Graphs Orientations : structures and algorithms

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Orientations des graphes : structures et algorithmes

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ABSTRACT

Orienting an undirected graph means replacing each edge by an arc with the same ends. We investigate the connectivity of the resulting directed graph. Orientations with arc-connectivity constraints are deeply understood but very few results are known in terms of vertex-connectivity. Thomassen conjectured that sufficiently highly vertex-connected graphs have a k-vertex-connected orientation while Frank conjectured a characterization of the graphs admitting such an orientation.

The results of this thesis are structures around the concepts of orientation, packing, connectivity and matroid. First, we disprove a conjecture of Recski on decomposing a graph into trees having orientations with specified indegrees. We also prove a new result on packing rooted arborescences with matroid constraints. This generalizes a fundamental result of Edmonds. Moreover, we show a new packing theorem for the bases of count matroids that induces an improvement of the only known result on Thomassen's conjecture.

Secondly, we give a construction and an augmentation theorem for a family of graphs related to Frank's conjecture. To conclude, we disprove the conjecture of Frank and prove that, for every integer $k \geqslant 3$, the problem of deciding whether a graph admits a k-vertex-orientation is NP-complete.

RÉSUMÉ

Orienter un graphe c'est remplacer chaque arête par un arc de mêmes extrémités. On s'intéresse à la connexité du graphe orienté ainsi obtenu. L'orientation avec des contraintes d'arc-connexité est comprise en profondeur mais très peu de résultats sont connus en terme de sommet-connexité. La conjecture de Thomassen avance que les graphes suffisament sommet-connexes ont une orientation k-sommet-connexe. De plus, la conjecture de Frank propose une caractérisation des graphes qui admettent une telle orientation.

Les résultats de cette thèse s'articulent autour des notions d'orientation, de packing, de connexité et de matroïde. D'abord, nous infirmons une conjecture de Recski sur la décomposition d'un graphe en arbres ayant des orientations avec degrés entrants prescrits. Nous prouvons également un nouveau résultat sur le packing d'arborescences enracinées avec contraintes de matroïdes. Ceci généralise un résultat fondamental d'Edmonds. Enfin, nous démontrons un nouveau théorème de packing sur les bases des matroïdes de dénombrement qui nous permet d'améliorer le seul résultat connu sur la conjecture de Thomassen.

D'autre part, nous donnons une construction et un théorème d'augmentation pour une famille de graphes liée à la conjecture de Frank. En conclusion, nous réfutons la conjecture de Frank et prouvons que, pour tout entier $k \geqslant 3$, décider si un graphe a une orientation k-sommet-connexe est un problème NP-complet.

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CONTENTS

1 INTRODUCTION

The notion of networks arises in many different forms. A classic one is the social network describing the friendship relations among a group of people. Another example is the natural food web depicting feeding connections between species of an ecosystem. There are many other examples, but all have a common pattern picturing a set of nodes linked to each other.

We may distinguish two types of networks depending on whether the links transpose two-way relations or one-way relations. In the first case networks are called undirected in opposition to directed in the second case. The social network is an example of undirected network since anyone is a friend of every friend of his/hers. On the other side, the food web is directed as preys do not eat their predators. The type of the road network of a city, where the nodes are crossings and each link is a street joining two crossings, is contingent to the fact that all the streets are two-way or one-way.

Graphs and directed graphs (or, for short, digraphs) are the mathematical tools capturing the common pattern of undirected networks and directed networks respectively. In both objects a set of vertices represents the set of nodes of the network. In graphs the links are called edges while in digraphs they are called arcs. The development of Graph Theory provides powerful approaches to address network design issues. For instance, city road networks are typically required to be fault tolerant, that is, any place in town should always be reachable from any other even if the traffic is cut in a few streets or crossings. In Graph Theory, this robustness property transforms into a connectivity property.

However, in an undirected road network any path joining two places may be used in both directions while this symmetric property does not hold in a directed one. So giving a direction to each street in an originally undirected road network would drastically change its robustness and one may try to find an assignment of directions such that the resulting directed road network achieves nice robustness properties. In graphs, this operation of changing each two-way link into a one-way link is called an orientation and the problem of finding a suitable assignment of directions becomes a problem of finding an orientation of a graph satisfying connectivity properties. This problem is the guiding principle of this document.

Connectivity is a fundamental notion in Graph Theory that is encountered in a countless number of practical or theoretical problems while orientation problems are more unusual. So this thesis is in a field where an essential notion and a less frequent concept meet.

LEGACY

As the connectivity notion is not symmetric in digraphs we differentiate the notions derived from the strong-connectivity defined by the existence of a directed path from any vertex to any other from the notions resulting from the rooted connectivity that focuses only on the existence of a directed path from a special vertex, called the root, to any other vertex.

In the field of graph orientations under connectivity constraints, the first contribution is due to Robbins [65] who proved in 1939 a characterization of graphs admitting a strongly-connected orientation. His proof technique is based on a decomposition of 2-edge-connected graphs called the ear decomposition. Twenty years later, Nash-Williams [60] settled the general problem of k-arc-connected orientation, and even more. He proved that every graph has a well-balanced orientation, that is, an orientation such that the resulting arc-connectivity from any vertex u to any other vertex v is at least half (rounded down) of the edge-connectivity between u and v in the original undirected graph. This proves that every $2k$-edge-connected graph admits a k-arc-connected orientation; proving the converse is trivial. The key idea of Nash-Williams is to reduce to the trivial Eulerian case by adding to the graph a suitable pairing of the vertices of odd degree called an odd-pairing. Now there exist alternative proofs of the characterization of graphs admitting a k-arc-connected orientation.

The splitting-off technique introduced by Lovász [50] together with a result of Lick [48] yields a new approach of the k-arc-connected orientation problem via a constructive characterization of $2k$-edge-connected graphs. Another approach is developed by Frank [21] who solved the problem of covering a supermodular set function. Frank [22] also showed how the submodular flows techniques introduced by Edmonds and Giles [15] properly addresses in polynomial time the initial k-arc-connected orientation problem with additional constraints such as minimizing a certain cost or bounding the indegree of vertices. Bernáth, Iwata, T. Király, Z. Király and Szigeti [3] showed that these two problems become NP-complete if one considers well-balanced orientations. This algorithmic point of view draw a clear line separating the problem of k-arc-connected orientation and the problem of well-balanced orientation.

For rooted arc-connectivity, a characterization of graphs having a rooted k-arc-connected orientation may be derived from the classic result of Tutte [69] and Nash-Williams [61] on packing spanning trees and the theorem of Edmonds [14] on packing spanning arborescences. A direct proof was given by Frank [19].

In terms of vertex-connectivity, Thomassen [68] conjectured that every sufficiently highly vertex-connected graph has a k-vertex-connected orientation and Frank [26] conjectured a characterization of the graphs having such an orientation. The basic case $k = 1$ is clearly settled by the theorem of Robbins. For $k = 2$, Jordán [41] proved a construction of the family of graphs given in Frank's conjecture which enabled Berg and Jordán [2] to show that the conjecture of Frank holds for weakly 4-connected Eulerian graphs. Using this result and proving a packing theorem on spanning rigid subgraphs, Jordán [40] settled the case $k = 2$ of Thomassen conjecture. Z. Király and Szigeti [46] gave a generalization of the Berg and Jordán's orientation theorem based on the odd-pairing theorem of Nash-Williams.

OUTLINE AND CONTRIBUTIONS

Chapter 2 gathers many definitions and results from Graph Theory and Matroid Theory. Although most of them are basic, the definitions of bi-sets, bi-set functions and count matroids are not so common. This chapter also defines the g-bounded connectivity together with the suitable Menger's theorem.

Chapter 3 offers a glimpse at two well understood areas of graphs orientation. The first one concerns with degree constrained orientation problems. We state the classic results of the field and present a very modest contribution resulting from a joint work with Klein, Nguyen[1] and Szigeti [8] that emphasis the link between Matroid Theory and graph orientation.

The second one is the k-arc-connected orientation problem. We present three different successful approaches for this problem: the approach of Lovász enabled by his splitting-off technique, the approach of Frank via a general orientation theorem on covering a supermodular function and the other approach of Frank that applies submodular flows techniques.

To conclude this chapter, we explain the original approach of Nash-Williams' for the k-arc-connected problem that leads to a deeper orientation theorem. We also answer negatively a modest question asked in [3] about a possible generalization of the odd-pairing technique.

Chapter 4 focuses on packing results. In Section 4.1, we state the classic result of Tutte and Nash-Williams on packing spanning trees and one of its recent generalization due to Katoh and Tanigawa [44] where the spanning property is replaced by a matroid constraint. Then we look into a variation of the spanning tree packing problem offered by Recski [64] where orientation constraints are added. We prove this variation to be NP-complete [7] and so, provided $P \neq NP$, we disprove a conjecture of Recski.

We mentioned that the characterization of graphs admitting a rooted k-arc-connected orientation problem is a direct corollary of the classic result of Tutte and Nash-Williams on packing spanning trees and its directed counterpart, the Edmond's theorem on packing spanning arborescences. Frank [19] gave a direct proof of the characterization and show how to derive the undirected packing of spanning trees from its directed counterpart. In Section 4.2, we mimic this approach to derive the Katoh and Tanigawa result from its directed counterpart via the theorem of Frank on covering supermodular functions. This material is from a joint work with Nguyen and Szigeti [9].

To settle the case $k = 2$ of Thomassen's conjecture, Jordán proved a sufficient connectivity condition for the existence of k edge-disjoint spanning rigid subgraphs that generalizes a classic result of Lovász and Yemini [51]. In Section 4.3, we weaken the sufficient condition given by Jordán and extend the packing to spanning trees as well. This work provides a simpler proof of a results of Jackson and Jordán [39] and enable us to improve upper bounds given by Jordán on a conjecture of Kriesell and the Thomassen's conjecture. This material results from a collaboration with Cheriyan and Szigeti [5]. Finally, we come to a more general result addressing the packing of bases of a quite general count matroid and spanning trees that comes from a joint work with Nguyen.

In Chapter 5, first we discuss the conjecture of Thomassen and the two conjectures of Frank arising questions on the k-vertex-connected orientations and rooted k-vertex-connected orientations problems.

To address these conjectures, the successful approach of Lovász for the k-arc-connected orientation problem motivates the study of the structure of highly connected graphs. Jordán followed this path and proved a constructive characterization of weakly 4-connected graphs that arise in the conjecture

1 To dispel any doubts, in this thesis Nguyen always refers to Nguyen Viet-Hang.

of Frank. As for the approach of Lovász, the result of Jordán lies on a new splitting-off theorem and a result of Kaneko and Ota [43] about the existence of vertices of degree 4. In a more general context, we investigate the existence of tight vertices that are of minimal degree with respect to the connectivity of the graph. More specifically, we prove that every minimally g-bounded k-connected graphs contains at least one tight vertex. The existence of tight vertices has also been studied in digraphs by Mader [53, 54]. We give a common generalization of a slight generalization of two results of Mader in this field.

In the last section of Chapter 5, we generalize the characterization of weakly 4-connected graphs to a larger class of graphs, namely the family of mixed $(2k, k)$-connected graphs. To that extend we prove a new splitting-off theorem settling also an augmentation problem. The material of this chapter results from a joint-work with Zoltán Szigeti.

Finally, in Chapter 6 we disprove the two above mentioned conjectures of Frank for $k \geqslant 3$ and prove that the problem of deciding whether a graph has a k-vertex-connected orientation and the problem of deciding whether a graph has a rooted k-vertex-connected orientation are both NP-complete for $k \geqslant 3$.

La notion de réseau apparaît de nombreux contextes. L'une des plus classiques est le réseau social qui décrit les relations d'amitié entretenues dans un groupe de personnes. Un autre exemple est la chaîne alimentaire qui représente les relations de prédation entre les espèces d'un écosystème. Il existe bien d'autres exemples mais tous dessinent ce motif consitiué de nœuds reliés entre eux.

On peut cependant distinguer deux types de réseaux suivant que les liens entre les nœuds traduisent des relations bilatérales ou unilatérales. Dans le premier cas le réseau est qualifié de non-orienté par opposition au second cas où il est qualifié d'orienté. Le réseau social est un exemple de réseau non-orienté puisque toute personne est l'amie de ses propres amis. En revanche, la chaîne alimentaire est orientée puisqu'une proie ne mange pas ses prédateurs. Le réseau routier d'une ville, dont les nœuds sont les carrefours et dont les liens sont les portions de rues reliant ces carrefours, est non-orienté si toutes les rues sont à double sens et orienté si toutes les rues sont à sens unique.

Les graphes et les graphes orientés sont les outils mathématiques qui capturent respectivement l'essence des réseaux non-orientés et orientés. Dans ces deux objets l'ensemble des nœuds du réseau est représenté par un ensemble de sommets. Les liens entres les nœuds sont appelés arêtes dans les graphes et arcs dans les graphes orientés. Le développement de la Théorie des Graphes fournit des méthodes efficaces pour aborder certaines problématiques concernant la conception de réseaux. Par exemple, une exigence typique est la résistance du réseau routier d'une ville, c'est à dire que n'importe quel lieu de la ville doit rester accessible depuis n'importe quel autre même si la circulation est coupée dans quelques rues ou carrefours. En Théorie des Graphes cette propriété de robustesse se traduit en une propriété de dite de connexité.

Cependant dans un réseau routier non orienté, un chemin reliant deux lieux peut être emprunter dans les deux sens alors que, dans un réseau routier orienté, cette propriété de symétrie n'est plus valable. Par conséquent donner un sens unique de circulation à chacune des rues dans un réseau routier dont les rues sont initialement à double sens va considérablement changer sa robustesse. On peut alors chercher à trouver un choix des sens uniques qui permette d'obtenir certaines propriétés de robustesse. Dans un graphe, l'opération consistant à mettre chaque rue en sens unique est appelée orientation et chercher un bon choix des sens uniques revient à chercher une orientation du graphe qui satisfait certaines propriétés de connexité. Ce problème est le fil rouge de cette thèse.

La connexité est une notion fondamentale en Théorie des Graphes qui se rencontre dans d'innombrables problèmes tant pratiques que théoriques, tandis que, les problèmes d'orientation sont moins habituels. Ainsi, cette thèse investit un champs à l'intersection d'une notion essentielle avec un concept plus exotique.

HÉRITAGE

Puisque la notion de connexité n'est pas symétrique dans les graphes orientés nous aurons soin de distinguer les notions dérivant de la forte connexité, définie comme l'existence d'un chemin allant de n'importe quel sommet à n'importe quel autre, des notions dérivant de la racine-connexité qui s'intéresse seulement à l'existence de chemins allant d'un sommet spécifié, appelé la racine, vers les autres sommets.

En terme d'orientation des graphes sous contraintes de connexité, la première contribution est due à Robbins [65] qui a prouvé en 1939 une caractérisation des graphes qui ont une orientation fortement connexe. Sa preuve s'appuie sur une décomposition des graphes 2-arête-connexes appelée décomposition en oreilles. Vingt ans plus tard, Nash-Williams [60] a résolu le problème plus général de l'orientation k-arc-connexe, et plus encore. Il a prouvé que tout graphe a une orientation bien équilibrée, c'est dire, une orientation dans laquelle l'arc-connexité d'un sommet quelconque u à un autre sommet quelconque v est au moins la moitié (arrondie inférieurement) de l'arête-connexité entre u et v dans le graphe non-orienté de départ. Cela prouve que tout graphe $2k$-arête connexe a une orientation k-arc-connexe; la réciproque est triviale. L'idée clef de Nash-Williams est de se ramener au cas simple des graphes eulériens en ajoutant au graphe un couplage approprié des sommets impairs. Il existe aujourd'hui d'autres preuves de la caractérisation des graphes ayant une orientation k-arc-connexe.

La technique de séparation introduite par Lovász [50] utilisée avec un résultat de Lick [48] donne une nouvelle approche pour le problème de l'orientation k-arc-connexe au travers d'une caracterisation constructive des graphes $2k$-arête-connexes. Une autre approche est développée par Frank [21] qui résoud le problème de couvrir un fonction sur-modulaire avec une orientation. Frank [22] a aussi expliqué comment la technique des flots sous-modulaires introduite par Edmonds and Giles [15] répond en temps polynomial au problème initial de trouver une orientation k-arc-connexe avec des contraintes additionnelles telles que minimiser un certain coût ou borner les degrés entrants des sommets. Bernáth, Iwata, T. Király, Z. Király and Szigeti [3] ont montré que ces deux problèmes deviennent NP-complets si on considère des orientations bien-équilibrées. Ce point de vue algorithmique dessine une séparation claire entre les problèmes liés à l'orientation k-arc-connexe et ceux issus de l'orientation bien-équilibrée.

Quant à la racine-connexité, une caractérisation des graphes qui ont une orientation racine k-arc-connexe peut être déduite des résultats classiques de

Tutte [69] and Nash-Williams [61] concernant le packing d'arbres couvrants et le théorème d'Edmonds [14] concernant le packing d'arborescences couvrantes. Une preuve directe a été données par Frank [19].

En termes de sommet-connexité, Thomassen [68] a conjecturé qu'un graphe dont la sommet-connexité est suffisament grande a une orientation k-sommet-connexe et Frank [26] a conjecturé une caractérisation des graphes qui ont une telle orientation. Le cas $k = 1$ est résolu par le théorème de Robbins. Pour $k = 2$, Jordán [41] a déterminé une construction de la famille de graphes en question dans la conjecture de Frank qui a ensuite permis à Berg et Jordán [2] de prouver que la conjecture de Frank est vraie pour les graphes eulériens et faiblement 4-connexes. En utilisant ce résultat ainsi qu'en prouvant un résultat de packing pour les sous-graphes rigides couvrant, Jordán [40] a résolu le cas $k = 2$ de la conjecture de Thomassen. Z. Király et Szigeti [46] ont donné une généralisation du théorème d'orientation de Berg et Jordán basée sur le théorème de couplage des sommets impairs de Nash-Williams.

PLAN ET CONTRIBUTIONS

Le Chapitre 2 rassemble de nombreuses définitions et résultats issus des la Théorie des Graphes et de la Théorie des Matroïdes. Bien que la plupart soit basique, les définitions de bi-ensembles, fonctions sur les bi-ensembles et count matroïdes ne sont pas si usuelles. Ce chapitre définit également la connexité g-bornée ainsi que le théorème de Menger correspondant.

Le Chapter 3 donne, en premier lieu, un aperçu rapide sur deux domaines bien compris de l'orientation des graphes. Le premier concerne l'orientation avec contraintes sur les degrés entrants. Nous énonçons le résultat classique du domaine et présentons une bien modeste contribution issue d'une collaboration avec Klein, Nguyen[2] et Szigeti [8] qui souligne le lien entre la Théorie des Matroïdes et l'orientation des graphes.

Le second concerne le problème de l'orientation k-arc-connexe. Nous présentons trois approches fructueuses pour ce problème : l'approche de Lovász rendue possible par la technique de séparation, l'approche de Frank via son théorème sur les orientations couvrant des fonctions sur-modulaires et l'autre approche de Frank qui utilise les techniques de flots sous-modulaires.

Pour conclure ce chapitre, nous expliquons l'approche originel de Nash-Williams pour le problème de l'orientation k-arc-connexe qui amène à un résultat encore plus profond. De plus, nous répondons par la négative à une question posée dans [3] et qui concerne une possible généralisation de la technique de couplage des sommets impaires.

Le Chapitre 4 se concentre sur des résultats de packing. Dans la section 4.1, nous énonçons le résultat classique de Tutte et Nash-Williams à propos du packing d'arbres couvrant et sa récente généralisation par Katoh and Tanigawa [44] où la propriété de couverture est remplacée par une contrainte de matroïde. Ensuite nous nous penchons sur autre variation du problème de packing d'arbres couvrants proposée par Recski [64] où une contrainte d'orientation est ajoutée. Nous montrons que cette variation est NP-complète [7] et nous infirmons donc une conjecture de Recski, si tant est que $P \neq NP$.

Nous avons mentionné précédement que la caractérisation des graphes ayant une orientation racine k-arc-connexe est un corollaire direct du ré-

2 Afin de dissiper tout doute possible, dans cette thèse Nguyen désigne toujours Nguyen Viet-Hang.

sultat classic de Tutte et Nash-Williams sur le packing d'arbres couvrants et du résultat orienté analogue, le théorème d'Edmonds sur le packing d'arborescences couvrantes. Frank a donné un preuve directe de cette caractérisation et a expliqué comment en déduire le packing dans le cas non-orienté à partir du cas orienté. Dans la Section 4.2, nous imitons cette approche pour déduire le résultat de Katoh and Tanigawa depuis le résultat orienté analogue via le théorème de Frank sur les orientations couvrant des fonctions sur-modulaires. La contenu de cette section provient d'une collaboration avec Nguyen et Szigeti [9].

Afin de résoudre le cas $k = 2$ de la conjecture de Thomassen, Jordán a prouvé une condition suffisante pour l'existence de k sous-graphes rigides couvrants arête-disjoints qui généralise un résultat classique de Lovász et Yemini [51]. Dans la section 4.3, nous affaiblissons la condition suffisante donnée par Jordán et étendons le packing aux arbres couvrants. Ce travail fournit une preuve simplifiée d'un résulat de Jackson and Jordán [39] et nous permet d'améliorer les bornes supérieures données par Jordán sur une conjecture de Kriesell ainsi que sur la conjecture de Thomassen. La matière de cette section résulte d'une collaboration avec avec Cheriyan et Szigeti [5]. Enfin, nous en venons à un résultat plus général qui aborde le packing des bases d'un count matroïde plus général et d'arbres couvrant provenant d'un travail avec Nguyen.

Le Chapitre 5 commence par une discussion sur la conjecture de Thomassen et les deux conjectures de Frank qui traitent des orientations k-sommet-connexes et racine k-sommet-connexes.

Pour aborder ces conjectures, l'approche fructueuse de Lovász du problème de l'orientation k-arc-connexe motive l'étude de la structure des graphes hautement connexes. En suivant cette direction, Jordán a montré une caractérisation constructive des graphes faiblement 4-connexes dont il est question dans la conjecture de Frank. Comme le résultat de Lovász, le résultat de Jordán s'appuie sur un nouveau théorème de séparation et sur un résultat de Kaneko et Ota [43] à propos de l'existence de sommets de degré 4. Dans un contexte plus général, nous étudions l'existence de sommets serrés dont le dégré est minimal en regard de la connexité du graphe. Plus précisément, nous montrons que tout graphe minimalement k-connexe g-borné contient au moins un sommet serré. L'existence de sommets serrés a aussi été étudiée dans les graphes orientés par Mader [53, 54]. Nous donnons une légère généralization de deux résultats de Mader dans ce domaine.

Dans la dernière section du Chapitre 5, nous généralisons la caractérisation des graphes faiblement 4-sommet connexes à une classe de graphes plus grande, à savoir la famille des graphes $(2k, k)$-connexes. A cette fin, nous montrons un nouveau théorème de séparation qui résoud aussi un problème d'augmentation. Le contenu de ce chapitre est issu d'une collaboration avec Zoltán Szigeti.

Enfin, dans le Chapitre 6 nous infirmons les deux conjectures de Frank mentionnées auparavant pour $k \geqslant 3$ et montrons que le problème de décider si un graphe a une orientation k-sommet-connexe et le problème de décider si un graphe a une orientation racine k-sommet-connexe sont tous les deux NP-complet pour $k \geqslant 3$.

PRELIMINARIES

2

2.1 GRAPHS AND DIGRAPHS

2.1.1 *Basics*

A *graph* G is a couple of sets. The first set, denoted by $V(G)$ defines the *vertices* of the graph. The second set, denoted by $E(G)$ contains pairs of vertices called the *edges* of the graph (see Figure 1 (a)). Given two vertices u and v the edge $\{u, v\}$ is denoted by uv or vu and u and v are called the *ends* of uv. Given $F \subseteq E(G)$, the set of the ends of the edges of F is denoted by $V(F)$. Two edges that have the same ends are called *parallel*. A graph containing no parallel edges is *simple*.

A *directed graph* (or, for short, *digraph*) D is a couple of sets. The first set, denoted by $V(D)$, is a vertex set. The second set, denoted by $A(V)$ is a set of couples of vertices called the *arcs* of the directed graph (see Figure 1 (b)). The first vertex of an arc is called the *tail* and the second one it called the *head*. An arc with tail u and head v is denoted by uv. Whether uv refers to an edge or an arc will be clear from the context. Given $F \subseteq A$, $V(F)$ denotes the set of heads and tails of the arcs of F. Two arcs having the same head and the same tail are *parallel* and the arcs uv and vu are *opposed*. *Reversing* an arc means replacing it by the opposed one.

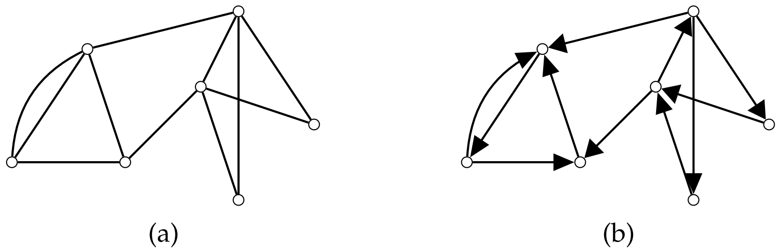


Figure 1: (a) A graph G with 7 vertices and 11 edges that contains two parallel edges. (b) A digraph D that contains two opposed arcs. Note that D is an orientation of G.

Observe that a unique graph can be obtained from a directed graph D by replacing each arc uv by an edge uv. This graph is called the *underlying undirected graph* of D. The inverse operation consists of replacing each edge uv of a graph G by either the arc uv or the arc vu. The resulting digraph is called an *orientation* of G (see Figure 1). Observe that G has 2^m possible orientations where m is the number of edges of G. If a digraph D' can be obtained from an other digraph D by reversing some arcs then D' is a *reorientation* of D. Note that a digraph with m arcs has $2^m - 1$ reorientations.

All the graphs and digraphs considered in this thesis have a finite number of vertices, edges and arcs and have no loop, that is, an arc or an edge of type vv.

In the following definitions $G = (V, E)$ is a graph and $D = (V, A)$ is a digraph. A *subgraph* of G is a graph $H = (U, F)$ where $U \subseteq V$ and $F \subseteq E$ (see Figure 2). Note that, since H is a graph, $V(F) \subseteq U$. When equality holds, H is the subgraph of G *induced by* F and we denote it $G[F]$. So we have

$V(F) = V(G[F])$. If F is the set of edges of E with both ends in U then H is the subgraph of G *induced by* U and we denote it $G[U]$.

A *subgraph* of D is a digraph $H = (U, F)$ where $U \subseteq V$ and $F \subseteq A$. Note that, since H is a digraph, $V(F) \subseteq U$. When equality holds, H is the subgraph of D *induced* by F and we denote it $D[F]$. Again, we have $V(F) = V(D[F])$.

A subgraph $H = (U, F)$ of G or D *spans* $W \subseteq V$ if $W \subseteq U$. If H spans V then H is simply called *spanning*.

