits of a work done with Michel Mollard cf.

[CM12]. The *eccentricity* $e(u)$ of a vertex u, is the greatest distance between u and any other vertex v in the graph. Notice that not all the vertices of Γ_n or Λ_n have the same eccentricity as it happens in Q_n where there are no restrictions of consecutive 1's.
We define the *eccentricity sequence* of G as the sequence $\{a_k\}_{k=0}^{diam(G)}$ of non-negative integers, where a_k is the number of vertices of eccentricity k in G. For example, in the next table we show the number of vertices of eccentricity k in Γ_n and in Λ_n for $n = 0$ to 5 which can be easily computed by hand with help of figure 1.1.

n	0	1		2			3				4					5					
k	0	0	1	0	1	2	0	1	2	3	0	1	2	3	4	0	1	2	3	4	5
$\Gamma:$	1	0	2	0	1	2	0	0	3	2	0	0	1	5	2	0	0	0	4	7	2
$\Lambda:$	1	1	0	0	1	2	0	1	3	0	0	0	1	4	2	0	0	1	5	5	0

Table 1.1: Number of vertices of eccentricity k in Γ_n and Λ_n.

We proceed to prove in section 4.2 that a vertex x in Γ_n can be written uniquely as the concatenation of particular strings. We give some results concerning the eccentricity of these substrings which lead us to compute $e(x)$ and to characterize the vertices y in Γ_n that satisfy $e(x)$. In the section 4.3, we consider the subsets $\mathcal{F}_{n,k}^{\cdot od}$ and $\mathcal{F}_{n,k}^{\cdot ev}$, which represent the set of strings of Γ_n with eccentricity k that end with an odd number of 0's for the first subset and that end with an even (eventually null) number of 0's for the second subset. Then we compute the generating function of the eccentricity sequence of each subset to conclude with a corollary that determines the value of $f_{n,k}$, the number of vertices of eccentricity k in Γ_n:

Corollary 1.0.5. *For all n, k such that $n \geq k \geq 1$,*

$$f_{n,k} = \binom{k}{n-k} + \binom{k-1}{n-k}$$

Furthermore, $f_{0,0} = 1$ and $f_{n,0} = 0$ for $n > 0$.

The results of the previous section and some observations that relate some sets of strings of the Lucas and the Fibonacci cubes, give us the eccentricity sequence of the Lucas cube's strings, $\ell(x, y)$. As a corollary we obtain the value of $\ell_{n,k}$, the number of vertices in Λ_n with eccentricity k.

The last section of this chapter shows an alternative proof for obtaining the generating function of the eccentricity sequence of the Lucas cube's strings with a direct approach.

Finally, chapter 5, presents a study of the Fibonacci and the Lucas cubes from the domination ($\gamma(G)$) and the 2-packing ($\rho(G)$) points of view. The results presented here are the product of the work done with Sandi Klavžar, Michel Mollard and Yoomi Rho cf. [CKMR11]. In the first section of this chapter, the automorphism groups of the Fibonacci and the Lucas cubes are determined in the following theorems.

Theorem 1.0.6. *For any* $n \geq 1$, $\mathrm{Aut}(\Gamma_n) \simeq \mathbb{Z}_2$.

Theorem 1.0.7. *For any* $n \geq 3$, $\mathrm{Aut}(\Lambda_n) \simeq D_{2n}$.

Section 5.2 considers the domination number of Fibonaci and Lucas cubes where a relation between the domination numbers of both cubes is shown. Then we discuss exact domination numbers for small dimensions following Pike and Zou ([PZ12]) who determined the minimum dominating sets of Γ_8. Thus we compute the domination numbers for Γ_9 and Λ_9 and conjecture that $\gamma(\Gamma_9) = 17$ and $\gamma(\Lambda_9) = 16$, hold. (Conjecture that was later confirmed by Ilić and Milošević in [IM]).
The section concludes with a lower bound for the domination number of the Lucas cubes:

Theorem 1.0.8. *For any* $n \geq 7$, $\gamma(\Lambda_n) \geq \left\lceil \frac{L_n - 2n}{n-3} \right\rceil$.

The last section, 5.3, is dedicated to the 2-packing number where it is proven the next lower bound:

Theorem 1.0.9. *For any* $n \geq 8$, $\rho(\Gamma_n) \geq \rho(\Lambda_n) \geq 2^{2^{\frac{\lfloor \lg n \rfloor}{2}-1}}$.

We present next the 2-packing numbers of of Γ_n and Λ_n for $n \leq 10$ found using computer:

n	0	1	2	3	4	5	6	7	8	9	10
$\gamma(\Gamma_n)$	1	1	1	2	3	4	5	8	12	≤17	-
$\rho(\Gamma_n)$	1	1	1	2	2	3	5	6	9	14	20
$\gamma(\Lambda_n)$	1	1	1	1	3	4	5	7	11	≤16	-
$\rho(\Lambda_n)$	1	1	1	1	2	3	5	6	8	13	18

Table 1.2: Domination numbers and 2-packing numbers of small cubes

We conclude the chapter with some conjectures that interrelate the domination and the 2-packing numbers of both the Fibonacci and the Lucas cubes.

Chapter 2

Fibonacci Cube

2.1 Motivation

Following J. Xu [Xu01], a *Computer network* is a system whose components are autonomus computers and other devices that are connected together usually over long physical distance in order to transfer information according to some pattern. A connection pattern of the components is called an *Interconnection network*. In other words, an interconnection network provides a specific way in which the components interact. An interconnection network can be represented by a graph $G = (V, E)$, where V denotes the processors and E the communication links for data exchange among the processors. Such graph is called the *Topological structure* of the interconnection network.

The hypercube of dimension n, also known as the *n-cube* is a popular interconnection network consisting in 2^n parallel processors, each one provided with its own memory and connected with n neighbors. The interconnection is achieved by sending a message from one processor to another processor. This message (or data) travels through a sequence of nearests neighbors.

The structural properties of the hypercube such as recurrency (an hypercube Q_n can be decomposed into two Q_{n-1} hypercubes), symmetry, vertex degree, diameter or connectivity are well-appreciated characteristics whithin an interconnection network. Embedding problems are concerned with finding mappings between two graphs that preserve certain topological properties. Many particular topological structures that arise while parallelizing an algorithm can be embedded into Q_n.

All of this make this graph very appealing for its implementation as an interconnection network. See [HHW88] for a survey of structural properties of the hypercube including hamiltoninan cycles and embeddings. We refer to [SS88] as well, for a wide examination of the hypercube properties that make this graph so attractive.

In the other hand, while implementing a parallel computer system, more processors and memories may be added as the budget permits it. In the case of the hypercube, the number of processors must be a power of 2. When the network is implemented with a smaller number of processors, many communications links may be unused. Thus, other topological structures are needed in order to allow the addition of a small number of nodes while minimizing the resource's wasting. [DYN03]

The Fibonacci cube was introduced by W-J. Hsu in [Hsu93] as a new interconnection network. This graph is an *isometric subgraph* of the hypercube. In other words, the set of vertices of the Fibonacci cube is a subset of the vertices of the hypercube and for every $u, v \in V(FibCube)$, $d_{FibCube}(u, v) = d_{Q_n}(u, v)$.

The Fibonacci cube, which is inspired in the Fibonacci numbers has also attractive recurrent structures such as its decomposition into disjoint subgraphs that are also Fibonacci cubes by themselves.

In the next section, we will define the Fibonacci cube and describe in more detail its *self-similar* structure.

2.2 Preliminaries

More than eight hundred years ago, Leonardo of Pisa, better known as Fibonacci, introduced the numbers that are known today as the Fibonacci numbers. These numbers were used to solve the problem of how many rabbits can be produced from an original pair through a year, supposing that each pair of rabbits will last all the year giving birth to a new pair each month and supposing also that they become fertile a month after they were born.

It is not clear though if Fibonacci invented the series of numbers that hold his name. Parmanand Singh wrote in [Sin85] that the same sequence had been studied and used by Indian scholars to write prosody in Sanskrit and Prakrit languages long time before Fibonacci wrote his book *Liber Abaci* where he presented the sequence. According to Singh, authorities on metrical sciences gave the rule for the Fibonacci numbers explicitly prior to 1200.

The Fibonacci numbers are often found in the nature, for example in the arrangement of the leaves around a plant stem or the number and arrangement of petals in flowers as daisies and sunflowers or in pine cones. These associations between nature and Fibonacci numbers are frequently related, as for the original rabbits problem, to counting problems.

The n^{th} *Fibonacci number*, $n \geq 2$, is determined by the following recurrence relation:

$$F_n = F_{n-1} + F_{n-2}$$

with the initial values

$$F_0 = 0,\ F_1 = 1.$$

In other words, each number in the sequence is the sum of the two preceding numbers.

We will denote by g the generating function of the Fibonacci sequence,

$$g(x) = \sum_{n \geq 0} F_n \; x^n.$$

Proposition 2.2.1.

$$g(x) = \frac{x}{1 - x - x^2}.$$

Proof.

$$\begin{aligned}
g(x) = \sum_{n \geq 0} F_n \; x^n &= F_0 + F_1 \; x + \sum_{n \geq 2} (F_{n-1} + F_{n-2}) \; x^n \\
&= x + \sum_{n \geq 2} F_{n-1} \; x^n + \sum_{n \geq 2} F_{n-2} \; x^n \\
&= x + \sum_{n \geq 2} (F_{n-1} \; x^{n-1}) x + \sum_{n \geq 2} (F_{n-2} \; x^{n-2}) x^2 \\
&= x + \sum_{n \geq 1} (F_{n-1} \; x^{n-1}) x + \sum_{n \geq 2} (F_{n-2} \; x^{n-2}) x^2 \\
&= x + x g(x) + x^2 g(x).
\end{aligned}$$

□

As time passed by, mathematicians continued to associate the Fibonacci sequence to a wide variety of problems and found different properties as well. As Edouard Zeckendorf observed, see [GKP94], each positive integer has a unique representation as a sum of nonconsecutive Fibonacci numbers. The result is a binary representation of the integer with no consecutive 1's.

Theorem 2.2.2 (Zeckendorf)**.** *Any integer i such that $0 \leq i < F_n$ can be uniquely represented as*

$$i = \sum_{j=2}^{n-1} a_j F_j,$$

where

$$a_j \cdot a_{j+1} = 0 \quad \textit{for} \quad 2 \leq j \leq (n-2)$$

with F_j is the j^{th} Fibonacci number and $a_j \in \{0, 1\}$.

Proof. Let $i = m - F_n$ with $F_n \leq m < F_{n+1}$. By inductive hypothesis, i has a unique representation as a sum of nonconsecutive Fibonacci numbers,

$i = \sum_{j=2}^{n-1} a_j F_j$.
Thus $m = \sum_{j=2}^{n-1} a_j F_j + (1 \cdot F_n)$. The representation of i does not have consecutive 1's and if the representation of m had, then $m \geq F_{n-1} + F_n = F_{n+1}$. Hence, m is expressed as a sum of nonconsecutive Fibonacci numbers. □

Note that by the Zeckendorf theorem, a number between 0 and $F_n - 1$ requires $(n-2)$ positions to be represented.

We say that a positive integer i is in its *Fibonacci representation* if it is expressed as a sum of two or more nonconsecutive Fibonacci numbers. Therefore, we can represent it as $(i)_F = (a_{n+1} \cdots a_3\, a_2)$ where $a_j \in \{0, 1\}$ and $a_j \cdot a_{j+1} = 0$; $2 \leq j \leq n+1$.

A greedy approach to find the Fibonacci representation of $(i)_F$ is described in the following way:
Assign 1 to a_{j_1} for the largest $F_{j_1} \leq i$. Then assign 1 to a_{j_2} for the largest $F_{j_2} \leq i - F_{j_1}$ and so on until the reminder is 0. The values of the unassigned a_j are 0.

Using this previous algorithm, and with the help of table 2.1, we show the first 13 positive integers in their Fibonacci representation (table 2.2).

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}
0	1	1	2	3	5	8	13	21	34	55

Table 2.1: First 11 Fibonacci numbers.

$(0)_F = (00000)$, $(1)_F = (00001)$, $(2)_F = (00010)$, $(3)_F = (00100)$,
$(4)_F = (00101)$, $(5)_F = (01000)$, $(6)_F = (01001)$, $(7)_F = (01010)$,
$(8)_F = (10000)$, $(9)_F = (10001)$, $(10)_F = (10010)$, $(11)_F = (10100)$,
$(12)_F = (10101)$.

Table 2.2: Fibonacci representation of the first 13 positive integers.

Notice that $(i)_F = (a_{n+1} \cdots a_3\, a_2)$ has length n.
Therefore, any string $(b_1\, b_2 \cdots b_n)$ where $b_j \in \{0, 1\}$ and $b_j \cdot b_{j+1} = 0$ will be defined as a *Fibonacci string of length n*.

The *Fibonacci cube*, Γ_n is the subgraph of Q_n induced by the Fibonacci strings of length n where adjacent vertices of Γ_n differ in one position. Taking into consideration the empty string, we define $\Gamma_0 = K_1$. Notice that $|V(\Gamma_n)| = F_{n+2}$. Thus, the generating function of the vertices of Γ_n is obtained in the following

Proposition 2.2.3. *The generating function of* $|V(\Gamma_n)|$ *is*

$$f(x) = \sum_{n\geq 0} |V(\Gamma_n)| x^n = \frac{1+x}{1-x-x^2}.$$

Proof.

$$\begin{aligned} f(x) &= \sum_{n\geq 0} F_{n+2}\ x^n \\ &= \frac{1}{x^2} \sum_{n\geq 0} F_{n+2}\ x^{n+2} \\ &= \frac{1}{x^2} \left(\sum_{n\geq 0} F_n\ x^n - F_0 - F_1\ x \right). \end{aligned}$$

Thus, by proposition 2.2.1,

$$f(x) = \frac{1}{x^2} \left(\frac{x}{1-x-x^2} - x \right) = \frac{1+x}{1-x-x^2}.$$

□

In figure 2.1, we show Γ_3, Γ_4 and Γ_5 as the subgraphs of Q_n induced by the Fibonacci strings of length $n = 3$, 4 and 5.

2.3 Recursive decompositions

Parallel computers are usually shared by several users at a time. It is desirable then, that the network traffic produced by other users does not affect other applications. Thus, it is well-appreciated if the network can be partitioned into smaller subnetworks. This recursion property may also be required for security reasons.

Fault-tolerance is the property that enables a system to continue operating properly in the event of the failure of one or more of its components. Fault-tolerance can also define the rules of interaction between machines. For example, when a failure occurs, other functioning components may be reassigned to a smaller and similar subgraph so that the system can continue operating. Therefore, a self-similar network is an appealing structure for fault-tolerant computing.

In this section we will discuss some recursive decompositions of the Fibonacci cube.

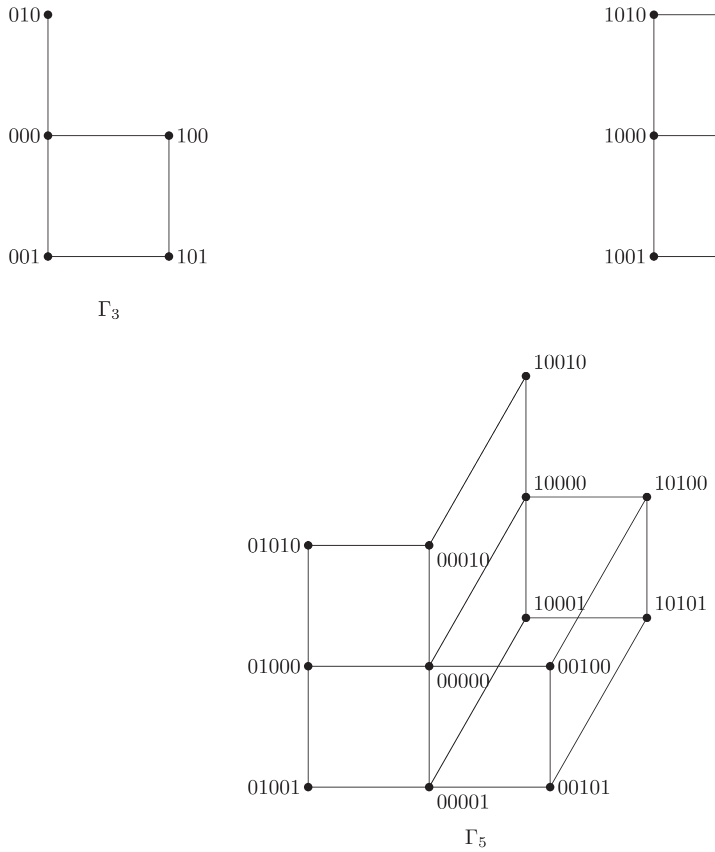


Figure 2.1: Γ_3, Γ_4 and Γ_5

Theorem 2.3.1 ([HPL93]). *The Fibonacci cube Γ_n, with $n \geq 2$, contains two disjoint subgraphs that are isomorphic to Γ_{n-1} and to Γ_{n-2}.*
Moreover, each vertex in the subgraph isomorphic to Γ_{n-2} has exactly one neighbor in the subgraph isomorphic to Γ_{n-1}.

Proof. Let $x = (b_1\, b_2 \cdots b_n) \in V(\Gamma_n)$. Thus x has two possible forms:

(i) If $b_1 = 0$,
thus the graph induced by $\{x \,|\, x = (0\, b_2 \cdots b_n)\}$ is a Fibonacci cube whose strings have length $(n-1)$.

(ii) If $b_1 = 1$, thus $b_2 = 0$,
then the graph induced by $\{x \,|\, x = (1\, 0\, b_3 \cdots b_n)\}$ is a Fibonacci cube whose strings have length $(n-2)$.

Furthermore, every string $x = (1\, 0\, b_3 \cdots b_n)$ in the subgraph isomorphic to

Γ_{n-2} has exactly one neighbor $x' = (0\,0\,b'_3 \cdots b'_n)$ in the subgraph isomorphic to Γ_{n-1} with $b_j = b'_j$ for $3 \leq j \leq n$. □

Let us denote by aX the concatenation of a string a to all the strings of a set of strings X.

Corollary 2.3.2. *The vertex set of Γ_n is defined recursively by*

$$V(\Gamma_n) = 0V(\Gamma_{n-1}) \uplus 10V(\Gamma_{n-2})$$

where $\uplus$ is the disjoint union of sets.

Figure 2.1, exemplifies the previous Theorem where it is clear that Γ_5 is composed of two disjoint subgraphs which are isomorphic to Γ_4 and to Γ_3.

We will continue to use Γ_5 through this chapter to exemplify the different recursive decompositions since it can illustrate a fair number of them. Therefore, in order to introduce the generalization of Theorem 2.3.1, we will use again this graph as follows:
As mentioned above, Γ_5 is decomposed into Γ_4 and Γ_3. We can decompose subsequently the subgraph Γ_4 into Γ_3 and Γ_2. By now, Γ_5 is decomposed in two subgraphs Γ_3 and one subgraph Γ_2.
Again, Γ_3 is the disjoint union of Γ_2 and Γ_1. Therefore, Γ_5 can be decomposed in three subgraphs Γ_2 and two subgraphs Γ_1.
Finally, with the decomposition of each Γ_2 into Γ_1 and Γ_0, we observe that Γ_5 can be decomposed in five Γ_1 and three Γ_0. Figure 2.2 illustrates these recursive decompositions.

Theorem 2.3.3 ([HPL93])**.** *For $2 \leq k \leq n$, the Fibonacci cube Γ_n can be decomposed in F_k disjoint subgraphs isomorphic to Γ_{n-k+1} and F_{k-1} disjoint subgraphs isomorphic to Γ_{n-k}.*

Proof. The statement is true for Γ_2. Let us assume that it is true for Γ_N and let us consider Γ_{N+1}. By Theorem 2.3.1, Γ_{N+1} can be decomposed in Γ_N and Γ_{N-1}. By hypothesis,
Γ_N can be decomposed into F_k subgraphs isomorphic to Γ_{N-k+1} and F_{k-1} subgraphs isomorphic to Γ_{N-k} for $2 \leq k \leq N$ and
Γ_{N-1} into F_{k-1} subgraphs isomorphic to $\Gamma_{(N-1)-(k-1)+1}$ and F_{k-2} subgraphs isomorphic to $\Gamma_{(N-1)-(k-1)}$ for $2 \leq k-1 \leq N-1$.
The case $k = 2$ is verified by Theorem 2.3.1. Thus, for $3 \leq k \leq N$,
Γ_{N+1} can be decomposed in F_k subgraphs isomorphic to Γ_{N-k+1}, F_{k-1} subgraphs isomorphic to Γ_{N-k} and

F_{k-1} subgraphs isomorphic to $\Gamma_{(N-1)-(k-1)+1}$ and F_{k-2} subgraphs isomorphic to $\Gamma_{(N-1)-(k-1)}$.
Notice that
$\Gamma_{N-k+1} = \Gamma_{(N-1)-(k-1)+1} = \Gamma_{(N+1)-(k+1)+1}$ and
$\Gamma_{N-k} = \Gamma_{(N-1)-(k-1)} = \Gamma_{(N+1)-(k+1)}$.
Therefore, Γ_{N+1} has F_{k+1} subgraphs isomorphic to $\Gamma_{(N+1)-(k+1)+1}$ and F_k subgraphs isomorphic to $\Gamma_{(N+1)-(k+1)}$.
For the case $k = N+1$, let us consider $k = N$. Then,
Γ_{N+1} can be decomposed in F_N subgraphs isomorphic to $\Gamma_{(N+1)-n+1}$ and F_{N-1} subgraphs isomorphic to $\Gamma_{(N+1)-N}$.
At the same time, each one of the F_N subgraphs isomorphic to $\Gamma_{(N+1)-N+1}$ can be decomposed into one subgraph isomorphic to $\Gamma_{(N+1)-N}$ and one subgraph isomorphic to $\Gamma_{(N+1)-(N+1)}$.
Hence, Γ_{N+1} is decomposed into F_N subgraphs isomorphic to $\Gamma_{(N+1)-N}$, F_N subgraphs isomorphic to $\Gamma_{(N+1)-(N+1)}$ and F_{N-1} subgraphs isomorphic to $\Gamma_{(N+1)-N}$.
Therefore, Γ_{N+1} can be decomposed in:
F_{N+1} subgraphs isomorphic to $\Gamma_{(N+1)-(N+1)+1}$ and F_N subgraphs isomorphic to $\Gamma_{(N+1)-(N+1)}$, which completes the proof. □

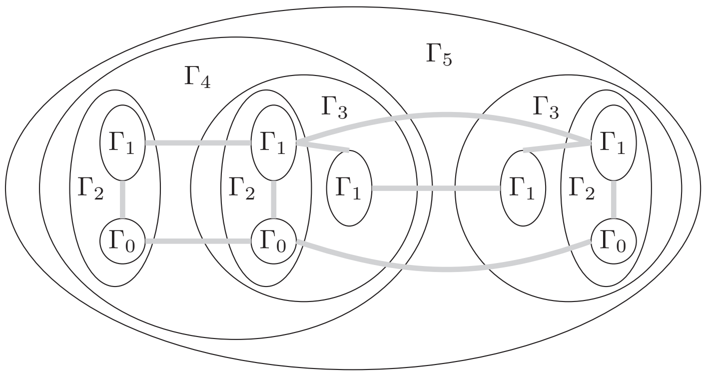


Figure 2.2: Theorem 2.3.3, recursive decompositions of Γ_5.

2.3.1 Quotient Fibonacci Cube

We will discuss a last recursive decomposition in this section, which plunges a Fibonacci cube into another Fibonacci cube of a smaller order. The main idea is to use Theorem 2.3.1 to decompose Γ_n into Γ_{n-1} and Γ_{n-2}. Then, proceed to merge the edges that link these two subgraphs into a unique edge.

The resulting graph is isomorphic to Γ_1. Applying this method iteratively to every subgraph, we obtain each time, a subgraph isomorphic to Γ_k.

Let $(b_1\, b_2 \ldots b_n) \in V(\Gamma_n)$ and let $(c_1\, c_2 \ldots c_k)$ be a string of Fibonacci of length k with $1 \leq k < n$.

Let $G_n(c_1\, c_2 \ldots c_k)$ be the subgraph of Γ_n induced by the set of vertices $\{(b_1\, b_2 \cdots b_n) \,|\, b_i = c_i;\ 1 \leq i \leq k\}$.

For $1 \leq k < n$, let $\mathcal{F}_k$ be the set of strings of Γ_k.

Then, $\Gamma_n/k = (V_n/k, E_n/k)$, is called the *Quotient Fibonacci Cube*, where

(i) $V_n/k = \{G_n(c_1\, c_2 \ldots c_k)$ for every $(c_1\, c_2 \ldots c_k) \in \mathcal{F}_k\}$ and

(ii) for two different vertices of V_n/k, namely $G_n(c_1\, c_2 \ldots c_k)$ and $G_n(c'_1\, c'_2 \ldots c'_k)$, the edge $(G_n(c_1\, c_2 \ldots c_k), G_n(c'_1\, c'_2 \ldots c'_k)) \in E_n/k$ if and only if there exists $(v_1, v_2) \in E(\Gamma_n)$ such that $v_1 \in G_n(c_1\, c_2 \ldots c_k)$ and $v_2 \in G_n(c'_1\, c'_2 \ldots c'_k)$.

In other words, at each iteration $k+1$, we will decompose each vertex $G_n(c_1\, c_2 \ldots c_k)$ into two new vertices, i.e. $G_n(c_1\, c_2 \ldots c_k\, 0)$ and $G_n(c_1\, c_2 \ldots c_k\, 1)$ when $c_k = 0$. Whenever $c_k = 1$, then $G_n(c_1\, c_2 \ldots c_{k-1}\, 1) = G_n(c_1\, c_2 \ldots c_{k-1}\, 1\, 0)$.

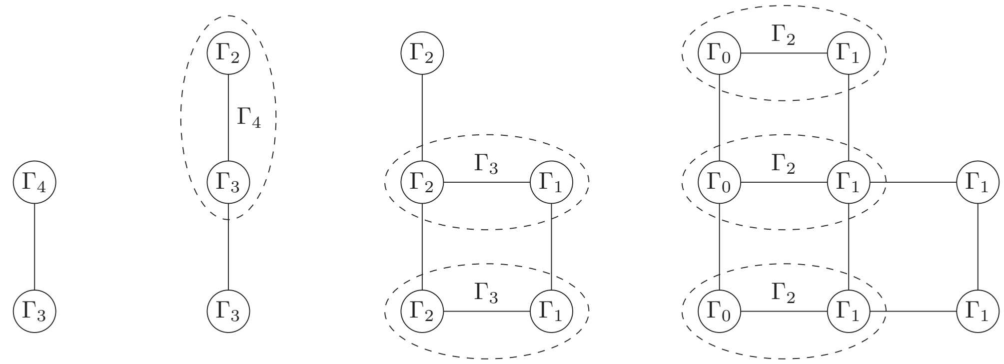
In table 2.3, we show the vertices of Γ_5 decomposed into the subsets $G_5(c_1 \ldots c_k)$ for every $(c_1 \ldots c_k) \in \mathcal{F}_k$; $1 \leq k < n$.

$k=1$	$k=2$	$k=3$	$k=4$	$V(\Gamma_5)$
$G_5(0)$	$G_5(00)$	$G_5(000)$	$G_5(0000)$	(00000), (00001)
			$G_5(0001)$	(00010)
		$G_5(001)$	$G_5(0010)$	(00100), (00101)
	$G_5(01)$	$G_5(010)$	$G_5(0100)$	(01000), (01001)
			$G_5(0101)$	(01010)
$G_5(1)$	$G_5(10)$	$G_5(100)$	$G_5(1000)$	(10000), (10001)
			$G_5(1001)$	(10010)
		$G_5(101)$	$G_5(1010)$	(10100), (10101)

Table 2.3: V_5/k

Figure 2.3 illustrates the Quotient Fibonacci cube Γ_5/k. Note that for $k = 1, 2, 3$ and 4, each Quotient cube Γ_n/k is isomorphic to Γ_k.

Theorem 2.3.4 ([HPL93]). *For $1 \leq k \leq n-1$, the Quotient Fibonacci cube Γ_n/k is isomorphic to Γ_k.*



$\Gamma_5/1 \simeq \Gamma_1$ $\Gamma_5/2 \simeq \Gamma_2$ $\Gamma_5/3 \simeq \Gamma_3$ $\Gamma_5/4 \simeq \Gamma_4$

Figure 2.3: Quotient Fibonacci cubes Γ_5/k.

Proof. By Corollary 2.3.2, $V(\Gamma_n) = 0V(\Gamma_{n-1}) \uplus 10V(\Gamma_{n-2})$ Consider all the vertices of Γ_{n-1} as a single node v_1 and the vertices of Γ_{n-2} as a single node v_2. Let the $|V(\Gamma_{n-2})|$ edges between Γ_{n-1} and Γ_{n-2} be merged into a single edge (v_1, v_2). The resulting graph is isomorphic to $\Gamma_n/1$. We assume then, that $\Gamma_n/K \simeq \Gamma_K$.
We will consider Γ_{K+1} with $2 \leq (K+1) \leq (n-1)$.
Again, by Corollary 2.3.2, $V(\Gamma_{K+1}) = 0V(\Gamma_K) \uplus 10V(\Gamma_{K-1})$. By hypothesis, $\Gamma_n/K \simeq \Gamma_K$ and $\Gamma_n/(K-1) \simeq \Gamma_{K-1}$. Therefore,

(i) $V_n/K = \{G_n(c_1\, c_2 \ldots c_K)$ for every $(c_1\, c_2 \ldots c_K) \in \mathcal{F}_K\}$,

(ii) for two different vertices of V_n/K, namely $G_n(c_1c_2 \ldots c_K)$ and $G_n(c'_1c'_2 \ldots c'_K)$, the edge $(G_n(c_1c_2 \ldots c_K), G_n(c'_1c'_2 \ldots c'_K)) \in E_n/K$ if and only if there exists $(v_1, v_2) \in E(\Gamma_n)$ such that $v_1 \in G_n(c_1c_2 \ldots c_K)$ and $v_2 \in G_n(c'_1c'_2 \ldots c'_K)$ and

(i') $V_n/(K-1) = \{G_n(c_1\, c_2 \ldots c_{K-1})$ for every $(c_1\, c_2 \ldots c_{K-1}) \in \mathcal{F}_{K-1}\}$,

(ii') for two different vertices of $V_n/(K-1)$, namely $G_n(c_1c_2 \ldots c_{K-1})$ and $G_n(c'_1c'_2 \ldots c'_{K-1})$, the edge $(G_n(c_1\, c_2 \ldots c_{K-1}), G_n(c'_1\, c'_2 \ldots c'_{K-1})) \in E_n/(K-1)$ if and only if there exists $(v_1, v_2) \in E(\Gamma_n)$ such that $v_1 \in G_n(c_1\, c_2 \ldots c_{K-1})$ and $v_2 \in G_n(c'_1\, c'_2 \ldots c'_{K-1})$.

Thus, $V_n/K \uplus V_n/(K-1)$ and the $|V(\Gamma_{K-1})|$ edges between Γ_K and Γ_{K-1} together with the edges defined by (ii) and (ii') give us $\Gamma_n/(K+1)$. □

2.4 Other structural results

Let G be a graph. Then a *median* of vertices u, v, w is a vertex that simultaneously lies on a shortest $(u, v)-$path, a shortest $(u, w)-$path and a shortest $(v, w)-$path. A connected graph is called a *median graph* if every triplet of its vertices has a unique median.

Theorem 2.4.1 ([Mul78]). *A graph G is a median graph if and only if G is a connected induced subgraph of an $n-$cube such that any three vertices of G, their median in the $n-$cube is also a vertex of G.*

A subgraph H is *median closed* if, with any triplet of vertices of H, their median is also in H. Klavžar proved in 2005 that the Fibonacci cube is a median closed subgraph of the $n-$cube:

Theorem 2.4.2 ([Kla05]). *For $n \geq 0$, Γ_n is a median graph.*

Proof. Let $u = (u_1, u_2, \ldots u_n)$, $v = (v_1, v_2, \ldots v_n)$ and $w = (w_1, w_2, \ldots w_n)$ be three arbitrary vertices of Γ_n embedded into Q_n. The median of a triplet in Q_n is obtained by the majority rule: the i^{th} coordinate is equal to the element that appears at least twice among u_i, v_i and w_i. Suppose that for some i the majorities of u_i, v_i, w_i and $u_{i+1}, v_{i+1}, w_{i+1}$ are both 1. Then there are two consecutive 1's in at least one of the vertices of u, v or w, say $u_i = u_{i+1}$ which is not possible. Therefore, the median of u, v and w does not contain two consecutive 1's and hence it is a vertex of Γ_n. Thus Γ_n is a median closed subgraph of Q_n and hence a median graph. □

Also, it is proved in [Kla05] the following proposition concerning the number of edges of the Fibonacci cube:

Proposition 2.4.3. *For any $n \geq 1$,*

$$|E(\Gamma_n)| = F_{n+1} + \sum_{i=1}^{n-2} F_i F_{n+1-i}.$$

Proof. The equality holds for $n = 1, 2$. Let $n \geq 3$ and assume that it holds for all indices smaller than n. Since $|E(\Gamma_n)| = |E(\Gamma_{n-1})| + |E(\Gamma_{n-2})| + |V(\Gamma_{n-2})|$

and $|V(\Gamma_{n-2})| = F_n$, then

$$\begin{aligned}
|E(\Gamma_n)| &= (F_n + \sum_{i=1}^{n-3} F_i F_{n-i}) + (F_{n-1} + \sum_{i=1}^{n-4} F_i F_{n-1-i}) + F_n \\
&= F_{n+1} + \sum_{i=1}^{n-4} F_i (F_{n-i} + F_{n-1-i}) + F_{n-3} F_3 + F_n \\
&= F_{n+1} + \sum_{i=1}^{n-4} F_i F_{n+1-i} + 2F_{n-3} + F_{n-1} + F_{n-2} \\
&= F_{n+1} + \sum_{i=1}^{n-4} F_i F_{n+1-i} + 3F_{n-3} + 2F_{n-2} \\
&= F_{n+1} + \sum_{i=1}^{n-4} F_i F_{n+1-i} + F_4 F_{n-3} + F_3 F_{n-2} \\
&= F_{n+1} + \sum_{i=1}^{n-2} F_i F_{n+1-i}.
\end{aligned}$$

□

In the other hand, Munarini, Cippo and Zagaglia Salvi found the next proposition, also concerning $|E(\Gamma_n)|$:

Proposition 2.4.4 ([MCS01]). *For any $n \geq 1$,*

$$|E(\Gamma_n)| = \frac{n\, F_{n+1} + 2(n+1)\, F_n}{5}.$$

Proof. The equality holds for $n = 1, 2$. Let $n \geq 3$ and assume that it holds for all indices smaller than n. Then

$$\begin{aligned}
|E(\Gamma_n)| &= |E(\Gamma_{n-1})| + |E(\Gamma_{n-2})| + |V(\Gamma_{n-2})| \\
&= \frac{(n-1)F_n + 2nF_{n-1}}{5} + \frac{(n-2)F_{n-1} + 2(n-1)F_{n-2}}{5} + F_n \\
&= \frac{(n+4)F_n + 2nF_{n-1} + (n-2)F_{n-1} + 2(n-1)F_{n-2}}{5} \\
&= \frac{(n+4)F_n + nF_{n-1} + (2n-2)F_{n-1} + (2n-2)F_{n-2}}{5} \\
&= \frac{(n+4)F_n + nF_{n-1} + (2n-2)(F_{n-1} + F_{n-2})}{5} \\
&= \frac{(n+4)F_n + nF_{n-1} + (2n-2)F_n}{5} \\
&= \frac{n(F_n + F_{n-1}) + (2n+2)F_n}{5} \\
&= \frac{n\, F_{n+1} + 2(n+1)\, F_n}{5}.
\end{aligned}$$

□

In more recent papers, other enumerative results have been obtained such as the following due to Klavžar, Mollard and Petkovšek concerning the number of vertices of a given degree in [KMP11].
Let $f_{n,k}$ denote the number of vertices of Γ_n having degree k.

Theorem 2.4.5 ([KMP11])**.** *For all $n \geq k \geq 0$,*

$$f_{n,k} = \sum_{i=0}^{k} \binom{n-2i}{k-i} \binom{i+1}{n-k-i+1}.$$

Let $f_{n,k,w}$ be the number of vertices in Γ_n having degree k and weight w, where the *weight* of a string u is the number of 1's in u.