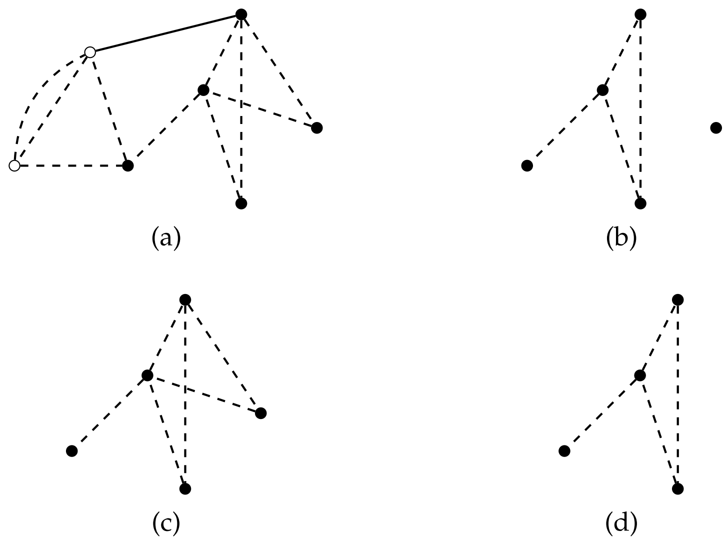


Figure 2: (a) A graph $G = (V, E)$ where the black vertices define a set $U \subset V$ and the dashed edges define a set $F \subset E$. (b) The subgraph (U, F) of G. (c) The subgraph $G[U]$ induced by U. (d) The subgraph $G[F]$ induced by F.

In G, a *path joining* u *and* v (or, for short, a uv-*path*) is a sequence of vertices $x_0, x_1, \cdots, x_k$ such that $x_0 = u$, $x_k = v$ and $x_i x_{i+1}$ is an edge of G for all $i \in \{0, \cdots, k-1\}$. The vertices x_0 and x_k are the *ends* of the path and $x_1, \cdots, x_{k-1}$ are the *inner-vertices* of the path. If the x_i are pairwise distinct except for $u = v$ then the path is called a *cycle*. Observe that reversing the order of the vertices in a uv-path results in a vu-path.

A set of vertices U is called *connected* if any two vertices of U are joined by a path in G. A *connected component* of G is an inclusionwise maximal connected set of vertices. We denote $c(G)$ the number of connected components of G. If G contains only one connected component (or, equivalently, V is connected) then G is called *connected*. Note that the graph defined in Figure 1 is connected.

In D, a *directed path from* u *to* v (or, for short, a uv-*dipath*) is a sequence of vertices $x_0, x_1, \cdots, x_k$ such that $x_0 = u$, $x_k = v$ and $x_i x_{i+1}$ is an arc of D for all $i \in \{0, \cdots, k-1\}$. If the x_i are pairwise distinct except for $u = v$ then the dipath is called a *circuit*. Unlike in graphs, the existence of a uv-dipaht does not imply the existence of a vu-dipath.

A set of vertices U is called *strongly connected* if, for any two vertices u, v of U, there exist a uv-dipath and a vu-dipath in D. A *strongly connected component* of D is an inclusionwise maximal strongly connected set of vertices. If V is strongly connected then D is called *strongly connected*. Note that the digraph defined in Figure 1 is not strongly connected. Its strongly connected componants are given in Figure 3.

Given a special vertex r, the digraph D is called *rooted connected at* r (or simply rooted connected if r is clear from the context) if there exists a rv-dipath for any $v \in V \setminus r$.

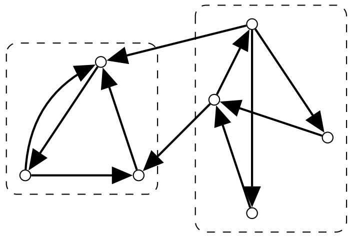


Figure 3: The two dashed rectangles represent the two strongly connected components of the digraph.

2.1.2 *Set Functions*

Let Ω be a ground set and X and Y be subsets of Ω. The *intersection* and the *union* of X and Y are denoted by $X \cap Y$ and $X \cup Y$ respectively. The sets X and Y are *intersecting* if $X \cap Y$ is not empty. The set of elements of Ω contained in X but not in Y is denoted by $X \setminus Y$. The sets X and Y are *properly intersecting* if they are intersecting and none of $X \setminus Y$ and $Y \setminus X$ is empty. The sets X and Y are *crossing* if they are properly intersecting and $\Omega \setminus (X \cup Y)$ is non empty. We denote $\overline{X} = V \setminus X$ the complement of a set X.

Let $f : 2^{\Omega} \mapsto \mathbb{Z}$ be a set function. The function f is called *submodular* if

$$f(X) + f(Y) \geqslant f(X \cap Y) + f(X \cup Y) \tag{2.1}$$

holds for every pair of sets $X, Y \subseteq \Omega$. If equality holds for every pair of sets then f is called *modular*. Note that a function f is modular if an only if there exists a vector $v \in \mathbb{R}^{\Omega}$ such that $f(X) = \sum_{x \in X} v(x)$. So we may consider vectors as modular functions and vice versa. For $a \in \mathbb{R}$ we denote a_{Ω} the vector of $\mathbb{R}^{\Omega}$ where all the coordinates have value a.

When f is symmetric and submodular note that (2.1) applied for $\overline{X}$ and Y gives

$$f(X) + f(Y) \geqslant f(X \setminus Y) + f(Y \setminus X). \tag{2.2}$$

If (2.1) is required only when X and Y are interesting (resp. crossing) then f is called *intersecting submodular* (resp. *crossing submodular*). The function f is called *skew submodular* if for every pair of sets $X, Y \subseteq \Omega$ at least one of (2.1) and (2.2) holds.

The function f is called is *supermodular* (resp. *intersecting supermodular*, resp. *crossing supermodular*, resp. *skew supermodular*) is $-f$ is submodular (resp. *intersecting submodular*, resp. *crossing submodular*, resp. *skew submodular*).

The complexity of algorithms involving a submodular function strongly depends on the method used to evaluate the function. In this document we make the very common assumption that arbitrary submodular functions are given by an oracle that determines the value of the function on subsets of Ω. The following theorem was proved by Iwata, Fleischer and Fujishige and, independently, by Schrijver. This theorem has deep consequences in many optimization problems.

Theorem 2.1 (Iwata, Fleisher and Fujishige [38], Schrijver [67]). *A submodular function can be minimized in polynomial time in $|\Omega|$ and the number of calls to the oracle.*

Goemans and Ramakrishnan [34] pointed out that minimizing a crossing submodular function can also be done in polynomial time since it reduces to $|\Omega|$ calls to the submodular function minimization algorithm.

In the following definitions $G = (V, E)$ is a graph and $D = (V, A)$ is a digraph. Given two disjoint vertex sets X and Y, $d_G(X, Y)$ denotes the number of edges of G with one end in $X \setminus Y$ and the other one in $Y \setminus X$ and $d_D(X, Y)$ denotes the number of arcs of D with the tail in $X \setminus Y$ and the head in $Y \setminus X$.

In G, an edge e *enters* a set $X \subset V$ if one end of e belongs to X and the other end belongs to $V \setminus X$. The set of edges entering X is denoted by $\Delta_G(X)$ and the number of such edges, denoted by $d_G(X) = d_G(X, V \setminus X)$, is called the *degree* of X (see Figure 4 (a)). The graph G is called *Eulerian* if the degree of each vertex is even. We denote T_G the set of vertices of odd degree in G. For $T \subseteq V$, a T-join is a set F of edges (not necessarily in E) such that the set of vertices of odd degree in (V, F) is T. Clearly adding a T_G-join to G results in an Eulerian graph. One may also easily see that, for every connected graph $G = (V, E)$, E contains a T_G-join.

An edge is *induced* by $X \subseteq V$ if both of its ends are in X. For $F \subseteq E$, $i_F(X)$ denotes the number of edges of F induced by X and $e_F(X)$ denotes the number of edges of F with at least one end in X. Note that,

$$e_F(X) - i_F(X) = d_{(V,F)}(X), \tag{2.3}$$

$$e_F(V \setminus X) + i_F(X) = |F|. \tag{2.4}$$

In D, an arc a *enters* a set $X \subset V$ if the head of a belongs to X and the tail of a belongs to $V \setminus X$ (see Figure 4 (b)). The set of arcs entering X is denoted by $\Delta_D^-(X)$ and the number of such arcs, denoted by $\rho_D(X) = d_D(V \setminus X, X)$, is called the *indegree* of X. An arc *leaves* X if it enters $V \setminus X$. The set of edges leaving X is denoted by $\Delta_D^+(X)$ and we denote $\delta_D(X) = d_D(X, V \setminus X)$ the *outdegree* of X. The digraph D is called *Eulerian* if, for each vertex v, $\rho_D(v) = \delta_D(v)$.

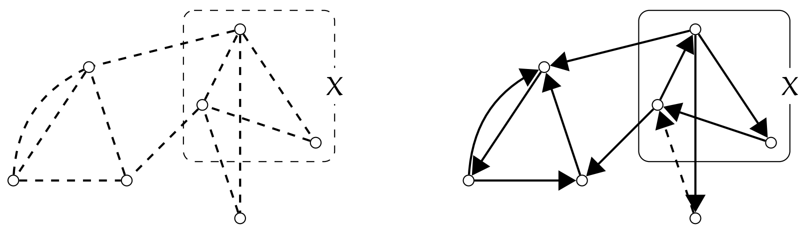


Figure 4: (a) The dashed edges represent the set $\Delta_G(X)$ of edges entering the set X in G and $d_G(X) = 4$. (b) The dashed arc represents the set $\Delta_G^-(X)$ of arcs entering the set X in D and $\rho_D(X) = 1$.

Degree functions have properties that are the foundations of many results. For any vertex sets X and Y,

$$d_G(X) + d_G(Y) = d_G(X \cap Y) + d_G(X \cup Y) + 2d_G(X, Y) \tag{2.5}$$

$$\begin{aligned}\rho_D(X) + \rho_D(Y) = {}& \rho_D(X \cap Y) + \rho_D(X \cup Y) \\ & + d_D(X, Y) + d_D(Y, X).\end{aligned} \tag{2.6}$$

We skip the proof of these equalities that consists of counting the contribution of each edge or arc to each side. Interestingly it follows from (2.5) and (2.6) that d_G and ρ_D are submodular.

2.1.3 *Bi-Sets Functions*

Let Ω be a ground set. For $X_I \subseteq X_O \subseteq \Omega$, the pair $X = (X_O, X_I)$ is called a *bi-set* of Ω [1]. The sets X_I, X_O and $w^b(X) = X_O \setminus X_I$ are the *inner-set*, the *outer-set* and the *wall* of X, respectively. If $X_I = \emptyset$ or $X_O = \Omega$ then the bi-set X is called *trivial*. The *intersection* and the *union* of two bi-sets $X = (X_O, X_I)$ and $Y = (Y_O, Y_I)$ are defined by $X \sqcap Y = (X_O \cap Y_O, X_I \cap Y_I)$ and $X \sqcup Y = (X_O \cup Y_O, X_I \cup Y_I)$. We say that X *is included in* Y, denoted by $X \sqsubseteq Y$, if $X_O \subseteq Y_O$ and $X_I \subseteq Y_I$. We say that X and Y are *innerly-disjoint* if $X_I \cap Y_I = \emptyset$. We extend the complement operation to bi-sets by defining the complement of X as $\overline{X} = (\overline{X_I}, \overline{X_O})$. For a family $\mathcal{F}$ of bi-sets of Ω, we denote $\Omega_I(\mathcal{F}) = \cup_{X \in \mathcal{F}} X_I$.

A function defined on the set of bi-sets of Ω is called a *bi-set function*. In this document, a tiny letter b is used the to prevent the confusion between set and bi-set functions. A bi-set function f^b is called *submodular* if, for all bi-sets X and Y,

$$f^b(X) + f^b(Y) \geqslant f^b(X \sqcap Y) + f^b(X \sqcup Y). \tag{2.7}$$

A bi-set function f^b is called *supermodular* if $-f^b$ is submodular.

An edge e of G *enters a bi-set* $X = (X_O, X_I)$ of V, if one end of e belongs to X_I and the other end of e does not belong to X_O which is equivalent to say that e enters both sets X_I and X_O. The *degree* of X, denoted by $d^b_G(X)$, is the number of edges entering X. We point out that d^b_G is a generalization of d_G since for a bi-set X such that the inner-set and the outer-set coincide we have $d^b_G(X) = d_G(X_I)$.

Fact 2.1. *The degree bi-set function* d^b_G *is symmetric with respect to the complement operation, submodular and, for every bi-sets* X, Y *of* V*,*

$$\begin{aligned} d^b_G(X) + d^b_G(Y) = d^b_G(X \sqcap Y) + d^b_G(X \sqcup Y) + d_G(\overline{X_O} \cap Y_O, X_I \cap \overline{Y_I}) \\ + d_G(\overline{Y_O} \cap X_O, Y_I \cap \overline{X_I}). \end{aligned} \tag{2.8}$$

The symmetry of d^b_G is obvious and the submodularity, pointed out by Frank and Jordán [30], is a consequence of (2.8) which can be proved by counting the contribution of each edge of G to each side. Given a family $\mathcal{F}$ of bi-sets we denote $e^b_G(\mathcal{F})$ the number of edges of G entering at least one element of $\mathcal{F}$.

An arc a of D *enters a bi-set* $X = (X_O, X_I)$ of V, if the head of a belongs to X_I and the tail of a does not belong to X_O which is equivalent to say that a enters both sets X_I and X_O. The *indegree* of X, denoted by $\rho^b_D(X)$, is the number of arcs entering X. An arc *leaves the bi-set* X if it enters $\overline{X}$. The number of arcs leaving X is denoted by $\delta^b_D(X)$. We point out that ρ^b_D is a generalization of ρ_D since for a bi-set X such that the inner-set and the outer-set coincide we have $\rho^b_D(X) = \rho_D(X_I)$.

Fact 2.2. *The indegree bi-set function* ρ^b_D *is submodular and, for every bi-sets* X, Y *of* V*,*

$$\begin{aligned} \rho^b_D(X) + \rho^b_D(Y) = \rho^b_D(X \sqcap Y) + \rho^b_D(X \sqcup Y) + d_D(\overline{X_O} \cap Y_O, X_I \cap \overline{Y_I}) \\ + d_D(\overline{Y_O} \cap X_O, Y_I \cap \overline{X_I}). \end{aligned} \tag{2.9}$$

1 As fas as I know, the definition of bi-sets given is this document is due to Frank [27] and a slightly different form was considered earlier by Frank and Jordán [30].

2.1.4 *Trees and Arborescences*

Let $G = (V, E)$ be a graph. A *tree* is a connected cycle free subgraph of G. The size $|T|$ of a tree T is the number of edges of T. It is easy to see by induction that $|T| = |V(T)| - 1$. Note that, for $u, v \in V(T)$, there exists a unique path joining u and v in T since T is connected and cycle free. A *forest* is a cycle free subgraph of G. Thus, each connected component of a forest is a tree. The size $|F|$ of a forest F is the sum of the sizes of its trees and is given by

$$|F| = |V(F)| - c(G[F]). \tag{2.10}$$

Let $D = (V, A)$ be a digraph and $r \in V$. An r-*arborescence* is a circuit-free subgraph of G in which every vertex except r has indegree exactly one. The special vertex r is called the *root* of the arborescence. The size $|T|$ of an arborescence T is defined as the number of arcs of T and is given by $|T| = \sum_{v \in V(T)} \rho_T(v) = |V(T)| - 1$. As we shall see, arborescences are somehow the directed counterparts of trees.

We claim and briefly prove that the underlying undirected graph of an arborescence is a tree. Let T be an arborescence and let T' be the underlying undirected graph. If there is cycle in T' then, by the indegree condition, this cycle is a circuit in T, a contradiction. So T' is a forest and, by (2.10), $|V(T')| - c(G[T']) = |T'| = |T| = |V(T)| - 1$. Hence, since $V(T) = V(T')$, T' is connected.

Conversely, given a tree T' and $r \in V(T')$, there exists a unique orientation of T' which is an r-arborescence. Such an orientation is obtained by replacing each edge incident to r by an arc with tail r and then, recursively, choose the head v of an arc and replace each (not yet oriented) edge incident to v by an arc leaving v. Hence a tree T' has $|V(T')|$ orientations into arborescences (depending on the root).

Suppose that given an r-arborescence T and $r' \in V(T) \setminus r$, we have to reorient some arcs of T to obtain an r' arboresence. By the discussion above, it can be done considering the underlying tree T' and finding an r'-arborescence orientation of T'. However it can be done more efficiently. In T' there exists a unique rr'-path and, by the indegree condition, this path is an rr'-dipath in T. Reversing each arc of this dipath results in an r'-arborescence. Indeed this operation decreases the indegree of r' by one and increases the indegree of r by one.

A *branching* is the union of pairwise vertex-disjoint arborescences. The size $|F|$ of a branching F, defined as the sum of the sizes of its arborescences, satisfies (2.10).

2.2 CONNECTIVITY IN GRAPHS AND DIGRAPHS

So far we defined connectivity in graphs and strong-connectivity and rooted connectivity in digraphs. These basic concepts are the foundations of classic deeper connectivity notions such as edge-connectivity, vertex-connectivity, and mixed-connectivity in graphs and (rooted) arc-connectivity and (rooted) vertex-connectivity in digraphs. These notions describe graphs and digraphs that keep a basic connectivity property even if some edges, arcs or vertices are removed. For each connectivity notion there exists a theorem characterizing graphs or digraphs that have this resiliance property with the existence, for each pair of vertices, of a certain number of disjoint paths between them. These results are variations of the essential theorem of Menger [57] who first proved such a characterization.

In the present chapter we introduce a general connectivity definition for graphs and digraphs called g-*bounded connectivity*. This definition captures each of the more classic connectivity notions mentioned above. In the undirected case as in the directed one, a Menger's type theorem comes along with g-bounded connectivity. From these general results formulated in terms of bi-sets we derive the classic characterizations concerning edge-connectivity, vertex-connectivity and mixed-connectivity in graphs and (rooted) arc-connectivity, and (rooted) vertex-connectivity in digraphs. Note that g-*bounded connectivity* was introduced by Nagamochi and Ibaraki [58] and also studied by Frank [27] in its rooted formulation for digraphs.

2.2.1 g-*Bounded Connectivity*

Let $G = (V, E)$ be a graph and $g : V \mapsto \mathbb{Z}_+$ a positive function. For $u, v \in V$, a set $\mathcal{P}$ of pairwise edge-disjoint uv-paths is called g-*bounded* if each vertex $w \in V \setminus \{u, v\}$ is contained in at most $g(w)$ paths of $\mathcal{P}$. We emphasize that g-boundedness automatically means that the paths are edge-disjoint. The g-*bounded connectivity* between u and v, denoted by $\mu_G(u, v, g)$ is the maximum number of g-bounded uv-paths. A graph G is called g-*bounded* k-*connected* if any pair of vertices are joined by k g-bounded paths and

$$g(V \setminus v) \geqslant k \tag{2.11}$$

holds for all $v \in V$. Since g is positive g-bounded 1-connectivity and connectivity coincide on graphs with at least two vertices. A graph is called *minimally* g-*bounded* k-*connected* if it is g-bounded k-connected and removing any edge ruins this property.

Let u and v be distinct vertices of G, $U \subseteq V \setminus \{u, v\}$ and $F \subseteq E$ such that u and v are in different connected components of $G - U - F$. Now consider a set of $\mu_G(u, v, g)$ g-bounded uv-paths. Each path contains at least one element of $U \cup F$, each edge of F is contained in at most one path and each vertex $w \in U$ is contained in at most $g(w)$ paths so $|F| + g(U) \geqslant \mu_G(u, v, g)$.

A bi-set X of V *separates* two vertices if one of these vertices is in X_I and the other one is not in X_O. By the previous observation, a bi-set X separating two vertices u, v satisfies $d^b_G(X) + g(w^b(X)) \geqslant \mu_G(u, v, g)$. The following variation of Menger's theorem states that equality holds if the bi-set X minimizes the left hand side of the inequality.

Theorem 2.2. *Let* $G = (V, E)$ *be a graph and* $g : V \mapsto \mathbb{Z}_+$ *a positive function. For* $u, v \in V$,

$$\mu_G(u, v, g) = \min\{d^b_G(X) + g(w^b(X))\} \tag{2.12}$$

where the minimum is taken over all the bi-sets X *separating* u *and* v. *Moreover, provided* (2.11), *the following assertions are equivalent,*

(i) G *is* g-*bounded* k-*connected*

(ii) $G - U - F$ *is connected for all* $F \subset E$, $U \subset V$ *satisfying* $|F| + g(U) < k$

(iii) $d^b_G(X) + g(w^b(X)) \geqslant k$ *for all non-empty* $X \subset V$.

Note that the function $f^b_G : X \mapsto d^b_G(X) + g(w^b(X))$ is submodular since d^b_G is submodular and $X \mapsto g(w^b(X))$ is modular. A bi-set X satisfying the inequality in (iii) with equality is called *tight*. Tight bi-sets play an important role when investigating properties resulting from g-bounded k-connectivity. A key property of those bi-sets is the following.

Fact 2.3. *Let $G = (V, E)$ be a graph and let $g : V \mapsto \mathbb{Z}_+$ be a positive function such that G is g-bounded k-connected. If two tight bi-sets X and Y of G are crossing then both $X \sqcap Y$ and $X \sqcup Y$ are tight and $d_G(\overline{X_O} \cap Y_O, X_I \cap \overline{Y_I}) = d_G(\overline{Y_O} \cap X_O, Y_I \cap \overline{X_I}) = 0$.*

Proof. By tightness of X and Y, (2.8), modularity of the bi-set function $w^b(.)$, and g-bounded k-connectivity of G, we have,

$$\begin{aligned}
k + k &= f^b_G(X) + f^b_G(Y) \\
&= d^b_G(X) + g(w^b(X) + d^b_G(Y) + g(w^b(Y) \\
&= d^b_G(X \sqcap Y) + g(w^b(X \sqcap Y)) + d^b_G(X \sqcup Y) + g(w^b(X \sqcup Y)) \\
&+ d_G(\overline{X_O} \cap Y_O, X_I \cap \overline{Y_I}) + d_G(\overline{Y_O} \cap X_O, Y_I \cap \overline{X_I}) \\
&\geqslant k + k + 0 + 0.
\end{aligned}$$

Hence equality holds in the last inequality and the Fact 2.3 follows. □

In digraphs the definition of g-bounded connectivity follows the same scheme. Let $D = (V, A)$ be a digraph and let $g : V \mapsto \mathbb{Z}_+$ be a positive function. For $u, v \in V$, a set of pairwise arc-disjoint uv-dipaths is called g-*bounded* if each vertex $w \in V \setminus \{u, v\}$ is contained in at most $g(w)$ dipaths. The g-*bounded connectivity* from u to v, denoted by $\mu_D(u, v, g)$ is the maximum number of g-bounded uv-dipaths. A digraph D is called g-*bounded* k-*connected* if (2.11) holds and, for any pair u, v of vertices, there exists k g-bounded uv-dipaths and k g-bounded vu-dipaths. As for graphs, g-bounded 1-connectivity and strong-connectivity coincide in digraphs with at least 2 vertices. Given a vertex $r \in V$, D is called *rooted* g-*bounded* k-*connected at* r (or simply rooted g-bounded connected if r is clear from the context) if (2.11) holds and $\mu_D(r, v, g) \geqslant k$ for all $v \in V \setminus r$.

As for g-bounded connectivity in graphs, the g-bounded connectivity $\mu_D(u, v, g)$ from u to v is upper-bounded by $|F| + g(U)$ if $U \subseteq V \setminus \{u, v\}$ and $F \subseteq A$ are such that there is no uv-path in $D - U - F$. Hence, for a bi-set X of V such that $v \in X_I$ and $u \notin X_O$, we have $\rho^b_D(X) + g(w^b(X)) \geqslant \mu_D(u, v, g)$. The following directed counterpart of Theorem 2.2 states that equality holds if X minimizes the left hand side.

Theorem 2.3. *Let $D = (V, A)$ be a digraph and $g : V \mapsto \mathbb{Z}_+$ a positive function. For $u, v \in V$,*

$$\mu_D(u, v, g) = \min\{\rho^b_D(X) + g(w^b(X))\} \tag{2.13}$$

where the minimum is taken over all the bi-sets X such that $v \in X_I$ and $u \notin X_O$. Moreover, provided that (2.11) holds, the following assertions are equivalent,

(i) *D is g-bounded k-connected*

(ii) *$D - U - F$ is strongly connected for all $F \subset A$, $U \subset V$ satisfying $|F| + g(U) < k$*

(iii) *$\rho^b_D(X) + g(w^b(X)) \geqslant k$ for every non trivial bi-set X of V.*

Furthermore D is rooted g-bounded k-connected at a vertex r if and only if $\rho^b_D(X) + g(w^b(X)) \geqslant k$ for every non trivial bi-set X of V such that $r \notin X_O$.

As for the undirected case, the function $f^b_D : X \mapsto \rho^b_D(X) + g(w^b(X))$ is submodular. A bi-set X satisfying the inequality in (iii) with equality is called *in-tight*. The complement of an in-tight bi-set is called *out-tight*. As in graphs, in digraphs tight bi-sets play an important role when investigating properties resulting from g-bounded k-connectivity. The directed counterpart of Fact 2.3 holds in digraphs.

Fact 2.4. *Let* $D = (V, E)$ *be a digraph and* $g : V \mapsto \mathbb{Z}_+$ *a positive function such that* D *is* g*-bounded* k*-connected. If two intight bi-sets* X *and* Y *of* D *are crossing then both* $X \sqcap Y$ *and* $X \sqcup Y$ *are in-tight and* $d_D(\overline{X_O} \cap Y_O, X_I \cap \overline{Y_I}) = d_D(\overline{Y_O} \cap X_O, Y_I \cap \overline{X_I}) = 0$.

2.2.2 *Edge-Connectivity and Arc-Connectivity*

Let $G = (V, E)$ be a graph and define $g : V \mapsto \mathbb{Z}_+$ as the constant function of value $|E|$. Clearly, for $u, v \in V$, a set of uv-paths is g-bounded if and only if the paths are pairwise edge-disjoint. Hence, in this case the g-bounded connectivity and the classic edge-connectivity coincide. We define the *local edge-connectivity between two vertices* u *and* v, denoted by $\lambda_G(u, v) = \mu_G(u, v, |E|)$, as the maximum number of pairwise edge-disjoint uv-paths. The graph G on at least two vertices is k*-edge-connected* if any two vertices of G are joined by k pairwise edge-disjoint paths or, equivalently, $\lambda_G(u, v) \geqslant k$ for all $u, v \in V$.

For $g = |E|$ note that (2.12) is minimized by a tight bi-set with an empty wall. Note also that in a k-edge-connected graph, condition (ii) of Theorem 2.2 is violated if U is not empty and condition (iii) trivially holds if $w^b(X)$ is not empty. It means that, considering edge-connectivity, we may consider vertex sets instead of bi-sets. Hence Theorem 2.2 can reformulated as the following classic results. The maximum number of pairwise edge-disjoint uv-paths is the minimum degree of a set separating u and v. Moreover, a graph G with at least two vertices is k-edge-connected if and only if removing less than k edges preserves the connectivity or, equivalently, the degree of each non trivial vertex set is at least k.