Theorem 2.4.6 ([KMP11])**.** *For all integers k, n, w with $k, w \leq n$,*

$$f_{n,k,w} = \binom{w+1}{n-w-k+1} \binom{n-2w}{k-w}.$$

Introduced in [BKv03], the *cube polynomial*

$$C(G,x) = \sum_{n \geq 0} c_n(G) x^n$$

of a graph G is the counting polynomial for the number of induced subgraphs isomorphic to Q_n. In [KM12a], Klavžar and Mollard showed the following

Theorem 2.4.7 ([KM12a]). *For any $n \geq 0$, $C(\Gamma_n, x)$ is of degree $\lfloor \frac{n+1}{2} \rfloor$ and*

$$C(\Gamma_n, x) = \sum_{a=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-a+1}{a} (1+x)^a.$$

Corollary 2.4.8. *For any $n \geq 0$, the number of induced Q_k, $k \geq 0$ in Γ_n is*

$$c_k = \sum_{i=k}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-i+1}{i} \binom{i}{k}.$$

Mollard determined in [Mol11], the number of maximal induced hypercubes Q_k in Γ_n, that is,

Theorem 2.4.9 ([Mol11]). *For any $k \geq 1$, the number of maximal hypercubes of dimension k in Γ_n is equal to*

$$\binom{k+1}{n-2k+1}.$$

Fibonacci cubes were introduced as interconnection networks and were later studied from other points of view. In particular, some applications in chemistry have been associated to these cubes. The *Wiener index* of a graph is a very studied invariant in mathematical chemistry. See [Kla]. An equivalent approach is to study the average distance of a graph. Klavžar and Mollard obtained this index in terms of the Fibonacci numbers in [KM12b].

2.5 Related Graphs

We will present a variant and two generalizations of the Fibonacci cube in this section. The former, namely, the Lucas cube is obtained by removing all the strings that begin and end with 1 from the Fibonacci cube which lead us to a more symmetric graph. Therefore, the Fibonacci and the Lucas cubes are frequently studied together as we will be doing in some of the following chapters. The latter ones unify other interconnection topologies as the Hypercube Q_n and the Fibonacci cube Γ_n among others. For further variations of the Fibonacci cube proposed in literature, we refer to [Kla].

2.5.1 Lucas cubes

Introduced by E. Munarini, C. P. Cippo and N. Zagaglia in [MCS01], the Lucas cube is a subgraph of the n-cube that has also attractive properties as its recursive decompositions into two disjoint subgraphs which are isomorphic to Γ_{n-1} and Γ_{n-3}.

A Fibonacci string of length n is a *Lucas string* if $b_1 \cdot b_n \neq 1$. That is, a Lucas string has no two consecutive 1's including the first and the last element of the string. The *Lucas cube* Λ_n is the graph induced by the Lucas strings of length n where two strings are adjacent if they differ in one position. Considering the empty string, we have that $\Lambda_0 = K_1$. Furthermore, $\Lambda_1 = K_1$.

The n^{th} Lucas number, L_n is defined as $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$.
As for the Fibonacci cube, where $|V(\Gamma_n)|$ is F_{n+2}, the $(n+2)^{th}$ Fibonacci number; the number of vertices of the Lucas cube, $|V(\Lambda_n)|$ is L_n for $n \geq 1$.

Proposition 2.5.1. *The generating function of $|V(\Lambda_n)|$ is*

$$l(x) = \sum_{n\geq 0} |V(\Lambda_n)|\ x^n = \frac{1+x^2}{1-x-x^2}.$$

Proof.

$$\begin{aligned}
l(x) &= \sum_{n\geq 1} L_n\ x^n + 1 \\
&= 1 + x + 3x^2 + \sum_{n\geq 3} L_n\ x^n \\
&= 1 + x + 3x^2 + \sum_{n\geq 3} (L_{n-1} + L_{n-2})\ x^n \\
&= 1 + x + 3x^2 + x\left(\sum_{n\geq 3} L_{n-1}\ x^{n-1}\right) + x^2\left(\sum_{n\geq 3} L_{n-2}\ x^{n-2}\right) \\
&= 1 + x + 3x^2 + x\ (l(x) - x - 1) + x^2\ (l(x) - 1) \\
&= 1 + x^2 + x\ l(x) + x^2\ l(x)
\end{aligned}$$

Thus,

$$l(x)(1-x-x^2) = 1 + x^2.$$

□

In figure 2.4, we show the Lucas cubes Λ_3, Λ_4 and Λ_5 as the subgraphs of Q_n induced by the Lucas strings of length $n = 3, 4$ and 5.

2.5.2 Fibonacci (p,r)-cubes

For positive integers p, $r \leq n$, we define the *Fibonacci (p,r)-string* of length n as the binary string of length n in which there are at most r consecutive

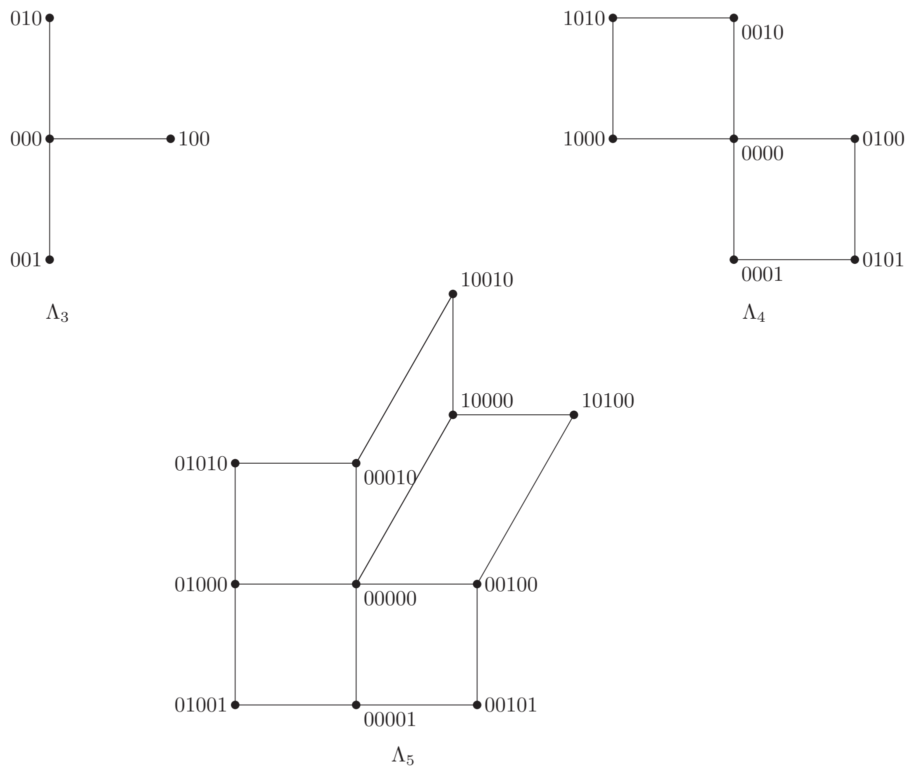


Figure 2.4: Λ_3, Λ_4 and Λ_5

1's, and at least p consecutive 0's between two substrings of (at most r) consecutive 1's. The *Fibonacci (p,r)-cube*, denoted $\Gamma_n^{(p,r)}$, is the graph induced by the Fibonacci (p,r)-strings of length n. Also, two adjacent strings differ in one position. This generalization was introduced by Karen Egiazarian and Jaakko Astola in [EA97] and contains the Hypercube and the Fibonacci cube as subgraphs since $\Gamma_n^{(1,n)} = Q_n$ and $\Gamma_n^{(1,1)} = \Gamma_n$.

2.5.3 Generalized Fibonacci cubes

Very recently, Aleksandar Ilič, Sandi Klavžar and Yoomi Rho, introduced in [IKR12] the Generalized Fibonacci cube $Q_n(f)$, as the graph obtained from removing all the vertices containing a given binary string f as a substring from the Hypercube. Notice that $Q_n(11) = \Gamma_n$. The authors study the question of which graphs of $Q_n(f)$ are embeddable into Q_n, proposing also several problems and conjectures around the Generalized Fibonacci cubes, opening them for further investigation.

Also using this term, J. Liu, W.-J. Hsu and M. J. Chung defined previously in [LHC94], the graphs $Q_n(1^s)$, where 1^s is the string of s consecutive 1's. Note that $Q_n(1^s)$ is included in the GFC defined by Ilič, Klavžar and Rho.

Chapter 3

Hamiltonicity

3.1 Hamiltonicity in the Fibonacci cubes

A graph G is called *pancyclic* if it contains a cycle of every length from 3 to $|V(G)|$. A restriction of the concept of pancyclicity was proposed for bipartite graphs whose cycles are necessarily of even length. Therefore, a bipartite graph G is called *bipancyclic* if it contains a cycle of every even lenght from 4 to $|V(G)|$.

Since Γ_n is bipartite, it can have a Hamiltonian cycle only if it has an even number of vertices. Thus $|V(\Gamma_n)| = F_{n+2}$ must be even and hence $n = 3k+1$ for some $k \geq 0$. Thus we have the next proposition which will be proven below:

Proposition 3.1.1. *For $n \geq 0$, $|V(\Gamma_n)|$ is even if and only if $n \equiv 1 (mod\, 3)$.*

In other words, Γ_n can have a Hamiltonian cycle if and only if $n \equiv 1 (mod\, 3)$.
We will show in Theorem 3.1.5 that when the order of Γ_n is even, then it is bipancyclic. Otherwise, the graph contains cycles of even lengths from 4 to $|V(\Gamma_n)| - 1$.

A *walk* in a graph G is a sequence of vertices $W = (w_1, w_2, \ldots, w_k)$ such that for $j = 1, 2, \ldots, k-1$, the vertices v_j and v_{j+1} are adjacent. We say that W is a *path* if it doesn't have repeated vertices.

For $n \geq 1$, we will denote a path whose strings have length n by $T_n = (t_1^n, t_2^n, \ldots, t_k^n)$ and by T_n^R the reversed sequence of T_n. Strictly speaking, $T_n^R = (t_1^n, t_2^n, \ldots, t_k^n)^R = (t_k^n, t_{k-1}^n \ldots, t_1^n)$.

Let $T_n^{'}, T_n^{''}$ be the concatenation of two paths.

Theorem 3.1.2 ([LHC94])**.** *For $n \geq 0$, Γ_n contains a Hamiltonian path.*

Proof. Consider $\mathcal{P}_n = (p_1^n, p_2^n, \ldots, p_{|V(\Gamma_n)|}^n)$, the sequence of strings of length n, defined by $\mathcal{P}_n = 0\,\mathcal{P}_{n-1}^R,\ 10\,\mathcal{P}_{n-2}^R$ for $n \geq 2$ with $\mathcal{P}_0 =$ the empty string and $\mathcal{P}_1 = (0, 1)$.
We will prove by induction that $\mathcal{P}_n$ is a Hamiltonian path of Γ_n and that $p_1^n = 010\,p_1^{n-3}$ for every $n \geq 3$.

Notice that for $n = 3$, the property is true since $\mathcal{P}_2 = (01, 00, 10)$ and $p_1^3 = 0\,p_2^3 = 010$. Thus $\mathcal{P}_3 = 0\,\mathcal{P}_2^R,\ 10\,\mathcal{P}_1^R = (010, 000, 001, 101, 100)$ is a Hamiltonian path in Γ_3.
Let us assume then that the statement is true for values up to $N \geq 3$.
Consider $\mathcal{P}_{N+1}$, which by Theorem 2.3.1, has exactly $|V(\Gamma_{N+1})|$ vertices. Notice also that $\mathcal{P}_{N+1}$ can be decomposed in the following way:

$$\begin{aligned}\mathcal{P}_{N+1} &= 0\,\mathcal{P}_N^R,\ 10\,\mathcal{P}_{N-1}^R\\ &= 0\,\mathcal{P}_N^R,\ 10\,(0\,\mathcal{P}_{N-2}^R,\ 10\,\mathcal{P}_{N-3}^R)^R)\\ &= 0\,\mathcal{P}_N^R,\ (1010\mathcal{P}_{N-3},\ 100\,\mathcal{P}_{N-2}).\end{aligned}$$

Since for every $N \geq 3$, $p_1^N = 010\,p_1^{N-3}$, then

$$\mathcal{P}_{N+1} = (0\,p^N_{|V(\Gamma_N)|}, \ldots, 0010\,p_1^{N-3}),\ (1010\,p_1^{N-3}, \ldots, 100\,p^{N-2}_{|V(\Gamma_{N-2})|})$$

is a Hamiltonian path of Γ_n and

$$p_1^{N+1} = 010\,(p^{N-2}_{|V(\Gamma_{N-2})|}) = 0\,p^N_{|V(\Gamma_N)|} = 010\,p_1^{N-2}$$

. □

Another proof for this theorem can be seen in [Vaj01], where the author constructs a Gray code for Γ_n.

Notice that from Corollary 2.3.2, when $n \geq 3$,

$$\begin{aligned}V(\Gamma_n) &= 0\,V(\Gamma_{n-1}) \uplus 10\,V(\Gamma_{n-2})\\ &= 00\,V(\Gamma_{n-2}) \uplus 010\,V(\Gamma_{n-3}) \uplus 10\,V(\Gamma_{n-2}).\end{aligned}$$

Thus Γ_n can be decomposed in two disjoint subraphs isomorphic to Γ_{n-2} and one subraph isomorphic to Γ_{n-3}.
Let us suppose that $|V(\Gamma_n)|$ is even, then $|V(\Gamma_{n-3})|$ must be even because $2|V(\Gamma_{n-2})|$ is even. Consequently, if $|V(\Gamma_n)|$ is odd, then $|V(\Gamma_{n-3})|$ is odd and hence we have the next proposition which will be proven below:

Proposition 3.1.3. *For every $n \geq 3$, $|V(\Gamma_n)|$ and $|V(\Gamma_{n-3})|$ have the same parity.*

Corollary 3.1.4. *For $n \geq 0$, $|V(\Gamma_n)|$ is even if and only if $n \equiv 1(mod\,3)$.*

Proof. $|V(\Gamma_1)|$ is even, $|V(\Gamma_0)|$ and $|V(\Gamma_2)|$ are odd. Thus by construction and using the previous proposition, for every $n \equiv 1(mod\,3)$, $|V(\Gamma_n)|$ is even.
□

Theorem 3.1.5 ([LHC94])**.** *For $n \geq 3$, and even l with $4 \leq l \leq |V(\Gamma_n)|$, a cycle of length l can be embedded in Γ_n.*

Proof. Let us define $\mathcal{S}_0$, as the subgraph induced by the nodes $0\,V(\Gamma_{n-1})$ and $\mathcal{S}_1$, the subgraph induced by the nodes $1\,(V(\Gamma_{n-1}) \setminus 10\,V(\Gamma_{n-3}))$.
$\mathcal{S}_0$ is isomorphic to Γ_{n-1} and $\mathcal{S}_1$ is isomorphic to a graph obtained by removing Γ_{n-3} from Γ_{n-1}, i.e. $\mathcal{S}_1$ is isomorphic to Γ_{n-2}.
By Theorem 3.1.2, $\mathcal{S}_0$ has a Hamiltonian path $\mathcal{P}_{n-1}$. Notice that by definition of $\mathcal{P}_{n-1}$, there exists the Hamiltonian path $0\,\mathcal{P}^R_{n-2} = \mathcal{P}_{n-1} \setminus 10\,V(\Gamma_{n-3})$ in $\mathcal{S}_1$.
Let L_0 be formed by $0\,\mathcal{P}^R_{n-1} = 0(0\,\mathcal{P}^R_{n-2},\, 10\,\mathcal{P}^R_{n-3})^R = 01(0\,\mathcal{P}_{n-3}),\, 00(\mathcal{P}_{n-2})$. Then using the fact that $0\,p^n_{|V(\Gamma_n)|} = p^{n+1}_1$, we have:
$L_0 = 01(0\,p^{n-3}_1, \ldots 0\,p^{n-3}_{|V(\Gamma_{n-3})|}),\, 00(0\,p^{n-3}_{|V(\Gamma_{n-3})|} = p^{n-2}_1, \ldots p^{n-2}_{|V(\Gamma_{n-2})|})$.
We can decompose L_0 in L_{01} and L_{00}. The former is a Hamiltonian path in $\Gamma_{n-1} \setminus 00\,V(\Gamma_{n-2})$ and the latter is a Hamiltonian path in $\Gamma_{n-1} \setminus 01\,V(\Gamma_{n-3})$.
In other words, L_{01} is a Hamiltonian path in Γ_{n-3} and L_{00} is a Hamiltonian path in Γ_{n-2}.
Let L_{10} be formed by $10\,\mathcal{P}^R_{n-2}$. Thus $L_{10} = 10(p^{n-2}_{|V(\Gamma_{n-2})|}, \ldots p^{n-2}_1)$.
Let $\mathcal{H}_n$, be the subraph of Γ_n formed by L_0 and L_{10}.
Notice that the path $L_0, L_{10} = 0\,\mathcal{P}^R_{n-1}\,10\,\mathcal{P}^R_{n-2}$ is a Hamiltonian path in Γ_n since L_0 and L_{10} can be linked together.
Furthermore, for every $1 \leq i \leq |V(\Gamma_{n-3})|$,

$$\begin{aligned} p^n_i =& 010\,p^{n-3}_i \text{ and} \\ p^n_{2|V(\Gamma_{n-3})|+1-i} =& 000\,p^{n-3}_i. \end{aligned}$$

Therefore, there exists the edge between every vertex $010\,p^{n-3}_i$ of L_{01} and $000\,p^{n-3}_1$ of L_{00} in $\mathcal{H}_n$.
In addition, for every $1 \leq i \leq |V(\Gamma_{n-2})|$,

$$\begin{aligned} p^n_{|V(\Gamma_{n-3})|+i} =& 00\,p^{n-2}_i \text{ and} \\ p^n_{|V(\Gamma_n)|+1-i} =& 10\,p^{n-2}_i. \end{aligned}$$

Thus, the edge between every vertex $00\,p^{n-2}_i$ of L_{00} and $10\,p^{n-2}_i$ of L_{10} exists in $\mathcal{H}_n$.
Figure 3.1 represents the subgraph $\mathcal{H}_n$ formed by L_{01}, L_{00} and L_{10} and the edges previously decribed between them.

We will next construct in $\mathcal{H}_n$ the cycles of even lenght that can be embedded in Γ_n.

Case 1: $4 \leq l \leq 2\,|V(\Gamma_{n-2})|$
The cycle of length l, $\mathcal{C}_{n,l}$ is constructed with the first $\frac{l}{2}$ strings of L_{00} and L^R_{10}.

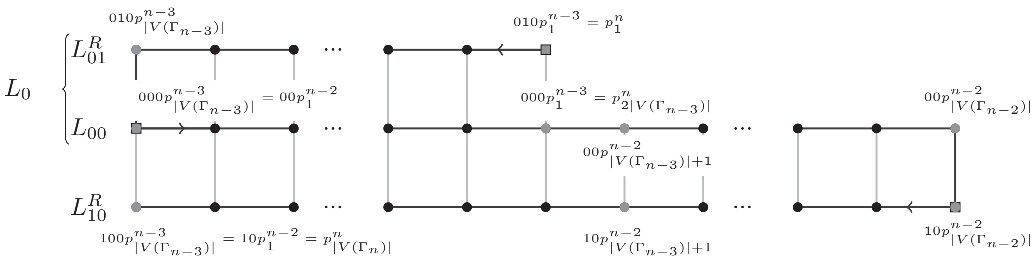


Figure 3.1: $\mathcal{H}_n$, subraph of Γ_n

Case 2: $2\,|V(\Gamma_{n-2})| < l \leq |V(\Gamma_n)|$

Taking back $\mathscr{C}_{n,l}$, the cycle of length $l = 2|V(\Gamma_{n-2})|$ constructed in Case 1, we will embedd the first pair of strings of L_{01}^R to obtain a new cycle of length $l+2$. By embedding the next pair of strings of L_{01}^R each time, we increase the length of the cycle by two (refer to figure 3.2) until having 1 string of L_{01}^R not been added if $|L_{01}| = |V(\Gamma_{n-3})|$ is odd, or 0 strings not been added if $|L_{01}|$ is even. Therefore, using corollary 3.1.4, the biggest cycle that we can embedd in $\mathcal{H}_n$, and hence in Γ_n is

$$\begin{cases} |V(\Gamma_n)| & \text{if } |V(\Gamma_n)| \text{ is even} \\ |V(\Gamma_n)| - 1 & \text{if } |V(\Gamma_n)| \text{ is odd.} \end{cases}$$

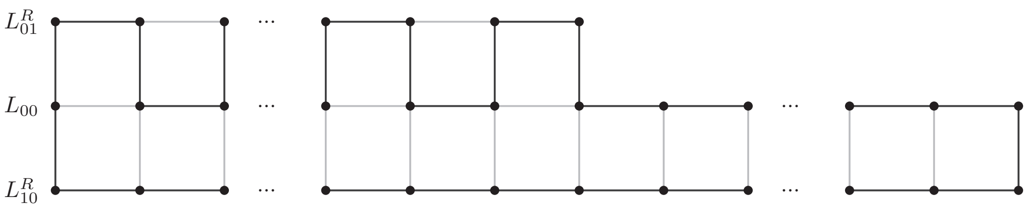


Figure 3.2: Cycle of length l with $2\,|V(\Gamma_{n-2})| < l \leq |V(\Gamma_n)|$ in Γ_n

Hence for $4 \leq l \leq |V(\Gamma_n)|$, a cycle of length l can be embedded in $\mathcal{H}_n$. Therefore, this cycle can also be embedded in Γ_n. □

From Theorem 3.1.5, we conclude the following

Theorem 3.1.6 ([LHC94]). *For $n \geq 3$, the Fibonacci cube Γ_n contains a Hamiltonian cycle if and only if $n \equiv 1(mod\,3)$; otherwise the longest cycle in Γ_n contains exactly $|V(\Gamma_n)| - 1$ vertices.*

An example of a cycle of length $|V(\Gamma_6)| - 1$ in Γ_6 is constructed next and depicted in figure 3.3.

Let $\mathcal{P}_5 = (01001, 01000, 01010, 00010, 00000, 00001, 00101, 00100, 10100, 10101, 10001, 10000, 10010)$ be a Hamiltonian path of Γ_5 constructed as in Theorem 3.1.2. In this path, we can recognize the Hamiltonian path of Γ_4, $0\,\mathcal{P}_4 = \mathcal{P}_5 \setminus 10\,V(\Gamma_3) = (01001, 01000, 01010, 00010, 00000, 00001, 00101, 00100)$.
Let $L_{01} = 01(0010, 0000, 0001, 0101, 0100)$,
$L_{00} = 00(0100, 0101, 0001, 0000, 0010, 1010, 1000, 1001)$ and
$L_{10} = 10(1001, 1000, 1010, 0010, 0000, 0001, 0101, 0100)$.
Thus,
$\mathscr{C}_{6,20} = (010100, 010101, 000101, 000001, 010001, 010000, 000000, 000010, 001010, 001000, 001001, 101001, 101000, 101010, 100010, 100000, 100001, 100101, 100100, 000100, 010100)$.

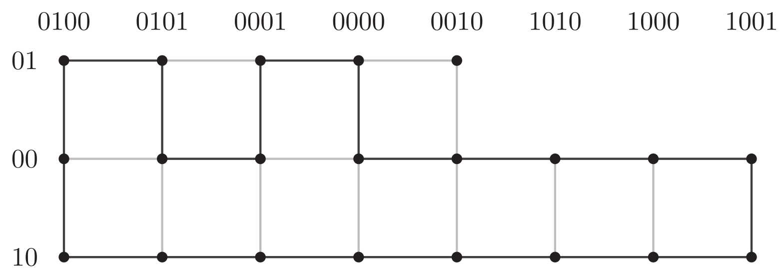


Figure 3.3: Cycle of length 20 in Γ_6

From Theorem 3.1.6, the longest cycle in Γ_n contains $|V(\Gamma_n)| - 1$ vertices when $|V(\Gamma_n)|$ is odd. Then, it rises the question of characterizing the vertices $v \in \Gamma_n$ for which the graph $\Gamma_n - v$ contains a Hamiltonian cycle. This problem is proposed by S. Klavžar in [Kla]. In that which follows, we give an answer to this question.

Consider the bipartition $V(\Gamma_n) = (V^{od}(\Gamma_n), V^{ev}(\Gamma_n))$ with
$V^{od}(\Gamma_n) = \{u \in V(\Gamma_n) \mid u \text{ has an odd number of 1's}\}$ and
$V^{ev}(\Gamma_n) = \{v \in V(\Gamma_n) \mid v \text{ has an even number of 1's}\}$.
Let a string $u \in V^{ev}(\Gamma_n)$. Then $00u$ belongs to $V^{ev}(\Gamma_{n+2})$, and $10u$ belongs to $V^{od}(\Gamma_{n+2})$. Accordingly, if $v \in V^{od}(\Gamma_n)$, therefore $00v$ belongs to $V^{od}(\Gamma_{n+2})$, and $10v$ belongs to $V^{ev}(\Gamma_{n+2})$.

Proposition 3.1.7. *For $n \geq 3$,*

$$|V^{ev}(\Gamma_n)| - |V^{od}(\Gamma_n)| = |V^{od}(\Gamma_{n-3})| - |V^{ev}(\Gamma_{n-3})|.$$

Proof. Let $(V^{ev}(\Gamma_n), V^{od}(\Gamma_n))$ be the bipartition of $V(\Gamma_n)$ with $n \geq 3$.
By Corollary 2.3.2, $V(\Gamma_n) = 010V(\Gamma_{n-3}) \cup 00V(\Gamma_{n-2}) \cup 10V(\Gamma_{n-2})$.
Thus,

$$\begin{aligned} V^{ev}(\Gamma_n) &= V^{ev}(010\,\Gamma_{n-3}) \cup V^{ev}(00\,\Gamma_{n-2} \cup 10\,\Gamma_{n-2}) \\ &= 010\,V^{od}(\Gamma_{n-3}) \cup V^{ev}(00\,\Gamma_{n-2} \cup 10\,\Gamma_{n-2}), \\ V^{od}(\Gamma_n) &= V^{od}(010\,\Gamma_{n-3}) \cup V^{od}(00\,\Gamma_{n-2} \cup 10\,\Gamma_{n-2}) \\ &= 010\,V^{ev}(\Gamma_{n-3}) \cup V^{od}(00\,\Gamma_{n-2} \cup 10\,\Gamma_{n-2}). \end{aligned}$$

Notice that

$$\begin{aligned} |V^{ev}(00\,\Gamma_{n-2} \cup 10\,\Gamma_{n-2})| &= |V^{ev}(\Gamma_{n-2})| + |V^{od}(\Gamma_{n-2})| \\ &= |V^{od}(10\,\Gamma_{n-2} \cup 00\,\Gamma_{n-2})|. \end{aligned}$$

Hence $|V^{ev}(\Gamma_n)| - |V^{od}(\Gamma_n)| = |V^{od}(\Gamma_{n-3})| - |V^{ev}(\Gamma_{n-3})|$. □

Proposition 3.1.8. *For $n \not\equiv 1\,(mod\,3)$, $n \geq 2$,*

$$|V^{ev}(\Gamma_n)| - |V^{od}(\Gamma_n)| = (-1)^{\lfloor \frac{n+2}{3} \rfloor}$$

Proof. If $n = 2$, then $|V^{ev}(\Gamma_2)| - |V^{od}(\Gamma_2)| = 1 - 2 = -1$.
If $n = 3$, then $|V^{ev}(\Gamma_3)| - |V^{od}(\Gamma_3)| = 2 - 3 = -1$.
Consider now Proposition 3.1.7 and let $n = 5$. Thus
$|V^{ev}(\Gamma_5)| - |V^{od}(\Gamma_5)| = |V^{od}(\Gamma_2)| - |V^{ev}(\Gamma_2)| = 2 - 1 = 1$.
Let us assume that the statement is true for $N \not\equiv 1\,(mod\,3)$, $N \geq 3$.
If $N + 1 \equiv 2\,(mod\,3)$, then $(N + 1) - 3 \equiv 2\,(mod\,3)$.
If $N + 1 \equiv 0\,(mod\,3)$, then $(N + 1) - 3 \equiv 0\,(mod\,3)$.
By hypothesis and Proposition 3.1.7,

$$(-1)^{\lfloor \frac{N}{3} \rfloor} = |V^{ev}(\Gamma_{(N+1)-3})| - |V^{od}(\Gamma_{(N+1)-3})| = |V^{od}(\Gamma_{N+1})| - |V^{ev}(\Gamma_{N+1})|$$

Notice that $(-1)^{\lfloor \frac{N}{3} \rfloor} = -1(-1)^{\lfloor \frac{(N+1)+2}{3} \rfloor}$ for $N + 1 \not\equiv 1\,(mod\,3)$.
Therefore,

$$|V^{ev}(\Gamma_{N+1})| - |V^{od}(\Gamma_{N+1})| = (-1)^{\lfloor \frac{(N+1)+2}{3} \rfloor}$$

□

As mentioned before, Γ_n has an odd number of vertices for $n \not\equiv 1\,(mod\,3)$. Thus one of the sets of $V^{ev}(\Gamma_n)$ or $V^{od}(\Gamma_n)$ has one more string than the other set. By Proposition 3.1.8, this set is $V^P(\Gamma_n)$ where

$$V^P(\Gamma_n) = \begin{cases} V^{ev}(\Gamma_n) & \text{if } \lfloor \frac{n+2}{3} \rfloor \text{ is even,} \\ V^{od}(\Gamma_n) & \text{if } \lfloor \frac{n+2}{3} \rfloor \text{ is odd.} \end{cases}$$

Proposition 3.1.9. *For $n \not\equiv 1\,(mod\,3)$, $n \geq 1$, the vertices $v \in V^P(\Gamma_n)$ are the vertices represented with the color gray in figure 3.4.*

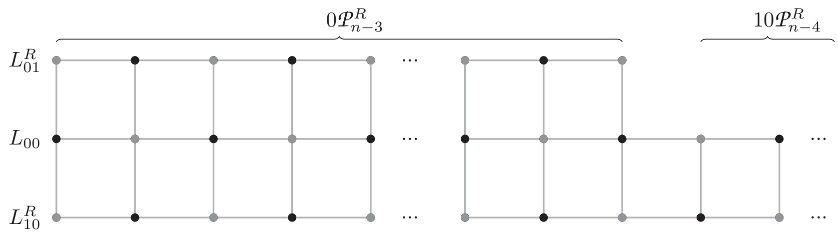


Figure 3.4: Vertices in $V^P(\Gamma_n)$

Proof. Recall that $|L_{01}| = |V(\Gamma_{n-3})|$, $|L_{00}| = |V(\Gamma_{n-2})|$ and $|L_{10}| = |V(\Gamma_{n-2})|$. Note that there are the same number of vertices of $V^P(\Gamma_n)$ in L_{00} than in L^R_{10}. Since we are considering the case where $|V(\Gamma_n)|$ is odd, then $|V(L^R_{01})|$ must be odd, hence $\{010p_j^{n-3}\,|\,j \textit{ is odd},\ 1 \leq j \leq |V(\Gamma_{n-3})|\}$ belong to $V^P(\Gamma_n)$.
Consequently, $\{00p_j^{n-2}\,|\,j \textit{ is even},\ 1 \leq j \leq |V(\Gamma_{n-2})|\}$ and
$\{10p_j^{n-2}\,|\,j \textit{ is odd},\ 1 \leq j \leq |V(\Gamma_{n-2})|\}$ belong to $V^P(\Gamma_n)$ as well.
Notice that when $|V(\Gamma_{n-2})|$ is even, then $00p^{n-2}_{|V(\Gamma_{n-2})|} \in V^P(\Gamma_n)$ and
$10p^{n-2}_{2|V(\Gamma_{n-2})|} \in V^P(\Gamma_n)$ when $|V(\Gamma_{n-2})|$ is odd. □

Theorem 3.1.10. *For $n \not\equiv 1\,(mod\,3)$, $n \geq 5$; let $v \in V^P(\Gamma_n)$. Then $\Gamma_n - v$ contains a Hamiltonian cycle. Moreover, $\Gamma_3 - (010)$ contains a Hamiltonian cycle.*
Furthermore, if $v \notin V^P(\Gamma_n)$, then $\Gamma_n - v$ does not contain a Hamiltonian cycle.

Proof. By definition, $V^{ev}(\Gamma_n)$ and $V^{od}(\Gamma_n)$ are independent sets. Thus, in order to have a Hamiltonian cycle, $|V^{ev}(\Gamma_n)|$ must be equal to $|V^{od}(\Gamma_n)|$. Hence, if a vertex $v \notin V^P(\Gamma_n)$, then $\Gamma_n - v$ does not contain a Hamiltonian cycle.
Consider the paths L_{01}, L_{00} and L_{10} described in Theorem 3.1.5. Then let
L^R_{01} be formed by $010\,\mathcal{P}^R_{n-3}$,
L_{00} be formed by $00\,\mathcal{P}_{n-2} = 000\,\mathcal{P}^R_{n-3}, 0010\,\mathcal{P}^R_{n-4}$ and
L^R_{10} formed by $10\,\mathcal{P}_{n-2} = 100\,\mathcal{P}^R_{n-3}, 1010\,\mathcal{P}^R_{n-4}$,
where $\mathcal{P}_{n-2}$, $\mathcal{P}_{n-3}$ and $\mathcal{P}_{n-4}$ are the Hamiltonian paths in Γ_{n-2}, Γ_{n-3} and Γ_{n-4}, constructed in Theorem 3.1.2, where $\mathcal{P}_n = 0\,\mathcal{P}^R_{n-1},\ 10\,\mathcal{P}^R_{n-2}$.
It isn't difficult to see that $\Gamma_3 - (010)$ contains a Hamiltonian cycle.
Let $v \in V^P(\Gamma_n)$ for $n \geq 5$. We will distinguish two cases. The first case considers the vertices of $V^P(\Gamma_n)$ in $\{\,010\,\mathcal{P}^R_{n-3} \cup 000\,\mathcal{P}^R_{n-3} \cup 100\,\mathcal{P}^R_{n-3}\} - \{$ last string of $100\,\mathcal{P}^R_{n-3}\}$ where the constructed Hamiltonian cycle has the

same pattern. The second case considers the vertices of $V^P(\Gamma_n)$ in $\{0010\,\mathcal{P}^R_{n-4} \cup 1010\,\mathcal{P}^R_{n-4}\}$. The last string of $\{100\,\mathcal{P}^R_{n-3}\}$, namely $10p^{n-2}_{|V(\Gamma_{n-3})|} = 100p^{n-3}_1$, is included as a variant of this case.

Case 1:

(a) $v \in L^R_{01}$

Thus $v = 010p^{n-3}_{2i+1}$ for some $0 \le i < m$. See Proposition 3.1.9. Consider the path $\mathcal{T}_n$, from $000p^{n-3}_1$ to $000p^{n-3}_{|V(\Gamma_{n-3})|}$ defined by the concatenation of

$$(000p^{n-3}_1, 010p^{n-3}_1, 010p^{n-3}_2, 000p^{n-3}_2, \ldots 000p^{n-3}_{2k-1}, 010p^{n-3}_{2k-1}, 010p^{n-3}_{2k},$$

$$000p^{n-3}_{2k}, \ldots 000p^{n-3}_{2i-1}, 010p^{n-3}_{2i-1}, 010p^{n-3}_{2i}, 000p^{n-3}_{2i}, 000p^{n-3}_{2i+1})$$

and

$$(000p^{n-3}_{2i+2}, 010p^{n-3}_{2i+2}, 010p^{n-3}_{2i+3}, \ldots 000p^{n-3}_{2k'}, 010p^{n-3}_{2k'}, 010p^{n-3}_{2k'+1},$$

$$000p^{n-3}_{2k'+1}, \ldots 000p^{n-3}_{2m}, 010p^{n-3}_{2m}, 010p^{n-3}_{2m+1}, 000p^{n-3}_{2m+1}).$$

for $1 \le k \le i$ and $i+1 \le k' \le m$.

See figure 3.5.

$\mathcal{T}_n$ can be rewritten as the concatenation of

$$(000p^{n-3}_{2k+1}, 010p^{n-3}_{2k+1}, 010p^{n-3}_{2k+2}, 000p^{n-3}_{2k+2}) \text{ for every } k = 0, \ldots i-1$$

with

$$(000p^{n-3}_{2i+1})$$

and

$$(000p^{n-3}_{2k'}, 010p^{n-3}_{2k'}, 010p^{n-3}_{2k'+1}, 000p^{n-3}_{2k'+1}) \text{ for every } k' = i+1, \ldots m.$$

Note that the length of $\mathcal{T}_n$ is $4i + 4(m-i) = 4m$.

Finally, we will concatenate $\mathcal{T}_n$ with L^R_{10} followed by $0010\,\mathcal{P}_{n-4}$.

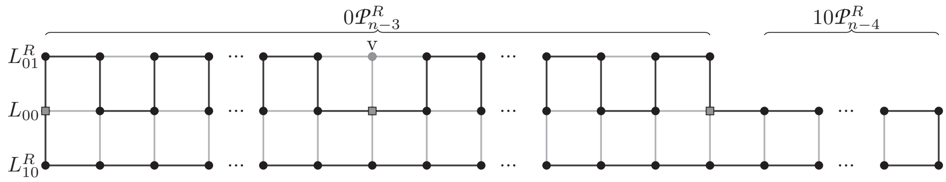


Figure 3.5: H. cycle in $\Gamma_n - v$: Case 1 (a)

(b) $v \in L_{00} \setminus 0010\,\mathcal{P}^R_{n-4}$
Thus $v = 000p^{n-3}_{2i}$ for some $1 \leq i \leq m$. See Proposition 3.1.9. We will describe a similar cycle as in the preceding case. Consider the path $\mathcal{T}_n$ from $000p^{n-3}_1$ to $000p^{n-3}_{|V(\Gamma_{n-3})|}$ formed by the concatenation of

$$(000p^{n-3}_1, 010p^{n-3}_1, 010p^{n-3}_2, 000p^{n-3}_2, \ldots 000p^{n-3}_{2k-1}, 010p^{n-3}_{2k-1}, 010p^{n-3}_{2k},$$
$$000p^{n-3}_{2k} \ldots 000p^{n-3}_{2i-1}, 010p^{n-3}_{2i-1}, 010p^{n-3}_{2i})$$

and

$$(010p^{n-3}_{2i+1}, 000p^{n-3}_{2i+1}, 000p^{n-3}_{2i+2}, 010p^{n-3}_{2i+2}, 010p^{n-3}_{2i+3}, 000p^{n-3}_{2i+3} \ldots 000p^{n-3}_{2k'},$$
$$010p^{n-3}_{2k'}, 010p^{n-3}_{2k'+1}, 000p^{n-3}_{2k'+1}, \ldots 000p^{n-3}_{2m}, 010p^{n-3}_{2m}, 010p^{n-3}_{2m+1}, 000p^{n-3}_{2m+1})$$

for $1 \leq k < i$ and $i + 1 \leq k' \leq m$.
See Figure 3.6.
$\mathcal{T}_n$ can be rewritten as the concatenation of:

$$(000p^{n-3}_{2k+1}, 010p^{n-3}_{2k+1}, 010p^{n-3}_{2k+2}, 000p^{n-3}_{2k+2}), \text{ for every } k = 0, \ldots i - 2$$

with

$$(000p^{n-3}_{2i-1}, 010p^{n-3}_{2i-1}, 010p^{n-3}_{2i}, 010p^{n-3}_{2i+1}, 000p^{n-3}_{2i+1})$$

and

$$(000p^{n-3}_{2k'}, 010p^{n-3}_{2k'}, 010p^{n-3}_{2k'+1}, 000p^{n-3}_{2k'+1}) \text{ for every } k' = i + 1, \ldots m.$$

Note that the length of $\mathcal{T}_n$ is $4i + 4 + 4(m - i - 1) = 4m$.
To complete the cycle, we will concatenate $\mathcal{T}_n$ with L^R_{10} followed by $0010\,\mathcal{P}_{n-4}$.

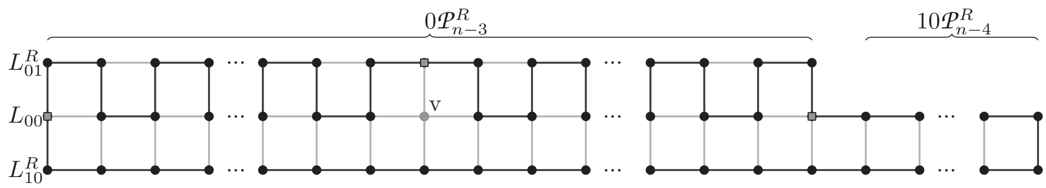


Figure 3.6: H. cycle in $\Gamma_n - v$: Case 1 (b)

(c) $v \in L^R_{10} \setminus \{1010\,\mathcal{P}^R_{n-4},\ 10p^{n-2}_{|V(\Gamma_{n-3})|} = 100p^{n-3}_1\}$
Then we have $v = 100p^{n-3}_{2i+1}$ for some $1 \leq i < m$. See Proposition 3.1.9. Consider $\mathcal{T}_n$, a path from $100p^{n-3}_2$ to $000p^{n-3}_{|V(\Gamma_{n-3})|}$ defined by the concatenation of

$$(100p^{n-3}_2, 000p^{n-3}_2, 000p^{n-3}_3, 100p^{n-3}_3, \ldots 100p^{n-3}_{2k}, 000p^{n-3}_{2k}, 000p^{n-3}_{2k+1},$$

$$100p^{n-3}_{2k+1}, \dots 100p^{n-3}_{2i}, 000p^{n-3}_{2i}, 000p^{n-3}_{2i+1})$$

and

$$(000p^{n-3}_{2i+2}, 100p^{n-3}_{2i+2}, 100p^{n-3}_{2i+3}, \dots 000p^{n-3}_{2k'}, 100p^{n-3}_{2k'}, 100p^{n-3}_{2k'+1},$$

$$000p^{n-3}_{2k'+1}, \dots 000p^{n-3}_{2m}, 100p^{n-3}_{2m}, 100p^{n-3}_{2m+1}, 000p^{n-3}_{2m+1}).$$

for $1 \leq k' < i$ and $i+1 \leq k' \leq m$.
See figure 3.7. Again, we can restate $\mathcal{T}_n$ as the concatenation of

$$(100p^{n-3}_{2k}, 000p^{n-3}_{2k}, 000p^{n-3}_{2k+1}, 100p^{n-3}_{2k+1}) \text{ for every } k = 1, \dots i-1$$

with

$$(100p^{n-3}_{2i}, 000p^{n-3}_{2i}, 000p^{n-3}_{2i+1})$$

and

$$(000p^{n-3}_{2k'}, 100p^{n-3}_{2k'}, 100p^{n-3}_{2k'+1}, 000p^{n-3}_{2k'+1}) \text{ for every } k' = i+1, \dots m.$$

$\mathcal{T}_n$ has length $4(i-1) + 2 + 4(m-i) = 4m - 2$.
Concatenate $\mathcal{T}_n$ with L^R_{01} followed by the vertex $(000p^{n-3}_1)$. Concatenate next $0010\,\mathcal{P}^R_{n-4}, 1010\,\mathcal{P}_{n-4}$ and the vertex $(100p^{n-3}_1)$.

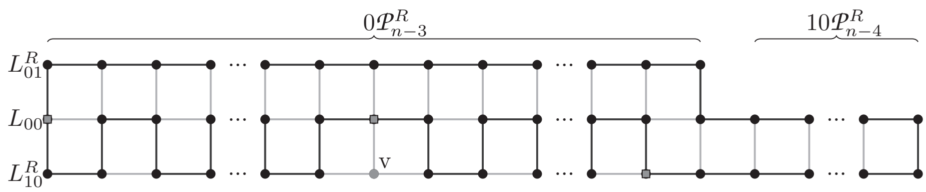


Figure 3.7: H. cycle in $\Gamma_n - v$: Case 1 (c)

Case 2:

(a) $v \in \{0010\,\mathcal{P}^R_{n-4} \cup 1010\,\mathcal{P}^R_{n-4}\}$
Let $v \in \{0010\,\mathcal{P}^R_{n-4}\}$. Thus $v = 00p^{n-2}_{2i}$, for some $2m+1 < 2i \leq |V(\Gamma_{n-2})|$. See Proposition 3.1.9. The case $2i = |V(\Gamma_{n-2})|$ will be considered at the end of this item. Assume then, $2m+1 < 2i < |V(\Gamma_{n-2})|$.
Recall that $\mathcal{P}_{n-2} = 0\mathcal{P}^R_{n-3}, 10\,\mathcal{P}^R_{n-4} = 010\mathcal{P}_{n-5}, 00\mathcal{P}_{n-4}, 10\mathcal{P}^R_{n-4}$.
Therefore any vertex of $10\mathcal{P}^R_{n-4}$ is adjacent to some vertex of $00\mathcal{P}_{n-4}$.
Let $u = 00p^{n-2}_{2i+1} \in \{10\,\mathcal{P}^R_{n-4}\}$. Thus $2i+1 = |V(\Gamma_{n-3})| + \delta$ with $\delta > 0$ and the element in the position $|V(\Gamma_{n-3})| - \delta + 1$ of L_{00} is adjacent to u. Therefore $u' = 00p^{n-2}_{2|V(\Gamma_{n-3})|-2i}$ is adjacent to u.

With the help of figure 3.8, we will construct a Hamiltonian cycle in $\Gamma_n - v$ as described next.
Consider $\mathcal{T}_n$, a path from $u' = 00p^{n-2}_{2|V(\Gamma_{n-3})|-2i} (= 00p^{n-2}_{2(2m+1)-2i})$ to $00p^{n-2}_1 (= 000p^{n-3}_{2m+1})$ defined by the concatenation of

$$(00p^{n-2}_{2(2m+1)-2i}, 10p^{n-2}_{2(2m+1)-2i}, 10p^{n-2}_{2(2m+1)-2i-1}, 00p^{n-2}_{2(2m+1)-2i-1)},$$

$$\ldots 00p^{n-2}_{2k}, 10p^{n-2}_{2k}, 10p^{n-2}_{2k-1}, 00p^{n-2}_{2k-1}, \ldots 00p^{n-2}_{2}, 10p^{n-2}_{2}, 10p^{n-2}_{1}, 00p^{n-2}_{1})$$

for $(2m+1) - i \leq k \leq 1$.

We can restate $\mathcal{T}_n$ as the concatenation of

$$(00p^{n-2}_{2k}, 10p^{n-2}_{2k}, 10p^{n-2}_{2k-1}, 00p^{n-3}_{2k-1})$$

for every $k = (2m+1) - i, \ldots 1$.
$\mathcal{T}_n$ has length $2[2(2m+1) - 2i] - 1 = 2(4m + 2 - 2i) - 1$.

Consider also $\mathcal{T}'_n$, a path from $10p^{n-2}_{|V(\Gamma_{n-3})|+1}$ to $10p^{n-2}_{2i}$ defined by

$$(10p^{n-2}_{2m+2}, 00p^{n-2}_{2m+2}, 00p^{n-2}_{2m+3}, 10p^{n-2}_{2m+3}, \ldots 10p^{n-2}_{2k}, 00p^{n-2}_{2k},$$

$$00p^{n-2}_{2k+1}, 10p^{n-2}_{2k+1}, \ldots 10p^{n-2}_{2i-2}, 00p^{n-2}_{2i-2}, 00p^{n-2}_{2i-1}, 10p^{n-2}_{2i-1}, 10p^{n-2}_{2i})$$

for $m + 1 \leq k \leq i - 1$.
We can also rewrite $\mathcal{T}'_n$ as the concatenation of

$$(10p^{n-2}_{2k}, 00p^{n-2}_{2k}, 00p^{n-2}_{2k+1}, 10p^{n-2}_{2k+1}) \text{ for every } k = m+1, \ldots i-1$$

and

$$(10p^{n-2}_{2i}).$$

Thus $\mathcal{T}'_n$ has length $2(2i - 2m - 2)$.

We will form the cycle concatenating:
$\mathcal{T}_n$ with L^R_{01} followed by $(00p^{n-2}_{2m+1}, 00p^{n-2}_{2m}, \ldots 00p^{n-2}_{2(2m+1)-2i+1})$,and by $(10p^{n-2}_{2(2m+1)-2i+1}, 10p^{n-2}_{2(2m+1)-2i+2}, \ldots 10p^{n-2}_{2m+1})$. Then $\mathcal{T}'_n$ and $(10p^{n-2}_{2i+1}, 10p^{n-2}_{2i+2}, \ldots 10p^{n-2}_{|V(\Gamma_{n-2})|}, 00p^{n-2}_{|V(\Gamma_{n-2})|}, 00p^{n-2}_{|V(\Gamma_{n-2})|-1}, \ldots$ $10p^{n-2}_{2i+1})$.

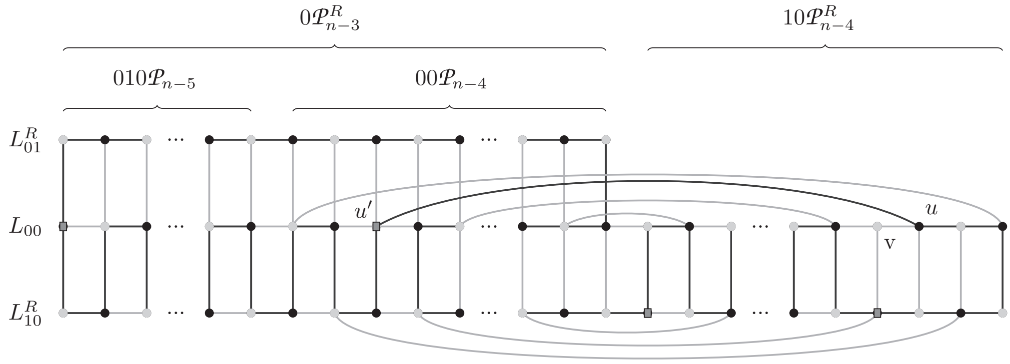


Figure 3.8: H. cycle in $\Gamma_n - v : v \in \{0010\, \mathcal{P}^R_{n-4}\}$

The cycle is similar if $v \in \{1010\, \mathcal{P}^R_{n-4}\}$. Thus $v = 10p^{n-2}_{2i+1}$, for some $2m+1 < 2i+1 \leq |V(\Gamma_{n-2})|$. See Proposition 3.1.9. Assume that $2m+1 < 2i+1 < |V(\Gamma_{n-2})|$ since we will consider the case $2i+1 = |V(\Gamma_{n-2})|$ below.
Let $u = 10p^{n-2}_{2i+2} \in \{10\, \mathcal{P}^R_{n-4}\}$. Thus $u' = 10p^{n-2}_{2|V(\Gamma_{n-3})|-(2i+1)}$ is adjacent to u.
See figure 3.9 In this case, let $\mathcal{T}_n$ be the path from $u' = 10p^{n-2}_{2|V(\Gamma_{n-3})|-(2i+1)}$ to $00p^{n-2}_1$ defined by

$$(10p^{n-2}_{2(2m+1)-(2i+1)}, 00p^{n-2}_{2(2m+1)-(2i+1)}, 00p^{n-2}_{2(2m+1)-(2i+2)}, 10p^{n-2}_{2(2m+1)-(2i+2)},$$

$$10p^{n-2}_{2(2m+1)-(2i+3)}, 00p^{n-2}_{2(2m+1)-(2i+3)}, \ldots 00p^{n-2}_{2k}, 10p^{n-2}_{2k}, 10p^{n-2}_{2k-1}, 00p^{n-2}_{2k-1},$$
$$\ldots 00p^{n-2}_2, 10p^{n-2}_2, 10p^{n-2}_1, 00p^{n-2}_1)$$

for $(2m+1)-(i+1) \leq k \leq 1$.
We can restate $\mathcal{T}_n$ as the concatenation of

$$(10p^{n-2}_{2(2m+1)-(2i+1)}, 00p^{n-2}_{2(2m+1)-(2i+1)})$$

and

$$(00p^{n-2}_{2k}, 10p^{n-2}_{2k}, 10p^{n-2}_{2k-1}, 00p^{n-3}_{2k-1})$$

for every $k = (2m+1)-(i+1), \ldots 1$.

Notice that $\mathcal{T}_n$ has length $2[2(2m+1)-2i]-1+2 = 2(4m+2-2i)+1$.

Let also $\mathcal{T}'_n$ be the path from $10p^{n-2}_{|V(\Gamma_{n-3})|+1}$ to $00p^{n-2}_{2i+1}$ defined by

$$(10p^{n-2}_{2m+2}, 00p^{n-2}_{2m+2}, 00p^{n-2}_{2m+3}, 10p^{n-2}_{2m+3}, \ldots 10p^{n-2}_{2k}, 00p^{n-2}_{2k},$$

$$00p^{n-2}_{2k+1}, 10p^{n-2}_{2k+1}, \ldots 10p^{n-2}_{2i-2}, 00p^{n-2}_{2i-2}, 00p^{n-2}_{2i-1}, 10p^{n-2}_{2i-1}, 10p^{n-2}_{2i}, 00p^{n-2}_{2i}, 00p^{n-2}_{2i+1})$$

for $m+1 \leq k \leq i-1$.
We can also rewrite $\mathcal{T}_n^{'}$ as the concatenation of

$$(10p^{n-2}_{2k}, 00p^{n-2}_{2k}, 00p^{n-2}_{2k+1}, 10p^{n-2}_{2k+1}) \text{ for every } k = m+1, \ldots i-1$$

and

$$(10p^{n-2}_{2i}, 00p^{n-2}_{2i}, 00p^{n-2}_{2i+1}).$$

Thus $\mathcal{T}_n^{'}$ has length $2(2i-2m-1)$.

The Hamiltonian cycle in $\Gamma_n - v$ is obtained by concatenating $\mathcal{T}_n$ with L^R_{01} followed by $(00p^{n-2}_{2m+1}, 00p^{n-2}_{2m}, \ldots 00p^{n-2}_{2(2m+1)-2i+2}, 10p^{n-2}_{2(2m+1)-2i+2}, \ldots 10p^{n-2}_{2m+1})$. Then $\mathcal{T}_n^{'}$ and $(00p^{n-2}_{2i+2}, \ldots 00p^{n-2}_{|V(\Gamma_{n-2})|}, 10p^{n-2}_{|V(\Gamma_{n-2})|}, 10p^{n-2}_{|V(\Gamma_{n-2})|-1}, \ldots 10p^{n-2}_{2i+2})$.

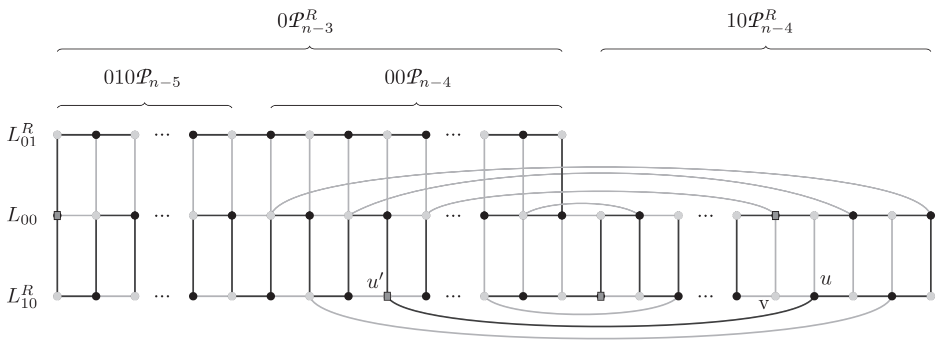


Figure 3.9: H. cycle in $\Gamma_n - v : v \in \{1010\,\mathcal{P}^R_{n-4}\}$

We have mentioned before that when $|V(\Gamma_{n-2})|$ is even, then $00p^{n-2}_{|V(\Gamma_{n-2})|}$ belongs to $V^P(\Gamma_n)$. In this case, let $u = 10p^{n-2}_{|V(\Gamma_{n-2})|}$. The Hamiltonian cycle of $\Gamma_n - 00p^{n-2}_{|V(\Gamma_{n-2})|}$ can be constructed as above. See figure 3.10.

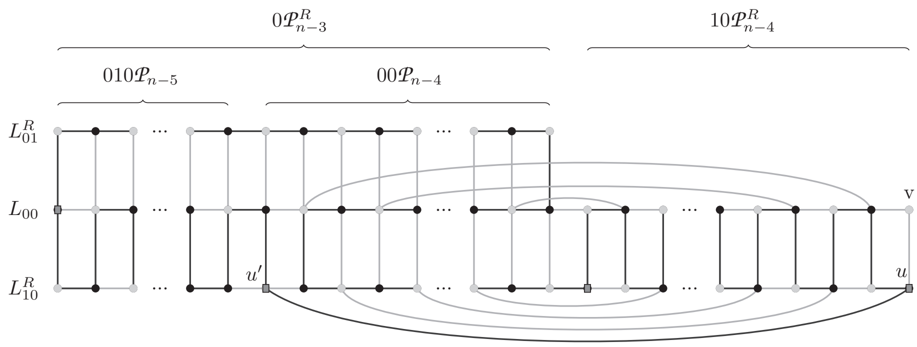


Figure 3.10: H. cycle in $\Gamma_n - 00p^{n-2}_{|V(\Gamma_{n-2})|}$

Whenever $|V(\Gamma_{n-2})|$ is odd, then $10p^{n-2}_{|V(\Gamma_{n-2})|}$ belongs to $V^P(\Gamma_n)$. In this case, let $u = 00p^{n-2}_{|V(\Gamma_{n-2})|}$ The Hamiltonian cycle in $\Gamma_n - 10p^{n-2}_{|V(\Gamma_{n-2})|}$ is constructed as previously done. Figure 3.11.

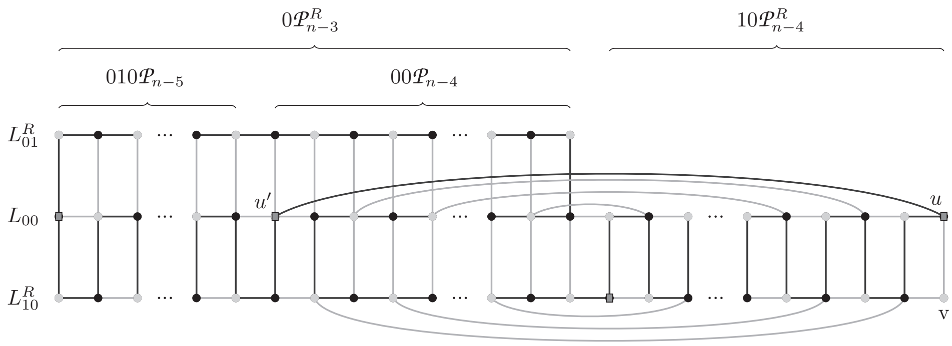


Figure 3.11: H. cycle in $\Gamma_n - 10p^{n-2}_{|V(\Gamma_{n-2})|}$

(b) $v = 10p^{n-2}_{|V(\Gamma_{n-3})|} = 100p^{n-3}_1$
In this case, let $u = 00p^{n-2}_{|V(\Gamma_{n-3})|+2}$ and proceed as in case 2(a). See figure 3.12.

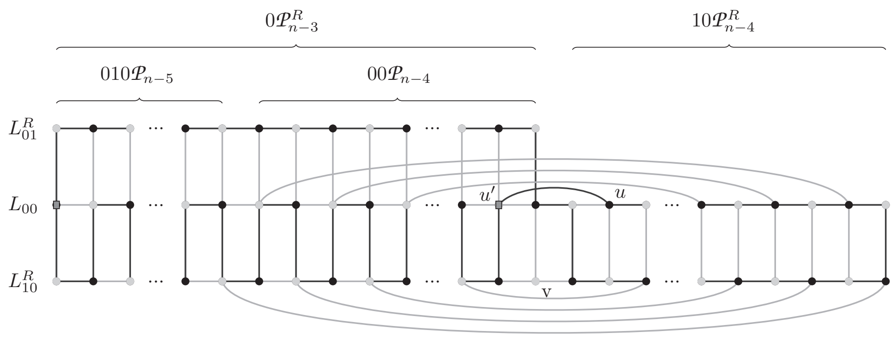


Figure 3.12: H. cycle in $\Gamma_n - 10p^{n-2}_{|V(\Gamma_{n-3})|}$

□

Corollary 3.1.11. *$\Gamma_n - v$ is bipancyclic with $v \in V^P(\Gamma_n)$; $n \not\equiv 1\,(mod\,3)$ and $n \geq 5$. Furthermore, $\Gamma_3 - (010)$ is bipancyclic as well.*

Proof. The only cycle in $\Gamma_3-(010)$ has length 4. Thus $\Gamma_3-(010)$ is bipancyclic. Let $v \in V^P(\Gamma_n)$; $n \not\equiv 1\,(mod\,3)$, $n \geq 5$. We will commence with $\mathscr{C}_{|V(\Gamma_n - v)|}$, the cycle constructed in Theorem 3.1.10 which contains $|V(\Gamma_n)| - 1$ vertices. Then we will remove recurrently a pair of vertices to obtain a smaller cycle each time. In order to do this, we will separate the vertices of $V^P(\Gamma_n)$ in four cases:

1. $v \in L_{01}^R \cup L_{10}^R \setminus \{00p_{|V(\Gamma_{n-3})|}^{n-2}, 1010\,\mathcal{P}_{n-4}^R\}$
From figure 3.5 (for $v \in L_{01}^R$) , we can see that by removing recurrently a pair of vertices from $L_{01}^R \in \mathscr{C}_{|V(\Gamma_n - v)|}$ the size of the cycle decreases by two each time until having a cycle of length $2|V(\Gamma_{n-2}|$. By deleting from the cycle the appropriate vertex $00p_k^{n-2}$ of L_{00} with $1 \le k < |V(\Gamma_{n-3})| - 2$ and its correspondent vertex $10p_k^{n-2}$ of L_{10}^R iteratively, we obtain the remaining cycles.
When $v \in L_{10}^R \setminus \{1010\,\mathcal{P}_{n-4}^R, 10p_{|V(\Gamma_{n-3})|}^{n-2}\}$, (see figure 3.7) the cycles are formed by deleting the appropriate vertex $00p_k^{n-2}$ of $0010P_{n-4}^R$ with $|V(\Gamma_{n-2})| \le k < |V(\Gamma_{n-3})| + 1$ and its correspondent vertex $10p_k^{n-2}$ of $1010P_{n-4}^R$ iteratively until having a cycle of length $3|V(\Gamma_{n-3}|$. Then we can remove recurrently a pair of vertices in the cycle from L_{10}^R until obtaining a cycle of length $2|V(\Gamma_{n-3}|$. Removing the suitable vertices $01p_k^{n-2}$ and $00p_k^{n-2}$ with $|V(\Gamma_{n-3}| \le k < 2$, we get the smaller cycles.
2. $v \in L_{00} \setminus \{0010\,\mathcal{P}_{n-4}^R\}$
For $v = 00p_j^{n-2},\ 2 \le j < |V(\Gamma_{n-3}|$, see figure 3.6. Then, as in case 1, we can remove the appropriate pair of vertices each time until having the cycle that surrounds $00p_j^{n-2}$, that is,
$\mathscr{C}_8 = (00p_{j-1}^{n-2}, 01p_{j-1}^{n-2}, 01p_j^{n-2}, 01p_{j+1}^{n-2}, 00p_{j+1}^{n-2}, 10p_{j+1}^{n-2}, 10p_j^{n-2}, 10p_{j-1}^{n-2}, 00p_{j-1}^{n-2})$.
The two remaining cycles:
$\mathscr{C}_6 = (00p_{|V(\Gamma_{n-3})|}^{n-2}, 00p_{|V(\Gamma_{n-3})|+1}^{n-2}, 00p_{|V(\Gamma_{n-3})|+2}^{n-2}, 10p_{|V(\Gamma_{n-3})|+2}^{n-2}, 10p_{|V(\Gamma_{n-3})|+1}^{n-2}, 10p_{|V(\Gamma_{n-3})|}^{n-2}, 00p_{|V(\Gamma_{n-3})|}^{n-2})$ and
$\mathscr{C}_4 = (00p_{|V(\Gamma_{n-3})|}^{n-2}, 00p_{|V(\Gamma_{n-3})|+1}^{n-2}, 10p_{|V(\Gamma_{n-3})|+1}^{n-2}, 10p_{|V(\Gamma_{n-3})|}^{n-2}, 00p_{|V(\Gamma_{n-3})|}^{n-2})$.
3. $v \in \{0010\,\mathcal{P}_{n-4}^R\} - 00p_{|V(\Gamma_{n-3})|+1}^{n-2} \ \cup\ \{1010\,\mathcal{P}_{n-4}^R\}$
Let $v = 00p_j^{n-2} \in \{0010\,\mathcal{P}_{n-4}^R - 00p_{|V(\Gamma_{n-3})|+1}^{n-2}\}$.
Then, for $j < j' \le |V(\Gamma_{n-2})|$, remove recurrently from the cycle the vertices $00p_{j'}^{n-2}$ and its correspondent $10p_{j'}^{n-2}$. See figure 3.13.

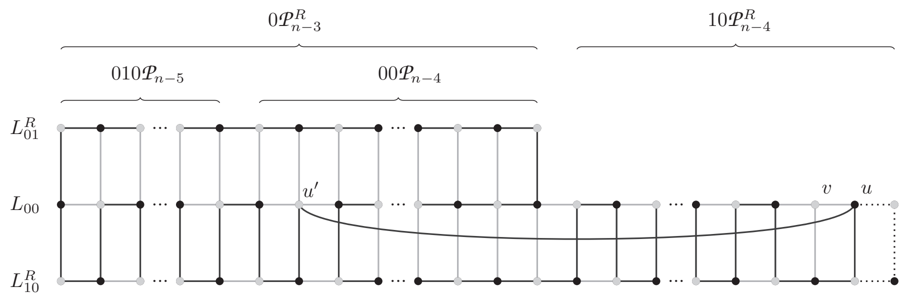


Figure 3.13: Case 3: $v \in \{0010\mathcal{P}_{n-4}^R\} - 00p_{|V(\Gamma_{n-3})|+1}^{n-2}$

Refer to figure 3.14 for the next cycle which does not contain the pair of the vertices $01p^{n-2}_{|V(\Gamma_{n-3})|}$ and $00p^{n-2}_{|V(\Gamma_{n-3})|}$.

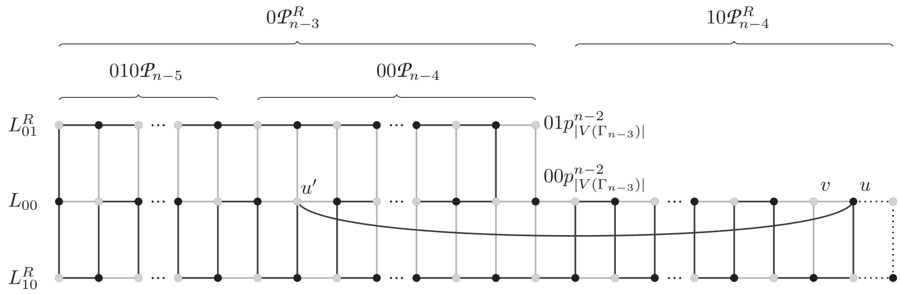


Figure 3.14: Case 3: $\mathscr{C} \setminus \{01p^{n-2}_{|V(\Gamma_{n-3})|}, 00p^{n-2}_{|V(\Gamma_{n-3})|}\}$

We will include again the vertex $00p^{n-2}_{|V(\Gamma_{n-3})|}$ and exclude the vertices $u = 00p^{n-2}_{j+1}$, $10p^{n-2}_{j+1}$ and $10p^{n-2}_{j}$ in order to construct a new cycle with two vertices less than the previous one, using the same structure as in case 1. See figure 3.15. Therefore we can find the remaining cycles in the same form.

For the case $v = 10p^{n-2}_{|V(\Gamma_{n-2})|}$, we commence with the cycle of length $|V(\Gamma_n)| - 1$. Thus, in this stage, we will delete vertex $01p^{n-2}_{|V(\Gamma_{n-3})|}$ and the vertex $u = 00p^{n-2}_{|V(\Gamma_{n-2})|}$, where we can use the structure of case 1 to find the remaining cycles.

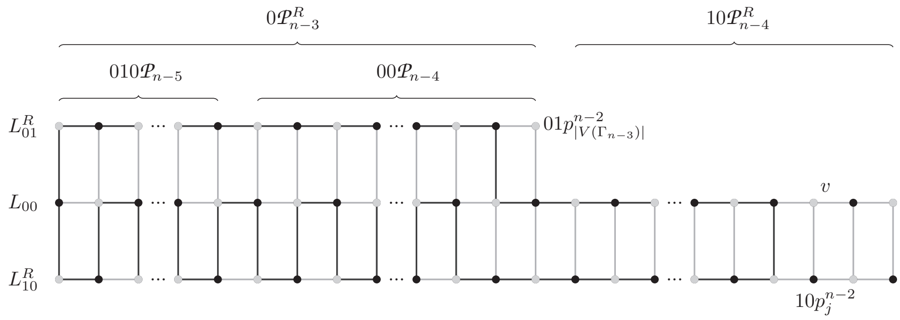


Figure 3.15: Case 3: $\mathscr{C} \setminus \{10p^{n-2}_{j}, 01p^{n-2}_{|V(\Gamma_{n-3})|}\}$

If $v = 10p^{n-2}_{j} \in \{1010\,\mathcal{P}^R_{n-4}\}$, the cycles are constructed similarly.
If $v = 00p^{n-2}_{|V(\Gamma_{n-2})|}$, we use the cycle of length $|V(\Gamma_n)| - 1$. We will delete vertex $01p^{n-2}_{|V(\Gamma_{n-3})|}$ and the vertex $u = 10p^{n-2}_{|V(\Gamma_{n-2})|}$, where we can use the structure of case 1 to find the remaining cycles.

4. $v = 00p^{n-2}_{|V(\Gamma_{n-3})|+1}$
For $|V(\Gamma_{n-3})|+2 < j' \leq |V(\Gamma_{n-2})|$, remove recurrently from the cycle the vertices $00p^{n-2}_{j'}$ and its correspondent $10p^{n-2}_{j'}$. Figure 3.16.

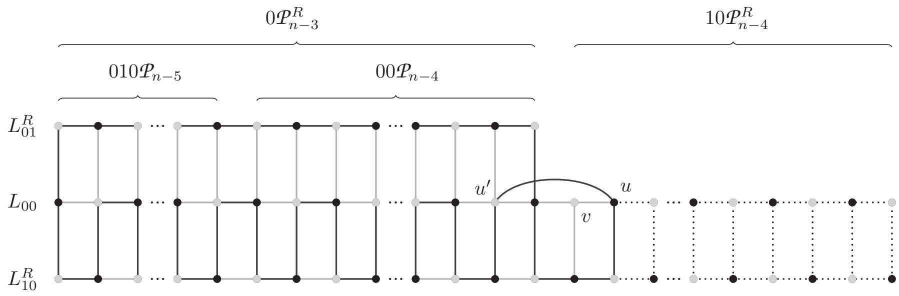


Figure 3.16: Case 4: $v = 00p^{n-2}_{|V(\Gamma_{n-3})|+1}$

We will construct the new cycle by removing the vertices $10p^{n-2}_{|V(\Gamma_{n-3})|}$ and
$10p^{n-2}_{|V(\Gamma_{n-3})|+1}$ and by replacing the edge $(00p^{n-2}_{|V(\Gamma_{n-3})|-1}, 10p^{n-2}_{|V(\Gamma_{n-3})|-1})$ by the edge $(10p^{n-2}_{|V(\Gamma_{n-3})|+2}, 10p^{n-2}_{|V(\Gamma_{n-3}|)-1})$. See figure 3.17.

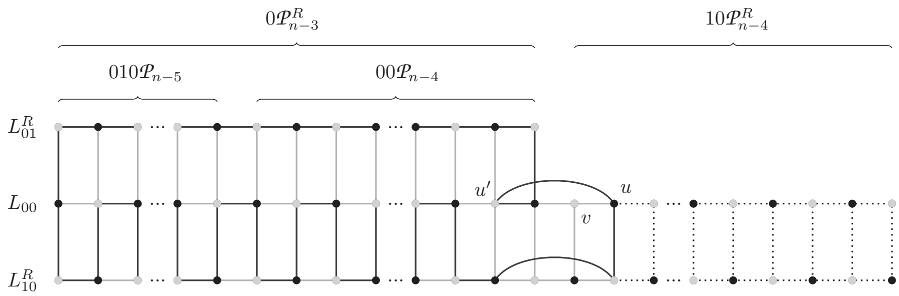


Figure 3.17: Case 4: $\mathscr{C} \setminus \{10p^{n-2}_{|V(\Gamma_{n-3})|}, 10p^{n-2}_{|V(\Gamma_{n-3})|+1}\}$

A smaller cycle is obtained by deleting $l^{00}_{|V(\Gamma_{n-3})|+2}$ and $l^{10}_{|V(\Gamma_{n-3})|+2}$. We recognize now the structure of case 1. Therefore the remaining cycles can be obtained.