Let $D = (V, A)$ be a graph and define $g : V \mapsto \mathbb{Z}_+$ as the constant function of value $|A|$. Again, for $u, v \in V$, a set of uv-dipaths is g-bounded if and only if the paths are pairwise arc-disjoint and the g-bounded connectivity and the classic arc-connectivity coincide on digraphs containing at least two vertices. We define the *local arc-connectivity from a vertex* u *to another vertex* v, denoted by $\lambda_D(u, v) = \mu_D(u, v, |A|)$, as the maximum number of pairwise arc-disjoint uv-dipaths. The digraph D on at least two vertices is k*-arc-connected* if, for every pair u, v of vertices, there exist k pairwise arc-disjoint uv-paths and k pairwise arc-disjoint vu-paths. Given a vertex $r \in V$, D is called *rooted* k*-arc-connected at* r if $\lambda_D(r, v) \geqslant k$ for all $v \in V \setminus r$.

As for graphs, the conditions given in Theorem 2.3 can be replaced by conditions on vertex-sets rather that bi-sets to obtain the Menger's type classic results about arc-connectivity. The maximum number of pairwise arc-disjoint uv-dipaths is the minimum indegree of a set containing v but not u. Moreover, a digraph is k-arc-connected if and only if removing less than k arcs results in a strongly-connected graph or, equivalently, the indegree of each non trivial vertex set is at least k. And a digraph is rooted k-arc-connected at r if and only if the indegree of every non-empty vertex-set not containing r is at least k.

2.2.3 *Vertex-Connectivity*

Let $G = (V, E)$ be a graph and define $g : V \mapsto \mathbb{Z}_+$ as the constant function of value 1. For $u, v \in V$, a pair of uv-paths is g-bounded if and only if the paths are pairwise *innerly-disjoint* that is the paths have no common inner-vertices. A set of k innerly-disjoint uv-paths is called a k*-fan joining* u *and* v. We define the *local vertex-connectivity between two vertices* u *and* v, denoted by $\kappa_G(u, v) = \mu_G(u, v, 1)$, as the maximum number of pairwise innerly-disjoint

uv-paths. The graph G is k-*vertex-connected* if $|V| > k$ and any two vertices of G are joined by a k-fan.

The classic formulation of Theorem 2.2 for vertex-connectivity states that, given two non adjacent vertices u, v, the maximum number of pairwise innerly-disjoint uv-paths is the minimum size of a vertex-set $U \subseteq \{u, v\}$ such that removing U disconnects u and v. It means that in (2.12) the minimum can be taken over all bi-sets separating u and v entered by no edge. Indeed, we choose X a bi-set minimizing the right hand side of (2.12) such that $d^{\flat}_G(X)$ is minimum and we show that $d^{\flat}_G(X) = 0$. Since, u and v are non-adjacent, if there exists an edge xy entering X we may assume that x is none of u and v. Hence the bi-set $Y = (X_O \cup x, X_I)$ separates u and v and satisfies $d^{\flat}_G(Y) + g(w^{\flat}(Y)) \leqslant d^{\flat}_G(X) + g(w^{\flat}(X))$ with $d^{\flat}_G(Y) < d^{\flat}_G(X)$, a contradiction to the choice of X.

In a digraph $D = (V, A)$, the vertex-connectivity is obtained from g-bounded connectivity by the same choice of $g : V \mapsto \mathbb{Z}_+$ as the constant function of value 1. Again, g-boundedness of dipaths corresponds to inner-disjointness and we define a k-*difan from a vertex* u *to a vertex* v as a set of k pairwise innerly-disjoint uv-dipaths. We define the *local vertex-connectivity from* u *to* v, denoted by $\kappa_D(u, v) = \mu_D(u, v, 1)$, as the maximum number of pairwise innerly-disjoint uv-dipaths. The graph D is k-*vertex-connected* if $|V| > k$ and, for every pair u, v of vertices, there exist a k-difan from u to v and a k-difan from v to u. Given a vertex $r \in V$, D is called *rooted* k-*vertex-connected at* r if there exists a k-difan from r to any other vertex.

As for graphs, Theorem 2.3 implies the following classic result. For two vertices u, v such that $uv \notin A$, the maximum number of pairwise innerly-disjoint uv-dipaths is the minimum size of a vertex-set $U \subseteq \{u, v\}$ such that, in $D - U$, v is not reachable from u.

We prove a very simple fact that concerns both graphs and digraphs.

Fact 2.5. *Every minimally* k-*vertex-connected graph or digraph is simple.*

Proof. Let $G = (V, E)$ be a minimally k-vertex-connected graph and suppose that there exists $u, v \in V$ such that $d_G(u, v) \geqslant 2$. By minimality there exists a tight bi-set X such that $u \in X_I$ and $v \notin X_O$. By $|V| > k$ and $d^{\flat}_G(X) \geqslant 2$, we may assume that $|X_I| \geqslant 2$. So the non trivial bi-set $Y = (X_O, X_I \setminus u)$ satisfies $k \leqslant d_G(Y) + |w^{\flat}(Y)| \leqslant d^{\flat}_G(X) - d_G(u, v) + |w^{\flat}(X)| + 1 < k$, a contradiction. For the directed case the proof is similar. □

2.2.4 *Mixed-Connectivity*

Mixed-connectivity is defined from g-bounded connectivity by defining g as a constant. So this concept generalizes both edge-connectivity and vertex-connectivity in graphs.

Let $G = (V, E)$ be a graph, ℓ a positive integer. *Mixed-*(k, ℓ)*-connectivity* is defined as g-bounded k-connectivity where g is the constant function of value ℓ. In case ℓ divides k, this notion was introduced by Egawa, Kaneko and Matsumoto [16] as follows: a pair u, v of vertices is mixed-(k, ℓ)-connected if there exists ℓ edge-disjoint $\frac{k}{\ell}$-fans joining u and v. They also proved the suitable version of Theorem 2.2.

A particular mixed connectivity is worth mentioning since it plays a role in a conjecture of Frank [26] that will be introduced in Chapter 5. Mixed-$(2k, 2)$-connectivity is called *weak* $2k$-*connectivity*. For example, we say that

G is weakly 4-connected if G is 4-edge-connected and, removing any vertex from G results in a 2-edge-connected graph.

2.3 MATROIDS

In this short section we define the notion of matroids introduced by Whitney [74] and we state the few results of this field that will be needed in this document.

2.3.1 *Definition*

Let S be a set and $\mathcal{I}$ a collection of subsets of S. The pair $\mathcal{M} = (S, \mathcal{I})$ is called a *matroid* if the following three properties, called *independence axioms*, are satisfied:

(I.1) $\emptyset \in \mathcal{I}$,

(I.2) if $U \in \mathcal{I}$ and $T \subseteq U$ then $T \in \mathcal{I}$,

(I.3) for each $U \subseteq S$, the maximal subsets of U which are in $\mathcal{I}$ have the same cardinality.

The elements of $\mathcal{I}$ are called *independent*. Axiom (I.2) is called the hereditary property. The maximal independent sets are called the *basis* of $\mathcal{M}$. Two elements $t, u \in S$ are called *parallel* if $\{t\} \in \mathcal{I}$, $\{u\} \in \mathcal{I}$ and $\{t, u\} \notin \mathcal{I}$.

The motivation of Whitney was to capture the notion of independence in linear spaces. Indeed, given a matrix M, if we define S as the set of columns of M and $\mathcal{I}$ as the subsets of columns linearly independent, then $\mathcal{M} = (S, \mathcal{I})$ is a matroid.

Another standard example of a matroid is the following. Let $G = (V, E)$ be a graph and let $\mathcal{I}$ be the sets of edges that induce a cycle free subgraph of G. In other words, $\mathcal{I}$ is the collection of the edge sets of the forests of G. The set system $\mathcal{C}(G) = (E, \mathcal{I})$ clearly satisfies axioms (I.1) and (I.2). So to prove that $\mathcal{C}(G)$ is a matroid it remains to prove (I.3). Let $F \subseteq E$ and F' be a maximal independent subset of F. By maximality of F', $V(F') = V(F)$ and F' induces a tree in each connected component of F. Hence, by (2.10), we have, $|F'| = |V(F')| - c(G[F']) = |V(F)| - c(G[F])$. Thus the size of F' depends only on F and axiom (I.3) is satisfied. The matroid $\mathcal{C}(G)$ is called the *circuit matroid* of G. Observe that two edges are parallel in the graph G if and only if they are parallel in $\mathcal{C}(G)$.

2.3.2 *Rank Function*

In a matroid $\mathcal{M} = (S, \mathcal{I})$, by axiom (I.3), all the maximal independent subsets of $U \subseteq S$ have the same cardinality. This number defines the *rank* of U and is denoted by $r_{\mathcal{M}}(U)$. Note that a matroid is characterized by its rank function since a set U is independent if and only if $|U| = r_{\mathcal{M}}(U)$.

When we showed that, for a graph $G = (V, E)$, $\mathcal{C}(G)$ is a matroid we actually proved that, for $F \subseteq E$,

$$r_{\mathcal{C}(G)}(F) = |V(F)| - c(G[F]). \tag{2.14}$$

Trivially the rank function of a matroid is a non-decreasing function. The following property is even more interesting.

Proposition 2.1. *The rank function of a matroid is submodular.*

Conversely, a matroid can be defined from a submodular function as follows. For a set function b on S, let

$$\mathcal{I}(b) = \{U \subseteq S \text{ such that } |T| \leqslant b(T) \text{ for all } T \subseteq U\}. \tag{2.15}$$

Theorem 2.4 (Edmonds [13])**.** *Let b be an integer-valued non-decreasing intersecting submodular function on S such that $b(\emptyset) = 0$. Then $\mathcal{M}(b) = (S, \mathcal{I}(b))$ is a matroid and the rank function of $\mathcal{M}(b)$ is given by*

$$r_{\mathcal{M}(b)}(U) = \min\left\{\sum_{i=1}^{t} b(T_i) + |U \setminus T|\right\}, \tag{2.16}$$

where the minimum is taken over all subsets T of $U \subseteq S$ and all partitions $\{T_1, \cdots, T_t\}$ of T. If b is fully submodular then the rank function is simply given by

$$r_{\mathcal{M}(b)}(U) = \min\{b(T) + |U \setminus T| : T \subseteq U\}. \tag{2.17}$$

2.3.3 *Examples*

Lorea and, independently, White and Whiteley introduced a class of matroids defined on the edge set of graphs.

Theorem 2.5 (Lorea [49] White and Whiteley [73])**.** *Let $G = (V, E)$ be a graph. Let m be an element of $(\mathbb{Z}_+)^V$ and ℓ an integer such that, for each edge $uv \in E$, $m(u) + m(v) \geqslant \ell$. Then*

$$\mathcal{F} = \{F \subseteq E : i_F(X) \leqslant m(X) - \ell,\ \forall X \subseteq V \text{ such that } i_F(X) \geqslant 1\} \tag{2.18}$$

satisfies the independence axioms of matroids.

Proof. Let b be the set function on E defined by, $b(\emptyset) = 0$ and $b(F) = m(V(F)) - \ell$ for each non-empty set F of edges. It is easy to check that $F \mapsto m(V(F))$ is submodular. Hence b is non-decreasing integer-valued and intersecting submodular and, by Theorem 2.4, defines a matroid $\mathcal{M}(b)$. We prove that $\mathcal{F}$ is the set of the independent sets of $\mathcal{M}(b)$.

Let $F \subseteq E$ be independent in $\mathcal{M}(b)$. Let $X \subseteq V$ such that $i_F(X) \geqslant 1$ and denote J the subset of F induced by X. We have

$$i_F(X) = |J| \leqslant b(J) = m(V(J)) - \ell \leqslant m(X) - \ell,$$

that is $F \in \mathcal{F}$.

Conversely, if $F \subseteq E$ is not independent in $\mathcal{M}(b)$ then there exists $J \subseteq F$ such that $|J| > b(J)$. Hence we have

$$i_F(V(J)) \geqslant |J| > b(J) = m(V(J)) - \ell$$

that is $F \notin \mathcal{F}$. □

The matroid $(E, \mathcal{F})$ is called a *count matroid* or, to be more specific, the (m, ℓ)-*count matroid*. The circuit matroid of $G = (V, E)$ is the $(1_V, 1)$-count matroid. Indeed, if $F \in \mathcal{F}$ contains a cycle on the vertex set X, then $i_F(X) \geqslant |X|$ which contradicts (2.18). Conversely, let F be a cycle free set of edges and let $X \subseteq V$ such that the set $F' \subseteq F$ induced by X is non-empty. Since $G[F']$ is a forest, by (2.10), $i_F(X) = |F'| = |V(F')| - c(G[F']) \leqslant |X| - 1$.

2.3.4 *Matroid Union*

Let $\mathcal{M}_1 = (S_1, \mathcal{I}_1), \cdots, \mathcal{M}_k = (S_k, \mathcal{I}_k)$ be matroids. The *union* of these matroids is the set system defined by $\mathcal{M} = (S_1 \cup \cdots \cup S_k, \mathcal{I})$ where

$$\mathcal{I} = \{I_1 \cup \cdots \cup I_k : I_1 \in \mathcal{I}_1, \cdots, I_k \in \mathcal{I}_k\}. \tag{2.19}$$

The following theorem is due to Edmonds [12]. The particular case where all $\mathcal{M}_i$ are equal was formulated by Nash-Williams [62].

Theorem 2.6 (Edmonds [12])**.** *Let $\mathcal{M}_1 = (S_1, \mathcal{I}_1), \cdots, \mathcal{M}_1 = (S_k, \mathcal{I}_k)$ be matroids with rank functions $r_1, \cdots, r_k$, respectively. Then the union $\mathcal{M}$ of these matroids is a matroid with rank function given by:*

$$r_{\mathcal{M}}(U) = \min_{T \subseteq U} (r_1(T \cap S_1) + \cdots + r_k(T \cap S_k) + |U \setminus T|) \tag{2.20}$$

for $U \subseteq S_1 \cup \cdots \cup S_k$.

Edmonds also proved algorithmic results on matroids.

Theorem 2.7 (Edmonds [12])**.** *Given a matroid $(S, \mathcal{I})$ by an oracle testing independence, a maximum number of disjoint bases can be found in polynomial time.*

3 DEGREE CONSTRAINED AND ARC-CONNECTED ORIENTATIONS

3.1 DEGREE CONSTRAINED ORIENTATION

3.1.1 m-*Orientations*

The paper Seven Bridges of Königsberg written by Euler in 1736 is regarded as the birth of Graph Theory. Interestingly, from the early result of Euler one may readily derive the following corollary on graph orientation.

Theorem 3.1 (Euler 1736)**.** *A graph G has an Eulerian orientation if and only if G is Eulerian.*

Actually, finding an Eulerian orientation is a particular case of a more general problem. Let $G = (V, E)$ be a graph and $m \in \mathbb{Z}_+^V$ an indegree vector. An m-orientation of G is an orientation D such that, for each $v \in V$, $\rho_D(v) = m(v)$. This generalizes the Eulerian orientations since an Eulerian orientation is an m-orientation where $m(v) = \frac{1}{2}d_G(v)$ for each vertex v. Note also that the specifying the outdegree rather than the indegree would yield an equivalent problem, since $\rho_D(v) + \delta_D(v) = d_G(v)$. Hakimi proved the following result.

Theorem 3.2 (Hakimi [36])**.** *Let $G = (V, E)$ be an undirected graph and $m \in \mathbb{Z}_+^V$ an indegree vector such that $m(V) = |E|$. Then G has an m-orientation if and only if $i_E(X) \leqslant m(X)$ for every $X \subseteq V$.*

The necessity of the condition is straightforward. Suppose that G has an m-orientation D. For $X \subseteq V$, the orientation of each edge induced by X in G results in an arc entering a vertex of X in D, so $i_E(X) \leqslant \sum_{v \in X} \rho_D(v) = m(X)$. Note also that by $m(V) = |E|$ and (2.4), the condition $i_E(X) \leqslant m(X)$ for every $X \subseteq V$ is equivalent to the condition $e_E(X) \geqslant m(X)$ for every $X \subseteq V$.

As corollary of Hakimi theorem one may obtain the following result of Ford and Fulkerson. The union of a graph (V, E) and a digraph (V, A) defines a *mixed graph* denoted $M = (V, E \cup A)$. So this object contains a set of vertices and a set of connections of two types: edges and arcs. An *orientation* of M, is a digraph $(V, E' \cup A)$ where E' is an orientation of E.

Theorem 3.3 (Ford and Fulkerson [18])**.** *Let $M = (V, E \cup A)$ be a mixed graph union of a graph $G = (V, E)$ and a digraph $D = (V, A)$. There exists an Eulerian orientation of M if and only if, $\delta_D(v) + \rho_D(v) + d_G(v)$ is even for all $v \in V$ and $d_G(X) \geqslant \rho_D(X) - \delta_D(X)$ for all $X \subseteq V$.*

3.1.2 *Bounding the indegree*

Hakimi actually proved a more general result that characterises the existence of an orientation where the indegrees satisfy a lower bound, instead of being explicitly specified. For the same reason as above, replacing the lower bound on the indegrees by an upper bound on the outdegrees gives an equivalent problem. Thus, by reversing all the arcs, the lower bounded indegree orientation problem is equivalent to the upper bounded indegree orientation problem. The problem of finding an orientation such that the indegrees satisfies both a lower bound and an upper bound is solved by Frank.

Theorem 3.4 (Hakimi [36] and Frank [25]). *Let $G = (V, E)$ be an undirected graph and $f, g \in \mathbb{Z}_+^V$ two vectors such that $f \geqslant g$. The graph G has an orientation D such that*

(A) $\rho_D(v) \geqslant f(v)$ *for each* $v \in V$ *if and only if, for all* $X \subseteq V$,

$$e_E(X) \geqslant f(X),$$

(B) $g(v) \geqslant \rho_D(v)$ *for each* $v \in V$ *if and only if, for all* $X \subseteq V$,

$$g(X) \geqslant i_E(X),$$

(C) $g(v) \geqslant \rho_D(v) \geqslant f(v)$ *for each* $v \in V$ *if and only if for all* $X \subseteq V$,

$$g(X) \geqslant i_E(X) \text{ and } e_E(X) \geqslant f(X).$$

Interestingly in this result the part (C) can be reformulated as follow: there exists an orientation satisfying the lower bound and the upper bound if and only if there exists an orientation satisfying the lower bound and an orientation satisfying the upper bound.

3.1.3 *Sandwich problem for degree orientation*

The graph sandwich problem for property Π is defined as follows: Given two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ such that $E_1 \subseteq E_2$, is there a graph $G = (V, E)$ such that $E_1 \subseteq E \subseteq E_2$ which satisfies property Π? In this section, we propose to study the sandwich problems for property Π being the existence of degree contrained orientation. This theorem resulting from joint work with Klein, Nguyen and Szigeti [8] aims more to glimpse at the link between matroid theory and graph orientation than to present new and deep results on orientation.

Theorem 3.5 (Durand de Gevigney, Klein, Nguyen and Szigeti [8]). *Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ be two undirected graphs such that $E_1 \subseteq E_2$, let $m \in \mathbb{Z}_+^V$ be an indegree vector and denote $\mathcal{M}$ the $(m, 0)$-count matroid in G_2. The following assertions are equivalent.*

(a) *There exists* $E_1 \subseteq E \subseteq E_2$ *such that* (V, E) *has an* m*-orientation.*

(b) E_1 *is independent in* $\mathcal{M}$ *and* $\mathcal{M}$ *has an independent set of size* $m(V)$.

(c) $r_{\mathcal{M}}(E_1) = |E_1|$ *and* $r_{\mathcal{M}}(E_2) \geqslant m(V)$.

(d) $i_{E_1}(X) \leqslant m(X) \leqslant e_{E_2}(X)$ *for all* $X \subseteq V$.

Proof. **(a) implies (d)** Let $X \subseteq V$. By necessity in Theorem 3.2, we have $i_{E_1}(X) \leqslant i_E(X) \leqslant m(X) \leqslant e_E(X) \leqslant e_{E_2}(X)$.

(d) implies (c). By (2.18), $i_{E_1}(X) \leqslant m(X)$ for all $X \subseteq V$, implies that E_1 is independent in $\mathcal{M}$ that is $r_{\mathcal{M}}(E_1) = |E_1|$. Now let F be a subset of E_2, the condition $m(\overline{V(F)}) \leqslant e_{E_2}(\overline{V(F)})$ implies that $m(V) \leqslant m(V(F)) + e_{E_2}(\overline{V(F)}) \leqslant m(V(F)) + |E_2 \setminus F|$. Hence, by Theorem 2.4, $r_{\mathcal{M}}(E_2) \geqslant m(V)$.

(c) implies (b). By definition.

(b) implies (a). By (I.3) in $\mathcal{M}$ there exists an independent set E of size $m(V)$ that contains E_1. Hence, by (2.18) and Theorem 3.2, the graph (V, E) admits an m-orientation. □

3.2 ARC-CONNECTED ORIENTATIONS

3.2.1 *Nash-Williams' Weak Theorem*

The question answered in this section is the following: can one characterises the graphs admitting a k-arc-connected orientation? For $k = 1$, the answer is given by Robbins.

Theorem 3.6 (Robbins [65]). *A graph G admits a strongly-connected orientation if and only if G is 2-edge-connected.*

The necessity easily derives from Menger theorems. Indeed, if D is a strongly connected orientation of a graph G, then, for any non trivial vertex set X, by Theorem 2.3, $d_G(X) = \rho_D(X) + \delta_G(X) \geqslant 1 + 1$, hence, by Theorem 2.2, G is 2-edge-connected. Robbins' proof of sufficiency relies on the following decomposition of 2-edge-connected graphs.

Theorem 3.7 (Robbins [65]). *A graph $G = (V, E)$ is 2-edge-connected if and only there exists a sequence of graphs $G_0 \subset G_1 \subset \cdots G_j = G$ such that G_0 is a cycle and each G_i is obtained from G_{i-1} by adding a path P_i such that the ends of P_i belong to $V(G_{i-1})$ and no inner-vertex of P_i belongs to G_{i-1}.*

The sequence $G_0, \cdots, G_j$ is called an *ears decomposition* of G and the paths P_i are the *ears* of the decomposition. Robbins observed that orienting G_0 as a circuit and each ear as a dipath results in a strongly connected orientation of G. Thus this observation proves the sufficiency in Theorem 3.6.

For general k, the charaterization of graphs admitting k-arc-connected orientations is given by Nash-Williams. The following result is hence a generalization of Theorem 3.6.

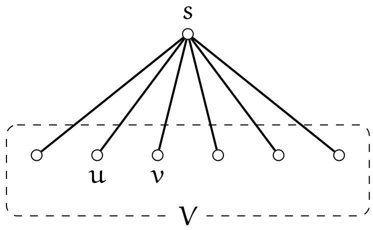
Theorem 3.8 (Nash-Williams [60]). *A graph G admits a k-arc-connected orientation if and only if G is $2k$-edge-connected.*

This theorem is refered to as Nash-Williams' weak orientation theorem as Nash-Williams actually shows a stronger orientation result that we postponed to Section 3.3. Again, in Theorem 3.8, the necessity directly derives from the Menger's theorem. So the difficulty remains in the sufficiency. In the next subsection we present Lovász' proof of sufficiency which relies on a decomposition of $2k$-edge-connected graphs. We will prove a similar decomposition for another family of graphs in Chapter 5.

3.2.2 *Structure of $2k$-Edge-Connected Graphs*

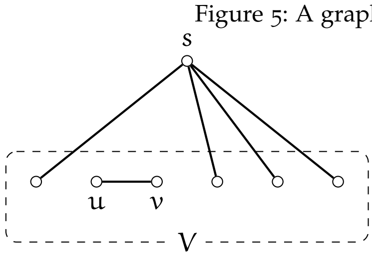
Let $G = (V \cup s, E)$ be a graph and let su and sv be two edges incident to s (see Fig. 5). *Splitting the pair* (su, sv) means deleting these two edges and adding the edge uv (see Fig. 6). The graph resulting from this operation is denoted $G_{u,v}$. When $d_G(s)$ is even, we define a *complete splitting-off at* s as a sequence of $\frac{1}{2}d_G(s)$ splitting-off of disjoint pairs of edges incident to s (see Fig. 7). Hence, in a graph obtained by such a complete splitting-off at s the degree of s is zero.

We are interested in splitting-off that preserves some connectivity properties. In the present chapter this property is the $2k$-edge-connectivity of V. Provided that V is k-edge-connected in G, a splitting-off is called *admissible* if V is k-edge-connected in the graph resulting from this operation. A complete splitting-off is called admissible if each of the $\frac{1}{2}d_G(s)$ splitting-off is admissible. One may easily see that each possible splitting-off is not necessarily



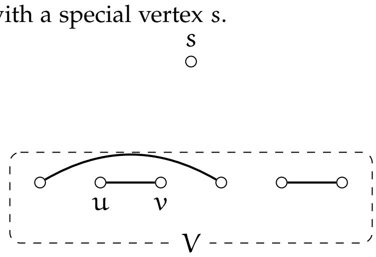
$G = (V \cup s, E)$

Figure 5: A graph with a special vertex s.



$G_{u,v}$

Figure 6: Splitting the pair su, sv.



G'

Figure 7: A complete splitting-off at s.

admissible but Lovász proved that, provided that $k \geqslant 2$ and $d(s)$ is even, there always exists a pair of edges incident to s that defines an admissible splitting-off. From this result he derived the following theorem.

Theorem 3.9 (Lovász [50]). *Let $G = (V \cup s, E)$ be a graph and k be a positive integer such that V is $2k$-edge-connected and $d_G(s)$ is even and $|V| \geqslant 3$. Then there exists an admissible complete splitting-off at s.*

The splitting-off has an inverse operation. *Pinching a set F of edges* means subdividing each edge with a new vertex and identifying the $|F|$ new vertices as a single one. In contrast with splitting-off, pinching enough edges preserves the edge-connectivity.

Fact 3.1. *Let $G = (V, E)$ be a $2k$-edge-connected graph and let $F \subseteq E$, denote G' the graph arising from the pinching of F and denote s the new vertex in G'. Then the vertex set V is $2k$-edge-connected in G'. Moreover G' is $2k$-edge-connected if and only if $|F| \geqslant k$.*

Since this pinching edges does not decrease the degree of subsets of V and by Menger's theorem, G' is $2k$-edge-connected if and only if $2|F| = d_{G'}(s) \geqslant 2k$.

The pinching operation is also defined in digraphs. In a digraph, *pinching a set F of arcs* consists of subdividing each arc with a new vertex and identifying the $|F|$ new vertices as a single one[1]. As for undirected graphs, in digraphs the pinching operation preserves the arc-connectivity.