When $v = l^{10}_{|V(\Gamma_{n-3})|}$, we will delete again, recurrently from $\mathscr{C}_{|V(\Gamma_n - v)|}$, the vertices $00p^{n-2}_{j'}$ and $10p^{n-2}_{j'}$ for $|V(\Gamma_{n-3})| + 1 < j' \leq |V(\Gamma_{n-2})|$ to obtain the cycles of length $l = |V(\Gamma_n)| - 1, \ldots |V(\Gamma_n)| - 2(|V(\Gamma_{n-2})| - 3)$.
We get a smaller cycle by removing $10p^{n-2}_{|V(\Gamma_{n-3})|+1}$ and $00p^{n-2}_{|V(\Gamma_{n-3})|+1}$, and adding the edge $(10p^{n-2}_{|V(\Gamma_{n-3})|+2}, 10p^{n-2}_{|V(\Gamma_{n-3})|-1})$. (Figure 3.18.)

As for the previous case, deleting $00p^{n-2}_{|V(\Gamma_{n-3})|+2}$ and $10p^{n-2}_{|V(\Gamma_{n-3})|+2}$ gives us the known structure of case 1.

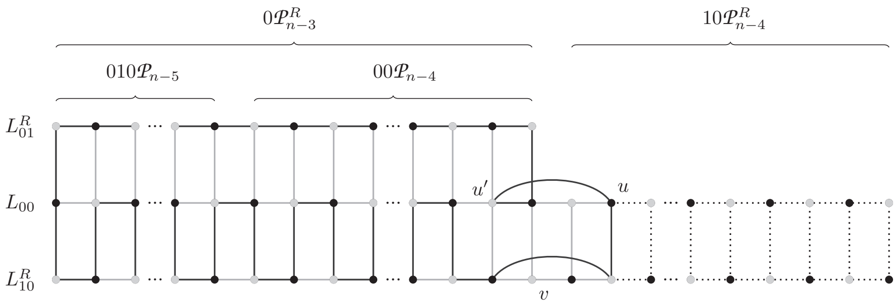


Figure 3.18: Case 5: $v = 10p^{n-2}_{|V(\Gamma_{n-3})|}$

□

3.2 Hamiltonicity in the Lucas cubes

We have mentioned before that the Lucas cubes have been widely studied due to the fact that these cubes are closely related to the Fibonacci cubes. The Lucas cubes are defined as the Fibonacci cubes but with the restriction of consecutive 1's applied in a circular manner.

In 2005, Jean-Luc Baril and Vincent Vajnovszki proved in [BV05] that the Lucas cube has a Hamiltonian path if and only if $n \not\equiv 0\,(mod\,3)$ constructing a Gray Code for the Lucas strings of length n which is an ordered list of strings such that the distance between two strings is 1.

In this section, we will construct nearly Hamiltonian cycles for the Lucas cubes in the same way as we did for the Fibonacci cubes. Therefore, we will be using many of the results of the previous section throughout this section.

Lemma 3.2.1. *For $n \geq 3$, $V(\Lambda_n) = 0\,V(\Gamma_{n-1}) \cup 10\,V(\Gamma_{n-3})\,0$.*

Proof. By definition, $V(\Lambda_n) = V(\Gamma_n) \setminus \{$ strings that begin and end with $1\}$ and since $V(\Gamma_n) = 0\,V(\Gamma_{n-1}) \cup 10\,V(\Gamma_{n-2}) = 0\,V(\Gamma_{n-1}) \cup 10\,V(\Gamma_{n-3})\,0 \cup 10\,V(\Gamma_{n-3})\,1$, the result is obtained. □

The parity difference relation is stated in the next known result:

Theorem 3.2.2. *For $n \geq 1$, then*

$$|V^{ev}(\Lambda_n)| - |V^{od}(\Lambda_n)| = \begin{cases} -(-1)^{\lfloor \frac{n+2}{3} \rfloor} & \text{if } n \equiv 1\,(mod\,3), \\ (-1)^{\lfloor \frac{n+2}{3} \rfloor} & \text{if } n \equiv 2\,(mod\,3), \\ 2(-1)^{\lfloor \frac{n+2}{3} \rfloor} & \text{if } n \equiv 0\,(mod\,3). \end{cases}$$

Proof. For $n = 1, 2$ and 3,

$$\begin{aligned} |V^{ev}(\Lambda_1)| - |V^{od}(\Lambda_1)| &= 1 - 0 &= 1 \\ |V^{ev}(\Lambda_2)| - |V^{od}(\Lambda_2)| &= 1 - 2 &= -1 \\ |V^{ev}(\Lambda_3)| - |V^{od}(\Lambda_3)| &= 1 - 3 &= -2. \end{aligned}$$

By Lemma 3.2.1, for $n \geq 3$, $V(\Lambda_n) = 00V(\Gamma_{n-2}) \cup 010V(\Gamma_{n-3}) \cup 10V(\Gamma_{n-3})0$. Notice that

$$\begin{aligned} |V^{ev}(010\,\Gamma_{n-3})| &= |V^{ev}(10\,\Gamma_{n-3}\,0)| = |V^{od}(\Gamma_{n-3})|, \\ |V^{od}(010\,\Gamma_{n-3})| &= |V^{od}(10\,\Gamma_{n-3}\,0)| = |V^{ev}(\Gamma_{n-3})| \text{ and} \\ |V^{ev}(00\,\Gamma_{n-2})| &= |V^{ev}(\Gamma_{n-2})|, \\ |V^{od}(00\,\Gamma_{n-2})| &= |V^{od}(\Gamma_{n-2})|. \end{aligned}$$

Hence,

$$\begin{aligned} |V^{ev}(\Lambda_n)| - |V^{od}(\Lambda_n)| = {} & 2(|V^{od}(\Gamma_{n-3})| - |V^{ev}(\Gamma_{n-3})|) \\ & + |V^{ev}(\Gamma_{n-2})| - |V^{od}(\Gamma_{n-2})|. \end{aligned}$$

Assume that the statement is true for Λ_N; $N \geq 3$. Thus, using Propositions 3.1.8 and 3.1.7 for Γ_{N+1}, we have that

If $N + 1 \equiv 1\,(mod\,3)$,
then $(N + 1) - 3 \equiv 1\,(mod\,3)$ and $|V^{od}(\Gamma_{N+1-3})| = |V^{ev}(\Gamma_{N+1-3})|$ and $(N + 1) - 2 \equiv 2\,(mod\,3)$.
Thus

$$\begin{aligned} |V^{ev}(\Lambda_{N+1})| - |V^{od}(\Lambda_{N+1})| &= |V^{ev}(\Gamma_{N-1})| - |V^{od}(\Gamma_{N-1})| \\ &= (-1)^{\lfloor \frac{N-1+2}{3} \rfloor}. \end{aligned}$$

Notice that for $N + 1 \equiv 1\,(mod\,3)$, then
$(-1)^{\lfloor \frac{N-1+2}{3} \rfloor} = -(-1)^{\lfloor \frac{(N+1)+2}{3} \rfloor}$.
Therefore,

$$|V^{ev}(\Lambda_{N+1})| - |V^{od}(\Lambda_{N+1})| = -(-1)^{\lfloor \frac{(N+1)+2}{3} \rfloor}.$$

If $N+1 \equiv 2\,(mod\,3)$,
then $(N+1)-3 \equiv 2\,(mod\,3)$ and $(N+1)-2 \equiv 1\,(mod\,3)$;
Hence

$$\begin{aligned}|V^{ev}(\Lambda_{N+1})| - |V^{od}(\Lambda_{N+1})| &= -2(|V^{ev}(\Gamma_{(N+1)-3})| - |V^{od}(\Gamma_{(N+1)-3})|) \\ &\quad + |V^{ev}(\Gamma_{(N+1)-2})| - |V^{od}(\Gamma_{(N+1)-2})| \\ &= -2(-1)^{\lfloor \frac{N}{3} \rfloor} + (-1)^{\lfloor \frac{N+1}{3} \rfloor}\end{aligned}$$

When $N+1 \equiv 2\,(mod\,3)$, then
$-(-1)^{\lfloor \frac{N}{3} \rfloor} = -(-1)^{\lfloor \frac{N+1}{3} \rfloor} = (-1)^{\lfloor \frac{(N+1)+2}{3} \rfloor}$.
Therefore,

$$|V^{ev}(\Lambda_{N+1})| - |V^{od}(\Lambda_{N+1})| = (-1)^{\lfloor \frac{(N+1)+2}{3} \rfloor}.$$

If $N+1 \equiv 0\,(mod\,3)$,
then $(N+1)-3 \equiv 0\,(mod\,3)$ and $(N+1)-2 \equiv 1\,(mod\,3)$ and
$|V^{ev}(\Gamma_{(N+1)-2})| = |V^{od}(\Gamma_{(N+1)-2})|$.
Hence

$$\begin{aligned}|V^{ev}(\Lambda_{N+1})| - |V^{od}(\Lambda_{N+1})| &= 2(|V^{od}(\Gamma_{N-2})| - |V^{ev}(\Gamma_{N-2})|) \\ &= 2(-1)^{\lfloor \frac{N}{3} \rfloor} \\ &= 2(-1)^{\lfloor \frac{(N+1)+2}{3} \rfloor}\end{aligned}$$

since $(-1)^{\lfloor \frac{N}{3} \rfloor} = (-1)^{\lfloor \frac{(N+1)+2}{3} \rfloor}$ for $N+1 \equiv 0\,(mod\,3)$. □

From Theorem 3.2.2, for any $n \geq 1$, one of the following occurs: if $n \not\equiv 0\,(mod\,3)$, the longest possible cycle in Λ_n contains $|V(\Lambda_n)|-1$ vertices and therefore, one of the sets of the partition $V(\Lambda_n) = (V^{ev}(\Lambda_n), V^{od}(\Lambda_n))$ has one more string than the other set; if $n \equiv 0\,(mod\,3)$, then the longest possible cycle in Λ_n contains $|V(\Lambda_n)|-2$ vertices and hence, one of the sets of the partition $(V^{ev}(\Lambda_n), V^{od}(\Lambda_n))$ has two more strings than the other set. In both cases, we will call this set, $V^P(\Lambda_n)$ where:
For every $n \geq 1$,

$$V^P(\Lambda_n) = \begin{cases} V^{ev}(\Lambda_n) \begin{cases} \text{if } n \equiv 1\,(mod\,3) \text{ and } & \lfloor \frac{n+2}{3} \rfloor \text{ is odd,} \\ \text{if } n \equiv 2\,(mod\,3) \text{ and } & \lfloor \frac{n+2}{3} \rfloor \text{ is even,} \\ \text{if } n \equiv 0\,(mod\,3) \text{ and } & \lfloor \frac{n+2}{3} \rfloor \text{ is even.} \end{cases} \\ V^{od}(\Lambda_n) \begin{cases} \text{if } n \equiv 1\,(mod\,3) \text{ and } & \lfloor \frac{n+2}{3} \rfloor \text{ is even,} \\ \text{if } n \equiv 2\,(mod\,3) \text{ and } & \lfloor \frac{n+2}{3} \rfloor \text{ is odd,} \\ \text{if } n \equiv 0\,(mod\,3) \text{ and } & \lfloor \frac{n+2}{3} \rfloor \text{ is odd.} \end{cases} \end{cases}$$

We will take back the Hamiltonian path constructed for the Fibonacci cubes, $\mathcal{P}_n = 0\,\mathcal{P}^R_{n-1}, 10\,\mathcal{P}^R_{n-2} = (p^n_1, p^n_2, \ldots p^n_{F_{n+2}})$.

Proposition 3.2.3. *Let $\mathcal{P}1_n$; $n \geq 1$ be a maximal subpath of $\mathcal{P}_n$ whose strings end with 1. Then we have that*

$(p^n_{F_{n+2}})$ *is the only* $\mathcal{P}1_n$ *subpath of length one*	*if* $n \equiv 1\,(mod\,3)$,
(p^n_1) *is the only* $\mathcal{P}1_n$ *subpath of length one*	*if* $n \equiv 2\,(mod\,3)$,
there are no $\mathcal{P}1_n$ *subpaths of length one*	*if* $n \equiv 0\,(mod\,3)$.

Moreover, any other $\mathcal{P}1_n$ subpath has length two and it is not found in the first nor in the last positions of $\mathcal{P}_n$.

Proof. Consider $\mathcal{P}_1 = (0, 1)$ and $\mathcal{P}_2 = (01, 00, 10)$. Then (p^1_2) and (p^2_1) are the only $\mathcal{P}1_1$ and $\mathcal{P}1_2$ subpaths and they both have length one.
Thus, $\mathcal{P}_3 = 0\mathcal{P}^R_2, 10\,\mathcal{P}^R_1$, and $\mathcal{P}1_3 = (0\,p^2_1, 10\,p^1_2)$ is the sole $\mathcal{P}1_3$ subpath.
We will now suppose that the statement is true for all the subpaths $\mathcal{P}1_N$ of $\mathcal{P}_N$. Let us consider $\mathcal{P}_{N+1}$ where we distinguish the three cases of $N+1$.

1. $N+1 \equiv 1\,(mod\,3)$:
   Thus $N \equiv 0\,(mod\,3)$ and $N-1 \equiv 2\,(mod\,3)$.
   By hypothesis of induction, $\mathcal{P}_N$ has no $\mathcal{P}1_N$ subpaths of length one and (p^{N-1}_1) is the only subpath of this length in $\mathcal{P}_{N-1}$. Therefore, $(10\,p^{N-1}_1)$, now labeled $(p^{N+1}_{F_{N+3}})$ is the only $\mathcal{P}1_{N+1}$ subpath of lenght one. The $\mathcal{P}1_N$ and $\mathcal{P}1_{N-1}$ subpaths of length two are maintained in $\mathcal{P}_{N+1}$.
2. $N+1 \equiv 2\,(mod\,3)$:
   Thus $N \equiv 1\,(mod\,3)$ and $N-1 \equiv 0\,(mod\,3)$.
   Again, by hypothesis of induction, $(p^N_{F_{N+2}})$ is the sole $\mathcal{P}1_N$ subpath of lenght one and $\mathcal{P}_{N-1}$ has no $\mathcal{P}1_{N-1}$ subpath with this length. Therefore, $(0\,p^N_{F_{N+2}}) = (p^{N+1}_1)$ is the only $\mathcal{P}1_{N+1}$ subpath with length one. As in the previous case, the $\mathcal{P}1_N$ and $\mathcal{P}1_{N-1}$ subpaths of length two are maintained in $\mathcal{P}_{N+1}$.
3. $N+1 \equiv 0\,(mod\,3)$:
   Thus $N \equiv 2\,(mod\,3)$ and $N-1 \equiv 1\,(mod\,3)$.
   (p^N_1) and $(p^{N-1}_{F_{N+1}})$ are the only $\mathcal{P}1_N$ and $\mathcal{P}1_{N-1}$ subpaths of length one of $\mathcal{P}_{N+1}$ and $\mathcal{P}_{N-1}$ respectively. Hence $\mathcal{P}_{N+1} = 0\,(p^n_{F_{N+2}}, \ldots p^N_1)$, $10\,(p^{N-1}_{F_{N+1}}, \ldots p^{N-1}_1)$ does not contain a $\mathcal{P}1_{N+1}$ subpath of length one. Notice that all the $\mathcal{P}1_N$ and $\mathcal{P}1_{N-1}$ subpaths of lenght two are preserved in $\mathcal{P}_{N+1}$ together with $(0\,p^N_1, 10\,p^{N-1}_{F_{N+1}})$. □

Proposition 3.2.4. *For every $n \geq 1$, $\mathcal{P}_n \setminus \{$strings that end with 1$\}$ is a Hamiltonian path in Γ_{n-1}.*

Proof. Let $\mathcal{S}_0$ be the subgraph induced by the vertices of $0\,\mathcal{P}_{n-1}$. Then $\mathcal{S}_0$ is isomorphic to Γ_{n-1}.
We will prove that $\mathcal{P}_n \setminus V(\Gamma_{n-2})\,01$ is a Hamiltonian path in $\mathcal{S}_0$.
From the preceding proposition, the strings of $\mathcal{P}_n$ that end with 1 are either the first or the last string of $\mathcal{P}_n$ or they form a subpath of two strings in $\mathcal{P}_n$.
Let $\mathcal{P}1_n = (p^n_j, p^n_{j+1})$ be one of these latter subpaths with $1 < j < |V(\Gamma_n|$. It suffices to show that p^n_{j-1} and p^n_{j+2} differ in one position. Notice that these two strings both end with 0. Thus they only differ in the position in which p^n_j and p^n_{j+1} differ and hence the edge between them exists in $\mathcal{S}_0$.
Therefore the path $\mathcal{P}_n \setminus V(\Gamma_{n-2})\,01$ is a Hamiltonian path in $\mathcal{S}_0$ and hence in Γ_{n-1}. □

We will next construct a subgraph $\mathcal{H}'_n$, isomorphic to Λ_n in the following way:
Let $\mathcal{S}_0$ be the subgraph isomorphic to Γ_{n-1} induced by the vertices $0\,\mathcal{P}_{n-1}$.
Let $\mathcal{S}'_1$ by induced by the vertices $10\,\mathcal{P}_{n-2} \setminus \{$strings that end with $1\}$. Then, by Proposition 3.2.4, $\mathcal{S}'_1$ is isomorphic to Γ_{n-3}.
Let $L_0 = 01(0\,\mathcal{P}_{n-3}), 00\,\mathcal{P}_{n-2}$. Thus, we can decompose L_0 in L_{01} and L_{00}, both Hamiltonian paths in Γ_{n-3} and Γ_{n-2} respectively.
Let also L'_{10} be the Hamiltonian path of Γ_{n-3} formed by $1\,\mathcal{P}_{n-2} \setminus \{$strings that end with $1\}$.
Finally, let $\mathcal{H}'_n$ be the subgraph of Γ_n formed by L_0 and L'_{10}.
By Lemma 3.2.1, $\mathcal{H}'_n$ is isomorphic to Λ_n. Also, as we proved in Theorem 3.1.5, there exists the edge between every vertex $010\,p^{n-3}_i$ of L_{01} and $000\,p^{n-3}_i$ of L_{00} in $\mathcal{H}'_n$. Also, the edge between every vertex $10\,p^{n-3}_i0$ of L'_{10} and $00\,p^{n-2}_i$ of L_{00} exists in $\mathcal{H}'_n$.

Notice that for an easy recognition of the elements of $\mathcal{H}'_n$, we will maintain the same subscript j in the labels of the strings $10p'^{n-2}_j$ of L'^R_{10} as the labels of the strings $00p^{n-2}_j$ in L_{00}.

We will next describe and show the figures of $\mathcal{H}'_n$ for every $n \geq 2$. For this purpose, we will use Proposition 3.2.3 as well as the next

Proposition 3.2.5. *Let $\mathcal{P}0_n$; $n \geq 1$ be a maximal subpath of $\mathcal{P}_n$ whose strings end with 0. Then*

(p^n_1) *is a* $\mathcal{P}0_n$ *subpath of length one*	*if* $n \equiv 1\,(mod\,3)$,
$(p^n_{F_{n+2}-1}, p^n_{F_{n+2}})$ *is a* $\mathcal{P}0_n$ *subpath of length two*	*if* $n \equiv 2\,(mod\,3)$,
(p^n_1, p^n_2) *is a* $\mathcal{P}0_n$ *subpath of length two and*	
$(p^n_{F_{n+2}})$ *is a* $\mathcal{P}0_n$ *subpath of length one*	*if* $n \equiv 0\,(mod\,3)$.

Moreover, there are no other $\mathcal{P}0_n$ subpaths of length one. Furthermore, any other $\mathcal{P}0_n$ subpath has length two or four and is not found in the first or in the last positions of $\mathcal{P}_n$.

Proof. Consider $\mathcal{P}_1 = (0, 1)$ and $\mathcal{P}_2 = (01, 00, 10)$. Then (p_1^1) is the only $\mathcal{P}0_1$ subpath of length one and (p_2^2, p_3^2) is the only $\mathcal{P}0_1$ of length two. For $\mathcal{P}_3 = 0\mathcal{P}_2^R, 10\,\mathcal{P}_1^R = (010, 000, 001, 101, 100)$,the sole $\mathcal{P}0_3$ subpath of length two is $(0\,p_3^2, 0\,p_2^2) = (p_1^3, p_2^3)$ and $(10\,p_1^1) = (p_5^3)$ is the only $\mathcal{P}0_3$ subpath of length one in $\mathcal{P}_3$.
We will now suppose that the statement is true for all the $\mathcal{P}0_N$ subpaths of $\mathcal{P}_N$. Let us consider $\mathcal{P}_{N+1}$ where we distinguish the three cases of $N + 1$.

1. $N + 1 \equiv 1\,(mod\,3)$:
   Thus $N \equiv 0\,(mod\,3)$ and $N - 1 \equiv 2\,(mod\,3)$.
   By hypothesis of induction, (p_1^N, p_2^N) and $(p_{F_{N+2}}^N)$ are $\mathcal{P}0_N$ subpaths of length two and one respectively and $(p_{F_{(N-1)+2}-1}^{N-1}, p_{F_{(N-1)+2}}^{N-1})$ is a $\mathcal{P}0_{N-1}$ subpath of length two.
   Therefore, $(0\,p_{F_{N+2}}^N) = (p_1^{N+1})$ is $\mathcal{P}0_{N+1}$ subpath of length one.
   Notice also that $(10\,p_{F_{(N-1)+2}}^{N-1}, 10\,p_{F_{(N-1)+2}-1}^{N-1}, 0\,p_2^N, 0\,p_1^N)$ form a $\mathcal{P}0_{N+1}$ subpath of length four.
   All the $\mathcal{P}0_N$ and $\mathcal{P}0_{N-1}$ subpaths of lengths two and four are maintained in $\mathcal{P}_{N+1}$.
2. $N + 1 \equiv 2\,(mod\,3)$:
   Thus $N \equiv 1\,(mod\,3)$ and $N - 1 \equiv 0\,(mod\,3)$.
   Again, by hypothesis of induction, (p_1^N) is a $\mathcal{P}0_N$ of length one and (p_1^{N-1}, p_2^{N-1}) is a $\mathcal{P}0_N$ of length two and $(p_{F_{(N-1)+2}}^{N-1})$ is a $\mathcal{P}0_{N-1}$ of length one.
   Therefore, $(10\,p_2^{N-1}, 10\,p_1^{N-1}) = (p_{F_{(N+1)+2}-1}^{N+1}, p_{F_{(N+1)+2}}^{N+1})$ is a $\mathcal{P}0_{N+1}$ of length two.
   Notice also that $(0\,p_1^N, 10\,p_{F_{(N-1)+2}}^{N-1}) = (p_{F_{N+2}}^{N+1}, p_{F_{N+2}+1}^{N+1})$ is a $\mathcal{P}0_{N+1}$ of length two. As in the previous case, the $\mathcal{P}0_N$ and $\mathcal{P}0_{N-1}$ subpaths of length two are maintained in $\mathcal{P}_{N+1}$.
3. $N + 1 \equiv 0\,(mod\,3)$:
   Thus $N \equiv 2\,(mod\,3)$ and $N - 1 \equiv 1\,(mod\,3)$.
   Hence $(p_{F_{N+2}-1}^N, p_{F_{N+2}}^N)$ is a $\mathcal{P}0_N$ subpath of length two and (p_1^{N-1}) is a $\mathcal{P}0_{N-1}$ subpath of length one.
   Therefore, $(0\,p_{F_{N+2}}^N, 0\,p_{F_{N+2}-1}^N) = (p_1^{N+1}, p_2^{N+1})$ is a $\mathcal{P}0_{N+1}$ subpath of length two and $(10\,p_1^{N-1}) = (p_{F_{(N+1)+2}}^{N+1})$ is a $\mathcal{P}0_{N+1}$ subpath of length one. Notice that all the $\mathcal{P}0_N$ and $\mathcal{P}0_{N-1}$ subpaths of lenght two and four are preserved in $\mathcal{P}_{N+1}$. $\square$

Notice that $|L_{00}|$ is even for $n \equiv 0\,(mod\,3)$ and $|L_{00}|$ is odd for $n \not\equiv 0\,(mod\,3)$.
Additionally, since $|L_{01}^R| = |L_{10}'^R|$, then there are the same number of vertices of $V^P(\Lambda_n)$ in L_{01}^R than in $L_{10}'^R$.

$n \equiv 1(mod\,3)$:
For $n \geq 3$ and considering Propositions 3.2.3 and 3.2.5, we depict in figure 3.19 the vertices of $V^P(\Lambda_n)$ in gray. In this case, $|L_{00}| = |V(\Gamma_{n-2})|$ is odd and $(n-2) \equiv 2\,(mod\,3)$. Also, $|L_{01}^R| = |L_{10}'^R| = |V(\Gamma_{n-3})|$ is even and $(n-3) \equiv 1\,(mod\,3)$. Then for each of these paths, the number of vertices that do not belong to $V^P(\Lambda_n)$ is the same as the number of vertices that belong to $V^P(\Lambda_n)$. Therefore, $\{00p_j^{n-2}\,|\,j\ \textit{is odd},\ 1 \leq j \leq |V(\Gamma_{n-2})|\}$, $\{01p_j^{n-2}\,|\,j\ \textit{is even},\ 1 \leq j \leq |V(\Gamma_{n-3})|\}$ and $\{10p_j'^{n-2}\,|\,j\ \textit{is even},\ 1 \leq j \leq |V(\Gamma_{n-3})|\}$ correspond to $V^P(\Lambda_n)$.

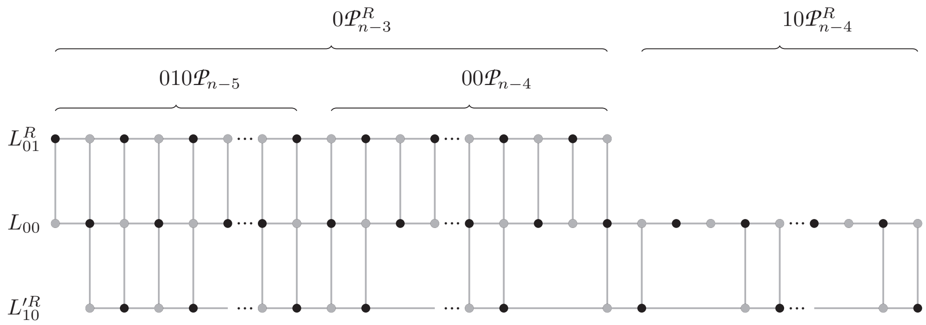


Figure 3.19: Vertices of $V^P(\Lambda_n)$ for $n \equiv 1(mod\,3)$

$n \equiv 2(mod\,3)$:
Considering again Propositions 3.2.3 and 3.2.5 and for $n \geq 3$, the vertices of $V^P(\Lambda_n)$ are drawn in gray in figure 3.20. Observe that $|L_{01}^R| = |L_{10}'^R| = |V(\Gamma_{n-3})|$ is odd and $(n-3) \equiv 2\,(mod\,3)$. Thus the set V^P has one more vertex of each of these paths.
Therefore $\{01p_j^{n-2}\,|\,j\ \textit{is odd},\ 1 \leq j \leq |V(\Gamma_{n-3})|\}$ and $\{10p_j'^{n-2}\,|\,j\ \textit{is odd},\ 1 \leq j \leq |V(\Gamma_{n-3})|\}$ belong to $V^P(\Lambda_n)$.
Furthermore, $|L_{00}|$ is odd, thus $\{00p_j^{n-2}\,|\,j\ \textit{is even},\ 1 \leq j \leq |V(\Gamma_{n-2})|\}$ belong to $V^P(\Lambda_n)$. Note that there is one more vertex of $|L_{00}|$ not in $V^P(\Lambda_n)$ than in $V^P(\Lambda_n)$ which, in addittion to the vertex of $|L_{01}^R|$ and $|L_{10}'^R|$ in $V^P(\Lambda_n)$ are consistent with Theorem 3.2.2.

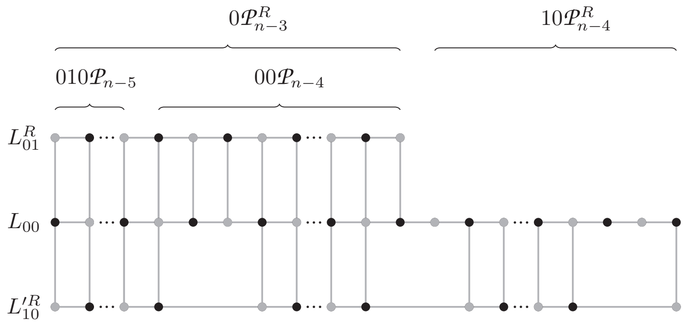


Figure 3.20: Vertices of $V^P(\Lambda_n)$ for $n \equiv 2(mod\,3)$

$n \equiv 0(mod\,3)$:

Figure 3.21 shows the isomorphic subgraph to the Lucas cube, $\mathcal{H}'_n$ formed by L^R_{01}, L_{00} and L'^R_{10} for $n \equiv 0\,(mod\,3)$, $n \geq 3$ and the vertices of $V^P(\Lambda_n)$ in gray. Note that in this case, $|L^R_{01}| = |L'^R_{10}| = |V(\Gamma_{n-3})|$ and $(n-3) \equiv 0\,(mod\,3)$. Thus by Proposition 3.1.8, the set V^P has one more vertex from each of these paths. Hence $\{01p^{n-2}_j \,|\, j \textit{ is odd, } 1 \leq j \leq |V(\Gamma_{n-3})|\}$ and $\{10p'^{n-2}_j \,|\, j \textit{ is odd, } 1 \leq j \leq |V(\Gamma_{n-3})|\}$ belong to $V^P(\Lambda_n)$. Also, since $|L_{00}|$ is even, $\{00p^{n-2}_j \,|\, j \textit{ is even, } 1 \leq j \leq |V(\Gamma_{n-2})|\}$ belong to $V^P(\Lambda_n)$ as well.

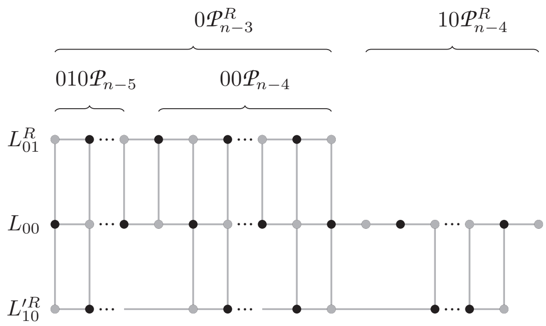


Figure 3.21: Vertices of $V^P(\Lambda_n)$ for $n \equiv 0(mod\,3)$

Theorem 3.2.6 ([BV05]). *Λ_n has a Hamiltonian path for $n \not\equiv 0\,(mod\,3), n \geq 2$.*

Proof. By construction of $\mathcal{H}'_n$, isomorphic to Λ_n, it is induced by the vertices of $L_{01} = 010\,\mathcal{P}_{n-3}$, $L_{00} = 00\,\mathcal{P}_{n-2}$ and $L'_{10} = 10\,\mathcal{P}_{n-3}\,0$. We've proved that in $\mathcal{H}'_n$, every $10p'^{n-2}_i$ of L'_{10} is adjacent to $00p^{n-2}_i$ of L_{00}.
Then, it suffices to show that the string $10p'^{n-2}_{|V(\Gamma_{n-2})|}$ exists for every $n \not\equiv$

$0\,(mod\,3)$. Indeed, when $n \equiv 1\,(mod\,3)$ then $(n-3) \equiv 1\,(mod\,3)$. When $n \equiv 2\,(mod\,3)$ then $(n-3) \equiv 2\,(mod\,3)$. In both cases, because $|L'_{10}| = |V(\Gamma_{n-3})|$ and by Proposition 3.2.3, the string exists. Therefore, $L^R_{01} \cup L_{00} \cup L'^R_{10}$ is a Hamiltonian path in Λ_n. See figures 3.19 and 3.20. $\square$

Theorem 3.2.7. *Let $v \in V^P(\Lambda_n)$ with $n \not\equiv 0\,(mod\,3)$ and $n \geq 5$. Then $\Lambda_n - v$ contains a Hamiltonian cycle.*
Furthermore, if $v \notin V^P(\Lambda_n)$, then $\Lambda_n - v$ does not contain a Hamiltonian cycle.

Proof. Let us consider the Lucas cube induced by L^R_{01}, L_{00} and L'^R_{10}. We will describe the Hamiltonian cycle in $\Lambda_n - v$ considering v within five main subsets of vertices.
The cycles where v belongs to $L^R_{01}, L_{00} \setminus \{$vertices of L_{00} in $0010\,\mathcal{P}^R_{n-4}\}$ and $L'^R_{10} \setminus \{$vertices of L'^R_{10} in $1010\,\mathcal{P}^R_{n-4}\}$ have the same structure with some slight differences. We consider as well, the cases of v in $L_{00} \cap 0010\,\mathcal{P}^R_{n-4}$ and v in $L'^R_{10} \cap 1010\,\mathcal{P}^R_{n-4}$; all together with some special subcases.
Let $v \in V^P(\Lambda_n)$.
Case $n \equiv 1(mod\,3)$: Since $|V(\Gamma_{n-3})|$ is even, then let $|V(\Gamma_{n-3})| = 2m$

1. $v \in L^R_{01} \cup L_{00} \setminus \{010p^{n-3}_{|V(\Gamma_{n-3})|-1}, 00p^{n-2}_1 = 000p^{n-3}_{|V(\Gamma_{n-3})|}, 0010\,\mathcal{P}^R_{n-4}\}$
   Let $v \in L^R_{01} - 010p^{n-3}_{|V(\Gamma_{n-3})|-1} (= 01p^{n-2}_2)$ Thus $v = 01p^{n-2}_{2i}$ for some $2 \leq i \leq m$. Consider the path $\mathcal{T}_n$, from $00p^{n-2}_{|V(\Gamma_{n-3})|} = 000p^{n-3}_1$ to $00p^{n-2}_4 = 000p^{n-3}_{|V(\Gamma_{n-3})|-3}$ defined by the concatenation of

   $$(00p^{n-2}_{2m}, 01p^{n-2}_{2m}, 01p^{n-2}_{2m-1}, 00p^{n-2}_{2m-1}, \ldots 00p^{n-2}_{2k}, 01p^{n-2}_{2k}, 01p^{n-2}_{2k-1}, 00p^{n-2}_{2k-1},$$

   $$\ldots 00p^{n-2}_{2i+2}, 01p^{n-2}_{2i+2}, 01p^{n-2}_{2i+1}, 00p^{n-2}_{2i+1}, 00p^{n-2}_{2i})$$

   and

   $$(00p^{n-2}_{2i-1}, 01p^{n-2}_{2i-1}, 01p^{n-2}_{2i-2}, 00p^{n-2}_{2i-2}, \ldots 00p^{n-2}_{2k'-1}, 01p^{n-2}_{2k'-1}, 01p^{n-2}_{2k'-2}, 00p^{n-2}_{2k'-2},$$

   $$\ldots 00p^{n-2}_5, 01p^{n-2}_5, 01p^{n-2}_4, 00p^{n-2}_4)$$

   for $m \geq k \geq i+1$ and $i \geq k' \geq 2$.
   See Figure 3.22.
   $\mathcal{T}_n$ can be rewritten as the concatenation of

   $$(00p^{n-2}_{2k}, 01p^{n-2}_{2k}, 01p^{n-2}_{2k-1}, 00p^{n-3}_{2k-1}) \text{ for every } k = m, \ldots i+1$$

   with

   $$(00p^{n-2}_{2i})$$

and

$$(00p_{2k'-1}^{n-2}, 01p_{2k'-1}^{n-2}, 01p_{2k'-2}^{n-2}, 00p_{2k'-2}^{n-3}) \text{ for every } k' = i, \ldots 2.$$

Note that the length of $\mathcal{T}_n$ is $2(2m - 2i + 2i - 1 - 3) = 4m - 8$.
We will finally concatenate $\mathcal{T}_n$ with $(00p_3^{n-2}, 01p_3^{n-2}, 01p_2^{n-2}, 01p_1^{n-2}, 00p_1^{n-2}, 00p_2^{n-2})$ followed by $L_{10}^{\prime R}$ and the vertices of $0010\,\mathcal{P}_{n-4}$.

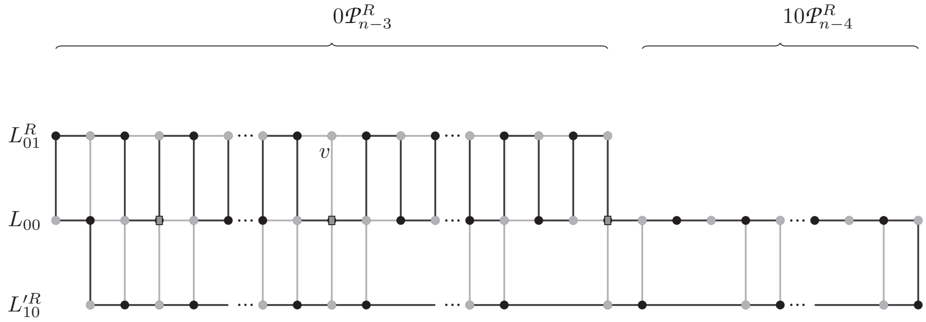


Figure 3.22: H. cycle in $\Lambda_n - v$ for $v \in L_{01}^R - 01p_2^{n-2}$ and $n \equiv 1(mod\,3)$

Let $v \in L_{00} \setminus \{00p_1^{n-2} = 000p_{|V(\Gamma_{n-3})|}^{n-3}, 0010\,\mathcal{P}_{n-4}^R\}$ Thus $v = 00p_{2i+1}^{n-2}$ for some $1 \leq i \leq m$. We will construct $\mathcal{T}_n$, from $00p_{|V(\Gamma_{n-3})|}^{n-2} = 000p_1^{n-3}$ to $01p_3^{n-2} = 010p_{|V(\Gamma_{n-3})|-2}^{n-3}$ concatenating

$$(00p_{2m}^{n-2}, 01p_{2m}^{n-2}, 01p_{2m-1}^{n-2}, 00p_{2m-1}^{n-2}, \ldots 00p_{2k}^{n-2}, 01p_{2k}^{n-2}, 01p_{2k-1}^{n-2}, 00p_{2k-1}^{n-2},$$

$$\ldots 00p_{2i+4}^{n-2}, 01p_{2i+4}^{n-2}, 01p_{2i+3}^{n-2}, 00p_{2i+3}^{n-2}, 00p_{2i+2}^{n-2}, 01p_{2i+2}^{n-2}, 01p_{2i+1}^{n-2})$$

and

$$(01p_{2i}^{n-2}, 00p_{2i}^{n-2}, 00p_{2i-1}^{n-2}, 01p_{2i-1}^{n-2}, 01p_{2i-2}^{n-2}, 00p_{2i-2}^{n-2}, \ldots 00p_{2k'-1}^{n-2}, 01p_{2k'-1}^{n-2},$$

$$01p_{2k'-2}^{n-2}, 00p_{2k'-2}^{n-2}, \ldots 00p_5^{n-2}, 01p_5^{n-2}, 01p_4^{n-2}, 00p_4^{n-2}, 00p_3^{n-2}, 01p_3^{n-2})$$

for $m \leq k \leq i+2$ and $i \leq k' \leq 3$.
See figure 3.23.
We can restate $\mathcal{T}_n$ as the concatenation of

$$(00p_{2k}^{n-2}, 01p_{2k}^{n-2}, 01p_{2k-1}^{n-2}, 00p_{2k-1}^{n-2}) \text{ for every } k = m, \ldots i+2$$

with

$$(00p_{2i+2}^{n-2}, 01p_{2i+2}^{n-2}, 01p_{2i+1}^{n-2}, 01p_{2i}^{n-2}, 00p_{2i}^{n-2})$$

and

$$(00p_{2k'-1}^{n-2}, 01p_{2k'-1}^{n-2}, 01p_{2k'-2}^{n-2}, 00p_{2k'-2}^{n-2} \text{ for every } k' = i, \cdots 3$$

with

$$(00p_3^{n-2}, 01p_3^{n-2}).$$

$\mathcal{T}_n$ has length $2(2m-2i-2)+4+2(2i-1-3)+2 = 4m-6$. The cycle is formed by $\mathcal{T}_n$ linked to $(01p_2^{n-2}, 01p_1^{n-2}, 00p_1^{n-2}, 01p_2^{n-2})$ and to L'^R_{10} and the vertices of $0010\,\mathcal{P}_{n-4}$.

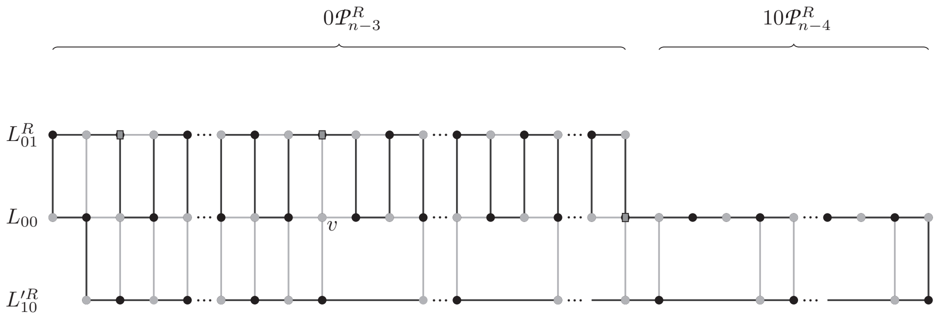


Figure 3.23: H. cycle in $\Lambda_n - v$ for $v \in L_{00} \setminus \{00p_1^{n-2}, 0010\,\mathcal{P}^R_{n-4}\}$ and $n \equiv 1(mod\,3)$

2. $v \in \{01p_2^{n-2},\ 00p_1^{n-2}\}$
   Recall that $L^R_{01} = 010\,\mathcal{P}^R_{n-3}$ where
   $0\,\mathcal{P}^R_{n-3} = 010\,\mathcal{P}_{n-5}, 00\,\mathcal{P}_{n-4} = 010\,\mathcal{P}_{n-5}, 000\,\mathcal{P}^R_{n-5}, 0010\,\mathcal{P}^R_{n-6}$.
   Let $2|V(\Gamma_{n-5})| = 2m'$ and notice that $\mathcal{P}^R_{n-6}$ has an even length.
   Let $v = 01p_2^{n-2}$, then let $u = 01p_1^{n-2} \in 010\,\mathcal{P}_{n-5}$. Thus u is adjacent to $u' = 01p^{n-2}_{2|V(\Gamma_{n-5})|} \in 000\,\mathcal{P}^R_{n-5}$.
   Let $\mathcal{T}_n$, the path from $00p^{n-2}_{|V(\Gamma_{n-3})|} = 000p_1^{n-3}$ to $00p^{n-2}_{2|V(\Gamma_{n-5})|+1}$ as the concatenation of

$$(00p^{n-2}_{2m}, 01p^{n-2}_{2m}, 01p^{n-2}_{2m-1}, 00p^{n-2}_{2m-1}, \ldots 00p^{n-2}_{2k}, 01p^{n-2}_{2k}, 01p^{n-2}_{2k-1}, 00p^{n-2}_{2k-1},$$

$$\ldots 00p^{n-2}_{2m'+2}, 01p^{n-2}_{2m'+2}, 01p^{n-2}_{2m'+1}, 00p^{n-2}_{2m'+1})$$

   for $m'+1 \le k \le m$.
   See Figure 3.24.
   $\mathcal{T}_n$ can be rewritten as the concatenation of

$$(00p^{n-2}_{2k}, 01p^{n-2}_{2k}, 01p^{n-2}_{2k-1}, 00p^{n-2}_{2k-1}) \text{ for every } k = m'+1, \ldots m.$$

   The Hamiltonian cycle in $\Lambda_n - 01p_2^{n-2}$ is constructed linking u to $00p_1^{n-2}$, followed by L'^R_{10}, $10\,\mathcal{P}_{n-4}, \mathcal{T}_n$, and by the vertices $(00p^{n-2}_{2|V(\Gamma_{n-5})|}, 00p^{n-2}_{2|V(\Gamma_{n-5})|-1}, \ldots 00p_3^{n-2}, 01p_3^{n-2}, 01p_4^{n-2}, \ldots 01p^{n-2}_{2|V(\Gamma_{n-5})|} = u')$.

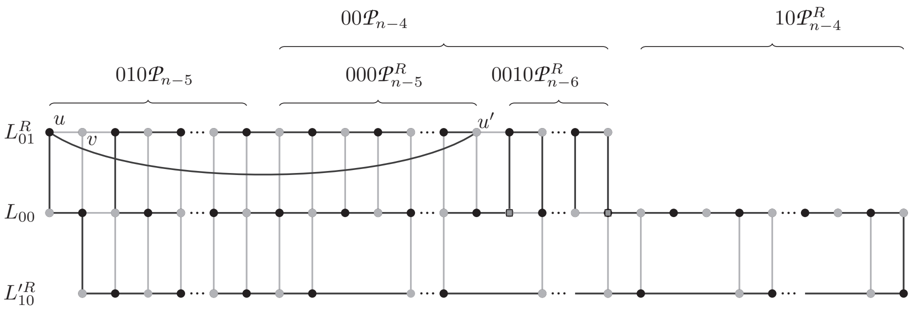


Figure 3.24: H. cycle in $\Lambda_n - 01p_2^{n-2}$ for $n \equiv 1(mod\,3)$

For $v = 00p_1^{n-2}$, let $u = 01p_1^{n-2}$. Thus $u' = 01p^{n-2}_{|V(\Gamma_{n-5})|}$. We will consider $\mathcal{T}_n$ as in the previous case. Hence, $\mathcal{T}_n$ is the concatenation of

$$(00p^{n-2}_{2k}, 01p^{n-2}_{2k}, 01p^{n-2}_{2k-1}, 00p^{n-2}_{2k-1}) \text{ for every } k = m' + 1, \ldots m.$$

See figure 3.25.
The Hamiltonian cycle in $\Lambda_n - 00p_1^{n-2}$ consists in $(u = 01p_1^{n-2}, 01p_2^{n-2}, \ldots 01p^{n-2}_{|V(\Gamma_{n-5})|-1}, 00p^{n-2}_{|V(\Gamma_{n-5})|-1}, 00p^{n-2}_{|V(\Gamma_{n-5})|-2}, 00p_3^{n-2}, 00p_2^{n-2})$, linked to L'^R_{10} and to $\mathcal{P}^R_{n-4}$. Then, $\mathcal{T}_n$ followed by $00p^{n-2}_{|V(\Gamma_{n-5})|}, 01p^{n-2}_{|V(\Gamma_{n-5})|} = u'$.

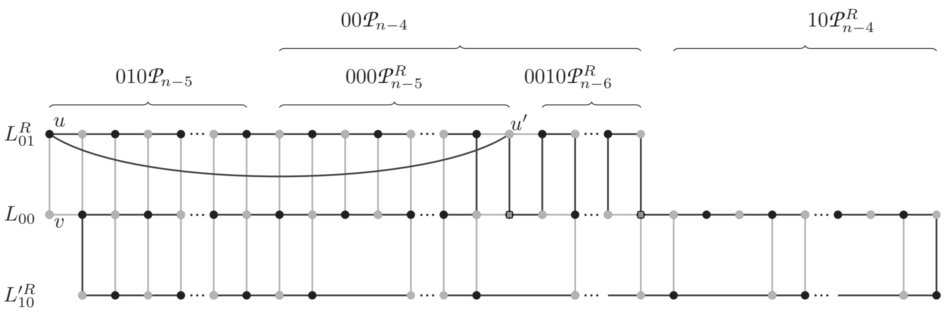


Figure 3.25: H. cycle in $\Lambda_n - 00p_1^{n-2}$ for $n \equiv 1(mod\,3)$

3. $v \in L'^R_{10} \setminus \{1010\,\mathcal{P}^R_{n-4},\ 10p'^{n-2}_{|V(\Gamma_{n-3})|}\}$
   Let $v = 10p'^{n-2}_{2j}$. We refer to Figure 3.26, which represents the cycle in $\mathcal{T}_n - v$ for this case which is similar to the cycle of this case for the Fibonacci cubes.

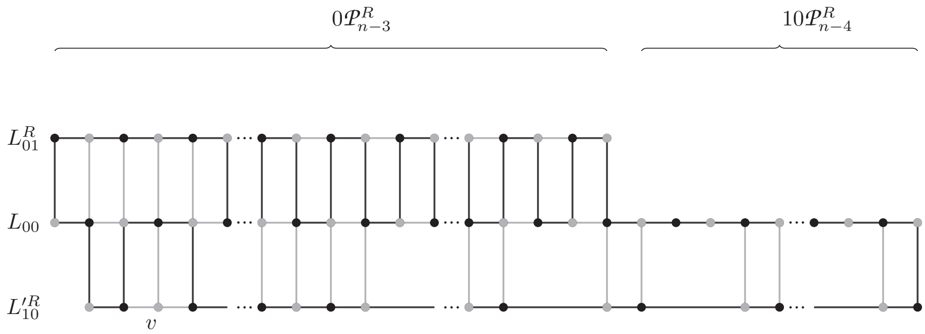


Figure 3.26: H. cycle in $\Lambda_n - v$ for $v \in L'^R_{10} \setminus 1010\mathcal{P}^R_{n-4}$ and $n \equiv 1(mod\, 3)$

4. $v \in \{1010\,\mathcal{P}^R_{n-4}, 00p^{n-2}_{|V(\Gamma_{n-3})|+1}, 00p^{n-2}_{|V(\Gamma_{n-2})|}, 10p'^{n-2}_{|V(\Gamma_{n-3})|}\}$
   Let $v = 10p'^{n-2}_j \in 1010\,\mathcal{P}^R_{n-4}$. Let $u = 10p'^{n-2}_{j+1} \in 1010\,\mathcal{P}^R_{n-4}$. Thus u is adjacent to $u' = 10p'^{n-2}_{2|V(\Gamma_{n-3})|-j}$ in $00\,\mathcal{P}_{n-4}$ since $L'^R_{10} = 0\,\mathcal{P}^R_{n-3}, 10\,\mathcal{P}^R_{n-4} = 010\,\mathcal{P}_{n-5}, 00\,\mathcal{P}_{n-4}, 10\,\mathcal{P}^R_{n-4}$. Again, Figure 3.27 represents the Hamiltonian cycle that is constructed in a similar way as in the Fibonacci cubes.
   If $v = 00p^{n-2}_{|V(\Gamma_{n-3})|+1}$ or $v = 10p'^{n-2}_{|V(\Gamma_{n-3})|}$, then let let $u = 00p^{n-2}_{|V(\Gamma_{n-3})|+2}$ in $10\,\mathcal{P}^R_{n-4}$ for both cases. See figures 3.28 and 3.29 respectively.
   If $v = 00p^{n-2}_{|V(\Gamma_{n-2})|}$ then let $u = 10p'^{n-2}_{|V(\Gamma_{n-2})|}$ and refer to Figure 3.30 for this case.

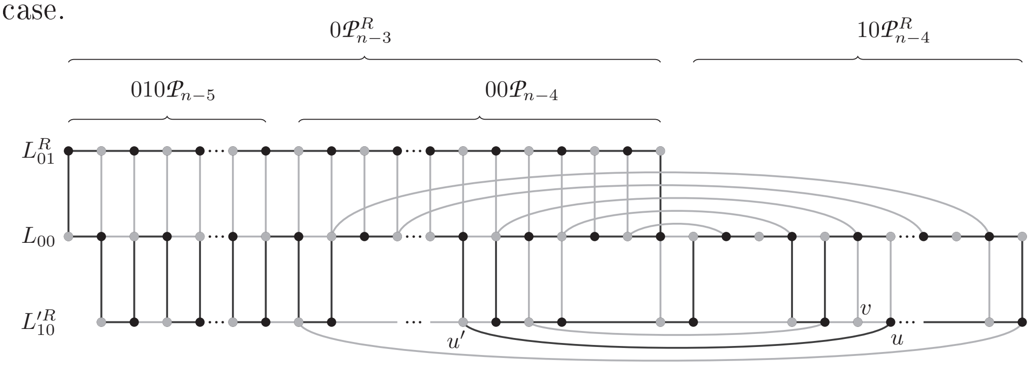


Figure 3.27: H. cycle in $\Lambda_n - v$ for $v \in 1010\,\mathcal{P}^R_{n-4}$ and $n \equiv 1(mod\, 3)$

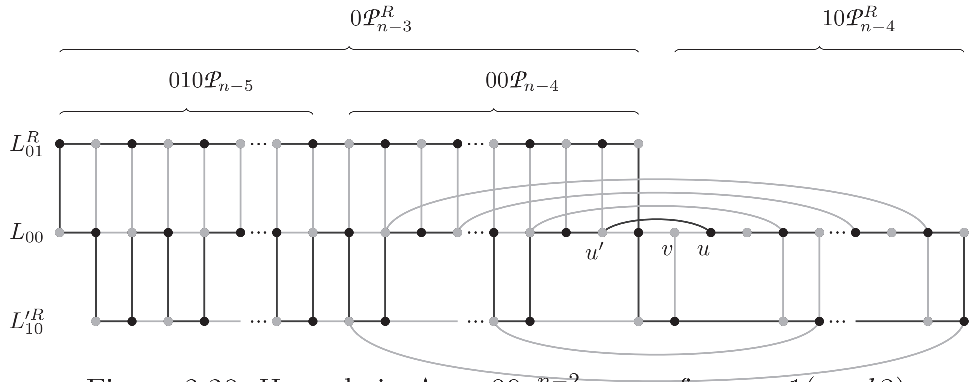


Figure 3.28: H. cycle in $\Lambda_n - 00p^{n-2}_{|V(\Gamma_{n-3})|+1}$ for $n \equiv 1(mod\, 3)$

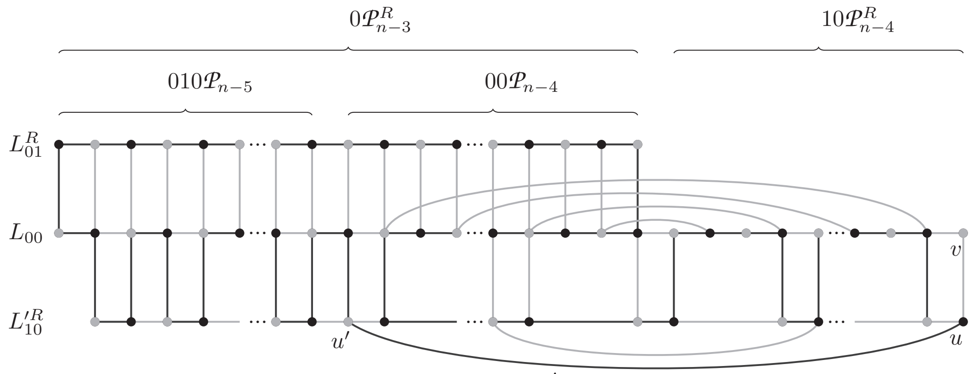


Figure 3.29: H. cycle in $\Lambda_n - 10p'^{n-2}_{|V(\Gamma_{n-3})|}$ for $n \equiv 1(mod\, 3)$

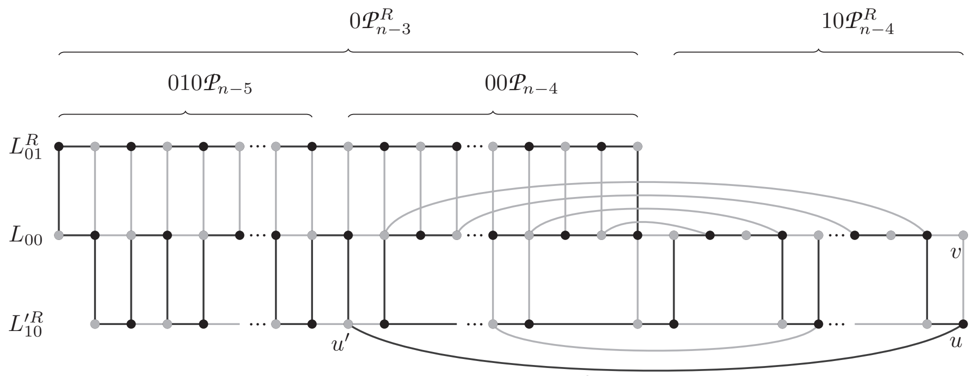


Figure 3.30: H. cycle in $\Lambda_n - 00p^{n-2}_{|V(\Gamma_{n-2})|}$ for $n \equiv 1(mod\, 3)$

5. $v \in 0010\,\mathcal{P}^R_{n-4} \setminus \{00p^{n-2}_{|V(\Gamma_{n-3})|+1}, 00p^{n-2}_{|V(\Gamma_{n-2})|}\}$
   If $v = 00p'^{n-2}_j$, let $w = 00p^{n-2}_{j-1}$ and $u = 00p^{n-2}j + 1^{00}$, both in $0010\,\mathcal{P}^R_{n-4}$. Therefore, there exist $w' = 00p^{n-2}_{2|V(\Gamma_{n-3})|-(j-2)}$ and $u' = 00p^{n-2}_{2|V(\Gamma_{n-3})|-j}$ which are adjacent to w and u respectively.
   Figure 3.31 shows the Hamiltonian cycle in $\Lambda_n - v$.

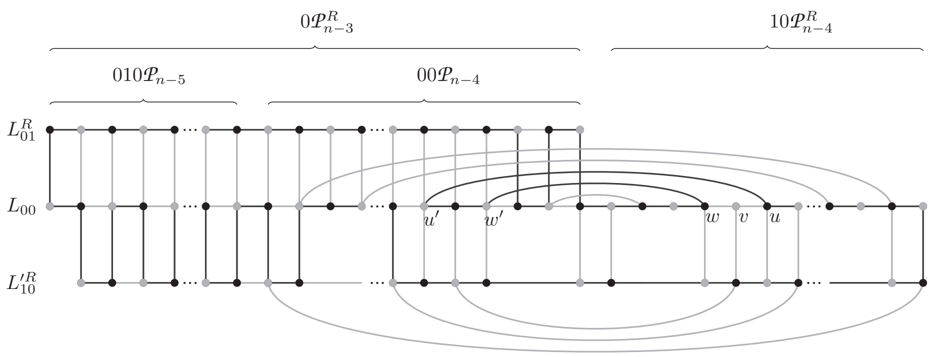


Figure 3.31: H. cycle in $\Lambda_n - v$ for $v \in 0010\,\mathcal{P}^R_{n-4} \setminus \{00p^{n-2}_{|V(\Gamma_{n-3})|+1}, 00p^{n-2}_{|V(\Gamma_{n-2})|}\}$ and $n \equiv 1(mod 3)$

Case $n \equiv 2\,(mod\,3)$: Because the similarity between the Hamiltonian cycles of the Lucas cubes for $n \equiv 1\,(mod\,3)$, we will not describe the Hamiltonian cycles for this case, and will refer to the correspondent figures.

1. $v \in L_{01}^R \cup L_{00} \setminus 0010\,\mathcal{P}_{n-4}^R$
   See Figure 3.32 for $v \in L_{01}^R$ and Figure 3.33 for $v \in L_{00} \setminus 0010\,\mathcal{P}_{n-4}^R$.

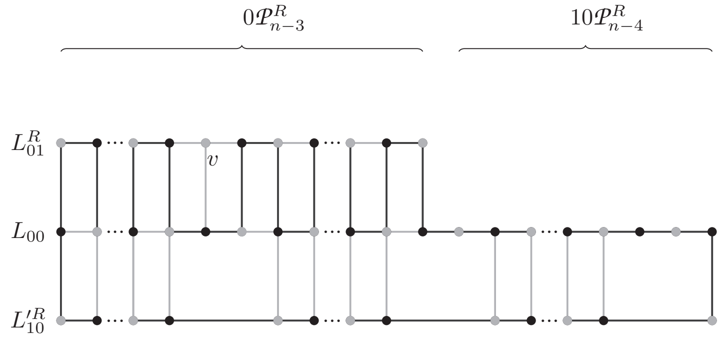


Figure 3.32: H. cycle in $\Lambda_n - v$ for $v \in L_{01}^R$ and $n \equiv 2\,(mod\,3)$

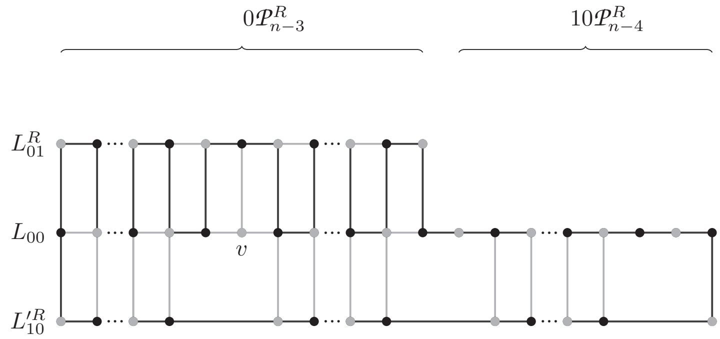


Figure 3.33: H. cycle in $\Lambda_n - v$ for $v \in L_{00} \setminus 0010\,\mathcal{P}_{n-4}^R$ and $n \equiv 2\,(mod\,3)$

2. $v \in L_{10}'^R \setminus 1010\,\mathcal{P}_{n-4}^R$
   See Figure 3.34.

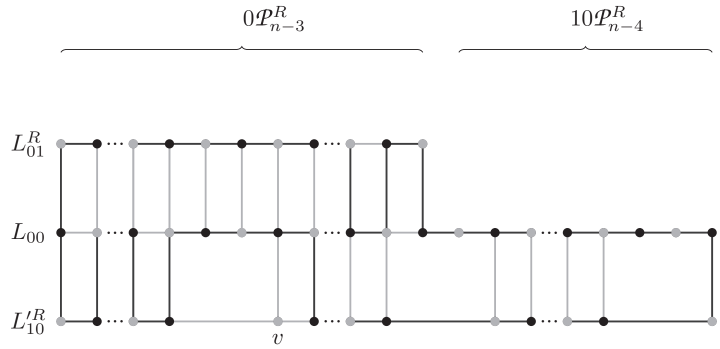


Figure 3.34: H. cycle in $\Lambda_n - v$ for $v \in L_{10}'^R \setminus 1010\,\mathcal{P}_{n-4}^R$ and $n \equiv 2\,(mod\,3)$

3. $v = 00p^{n-2}_{|V(\Gamma_{n-3})|+1}$
   Refer to figure 3.35 which depicts the cycle for this case, then let $u = 00p^{n-2}_{|V(\Gamma_{n-3}|+2}$. Thus $u' = 00p^{n-2}l_{|V(\Gamma_{n-3})|-1}$ is adjacent to u since L_{00} is formed by $00\,\mathcal{P}_{n-2}$, where $\mathcal{P}_{n-2} = 0\,\mathcal{P}^R_{n-3},\ 10\,\mathcal{P}^R_{n-4} = 010\,\mathcal{P}_{n-5}, 00\,\mathcal{P}_{n-4},\ 10\,\mathcal{P}^R_{n-4}$.

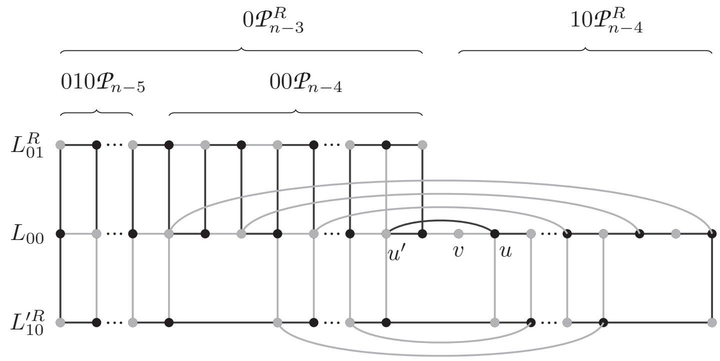


Figure 3.35: H. cycle in $\Lambda_n - 00p^{n-2}_{|V(\Gamma_{n-3})|+1}$ for $n \equiv 2\,(mod\,3)$

4. $v \in 0010\,\mathcal{P}^R_{n-4} - 00p^{n-2}_{|V(\Gamma_{n-3})|+1}$
   Let $v = 00p^{n-2}_j$ thus $|V(\Gamma_{n-3})| + 3 < j < |V(\Gamma_{n-2})| - 1$.
   Let $u = 00p^{n-2}_{j+1}$ and $w = 00p^{n-2}_{j-1}$, both in $0010\,\mathcal{P}^R_{n-4}$.
   Therefore $j + 1 = |V(\Gamma_{n-3})| + \delta$ and $j - 1 = |V(\Gamma_{n-3})| + \delta - 2$ for $\delta > 0$.
   Hence, there exist $u' = 00p^{n-2}_{2|V(\Gamma_{n-3})|-j}$ and $w' = 00p^{n-2}_{2|V(\Gamma_{n-3})|-j+2}$ which are adjacent to u and w respectively since u' is the element $|V(\Gamma_{n-3})| - \delta + 1$ and w' is the element $(|V(\Gamma_{n-3})| - \delta + 1) + 2$ of L_{00}.
   In figure 3.36, we show the cycle corresponding to this case.

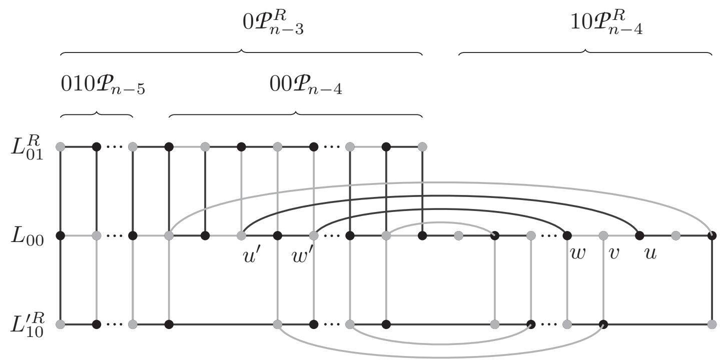


Figure 3.36: H. cycle in $\Lambda_n - v$ for $v \in 0010\,\mathcal{P}^R_{n-4} - 00p^{n-2}_{|V(\Gamma_{n-3})|+1}$ and $n \equiv 2(mod\,3)$

5. $v \in 1010\,\mathcal{P}^R_{n-4}$
   Let $v = 10p'^{n-2}_j$. $|V(\Gamma_{n-3})| < j \leq |V(\Gamma_{n-2})|$. The case $j = |V(\Gamma_{n-2})|$ will be considered at the end of this item.

Assume then, $|V(\Gamma_{n-3})| < j < |V(\Gamma_{n-2})|$.
Let $u = 10p'^{n-2}_{j+1}$ in $1010\,\mathcal{P}^R_{n-4}$. Therefore, $u' = 10p'^{n-2}_{2|V(\Gamma_{n-3})|-j}$ is adjacent to u. The cycle depicted in figure 3.37, is a Hamiltonian cycle in $\Lambda_n - v$.
When $v = 10p'^{n-2}_{|V(\Gamma_{n-2})|}$, then let $u = 10p'^{n-2}_{|V(\Gamma_{n-2})|}$. The Hamiltonian cycle is similar to the one described in this case. See figure 3.38.

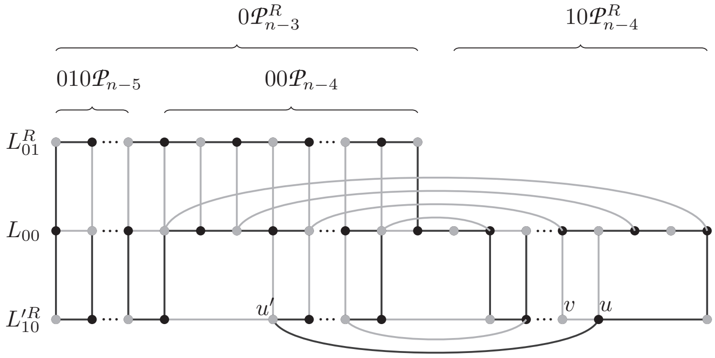


Figure 3.37: H. cycle in $\Lambda_n - v$ for $v \in 1010\,\mathcal{P}^R_{n-4}$ and $n \equiv 2(mod\,3)$

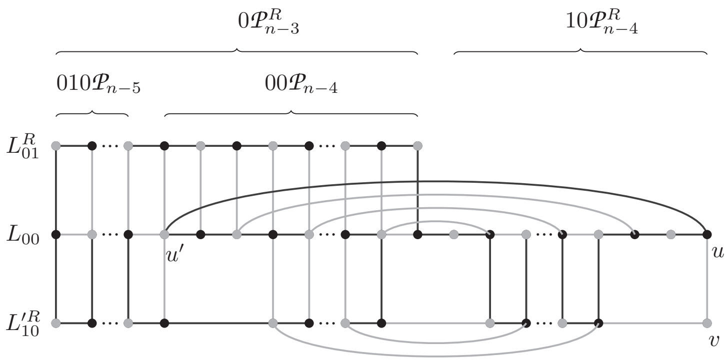


Figure 3.38: H. cycle in $\Lambda_n - 10p'^{n-2}_{|V(\Gamma_{n-2})|}$ for $n \equiv 2(mod\,3)$

□

Proposition 3.2.8. *For $n \geq 0$, $n \equiv 0\,(mod\,3)$, Λ_n has a path of length $|\Lambda_n| - 1$.*

Proof. By construction, the subgraph $\mathcal{H'}_n$, isomorphic to Λ_n is induced by the vertices of $L_0 = 0\,\mathcal{P}_{n-1} = 01(0\,\mathcal{P}_{n-3}), 00\,\mathcal{P}_{n-2}$ and by the vertices of $L'_{10} = 10\,\mathcal{P}_{n-2} \setminus \{\text{strings that end with } 1\} = 10\,\mathcal{P}_{n-3}\,0$. Notice that by Proposition 3.2.3, the last string of $\mathcal{P}_{n-2}$, namely $00p^{n-2}_{|V(\Gamma_{n-2})|}$ is a a $\mathcal{P}1_{n-2}$ and hence the string $10p'^{n-2}_{|V(\Gamma_{n-2})|}$ does not exist in $L'_{10} = 10\,\mathcal{P}_{n-3}\,0$.
We have already proved that every $10p'^{n-2}_i$ of L'_{10} is adjacent to $00p^{n-2}_i$ of L_{00} in $\mathcal{H'}_n$.

Therefore, the path $01(0\,\mathcal{P}_{n-3}), (00\,\mathcal{P}_{n-2} - 00p^{n-2}_{|V(\Gamma_{n-2})|}), 10\,\mathcal{P}^R_{n-3}\,0$ is a path of length $|\Lambda_n| - 1$. □

Theorem 3.2.9. *For $n \equiv 0\,(mod\,3)$; $n \geq 6$ and even l with $4 \leq l \leq |V(\Lambda_n)| - 2$, a cycle of length l can be embedded in Λ_n.*

Proof. By definition, the only vertices that can be removed in order to find a Hamiltonian cycle in $\Lambda_n \setminus \{v_1, v_2\}$ are those of $V^P(\Lambda_n)$. Also, $|V(\Gamma_{n-3})|$ is odd. Thus $|V(\Gamma_{n-3})| = 2m+1$. Let $v_1 = 01p^{n-2}l_{|V(\Gamma_{n-3})|}$ and $v_2 = 00p^{n-2}_{|V(\Gamma_{n-2})|}$. Consider $\mathcal{T}_n$ from $00p^{n-2}_2$ to $00p^{n-2}_{|V(\Gamma_{n-3})|}$ defined as the concatenation of

$$(00p^{n-2}_2, 01p^{n-2}_2, 01p^{n-2}_3, 00p^{n-2}_3, \ldots 00p^{n-2}_{2k}, 01p^{n-2}_{2k}, 01p^{n-2}_{2k+1}, 00p^{n-2}_{2k+1}, \ldots$$
$$00p^{n-2}_{2m}, 01p^{n-2}_{2m}, 01p^{n-2}_{2m+1}, 00p^{n-2}_{2m+1}).$$

$\mathcal{T}_n$ can be rewritten as the concatenation of

$$(00p^{n-2}_{2k}, 01p^{n-2}_{2k}, 01p^{n-2}_{2k+1}, 00p^{n-2}_{2k+1}) \text{ for every } k = 1, \ldots m.$$

Hence, the cycle $\mathscr{C}'$ shown in figure 3.39 is a Hamiltonian cycle in $\Lambda_n \setminus \{v_1, v_2\}$, which begins with $\mathcal{T}_n$ followed by L^R_{01} and by $10\,\mathcal{P}_{n-4}$.