Fact 3.2. *Let $D = (V, A)$ be a k-arc-connected digraph, let $F \subseteq A$, denote D' the digraph arising from the pinching of F and denote s the new vertex in D'. Then the vertex set V is k-arc-connected in D'. Moreover D' is k-arc-connected if and only if $|F| \geqslant k$.*

Before we prove the constructive charaterization of $2k$-edge-connected graphs, leading to the proof of Theorem 3.8, we need another initially proved by Lick, that will be generalized in Chapter 5. We recall that a graph G is

1 The splitting-off may also be defined in digraphs but we skip this operation that will not be needed.

called minimally k-edge-connected if G is k-edge-connected and the deletion of any edge ruins this property.

Fact 3.3 (Lick [48]). *Every minimally k-edge-connected graph contains a vertex of degree k.*

Now we can prove the promised decomposition theorem.

Theorem 3.10 (Lovász [50]). *A graph $G = (V, E)$ is $2k$-edge-connected if and only if it can be constructed from a pair of vertices joined by $2k$ edges by a sequence of the following operations:*

(I) add an edge between existing vertices,

(II) pinch a set of k existing edges.

Proof. Since adding edges and, by fact 3.1, pinching k edges preserves $2k$-edge-connectivity, the sufficiency follows.

We prove the necessity by induction on $|E|$. The base case is trivial since a pair of vertices joined by $2k$ edges is clearly a $2k$-edge-connected. We have to prove that any $2k$-edge-connected graph such that $|E| > 2k$ is obtained from a $2k$-edge-connected graph by the operation (I) or (II). If there exists an edge e such that $(V, E \setminus e)$ is $2k$-edge-connectivited then this is done. So we may assume that G is minimally $2k$-edge-connected. Thus, by Fact 3.3 and Theorem 3.9, there exist a vertex s of degree $2k$ in G and an admissible complete splitting-off at s. We denote G' and F the graph and the edges resulting from this operation respectively. So G is obtained from the $2k$-edge-connected graph G' by pinching the k edges of F. □

This insight into the structure of $2k$-edge-connected graphs enables us to easily prove Nash-Williams' weak orientation theorem.

Proof of Theorem 3.8. Suppose that $G = (V, E)$ has a k-arc-connected orientation D. Then, for any non-empty $X \subset V$, $d_G(X) = \rho_D(X) + \delta_D(X) \geqslant k + k = 2k$. Hence, by Theorem 2.2, G is $2k$-edge-connected.

We prove by induction on the number of edges that every $2k$-edge-connected graph has a k-arc-connected orientation. If $|E| = 2k$ then the graph is a pair of vertices joined by $2k$ edges and orienting half of the edges is one way and the other half the other way results in a k-arc-connected digraph. Let $G = (V, E)$ be a $2k$-edge-connected graph such that $|E| > 2k$. By Theorem 3.10, G is obtained from a smaller $2k$-edge-connected graph G' by operation (I) or (II). By induction G' admits a k-arc-connected orientation D'. If G is obtained from G' by adding an edge e then the orientation of G resulting from D' by giving e an arbitrary orientation is k-arc-connected. If G is obtained from G' by pinching a set F of k-edge then, by Fact 3.2, the orientation of G obtained from D' by pinching the set of arcs corresponding to F is k-arc-connected. □

3.2.3 *Abstract Connectivity Requirements*

Fank initiated two approaches for graph orientations; each provides a deep insight into the k-arc-connected orientation problem. The first one characterizes the existence of an orientation covering a supermodular function. This result is far reaching as it answers orientation questions about abstract arc-connectivity requirements. It will be used in Chapter 4.2.

The second one derives from the submodular flows polyhedron introduced by Edmonds and Giles. From an algorithmical point of view this approach is very efficient. It provides algorithmical tools to solve in polynomial time two variations of the k-arc-connected orientation problem: find such an orientation with minimum cost or with bounded degrees.

3.2.3.1 *Covering Supermodular Functions*

A digraph $D = (V, A)$ *covers* a set function $p : 2^V \mapsto \mathbb{R}$ if, for all $X \subseteq V$,

$$\rho_D(X) \geqslant p(X). \tag{3.1}$$

Theorem 3.11 (Frank [21])**.** *Let $G = (V, E)$ be a graph and $p : 2^V \rightarrow \mathbb{Z}_+$ a non-negative crossing supermodular function such that $p(\emptyset) = p(V) = 0$. There exists an orientation D covering p if and only if, for every partition $\mathcal{P}$ of V,*

$$e_G(\mathcal{P}) \geqslant \sum_{X \in \mathcal{P}} p(X), \text{ and} \tag{3.2}$$

$$e_G(\mathcal{P}) \geqslant \sum_{X \in \mathcal{P}} p(V \setminus X). \tag{3.3}$$

The condition (3.2) alone is sufficient if p is non-increasing. If p is non-increasing and symmetric then (3.2) reduces to

$$d_G(X) \geqslant 2p(X) \tag{3.4}$$

for all non-empty $X \subset V$.

This theorem readily implies the weak orientation theorem of Nash-Williams. Indeed define p by $p(\emptyset) = p(V) = 0$ and $p(X) = k$ for all non-empty $X \subset V$. Clearly p is non-increasing, non-negative, symmetric and crossing submodular. Hence, by the above theorem, there exists an orientation D of G such that $\rho_D(X) \geqslant k$ for all non-empty $X \subset V$, if and only if $d_G(X) \geqslant 2k$ for all non-empty $X \subset V$.

We note that from Theorem 3.11, Frank also derived a characterization of a k-arc-connected orientation with bounded indegrees.

3.2.3.2 *Submodular Flows*

Let $D = (V, A)$ be a digraph and let $x : A \mapsto \mathbb{R}$. For $Y \subseteq V$, we denote $\rho^x_D(Y) = x(\Delta^-_D(Y))$ and $\delta^x_D(Y) = x(\Delta^+_D(Y))$. Given a crossing submodular function $b : 2^V \mapsto \mathbb{R}$, the function x is called a *submodular flow* if, for every $Y \subseteq V$,

$$\rho^x_D(Y) - \delta^x_D(Y) \leqslant b(Y). \tag{3.5}$$

Given $f, g : A \mapsto \mathbb{R}$ such that $f \leqslant g$, a submodular flow x is *feasible* if

$$f \leqslant x \leqslant g. \tag{3.6}$$

Submodular flows were introduced by Edmonds and Giles [15] who proved that the system defined by (3.5) and (3.6) is totally dual integral[2]. A direct consequence is the following.

Theorem 3.12 (Edmonds and Giles [15])**.** *Let $D = (V, A)$ be a digraph, $b : 2^V \mapsto \mathbb{Z}$ be a crossing submodular function and $f, g : A \mapsto \mathbb{Z}$ such that $f \leqslant g$. Then the polyhedron $P_{D,b}$ of $\mathbb{R}^A$ defined by (3.5) and (3.6) is integer.*

Frank [22] pointed out that the weak orientation of Nash-Williams can be derived from Theorem 3.12.

2 We do not present any linear programming theory in this document so the definition of TDI system is skipped.

Proof of Theorem 3.8. Let $G = (V, E)$ be $2k$-edge-connected graph. Choose an arbitrary orientation D of G and define the set function $b : 2^V \mapsto \mathbb{R}$ by $b(X) = \rho_D(X) - k$ for all non-empty $X \subset V$ and $b(\emptyset) = b(V) = 0$. By submodularity of ρ_D, b is crossing submodular. Let $f = 0_A$ and $g = 1_A$ the constant functions over A. Observe that, by $2k$-edge-connectivity of G, the polyhedron of $\mathbb{R}^A$ defined by (3.5) and (3.6) contains the vector $x = \frac{1}{2}1_A$. Indeed, by the choice of b and x, (3.5) is equivalent to

$$k \leqslant \rho_D^{\frac{1}{2}}(X) + \delta_D^{\frac{1}{2}}(X) = \frac{1}{2}(\rho_D(X) + \delta_D(X)) = \frac{1}{2}d_G(X).$$

Hence P is not empty and, by Theorem 3.12, it contains an integer point y. Let D' be the orientation of G obtained from D by reversing all the arcs a such that $y(a) = 1$. The digraph D' is k-arc-connected since, for any non empty $X \subset V$, $\rho_{D'}(X) = \rho_D(X) - \rho_D^y(X) + \delta_D^y(X) \geqslant k$. □

By crossing submodularity of $b - (\rho_D^x - \delta_D^x)$ and Theorem 2.1, the polyhedron $P_{D,b}$ can be separated in polynomial time. Thus, by the ellipsoid method of Grötschel, Lovász, and Schrijver [35], one may optimize in polynomial time over this polyhedron. Yet this method does not provide a very practical algorithm. In [22], Frank provided a combinatorial algorithm to optimize over $P_{D,b}$ with bounds $f = 0_A$ and $f = 1_A$. He pointed out that this approach also allows to solve two variations of the k-arc-connected orientation problem.

Theorem 3.13 (Frank [22]). *Let G be a $2k$-edge-connected graph. Given, for each edge e of G, a non negative cost for each possible orientation of e, a k-arc-connected orientation of G with minimum cost can be found in polynomial time.*

Theorem 3.14 (Frank [22]). *Let G be a $2k$-edge-connected graph. Given, for each vertex v of G, a lower bound $\alpha(v)$ and an upper bound $\beta(v)$ a k-arc-connected orientation D of G such that $\alpha(v) \leqslant \rho_D(v) \leqslant \beta(v)$ or a certificate that such an orientation does not exist can be found in polynomial time.*

3.3 WELL-BALANCED ORIENTATIONS

3.3.1 *Nash-Williams' Strong Orientation Theorem*

In Section 3.2 we mentioned that, in [60], Nash-Williams proved a stronger result than Theorem 3.8. This section is dedicated to this result which answers the following problem: given a graph $G = (V, E)$ and a symmetric arc-connectivity requirement $r : V^2 \mapsto \mathbb{Z}_+$ find an orientation D of G such that $\lambda_D(u, v) \geqslant r(u, v)$ for all $u, v \in V$. The previous chapter answers this question for the particular case where r is a constant.

When the general question has a positive answer one may easily derive an upper bound on r. Indeed, if G has such an orientation D then, for $u, v \in V$ and a vertex set X containing u but not v minimizing (2.12), we have $\lambda_G(u, v) = d_G(X) = \rho_D(X) + \delta_D(X) \geqslant \lambda_D(v, u) + \lambda_D(u, v) \geqslant 2r(u, v)$ hence $r(u, v) \leqslant \lfloor \frac{1}{2}\lambda_G(u, v) \rfloor$. The strong orientation theorem of Nash-Williams proves that this upper bound can be reached. A *well-balanced* orientation D of a graph $G = (V, E)$ is an orientation such that, for all $u, v \in V$,

$$\lambda_D(u, v) \geqslant \left\lfloor \frac{1}{2}\lambda_G(u, v) \right\rfloor. \tag{3.7}$$

Theorem 3.15 (Nash-Williams [60]). *Every graph has a well-balanced orientation.*

This theorem, called the Nash-Williams' strong orientation theorem, directly implies Theorem 3.8. One may easily observe that, by Theorem 3.1 and the following fact, Theorem 3.15 holds for Eulerian graphs.

Fact 3.4. *Every Eulerian orientation of an Eulerian graph is well-balanced.*

Proof. Let $G = (V, E)$ be an Eulerian graph and D an Eulerian orientation of D. Let $u, v \in V$. By Theorem 2.3 there exists a set X containing v but not u such that $\lambda_D(u,v) = \rho_D(X)$. Observe that, $\rho_D(X) - \delta_D(X) = \sum_{v \in X}(\rho_D(v) - \delta_D(v)) = 0$. Hence, by choice of X and $d_G(X) = \rho_D(X) + \delta_D(X)$, we have $\lambda_D(u,v) = \rho_D(X) = \frac{1}{2}d_G(X) \geqslant \lfloor \frac{1}{2}\lambda_G(u,v) \rfloor$. This proves that D is well-balanced. □

3.3.2 *Odd-Pairing*

It is a basic exercise of Graph Theory to check that, in a graph, the number of vertices of odd degree is even. For a graph $G = (V, E)$ containing 2ℓ vertices of odd degree, an *odd-pairing* is a set M of ℓ edges (not necessarily belonging to G) such that each vertex of odd degree is incident to an edge of M. Clearly adding an odd-pairing to a graph results in a Eulerian graph. The idea of Nash-Williams to find a well-balanced orientation of a graph G is to add to G an odd-pairing M, take any Eulerian orientation of $G + M$, and then remove from this digraph the arcs from M. However to be certain that the resulting orientation of G will be well-balanced the odd-pairing must satisfies a certain property.

For a non-empty subset X of V, we denote $R_G(X) = \max \lfloor \frac{1}{2}\lambda_G(u,v) \rfloor$ where the maximum is taken over all the pairs of vertices u,v separated by X. Thus, by Theorem 2.3 an orientation D is well-balanced if and only if, for every non-empty proper subset X of V,

$$\rho_D(X) \geqslant R_G(X). \tag{3.8}$$

An odd-pairing M is called *admissible* if, for all non-empty $X \subset V$,

$$d_M(X) \leqslant d_G(X) - 2R_G(X). \tag{3.9}$$

Claim 3.1 (Nash-Williams [60])**.** *Let* G *be a graph and* M *an admissible odd-pairing of* G*. Deleting the arc from* M *in any Eulerian orientation of* $G + M$ *results in a well-balanced orientation of* G.

Proof. Let D and M' be the orientations of G and M respectively resulting from an arbitrary Eulerian orientation D' of $G + M$. For any non-empty proper subset X of V, we have

$$\begin{aligned}
\rho_D(X) &= \rho_{D'}(X) - \rho_{M'}(X) \\
&\geqslant \frac{1}{2}d_{G+M}(X) - d_M(X) \\
&= \frac{1}{2}(d_G(X) - d_M(X)) \\
&\geqslant R_G(X).
\end{aligned}$$

Hence (3.8) is satisfied, that is, D is well-balanced. □

So Theorem 3.15 derives from Claim 3.1 and the following deep result of Nash-Williams called the odd-pairing theorem.

Theorem 3.16 (Nash-Williams [60])**.** *Every graph has an admissible odd-pairing.*

Note that in every orientation D obtained in Claim 3.1, for every vertex v, $|\rho_D(v) - \delta_D(v)| \leqslant 1$. An orientation with this property is called *smooth* and well-balanced smooth orientations are called *best-balanced*. Hence Nash-Williams actually proved that every graphs has a best-balanced orientation. From the odd-pairing theorem, Z. Király and Szigeti derived a generalization of Theorem 3.15.

Theorem 3.17 (Z. Király, Szigeti [46]). *Let $G = (V, E)$ be graph and $\{V_1, \cdots, V_k\}$ a partition of V. Then there exists a best-balanced orientation D of G such that, for each i, $D[V_i]$ is a best-balanced orientation of $G[V_i]$.*

3.3.3 *Ingredients for Proof*

In addition to the original proof of Nash-Williams two proofs of Theorem 3.16 are known (Mader [53] and Frank [25]). In this document we do not outline any and we refer the interested reader to the proof in the recent book of Frank [28]. Yet, in this section, we present two essential ingredients of these proofs. The first one is an observation made by Nash-Williams.

Fact 3.5 (Nash-Williams [60]). *For every graph G, the function R_G is skew supermodular.*

The second ingredient is a strengthening of the splitting-off result seen in Section 3.2 yielding the constructive characterization of 2k-edge-connected graphs. In contrast with the previous section, presently we are interested in splitting-off operations that preserve local edge-connectivity for each pair vertices rather than the global edge-connectivity. Mader proved that such an operation exists under a weak assumption.

Theorem 3.18 (Mader [53]). *Let $G = (V \cup s, E)$ be a graph such that no cut-edge of G is incident to s and $d_G(s) \neq 3$. Then there exists a pair of edges incident to s that can be split preserving the local edge-connectivity between each pair of vertices of V.*

Frank proved that Mader's result is equivalent to the following formulation.

Theorem 3.19 (Frank [24]). *Let $G = (V \cup s, E)$ be a graph such that $d_G(s)$ is even and no cut-edge of G is incident to s. Then there exists a complete splitting-off at s that preserves the local edge-connectivity between each pair of vertices of V.*

Provided an efficient method for finding a complete splitting-off that preserves local edge-connectivity, both proofs of Mader and Frank mentioned above can be turned into an algorithm for finding an admissible pairing. In [32], Gabow developed such a method and stated that, consequently, an admissible odd-pairing can be found in time $O(|V|^6)$. But the following question of Frank remains open.

Question 3.1. *Given an odd-pairing M of a graph G can one decide in polynomial time whether M is admissible?*

3.3.4 *Constrained Well-Balanced Orientations*

In Subsection 3.2.3, we saw how crossing submodular functions can be considered as a generalization of global arc-connectivity. The notion that corresponds to local arc-connectivity is skew-submodular function. That explains why the two approaches introduced in Subsection 3.2.3, are not suitable for the well-balanced orientation problem. It is unlikely that similar approaches exist since the two variations of the k-arc-connected orientations problems solved by submodular flows turn out to be NP-complete when one considers well-balanced orientations.

Theorem 3.20 (Bernáth, Iwata, T. Király, Z. Király, and Szigeti [3]). *Let G be a graph. Given, for each edge e of G, a non negative cost for each possible orientation of e and an integer bound K, deciding whether there exists a well-balanced orientation of G with cost at most K is NP-complete.*

Theorem 3.21 (Bernáth et al. [3]). *Let G be a graph. Given, for each vertex v of G, a lower bound $\alpha(v)$ and an upper bound $\beta(v)$, deciding whether there exists a well-balanced orientation D of G such that $\alpha(v) \leqslant \rho_D(v) \leqslant \beta(v)$ is NP-complete.*

3.3.5 *Abstract Odd-Pairing*

In [3], the authors investigated a possible generalization of the odd-pairing theorem of Nash-Williams. For a set function $b : 2^V \mapsto \mathbb{Z}$ we denote T_b the set of elements $v \in V$ such that $b(v)$ is odd. Given a graph $G = (V, E)$, the function $b_G : X \mapsto d_G(X) - R_G(X)$ is symmetric skew-submodular since d_G is submodular and, as we mentioned, R_G is skew supermodular. Moreover T_{b_G} is exactly the set of vertices of G with odd degree since R_G only takes even values. This is also easy to see that $b_G(X)$ has the parity of $|X \cap T_{b_G}|$ for every $X \subseteq V$. Hence Theorem 3.16 answers positively the following question for the special case $b = b_G$.

Question 3.2. *For every symmetric skew-submodular function $b : 2^V \mapsto \mathbb{Z}_+$ such that $b(\emptyset) = 0$ and $b(X)$ has the parity of $|X \cap T_b|$ for every $X \subseteq V$ does there exist a pairing M of T_b such that, for every $X \subseteq V$,*

$$d_M(X) \leqslant b(X)? \tag{3.10}$$

However Bernáth et al. disproved this statement. So they suggested a generalization of the following corollary of Theorem 3.16.

Corollary 3.1. *In every $2k$-edge-connected graph $G = (V, E)$ there exists an odd-pairing M such that, for all $X \subseteq V$,*

$$d_M(X) \leqslant d_G(X) - 2k. \tag{3.11}$$

In a $2k$-edge-connected graph G, the set function b'_G defined by $b'_G(\emptyset) = b'_G(V) = 0$ and $b'_G(X) = d_G(X) - 2k$ for non-empty $X \subset V$ is non-negative and crossing submodular. Moreover b'_G satisfies the same parity constraint as b_G. So Corollary 3.1 answers positively the following question in a particular case.

Question 3.3 (Bernáth et al. [3]). *For every symmetric crossing submodular function $b : 2^V \mapsto \mathbb{Z}_+$ such that $b(\emptyset) = 0$ and $b(X)$ has the parity of $|X \cap T_b|$ for all $X \subseteq V$ does there exists a pairing M of T_b satisfying (3.10)?*

Note that the orientation result associated to the statement of Question 3.3 can be easily proved by Theorem 3.12 or Theorem 3.11.

Theorem 3.22 (Bernáth et al. [3]). *Let $G = (V, E)$ be a graph and $b : 2^V \mapsto \mathbb{Z}_+$ a symmetric crossing submodular function such that T_b is the set odd degree vertices of G. Then there exists an orientation D of G such that, for all $X \subseteq V$, $\rho_D(X) - \delta_D(X) \leqslant b(X)$.*

In the rest of this subsection we give a negative answer to Question 3.3 which remained opened in [3].

Let V be a set of 8 elements, say $v_1, \cdots, v_8$. We define a collection $\mathcal{H}$ of 7 subsets of V as

$$\mathcal{H} = \{\{v_1, v_2, v_3, v_4\}, \{v_1, v_2, v_5, v_6\}, \{v_1, v_2, v_7, v_8\}, \{v_1, v_3, v_5, v_7\},$$
$$\{v_1, v_3, v_6, v_8\}, \{v_1, v_4, v_5, v_8\}, \{v_1, v_4, v_6, v_7\}\}.$$

The set $\mathcal{H}$ is drawn in Figure 8. Let $b : 2^V \mapsto \mathbb{Z}_+$ be defined by

$$b(X) = \begin{cases} 2 & \text{if } X \in \mathcal{H} \text{ or } (V \setminus X) \in \mathcal{H}, \\ \min(|X|, 8 - |X|) & \text{otherwise.} \end{cases}$$

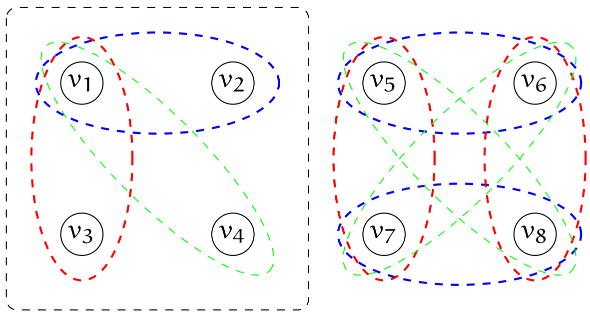


Figure 8: Each set of the collection $\mathcal{H}$ is either the black one or the union of a dashed colored set and a plain set of the same color.

Fact 3.6. *The set function* b *is symmetric crossing submodular and, for all* $X \subseteq V$, $b(X)$ *and* $|X \cap T_b|$ *have the same parity.*

Proof. The symmetry of b is obvious and, since $T_b = V$, the parity conditions is trivial. To prove the crossing submodularity property we define $\mathcal{H}'$ as the set of the complements of the elements of $\mathcal{H}$. Let X and Y be two crossing subsets of V.

If both X and Y belong to $\mathcal{H} \cup \mathcal{H}'$ then observe that $|X \cap Y| = 2$ and $|X \cup Y| = 6$ thus $b(X) + b(Y) = 2 + 2 = b(X \cap Y) + b(X \cup Y)$.

Suppose that X belongs to $\mathcal{H} \cup \mathcal{H}'$ but Y does not. So we have $|X| = 4$ and none of $X \setminus Y$ or $Y \setminus X$ is empty. Hence

$$\begin{aligned} |X \cap Y| + (8 - |X \cup Y|) &= |X| - |Y| - 2|X \setminus Y| + 8 \\ &\leqslant 4 - |Y| - 2 + 8 \\ &= b(X) + (8 - |Y|) \text{ and} \\ |X \cap Y| + (8 - |X \cup Y|) &= |Y| - |X| - 2|Y \setminus X| + 8 \\ &\leqslant |Y| - 4 - 2 + 8 \\ &= b(X) + |Y|. \end{aligned}$$

Thus, $b(X \cap Y) + b(X \cup Y) = |X \cap Y| + (8 - |X \cup Y|) \leqslant b(X) + b(Y)$.

Now suppose that none of X or Y belongs to $\mathcal{H} \cup \mathcal{H}'$. Note that $b(X \cap Y) \leqslant \min\{|X \cap Y|, 8 - |X \cap Y|\}$ and $b(X \cup Y) \leqslant \min\{|X \cup Y|, 8 - |X \cup Y|\} = \min\{|X| + |Y| - |X \cap Y|, 8 - |X| - |Y| + |X \cap Y|\}$. Hence

$$\begin{aligned} b(X \cap Y) + b(X \cup Y) &\leqslant \min\{|X| + |Y|, 8 - |X| + 8 - |Y|, \\ &\quad 8 - |X| - |Y| + 2|X \cap Y|\} \\ &\leqslant \min\{|X| + |Y|, 8 - |X| + 8 - |Y|, \\ &\quad 8 - |X| + |Y|, |X| + 8 - |Y|\} \\ &\leqslant b(X) + b(Y). \end{aligned}$$

□

Fact 3.7. *There exists no pairing M of T_b such that (3.10) holds for all $X \subseteq V$.*

Proof. Now suppose by contradiction that there exists a pairing M of V such that (3.10) holds. Since $T_b = V$ we have $|M| = 4$. Each $X \in \mathcal{H}$ induces at least one edge of M since $d_M(X) \leqslant b(X) = 2$. Thus, since $|M| = 4 < 7 = |\mathcal{H}|$, there exists an edge e induced by more than one element of $\mathcal{H}$. Denote $\mathcal{H}'$ the set of elements of $\mathcal{H}$ that induce e. Observe that e is incident to v_1, so e is unique and $|\mathcal{H}'| \leqslant 3$. Hence we have $3 = |M \setminus e| \geqslant |\mathcal{H} \setminus \mathcal{H}'| \geqslant 4$, a contradiction. □

4 PACKING IN HIGHLY CONNECTED GRAPHS

In this chapter we investigate relations between graphs orientation problems and packing problems. First we consider a problem of Recski that combines the well understood m-orientation problem (recall subsection 3.1.1) with an other classic problem on packing spanning trees. We prove that the combination of these two polynomial time solvable problems turns to be NP-complete.

An other relation is given by Frank [19] who emphasized that, provided the spanning arborescences packing theorem of Edmonds [14], the classic result on packing spanning trees [69, 61] and the characterization of graphs having a rooted k-arc-connected orientation are equivalent. As we will see, this equivalence still holds if we consider more general problems.

Finally, we look into a relation given by Jordán [40] who proved a sufficient condition for the existence of a packing of rigid spanning subgraphs to settle a special case of Thomassen's conjecture on vertex-connected orientations. We will generalize the packing theorem of Jordán to improve the orientation result.

4.1 PACKING OF TREES

In the first subsection of this section we state and prove the classic result of Tutte [69] and Nash-Williams [61] that characterizes the existence of a packing of spanning trees. The key point is that a spanning tree is the base of the circuit matroid; so the problem is a special case of finding disjoint bases of a matroid. Consequently, the results of Edmonds [12] solve both theoretical and algorithmic aspects of the problem since the characterization can be deduced from Theorem 2.6 and, by Theorem 2.7, the trees can be found in polynomial.

In the second subsection, we introduce a recent theorem of Katoh and Tanigawa [44] that generalizes the characterization of Tutte and Nash-Williams and discuss other formulations of it. In their variation of the problem, the spanning property is replaced by a matroid constraint. In the next section we will derive the theorem of Katoh and Tanigawa from its directed counterpart proved by Nguyen, Szigeti and I [9].

In [64] Recski, motivated by the study of rigidity of framework in d-dimensional spaces, introduced an other variation of the spanning tree problem. In this variation each spanning tree is required to admit an orientation with given indegrees. He conjectured that given the orientation constraints, the existence of two edge-disjoint spanning trees that admit degree constrained orientations can be decided in polynomial time. In the last subsection, we disprove this conjecture provided that $P \neq NP$.