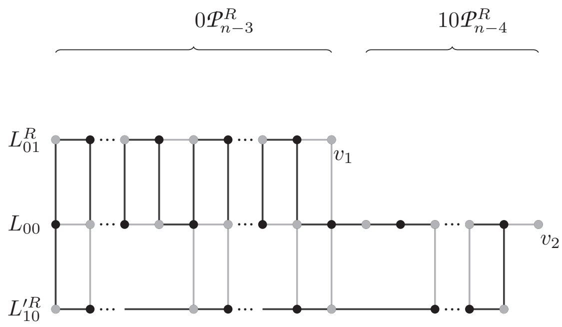


Figure 3.39: H. cycle in $\Lambda_n \setminus \{v_1, v_2\}$ for $n \equiv 0\,(mod\,3)$

Removing a pair of vertices of L^R_{01} from $\mathscr{C}'$, decreases the lenght of the cycle by two. Thus the cycles of length s with $|V(\Lambda_n)| - 2 \leq s \leq 2|V(\Gamma_{n-2})|$ are obtained. Cycles of length s' with $2|V(\Gamma_{n-2})| < s' \leq 4$ are constructed by removing two suitable vertices of the current cycle each time. □

A natural question that would be interesting to respond is for which pair of vertices v_1 and v_2 of $V^P(\Lambda_n), n \equiv 0\,(mod\,3)$ is always possible to have a Hamiltonian cycle in $\Lambda_n \setminus \{v_1, v_2\}$? In a partial response, we let $v_1 = 00p^{n-2}_{|V(\Gamma_{n-2})|}$ and $v_2 \in V^P(\Lambda_n)$ which seems to work properly. Unfortunately we still have not found a complete answer to this question.

Chapter 4

Eccentricity Sequences of the Fibonacci and Lucas cubes

The *eccentricity* of a vertex u, denoted $e_G(u)$ is the greatest distance between u and any other vertex v in the graph. When no confusion is possible we will shorten this notation to $e(u)$. We say that v *satisfies* the eccentricity of u when $d(u, v) = e(u)$. Clearly, not all the vertices of Γ_n or Λ_n have the same eccentricity as it happens in Q_n where there are no restrictions of consecutive 1's. For example, let us consider the vertex $u = (01010)$ that belongs to Γ_5. We can come back to figure 2.1 to see that $e(u) = 5$ since the vertex $v = (10101)$ has the gratest distance between u and any other vertex among all the vertices of Γ_5. In the other hand, we can see that the vertex $v = (10101)$ also satisfies the eccentricity of $u' = (00000)$ and $e(00000) = 3$.
Now, let us consider the same vertex $u = (01010)$ in Λ_5. The vertex $v = (10101)$ does not belong to Λ_5 because it begins and ends with 1. Thus, we can return to figure 2.4 to verify that the vertices $v_1 = (10100)$ and $v_2 = (00101)$ satisfy both, the eccentricity of u and that $e(u) = 4$. Also, the vertex that satisfy the eccentricity of $u' = (00000)$ in Γ_n does not belong to Λ_n. Therefore, the vertices $(10100), (10010), (01010), (01001)$ and (00101) satisfy all of them, the eccentricity of u' in Λ_n and $e(00000) = 2$.

The *radius* of a graph G, denoted $rad(G)$, is the minimum eccentricity among the vertices of G, while the *diameter* of G, denoted $diam(G)$ is the maximum eccentricity among the vertices of the graph.
The radius, $rad(\Gamma_n) = \left\lceil \frac{n}{2} \right\rceil$ and diameter, $diam(\Gamma_n) = n$ of the Fibonacci cubes are obtained in [MS02]. Similarly $rad(\Lambda_n) = \left\lfloor \frac{n}{2} \right\rfloor$ and $diam(\Lambda_n) = 2 \left\lfloor \frac{n}{2} \right\rfloor$ are determined in [MCS01].

We define the *eccentricity sequence* of G as the sequence $\{a_k\}_{k=0}^{diam(G)}$ of nonnegative integers, where a_k is the number of vertices of eccentricity k in G.

In the next table, we show the number of vertices of eccentricity k in Γ_n and in Λ_n for $n = 0$ to 10 which can be computed by hand.

n	0	1		2			3				4				
k	0	0	1	0	1	2	0	1	2	3	0	1	2	3	4
Γ :	1	0	2	0	1	2	0	0	3	2	0	0	1	5	2
Λ :	1	1	0	0	1	2	0	1	3	0	0	0	1	4	2

n	5						6						
k	0	1	2	3	4	5	0	1	2	3	4	5	6
Γ :	0	0	0	4	7	2	0	0	0	1	9	9	2
Λ :	0	0	1	5	5	0	0	0	0	1	9	6	2

n	7								8								
k	0	1	2	3	4	5	6	7	0	1	2	3	4	5	6	7	8
Γ :	0	0	0	0	5	16	11	2	0	0	0	0	1	14	25	13	2
Λ :	0	0	0	1	7	14	7	0	0	0	0	0	1	16	20	8	2

n	9										10										
k	0	1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6	7	8	9	10
Γ :	0	0	0	0	0	6	30	36	15	2	0	0	0	0	0	1	20	55	49	17	2
Λ :	0	0	0	0	1	9	30	27	9	0	0	0	0	0	0	1	25	50	35	10	2

Table 4.1: Number of vertices of eccentricity k in Γ_n and Λ_n.

4.1 Notation of Fibonacci cubes

Let $\mathcal{F}_n$ be the set of strings of Γ_n.
Let $\mathcal{F}_n^{od\,\cdot}$ be the set of strings of Γ_n that begin with an odd number of 0's,
$\mathcal{F}_n^{ev\,\cdot}$ the set of strings of Γ_n that begin with an even number (eventually null) of 0's,
$\mathcal{F}_n^{ev^*\cdot}$ the set of strings of Γ_n that begin with an even number, not null of 0's and
$\mathcal{F}_n^{\varnothing\,\cdot}$ the set of strings of Γ_n that do not begin with a 0.
We have thus $\mathcal{F}_n = \mathcal{F}_n^{od\,\cdot} \uplus \mathcal{F}_n^{ev\,\cdot} = \mathcal{F}^{od\,\cdot} \uplus \mathcal{F}^{ev^*\cdot} \uplus \mathcal{F}_n^{\varnothing\,\cdot}$, where $\uplus$ is the disjoint union of sets.
Let $\mathcal{F}_n^{\cdot\,od}$ be the set of strings in Γ_n that end with an odd number of 0's.
Similarly, we define $\mathcal{F}_n^{\cdot\,b}$ where $b \in \{ev, ev^*, \varnothing\}$.
Let $\mathcal{F}_n^{od\,od}$ be the set of strings in Γ_n that begin and end with an odd number of 0's.
In the same way, we define $\mathcal{F}_n^{ab}$ where $a, b \in \{od, ev, ev^*, \varnothing, \cdot\}$.
Note that $\mathcal{F}_n^{\cdot\,\cdot} = \mathcal{F}_n$. Let $\mathcal{F}_{n,k}$ the set of strings of Γ_n with eccentricity k.
For any $a, b \in \{od, ev, ev^*, \varnothing, \cdot\}$, let $\mathcal{F}_{n,k}^{a\,b} = \mathcal{F}_n^{a\,b} \cap \mathcal{F}_{n,k}$ and $f_{n,k}^{a\,b}$ be $|\mathcal{F}_{n,k}^{a\,b}|$.
We will denote by $f^{a\,b}$ the generating function

$$f^{a\,b}(x,y) = \sum_{n,k \geq 0} f_{n,k}^{a\,b}\, x^n y^k$$

4.2 Eccentricity of a vertex of Γ_n

In this section, we show that a vertex x in Γ_n can be written uniquely as the concatenation of particular strings. We give some results concerning the eccentricity of these substrings. These results lead us to compute $e(x)$ and to characterize the vertices y in Γ_n that satisfy $e(x)$. Finally, we determine the last character of the strings y at distance $e(x)$ (Corollary 4.2.9). This latter result will be very useful through this section.

Let us recall that Γ_n is an isometric subgraph of Q_n, i.e.:

Proposition 4.2.1. *The distance $d_{\Gamma_n}(a,b)$ between a and b in Γ_n is $d_{Q_n}(a,b)$, the number of positions in which the two strings a and b differ.*

Proof. Let $a = (a_1 a_2 \cdots a_n)$, $b = (b_1 b_2 \cdots b_n) \in \Gamma_n$ and let $z = (z_1 z_2 \cdots z_n) \in Q_n$ be defined as

$$z_i = \begin{cases} a_i \text{ if } a_i = b_i \\ 0 \text{ if } a_i \neq b_i, \end{cases}$$

Note first that z is a Fibonacci string. Indeed $z_i = z_{i+1} = 1$ would imply $a_i = a_{i+1} = 1$. Consider now a shortest path in Q_n from a to b, $s = (a = s_0, s_1, \cdots, z, \cdots, s_j = b)$, obtained by concatenation of a shortest path from a to z and a shortest path from z to b. It is easy to see that all the vertices of s belong to Γ_n as well thus s is also a path in Γ_n. Furthermore s is a shortest path in Γ_n because, as a subgraph, $d_{\Gamma_n}(a,b) \geq d_{Q_n}(a,b)$. □

We will thus shorten the notation $d_{\Gamma_n}(a,b)$ to $d(a,b)$ in this section. Let us denote by $x = (ab)$ the concatenation of two strings a and b.

Proposition 4.2.2. *Let $z \in \mathcal{F}_n$ such that $z = (xy)$ with $x \in \mathcal{F}_{n_1}$, $y \in \mathcal{F}_{n_2}$ and $n_1 + n_2 = n$, then*

$$e(z) \leq e(x) + e(y)$$

Proof. Let $c \in \mathcal{F}_n$ such that $d(z,c) = e(z)$. Then $c = (ab)$ with $a \in \mathcal{F}_{n_1}$ and $b \in \mathcal{F}_{n_2}$.
By the definition of eccentricity, $d(x,a) \leq e(x)$ and $d(y,b) \leq e(y)$.
Then $e(xy) = d(xy, ab) = d(x,a) + d(y,b) \leq e(x) + e(y)$. □

Proposition 4.2.3. *Let $z \in \mathcal{F}_n$ such that $z = (xy)$ with $x \in \mathcal{F}_{n_1}$, $y \in \mathcal{F}_{n_2}$ and $n_1 + n_2 = n$. If $e(xy) = e(x) + e(y)$, then any string $u \in \mathcal{F}_n$ that satisfies $d(u,z) = e(z)$, can be decomposed in $u = (vw)$ with $v \in \mathcal{F}_{n_1}$, $w \in \mathcal{F}_{n_2}$ such that $d(v,x) = e(x)$ and $d(w,y) = e(y)$.*

Proof. Consider a string $u \in \mathcal{F}_n$ that verifies the eccentricity of z, then $u = (vw)$ with $v \in \mathcal{F}_{n_1},\ w \in \mathcal{F}_{n_2}$ and $e(xy) = d(vw, xy) = d(v, x) + d(w, y)$. But $d(v, x) \leq e(x)$ and $d(w, y) \leq e(y)$.
Thus, we must have $d(v, x) = e(x)$ and $d(w, y) = e(y)$. □

Because a Fibonacci string of length n is a binary string with no consecutive 1's, the next proposition is clear

Proposition 4.2.4. *The strings of $\mathcal{F}_n$ with $n \geq 0$, can be uniquely written as*

$$x = 0^{l_0}10^{l_1}10^{l_2}\cdots 10^{l_p}$$

with $p \geq 0,\ l_0, l_p \geq 0$ and $l_1, \cdots, l_{p-1} \geq 1$.

Proposition 4.2.5. *For $l \geq 0$,*

$$e(0^{l+2}) = e(0^l) + 1$$

Proof. 0^{l+2} is the concatenation of 0^l and 0^2, then by Proposition 4.2.2, $e(0^{l+2}) \leq e(0^l) + 1$. Furthermore, if $y \in \mathcal{F}_l$ is a string that satisfy the eccentricity of 0^l, then the string $(y01)$ belongs to $\mathcal{F}_{l+2}$ and is at distance $e(0^l) + 1$ of 0^{l+2}. □

Proposition 4.2.6. *For $l \geq 0$,*

$$e(10^l) = e(0^l) + 1$$

Proof. Again, by Proposition 4.2.2, $e(10^l) \leq 1 + e(0^l)$. Assume that $y \in \mathcal{F}_l$ is a string that satisfy the eccentricity of 0^l, then $(0y) \in \mathcal{F}_{l+1}$ is at distance $e(0^l) + 1$ of 10^l. □

We associate next, to every string $0^l \in \mathcal{F}_l$, a set of strings $W(0^l)$ of $\mathcal{F}_l$ in the following way:

$$W(0^l) = \begin{cases} \{1(01)^{\lfloor \frac{l-1}{2} \rfloor}\} \text{ if } l \text{ is odd} \\ \{(10)^a(01)^b / 2a + 2b = l, a, b \geq 0\} \text{ if } l \text{ is even} \end{cases}$$

Proposition 4.2.7. *For $l \geq 0$,*

$$e(0^l) = \lfloor \frac{l+1}{2} \rfloor$$

Furthermore, the strings of $W(0^l)$ are the only strings that satisfy the eccentricity of 0^l.

Proof. Notice that the property is true when $l \leq 1$. When $l = 2$, the property is also true and $W(00) = \{10, 01\}$. Assume by induction that the proposition is true for l.
$e(0^{l+2}) = e(0^l) + 1$ by proposition 4.2.5. Furthermore, by Proposition 4.2.3, a string of $\mathcal{F}_{l+2}$ that satisfies $e(0^{l+2})$ must be $(w01)$ or $(w10)$ with $w \in W(0^l)$. Let W be the set of strings of $\mathcal{F}_{l+2}$ that satisfy the eccentricity of 0^{l+2}.

(i) If l is odd, then $W(0^l) = \{1(01)^{\lfloor \frac{l-1}{2} \rfloor}\}$. Then w ends with 1 and only $(w01)$ belongs to $\mathcal{F}_{l+2}$, thus $W = \{1(01)^{\lfloor \frac{l+1}{2} \rfloor}\} = W(0^{l+2})$.

(ii) If l is even, then $W(0^l) = \{(10)^a(01)^b \ /2a + 2b = l;\ a, b \geq 0\}$.
If $b = 0$, then both $(10)^{\frac{l}{2}}01$ and $(10)^{\frac{l}{2}}10$ satisfy the eccentricity of 0^{l+2}. If $b \neq 0$, then only the string $(10)^a(01)^b(01) = (10)^a(01)^{b+1}$ verifies the eccentricity of 0^{l+2}.
Then $W = \{(10)^{\frac{l}{2}}01,\ (10)^{\frac{l}{2}}10\} \cup \{(10)^a(01)^{b+1} \ /2a + 2b = l;\ a \geq 0, b \geq 1\} = \{(10)^a(01)^{b'} \ /2a + 2b' = l + 2;\ a, b' \geq 0\} = W(0^{l+2})$.

□

Theorem 4.2.8 ([CM12]). *For every $x = 0^{l_0}10^{l_1}10^{l_2} \cdots 10^{l_p}$ in $\mathcal{F}_n$, with $p,\ l_0,\ l_p \geq 0;\ l_1, \cdots, l_{p-1} \geq 1$,*

$$e(x) = p + \sum_{i=0}^{p} \lfloor \frac{l_i + 1}{2} \rfloor$$

Furthermore, the strings that verify the eccentricity of x are the strings

$$y = w_0 0 w_1 0 \cdots w_{p-1} 0 w_p$$

where $w_i \in W(0^{l_i})$ for $i = 0, 1, \cdots, p$.

Proof. Let $x = 0^{l_0}10^{l_1}10^{l_2} \cdots 10^{l_p} \in \mathcal{F}_n$, with $p,\ l_0,\ l_p \geq 0;\ l_1, \cdots, l_{p-1} \geq 1$.
Then, from Proposition 4.2.2, $e(x) \leq e(0^{l_0}) + e(10^{l_1}) + e(10^{l_2}) + \cdots + e(10^{l_p})$.
Combining Propositions 4.2.6 and 4.2.7, $e(x) \leq \lfloor \frac{l_0+1}{2} \rfloor + \sum_{i=1}^{p}(\lfloor \frac{l_i+1}{2} \rfloor + 1)$.
Hence $e(x) \leq p + \sum_{i=0}^{p} \lfloor \frac{l_i+1}{2} \rfloor$.
Furthermore, any string $y = w_0 0 w_1 0 \cdots w_{p-1} 0 w_p$ with $w_i \in W(0^{l_i})$ satisfies $d(x, y) = p + \sum_{i=0}^{p} \lfloor \frac{l_i+1}{2} \rfloor$, then we have the equality for the eccentricity.
Given that the strings of $W(0^{l_i})$ are the only ones that verify the eccentricity of 0^{l_i}, by Proposition 4.2.3, the only strings $z \in \mathcal{F}_n$ that satisfy $d(x, z) = e(x)$ are those of the form of y. □

We will use frequently the following consequence:

Corollary 4.2.9. *For every $x = 0^{l_0}10^{l_1}10^{l_2} \cdots 10^{l_p} \in \mathcal{F}_n$, with $p,\ l_0,\ l_p \geq 0;\ l_1, \cdots, l_{p-1} \geq 1,\ n \geq 1$, the following are true:*

(i) *if* l_p *is an odd number and* $y \in \mathcal{F}_n$ *satisfies the eccentricity of* x, *then* $y = (y'1)$ *with* $y' \in \mathcal{F}_{n-1}$,

(ii) *if* l_p *is a not null even number, then there exist* y', $y'' \in \mathcal{F}_{n-1}$, *such that* $y = (y'0)$ *and* $y = (y''1)$, *both satisfy* $e(x)$,

(iii) *if* $l_p = 0$ *and* $y \in \mathcal{F}_n$ *satisfy the eccentricity of* x, *then* $y = (y'0)$ *with* $y' \in \mathcal{F}_{n-1}$.

Proof. Consider $y \in \mathcal{F}_n$ such that $d(x, y) = e(x)$.

(i) Since l_p is odd, the only string of $W(0^{l_p})$ is $1(01)^{\lfloor \frac{l_i-1}{2} \rfloor}$. Thus, $y = (y'1)$.

(ii) Because l_p is a not null even number, $W(0^{l_p}) = \{(10)^a(01)^b/2a + 2b = l_p,\ a, b \geq 0\}$. When $b = 0$ then $a \geq 1$ and y takes the form $y = (y'0)$. When $b \geq 1$ then $y = (y''1)$. The two cases are possible since l_p is not null.

(iii) Given that $l_p = 0$, it follows from Theorem 4.2.8 that $y = (y'0)$.

□

Notice that if we consider the beginning of a word $x = 0^{l_0}10^{l_1}10^{l_2} \cdots 10^{l_p} \in \mathcal{F}_n$ rather than the end, then the symmetrical of Corollary 4.2.9 occurs. In this case $(i), (ii)$ and (iii) will be satisfied according to the parity of l_0.

4.3 Eccentricity sequence of Fibonacci cubes

Considering two subsets, namely, $\mathcal{F}_{n,k}^{\cdot\, od}$ and $\mathcal{F}_{n,k}^{\cdot\, ev}$, we will compute $f(x, y)$, the generating function of the eccentricity sequence of the Fibonacci cube's strings. As a corollary, the value of $f_{n,k}$ is also determined.

Proposition 4.3.1. *For* $n \geq 1$, $k \geq 1$,

$$f_{n,k}^{\cdot\, od} = f_{n-1,k-1}^{\cdot\, ev}$$

Proof. Let $x = 0^{l_0}10^{l_1}10^{l_2} \cdots 10^{l_p} \in \mathcal{F}_{n,k}^{\cdot\, od}$, thus p, $l_0 \geq 0$; $l_1, \cdots, l_{p-1}$, $l_p \geq 1$; $n \geq 1$, $k \geq 1$ and assume that l_p is an odd number. Notice that $l_p - 1$ is a possibly null even number. Then $x = (\theta(x)0)$ with $\theta(x) \in \mathcal{F}_{n-1}^{\cdot\, ev}$ such that

$$\theta(x) = \begin{cases} 0^{l_0}10^{l_1} \cdots 10^{l_p-1} \text{ if } l_p \geq 3 \\ 0^{l_0}10^{l_1} \cdots 10^{l_{p-1}}1 \text{ if } l_p = 1. \end{cases}$$

We have $e(x) \leq e(\theta(x)) + 1$. Furthermore, by Corollary 4.2.9, (ii) and (iii), there exists a vertex $y = (y'0)$ with $d(y, \theta(x)) = e(\theta(x))$. Since $d((y'01), x) = e(\theta(x)) + 1$, we have $e(x) = e(\theta(x)) + 1$, and θ is a 1 to 1 mapping between $\mathcal{F}_{n,k}^{\cdot\, od}$ and $\mathcal{F}_{n-1,k-1}^{\cdot\, ev}$. □

Proposition 4.3.2. *For $n \geq 3,\ k \geq 2$,*

$$f^{\cdot\, ev}_{n,k} = f^{\cdot\, ev}_{n-2,k-1} + f^{\cdot\, ev}_{n-2,k-2} + f^{\cdot\, ev}_{n-3,k-2}.$$

Proof. Let $x = 0^{l_0}10^{l_1}10^{l_2}\cdots 10^{l_p} \in \mathcal{F}^{\cdot\, ev}_{n,k}$, hence $p,\ l_0,\ l_p \geq 0;\ l_1, \cdots, l_{p-1} \geq 1;\ n \geq 3,\ k \geq 2$. As l_p is an even number, we will distinguish two cases:

(i) If $l_p \geq 2$, then $x = (x'00)$ with $x' \in \mathcal{F}^{\cdot\, ev}_{n-2}$. Furthermore, by theorem 4.2.8, $e(x') = e(x) - 1 = k - 1$ thus $x' \in \mathcal{F}^{\cdot\, ev}_{n-2,k-1}$.

(ii) If $l_p = 0$ then let us consider l_{p-1}.
If l_{p-1} is odd, then $x = (x'1)$ with $x' \in \mathcal{F}^{\cdot\, od}_{n-1}$. If y satisfies $e(x')$, then $d((y0), x) = e(x') + 1$. Therefore, $e(x') = e(x) - 1$ and $x' \in \mathcal{F}^{\cdot\, od}_{n-1,k-1}$.
If l_{p-1} is even, then since l_{p-1} cannot be null, $x = (x'001)$ with $x' \in \mathcal{F}^{\cdot\, ev}_{n-3}$. Because $e(001) = 2$, then $e(x) \leq e(x') + 2$. The equality is reached because if y is such that $d(x', y) = e(y)$, then $d((y010), x) = e(y) + 2$. Then $x' \in \mathcal{F}^{\cdot\, ev}_{n-3,k-2}$.

Then $x \to x'$ is a 1 to 1 mapping between $\mathcal{F}^{\cdot\, ev}_{n,k}$ and $\mathcal{F}^{\cdot\, ev}_{n-2,k-1} \cup \mathcal{F}^{\cdot\, od}_{n-1,k-1} \cup \mathcal{F}^{\cdot\, ev}_{n-3,k-2}$. By the previous proposition, $f^{\cdot\, od}_{n-1,k-1} = f^{\cdot\, ev}_{n-2,k-2}$ and we are done. □

Theorem 4.3.3 ([CM12])**.**

$$f^{\cdot\, ev}(x, y) = f^{ev\,\cdot}(x, y) = \frac{1}{1 - x(x+1)y}, \tag{4.3.1}$$

$$f^{\cdot\, od}(x, y) = f^{od\,\cdot}(x, y) = \frac{xy}{1 - x(x+1)y}, \tag{4.3.2}$$

thus the generating function for the eccentricity sequence is

$$\sum_{n,k \geq 0} f_{n,k}\ x^n y^k = \frac{1 + xy}{1 - x(x+1)y}.$$

Proof. Let $x = 0^{l_0}10^{l_1}1\cdots 10^{l_p} \in \mathcal{F}^{\cdot\, ev}$, thus $p \geq 0;\ l_0, l_p \geq 0;\ l_1, \cdots, l_{p-1} \geq 1$ and p is even.
Let $r(x) = 0^{l_p}10^{l_{p-1}}\cdots 10^{l_0}$ in $\mathcal{F}^{ev\,\cdot}$. Then r is a 1 to 1 mapping between $\mathcal{F}^{\cdot\, ev}$ and $\mathcal{F}^{ev\,\cdot}$.
Hence for any $n,\ k \geq 0$, $f^{\cdot\, ev}_{n,k} = f^{ev\,\cdot}_{n,k}$ and $f^{\cdot\, ev}(x, y) = f^{ev\,\cdot}(x, y)$.
The same applies for $x \in \mathcal{F}^{\cdot\, od}$, therefore $f^{\cdot\, od}(x, y) = f^{od\,\cdot}(x, y)$.

We will first demonstrate the equality (4.3.1), considering the linear recurrence given by Proposition 4.3.2, and the following initial values:

$$f^{\cdot\, ev}_{0,0} = f^{\cdot\, ev}_{1,1} = f^{\cdot\, ev}_{2,1} = f^{\cdot\, ev}_{2,2} = 1 \text{ and}$$

$$f^{\cdot\, ev}_{n,0} = 0 \text{ for } n \geq 1,\ f^{\cdot\, ev}_{n,1} = 0 \text{ for } n \geq 3,$$
$$f^{\cdot\, ev}_{n,k} = 0 \text{ for } k > n.$$

The generating function

$$f^{\cdot\, ev}(x,y) = \sum_{n,k\geq 0} f^{\cdot\, ev}_{n,k} x^n y^k$$

satisfies the equation

$$f^{\cdot\, ev}(x,y) = 1 + xy + x^2y + x^2y^2 + \sum_{n\geq 3,\ k\geq 2} f^{\cdot\, ev}_{n,k} x^n y^k.$$

Then

$$\begin{aligned}
f^{\cdot\, ev}(x,y) &= 1 + xy + x^2y + x^2y^2 + \sum_{n\geq 3,\ k\geq 2} (f^{\cdot\, ev}_{n-2,k-1} + f^{\cdot\, ev}_{n-2,k-2} + f^{\cdot\, ev}_{n-3,k-2}) x^n y^k \\
&= 1 + xy + x^2y + x^2y^2 + \sum_{n\geq 3,\ k\geq 2} (f^{\cdot\, ev}_{n-2,k-1} x^{n-2} y^{k-1}) x^2 y \\
&\qquad + \sum_{n\geq 3,\ k\geq 2} (f^{\cdot\, ev}_{n-2,k-2} x^{n-2} y^{k-2}) x^2 y^2 \\
&\qquad + \sum_{n\geq 3,\ k\geq 2} (f^{\cdot\, ev}_{n-3,k-2} x^{n-3} y^{k-2}) x^3 y^2 \\
&= 1 + xy + x^2y + x^2y^2 + (f^{\cdot\, ev}(x,y) - 1)x^2y + (f^{\cdot\, ev}(x,y) - 1)x^2y^2 + f^{\cdot\, ev}(x,y)x^3y^2.
\end{aligned}$$

Hence

$$f^{\cdot\, ev}(x,y) = \frac{1}{1 - x(x+1)y}.$$

For the equality (4.3.2), we will use the relation given by Proposition 4.3.1 and the initial values

$$f^{\cdot\, od}_{0,k} = f^{\cdot\, od}_{n,0} = 0 \text{ for } n,\ k \geq 0.$$

Thus

$$\begin{aligned}
f^{\cdot\, od}(x,y) &= \sum_{n,k\geq 0} f^{\cdot\, od}_{n,k} x^n y^k = \sum_{n,k\geq 1} f^{\cdot\, od}_{n,k} x^n y^k \\
&= xy \sum_{n,k\geq 1} f^{\cdot\, ev}_{n-1,k-1} x^{n-1} y^{k-1} = xy f^{\cdot\, ev}(x,y).
\end{aligned}$$

Therefore,

$$f^{\cdot\, od}(x,y) = \frac{xy}{1 - x(x+1)y}.$$

□

Corollary 4.3.4 ([CM12]). *For all n, k such that $n \geq k \geq 1$,*

$$f_{n,k} = \binom{k}{n-k} + \binom{k-1}{n-k}$$

Furthermore, $f_{0,0} = 1$ and $f_{n,0} = 0$ for $n > 0$.

Proof.

$$\begin{aligned} f^{\cdot\, ev}(x,y) &= \frac{1}{1 - x(x+1)y} = \sum_{b \geq 0} (xy(1+x))^b \\ &= \sum_{b \geq 0} \left[x^b y^b \sum_{a=0}^{b} x^a \binom{b}{a} \right] = \sum_{b \geq 0} \sum_{a=0}^{b} x^{a+b} y^b \binom{b}{a} \\ &= \sum_{n \geq 0} \sum_{k=0}^{n} x^n y^k \binom{k}{n-k}. \end{aligned}$$

Therefore,

$$f^{\cdot\, ev}_{n,k} = \binom{k}{n-k}.$$

The proof for $f^{\cdot\, od}_{n,k}$ is similar to the proof of $f^{\cdot\, ev}_{n,k}$ since $f^{\cdot\, od}(x,y)$ is xy times $f^{\cdot\, ev}(x,y)$. Hence

$$\begin{aligned} f^{\cdot\, od}(x,y) &= \frac{xy}{1 - x(x+1)y} = xy \sum_{b \geq 0} (xy(1+x))^b \\ &= xy \sum_{b \geq 0} \sum_{a=0}^{b} x^{a+b} y^b \binom{b}{a} = \sum_{b \geq 0} \sum_{a=0}^{b} x^{a+b+1} y^{b+1} \binom{b}{a} \\ &= \sum_{n \geq 1} \sum_{k=1}^{n} x^n y^k \binom{k-1}{n-k}. \end{aligned}$$

Thus $f^{\cdot\, od}_{n,k} = \binom{k-1}{n-k}$ when $n \geq k \geq 1$, and $f^{\cdot\, od}_{n,0} = 0$ for $n \geq 0$. In conclusion $f_{n,k} = f^{\cdot\, ev}_{n,k} + f^{\cdot\, od}_{n,k} = \binom{k}{n-k} + \binom{k-1}{n-k}$ □

Using the precedent corollary, it is immediate to deduce the value of $rad(\Gamma_n)$ determined in [MS02]:

Corollary 4.3.5. *The value of $k \geq 0$ that satisfies $\min_k \{f_{n,k} \mid f_{n,k} > 0\}$ is $k = rad(\Gamma_n) = \lceil \frac{n}{2} \rceil$.*

Notice that using

$$\sum_{i=0}^{m} \binom{m-i}{i} = F_{m+1}$$

(see [GKP94], pg. 289, equation 6.130), we obtain

$$\begin{aligned}\sum_{i=0}^{n} f_{n,k} &= \sum_{k=1}^{n} \left(\binom{k}{n-k} + \binom{k-1}{n-k} \right) \\ &= \sum_{i=0}^{n} \binom{n-i}{i} + \sum_{i=0}^{n-1} \binom{n-1-i}{i} = F_{n+1} + F_n = F_{n+2}\end{aligned}$$

which is consistent with

$$|V(\Gamma_n)| = F_{n+2}.$$

4.4 Eccentricity sequence of Lucas cubes

We will use the same notation for the strings in the Fibonacci cube to define the strings in the Lucas cube. In all the previous sections, when we referred to Fibonacci sets, we used the letter $\mathcal{F}$. For the Lucas sets, we will use the letter $\mathcal{L}$.
Accordingly, the functions for the Lucas cube will be defined in the same way as in the Fibonacci cube, but with a different letter, ℓ.

In this section, we will compute the generating function of the eccentricity sequence of the Lucas cube's strings, $\ell(x, y)$. For this aim, we will prove that the sets $\mathcal{L}^{a\,b}_{n,k}$ and $\mathcal{F}^{a\,b}_{n,k}$ are the same for all (a, b) excluding two sets, namely, $\mathcal{L}^{od\,od}$ and $\mathcal{L}^{\varnothing\,\varnothing}$. We proceed to compute the values of $\ell^{od\,od}_{n,k}$ and $\ell^{\varnothing\,\varnothing}_{n,k}$ as well as the values of $f^{od\,od}_{n,k}$ and $f^{\varnothing\,\varnothing}_{n,k}$. These results and Theorem 4.3.3 will give us the eccentricity sequence that we search. As a corollary we obtain the value of $\ell_{n,k}$.

Note further that Λ_n is an isometric subgraph of Γ_n and Q_n, i.e.:

Proposition 4.4.1. *For all $x, y \in \mathcal{L}_n, n \geq 1$,*

$$d_{\Lambda_n}(x, y) = d_{\Gamma_n}(x, y) = d_{Q_n}(x, y)$$

Proof. We will prove this proposition in the same way that we proved that $d_{\Gamma_n}(x, y) = d_{Q_n}(x, y)$ at the beginning of Section 4.2.
We have $d_{\Lambda_n}(x, y) \geq d_{Q_n}(x, y)$. Assume $x = (x_1 x_2 \cdots x_n)$, $y = (y_1 y_2 \cdots y_n)$ and let $z = (z_1 z_2 \cdots z_n) \in Q_n$ be defined as

$$z_i = \begin{cases} x_i \text{ if } x_i = y_i \\ 0 \text{ if } x_i \neq y_i, \end{cases}$$

then the path $s = (x = s_0, s_1, \cdots, z, \cdots, s_j = y)$ considered in proposition 4.2.1 is a shortest path in Q_n from x to y using only vertices of Λ_n, thus the equality is obtained. □

Proposition 4.4.2. *For $x \in \mathcal{L}_n, n \geq 1$,*

$$e_{\Lambda_n}(x) \leq e_{\Gamma_n}(x)$$

Proof. Let $x \in \mathcal{L}_n$. Then using proposition 4.4.1 and the fact that $\mathcal{L}_n \subset \mathcal{F}_n$, we have

$$e_{\Lambda_n}(x) = \max_{z \in \mathcal{L}_n}\{d_{\Lambda_n}(x,z)\} = \max_{z \in \mathcal{L}_n}\{d_{\Gamma_n}(x,z)\} \leq \max_{y \in \mathcal{F}_n}\{d_{\Gamma_n}(x,y)\} = e_{\Gamma_n}(x).$$

□

Proposition 4.4.3. *For $x \in \mathcal{L}_n \setminus \mathcal{L}_n^{od\,od}, n \geq 1$,*

$$e_{\Lambda_n}(x) = e_{\Gamma_n}(x).$$

Proof. Let $x \in \mathcal{L}_n \setminus \mathcal{L}_n^{od\,od}$ and without loss of generality, let us assume that x ends with an even (eventually null) number of 0's. By Corollary 4.2.9 (ii) and (iii), there exists $y \in \mathcal{F}_n$ such that $d_{\Gamma_n}(x,y) = e_{\Gamma_n}(x)$ and y ends with a 0. Therefore, $y \in \mathcal{L}_n$ and

$$d_{\Lambda_n}(x,y) = d_{\Gamma_n}(x,y) = e_{\Gamma_n}(x).$$

□

Let us observe that $\ell_{n,k}$ can be decomposed as follows:

$$\ell_{n,k} = \ell_{n,k}^{od\,od} + \ell_{n,k}^{od\,ev^*} + \ell_{n,k}^{od\,\varnothing} + \ell_{n,k}^{ev^*od} + \ell_{n,k}^{ev^*ev^*} + \ell_{n,k}^{ev^*\varnothing} + \ell_{n,k}^{\varnothing\,od} + \ell_{n,k}^{\varnothing\,ev^*} + \ell_{n,k}^{\varnothing\,\varnothing}.$$

Corollary 4.4.4. *For $n \geq 0$, $k \geq 0$,*

$$\begin{aligned}
\ell_{n,k}^{od\,ev^*} &= \ell_{n,k}^{ev^*od} = f_{n,k}^{od\,ev^*},\\
\ell_{n,k}^{od\,\varnothing} &= \ell_{n,k}^{\varnothing\,od} = f_{n,k}^{od\,\varnothing},\\
\ell_{n,k}^{ev^*ev^*} &= f_{n,k}^{ev^*ev^*},\\
\ell_{n,k}^{ev^*\varnothing} &= \ell_{n,k}^{\varnothing\,ev^*} = f_{n,k}^{\varnothing\,ev^*}.
\end{aligned}$$

Proof. When $n=0$ all these numbers are null. Assume $n \geq 1$ and let $x \in \mathcal{F}_n^{a\,b}$ with $(a,b) \neq (\varnothing,\varnothing)$ then $x \in \mathcal{L}_n^{a\,b}$.
Furthermore if $(a,b) \neq (od,od)$ we have, by Proposition 4.4.3, $e_{\Lambda_n}(x) = e_{\Gamma_n}(x)$ and

$$\mathcal{L}_{n,k}^{a\,b} = \mathcal{F}_{n,k}^{a\,b}.$$

□

In order to obtain $\ell_{n,k}$, we will compute the values of the functions $\ell_{n,k}^{a\,b}$ in terms of $f_{n,k}^{a\,b}$. For this reason, we will come again to the Fibonacci cube in this part of the section.

Proposition 4.4.5. *For $n, k \geq 2$,*

$$f_{n,k}^{od\,od} = f_{n-2,k-2}^{od\,od} + f_{n-2,k-2}^{od\,ev^*} + f_{n-2,k-1}^{od\,od}$$

Proof. Let $x = 0^{l_0}10^{l_1}1\cdots10^{l_p} \in \mathcal{F}_{n,k}^{od\,od}$, $n,k \geq 2$, thus p, l_0, $l_p \geq 0$; $l_1,\cdots,l_{p-1} \geq 1$ and l_0, l_p are odd numbers. Let us consider l_p. We distinguish 2 cases:

(i) If $l_p = 1$, then $p \neq 0$ and $x = (x'10)$ where x' is either in $\mathcal{F}_{n-2}^{od\,ev^*}$ or in $\mathcal{F}_{n-2}^{od\,od}$.
Let $y \in \mathcal{F}_{n-2}$ such that $d(x',y) = e(x')$, then $d(y01, x'10) = e(x')+2$ and since $e(10) = 2$ then $e(x) \leq e(x')+2$.
Therefore $e(x) = e(x')+2$ and $x' \in \mathcal{F}_{n-2,k-2}^{od\,ev^*}$ or $x' \in \mathcal{F}_{n-2,k-2}^{od\,od}$.

(ii) If $l_p \geq 3$, then $x = (x'00)$ with $x' \in \mathcal{F}_{n-2}^{od\,od}$. There exists $y \in \mathcal{F}_{n-2}$ such that $d(y,x') = e(x')$ then $d(y01, x'00) = e(x')+1$ and $e(x) \leq e(x')+1$.
Therefore $e(x) = e(x')+1$ and $x' \in \mathcal{F}_{n-2,k-1}^{od\,od}$.

Then $x \to x'$ is a 1 to 1 mapping between $\mathcal{F}_{n,k}^{od\,od}$ and $\mathcal{F}_{n-2,k-2}^{od\,od} \cup \mathcal{F}_{n-2,k-2}^{od\,ev^*} \cup \mathcal{F}_{n-2,k-1}^{od\,od}$. □

Consider a string $x = 0^{l_0}10^{l_1}1\cdots10^{l_p} \in \mathcal{F}_{n,k}^{od\,ev^*}$. We will demonstrate next, that when we remove a 0 from 0^{l_p}, we obtain a string that belongs to $\mathcal{F}_{n-1,k}^{od\,od} \setminus \{$ words composed by an odd number $(n-1)$ of 0's $\}$.
For this purpose, for even n and eccentricity k, let $g_{n,k}^{even}$ be the number of strings in $\mathcal{F}_n$ composed only by 0's. Notice that by Proposition 4.2.7, $n = 2k$, then

$$g_{n,k}^{even} = \begin{cases} 1 & \text{if } n = 2k \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.4.6. *For $n \geq 1$, $k \geq 0$,*

$$f_{n,k}^{od\,ev^*} = f_{n-1,k}^{od\,od} - g_{n,k}^{even}.$$

Proof. Let $x = 0^{l_0}10^{l_1}1\cdots 10^{l_p} \in \mathcal{F}_{n,k}^{od\,ev^*}$, $n \geq 1$, $k \geq 0$, thus $p \geq 1$; $l_0, l_1, \cdots, l_{p-1} \geq 1$ and $l_p \geq 2$. Then $x = (x'0)$ with $x' \in \mathcal{F}_{n-1}^{od\,od}$ such that

$$x' = 0^{l_0}10^{l_1}1\cdots 10^{l_p-1}.$$

Then by Corollary 4.2.9 (i), all the strings of $\mathcal{F}_{n-1}$ that satisfy the eccentricity of x' have the form $y = (y'1)$. Thus $e(x') = e(x)$, and $x' \in \mathcal{F}_{n-1,k}^{od\,od}$. Conversely, for any string $z \in \mathcal{F}_{n-1}^{od\,od}$ that is not composed only by 0's, the string $(z0) \in \mathcal{F}_{n}^{od\,ev^*}$.
Therefore, $x \to x'$ is a 1 to 1 mapping between $\mathcal{F}_{n,k}^{od\,ev^*}$ and $\mathcal{F}_{n-1,k}^{od\,od} \setminus \{$ words composed by an odd number $(n-1)$ of 0's $\}$. □

Proposition 4.4.5 can be rewritten in terms of $f^{od\,od}$ using the result of Proposition 4.4.6, which gives us the next

Proposition 4.4.7. *For $n \geq 3$, $k \geq 2$,*

$$f_{n,k}^{od\,od} = f_{n-2,k-2}^{od\,od} + f_{n-2,k-1}^{od\,od} + f_{n-3,k-2}^{od\,od} - g_{n-2,k-2}^{even}.$$

Notice that

$$\begin{aligned} g^{even}(x,y) &= \sum_{n,k\geq 0} g_{n,k}^{even} x^n y^k \\ &= \sum_{n,k\geq 0} x^{2k} y^k = 1 + x^2y + x^4y^2 + x^6y^3 + \cdots \\ &= \frac{1}{1-x^2y} \end{aligned} \tag{4.4.1}$$

Proposition 4.4.8.

$$f^{od\,od}(x,y) = \frac{xy(1-x^2y-x^3y^2)}{(1+xy)(1-x^2y)(1-xy-x^2y)}, \tag{4.4.2}$$

$$f^{od\,ev^*}(x,y) = \frac{x^4y^3}{(1+xy)(1-x^2y)(1-xy-x^2y)}. \tag{4.4.3}$$

Proof. Considering that

$$f_{1,1}^{od\,od} = 1 \text{ and } f_{n,k}^{od\,od} = 0 \text{ for other values } n \leq 2 \text{ or } k \leq 1, \text{ then}$$

$$\begin{aligned} f^{od\,od}(x,y) &= \sum_{n,k\geq 0} f_{n,k}^{od\,od} x^n y^k \\ &= xy + \sum_{n\geq 3,\ k\geq 2} f_{n,k}^{od\,od} x^n y^k \end{aligned}$$

Therefore by Proposition 4.4.7

$$\begin{aligned} f^{od\,od}(x,y) - xy &= \sum_{n\geq 3,\ k\geq 2} (f^{od\,od}_{n-2,k-2} + f^{od\,od}_{n-2,k-1} + f^{od\,od}_{n-3,k-2} - g^{even}_{n-2,k-2})x^n y^k \\ &= x^2y^2 \sum_{n\geq 1,\ k\geq 0} f^{od\,od}_{n,k} x^n y^k + x^2 y \sum_{n,k\geq 1} f^{od\,od}_{n,k} x^n y^k \\ &\quad + x^3y^2 \sum_{n,k\geq 0} f^{od\,od}_{n,k} x^n y^k - x^2y^2 \sum_{n\geq 1,\ k\geq 0} g^{even}_{n,k} x^n y^k \end{aligned}$$

thus

$$f^{od\,od}(x,y) - xy = x^2y^2 f^{od\,od}(x,y) + x^2 y f^{od\,od}(x,y) + x^3y^2 f^{od\,od}(x,y) - x^2y^2(g^{even}(x,y) - 1)$$

and by relation (4.4.1),

$$f^{od\,od}(x,y)(1 - x^2y^2 - x^2y - x^3y^2) = xy + x^2y^2 + \frac{x^2y^2}{1-x^2y},$$

thus

$$f^{od\,od}(x,y) = \frac{xy(1-x^2y-x^3y^2)}{(1+xy)(1-x^2y)(1-xy-x^2y)}.$$

Now we will prove equation (4.4.3). First we observe that $f^{od\,ev^*}_{0,0} = 0$ then

$$f^{od\,ev^*}(x,y) = \sum_{n\geq 1,\ k\geq 0} f^{od\,ev^*}_{n,k} x^n y^k$$

and by Proposition 4.4.6,

$$\begin{aligned} f^{od\,ev^*}(x,y) &= \sum_{n\geq 1,\ k\geq 0} (f^{od\,od}_{n-1,k} - g^{even}_{n,k})x^n y^k = \sum_{n,k\geq 0} f^{od\,od}_{n,k} x^{n+1} y^k - \sum_{n\geq 1,\ k\geq 0} g^{even}_{n,k} x^n y^k \\ &= x f^{od\,od}(x,y) - (g^{even}(x,y) - 1). \end{aligned}$$

Therefore, by relation (4.4.1),

$$\begin{aligned} f^{od\,ev^*}(x,y) &= \frac{x^2y(1-x^2y-x^3y^2)}{(1+xy)(1-x^2y)(1-xy-x^2y)} - \frac{x^2y}{1-x^2y} \\ &= \frac{x^4y^3}{(1+xy)(1-x^2y)(1-xy-x^2y)}. \end{aligned}$$

□

Proposition 4.4.9. *For $n, k \geq 1$,*

$$f^{od\,\varnothing}_{n,k} = f^{od\,ev^*}_{n-1,k-1} + f^{od\,od}_{n-1,k-1}.$$

Proof. Let $x = 0^{l_0}10^{l_1}1\cdots10^{l_p} \in \mathcal{F}^{od\,\varnothing}_{n,k}$; $n,\ k \geq 1$.
Thus $p \geq 1$; $l_0, l_1, \cdots, l_{p-1} \geq 1$ and $l_p = 0$. We have therefore, $x = (x'1)$ with x' either in $\mathcal{F}^{od\,od}_{n-1}$ or in $\mathcal{F}^{od\,ev^*}_{n-1}$.
Let $y \in \mathcal{F}_{n-1}$ such that $d(x', y) = e(x')$.
Then $d((x'1), (y0)) = e(x') + 1$ and $e(x) \leq e(x') + 1$, thus $e(x) = e(x') + 1$.
Therefore, x' belongs to $\mathcal{F}^{od\,od}_{n-1,k-1}$ or to $\mathcal{F}^{od\,ev^*}_{n-1,k-1}$.
Then $x \to x'$ is a 1 to 1 mapping between $\mathcal{F}^{od\,\varnothing}_{n,k}$ and $\mathcal{F}^{od\,od}_{n-1,k-1} \cup \mathcal{F}^{od\,ev^*}_{n-1,k-1}$. $\square$

Proposition 4.4.10.

$$f^{od\,\varnothing}(x, y) = \frac{x^2y^2}{(1 + xy)(1 - xy - x^2y)}.$$

Proof. Considering that

$$f^{od\,\varnothing}_{n,0} = f^{od\,\varnothing}_{0,k} = 0 \text{ for } n,\ k \geq 0, \text{ we have}$$

$$f^{od\,\varnothing}(x, y) = \sum_{n,k\geq1} f^{od\,\varnothing}_{n,k} x^n y^k.$$

Then from Proposition 4.4.9,

$$\begin{aligned}
f^{od\,\varnothing}(x, y) &= \sum_{n,k\geq1} (f^{od\,ev^*}_{n-1,k-1} + f^{od\,od}_{n-1,k-1}) x^n y^k \\
&= \sum_{n,k\geq1} (f^{od\,ev^*}_{n-1,k-1} x^{n-1} y^{k-1}) xy + \sum_{n,k\geq1} (f^{od\,od}_{n-1,k-1} x^{n-1} y^{k-1}) xy \\
&= f^{od\,ev^*}(x, y) xy + f^{od\,od}(x, y) xy.
\end{aligned}$$

Thus by Proposition 4.4.8,

$$\begin{aligned}
f^{od\,\varnothing}(x, y) &= \frac{x^4y^3xy}{(1 + xy)(1 - x^2y)(1 - xy - x^2y)} + \frac{xy(1 - x^2y - x^3y^2)xy}{(1 + xy)(1 - x^2y)(1 - xy - x^2y)} \\
&= \frac{x^2y^2(1 - x^2y)}{(1 + xy)(1 - x^2y)(1 - xy - x^2y)} \\
&= \frac{x^2y^2}{(1 + xy)(1 - xy - x^2y)}.
\end{aligned}$$

$\square$

Proposition 4.4.11. *For $n \geq 1$, $k \geq 0$,*

$$f^{ev^*\varnothing}_{n,k} = f^{od\,\varnothing}_{n-1,k},$$

thus

$$f^{ev^*\varnothing}(x, y) = x f^{od\,\varnothing}(x, y).$$

Proof. The equality is true when $n = 1$ or $n = 2$.
Then let $x = 0^{l_0}10^{l_1}10^{l_2}\cdots 10^{l_p} \in \mathcal{F}_{n,k}^{ev^*\varnothing}$, with $n \geq 3$ and $k \geq 0$. Thus $p \geq 1$; $l_0 \geq 2$; $l_1, \cdots, l_{p-1} \geq 1$; $l_p = 0$.
Because $l_0 > 0$, then $x = (0x')$ with $x' \in \mathcal{F}_{n-1}^{od\varnothing}$.
By Proposition 4.2.2,

$$e(x) \leq e(x') + 1.$$

Let's suppose that $e(x) = e(x') + 1$, then there exists $y = (1y')$ such that $d(y', x') = e(x')$. By a symmetry argument and Corollary 4.2.9, y' must begin with 1 which leads us to a contradiction.
Therefore, $e(x) = e(x')$. Thus $x \to x'$ is a 1 to 1 mapping between $\mathcal{F}_{n,k}^{ev^*\varnothing}$ and $\mathcal{F}_{n-1,k}^{od\varnothing}$.
Considering the fact that $f_{0,k}^{ev^*\varnothing} = 0$ for $k \geq 0$, we have:

$$\begin{aligned} f^{ev^*\varnothing}(x,y) &= \sum_{n,k\geq 0} f_{n,k}^{ev^*\varnothing} x^n y^k = \sum_{n\geq 1,k\geq 0} f_{n,k}^{ev^*\varnothing} x^n y^k \\ &= \sum_{n\geq 1,k\geq 0} x f_{n-1,k}^{od\varnothing} x^{n-1} y^k = x f^{od\varnothing}(x,y). \end{aligned}$$

□

Proposition 4.4.12. *For $n \geq 3$, $k \geq 1$,*

$$f_{n,k}^{\varnothing\,\varnothing} = f_{n-1,k-1}^{\varnothing\,ev^*} + f_{n-1,k-1}^{\varnothing\,od}.$$

Proof. Let $x = 0^{l_0}10^{l_1}1\cdots 10^{l_p} \in \mathcal{F}_{n,k}^{\varnothing\,\varnothing}$ with $n \geq 3$, $k \geq 1$. Thus $p \geq 2$; $l_1, \cdots, l_{p-1} \geq 1$ and $l_0 = l_p = 0$.
Then $x = (x'1)$ with $x' \in \mathcal{F}_{n-1}^{\varnothing\,ev^*}$ if l_{p-1} is an even number and $x' \in \mathcal{F}_{n-1}^{\varnothing\,od}$ if l_{p-1} is odd.
By Proposition 4.2.2, $e(x) \leq e(x') + 1$.
Let $y' \in \mathcal{F}_{n-1}$ such that $d(x', y') = e(x')$, then $d((y'0), (x'1)) = e(x') + 1$.
Hence $e(x) = e(x') + 1$. Thus $x \to x'$ is a 1 to 1 mapping between $\mathcal{F}_{n,k}^{\varnothing\,\varnothing}$ and $\mathcal{F}_{n-1,k-1}^{\varnothing\,ev^*} \cup \mathcal{F}_{n-1,k-1}^{\varnothing\,od}$. □

Proposition 4.4.13.

$$f^{\varnothing\,\varnothing}(x,y) = 1 + xy + \frac{xy(x^3y^2 + x^2y^2)}{(1+xy)(1-xy-x^2y)}.$$

Proof. Let us consider the next initial values:

$$f_{0,0}^{\varnothing\,\varnothing} = f_{1,1}^{\varnothing\,\varnothing} = 1 \text{ and } f_{n,k}^{\varnothing\,\varnothing} = 0 \text{ for other values } n \leq 2 \text{ or } k = 0.$$

Then

$$f^{\varnothing\,\varnothing}(x,y) = \sum_{n,k\geq 0} f^{\varnothing\,\varnothing}_{n,k} x^n y^k$$
$$= 1 + xy + \sum_{n\geq 3,k\geq 1} f^{\varnothing\,\varnothing}_{n,k} x^n y^k.$$

Then by Proposition 4.4.12,

$$f^{\varnothing\,\varnothing}(x,y) - 1 - xy = \sum_{n\geq 3,k\geq 1} (f^{\varnothing\,ev*}_{n-1,k-1} + f^{od\,\varnothing}_{n-1,k-1}) x^n y^k$$
$$= xy \sum_{n\geq 2,k\geq 0} (f^{\varnothing\,ev*}_{n,k} + f^{od\,\varnothing}_{n,k}) x^n y^k$$

Observe that when $n \leq 1$

$$f^{\varnothing\,ev*}_{n,k} + f^{od\,\varnothing}_{n,k} = 0.$$

Hence

$$f^{\varnothing\,\varnothing}(x,y) - 1 - xy = xy(f^{\varnothing\,ev*}(x,y) + f^{od\,\varnothing}(x,y)).$$

From Proposition 4.4.11,

$$f^{\varnothing\,\varnothing}(x,y) = 1+xy+xy(xf^{od\,\varnothing}(x,y)+f^{od\,\varnothing}(x,y)) = 1+xy+xy(1+x)f^{od\,\varnothing}(x,y).$$

Substituting $f^{od\,\varnothing}(x,y)$ from Proposition 4.4.10, we obtain the desired result. □

Proposition 4.4.14. *For $n \geq 3$, $k \geq 1$,*

$$\ell^{od\,od}_{n,k} = f^{od\,od}_{n,k+1}$$

thus

$$\ell^{od\,od}(x,y) = y^{-1} f^{od\,od}(x,y).$$

Proof. Let $x = 0^{l_0}10^{l_1}1\cdots 10^{l_p} \in \mathcal{L}^{od\,od}_{n,k}$, $n \geq 3$, $k \geq 1$. Thus $p \geq 0$; $l_0, l_1 \cdots, l_{p-1}, l_p \geq 1$
By Corollary 4.2.9 (i) and by symmetry, every y such that $d(x,y) = e_{\Gamma_n}(x)$ has the form $y = (1y'1)$, with $y' \in \mathcal{F}_{n-2}$. Then, $y \notin \mathcal{L}_n$ and $e_{\Lambda_n}(x) < e_{\Gamma_n}(x)$. Furthermore, note that the string $(1y'0) \in \mathcal{L}_n$. Thus $d((1y'0),x) = e_{\Gamma_n}(x) - 1$. Hence $e_{\Lambda_n}(x) = e_{\Gamma_n}(x) - 1$.
Thus, $x \to x$ maps $\mathcal{L}^{od\,od}_{n,k}$ into $\mathcal{F}^{od\,od}_{n,k+1}$.
For the second part of the Proposition, consider the initial values

$$\ell_{1,0}^{od\,od} = 1 \text{ and } \ell_{n,k}^{od\,od} = 0 \text{ for other values } n \leq 2 \text{ or } k = 0.$$

Thus

$$\begin{aligned}\ell^{od\,od}(x,y) &= \sum_{n,k\geq 0} \ell_{n,k}^{od\,od} x^n y^k = x + \sum_{n\geq 3,k\geq 1} f_{n,k+1}^{od\,od} x^n y^k \\ &= x + y^{-1} \sum_{n\geq 3,k\geq 1} f_{n,k+1}^{od\,od} x^n y^{k+1}.\end{aligned}$$

But

$$f^{od\,od}(x,y) = xy + \sum_{n\geq 3,k\geq 2} f_{n,k}^{od\,od} x^n y^k,$$

thus

$$\ell^{od\,od}(x,y) = x + y^{-1}(f^{od\,od}(x,y) - xy) = y^{-1} f^{od\,od}(x,y).$$

□

Proposition 4.4.15.

$$\ell^{\varnothing\,\varnothing}(x,y) = 1.$$

Proof. The empty word is the only string that belongs to some $\mathcal{L}_n$ that neither begins nor ends with a 0. Thus $\ell_{n,k}^{\varnothing\,\varnothing} = 0$ for $n \geq 1$. □

Theorem 4.4.16 ([CM12])**.** *The generating function for the eccentricity sequence of Lucas cube is*

$$\ell(x,y) = \sum_{n,k\geq 0} \ell_{n,k} x^n y^k = \frac{1 + x^2 y}{1 - xy - x^2 y} + \frac{1}{1+xy} - \frac{1-x}{1-x^2 y}.$$

Proof. Recall that

$$\ell_{n,k} = \ell_{n,k}^{od\,od} + \ell_{n,k}^{od\,ev^*} + \ell_{n,k}^{od\,\varnothing} + \ell_{n,k}^{ev^*od} + \ell_{n,k}^{ev^*ev^*} + \ell_{n,k}^{ev^*\varnothing} + \ell_{n,k}^{\varnothing\,od} + \ell_{n,k}^{\varnothing\,ev^*} + \ell_{n,k}^{\varnothing\,\varnothing}.$$

and we have the same decomposition for $f_{n,k}$.

From Corollary 4.4.4, when $(a,b) \neq (od, od)$ and $(a,b) \neq (\varnothing, \varnothing)$, then $\ell_{n,k}^{a\,b} = f_{n,k}^{a\,b}$. Thus

$$\ell_{n,k} = f_{n,k} - f_{n,k}^{od\,od} - f_{n,k}^{\varnothing\,\varnothing} + \ell_{n,k}^{od\,od} + \ell_{n,k}^{\varnothing\,\varnothing}.$$

Thus, the generating function

$$\ell(x,y) = \sum_{n,k\geq 0} \ell_{n,k} x^n y^k$$

satisfies the equation

$$\ell(x,y) = \sum_{n,k\geq 0} (f_{n,k} - f_{n,k}^{od\,od} - f_{n,k}^{\varnothing\,\varnothing} + \ell_{n,k}^{od\,od} + \ell_{n,k}^{\varnothing\,\varnothing}).$$

By Theorem 4.3.3 and Propositions 4.4.8, 4.4.13, 4.4.14 and 4.4.15, we conclude that

$$\begin{aligned}
\ell(x,y) &= \frac{1+xy}{1-xy-x^2y} - \frac{xy(1-x^2y-x^3y^2)}{(1+xy)(1-x^2y)(1-xy-x^2y)} \\
&\quad - \left(1+xy+\frac{xy(x+1)x^2y^2}{(1+xy)(1-xy-x^2y)}\right) \\
&\quad + y^{-1}\left(\frac{xy(1-x^2y-x^3y^2)}{(1+xy)(1-x^2y)(1-xy-x^2y)}\right) + 1 \\
&= \frac{1+x-x^2y+x^2y^2-x^3y+x^3y^2-x^4y^2-x^5y^3}{(1+xy)(1-x^2y)(1-xy-x^2y)} \\
&= \frac{1}{1+xy} - \frac{1-x}{1-x^2y} + \frac{1+x^2y}{1-xy-x^2y}.
\end{aligned}$$

□

Corollary 4.4.17 ([CM12]). *For all n,k with $n > k \geq 1$,*

$$\ell_{n,k} = \binom{k}{n-k} + \binom{k-1}{n-k-1} + \varepsilon_{n,k} \tag{4.4.4}$$

where

$$\varepsilon_{n,k} = \begin{cases} -1 & \text{if } n = 2k, \\ 1 & \text{if } n = 2k+1, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, $\ell_{0,0} = \ell_{1,0} = 1$, $\ell_{n,0} = 0$ for $n > 1$ and

$$\ell_{n,n} = \begin{cases} 2 & \text{if } n \text{ is even } (n \geq 2), \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Proof. By the previous theorem,

$$\ell(x,y) = \frac{1}{1-xy-x^2y} + \frac{x^2y}{1-xy-x^2y} + \frac{1}{1+xy} - \frac{1-x}{1-x^2y}.$$

We will analyse each term of this sum separately.

$$\begin{aligned}\frac{1}{1-xy-x^2y} &= \sum_{b\geq 0}(xy(1+x))^b = \sum_{b\geq 0} x^b y^b \sum_{a=0}^{b} x^a \binom{b}{a} \\ &= \sum_{b\geq 0}\sum_{a=0}^{b} x^{a+b} y^b \binom{b}{a} = \sum_{n\geq 0}\sum_{k=0}^{n} \binom{k}{n-k} x^n y^k \qquad (4.4.5)\end{aligned}$$

$$\begin{aligned}\frac{x^2y}{1-xy-x^2y} &= x^2y\sum_{b\geq 0}(xy(1+x))^b = x^2y\sum_{b\geq 0}\sum_{a=0}^{b} x^{a+b}y^b\binom{b}{a} \\ &= \sum_{b\geq 0}\sum_{a=0}^{b} x^{a+b+2}y^{b+1}\binom{b}{a} = \sum_{n\geq 2}\sum_{k=1}^{n-1}\binom{k-1}{n-k-1}x^n y^k. \qquad (4.4.6)\end{aligned}$$

The third term of the sum is

$$\frac{1}{1+xy} = \sum_{b\geq 0}(-xy)^b = \sum_{n\geq 0}(-1)^n x^n y^n. \qquad (4.4.7)$$

Finally, the last term will be decomposed as follows:

$$-\frac{1-x}{1-x^2y} = \frac{x}{1-x^2y} - \frac{1}{1-x^2y},$$

$$\frac{x}{1-x^2y} = x\sum_{a\geq 0}(x^2y)^a = \sum_{k\geq 0}(x^{2k+1})y^k, \qquad (4.4.8)$$

and the second sub-term

$$\frac{-1}{1-x^2y} = -\sum_{a\geq 0}(x^2y)^a = -\sum_{k\geq 0}(x^{2k})y^k. \qquad (4.4.9)$$

Equations (4.4.5), (4.4.6), (4.4.8) and (4.4.9) give us the desired result when $k \neq 0$, $k \neq n$.
When $k = 0$, equation (4.4.5) contributes with 1 when $n = 0$; equation (4.4.7) contributes with 1 when $n = 0$; equation (4.4.8) contributes with 1 when $n = 1$ and equation (4.4.9) contributes with -1 for $n = 0$.
When $k = n \geq 1$, equation (4.4.5) contributes with 1 and equation (4.4.7) contributes with $(-1)^n$. □

Notice that for $n \geq 2$,

$$\sum_{k=0}^{n}\ell_{n,k} = \sum_{k=1}^{n-1}\left[\binom{k}{n-k} + \binom{k-1}{n-k-1}\right] + \varepsilon_{n,\lfloor\frac{n}{2}\rfloor} + \ell_{n,0} + \ell_{n,n},$$

where

$$\varepsilon_{n,\lfloor \frac{n}{2} \rfloor} = (-1)^{n+1}, \quad \ell_{n,0} = 0 \quad \text{and} \quad \ell_{n,n} = 1 + (-1)^n.$$

Therefore,

$$\begin{aligned}\sum_{k=0}^{n} \ell_{n,k} &= \sum_{k=0}^{n} \binom{k}{n-k} + \sum_{k=0}^{n-2} \binom{k}{n-2-k} \\ &= F_{n+1} + F_{n-1} = L_n = |V(\Lambda_n)|.\end{aligned}$$

4.5 Eccentricity sequence of Lucas cubes Second version

In this section, we give an alternative proof of Theorem 4.4.16, obtaining the generating function of the eccentricity sequence of the Lucas cube's strings with a direct approach.

Let us recall that in section 4.3, we obtained the generating function of all the strings that end with an even number of 0's (eventually null) along with the generating function of all the strings that end with an odd number of 0's. That is, $f^{\cdot\, ev}(x,y)$ and $f^{\cdot\, od}(x,y)$ respectively. (Theorem 4.3.3.)
Recall further that

$$\begin{aligned}f_{n,k} &= f^{ev\,\cdot}_{n,k} + f^{od\,\cdot}_{n,k} = f^{\cdot\, ev}_{n,k} + f^{\cdot\, od}_{n,k} \\ &= f^{od\, od}_{n,k} + f^{od\, ev^*}_{n,k} + f^{od\, \varnothing}_{n,k} + f^{ev^*\, od}_{n,k} + f^{ev^*\, ev^*}_{n,k} + f^{ev^*\, \varnothing}_{n,k} + f^{\varnothing\, od}_{n,k} + f^{\varnothing\, ev^*}_{n,k} + f^{\varnothing\, \varnothing}_{n,k}\end{aligned}$$

and that the same decomposition applies for $\ell_{n,k}$.

In section 4.4, we used the fact that $\ell^{a\, b}_{n,k} = \ell^{b\, a}_{n,k} = f^{a\, b}_{n,k}$ for $(a,b) \neq (od, od)$ and $(a,b) \neq (\varnothing, \varnothing)$ (Corollary 4.4.4), which gave us the next equation:

$$\ell_{n,k} = f_{n,k} - f^{od\, od}_{n,k} - f^{\varnothing\, \varnothing}_{n,k} + \ell^{od\, od}_{n,k} + \ell^{\varnothing\, \varnothing}_{n,k}.$$

Then, using $f_{n,k}$ and computing the missing generating functions of this equation, we obtained $\ell_{n,k}$.

We will now compute $\ell_{n,k}$ directly, using each of the nine terms that compose it. For this purpose, we will take back the terms already obtained in the previous section, namely:

$$\begin{aligned}\ell^{od\, od}(x,y) &= y^{-1}\ f^{od\, od}(x,y), \ \text{Proposition 4.4.14}, \\ &= \frac{xy(1-x^2y-x^3y^2)}{(1+xy)(1-x^2y)(1-xy-x^2y)}, \ \text{Proposition 4.4.8},\end{aligned}$$

$$\ell^{od\,ev^*}(x,y) = f^{od\,ev^*}(x,y) = \frac{x^4y^3}{(1+xy)(1-x^2y)(1-xy-x^2y)}, \text{ Proposition 4.4.3,}$$

$$\ell^{od\,\varnothing}(x,y) = f^{od\,\varnothing}(x,y) = \frac{x^2y^2}{(1+xy)(1-xy-x^2y)}, \text{ Proposition 4.4.10,}$$

$$\ell^{ev^*\varnothing}(x,y) = f^{ev^*\varnothing}(x,y) = x\,\ell^{od\,\varnothing}(x,y), \text{ Proposition 4.4.11 and}$$

$$\ell^{\varnothing\,\varnothing}(x,y) = 1, \text{ Proposition 4.4.15.}$$

In order to compute $\ell^{ev^*ev^*}(x,y)$, the generating function that has not been obtained up to now, we will consider a string $x = 0^{l_0}10^{l_1}1\cdots10^{l_p} \in \mathcal{F}_{n,k}^{ev^*\,ev^*} \setminus \{$words composed by an even number n of 0's$\}$, and demonstrate that when we remove a 0 from 0^{l_p}, we obtain a string that belongs to $\mathcal{F}_{n-1,k}^{ev^*\,od}$. Recall that $g_{n,k}^{even}$ was defined as the number of strings in $\mathcal{F}_n$ composed only by 0's for even n and eccentricity k and that by Proposition 4.2.7, $n = 2k$, then

$$g_{n,k}^{even} = \begin{cases} 1 & \text{if } n = 2k \\ 0 & \text{otherwise.} \end{cases}$$

Recall as well that

$$g^{even}(x,y) = \frac{1}{1-x^2y}.$$

Proposition 4.5.1. *For $n \geq 1$ and $k \geq 0$,*

$$f_{n,k}^{ev*\,ev*} = f_{n-1,k}^{ev*\,od} + g_{n,k}^{even},$$

thus

$$f^{ev*\,ev*}(x,y) = \frac{x^2y - x^4y^2 - x^4y^3}{(1+xy)(1-x^2y)(1-xy-x^2y)}.$$

Proof. Let $x = 0^{l_0}10^{l_1}1\cdots10^{l_p} \in \mathcal{F}_{n,k}^{ev*\,ev^*}$ that is not composed only by 0's with $n \geq 1$, $k \geq 0$, thus $p \geq 0$; l_0, $l_p \geq 2$ and $l_1, \cdots l_{p-1} \geq 1$. Then $x = (x'0)$ with $x' \in \mathcal{F}_{n-1}^{ev*\,od}$ such that $x' = 0^{l_0}10^{l_1}1\cdots10^{l_p-1}$. Then, by Corollary 4.2.9 (i), all the strings of $\mathcal{F}_{n-1}$ that satisfy the eccentricity of x' have the form $y = (y'1)$.

On the other hand, for any string $y \in \mathcal{F}_{n-1}^{ev*\,od}$, the string $(y0) \in \mathcal{F}^{ev*\,ev*}$.
Therefore, $x \to x'$ is a 1 to 1 mapping between $\mathcal{F}_{n,k}^{ev*\,ev*} \setminus \{$ words composed by an even number n of 0's $\}$ and $\mathcal{F}_{n-1,k}^{ev*\,od}$.
For the second part of the proposition, let us consider the following initial conditions:

$$f_{0,k}^{ev*\,ev*} = f_{1,k}^{ev*\,ev*} = 0, \text{ for } k \geq 0, \text{ then we have}$$

$$\begin{aligned} f^{ev*\,ev*}(x,y) &= \sum_{n\geq 2, k\geq 0} f_{n,k}^{ev*\,ev*} x^n y^k \\ &= \sum_{n\geq 2, k\geq 0} (f_{n-1,k}^{ev*\,od} + g_{n,k}^{even}) x^n y^k \\ &= x \sum_{n\geq 1, k\geq 0} f_{n,k}^{ev*\,od} x^n y^k + \sum_{n\geq 2, k\geq 0} g_{n,k}^{even} x^n y^k, \end{aligned}$$

thus

$$\begin{aligned} f^{ev*\,ev*}(x,y) &= x f^{ev*\,od}(x,y) + (g^{even}(x,y) - 1) \\ &= \frac{x(x^4y^3)}{(1+xy)(1-x^2y)(1-xy-x^2y)} + \left(\frac{1}{1-x^2y} - 1\right) \\ &= \frac{x^2y - x^4y^2 - x^4y^3}{(1+xy)(1-x^2y)(1-xy-x^2y)}. \end{aligned}$$

□

Theorem 4.5.2 ([CM12]). *The generating function for the eccentricity sequence of the Lucas cube is*

$$\ell(x,y) = \sum_{n,k\geq 0} \ell_{n,k} x^n y^k = \frac{1+x^2y}{1-xy-x^2y} + \frac{1}{1+xy} - \frac{1-x}{1-x^2y}.$$

Proof. The generating function

$$\ell(x,y) = \sum_{n,k\geq 0} \ell_{n,k} x^n y^k$$

satisfies the equation

$$\ell(x,y) = \sum_{n,k\geq 0} (\ell_{n,k}^{od\,od} + f_{n,k}^{od\,ev^*} + f_{n,k}^{od\,\varnothing} + f_{n,k}^{ev^*od} + f_{n,k}^{ev^*ev^*} + f_{n,k}^{ev^*\varnothing} + f_{n,k}^{\varnothing\,od} + f_{n,k}^{\varnothing\,ev^*} + \ell_{n,k}^{\varnothing\,\varnothing}).$$

By Propositions 4.4.14, 4.4.8, 4.4.3, 4.5.1, 4.4.10, 4.4.11 and 4.4.15, we conclude that

$$\begin{aligned}\ell(x,y) &= \frac{x(1-x^2y-x^3y^2)}{(1+xy)(1-x^2y)(1-xy-x^2y)} + \frac{x^4y^3}{(1+xy)(1-x^2y)(1-xy-x^2y)} \\ &+ \frac{x^2y^2}{(1+xy)(1-xy-x^2y)} + \frac{x^4y^3}{(1+xy)(1-x^2y)(1-xy-x^2y)} \\ &+ \frac{x^2y-x^4y^2-x^4y^3}{(1+xy)(1-x^2y)(1-xy-x^2y)} + \frac{x(x^2y^2)}{(1+xy)(1-xy-x^2y)} \\ &+ \frac{x^2y^2}{(1+xy)(1-xy-x^2y)} + \frac{x(x^2y^2)}{(1+xy)(1-xy-x^2y)} + 1 \\ &= \frac{1+x-x^2y+x^2y^2-x^3y+x^3y^2-x^4y^2-x^5y^3}{(1+xy)(1-x^2y)(1-xy-x^2y)}.\end{aligned}$$

Therefore,

$$\ell(x,y) = \frac{1+x^2y}{1-xy-x^2y} + \frac{1}{1+xy} - \frac{1-x}{1-x^2y}.$$

□

As a Corollary, which has been proven in the previous section, the number of vertices of Λ_n with eccentricity k is stated below.