4.1.1 *Packing of Spanning Trees*

Let $G = (V, E)$ be a graph. For a partition $\mathcal{P}$ of V, we recall that $e_G(\mathcal{P})$ denotes the number of edges of G between the different members of $\mathcal{P}$. We always suppose that the members of $\mathcal{P}$ are not empty. Following Frank [28], G is called k-*partition-connected* if, for every partition $\mathcal{P}$ of V,

$$e_G(\mathcal{P}) \geqslant k(|\mathcal{P}| - 1). \tag{4.1}$$

Tutte and Nash-Williams independently showed that partition-connectivity characterizes the existence of a packing of spanning trees.

Theorem 4.1 (Nash-Williams [61], Tutte [69])**.** *A graph* G *contains* k *edge-disjoint spanning trees if and only if* G *is* k*-partition connected.*

We give the proof of this result for two reasons. First it emphasizes the help of orienting trees into arborescences to prove the necessity. Second it gives a first application of Theorem 2.6. Indeed, a spanning tree is a base of $\mathcal{C}(G)$ the circuit matroid of G. Hence G contains k edge-disjoint spanning trees if and only if the matroid $\mathcal{C}_k(G)$ defined as the union of k copies of $\mathcal{C}(G)$ has rank $k(|V| - 1)$ where $|V| - 1$ is the size of a base of $\mathcal{C}(G)$.

Proof of Theorem 4.1. Suppose that G contains k edge-disjoint spanning trees. Choose arbitrarily a vertex $r \in V$ and take an orientation D of G such that each spanning tree results in an r-arborescence. For $v \in V \setminus r$, in each r-arborescence there is a dipath from r to v and the dipaths are arc-disjoint since the trees are edge-disjoint. Hence, D is rooted k-arc-connected at r. So, for a partition $\mathcal{P}$ of V we have $e_G(\mathcal{P}) \geqslant \sum_{X \in \mathcal{P}, r \notin X} \rho_D(X) \geqslant k(|\mathcal{P}| - 1)$, and the necessity follows.

To see the sufficiency suppose that G is k-partition-connected. As we pointed out above, we have to prove that

$$r_{\mathcal{C}_k(G)}(E) = k(|V| - 1). \tag{4.2}$$

By Theorem 2.6 and (2.14),

$$r_{\mathcal{C}_k(G)}(E) = \min_{F \subseteq E} k(|V| - c(G_F)) + |E \setminus F|,$$

where $c(G_F)$ is the number of connected components of $G_F = (V, F)$. Note that $r_{\mathcal{C}_k}(E) \leqslant k(|V| - 1)$. Let $F \subseteq E$ and define $\mathcal{P}$ as the connected components of G_F. By k-partition-connectivity of G, we have $|E \setminus F| \geqslant e_G(\mathcal{P}) \geqslant k(|\mathcal{P}| - 1) = k(c(G_F) - 1)$ thus $k(|V| - c(G_F)) + |E \setminus F| \geqslant k(|V| - 1)$ and (4.2) follows. □

4.1.2 *Matroid-Based Packing of Rooted-Trees*

In this subsection we present a generalization of Theorem 4.1 due to Katoh and Tanigawa [44].

Let $G = (V, E)$ be a graph, $\mathcal{M}$ a matroid on a ground set S with rank function $r_{\mathcal{M}}$, and $\pi : S \mapsto V$. In this subsection t will always denote $|S|$ and the elements of S will be denoted $s_1, \cdots, s_t$. The elements of S are called the *roots*. The function π is a *placement* of the roots on the set of vertices. Note that different roots may be placed at the same vertex. We denote $S_X = \pi^{-1}(X)$ the set of roots placed in X. The quadruple $(G, \mathcal{M}, S, \pi)$ is called a *matroid-based rooted-graph*.

A *rooted tree* is a pair (T, s) where T is a tree of G and $s \in S$ is placed on a vertex of T. The element s is called the *root of the rooted tree* (T, s). Note that T may consist of a single vertex $\pi(s)$ with no edge.

The following definition was introduced by Katoh and Tanigawa [44]. A *matroid-based packing of rooted-trees* of $(G, \mathcal{M}, S, \pi)$ is a set $\{(T_1, s_1), \ldots, (T_t, s_t)\}$ of pairwise edge-disjoint rooted-trees such that for each $v \in V$, the set

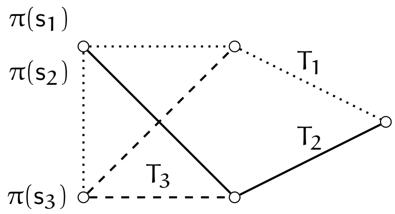


Figure 9: A matroid-based packing of rooted-trees where the set of the independent sets of the matroid on $S = \{s_1, s_2, s_3\}$ is $2^S \setminus S$.

$\{s_i \in S : v \in V(T_i)\}$ of the elements of the matroid whose tree contains v forms a base of $\mathcal{M}$ (see Figure 9). Note that the trees are not necessarily spanning and each vertex of G belongs to exactly $r_{\mathcal{M}}(S)$ trees.

Matroid-based packings of rooted-trees is a generalization of packings of spanning trees. Indeed, if $\mathcal{M}$ is the free matroid on k elements then, since S is the only base of $\mathcal{M}$, each vertex is contained by the k trees. Note that in this particular case the placement π of the roots is not relevant.

The following result characterizes matroid-based rooted-graphs that have a matroid-based packing of rooted-trees. We say that the map π is $\mathcal{M}$-*independent* if S_v is independent in $\mathcal{M}$ for all $v \in V$. The quadruple $(G, \mathcal{M}, S, \pi)$ is called *partition-connected* if, for every partition $\mathcal{P}$ of V,

$$e_G(\mathcal{P}) \geqslant r_{\mathcal{M}}(S)|\mathcal{P}| - \sum_{X \in \mathcal{P}} r_{\mathcal{M}}(S_X). \tag{4.3}$$

Theorem 4.2 (Durand de Gevigney, Nguyen, Szigeti [9]). *Let $(G, \mathcal{M}, S, \pi)$ be a matroid-based rooted-graph. There exists a matroid-based packing of rooted-trees in $(G, \mathcal{M}, S, \pi)$ if and only if π is $\mathcal{M}$-independent and $(G, \mathcal{M}, S, \pi)$ is partition-connected.*

Since matroid-based packing of rooted-trees is a generalization of packing of spanning trees, Theorem 4.2 is a proper extension of Theorem 4.1. We may also check that, if $\mathcal{M}$ is the free matroid on k elements then $r_{\mathcal{M}}(S) = k$ and, for every partition $\mathcal{P}$ of V, $\sum_{X \in \mathcal{P}} r_{\mathcal{M}}(S_X) = k$ thus (4.3) is exactly (4.1).

In [44], Theorem 4.2 is not explicitly stated. Katoh and Tanigawa did not formulated the problem in terms of packing but in terms of decomposition, that is finding subgraphs of G whose edge sets partition E. A *rooted-component* of $(G, \mathcal{M}, S, \pi)$ is a pair (C, s) where C is a connected subgraph of G and $s \in S_{V(C)}$.

Theorem 4.3 (Katoh, Tanigawa [44]). *Let $(G, \mathcal{M}, S, \pi)$ be a matroid-based rooted-graph. Then $(G, \mathcal{M}, S, \pi)$ can be decomposed into rooted-components $(C_1, s_1), \dots, (C_t, s_t)$ such that the set $\{s_i \in S : v \in V(C_i)\}$ is a spanning set of $\mathcal{M}$ for every $v \in V$ if and only if $(G, \mathcal{M}, S, \pi)$ is partition-connected.*

In the paper of Katoh and Tanigawa, Theorem 4.3 is derived from the following formulation and the proof implicitly proves Theorem 4.2. Here we prove that Theorem 4.2 implies Theorem 4.4.

Theorem 4.4 (Katoh, Tanigawa [44]). *Let $(G, \mathcal{M}, S, \pi)$ be a matroid-based rooted-graph and denote k and $r_{\mathcal{M}}$ the rank and the rank function of $\mathcal{M}$ respectively. Then $(G, \mathcal{M}, S, \pi)$ admits a matroid-based rooted-tree decomposition if and only if π is $\mathcal{M}$-independent, $|E| + |S| = k|V|$ and $i_E(X) + |S_X| \leqslant k|X| - k + r_{\mathcal{M}}(S_X)$ for all non-empty $X \subseteq V$.*

Proof. The necessity is proved using orientation of the rooted-trees into rooted-arborescences. We skip this part since, as one can see in [44], this is quite straightforward and a similar argument will be detailed in the proof of Theorem 4.2 in the next section.

Now suppose that the conditions hold. For every partition $\mathcal{P}$ of V, by the inequality applied for $X \in \mathcal{P}$ and by $|E| + |S| = k|V|$, we have

$$\begin{aligned} e_G(\mathcal{P}) &= |E| - \sum_{X \in \mathcal{P}} i_E(X) \\ &\geqslant |E| - \sum_{X \in \mathcal{P}} (k|X| - k + r_{\mathcal{M}}(S_X) - |S_X|) \\ &= k|\mathcal{P}| - \sum_{X \in \mathcal{P}} r_{\mathcal{M}}(S_X). \end{aligned}$$

Hence $(G, \mathcal{M}, S, \pi)$ is partition-connected. Then, since π is $\mathcal{M}$-independent, Theorem 4.2 implies that $(G, \mathcal{M}, S, \pi)$ admits a matroid-based packing of rooted-trees $\{(T_1, s_1), \cdots, (T_t, s_t)\}$. By (2.10), we have,

$$\begin{aligned} \sum_{i=1}^{t} |T_i| &= \sum_{i=1}^{t} (|V(T_i)| - 1) \\ &= \sum_{v \in V} |\{s_i \in S : v \in V(T_i)\}| - t \\ &= k|V| - |S|. \end{aligned}$$

Hence every edge of E belongs to a T_i, so $\{(T_1, s_1), \cdots, (T_t, s_t)\}$ is a matroid-based rooted-tree decomposition of $(G, \mathcal{M}, S, \pi)$. $\square$

4.1.3 *Decomposition into Two Trees with Orientation Constraints*

By Theorem 2.7, finding the maximum number of edge-disjoint spanning trees a graph contains can be done in polynomial time. Especially, deciding whether the edge set of a graph can be partitioned into two trees is tractable in polynomial time. In this subsection we show that if we ask each partition to have an indegree constrained orientation then the problem becomes NP-complete. The material of this subsection is from [7].

Let $G = (V, E)$ be a graph. For $m \in \mathbb{Z}_+^V$, we recall from Subsection 3.1 that an m-*orientation* of $F \subseteq E$ is an orientation of the edges in F such that the number of arcs of F entering v is $m(v)$ for each $v \in V$. Given $b, r \in \mathbb{Z}_+^V$, a (b, r)-*partition* of E is a partition of E into two spanning trees, say a blue one and a red one, such that the blue tree has a b-orientation and the red tree has an r-orientation. Recski [64] proved that the problem of finding a (b, r)-partition is a special case of the 3 matroids intersection problem that is known to be NP-hard [72] and he conjectured that deciding the existence of a (b, r)-partition can be done in polynomial time. The following theorem answers negatively this question if $P \neq NP$.

Theorem 4.5. *Let* $G = (V, E)$ *be a graph and* $b, r \in \mathbb{Z}_+^V$. *Deciding whether there exists a* (b, r)*-partition of* E *is NP-complete.*

To prove this result we give a reduction of an instance of NotAllEqual 3-Sat. In such an instance, each clause consists of three non-negated variables and an assignment is a coloring of the variables with blue or red. A clause is satisfied if it contains both a blue and a red variable. Schaefer [66] proved that this variation of Sat is NP-complete.

Theorem 4.6 (Schaefer [66]). NotAllEqual 3-Sat *is NP-complete.*

Let Π be an instance of NotAllEqual 3-Sat and denote n the number of clauses. We will define a graph $G(\Pi) = (V, E)$ on $12n + 1$ vertices and two indegree vectors $b, r \in \mathbb{Z}_+^V$ with the following property.

Claim 4.1. *There exists a* (b, r)*-partition of* E *if and only if there exists a coloring of the variables satisfying* Π*.*

Hence Theorem 4.5 will follow from Theorem 4.6 and Claim 4.1.

For each clause C of Π we add a copy C' of that clause. Hereinafter C will always denote a clause that is originally in Π and C' will always denote a copy.

For each clause $C = (x, y, z)$ we construct a C-gadget on six vertices $u_x^C, u_y^C, u_z^C, v_x^C, v_y^C, v_z^C$. This gadget consists of the triangle on the vertex set $U^C = \{u_x^C, u_y^C, u_z^C\}$ and the edge set $E^C = \{u_x^C v_x^C, u_y^C v_y^C, u_z^C v_z^C\}$ (see Figure 10). This construction is also done for the copy C' of C. The indegree vectors b and r are defined by $b = r = 1$ for each of the 6 vertices denoted by the letter u in the C-gadget and the C'-gadget, $b = 1$ and $r = 0$ for each of the 3 vertices denoted by the letter v in the C-gadget, and $b = 0$ and $r = 1$ for each of the 3 vertices denoted by the letter v in the C'-gadget.

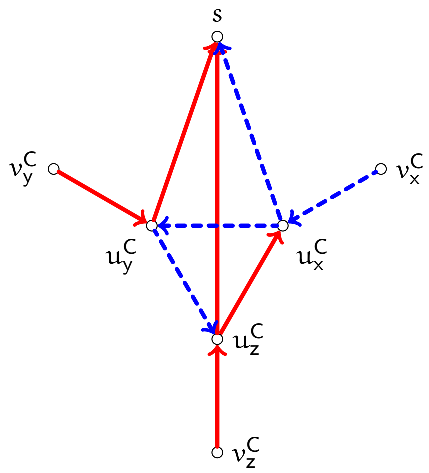


Figure 10: A clause gadget for $C = (x, y, z)$. The coloring (dashed is blue and plain is red) and the orientation of the edges corresponds to a blue coloring of x and a red coloring of y and z.

We add a special vertex s and, for each vertex u of the $6n$ vertices denoted by the letter u we add the edge su. The indegree vectors are defined on s by $b = r = 3n$.

For each variable x, we add a cycle Δ_x going trough all the vertices of type v_x^C and $v_x^{C'}$ where C contains x. This cycle alternates vertices from C-gadgets and vertices from C'-gadgets (see Figure 11). This ends the definition of $G(\Pi)$.

Proposition 4.1. *In every* b*-orientation of the blue tree and in every* r*-orientation of the red tree of a* (b, r)*-partition of* E*, for every clause* $C = \{x, y, z\}$*, each arc resulting from the orientation of an edge of* E^C *enters* U^C*,* E^C *contains both a blue and a red edge and both trees restricted to* $s \cup U^C$ *are connected. This holds also for copies* C' *of the original clauses.*

Proof. Let $\{T_b, T_r\}$ be a (b, r)-partition of E and D be an orientation of G resulting from a b-orientation of T_b and an r-orientation of T_r.

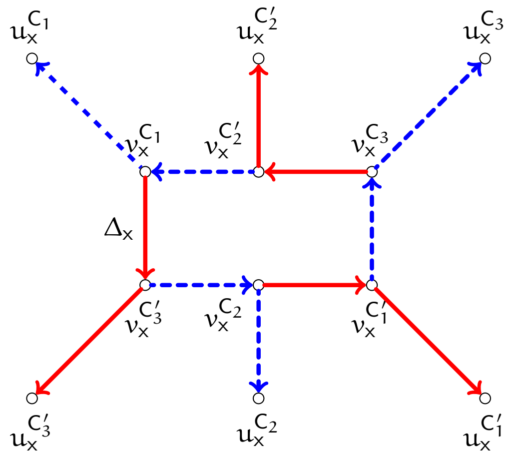


Figure 11: A variable gadget where the original clauses C_1, C_2 and C_3 contain the variable x. The coloring (dashed is blue and plain is red) of the edges corresponds to a blue coloring of x.

Observe that the neighbors of s are the vertices denoted by the letter u in the C-gadgets and the C'-gadgets and there are 3 neighbors in each of the $2n$ gadgets. So we have $d_G(s) = 6n = b(s) + r(s)$ hence

$$\text{all the arcs of } D \text{ incident to } s \text{ enter } s. \tag{$\star$}$$

Hence, by $r(U^C) = b(U^C) = 3$, each of the 6 arcs of D incident to U^C in $G - s$ enters a vertex of U^C, exactly 3 are blue and exactly 3 are red. So the arcs resulting from the orientation of E^C enter U^C. The set E^C contains both a blue and a red edge otherwise one of the trees would contain the triangle $u_x^C u_y^C u_z^C$.

Hence, by permuting x, y and z if necessary, we may assume that the edge $u_x^C v_x^C$ is blue and $u_y^C v_y^C$ and $u_z^C v_z^C$ are red. Thus the triangle $u_x^C u_y^C u_z^C$ contains exactly two blue edges and, by ($\star$) and $r(u_x^C) = 1$, the common end vertex of the two blue edges is not u_x^C. By permuting y and z if necessary, we may assume that $u_x^C u_y^C$ and $u_y^C u_z^C$ are blue and $u_x^C u_z^C$ is red. One of the edges su_x^C, su_z^C is blue, otherwise the red tree would contain the triangle $su_x^C u_z^C$, and there is at most one blue edge from s to U^C, otherwise the blue tree would contain one of the cycles $su_x^C u_y^C s$, $su_y^C u_z^C s$, $su_x^C u_y^C u_z^C s$. So either su_x^C is blue and su_y^C, su_z^C are red or su_z^C is blue and su_x^C, su_y^C are red. In both cases each of the trees restricted to $s \cup U^C$ is connected. □

Proposition 4.2. *Let x be a variable. In every (b, r)-partition of E, all the edges $u_x^C v_x^C$, where C is an original clause containing x, have the same color and all the edges $u_x^{C'} v_x^{C'}$, where C' is a copy of an original clause containing x, have the other color.*

Proof. By Proposition 4.1, in a b-orientation of the blue tree and an r-orientation of the red tree, the arcs of type $v_x^C u_x^C$ or $v_x^{C'} u_x^{C'}$ leave the cycle Δ_x. Hence, since in Δ_x vertices with $r = 1$ and $b = 0$ and vertices with $r = 0$ and $b = 1$ alternate, Δ_x has a circuit orientation and the color of the edges alternates.

Let C_i be an original clause containing x and suppose that the edge $u_x^{C_i} v_x^{C_i}$ is blue (for instance $i = 1$ in Figure 11). Denote by $v_x^{C_j'}$ the neighbor of $v_x^{C_i}$ in Δ_x such that the edge $v_x^{C_i} v_x^{C_j'}$ is blue ($j = 1$ in Figure 11). By Proposition 4.1, there exist a blue path joining s and $u_x^{C_i}$ in $s \cup U^{C_i}$ and a blue path joining s and $u_x^{C'_j}$ in $s \cup U^{C'_j}$. Thus the edge $u_x^{C'_j} v_x^{C'_j}$ is red, otherwise the blue tree would contain a cycle including the two paths and the path $u_x^{C_i} v_x^{C_i} v_x^{C'_j} u_x^{C'_j}$. The same argument shows that the edge $u_x^{C_k} v_x^{C_k}$ is red where $v_x^{C_k}$ is the other neighbor of $v_x^{C'_j}$ in Δ_x ($k = 2$ in Figure 11). Since Δ_x is even, a repeated application of this argument proves the proposition. □

Proof of Claim 4.1. Suppose there exists a (b, r)-partition of E. By Proposition 4.2, for each variable x, all the edges of type $u_x^C v_x^C$, where C is an original clause containing x, have the same color. Hence it is consistent to color a variable x with the color of an edge $u_x^C v_x^C$ for an original clause C containing x. By Proposition 4.1, for each original clause C of Π, E^C contains a blue and a red edge, thus C contains a blue and a red variable, that is, C is satisfied. It follows that this coloring satisfies Π.

Now suppose that there exists a coloring satisfying Π. For each variable x and each original clause C containing x, color $u_x^C v_x^C$ with the color of x and color $u_x^{C'} v_x^{C'}$ with the other color. Since each clause C contains a blue and a red variable, the coloring of the edges induced by $s \cup U^C$ can be done as in Figure 11 (permute the variables and the colors if necessary). Do the same for the coloring of the edges in the C'-gadgets. For each variable x alternate the color along the cycle Δ_x. So far we obtained a partition of E into a blue and a red spanning tree.

Now orient the edges incident to every C-gadget or C'-gadget as in Figure 11 (or the inverse coloring of that figure). Observe that the multiplicity of colors is the same in E^C and in the arcs from U^C to s. Hence, for each clause C, there are exactly 3 blue edges and 3 red edges from $U^C \cup U^{C'}$ to s and the indegree of s is $\frac{1}{2} d_G(s) = 3n$ in each tree. For each variable x orient the bicolored cycle Δ_x to obtain a circuit that satisfies the indegree constraints on vertices of type v_x^C and $v_x^{C'}$. Hence we obtain a b-orientation of the blue tree and an r-orientation of the red tree. □

4.2 PACKING OF ARBORESCENCES

In this section we state a well know result of Edmonds [14] that is the directed counterpart of Theorem 4.1. These two results characterize the graphs admitting a rooted k-arc-connected orientation. In [19], Frank initiated a new approach: he gave a direct proof of this characterization to derive easily Theorem 4.1 from the theorem of Edmonds.

In this subsection we follow the approach of Frank for the matroid-based packing of rooted trees problem. That is, we prove the directed counterpart of Theorem 4.2 and we show how the undirected result can be immediately obtained from the directed one and the general orientation theorem of Frank on covering supermodular functions. The material of the second section is from a joint work with Nguyen and Szigeti [9].

4.2.1 *Packing of Spanning Arborescences*

The theorem of Tutte and Nash-Williams states that partition-connectivity characterizes the existence of a packing of spanning trees in undirected graphs. Edmonds proved a similar result in digraphs: rooted-arc-connectivity characterizes the existence of a packing of spanning arborescences.

Theorem 4.7 (Edmonds [14])**.** *A digraph $D = (V, A)$ contains k arc-disjoint spanning r-arborescences if and only if D is rooted k-arc-connected at r.*

Note that the necessity of the connectivity condition is straightforward. Indeed, if there exist k arc-disjoint spanning r-arborescences then each arborescence contains a dipath from r to any other vertex and the k dipaths are pairwise arc-disjoint.

The link between Theorems 4.1 and 4.7 is an orientation result for which Frank gave a direct proof.

Theorem 4.8 (Frank [19]). *A graph $G = (V, E)$ has a rooted k-arc-connected orientation if and only if G is k-partition-connected.*

At first sight it may appear surprising that the position of the root is not specified in the characterization. Actually the position of the root is not relevant. Indeed, suppose that G has a rooted k-arc-connected orientation D rooted at $r \in V$ and choose $r' \in V \setminus r$. One may observe that, for $X \subset V$, reversing k arc-disjoint paths from r to r' in D increases $\rho_D(X)$ by k if $r \in X$ and $r' \notin X$ and does not decrease $\rho_D(X)$ if $r' \notin X$. Hence, by Menger's theorem, this operation yields a rooted k-arc-connected orientation of G rooted at r'.

Provided the packing result of Edmonds, this orientation theorem and the packing result of Tutte and Nash-Williams are equivalent since one can be deduced from the other. The point of Frank, was to derive Theorem 4.1 from Theorem 4.8. More generally, to prove a result in undirected graphs, the approach of Frank consists of proving an orientation result and the oriented counterpart of the aimed result in digraphs. We follow this approach in the next two subsections.

4.2.2 *Matroid-Based Packing of Rooted-Arborescences*

In this subsection we mimic Frank's approach for matroid-based packing of rooted-trees. We provide Theorem 4.9, the directed counterpart of Theorem 4.2, and show that this new result implies Theorem 4.2 via an orientation theorem of Frank [21]. The proof of Theorem 4.9 is postponed to the next subsection.

A *matroid-based rooted-digraph* is a quadruple $(D, \mathcal{M}, S, \pi)$ where D is a digraph, S is a set, $\mathcal{M}$ is a matroid on ground set S and π is a placement of the element of S on the vertices of D.

A *rooted-arborescence* is a pair (T, s) where s is an element of S and T is a $\pi(s)$-arborescence. We say that s is the *root* of the rooted-arborescence (T, s). Note that T may consist of the single vertex $\pi(s)$ and no arcs.

A *matroid-independent packing of rooted-arborescences* of $(D, \mathcal{M}, S, \pi)$ is a set $\{(T_1, s_1), \dots, (T_t, s_t)\}$ of pairwise arc-disjoint rooted-arborescences such that, for each $v \in V$, the set $\{s_i \in S : v \in V(T_i)\}$ of the roots of the rooted arborescences containing v is independent.

A *matroid-based packing of rooted-arborescences* is a matroid-independent packing of rooted-arborescences such that, for each $v \in V$, $\{s_i \in S : v \in V(T_i)\}$ is a base of $\mathcal{M}$ (see Figure 12). For a better understanding, let us mention that the rooted-arborescences are not necessarily spanning and each vertex of D belongs to exactly $r_{\mathcal{M}}(S)$ rooted-arborescences.

Matroid-based packings of rooted-arborescences is a generalization of packings of spanning arborescences. Indeed, if $\mathcal{M}$ is the free matroid on k elements and all the elements of S are placed at a vertex r then, since S is the only base of $\mathcal{M}$, each vertex is spanned by the k r-arborescences. Note that in

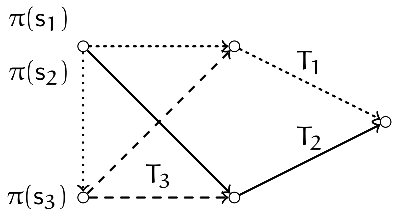


Figure 12: A matroid-based packing of rooted-arborescences where the set of the independent sets of the matroid on $S = \{s_1, s_2, s_3\}$ is $2^S \setminus S$.

contrast with rooted-trees, in this particular case the placement π does matter.

The following result is the directed counterpart of Theorem 4.2. The quadruple $(D, \mathcal{M}, S, \pi)$ is called *rooted-connected* if

$$\rho_D(X) \geqslant r_{\mathcal{M}}(S) - r_{\mathcal{M}}(S_X) \qquad \text{for all non-empty } X \subseteq V. \qquad (4.4)$$

Theorem 4.9 (Durand de Gevigney, Nguyen, Szigeti[9])**.** *Let $(D, \mathcal{M}, S, \pi)$ be a matroid-based rooted-digraph. There exists a matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, S, \pi)$ if and only if π is $\mathcal{M}$-independent and $(D, \mathcal{M}, S, \pi)$ is rooted-connected.*

Since matroid-based packing of rooted-arborescences is an generalization of packing of spanning arborescences, Theorem 4.9 is a proper extension of Theorem 4.7. Actually, Edmonds proved a more general result that is also a corollary of ours. The root set of a branching B is the set of the roots of the arborescences of B.

Theorem 4.10 (Edmonds [14])**.** *Let $D = (V, A)$ be a digraph and let $R_1, \ldots, R_k$ be subsets of V. Then there exist arc-disjoint branchings $B_1, \ldots, B_k$ such that, for each $i \in \{1, \ldots, k\}$, R_i is the root set of B_i if and only if*

$$\rho_D(X) \geqslant |\{i : R_i \cap X = \emptyset\}|$$

for all $X \subseteq V$.

Proof. Consider the matroid $\mathcal{M}$ obtained from the free matroid on $\{s_1, \ldots, s_k\}$ by adding to the ground set, for each $i \in \{1, \ldots, k\}$, $|R_i| - 1$ elements parallel to s_i. So the ground set S of $\mathcal{M}$ contains exactly $\sum_{i=1}^{k} |R_i|$ elements. Define π by placing on each vertex v one copy of s_i for each i such that $v \in R_i$.

The required packing in Theorem 4.10 is exactly a matroid-base packing of rooted-arborescences in the matroid-based rooted-digraph $(D, \mathcal{M}, S, \pi)$. Clearly π is $\mathcal{M}$-independent hence, by Theorem 4.9, the existence of such a packing is equivalent to

$$\rho_D(X) \geqslant r_{\mathcal{M}}(S) - r_{\mathcal{M}}(S_X) = k - |\{i : R_i \cap X \neq \emptyset\}|.$$

for all non-empty $X \subseteq V$. Hence Theorem 4.10 follows. □

The link between Theorem 4.2 and Theorem 4.9 is an orientation result deriving from the general orientation theorem of Frank, Theorem 3.11.

Corollary 4.1. *Let $(G, \mathcal{M}, S, \pi)$ be a matroid-based rooted-graph. There exists an orientation D of G such that $(D, \mathcal{M}, S, \pi)$ is rooted-connected if and only if $(G, \mathcal{M}, S, \pi)$ is partition-connected.*

Proof. Define the integer valued set function p on 2^V by

$$p(X) = \begin{cases} 0 & \text{if } X \text{ is empty} \\ r_{\mathcal{M}}(S) - r_{\mathcal{M}}(S_X) & \text{otherwise.} \end{cases}$$

Since $X \mapsto r_{\mathcal{M}}(S_X)$ is submodular non-decreasing and upper bounded by $r_{\mathcal{M}}(S)$, p is crossing supermodular non-negative non-increasing and satisfies $p(\emptyset) = p(V) = 0$. The required orientation is exactly an orientation covering p. By Theorem 3.11, such an orientation exists if an only if (4.3) is satisfied, that is, $(G, \mathcal{M}, S, \pi)$ is partition-connected. □

Let us show that Corollary 4.1 and Theorem 4.9 imply Theorem 4.2.

Proof of Theorem 4.2. First suppose that there exists a matroid-based packing $\{(T_1, s_1), \dots, (T_t, s_t)\}$ of rooted-trees in $(G, \mathcal{M}, S, \pi)$. Let D be an orientation of G where each rooted-tree (T_i, s_i) becomes a rooted-arborescence (T_i', s_i). Then $\{(T_1', s_1), \dots, (T_t', s_t)\}$ is a matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, S, \pi)$. By Theorem 4.9, π is $\mathcal{M}$-independent and $(D, \mathcal{M}, S, \pi)$ is rooted-connected and hence, by Corollary 4.1, $(G, \mathcal{M}, S, \pi)$ is partition-connected.

Now suppose that π is $\mathcal{M}$-independent and $(G, \mathcal{M}, S, \pi)$ is partition-connected. By Corollary 4.1, there exists an orientation D of G such that $(D, \mathcal{M}, S, \pi)$ is rooted-connected. Then, by Theorem 4.9, there exists a matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, S, \pi)$ which provides, by forgetting the orientation, a matroid-based packing of rooted-trees in $(G, \mathcal{M}, S, \pi)$. □

Recently, Cs. Király gave a common generalization of Theorem 4.9 and of other extensions of Theorem 4.7 due to Kamiyama, Katoh and Takizawa [42] and Fujishige [31]. We recall that $\sigma_D(X) \supseteq X$ is the set of vertices from where X can be reached in D. Clearly in any matroid-independent packing of rooted-arborescences, for each $v \in V$, $r_{\mathcal{M}}(\{s_i \in S : v \in V(T_i)\}) \leqslant r_{\mathcal{M}}(S_{\sigma_D(v)})$. When equality holds for each $v \in V$, the matroid-independent packing of rooted-arborescences is called *maximal*.

Theorem 4.11 (Cs. Király [45])**.** *Let $(D, \mathcal{M}, S, \pi)$ be a matroid-based rooted-digraph. There exists a maximal matroid-independent packing of rooted-arborescences in $(D, \mathcal{M}, S, \pi)$ if and only if π is $\mathcal{M}$-independent and*

$$\rho_D(X) \geqslant r_{\mathcal{M}}(S_{\sigma_D(X)}) - r_{\mathcal{M}}(S_X) \tag{4.5}$$

holds for each $X \subseteq V$.

One may easily derive the sufficiency in Theorem 4.9 from Theorem 4.11. Indeed, if $(D, \mathcal{M}, S, \pi)$ is rooted-connected then, for $X \subseteq V$,

$$r_{\mathcal{M}}(S) \geqslant r_{\mathcal{M}}(S_{\sigma_D(X)}) \geqslant r_{\mathcal{M}}(S) - \rho_D(\sigma_D(X)) = r_{\mathcal{M}}(S).$$

Hence (4.4) implies (4.5) and a maximal matroid-independent packing of rooted-arborescences is a matroid-based packing of rooted-arborescences.

4.2.3 *Proof of the Main Theorem*

First we prove the necessity of the $\mathcal{M}$-independence of π and the rooted-connectivity of $(D, \mathcal{M}, S, \pi)$, that are quite straightforward.

Proof of necessity in Theorem 4.9. Suppose that there exists a matroid-based packing $\{(T_1, s_1), \dots, (T_t, s_t)\}$ of rooted-arborescences in $(D, \mathcal{M}, S, \pi)$. Let v be an arbitrary vertex of V and X a vertex set containing v. Then $B := \{s_i \in S :$

$v \in V(T_i)\}$ forms a base of $\mathcal{M}$. Let $B_1 = B \cap S_X$ and $B_2 = B \setminus S_X$. Then, since B_1 is independent in $\mathcal{M}$ and $S_v \subseteq B_1$, π is $\mathcal{M}$-independent. Moreover, since $r_\mathcal{M}$ is monotone, $|B_1| = r_\mathcal{M}(B_1) \leqslant r_\mathcal{M}(S_X)$. For each root $s_i \in B_2$, there exists an arc of T_i that enters X. Since the rooted-arborescences are arc-disjoint, we have $\rho_D(X) \geqslant |B_2| = |B| - |B_1| \geqslant r_\mathcal{M}(S) - r_\mathcal{M}(S_X)$ that is $(D, \mathcal{M}, S, \pi)$ is rooted-connected. □

Before proving the sufficiency of the conditions we establish a technical claim. A vertex set X is called $\mathcal{M}$-*tight* if $\rho_D(X) = r_\mathcal{M}(S) - r_\mathcal{M}(S_X)$. For vertex sets X and Y, we say that Y *dominates* X if $S_X \subseteq \text{Span}_\mathcal{M}(S_Y)$. Note that since, for $Q \subseteq S$, $\text{Span}_\mathcal{M}(\text{Span}_\mathcal{M}(Q)) = \text{Span}_\mathcal{M}(Q)$, domination is a transitive relation. We say that an arc uv is *bad* if v dominates u, otherwise it is *good*. Note that in a matroid-based packing of rooted-arborescences only good arcs uv can be used in a rooted-arborescence whose root is placed at u, since there must exist $s \in S_u$ such that $S_v \cup s$ is independent in $\mathcal{M}$.

Claim 4.2. *Suppose that* $(D, \mathcal{M}, S, \pi)$ *is rooted-connected. Let* X *be a* $\mathcal{M}$*-tight set and* v *a vertex of* X.

(a) If Y *is a* $\mathcal{M}$*-tight set that contains* v*, then* $X \cap Y$ *and* $X \cup Y$ *are* $\mathcal{M}$*-tight. Moreover, if* $s \in \text{Span}_\mathcal{M}(S_X) \cap \text{Span}_\mathcal{M}(S_Y)$*, then* $s \in \text{Span}_\mathcal{M}(S_{X\cap Y})$.

(b) If no good arc exists in $D[X]$*, then* v *dominates* X.

Proof. (a) If we have s, then let $\alpha = s$, otherwise let $\alpha = \emptyset$. By the monotonicity and the submodularity of $r_\mathcal{M}$, $s \in \text{Span}_\mathcal{M}(S_X) \cap \text{Span}_\mathcal{M}(S_Y)$, the $\mathcal{M}$-tightness of X and Y, the submodularity of ρ_D, $X \cap Y \neq \emptyset$ and (4.4), we have

$$\begin{aligned}
r_\mathcal{M}(S_{X\cap Y}) + r_\mathcal{M}(S_{X\cup Y}) &= r_\mathcal{M}(S_X \cap S_Y) + r_\mathcal{M}(S_X \cup S_Y) \\
&\leqslant r_\mathcal{M}((S_X \cap S_Y) \cup \alpha) + r_\mathcal{M}((S_X \cup S_Y) \cup \alpha) \\
&\leqslant r_\mathcal{M}(S_X \cup \alpha) + r_\mathcal{M}(S_Y \cup \alpha) \\
&= r_\mathcal{M}(S_X) + r_\mathcal{M}(S_Y) \\
&= r_\mathcal{M}(S) - \rho_D(X) + r_\mathcal{M}(S) - \rho_D(Y) \\
&\leqslant r_\mathcal{M}(S) - \rho_D(X \cap Y) + r_\mathcal{M}(S) - \rho_D(X \cup Y) \\
&\leqslant r_\mathcal{M}(S_{X\cap Y}) + r_\mathcal{M}(S_{X\cup Y}).
\end{aligned}$$

Hence equality holds everywhere and (a) follows.

(b) Let us denote by Y the set $\sigma_{D[X]}(v)$ of vertices from which v is reachable in $D[X]$. We show that v dominates Y and Y dominates X and then, since domination is transitive, (b) follows.

For all $y \in Y$, there exists a directed path from y to v in $D[X]$. By assumption there is no good arc in this path so each vertex of the path is dominated by the next one and, by transitivity of domination, v dominates y. Hence $S_Y = \bigcup_{y\in Y} S_y \subseteq \text{Span}_\mathcal{M}(S_v)$ that is v dominates Y.

By the definition of Y, every arc of D that enters Y enters X as well. Then, by (4.4), the $\mathcal{M}$-tightness of X and the monotonicity of $r_\mathcal{M}$, we have $r_\mathcal{M}(S) - r_\mathcal{M}(S_Y) \leqslant \rho_D(Y) \leqslant \rho_D(X) = r_\mathcal{M}(S) - r_\mathcal{M}(S_X) \leqslant r_\mathcal{M}(S) - r_\mathcal{M}(S_Y)$. Thus equality holds everywhere and Y dominates X. □

Now we can prove the main result.

Proof of sufficiency in Theorem 4.9. We prove it by induction on the number of good arcs.

Base Case: No good arc exists.

Then $\{(v,s) : v \in V, s \in S_v\}$ forms a matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, S, \pi)$. Indeed, since V is $\mathcal{M}$-tight, (b) in Claim 4.2 implies that S_v is a spanning set of $\mathcal{M}$ and hence, since π is $\mathcal{M}$-independent, S_v is a base of $\mathcal{M}$ for all $v \in V$.

Induction Step: At least one good arc exists.

For a good arc $uv \in A$ and $s \in S_u \setminus \mathrm{Span}(S_v)$, let $D' = D - uv$, S' the set obtained by adding a new element s' to S, $\mathcal{M}'$ the matroid on S' obtained from $\mathcal{M}$ by considering s' as an element parallel to s and π' the placement of S' in V obtained from π by placing the new element s' at v (see Figure 13).

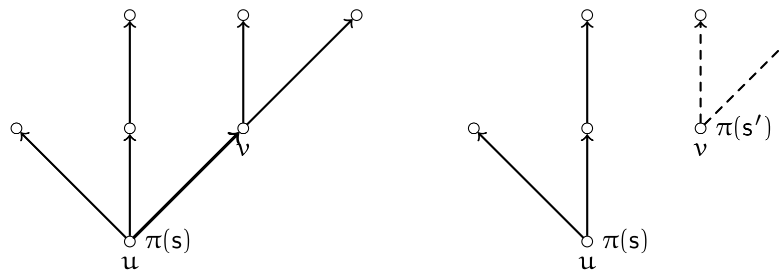


Figure 13: The arc uv is removed and a copy s' of s is placed at v.

By the choice of s and since π is $\mathcal{M}$-independent, it follows that π' is $\mathcal{M}'$-independent. If the matroid-based rooted-digraph $(D', \mathcal{M}', S', \pi')$ is rooted-connected, then, by induction, there exists a matroid-based packing $\mathcal{P}'$ of rooted-arborescences in $(D', \mathcal{M}', S', \pi')$. Since s and s' are parallel in $\mathcal{M}'$, the rooted-arborescences (T, s) and (T', s') of $\mathcal{P}'$ are vertex disjoint, so $(T'', s) = (T \cup T' \cup uv, s)$ is a rooted-arborescence (see Figure 13). Then the collection of rooted arborescences obtained from $\mathcal{P}'$ by substituting (T'', s) for (T, s) and (T', s') is a matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, S, \pi)$. Hence the proof of the theorem is reduced to the proof of the following claim.

Claim 4.3. *There exists a good arc uv and $s \in S_u \setminus \mathrm{Span}(S_v)$ such that $(D', \mathcal{M}', S', \pi')$ is rooted-connected.*

Proof. Assume that the claim is false. Let $uv \in A$ be a good arc and $s \in S_u \setminus \mathrm{Span}(S_v)$. By assumption, there exists $\emptyset \neq X_s \subset V$ such that $\rho_{D'}(X_s) < r_{\mathcal{M}}(S) - r_{\mathcal{M}'}(S'_{X_s})$. Hence, by (4.4) and the monotonicity of $r_{\mathcal{M}'}$, we have

$$\begin{aligned}
\rho_{D'}(X_s) + 1 &\geqslant \rho_{D'}(X_s) + \rho_{uv}(X_s) \\
&= \rho_D(X_s) \\
&\geqslant r_{\mathcal{M}}(S) - r_{\mathcal{M}}(S_{X_s}) \\
&\geqslant r_{\mathcal{M}}(S) - r_{\mathcal{M}'}(S'_{X_s}) \\
&\geqslant \rho_{D'}(X_s) + 1,
\end{aligned}$$

so equality holds everywhere and thus uv enters X_s, X_s is $\mathcal{M}$-tight in $(D, \mathcal{M}, S, \pi)$ and $s \in \mathrm{Span}_{\mathcal{M}}(S_{X_s})$. Thus, defining $X = \cup_{s \in S_u \setminus \mathrm{Span}(S_v)} X_s$, we have $S_u \setminus \mathrm{Span}(S_v) \subseteq \mathrm{Span}(S_X)$ and X dominates u since $v \in X$. By (a) in Claim 4.2, X is also $\mathcal{M}$-tight. So we proved that

$$\text{every good arc } uv \text{ enters a } \mathcal{M}\text{-tight set } X \text{ that dominates } u. \tag{4.6}$$

Among all pairs (uv, X) satisfying (4.6) choose one with X minimal. Since X dominates u but v does not dominate u, v does not dominate X. Then, by (b) in Claim 4.2, there exists a good arc $u'v'$ in $D[X]$. By (4.6), $u'v'$ enters a

$\mathcal{M}$-tight set Y that dominates u'. By $v' \in X \cap Y$, the $\mathcal{M}$-tightness of X and Y, $u' \in X$, $S_{u'} \subseteq \mathrm{Span}_{\mathcal{M}}(S_Y)$ and (a) in Claim 4.2, we have that $X \cap Y$ is $\mathcal{M}$-tight and $S_{u'} \subseteq \mathrm{Span}_{\mathcal{M}}(S_{X\cap Y})$. Since the good arc $u'v'$ enters the $\mathcal{M}$-tight set $X \cap Y$ that dominates u' and $X \cap Y$ is a proper subset of X (since $u' \in X \setminus Y$), this contradicts the minimality of X. □□

4.2.4 *Polyhedral and Algorithmic Aspects*

In this subsection we first study a polyhedron describing the matroid-based packings of rooted-arborescences. Then we prove that finding a matroid-based packing of rooted-arborescences can be done in polynomial time.

We need the following general result of Frank.

Theorem 4.12 (Frank [20]). *Let $D = (V, A)$ be a digraph, $p : 2^V \to \mathbb{Z}_+$ a non-negative intersecting supermodular set-function such that $\rho_D(Z) \geqslant p(Z)$ for every $Z \subseteq V$. Then the polyhedron defined by the following linear system is integer:*

$$\begin{aligned} 1 \geqslant x(a) &\geqslant 0 && \textit{for all } a \in A, \\ \rho_D^x(X) &\geqslant p(X) && \textit{for all non-empty } X \subseteq V. \end{aligned}$$

The following theorem is a corollary of Theorems 4.9 and 4.12.

Theorem 4.13. *Let $(D = (V, A), \mathcal{M}, S, \pi)$ be a matroid-based rooted-digraph where $\mathcal{M}$ is of rank k with rank function $r_{\mathcal{M}}$. There exists a matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, S, \pi)$ if and only if the polyhedron $P_{\mathcal{M},D}$ defined by the linear system*

$$1 \geqslant x(a) \geqslant 0 \qquad \textit{for all } a \in A, \tag{4.7}$$
$$\rho_D^x(X) \geqslant k - r_{\mathcal{M}}(S_X) \qquad \textit{for all non-empty } X \subseteq V, \tag{4.8}$$
$$x(A) = k|V| - |S| \tag{4.9}$$

is not empty. In this case, $P_{\mathcal{M},D}$ is integer and its vertices are the characteristic vectors of the arc sets of the matroid-based packings of rooted-arborescences in $(D, \mathcal{M}, S, \pi)$.

Proof. Suppose there exists a matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, S, \pi)$ and call $A' \subseteq A$ its arc set. Let x be the characteristic vector of A'. We have $x(A) = |A'| = \sum_{v\in V} \rho_{A'}(v) = \sum_{v\in V}(k - |S_v|) = k|V| - |S|$ and $\rho_D^x(X) = \rho_{A'}(X) \geqslant k - r_{\mathcal{M}}(S_X)$ for all non-empty $X \subseteq V$ by (4.4). So $x \in P_{\mathcal{M},D}$.

Now suppose that $P_{\mathcal{M},D}$ is not empty and pick arbitrarily $x \in P_{\mathcal{M},D}$. As we already pointed out in the proof of 4.1, $p : X \subseteq V \mapsto k - r_{\mathcal{M}}(S_X)$ is non-negative intersecting supermodular and, by (4.7) and (4.8), we have $\rho_D(X) \geqslant \rho_D^x(X) \geqslant p(X)$ for all non-empty $X \subseteq V$. Hence Theorem 4.12 applies, that is the polyhedron P described by (4.7) and (4.8) is integer.

By (4.8), for all $x \in P$,

$$x(A) = \sum_{v\in V} \rho_D^x(v) \geqslant \sum_{v\in V}(k - r_{\mathcal{M}}(S_v)) \geqslant \sum_{v\in V}(k - |S_v|) = k|V| - |S|, \tag{4.10}$$

that is, $x(A) \geqslant k|V| - |S|$ is a valid inequality for P. Then, by (4.9), $P_{\mathcal{M},D}$ is a face of the integer polyhedron P and hence $P_{\mathcal{M},D}$ is also integer.

Furthermore, for $x \in P_{\mathcal{M},D}$, equality holds everywhere in (4.10), thus, $|S_v| = r_{\mathcal{M}}(S_v)$ for all $v \in V$ and hence π is $\mathcal{M}$-independent. A vertex x

of $P_{\mathcal{M},D}$ defines an arc set $A' = \{a \in A, x(a) = 1\}$. By (4.8), the matroid-based rooted-digraph $((V, A'), \mathcal{M}, S, \pi)$ is rooted-connected. Therefore, by Theorem 4.9, there exists a matroid-based packing of rooted-arborescences in $((V, A'), \mathcal{M}, S, \pi)$ whose arc set is, by (4.9), equal to A', and the theorem follows. $\square$

This polyhedral description of the problem yield a polynomial time algorithm to solve it provided that the matroid is given by an oracle for the rank function.

Theorem 4.14. *Let $(D, \mathcal{M}, S, \pi)$ be a matroid-based rooted-digraph. A matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, S, \pi)$ or a vertex v certifying that π is not $\mathcal{M}$-independent or a vertex set X certifying that $(D, \mathcal{M}, S, \pi)$ is not rooted-connected can be found in polynomial time.*

Proof. By the submodularity of $\rho_D(X) + r_{\mathcal{M}}(S_X)$, Theorem 2.1, using the oracle on $\mathcal{M}$ and Theorem 4.9, we can either find a set violating (4.4) or a vertex certifying that π is not $\mathcal{M}$-independent or certify that there exists a matroid-based packing of rooted-arborescences in polynomial time.

In the latter case, a matroid-based packing of rooted-arborescences can be found in polynomial time following the proof of Theorem 4.9. Using the oracle, test whether each arc is bad or good. When an arc uv is good, for each $s \in S_u \setminus \mathrm{Span}(S_v)$, determine in polynomial time whether $(D', \mathcal{M}', S', \pi')$ is rooted-connected using the submodularity of $\rho_{D'}(X) + r_{\mathcal{M}'}(S'_X)$, the oracle for the rank function $r_{\mathcal{M}'}$ (that is easily computed from $r_{\mathcal{M}}$) and Theorem 2.1. Either all arcs are bad or we find a good arc uv and $s \in S_u \setminus \mathrm{Span}(S_v)$ satisfying Claim 4.3. In the first case, $\{(v, s) : v \in V, s \in S_v\}$ is the required packing. In the second case, it leads to the computation of a matroid-based packing of rooted-arborescences in $(D', \mathcal{M}', S', \pi')$ where D' contains less arcs than D. $\square$

By the submodularity of $\rho^x_D(X) + r_{\mathcal{M}}(S_X)$ and by Theorem 2.1, $P_{\mathcal{M},D}$ can be separated in polynomial time. Thus, using the ellipsoid method, by Grötschel, Lovász and Schrijver [35], and by Theorem 4.14, we have the following result.

Theorem 4.15. *Let $(D, \mathcal{M}, S, \pi)$ be a matroid-based rooted-digraph and c a cost function on the set of arcs of D. If there exists a matroid-based packing of rooted-arborescences in $(D, \mathcal{M}, S, \pi)$ then one of minimum cost can be found in polynomial time.*

The ellipsoid method does not provide a very practical algorithm hence the above result raises the following question.

Question 4.1. *Provide a combinatorial algorithm that finds in polynomial time a minimum cost matroid-based packing of arborescences in a matroid-based rooted-digraph when it exists.*

Now we point out another variation of the initial problem. Let $(D, \mathcal{M}, S, \pi)$ be a matroid-based rooted-digraph and $b : V \to \mathbb{Z}$ a lower bound. A *b-matroid-independent packing of rooted-arborescences* is a matroid-independent packing of rooted-arborescences such that $r_{\mathcal{M}}(\{s_i \in S : v \in V(T_i)\}) \geqslant b(v)$ for all $v \in V$. When the function b is constant, using Theorem 4.9 and matroid truncation, one can derive a characterization of matroid-based rooted-digraphs admitting a b-matroid-independent packing of rooted-arborescences. On the other hand, for general b, the problem turns out to be NP-complete since it contains the disjoint Steiner arborescences problem that is to find 2 arc-disjoint r-arborescences both covering a specified subset of vertices ([28] page 342).

4.3 PACKING OF SPANNING COUNT MATROID BASES

As a direct corollary of the Theorem 4.1 every $2k$-edge-connected graph G contains k edge-disjoint bases of the circuit matroid. Similar results [51, 39, 40] state the existence of a packing of another count matroid, namely the rigidity matroid, in highly connected graphs. In this section we give a common generalization of these three results that comes from a joint work with Cheriyan, Szigeti [5]. As I was writing the present document I reformulated our proof as a discharging method. Using this approach Nguyen and I could prove a packing result for a larger class of count matroids.

4.3.1 *Rigidity of Graphs*

In this subsection we are interesting in a particular count matroid called the *rigidity matroid*. Let $G = (V, E)$ be a graph. The rigidity matroid of G, denoted $\mathcal{R}(G)$, is the $(2_V, 3)$-count matroid, i.e., a set of edges $F \subseteq E$ is independent in $\mathcal{R}(G)$ if, for all $X \subseteq V$ of size at least 2,

$$i_F(X) \leqslant 2|X| - 3. \tag{4.11}$$

A graph is called *rigid* if it contains an independent set of edges of maximal theoretical size, i.e. $r_{\mathcal{R}(G)}(E) = 2|V| - 3$.

This definition of rigidity may seem very abstract but Lovász and Yemini [51] proved that it transposes the natural meaning of rigidity. When speaking about rigidity, the idea that should pop into the reader's mind is the following. Draw a graph G on the plane and imagine that the edges are pieces of metal connected to each others by vertices made of magnets. If one cannot deform the construction without breaking any piece of metal or disconnecting an edge from a vertex then the drawing of the graph is called rigid. Roughly speaking a graph G is rigid if G has a rigid drawing in the plane. As an example a square is not rigid since it can be deformed into a diamond but adding a diagonal to the square makes it rigid (in dimension 2).

Rigid graphs draw our attention because they have an interesting connectivity property.

Proposition 4.3. *A rigid graph with at least* 3 *vertices is 2-vertex-connected.*

This result is not surprising if we consider the intuition of rigidity given above. If a graph $G = (V, E)$ contains a vertex cut, say $\{v\}$, then every drawing of G can be deformed rotating a connected component U of $G - v$ while keeping $V \setminus U$ fixed.

Lovász and Yemini proved that sufficiently highly connected graphs are rigid.

Theorem 4.16 (Lovász, Yemini [51])**.** *Every 6-vertex-connected graph is rigid.*

They also showed the existence of a non-rigid 5-vertex-connected graph. Hence the number 6 is the best possible in terms of vertex-connectivity. However using the mixed connectivity Jackson and Jordán weakened the condition.

Theorem 4.17 (Jackson, Jordán [39])**.** *Every* $(6, 2)$*-connected simple graph is rigid.*

To see that Theorem 4.17 proves Theorem 4.16, one may observe that 6-vertex-connectivity implies $(6, 2)$-connectivity and removing all the edges but one in every set of parallel edges preserves 6-vertex-connectivity (Fact 2.5).

Again this mixed connectivity is the best possible. Indeed, the graph given in [51] is $(5, 2)$-connected and the graph given in Figure 14 is non-rigid and $(6, 3)$-connected.