Corollary 4.5.3 ([CM12]). *For all n, k with $n > k \geq 1$,*

$$\ell_{n,k} = \binom{k}{n-k} + \binom{k-1}{n-k-1} + \varepsilon_{n,k}$$

where

$$\varepsilon_{n,k} = \begin{cases} -1 & \text{if } n = 2k, \\ 1 & \text{if } n = 2k+1 \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, $\ell_{0,0} = \ell_{1,0} = 1$, $\ell_{n,0} = 0$ for $n > 1$ and

$$\ell_{n,n} = \begin{cases} 2 & \text{if } n \text{ is even } (n \geq 2), \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Chapter 5

Domination number and 2-packing number

An automorphism of a graph is a permutation α of its vertex set which preserves adjacency: if (uv) is an edge, then so is $(\alpha(u), \alpha(v))$.
The automorphism of a graph reflect its symmetries. For example, $u, v \in V(G)$ if there exists the automorphism α which maps u to v, then u and v are alike in the graph and are referred to as similar vertices. See [BM08].
In this sense, while we remove certain 'non-symmetric' vertices of the Fibonacci cube to generate the Lucas cube, we get graphs with more symmetries.
In this chapter, we study these cubes from the domination and 2-packing points of view. While searching for subsets of a graph, it is useful to know symmetries of the graph, hence we describe the automorphism groups of these graphs. Next, we investigate the domination number of the Fibonacci cubes as initiated by Pike and Zou in [PZ12], and also study that of the Lucas cubes. We give some connections between the domination number of Fibonacci cubes and Lucas cubes. Then we construct dominating sets for Γ_9 and for Λ_9 and a lower bound on the domination number of Λ_n. A graph invariant closely related to the domination number is the 2-packing number. First, we obtain a lower bound for the 2-packing number of the Lucas cubes and therefore, for the Fibonacci cubes. We also give the values for the 2-packing number of both cubes including those of dimension Γ_{10} and Λ_{10}.
Notice that the most part of this chapter is published in [CKMR11].

For $0 \leq k \leq n$, let $\Gamma_{n,k}$ be the set of vertices of Γ_n that contain k 1's. Hence $\Gamma_{n,k}$ is the set of vertices of Γ_n at distance k from 0^n. See figure 5.1. $\Lambda_{n,k}$ is defined analogously. In particular, $\Gamma_{n,0} = \Lambda_{n,0} = \{0^n\}$ and $\Gamma_{n,1} = \Lambda_{n,1} = \{10^{n-1}, 010^{n-2}, \ldots, 0^{n-1}1\}$. If $uv \in E(\Gamma_n)$, where $u \in \Gamma_{n,k}$ and $v \in \Gamma_{n,k-1}$ $(k \geq 1)$, then we say that v is a *down-neighbor* of u and that u is an *up-neighbor* of v. The same terminology again applies to Lucas cubes.

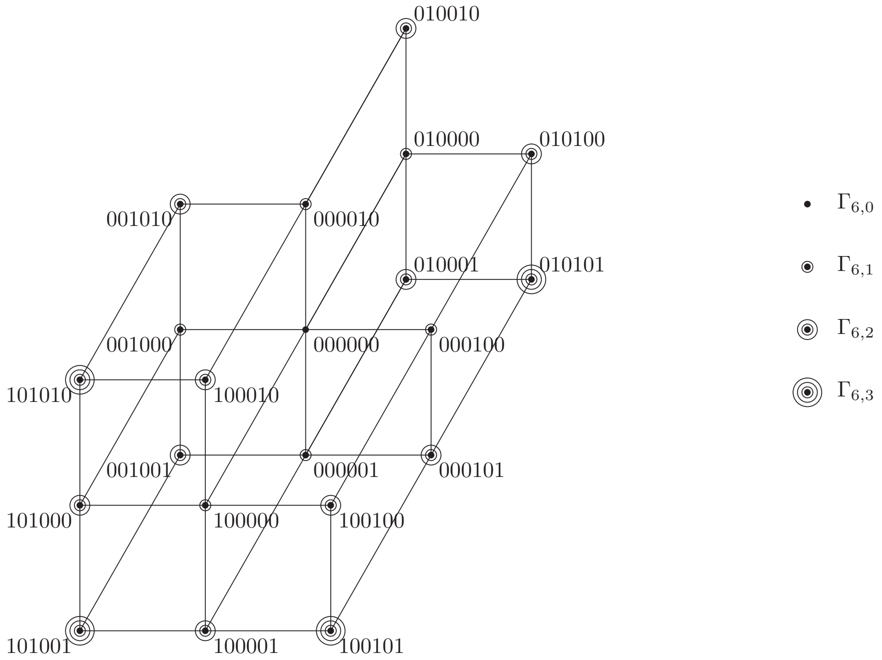


Figure 5.1: $\Gamma_{6,k}, k = 0, 1, 2, 3$.

For a binary string $b = b_1 b_2 \dots b_n$, let $\bar{b}$ be the binary complement of b and let $b^R = b_n b_{n-1} \dots b_1$ be the reverse of b. For binary strings b and c of equal length, let $b+c$ denote their sum computed bitwise modulo 2. For $1 \le i \le n$, let e_i be the binary string of length n with 1 in the i-th position and 0 elsewhere. According to this notation, $\Gamma_{n,1} = \Lambda_{n,1} = \{e_1, e_2, \dots, e_n\}$.

Let G be a graph. Then $D \subseteq V(G)$ is a *dominating set* if every vertex from $V(G) \setminus D$ is adjacent to some vertex from D. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A domination set $D(\Gamma_6) = \{010000, 001010, 001001, 000101, 100000\}$ of Γ_6 can be seen in figure 5.1.

A set $X \subseteq V(G)$ is called a *2-packing* if $d(u, v) > 2$ for any different vertices u and v of X. The *2-packing number* $\rho(G)$ is the maximum cardinality of a 2-packing of G. Notice that $\{010010, 010101, 001001, 101010, 100100\}$ is a 2-packing set in Γ_6 and therefore, $\gamma(\Gamma_6) = \rho(\Gamma_6) = 5$.
It is well known that for any graph G, $\gamma(G) \ge \rho(G)$. See [HHS98].

The automorphism group of a graph G is denoted by $\mathrm{Aut}(G)$. For instance, $\mathrm{Aut}(C_n) = D_{2n}$, where C_n is the n-cycle and D_{2n} is the *dihedral group* on n elements. Recall that D_{2n} can be represented as $\langle x, y \mid x^2 = 1, y^n = 1, (xy)^2 = 1\rangle$.

5.1 Automorphism groups

In this section we determine the automorphism groups of Fibonacci cubes and Lucas cubes.

Let $n \geq 1$ and define the *reverse map* $r : \Gamma_n \to \Gamma_n$ with:

$$r(b_1 b_2 \dots b_n) = b^R = b_n b_{n-1} \dots b_1 \,. \tag{5.1.1}$$

It is easy to observe that r is an automorphism of Γ_n. We are going to prove that r is the only nontrivial automorphism of Γ_n. For this sake, the following lemma is useful.

Lemma 5.1.1. *Let $n \geq 3$ and $k \geq 2$. Then $u, v \in \Gamma_{n,k}$ have different sets of down-neighbors.*

Proof. Since $u, v \in \Gamma_{n,k}$, $d(u, v) \geq 2$. We distinguish two cases.

Suppose first $d(u, v) = 2$ and let u and v differ in positions i and j. Since $u, v \in \Gamma_{n,k}$, we may assume without loss of generality that $u_i = v_j = 1$ and $u_j = v_i = 0$. Moreover, u and v agree in all the other positions. Since $k \geq 2$, there exists an index $\ell \neq i, j$ such that $u_\ell = v_\ell = 1$. Then $u + e_\ell$ is a down-neighbor of u but not a down-neighbor of v.

Assume now $d(u, v) \geq 3$. Let i be an arbitrary index such that $u_i \neq v_i$. We may assume that $u_i = 1$. Then $u + e_i$ is a down-neighbor of u but not of v. □

Theorem 5.1.2. *For any $n \geq 1$, $\mathrm{Aut}(\Gamma_n) \simeq \mathbb{Z}_2$.*

Proof. The assertion is clear for $n \leq 2$, hence assume in the rest that $n \geq 3$. Let $\alpha \in \mathrm{Aut}(\Gamma_n)$. Since 0^n is the only vertex of degree n, $\alpha(0^n) = 0^n$. Therefore, α maps $\Gamma_{n,1}$ onto $\Gamma_{n,1}$. Let $\Gamma'_{n,1} = \{10^{n-1}, 0^{n-1}1\}$ and $\Gamma''_{n,1} = \Gamma_{n,1} \setminus \Gamma'_{n,1}$. Since 10^{n-1} and $0^{n-1}1$ are the only vertices of degree $n-1$, α maps $\Gamma'_{n,1}$ and $\Gamma''_{n,1}$ onto $\Gamma'_{n,1}$ and $\Gamma''_{n,1}$, respectively. We distinguish two cases.

Case 1: $\alpha(10^{n-1}) = 10^{n-1}$.
Then, because α maps $\Gamma'_{n,1}$ onto $\Gamma'_{n,1}$, we have $\alpha(0^{n-1}1) = 0^{n-1}1$. Among the vertices of $\Gamma''_{n,1}$, only 010^{n-2} has no common up-neighbor with 10^{n-1}. Therefore, $\alpha(010^{n-2}) = 010^{n-2}$. In turn, among the remaining vertices of $\Gamma''_{n,1}$, only 0010^{n-3} has no common up-neighbor with 010^{n-2}. Therefore $\alpha(0010^{n-3}) = 0010^{n-3}$. By proceeding with the same argument, α fixes $\Gamma''_{n,1}$ pointwise and hence fixes $\Gamma_{n,1}$ pointwise. Now apply Lemma 5.1.1 and induction on k to conclude that α fixes $\Gamma_{n,k}$ pointwise for all k. Therefore $\alpha = \mathrm{id}$ in this case.

Case 2: $\alpha(10^{n-1}) = 0^{n-1}1$.
Now $\alpha(0^{n-1}1) = 10^{n-1}$. Among the vertices of $\Gamma''_{n,1}$, only 010^{n-2} has no common up-neighbor with 10^{n-1}. Thus $\alpha(010^{n-2}) = 0^{n-2}10$, which is the only element of $\Gamma''_{n,1}$ with no common up-neighbor together with $\alpha(10^{n-1}) = 0^{n-1}1$. By proceeding with the same argument, α reverses all the elements of $\Gamma''_{n,1}$, that is, $\alpha_{\Gamma''_{n,1}} = r_{\Gamma''_{n,1}}$ and consecutively $\alpha_{\Gamma_{n,1}} = r_{\Gamma_{n,1}}$. By Lemma 5.1.1 and induction on k, the same holds for any $\Gamma_{n,k}$, $k \geq 2$. Therefore $\alpha = r$ in this case. □

Let $n \geq 1$. An equivalent way to define Λ_n is that it is the subgraph of Q_n induced on all the binary strings of length n that have no two consecutive 1's in circular manner. This definition is more symmetric than the definition of the Fibonacci strings, so it is reasonable to expect that $\mathrm{Aut}(\Lambda_n)$ is richer than $\mathrm{Aut}(\Gamma_n)$. This is indeed the case. Define $\varphi : \Lambda_n \to \Lambda_n$ by

$$\varphi(b_1 b_2 \ldots b_n) = b_n b_1 \ldots b_{n-1} \,. \tag{5.1.2}$$

By the above remark it is clear that $\varphi \in \mathrm{Aut}(\Lambda_n)$. Zagaglia Salvi [Zag02] proved that the automorphism groups of the Lucas semilattices are the dihedral groups. The arguments that determine the automorphism group of the Lucas cubes are in a way parallel to the arguments from [Zag02], hence we next give just a sketch of them.

Note first that Lemma 5.1.1 (with the same proof) applies to Lucas cubes as well. Let $\alpha \in \mathrm{Aut}(\Lambda_n)$. Suppose that for some $a, b \in \{0, 1, \ldots, n-1\}$, $\alpha(10^{n-1}) = 0^a 10^{n-a-1}$ and $\alpha(0^{n-1}1) = 0^b 10^{n-b-1}$, where computations are mod n. Then either $b = a - 1$ or $b = a + 1$ because $\alpha(10^{n-1})$ and $\alpha(0^{n-1}1)$ cannot have a common up-neighbor. When $b = a - 1$ we get $\alpha = \varphi^a$ and in the other case $\alpha = \varphi^{a+1} \circ r$. We conclude that $\mathrm{Aut}(\Lambda_n)$ is generated by r and φ^a for $0 \leq a \leq n-1$, and hence:

Theorem 5.1.3. *For any $n \geq 3$, $\mathrm{Aut}(\Lambda_n) \simeq D_{2n}$.*

5.2 The domination number

In this section we consider the domination number of Fibonaci and Lucas cubes. We first interrelate their domination numbers. Then we discuss exact domination numbers for small dimensions. The section is conluded by establishing a general lower bound on the domination number of Lucas cubes.

Proposition 5.2.1. *Let $n \geq 4$, then*

(i) $\gamma(\Lambda_n) \leq \gamma(\Gamma_{n-1}) + \gamma(\Gamma_{n-3})$,

(ii) $\gamma(\Lambda_n) \leq \gamma(\Gamma_n) \leq \gamma(\Lambda_n) + \gamma(\Gamma_{n-4})$.

Proof. (i) $V(\Lambda_n)$ can be partitioned into vertices that start with 0 and vertices that start with 1. The latter vertices are of the form $10\ldots0$ and hance can be dominated by $\gamma(\Gamma_{n-3})$ vertices while the former vertices can be dominated by $\gamma(\Gamma_{n-1})$ vertices.

(ii) Let D be a minimum dominating set of Γ_n and set

$$D' = \{u \mid u \text{ is a Lucas string from } D\} \cup \{0b_2\ldots b_{n-1}0 \mid 1b_2\ldots b_{n-1}1 \in D\}.$$

A vertex $1b_2\ldots b_{n-1}1$ dominates two Lucas vertices, namely $0b_2\ldots b_{n-1}1$ and $1b_2\ldots b_{n-1}0$. Since these two vertices are dominated by $0b_2\ldots b_{n-1}0$, we infer that D' is a dominating set of Λ_n. It follows that $\gamma(\Lambda_n) \leq \gamma(\Gamma_n)$.

A dominating set of Λ_n dominates all vertices of Γ_n but the vertices of the form $10b_3\ldots b_{n-2}01$. These vertices can be dominated by $\gamma(\Gamma_{n-4})$ vertices. □

It can be easily checked that Proposition 5.2.1 (i) holds for any $n \geq 2$, and that the first inequality of Proposition 5.2.1 (ii) holds for any $n \geq 0$.

Pike and Zou [PZ12] obtained exact values of $\gamma(\Gamma_n)$ for $n \leq 8$, see Table 5.2. By computer search they found 509 minimum dominating sets of Γ_8. Following their approach we have computed the domination numbers of Λ_n, $n \leq 8$, see Table 5.2 again.

Hence the smallest Fibonacci cube and Lucas cube for which the domination numbers are not known are Γ_9 and Λ_9. Since $\gamma(\Gamma_n) \leq \gamma(\Gamma_{n-1}) + \gamma(\Gamma_{n-2})$, it follows that $\gamma(\Gamma_9) \leq 20$, cf. [PZ12, Lemma 3.1]. In order to find a smaller dominating set we have used a local search procedure, that is, to get a new dominating set we have replaced one or more vertices in the current dominating set with one or more vertices in their neighborhood. In this way we were able to construct a dominating set of Γ_9 of size 17 given on the left-hand side of Table 5.1. Similarly we have found a dominating set of Λ_9 of order 16 given on the right-hand side of Table 5.1. Hence:

Proposition 5.2.2. $\gamma(\Gamma_9) \leq 17$ *and* $\gamma(\Lambda_9) \leq 16$.

While we conjecture that $\gamma(\Gamma_9) = 17$ and $\gamma(\Lambda_9) = 16$ hold, Ilić and Milošević confirmed it later in [IM]. Furthermore, they established that $\gamma(\Gamma_{10}) = 25$.

Pike and Zou [PZ12] also proved that for any $n \geq 4$,

$$\gamma(\Gamma_n) \geq \left\lceil \frac{F_{n+2} - 3}{n - 2} \right\rceil.$$

010000000	000000000
100100000	000010000
010100000	000000100
001000100	000100100
000010010	000100010
000001010	000010010
000001001	101000010
101001000	100101000
101000010	010100001
100010100	010001010
100000101	001001001
001010001	101010100
000101001	101001010
000101010	010101001
000100101	010010101
101010001	001010101
010010101	

Table 5.1: A dominating set of Γ_9 and a dominating set of Λ_9

We next prove a parallel lower bound for the domination number of Lucas cubes. For this sake we first consider degrees of some specific vertices in Lucas cubes.

Recall that $\Lambda_{n,1}$ is the set of all the vertices with exactly one 1. In addition, set

$$\Lambda'_{n,2} = \{0^a 1010^{n-a-3} \mid 0 \le a \le n-1\},$$

where we again compute by modulo n. Hence $\Lambda'_{n,2}$ is the subset of $\Lambda_{n,2}$ consisting of the Lucas strings containing (in circular manner) 101 as a substring.

Lemma 5.2.3. *Let $n \ge 7$. Then for the Lucas cube Λ_n the followings hold.*

(i) The vertex 0^n is the only vertex of the maximum degree n.

(ii) The vertices of $\Lambda_{n,1}$ have degree $n-2$.

(iii) Among the vertices with at least two 1's, only the vertices of $\Lambda'_{n,2}$ have degree $n-3$ and all the other vertices have degree at most $n-4$.

Proof. (i) and (ii) are clear.

(iii) Let $u \in \Lambda_{n,k}$ for some $k \ge 2$. Then u has k down-neighbors. The up-neighbors of u are obtained by switching a bit 0 into 1. Let $i_1 < i_2 < \cdots < i_k$ be the positions in which u contains 1. Throughout the proof, the indices of

i's will be considered by modulo k and i_j by modulo n. As no consecutive bits of 1's are allowed, $i_{j+1} - i_j \geq 2$ for all $1 \leq j \leq k$. Let $I_j = \{i_j - 1, i_j + 1\}$ be the set of the positions which are adjacent to i_j for each $1 \leq j \leq k$ and let $I = \bigcup_{1 \leq j \leq k} I_j$. Then any bit which is not in I can be switched to 1 and hence the number of up-neighbors of u is $n - k - |I|$. Therefore, $\deg(u) = n - |I|$. Note that $I_j \cap I_{j'} = \emptyset$ if $|j - j'| \geq 2$, therefore by pigeon-hole principle, $|I| \geq k$. The equality holds if and only if $I_j \bigcap I_{j+1} \neq \emptyset$ for all $1 \leq j \leq k$, which occurs if and only if $i_{j+1} = i_j + 2$ for all $1 \leq j \leq k$, which is in turn if and only if n is even and $k = \frac{n}{2}$. But in this case, $\deg(u) = \frac{n}{2} \leq n - 4$ as $n \geq 8$. In the other cases, $|I| \geq k + 1$ and hence $\deg(u) \leq n - k - 1$. If $k \geq 3$, then $\deg(u) \leq n - 4$. Assume $k = 2$. Then $\deg(u) \leq n - 3$, where the equality holds exactly when $|I| = 3$ and $I_1 \bigcap I_2 \neq \emptyset$ which means that $u \in \Lambda'_{n,2}$. □

Lemma 5.2.4. *Any l vertices from $\Lambda'_{n,2}$ has at least l down-neighbors, that is, at least l neighbors in $\Lambda_{n,1}$.*

Proof. For $1 \leq i \leq l$, let A_i be the set of down-neighbors of $v_i \in \Lambda'_{n,2}$. Then $|A_i| = 2$ for each i. Considering bits by modulo n, each vertex $0^a 10^{n-a-1}$ in $\Lambda_{n,1}$ can be a down-neighbor of at most two vertices $0^a 1010^{n-a-3}$ and $0^{a-2} 1010^{n-a-1}$, and hence at most two of $v_1, \ldots, v_l$. By pigeon-hole principle, the assertion is true. □

To establish the announced lower bound, we will apply the natural concept of over-domination, just as it is done in [PZ12]. It is defined as follows. Let D be a dominating set of a graph G. Then the *over-domination* of G with respect to D is:

$$OD_G(D) = \sum_{v \in D} (\deg_G(v) + 1) - |V(G)| . \tag{5.2.1}$$

Note that $OD_G(D) = 0$ if and only if D is a perfect dominating set [LS90, HH07], that is, a dominating set such that each vertex is dominated exactly once.

Theorem 5.2.5. *For any $n \geq 7$, $\gamma(\Lambda_n) \geq \left\lceil \frac{L_n - 2n}{n - 3} \right\rceil$.*

Proof. Let D be a minimum dominating set of Λ_n. Set $D_1 = D \cap \Lambda_{n,1}$ and $D_2 = D \cap \Lambda'_{n,2}$, and let $k = |D \cap \Lambda_{n,1}|$ and $l = |D \cap \Lambda'_{n,2}|$. Then clearly $0 \leq k, l \leq n$. Note that the over-domination of G with respect to D can be rewritten as

$$OD(G) = \sum_{u \in V(\Lambda_n)} (|\{v \in D \mid d(u,v) \leq 1\}| - 1) . \tag{5.2.2}$$

For a vertex u of Λ_n, set $t(u) = |\{v \in D \mid d(u,v) \leq 1\}| - 1$. As D is a dominating set, $t(u) \geq 0$ for all $u \in V(\Lambda_n)$. We now distinguish two cases.

Case 1: $0^n \in D$.
Combining Lemma 5.2.3 with Equation (5.2.1) we get

$$\begin{aligned} OD(D) &\leq (n+1) + k(n-1) + l(n-2) + (\gamma(\Lambda_n) - k - l - 1)(n-3) - L_n \\ &= \gamma(\Lambda_n)(n-3) + 2k + l + 4 - L_n \,. \end{aligned}$$

Also as $t(u) \geq 0$ for all $u \in V$, Equation (5.2.2) implies

$$OD(D) \geq t(0^n) + \sum_{v \in D_1} t(v) \geq 2k \,.$$

Therefore $\gamma(\Lambda_n) \geq \lceil \frac{L_n - l - 4}{n-3} \rceil \geq \lceil \frac{L_n - n - 4}{n-3} \rceil$.

Case 2: $0^n \notin D$.
Again, combining Lemma 5.2.3 with Equation (5.2.1) we infer

$$\begin{aligned} OD(D) &\leq k(n-1) + l(n-2) + (\gamma(\Lambda_n) - k - l)(n-3) - L_n \\ &= \gamma(\Lambda_n)(n-3) + 2k + l - L_n \,. \end{aligned}$$

Let A be the set of down-neighbors of D_2. Then for $u \in D_1 \cap A$, $t(u) \geq 1$. By Lemma 5.2.4, $|A| \geq l$ and hence $|D_1 \bigcap A| \geq k + l - n$. Therefore by Equation (5.2.2),

$$OD(D) \geq \sum_{v \in D_1 \bigcap A} t(v) \geq k + l - n \,.$$

Thus $\gamma(\Lambda_n) \geq \lceil \frac{L_n - k - n}{n-3} \rceil \geq \lceil \frac{L_n - 2n}{n-3} \rceil$.

By Case 1 and Case 2, $\gamma(\Lambda_n) \geq \lceil \frac{L_n - 2n}{n-3} \rceil$. □

5.3 The 2-packing number

We now turn to the 2-packing number and first prove the following asymptotical lower bound.

Theorem 5.3.1. *For any $n \geq 8$, $\rho(\Gamma_n) \geq \rho(\Lambda_n) \geq 2^{2^{\frac{\lfloor \lg n \rfloor}{2} - 1}}$.*

Proof. Since for any $n \geq 1$, Λ_n is an isometric subgraph of Γ_n, cf. [Kla05], a 2-packing of Λ_n is also a 2-packing of Γ_n. Therefore $\rho(\Gamma_n) \geq \rho(\Lambda_n)$.

Let $r, s \geq 1$ and let X and Y be maximum 2-packings of Λ_r and Λ_s, respectively. Then $\{x0y \mid x \in X, y \in Y\}$ is a 2-packings of Λ_{r+s+1} of size $\rho(\Lambda_s)\rho(\Lambda_s)$. It follows that

$$\rho(\Lambda_{r+s+1}) \geq \rho(\Lambda_r)\rho(\Lambda_s)\,.$$

Set now $k = \lfloor \lg n \rfloor$. Then $\rho(\Lambda_{2^k}) \geq \rho(\Lambda_{2^{k-1}+1}) \geq \rho(\Lambda_{2^{k-2}})^2$. By repeatedly applying this argument we get

$$\rho(\Lambda_n) \geq \rho(\Lambda_{2^k}) \geq \rho(\Lambda_{2^{k-2l}})^{2^l}\,.$$

When k is even, take $l = \frac{k-2}{2}$ to get $\rho(\Lambda_n) \geq \rho(\Lambda_4)^{2^{\frac{k-2}{2}}} = 2^{2^{\frac{k-2}{2}}}$. When k is odd, take $l = \frac{k-3}{2}$ to get $\rho(\Lambda_n) \geq \rho(\Lambda_8)^{2^{\frac{k-3}{2}}} \geq 8^{2^{\frac{k-3}{2}}} = 2^{3\times 2^{\frac{k-3}{2}}} \geq 2^{2^{\frac{k-2}{2}}}$. $\square$

Using computer we obtained the 2-packing numbers of Γ_n and Λ_n for $n \leq 10$ given in Table 5.2.

n	0	1	2	3	4	5	6	7	8	9	10
$\gamma(\Gamma_n)$	1	1	1	2	3	4	5	8	12	$\leq$17	-
$\rho(\Gamma_n)$	1	1	1	2	2	3	5	6	9	14	20
$\gamma(\Lambda_n)$	1	1	1	1	3	4	5	7	11	$\leq$16	-
$\rho(\Lambda_n)$	1	1	1	1	2	3	5	6	8	13	18

Table 5.2: Domination numbers and 2-packing numbers of small cubes

Table 5.2 needs several comments.

- The computer search found exactly ten 2-packings of size 20 in Γ_{10}. This already implies that $\rho(\Gamma_{10}) = 20$. Indeed, if Γ_{10} would contain a 2-packing of size 21, then it would contain twenty-one 2-packings of size 20.
- By exhaustive search with computer no 2-packing of size 19 in Λ_{10} was found, hence $\rho(\Lambda_{10}) = 18$.
- There is only one (up to isomorphisms of the graphs considered) maximum 2-packing of Λ_5, Λ_6, Λ_7, Λ_9, as well as Γ_6. There are two non-isomorphic 2-packings of maximum cardinality of Γ_9, they are presented in the first two columns of Table 5.3.

Since the reverse map given in (5.1.1) is an automorphism of Fibonacci cubes, the reverse of a 2-packing is also a 2-packing. Interestingly, the maximum 2-packing of Γ_9 shown on the left-hand side of Table 5.3, denoted X, is also invariant under the reverse map. That is, $r(X) = X$.

Similarly, the shifts φ^i, where φ is given in (5.1.2) and are automorphisms of Lucas cubes, hence they map 2-packings into 2-packings. Now consider the 2-packing of Λ_9 shown on the right-hand side of Table 5.3, denote it Y. Then it can be checked that $\varphi^3(Y) = Y$. As a consequence, $\varphi^6(Y) = Y$.

000 001 010	000 001 000	100 100 100
010 100 000	000 100 100	000 010 001
000 100 101	001 000 010	000 101 001
101 001 000	001 010 001	001 000 010
001 000 001	010 000 101	001 000 101
100 000 100	010 010 000	010 001 000
010 001 001	010 100 010	010 010 100
100 100 010	010 101 001	010 100 010
010 010 101	100 010 010	010 100 101
101 010 010	100 010 101	100 010 010
001 010 100	100 100 001	100 101 010
010 010 010	100 101 010	101 001 000
100 010 001	101 000 100	101 010 100
100 101 001	101 001 001	

Table 5.3: Maximum 2-packings of Γ_9 and of Λ_9

In [CKMR11], we propose the following conjectures:

(i) $\gamma(\Gamma_n) - \rho(\Gamma_n) \geq \gamma(\Lambda_n) - \rho(\Lambda_n)$ for $n \geq 0$?

(ii) $\gamma(\Lambda_n) \geq \rho(\Gamma_n)$ for $n \geq 4$?

(iii) $\gamma(\Lambda_n) \leq \gamma(\Gamma_{n-1}) + \gamma(\Gamma_{n-3}) - 1$ for $n \geq 6$?

Note that the last question, if it has an affirmative answer, reduces the bound of $\gamma(\Lambda_n)$ in Proposition 5.2.1 (i) by 1. Moreover, if (iii) is true, then one can also ask whether $\gamma(\Lambda_n) \leq \gamma(\Gamma_{n-1}) + \gamma(\Gamma_{n-4})$ holds for $n \geq 6$.

CHAPTER 6

Conclusions

This work studies some structural and enumerative properties of the Fibonacci and the Lucas cubes. Originally introduced by W. -J. Hsu as an interconnection network, the Fibonacci cube is an isometric subgraph of the hypercube. Closely related to this class of graphs is the Lucas cube, introduced by Munarini, Cippo and Zagaglia Salvi as a subgraph of the Fibonacci cube and hence of the hypercube. The Fibonacci and the Lucas cubes have been a subject of research because of their structural properties. Some of the structural results that we discuss are the recursive decompositions, mainly, the fundamental decomposition of the Fibonacci cube which leads to many important structural properties.

Based on the fundamental decomposition, Liu, Hsu and Chung constructed cycles of every even length from 4 to $|V(\Gamma_n)|$ when Γ_n has even order. Using classical graph theory techniques, and in response to a question posed in the literature, we characterized the vertex set such that removing one of its vertices from the graph, it admits a Hamiltonian cycle (when Γ_n has odd order). As a corollary to this result, it was proved that $\Gamma_n - v$ is furthermore bipancyclic.

Turning to the Lucas cube Hamiltonicity, similar results are shown to characterize the vertex set such that removing any vertex (or two vertices in one case) of the set from the graph, it contains a Hamiltonian cycle. Baril and Vajnovszki previously showed the existence of a Hamiltonian path when $n \not\equiv 0\,(mod\,3)$. Extending this result, Hamiltonian cycles are constructed for $\Lambda_n - v$ when $n \not\equiv 0\,(mod\,3)$.

In the case of $n \equiv 0\,(mod\,3)$, bipancyclicity is proven for the graph $\Lambda_n \setminus \{v_1, v_2\}$ for particular vertices v_1 and v_2. Thus it would be interesting to know which pair of vertices we can remove in order to have a Hamiltonian cycle.

As stated by Bertsekas and Tsitsiklis, interconnection networks are usually evaluated in terms of their suitability for some standard communication tasks. Some typical criteria include the diameter and the connectivity of the network. While the diameter and the radius of the Fibonacci cube were obtained by Munarini and Zagaglia, the same invariants for the Lucas cube were determined by Munarini, Cippo and Zagaglia.

In this work, we characterized the vertices of Γ_n that satisfy the eccentricity of a given vertex x as well as the eccentricity of x. We computed afterwards, the generating function of the eccentricity sequence of the Fibonacci cube's strings. As a corollary, the number of vertices of eccentricity k of a given Fibonacci cube was obtained. Regarding the Lucas cube, similar results were shown. Bijective proofs to show that two sets have the same cardinality were used as well as the generating functions technique that allowed us to obtain an exact formula of the members of the eccentricity sequences.

In the last part of this work, the automorphism groups of the Fibonacci and the Lucas cubes are determined.

Broading a previous result due to Pike and Zou, we determined the domination numbers for Γ_9 and Λ_9, using computing techniques. These results were later extended by Ilić and Milošević.

Also, different upper and lower bounds were proven for the domination number and for the 2-packing number of the Fibonacci Lucas cubes. This work concluded with some conjectures that associate the domination and the 2-packing numbers of both the Fibonacci and the Lucas cubes which makes the subject wide open for further research.

Appendix A
Definitions

In that which follows, we present some basic graph theory definitions, following Bondy cf. [BM08] and Harary cf. [Har94].

A *graph* G is a set of vertices V and set E of unordered pairs of elements of V called edges.

The *order* of a graph is the number of vertices $|V|$ and the graph's size $|E|$ is the number of edges.

A graph with only one vertex and no edges is called the *trivial graph*.

An edge with identical ends is called a *loop* and two or more edges with the same pair of ends are said to be *parallel edges*. A graph is *simple* if it has no loops or parallel edges.

A *complete graph* is a simple graph in which any two vertices are adjacent. The complete graph of order n is denoted by K_n.

The *degree* of a vertex is the number of edges that connect to it. The ends of an edge are said to be *incident* with the edge, two vertices which are incident with a common edge are adjacent and two distinct adjacent vertices are *neighbors*. The set of neighbors of a vertex $v \in V$ is denoted by $N(v)$.

If $V' \subseteq V$, thus G[V'] is the *subgraph of G induced by V'* whose vertex set is V' and whose edge set consists of all edges of G which have both ends in V'.

A *path* is a simple graph whose vertices can be arranged in a linear sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise. If there is a path between any two vertices of G, then G is *connected*, otherwise *disconnected*. A *cycle* is a connected graph where every vertex has exactly two neighbors. The *length of a path or a cycle* is the number of its edges.

A graph is *bipartite* if its vertex set can be partitioned into two subsets V_1 and V_2 so that every edge has one end in V_1 and one end in V_2; such a partition (V_1, V_2) is called a bipartition of the graph, and V_1 and V_2 its parts.

The *cartesian product* of simple graphs G and H is the graph $G \square H$ whose vertex set is $V(G) \times V(H)$ and whose edge set is the set of all pairs $((u_1, v_1), (u_2, v_2))$ such that either $(u_1, u_2) \in E(G)$ and $v_1 = v_2$, or $(v_1, v_2) \in E(H)$ and $u_1 = u_2$.

The *n-cube* or *n-dimensional hypercube* Q_n is defined recursively in terms of the cartesian product of two graphs as follows:

$$Q_1 = K_2$$

$$Q_n = K_2 \square Q_{n-1}$$

The n-cube, Q_n may also be defined as a graph whose node set V, consists of the 2^n n-dimensional boolean vectors, i.e., vectors with binary coordinates 0 or 1, where two nodes are adjacent whenever they differ in exactly one coordinate.

A *Hamiltonian path* is a path in a graph that visits each vertex exactly once. A *Hamiltonian cycle* is a Hamiltonian path that is a cycle. A *Hamiltonian graph* is a graph having a Hamiltonian cycle.

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Codes de Gray généralisés à l'énumération des objets d'une structure combinatoire sous contrainte

Résumé : Le cube de Fibonacci est un sous-graphe isométrique de l'hypercube ayant un nombre de Fibonacci de sommets. Le cube de Fibonacci a été initialement introduit par W-J. Hsu comme un réseau d'interconnexion et, comme l'hypercube, il a des propriétés topologiques très attractives, mais avec une croissance plus modérée. Parmi ces propriétés, nous discutons de l'hamiltonicité dans le cube de Fibonacci et aussi dans le cube de Lucas qui est obtenu à partir du cube de Fibonacci en supprimant toutes les chaînes qui commencent et finissent avec 1. Nous trouvons également le nombre de sommets des cubes de Fibonacci et Lucas ayant une certaine excentricité. Enfin, nous présentons une étude de deux cubes du point de vue de la domination et du 2-packing.
Mots clés : Cube de Fibonacci, Cube de Lucas, Hypercube, Hamiltonicité, Excentricité, Domination, 2-packing, Code de Gray

Generalised Gray codes for the enumeration of the objects of a combinatorial structure under certain restrictions

Abstract: The Fibonacci cube is an isometric subgraph of the hypercube having a Fibonacci number of vertices. The Fibonacci cube was originally proposed by W-J. Hsu as an interconnection network and like the hypercube it has very attractive topological properties but with a more moderated growth. Among these properties, we discuss the hamiltonicity in the Fibonacci cube and also in the Lucas cube which is obtained by removing all the strings that begin and end with 1 from the Fibonacci cube. We give also the eccentricity sequences of the Fibonacci and the Lucas cubes. Finally, we present a study of both cubes from the domination and the 2-packing points of view.
Keywords: Fibonacci cube, Lucas cube, Hypercube, Hamiltonicity, Eccentricity Sequence, Vertex Domination, 2-packing, Gray code