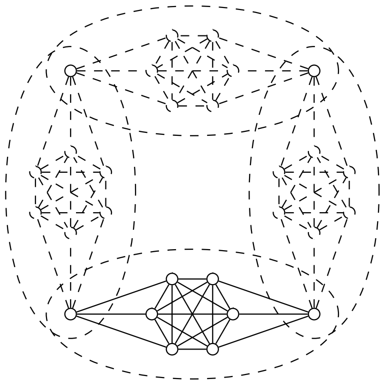


Figure 14: A $(6, 3)$-connected non-rigid graph $G = (V, E)$. To prove it, consider the collection $\mathcal{H}$ of the four dashed vertex sets and observe that, by Theorem 4.21, $r_{\mathcal{R}(G)}(E) \leqslant \sum_{X \in \mathcal{H}}(2|X| - 3) = 4 * (2 * 8 - 3) = 52 < 53 = 2 * 28 - 3 = 2|V| - 3$.

4.3.2 *Packing of Count Matroids Bases*

The following result is an easy consequence of Theorem 4.1.

Corollary 4.2. *Every* $2k$*-edge-connected graph* G *contains* k *edge-disjoint spanning trees.*

Proof. Let G be a $2k$-edge-connected graph and let $\mathcal{P}$ be a partition of V. We have $e_G(\mathcal{P}) = \frac{1}{2}\sum_{X \in \mathcal{P}} d_G(X) \geqslant \frac{1}{2}\sum_{X \in \mathcal{P}} 2k = k|\mathcal{P}|$. Hence G is k-partition-connected and, by Theorem 4.1, the result follows. □

Jordán proved a generalization of Theorem 4.16 that has a similar statement.

Theorem 4.18 (Jordán [40])**.** *Let* k *be an integer. Every* $6k$*-vertex-connected graph contains* k *edge-disjoint spanning rigid subgraphs.*

Jordán was motivated to settle the base cases of two conjectures. The statement of one of these conjectures is postponed to Chapter 5. The other one is the following statement of Kriesell.

Conjecture 4.1 (Kriesell in [40])**.** *For every integer* k*, there exists a least integer* $f(k)$ *such that in every* $f(k)$*-vertex-connected graph* G *there exists a spanning tree* T *such that* $G - T$ *is* k*-vertex-connected.*

Theorem 4.18 yields the existence of $f(2)$ and the upper bound $f(2) \leqslant 12$. Indeed, in every 12-vertex-connected graph there exist 2 edge-disjoint spanning rigid subgraphs R_1 and R_2. By Proposition 4.3, R_2 contains a spanning tree T and $G - T \supseteq R_1$ is 2-connected.

Clearly is this argument the 2-vertex-connectivity of R_2 is redundant since only the simple connectivity is useful. So in his paper Jordán mentioned that proving a packing theorem for spanning rigid subgraphs and spanning trees may improve the upper bound. Cheriyan, Szigeti, and I successfully followed this track and proved the following.

Theorem 4.19 (Cheriyan, Durand de Gevigney, Szigeti [5])**.** *Let $k \geqslant 1$ and $j \geqslant 0$ be two integers. Every $(6k+2j, 2k)$-connected simple graph contains k rigid and j connected edge-disjoint spanning subgraphs.*

According to plan, Theorem 4.19 proves $f(2) \leqslant 8$. Indeed, every 8-vertex-connected graph G is $(8,2)$-connected thus contains a spanning tree T and a rigid spanning rigid subgraph R that are edge-disjoint; hence $G - T \supseteq R$ is 2-vertex-connected by Proposition 4.3.

Note also that Theorem 4.19 is a common generalization of Theorem 4.17 ($k = 1$ and $j = 0$) and Theorem 4.18 ($j = 0$) since $6k$-vertex-connectivity implies $(6k, 2k)$-connectivity and removing all the edges but one in every set of parallel edges preserves 6-vertex-connectivity (Fact 2.5).

Very recently Nguyen and I generalized Theorem 4.19 to a larger class of count matroids.

Theorem 4.20 (Durand de Gevigney, Nguyen)**.** *Let m, ℓ be integers such that $2 \leqslant m \leqslant \ell \leqslant 2m-1$ and let $k \geqslant 1$ and $j \geqslant 0$ be two integers. In every $(2k\ell+2j, km)$-connected simple graph the rank of the (m_V, ℓ)-count-matroid is $m|V| - \ell$ and there exist k bases of the (m_V, ℓ)-count-matroid and j spanning trees pairwise edge-disjoint.*

4.3.3 *Proof by Discharging*

Before we start the proof Theorem 4.20 we derive from Theorem 2.4 the following expression of the rank function of count matroids.

Theorem 4.21. *Let m, ℓ be integers such that $2 \leqslant m \leqslant \ell \leqslant 2m-1$ and let $G = (V, E)$ be a simple graph. The rank of the (m_V, ℓ)-count matroid $\mathcal{M}$ is given, for $F \subseteq E$, by*

$$r_{\mathcal{M}}(F) = \min_{T \subseteq F} \sum_{X \in \mathcal{H}} (m|X| - \ell) + |F \setminus T| \tag{4.12}$$

where $\mathcal{H}$ is a collection of subsets of V such that $\{F(X); X \in \mathcal{H}\}$ partitions T and $|X| \geqslant 2m$ for all $X \in \mathcal{H}$.

Proof. We recall from the proof of Theorem 2.5 that the (m_V, ℓ)-matroid is defined by the non-decreasing integer-valued intersecting-submodular function $F \subseteq E \mapsto m(V(F)) - \ell$. Hence, by Theorem 2.4, the rank is given, for $F \subseteq E$, by,

$$r_{\mathcal{M}}(F) = \min \sum_{i=1}^{t} (m|V(T_i)| - \ell) + |F \setminus T|, \tag{4.13}$$

where the minimum is taken over all subsets T of F and all partitions $\{T_1, \cdots, T_t\}$ of T. Let $T \subseteq F$ and $\{T_1, \ldots, T_t\}$ be a partition of T that minimize the right hand side of (4.13) such that t is minimal and denote $X_i = V(T_i)$ for $i \in \{1, \cdots, t\}$. The proof is concluded by the following two propositions.

Proposition 4.4. *For each $i \in \{1, \cdots, t\}$, $|X_i| \geqslant 2m$.*

Proof. Consider the quadratic function $f : k \in \mathbb{Z}_+ \mapsto mk - \ell - \frac{k(k-1)}{2}$. Since f is concave, for every $2 \leqslant k \leqslant 2m-1$,

$$f(k) \geqslant \min\{f(2), f(2m-1)\} = 2m - \ell - 1 \geqslant 0. \tag{4.14}$$

Now suppose that there exists i such that $|X_i| < 2m$ and consider $T' = T \setminus T_i$. Then, since G is simple and by (4.14) for $|X_i|$, $|T_i| \leqslant \frac{|X_i|(|X_i|-1)}{2} \leqslant m|X_i| - \ell$. Hence, by (4.13) and choice of T,

$$\begin{aligned} r_{\mathcal{M}}(F) &\leqslant \sum_{j=1, j\neq i}^{t} (m|V(T_j)| - \ell) + |F \setminus T'| \\ &\leqslant \sum_{j=1}^{t} (m|V(T_j)| - \ell) + |F \setminus T| \\ &= r_{\mathcal{M}}(F). \end{aligned}$$

It means that T' and the partition $\{T_1, \cdots, T_{i-1}, T_{i+1}, \cdots, T_t\}$ minimizes the right hand side of (4.13). This contradicts the minimality of t. $\square$

Proposition 4.5. *For each* $i \in \{1, \cdots, t\}$, $T_i = F(X_i)$.

Proof. No edge of $F \setminus T$ is induced by an X_i otherwise adding this edge into T would decrease the right hand side of (4.13). Hence, for each i, $F(X_i) \subseteq T$. Now we prove that $|X_i \cap X_j| \leqslant 1$ for $i \neq j$. It follows that $F(X_i) \cap T_j = \emptyset$, thus $F(X_i) \subseteq T_i$ and the proposition follows.

Suppose there exist $i \neq j$ such that $|X_i \cap X_j| > 1$. By $\ell \leqslant 2m - 1$ we have

$$\begin{aligned} m|X_i \cup X_j| - \ell &= m|X_i| + m|X_j| - m|X_i \cap X_j| - \ell \\ &\leqslant m|X_i| + m|X_j| - 2m - \ell \\ &< m|X_i| - \ell + m|X_j| - \ell. \end{aligned}$$

Hence substituting $T_i \cup T_j$ for T_i and T_j in the partition of T would decrease the right hand side of (4.13), a contradiction. $\square\square$

Let $G = (V, E)$ be a graph and m, ℓ be integers such that $2 \leqslant m \leqslant \ell \leqslant 2m - 1$. For two integers $k \geqslant 1$ and $j \geqslant 0$, we consider the matroid $\mathcal{M}_{k,j}$ which is the union of k copies of the (m_V, ℓ)-matroid $\mathcal{M}$ and j copies of the graphic matroid $\mathcal{C}(G)$. There exist k edge-disjoint bases of the (m_V, ℓ)-matroid of sizes $m|V| - \ell$ and j spanning trees if and only if

$$r_{\mathcal{M}_{k,j}}(E) = k(m|V| - \ell) + j(|V| - 1). \tag{4.15}$$

Theorem 2.6 gives an expression for the left hand side of (4.15) and the proof of Theorem 4.20 consists of finding a lower bound for this expression.

Proof of Theorem 4.20. Let m, ℓ, k, j be integers as in the statement and suppose by contradiction that there exists a simple $(2k\ell + 2j, km)$-connected graph G that contradicts the theorem. By the above discussion and Theorem 2.6, we have

$$\begin{aligned} k(m|V| - \ell) + j(|V| - 1) &> r_{\mathcal{M}_{k,j}}(E) \\ &= \min_{F \subseteq E} k r_{\mathcal{M}}(F) + j r_{\mathcal{C}(G)}(F) + |E \setminus F|. \end{aligned} \tag{4.16}$$

Let F be a minimal set of edges minimizing the right hand side of (4.16). By Theorem 4.21, $r_{\mathcal{M}}(F) = \sum_{X \in \mathcal{H}} (m|X| - \ell) + |F \setminus T|$ where $T \subseteq F$ and $\mathcal{H}$ is a collection of subsets of V such that $\{F(X), X \in \mathcal{H}\}$ partitions T and $|X| \geqslant 2m$ for all $X \in \mathcal{H}$.

Fact 4.1. *We may assume that* V *does not belong to* $\mathcal{H}$.

Proof. Since mixed $(2k\ell+2j, km)$-connectivity implies 2j-edge-connectivity and consequently j-partition-connectivity, we have, $jr_{\mathcal{C}(G)}(F)+|E\setminus F| \geqslant j(|V|-1)$ (see proof of Theorem 4.1). Hence, if $V \in \mathcal{H}$ then $r_{\mathcal{M}}(F) = m|V|-\ell$ and (4.16) is violated and the theorem follows. □

Proposition 4.6. $T = F$ *that is* $\{F(X), X \in \mathcal{H}\}$ *partitions* F.

Proof. Suppose by contradiction that there exists $e \in F\setminus T$ and denote $F' = F\setminus e$. By Theorem 4.21, $r_{\mathcal{M}}(F') \leqslant \sum_{X\in\mathcal{H}}(m|X|-\ell)+|F'\setminus T| = r_{\mathcal{M}}(F)-1$ and, since the rank function of a matroid is non-decreasing, $r_{\mathcal{C}(G)}(F') \leqslant r_{\mathcal{C}(G)}(F)$. Hence we have

$$\begin{aligned} r_{\mathcal{M}_{k,j}}(E) &\leqslant kr_{\mathcal{M}}(F')+jr_{\mathcal{C}(G)}(F')+|E\setminus F'| \\ &\leqslant k(r_{\mathcal{M}}(F)-1)+jr_{\mathcal{C}(G)}(F)+|E\setminus F|+1 \\ &\leqslant r_{\mathcal{M}_{k,j}}(E), \end{aligned}$$

where the last inequality holds by $k \geqslant 1$ and choice of F. Hence F' also minimizes the right hand side of (4.16) and contradicts the minimality of F. □

It follows from (4.16) and the above proposition that

$$k(m|V|-\ell)+j(|V|-1) > k\sum_{X\in\mathcal{H}}(m|X|-\ell)+j(|V(F)|-c(G[F]))+|E\setminus F|. \tag{4.17}$$

Fact 4.2. *For each* $X \in \mathcal{H}$, $m|X| \geqslant 2\ell$.

Proof. By $|X| \geqslant 2m$, $2 \leqslant m$ and $\ell \leqslant 2m-1$, we have $m|X| \geqslant 2m^2 \geqslant 4m \geqslant 2\ell$. □

For each $X \in \mathcal{H}$, we define the border X_B of X as the vertices of X that belong to at least two elements of $\mathcal{H}$ and the proper part of X as $X_P = X\setminus X_B$. Denote by $\mathcal{H}'$ the set of elements of $\mathcal{H}$ that have a non-empty proper part. We denote by $\mathcal{K}$ the set of connected components of F that intersect no element of $\mathcal{H}'$. Note that

$$c(G[F]) \leqslant |\mathcal{K}|+|\mathcal{H}'|. \tag{4.18}$$

Now comes the discharging method: we define a weight for each element of V and $E\setminus F$ that is initialized by 0 and we do the following operations.

(I) For each $X \in \mathcal{H}$. For each $v \in X_P$ we increase $w(v)$ by km and for each $v \in X_B$ we increase $w(v)$ by $\frac{1}{2}km$. For each edge e entering X_P in $G-X_B$ we decrease e by $\frac{1}{2}$. Note that such an edge does not belong to F.

Denote $\Delta^X w$ the total weight added for X. If $X \notin \mathcal{H}'$ then $\Delta^X w = \frac{1}{2}km|X_B| = \frac{1}{2}km|X| \leqslant k(m|X|-\ell)$ by Fact 4.2. If $X \in \mathcal{H}'$ then since $X_P \neq \emptyset$ and $X_P\cup X_B = X \neq V$ by Proposition 4.1, by $(2k\ell+2j, km)$-connectivity of G, the total weight added during this step is

$$\begin{aligned} \Delta^X w &= km|X_P|+\frac{1}{2}km|X_B|-\frac{1}{2}d_{G-X_B}(X_P) \\ &\leqslant km|X_P|+\frac{1}{2}km|X_B|-\frac{1}{2}(2k\ell+2j-km|X_B|) \\ &= k(m|X|-\ell)-j. \end{aligned}$$

(II) For each $K \in \mathcal{K}$, we decrease by $\frac{1}{2}$ the weight of each edge entering K. By $(2k\ell+2j)$-edge-connectivity of G, the total weight added for K satisfies $\Delta^K w = -\frac{1}{2}d_G(K) \leqslant -\frac{1}{2}(2k\ell+2j) \leqslant -j$.

(III) For each edge $e = uv$ in $E \setminus F$.

(i) If none of u or v belongs to an element of $\mathcal{H}$ then we increase $w(u)$ and $w(v)$ by $\frac{1}{2}$.

(ii) If exactly one of u or v, say u, belongs to an element of $\mathcal{H}$ then we increase $w(v)$ and $w(e)$ by $\frac{1}{2}$.

(iii) If both of u and v belongs to an element of $\mathcal{H}$ then we increase $w(e)$ by one.

For each $e \in E \setminus F$, the total weight added is $\Delta^e w = 1$.

Hence, by (4.18), the total weight Δw added during all the steps satisfies

$$\begin{aligned}\Delta w &= \sum_{X \in \mathcal{H}} \Delta^X w + \sum_{K \in \mathcal{K}} \Delta^K w + \sum_{e \in E \setminus F} \Delta^e w \\ &\leqslant k \sum_{X \in \mathcal{H}} (m|X| - \ell) - j|\mathcal{H}'| - j|\mathcal{K}| + |E \setminus F| \\ &\leqslant k \sum_{X \in \mathcal{H}} (m|X| - \ell) - jc(G[F]) + |E \setminus F| \end{aligned} \tag{4.19}$$

We now prove a lower bound on the total weight W. Let $v \in V$. If v belongs to no element of $\mathcal{H}$ then each edge incident to v is in $E \setminus F$ and increases $w(v)$ by $\frac{1}{2}$ during steps (i) and (ii). Hence, by $(2k\ell + 2j)$-edge-connectivity of G, $w(v) = \frac{1}{2} d_G(v) \geqslant \frac{1}{2}(2k\ell + 2j) \geqslant km + j$. If v belongs to the proper part of an element of $\mathcal{H}$ then $w(v) = km$ by step (I). In the last case v belongs to the border of at least two elements of $\mathcal{H}$ thus $w(v)$ is increased by $\frac{1}{2}km$ at least twice during step (I) and $w(v) \geqslant km$.

Let $e \in E \setminus F$. Note that if $w(e)$ is decreased once by steps (I) and (II) then (ii) occurs and if $w(e)$ is decreased twice by steps (I) and (II) then case (iii) occurs. So $w(e) \geqslant 0$ for each $e \in E \setminus F$. Hence the total weight W satisfies

$$\begin{aligned} W &= \sum_{v \in V} w(v) + \sum_{e \in E \setminus F} w(e) \\ &\geqslant |V \setminus V(\mathcal{H})|(km + j) + |V(\mathcal{H})|km + 0 \\ &= km|V| + j|V \setminus V(F)|. \end{aligned} \tag{4.20}$$

Since the initial weight is null we have $W = \Delta w$, thus, by (4.19) and (4.20),

$$k \sum_{X \in \mathcal{H}} (m|X| - \ell) + j(|V(F)| - c(G[F])) + |E \setminus F| \geqslant km|V| + j|V|,$$

a contradiction to (4.17). □

Note that the arguments actually show that Theorem 4.20 still hold even if at most $k\ell + j$ edges are removed from E. But this is not surprising since Lovász and Yemini [51], Jackson and Jordán [39] and Jordán [40] gave a similar redundancy property.

5 STRUCTURE OF HIGHLY CONNECTED GRAPHS

Most of the orientation problems addressed so far in this document are solved by Frank's theorem on covering supermodular set-function[1]. Orientation problems that require vertex-connectivity in the resulting digraph fall out of the scope of this deep orientation theorem.

This chapter is motivated by three conjectures on vertex-connected orientations. One is due to Thomassen [68] and the two others are due to Frank [26, 29]. We state and discuss in detail these conjectures in the first subsection.

The approach of Robbins for the strongly-connected orientation problem and the approach of Lovász for the k-arc-connected problem suggest that a deep insight into the structure of highly connected graphs may help to solve these conjectures. This motivates the next two sections.

The first one focuses on the existence of vertices in graphs and digraphs that have a minimum degree with respect to the connectivity. We prove that every minimally g-bounded k-connected undirected graph contains a tight vertex and give a common generalization to two results of Mader [53, 54] in the directed case.

In the last section we generalize a constructive characterization of Jordán [41] of weakly 4-connected graphs to $(2k, k)$-connected graphs for k even. Our approach that follows the path of Lovász and Jordán is based on a new splitting-off theorem that enables us to solve an augmentation problem as well.

The material of this chapter is based on a joint work with Szigeti [10].

5.1 CONJECTURES OF THOMASSEN AND FRANK

In the area of vertex-connected orientation, the starting point is the conjecture of Thomassen stating that any sufficiently highly vertex-connected graph admits a k-vertex-connected orientation.

Conjecture 5.1 (Thomassen [68])**.** *For every integer k, there exists a least integer $h(k)$ such that every $h(k)$-vertex-connected graph admits a k-vertex-connected orientation.*

An easy observation is that if $h(k)$ exists then $h(k) \geqslant 2k$. Indeed, consider two disjoint copies of K_{2k} and a matching between $2k-1$ vertices of the first copy and $2k-1$ vertices of the second copy. The resulting graph G is $(2k-1)$-vertex-connected since each copy of K_{2k} is clearly $(2k-1)$-vertex-connected and the two copies are joined by $2k-1$ vertex-disjoint paths. However G has no k-vertex-connected orientation since in such an orientation there would be at least k arcs from each copy to the other one, that is, $2k$ edges between the two copies of K_{2k} in G.

In the above argument we actually proved that every graph G admitting a k-vertex-connected orientation D is $2k$-edge-connected. We may push further the necessary condition of the connectivity of G. For any non-trivial bi-set X of the vertex set, by Theorem 2.3 we have, $d^{\flat}_G(\mathsf{X}) = \rho^{\flat}_D(\mathsf{X}) + \delta^{\flat}_D(\mathsf{X}) \geqslant 2(k - |w^{\flat}(\mathsf{X})|)$, that is, by Theorem 2.2, G is $(2k, 2)$-connected. We recall that this special mixed-connectivity is also called weak $2k$-connectivity. Frank

1 the best-balanced orientation problem is a notable exception

conjectured that this necessary connectivity condition for the existence of a k-vertex-connected orientation is sufficient.

Conjecture 5.2 (Frank [26]). *A graph G admits a k-vertex-connected orientation if and only if G is weakly $2k$-connected.*

The conjecture of Frank may be considered as a sharpening of the conjecture of Thomassen since, Conjecture 5.2 would imply $h(k) \leqslant 2k$ in Conjecture 5.1 and thus $h(k) = 2k$ by the lower bound discussed above.

The base case of both conjectures is settled by Robbins' theorem (Theorem 3.6) stating that every 2-edge-connected graphs has a strongly connected orientation. Gerards [33] proved the case $k = 2$ of Conjecture 5.2 for 4-regular graphs. Then Berg and Jordán settled this case for Eulerian graphs.

Theorem 5.1 (Berg, Jordán [2]). *Every Eulerian weakly 4-connected graph has a 2-vertex-connected orientation.*

The constructive characterization of weakly 4-connected graphs given by Jordán [41] is a key element of their proof. In Section 5.3, we will investigate a generalization of this construction to $(2k, k)$-connected-graphs.

Using Nash-Williams' odd pairing theorem (Theorem 3.16), Z. Király and Szigeti found a simpler proof of Theorem 5.1 and actually showed a stronger result. However their argument is unlikely to extend to higher connectivity.

Theorem 5.2 (Z. Király, Szigeti [46]). *Let $k \geqslant 2$ be an integer. An Eulerian graph $G = (V, E)$ has a k-arc-connected orientation D such that $D - v$ is $(k-1)$-arc-connected for all $v \in V$ if and only if G is $2k$-edge-connected and $G - v$ is $(2k-2)$-edge-connected for all $v \in V$.*

Since the result of Berg and Jordán is restricted to Eulerian graphs it does not readily imply the case $k = 2$ of Thomassen's conjecture. Jordán overcame this difficulty by proving his spanning rigid subgraphs packing theorem (Theorem 4.18) that enabled him to prove that $h(2)$ exists and is upper bounded by 18.

Theorem 5.3 (Jordán [40]). *Every 18-vertex-connected graph has a 2-vertex-connected orientation.*

Proof. Let $G = (V, E)$ be 18-vertex-connected graph. By Theorem 4.18, there exist 3 edge-disjoint spanning rigid subgraphs of G say (V, E_1), (V, E_2) and (V, E_3). By Proposition 4.3, each of the two subgraphs is 2-vertex-connected. Hence $G' = (V, E_1 \cup E_2)$ is weakly 4-connected and there exists F a $T_{G'}$-join in (V, E_3). By Theorem 5.1, the Eulerian graph $(V, E_1 \cup E_2 \cup F)$ has a 2-vertex-connected orientation D. Adding into D all the arcs resulting from an arbitrary orientation of $E \setminus (E_1 \cup E_2 \cup F)$ yields a 2-vertex-connected orientation of G. □

As for settling the case $k = 2$ of Kriesell's conjecture, in the above proof, the 2-vertex-connectivity of G_3 is redundant since the simple connectivity is sufficient to find a $T_{G'}$-join. Hence packing 2 spanning rigid graphs and a spanning tree is sufficient to apply Theorem 5.1. So Theorem 4.19 allows to decrease the upper bound on $h(2)$ to 14.

Theorem 5.4 (Cheriyan, Durand de Gevigney, Szigeti [5]). *Every 14-vertex-connected graphs has a 2-vertex-connected orientation.*

The problem of finding rooted k-vertex-connected orientations as not been much studied although it may seem easier. Frank conjectured a characterization of graphs admitting a rooted k-vertex-connected orientation.

Conjecture 5.3 (Frank [29]). *A graph* $G = (V, E)$ *admits an orientation that is rooted* k*-vertex-connected at a vertex* r *if and only if*

$$e^{\flat}_G(\mathcal{F}) \geqslant \sum_{X \in \mathcal{F}} (k - |w^{\flat}(X)|) \qquad (5.1)$$

holds for every family $\mathcal{F}$ *of pairwise innerly-disjoint bi-sets of* $V \setminus r$.

One may easily see that the condition (5.1) is necessary. Suppose that D is a rooted k-connected orientation of G rooted at r. Then, by Theorem 2.3, for any non trivial bi-set X such that $r \notin X_O$, $\rho^{\flat}_D(X) \geqslant k - |w^{\flat}(X)|$. Hence, for any family $\mathcal{F}$ of pairwise innerly-disjoint bi-sets of $V \setminus r$, we have $e^{\flat}_G(\mathcal{F}) \geqslant \sum_{X \in \mathcal{F}} \rho^{\flat}_D(X) \geqslant \sum_{X \in \mathcal{F}} (k - |w^{\flat}(X)|)$. This condition also seems to be a natural generalization of the partition connectivity condition given in Theorem 4.8.

5.2 TIGHT VERTICES

5.2.1 *Tight Vertices in Graphs*

It is trivial to see that in a g-bounded k-connected graph the degree of each vertex is at least k. In such a graph G we call a vertex v *tight* if $d_G(v) = k$. The study of tight vertices is a tool to get a better insight into the structure of g-bounded k-connected graphs. For instance, in Subsection 3.2.2, the existence of a tight vertex (Fact 3.3) was an imperative ingredient to prove the Lovász' constructive characterization of $2k$-edge-connected graphs (Theorem 3.10). Jordán [41] used the existence of a tight vertex in every tight bi-set in order to prove a similar characterization for weakly 4-connected graphs.

Since adding an edge may ruin the tightness of a vertex, studying the existence of tight vertices in arbitrary g-bounded k-connected graphs makes little sense. However, it does in minimally g-bounded k-connected graphs. When tight vertices exist, the natural question is to find a lower bound on their number. This question has been studied for special values of g. The starting point is a theorem of Halin [37] stating that every minimally k-vertex-connected graph contains a tight vertex. A few years later Mader [52] improved this result showing that every minimally k-vertex connected graphs contains at least $k+1$ tight vertices. In minimally k-edge-connected graphs Lick [48] first proved the existence of a tight vertex. Kaneko and Ota [43] showed that every minimally $(k\ell, k)$-connected graphs contains at least ℓ vertices of degree $k\ell$ which has been improved to $\ell + 1$ by Mader [55]. These results suggest the following statement.

Question 5.1. *Let* G *be a minimally* g*-bounded* k*-connected graph and denote* U *the set of tight vertices of* G*. Does* $g(U) \geqslant k$ *hold?*

We do not have the answer to this question, but we will prove the following weaker statement.

Theorem 5.5. *Every minimally* g*-bounded* k*-connected graph contains at least one tight vertex.*

To answer positively Question 5.1, it may be profitable to understand the location of tight vertices in the graph. For edge-connectivity, vertex-connectivity and more generally mixed-connectivity, the inner-set of every tight bi-set contains a tight vertex [43]. But this may not hold for arbitrary g-bounded connectivity. However, we will prove such a statement for the special g-bounded $2k$-connectivity where $g(v)$ has the two possible values k or $2k$ for each vertex v.

5.2.1.1 *Tight Vertices in Minimally* g*-Bounded Graphs*

One key tool in the proof of Kaneko and Ota is the following result.

Claim 5.1 (Kaneko and Ota [43])**.** *Let* $G = (V, E)$ *be a minimally* (k, ℓ)*-connected graph such that* ℓ *divides* k*. Then* $d_G(u, v) \leqslant \ell$ *for every pair* u, v *of vertices.*

A somehow equivalent upper bound for g-bounded connectivity would be

$$d_G(u, v) \leqslant g(u) \tag{5.2}$$

for every pair of vertices u, v. But the graph given in Figure 15 shows the existence of minimally g-bounded connected graphs where (5.2) is violated.

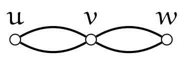


Figure 15: A minimally g-bounded 2-connected graph where $g(u) = g(w) = 1$ and $g(v) = 2$ containing exactly 2 tight vertices.

However we can prove that (5.2) is not violated at both ends of uv.

Claim 5.2. *Let* $G = (V, E)$ *be a minimally* g*-bounded* k*-connected graph. If there exists* $u, v \in V$ *such that* $d_G(u, v) > g(u)$ *then* $d_G(u, v) \leqslant g(v)$*.*

Note that this result implies Claim 5.1 since for mixed-connectivity g is a constant. This claim is derived from the following fact.

Fact 5.1. *Let* $G = (V, E)$ *be a* g*-bounded* k*-connected graph and* X *be a tight bi-set of* G*. Then for every* $w \in w^\flat(X)$*,*

$$d_G(w, X_I) \geqslant g(w) \tag{5.3}$$

hold and the inequality is strict if X *is inclusion wise minimal. Moreover, if* $|X_I| \geqslant 2$ *then, for every* $v \in X_I$*,*

$$d_G(v, \overline{X_O}) \leqslant g(v). \tag{5.4}$$

Proof. Let $w \in w^\flat(X)$ and consider the non trivial bi-set $Y = (X_O \setminus w, X_I)$. By g-bounded k-connectivity of G and tightness of X we have

$$\begin{aligned} k &\leqslant d^\flat_G(Y) + g(w^\flat(Y)) \\ &= d^\flat_G(X) + d_G(w, X_I) + g(w^\flat(X)) - g(w) \\ &= k + d_G(w, X_I) - g(w). \end{aligned}$$

So (5.3) follows and the inequality is strict if X is inclusion wise minimal since $Y \sqsubset X$.

Let $v \in X_I$. If $|X_I| \geqslant 2$ then the bi-set $Z = (X_O, X_I \setminus v)$ is non trivial and, by g-bounded k-connectivity of G and tightness of X,

$$\begin{aligned} k &\leqslant d^\flat_G(Z) + g(w^\flat(Z)) \\ &= d^\flat_G(X) - d_G(v, \overline{X_O}) + g(w^\flat(X)) + g(v) \\ &= k - d_G(v, \overline{X_O}) + g(v). \end{aligned}$$ □

Proof of Claim 5.2. Suppose there exist two vertices u, v contradicting the claim. By minimality of G, there exists a tight bi-set X such that $u \in X_I$ and $v \notin X_O$. By (5.4) for u and X, $X_I = \{u\}$ and, by (5.4) for u and $\overline{X}$, $\overline{X_O} = \{v\}$. Hence we have the following contradiction to (2.11), $g(V \setminus v) = g(X_O) = g(w^\flat(X)) + g(u) < g(w^\flat(X)) + d_G(u, v) = f^\flat(X) = k$. □

Following the proof of Theorem 4 in [43] we prove a stronger property in graphs where the degree function satisfies the upper bounded (5.2) suggested above.

Lemma 5.1. *Let* G *be a minimally* g*-bounded* k*-connected graph satisfying* (5.2)*. Denote* U *the set of tight vertices of* G *and* $W = V \setminus U$*. Then, for any* $W' \subseteq W$*, there exists* $w \in W'$ *such that* $d_{G[W']}(w) \leqslant g(w)$*.*

Proof. By contradiction we assume that there exists $W' \subseteq W$ contradicting the theorem. So W' induces at least one edge and we choose an edge uv of $G[W']$ and a tight bi-set X entered by uv such that $g(X_I)$ is minimum and, subject to that, X is minimal. We may assume that $u \in X_I$.

The set X_I is not reduced to the singleton $\{u\}$ otherwise, by $u \in W$, (5.2), and tightness of X we would have $k < d_G(u) = d^b_G(X) + d_G(u, w^b(X)) \leqslant d^b_G(X) + g(w^b(X)) = k$. Hence, by (5.4), there is a neighbor w of u in $W' \cap X_O$. By minimality of G there exists a tight bi-set Y such that $u \in Y_I$ and $w \notin Y_O$. Clearly, $X \sqcap Y$ is non trivial, so, if $X \sqcup Y$ is non-trivial then, by Fact 2.3, $X \sqcap Y$ is tight which contradicts the minimality of X. Thus we proved that $X \sqcup Y$ is trivial, that is, $X_O \cup Y_O = V$.

As above, by $u \in W$, (5.2), and tightness of Y, Y_I is not reduced to the singleton $\{u\}$. Hence, by $X_O \cup Y_O = V$, (5.4) applied to Y and tightness and minimality of X, we have

$$
\begin{aligned}
g(X_I) - g(\overline{Y_O}) &= g(X_I \cap Y_I) + g(w^b(X \sqcap Y)) - g(w^b(X)) \\
&\geqslant d_G(X_I \cap Y_I, \overline{Y_O}) + g(w^b(X \sqcap Y)) - g(w^b(X)) \\
&\geqslant d^b_G(X \sqcap Y) - d^b_G(X) + g(w^b(X \sqcap Y)) - g(w^b(X)) \\
&= f^b_G(X \sqcap Y) - f^b_G(X) > 0.
\end{aligned}
$$

Since uw enters $\overline{Y}$ this contradicts the choice of X. □

To prove Theorem 5.5, we actually use Lemma 5.1 only in the case $W' = W$.

Proof of Theorem 5.5. Let G be a minimally g-bounded k-connected graph. First assume that (5.2) is violated in G and choose u and v satisfying $d_G(u, v) > g(u)$ such that $g(u)$ is minimum. We prove that u is tight. Let X be an inclusion wise minimal tight bi-set such that $u \in X_I$ and $v \notin X_O$. By Fact 5.1, X_I is reduced to the singleton $\{u\}$. By minimality of X, (5.3) is strict, so if there exists $w \in w(X)$ then $d_G(w, u) = d_G(w, X_I) > g(w)$ and, by Claim 5.2, $g(u) \geqslant d_G(w, u) > g(w)$. Thus the pair w, u violates (5.2) with $g(w) < g(u)$, a contradiction to the choice of u. So we proved that $w^b(X) = \emptyset$, that is, u is a tight vertex.

Now we may assume that (5.2) is satisfied in G. We claim that, for any vertex u such that $g(u) > k$, decreasing $g(u)$ by one preserves the minimal g-bounded k-connectivity of G. Indeed, this operation preserves inequality (iii) of Theorem 2.2 and (2.11) hence the g-bounded k-connectivity still holds. Furthermore, the tightness of bi-sets is also preserved since the wall of no tight bi-set contains a vertex of g-value greater than k.

So we may suppose that $g(v) \leqslant k$ for any vertex. Denote U the set of tight vertices of G and $W = V \setminus U$. If W is empty then we are done, so we assume that W is not empty and, by Lemma 5.1, we choose $w \in W$ such that $k \geqslant g(w) \geqslant d_{G[W]}(w) = d_G(w, W)$. Hence, by $w \in W$, we have $d_G(w, U) = d_G(w) - d_G(w, W) > k - k = 0$ which proves that U is not empty. □

5.2.1.2 *On Minimally* 2k-T-*Connected Graphs*

Let $G = (V, E)$ be a graph and $T \subseteq V$. The graph G is called 2k-T-*connected* if G is g-bounded 2k-connected where g is defined by

$$g(v) = \begin{cases} k & \text{if } v \in T \\ 2k & \text{if } v \notin T. \end{cases} \tag{5.5}$$

Lemma 5.2. *Let $G = (V, E)$ be a minimally 2k-T-connected graph where $T \subseteq V$ and k is a positive integer. Then every inclusion wise tight bi-set of G has an empty wall and its inner-set is reduced to a single tight vertex.*

Proof. Let $X = (X_O, X_I)$ be a minimal tight bi-set of G. Assume that X_I is not a singleton. We claim that $G[X_i]$ is connected. Otherwise, there exists a connected component $Z \subset X_I$ of $G[X_I]$. Then the non-trivial bi-set $Y = (X_O \setminus Z, X_I \setminus Z)$ is tight since $2k = d^\flat_G(X) + g(w^\flat(X)) \geqslant d^\flat_G(Y) + g(w^\flat(Y)) \geqslant 2k$ by tightness of X and 2k-T-connectivity of G. This contradicts the minimality of X.

So there exists an edge uv induced by X_I. By minimality of G, this edge enters a tight bi-set Y of G. None of $X \sqcap Y$ and $X \sqcap \overline{Y}$ is trivial, thus, by Fact 2.3 and minimality of X, both $X \sqcup Y$ and $X \sqcup \overline{Y}$ are trivial. Hence, since X is not trivial, $\overline{X_O} \cap w^\flat(Y)$ is not empty. Moreover, since uv enters Y, $d^\flat(Y) \geqslant 1$, so $w^\flat(Y) = \overline{X_O}$ is reduced to a singleton of g-value k. Hence, by 2k-T-connectivity of G and minimality of X, submodularity of $d^\flat_G$ and tightness of X and Y, we have the following contradiction

$$\begin{aligned} 2k + 2k &< f^\flat(X \sqcap Y) + f^\flat(X \sqcap \overline{Y}) \\ &= g(w^\flat(X)) + d^\flat_G(X \sqcap Y) + d^\flat_G(X \sqcap \overline{Y}) \\ &\leqslant g(w^\flat(X)) + d^\flat_G(X) + 2d^\flat_G(Y) \\ &= 2k + 2k. \end{aligned}$$

This proves that X_I is reduced to a singleton say $\{u\}$. If u is tight then we are done. So in the following we assume that $d_G(u) > 2k$.

Observe that $d_G(u, w^\flat(X)) = d_G(u) - d^\flat_G(X) > 2k - f^\flat_G(X) + g(w^\flat(X)) = g(w^\flat(X))$. Hence there exists $w \in w^\flat(X)$ such that $d(u, w) > g(w)$. By minimality of G, there exists a tight bi-set Z of V such that $u \in Z_I$ and $w \notin Z_O$. Note that $w^\flat(Z)$ is empty since $d^\flat_G(Z) \geqslant d_G(u, w) > g(w) \geqslant k$. Note also that $\overline{X_O} \cap \overline{Z_O}$ is empty by Fact 2.3 since $X \sqcap Z \sqsubset X$ is non trivial and X is minimal. Hence, since X is non trivial $\overline{X} \sqcap Z$ is not trivial and, by (2.8), we have the following contradiction,

$$\begin{aligned} 2k &\leqslant f^\flat_G(\overline{X} \sqcap Z) \\ &\leqslant f^\flat_G(\overline{X}) + f^\flat_G(Z) - g(w^\flat(\overline{X} \sqcup Z)) - d_G(\overline{Z_O} \cap \overline{X_I}, Z_I \cap X_O) \\ &\leqslant 4k - g(w) - d_G(u, w) \\ &< 4k - 2g(w) \\ &\leqslant 2k. \end{aligned}$$

□

This lemma may also be formulated as follows: in a minimally 2k-T-connected graph, the inner-set of every tight bi-set contains a tight vertex. Since the complement of a tight bi-set is tight, the lemma readily implies the following result that answers positively a very special case of Question 5.1.

Corollary 5.1. *In every minimally* 2k-T-*connected graph there exists at least* 2 *vertices of degree* 2k.

The graph given in Figure 15 shows that the lower bounds in Question 5.1 and Corollary 5.1 are tight for 2k-T-connectivity.

5.2.2 *Tight Vertices in Digraphs*

In this subsection we investigate the existence of vertices with indegree and outdegree k in g-bounded k-connected digraphs. As for the undirected case, we call these vertices *tight*. Mader [53] proved that every minimally k-arc-connected digraph has a tight vertex and he conjectured the following.

Conjecture 5.4 (Mader [54]). *Every minimally k-vertex-connected digraph contains a tight vertex.*

In [56] Mader settled the case $k = 2$ proving that every minimally 2-vertex-connected graphs contains at least 2 tight vertices. For $k \geqslant 3$, the conjecture is still open, so proving the directed counterpart of Theorem 5.5 seems really challenging (if possible). However, for the special case of g-bounded 2-connectivity and under an assumption similar to (5.2), the existence of a tight vertex is confirmed.

Theorem 5.6 (Durand de Gevigney, Szigeti [11]). *Every minimally g-bounded 2-connected digraph such that, for every $uv \in A$,*

$$d_D(u, v) \leqslant g(u) \tag{5.6}$$

holds, contains a tight vertex.

This result is actually a common ground for both results of Mader. Indeed, in any minimally g-bounded 2-connected graph the multiplicity of arcs is trivially in upper bounded by 2. So (5.6) holds in minimally 2-arc-connected digraphs where g is defined by a constant function of value at least 2. By Fact 2.5, minimally 2-vertex-connected digraphs are simple, so (5.6) is also satisfied since for vertex-connectivity g is defined as 1_V.

The proof of Theorem 5.6 follows the approach of Mader given in [56] and requires a few notations. We denote V^+ the set of vertices $v \in V$ such that $\rho_D(v) > 2$ and $\delta_D(v) = 2$. Let A_0 be the set of arcs $uv \in A$ such that $\rho_D(v) > 2$ and $\delta_D(u) > 2$. Note that if D is minimally g-bounded 2-connected then every arc $a \in A \setminus A_0$ enters a vertex v of indegree 2 or leaves a vertex v of outdegree 2; in both cases we say that the vertex v *covers* a.

Proof. By contradiction, suppose that the theorem is false and let the digraph $D = (V, A)$ be a counterexample. Since D is a counterexample every vertex v covers at most 2 arcs, hence $|A| - |A_0| \leqslant 2|V|$, and satisfies $\rho_D(v) + \delta_D(v) > 4$, thus $|A| = \frac{1}{2}\sum_{v \in V}(\rho_D(v) + \delta_D(v)) > 2|V|$. This proves that A_0 is not empty.

By minimality of D, every arc of A_0 enters an in-tight bi-set or leaves an out-tight bi-set. Among all these bi-sets, we choose X such that $|X_O|$ is minimum and, with respect to that, $|X_I|$ is minimum. Without loss of generality we may assume that X is in-tight and an arc ab of A_0 enters X.

Claim 5.3. *Every arc $uv \in A_0$ such that $v \in X_I$ enters X.*

Proof. By contradiction, assume that there exists $uv \in A_0$ such that $v \in X_I$ and $u \in X_O$. By minimality of D, there exists a tight bi-set of V such that $v \in Y_I$ and $u \notin Y_O$. The bi-set $X \sqcup Y$ is trivial otherwise, by Fact 2.4, $X \sqcap Y \sqsubset X$ is in-tight and entered by $uv \in A_0$ which contradicts the minimality of X. This means that $\overline{X_O} \cap \overline{Y_O}$ is empty.

Thus, by tightness of $\overline{Y}$ and choice of X we have $|\overline{Y_I}| \geqslant |X_O|$, that is, $|\overline{X_O} \cap w^b(Y)| \geqslant |X_O \cap Y_I|$. Hence, by $v \in X_I \cap Y_I$, we have

$$\begin{aligned} 2 - 1 &\geqslant f^b_D(Y) - \rho^b_D(Y) \\ &= g(w^b(Y)) \\ &\geqslant |\overline{X_O} \cap w^b(Y)| \\ &\geqslant |X_O \cap Y_I| \\ &= |w^b(X) \cap Y_I| + |X_I \cap Y_I| \\ &\geqslant 0 + 1. \end{aligned}$$

Hence equality holds everywhere, that is, $w^b(Y)$ is a single vertex contained in $\overline{X_O}$ and $|\overline{Y_I}| = |X_O|$. Hence, by tightness of Y and choice of X, $|\overline{Y_O}| \geqslant |X_I|$, that is $|w^b(X) \cap \overline{Y_O}| \geqslant |X_I \cap Y_O| \geqslant 1$. So $w^b(X)$ is a singleton contained in $\overline{Y_O}$ and $|w^b(X \sqcup Y)| = 2$. Hence, since $X \sqcap Y$ is not trivial, by minimality of X, (2.9), tightness of X and Y, and positivity of g, we have the following contradiction,

$$\begin{aligned} 2 &< f^b_D(X \sqcap Y) \\ &\leqslant f^b_D(X) + f^b_D(Y) - g(w^b(X \sqcup Y)) \\ &\leqslant 4 - |w^b(X \sqcup Y)| \\ &= 2. \end{aligned}$$

□

Now we prove that, in $D[X_O]$, b can be reached from any other vertex. Actually it is sufficient to prove this statement in $D[X_I]$ since if $w^b(X)$ is not empty then it is reduced to a singleton such that $d_D(w^b(X), X_I) \geqslant 1$. Suppose by contradiction that there exists a proper subset U of X_I containing b such that $\rho_{D[X_I]}(U) = 0$. Then the non-trivial bi-set set $Z = (U \cup w^b(X), U)$ satisfies $2 \leqslant \rho^b(Z) + g(w^b(Z)) \leqslant \rho^b(X) + g(w^b(X)) = 2$. Hence Z is an in-tight bi-set entered by ab such that $|Z_O| < |X_O|$, a contradiction to the choice of X.

Since $\rho_D(b) > 2$ and D contains no tight vertex, every vertex from which b is reachable in $D - A_0$ belongs to V^+. Hence, by Claim 5.3, we have $X_O \subseteq V^+$, and thus, the following inequalities hold

$$\begin{aligned} \rho_D(X_I) &= \rho^b_D(X) + d_D(w^b(X), X_I) \\ &\leqslant \rho^b_D(X) + 2|w^b(X)| \\ &\leqslant \rho^b_D(X) + g(w^b(X)) + |w^b(X)| \\ &\leqslant 2 + 1 \text{ and} \\ \rho_D(X_I) &= \sum_{v \in X_I} (\rho_D(v) - \delta_D(v)) + \delta_D(X_I) \\ &\geqslant |X_I| + 2 \\ &\geqslant 3. \end{aligned}$$

So equality holds everywhere in both inequalities. In particular, $X_I = \{b\}$ and $w^b(X)$ is a single vertex w such that $d_G(w, b) = 2 > 1 = g(w)$. This contradicts (5.6). □

5.3 ON $(2k, k)$-CONNECTED GRAPHS

In Section 5.1 we mentioned that Jordán proved a constructive characterization of $(4, 2)$-connected graphs in [41]. The material of this section resulting from a joint work with Szigeti [10] generalizes this result to $(2k, k)$-connected graphs. As for the constructive characterization of 2k-edge-connected graphs of Lovász given in Section 3.2.2, the approach of Jordán consists of proving a splitting-off theorem for vertices of degree 4. Following the same path the natural generalization would be to prove a splitting-off theorem on vertices of degree $2k$. We actually proved a more general result that stands for every vertex of even degree. This further generalization is not necessary for the characterization since, as we saw in Section 5.2, every tight bi-set of a minimally $(2k,k)$-connected graph contains a vertex of degree $2k$. However, it enables us to address the following augmentation problem: given a graph G and an integer $k \geqslant 2$, what is the minimum number of edges to be added to make G $(2k,k)$-connected.

5.3.1 *Preliminaries*

We recall that a graph $G=(V,E)$ is $(2k,k)$-connected if and only if

$$f^\flat_G(X) := d^\flat_G(X) + k|w^\flat(X)| \geqslant 2k, \tag{5.7}$$

for all non-trivial bi-set X of V and that a bi-set is called tight if equality holds in (5.7). We call G *minimally* $(2k,k)$-*connected* if G is $(2k,k)$-connected and removing any edge destroys this property. Clearly, in a minimally $(2k,k)$-connected graph every edge enters a tight bi-set.

Let $H=(V\cup s,E)$ be a graph with a special vertex s. For convenience, in this chapter H will always denote a graph with a special vertex s. The graph H is called $(2k,k)$-*connected in* V if (5.7) holds for every non-trivial bi-set of V. Note that, considering the graph H, the complement of a bi-set is taken relatively to the ground set $V\cup s$.

Claim 5.4. *Let* $H=(V+s,E)$ *be a* $(2k,k)$-*connected graph in* V. *For every non-trivial bi-set* X *of* V,

$$2k - f^\flat_H(X) \leqslant d_H(s,\overline{X_O}) - d_H(s,X_I), \tag{5.8}$$

$$d_H(s,X_I) \leqslant \left\lfloor \frac{1}{2}(d_H(s) - d_H(s,w^\flat(X)) + f^\flat_H(X) - 2k) \right\rfloor. \tag{5.9}$$

Moreover, for every pair of bi-sets X *and* Y *of* $V\cup s$ *such that* $|w^\flat(X\sqcup Y)| \geqslant 2$ *and* $X\sqcap Y$ *is a non-trivial bi-set of* V,

$$\begin{aligned}(f^\flat_H(X)-2k)+(f^\flat_H(Y)-2k) \geqslant d^\flat_H(X\sqcup Y) + d_H(\overline{X_O}\cap Y_O, X_I\cap\overline{Y_I})\\ + d_H(\overline{Y_O}\cap X_O, Y_I\cap\overline{X_I}).\end{aligned} \tag{5.10}$$

Proof. For a non-trivial bi-set X of V, $X' = (\overline{X_I}-s,\overline{X_O}-s)$ is also a non-trivial bi-set of V and hence, by (5.7), $d_H(s,\overline{X_O}) - d_H(s,X_I) = f^\flat_H(X') - f^\flat_H(X) \geqslant 2k - f^\flat_H(X)$. Then, (5.9) follows from (5.8) and $d_H(s,\overline{X_O}) - d_H(s,X_I) = d_H(s) - d_H(s,w^\flat(X)) - 2d_H(s,X_I)$.

The inequality (5.10) is just a combination of (2.8), $f^\flat_H(X\sqcap Y) \geqslant 2k$ and $k|w^\flat(X\cup Y)| \geqslant 2k$. □

5.3.2 *Blocking Bi-sets*

In this chapter we will use notations concerning the splitting-off operation that were introduced in Subsection 3.2.2. Let $H=(V+s,E)$ be a $(2k,k)$-connected graph in V with a special vertex s and (su,sv) a pair of edges. In the current chapter we call the pair (su,sv) *admissible* if $H_{u,v}$ is $(2k,k)$-connected in V. Let X be a bi-set of V and observe that when splitting the pair (su,sv) three cases may occur:

(a) If both u and v belong to X_I then the degree of X decreases by 2.

(b) If one of u and v belongs to X_I and the other one belongs to $w^\flat(X)$ then the degree of X decreases by 1.

(c) If none of the two above cases occurs then the degree of X is preserved.

A non-trivial bi-set X of V is called a *blocking bi-set* for the pair (su, sv) (for short we say that X *blocks* (su, sv)) if either (a) occurs and $f^\flat_H(X) \leqslant 2k+1$ or (b) occurs and $f^\flat_H(X) \leqslant 2k$. A blocking bi-set is called *dangerous* in the first case and *critical* in the second case. Note that a critical bi-set X blocking (su, sv) is tight and $w^\flat(X)$ is either $\{u\}$ or $\{v\}$. Clearly, a blocked pair is not admissible. The following lemma states the converse.

Lemma 5.3. *Let $H = (V + s, E)$ be a $(2k, k)$-connected graph in V. A pair (su, sv) is non-admissible if and only if there exists a bi-set of V blocking (su, sv).*

Proof of necessity. Since (su, sv) is non-admissible, there exists a non-trivial bi-set X of V which violates (5.7) in $H_{u,v}$. Since $f^\flat_H(X) \geqslant 2k$, either the degree of X decreases by 2 (that is (a) occurs) and $f^\flat_H(X) \leqslant 2k+1$ or the degree of X decreases by 1 (that is (b) occurs) and $f^\flat_H(X) \leqslant 2k$. ☐

The following obvious remarks will be used later. By definition every blocking bi-set X satisfies

$$f^\flat_H(X) - 2k \leqslant d_H(s, X_I) - 1, \tag{5.11}$$
$$|w^\flat(X)| \leqslant 1. \tag{5.12}$$

Observe also that if X blocks (su, sv) then, after any sequence of splitting-off containing none of su or sv, X still blocks (su, sv). Hence, by Lemma 5.3, a non-admissible pair in H remains non-admissible in any graph arising from H by a sequence of splitting-off.

The following lemma gathers many technical details and will be used extensively in the proofs of the further sections.

Lemma 5.4. *Let $H = (V + s, E)$ be a $(2k, k)$-connected graph in V with $d_H(s)$ even. Let X be a maximal blocking bi-set for a pair (su, sv) with $u \in X_I$. Let $w \in N_H(s) \setminus X_I$ and Y a blocking bi-set for the pair (su, sw). Then $w^\flat(X)$ and $w^\flat(Y)$ coincide and are reduced to a singleton.*

Proof. Note that $u \in Y_O \cap X_I$ and $w \in Y_O \setminus X_I$.

Fact 5.2. *The bi-sets X and Y satisfy the following.*

(i) If $w^\flat(Y) \cap X_I$ is empty then $X \sqcap Y$ is a non-trivial bi-set of V.

(ii) If both $w^\flat(X) \cap Y_I$ and $w^\flat(Y) \cap \overline{X_I}$ are empty then $\overline{X} \sqcap Y$ is a non-trivial bi-set of V.

(iii) If both $w^\flat(X) \cap \overline{Y_I}$ and $w^\flat(Y) \cap X_I$ are empty then $X \sqcap \overline{Y}$ is a non-trivial bi-set of V.

(iv) If both $w^\flat(X) \cap \overline{Y_I}$ and $w^\flat(Y) \cap \overline{X_I}$ are empty then $X \sqcup Y$ is a non-trivial bi-set of V strictly containing X.

Proof. Since none of X_O and Y_O contains V, to prove (i), (ii) and (iii), we may only check that the inner-set of the bi-set resulting from the intersection is non-empty.

(i) Since $u \in X_I \cap Y_O$, by $w^\flat(Y) \cap X_I = \emptyset$, we have $u \in X_I \cap Y_O = X_I \cap Y_I$.

(ii) Since $w \in Y_O \setminus X_I$, by $w^\flat(Y) \cap \overline{X_I} = \emptyset$ and $w^\flat(X) \cap Y_I = \emptyset$, we have $w \in Y_O \cap \overline{X_I} = Y_I \cap \overline{X_I} = Y_I \cap \overline{X_O}$.

(iii) Suppose that $X \sqcap \overline{Y}$ is trivial, that is $X_I \cap \overline{Y_O} = \emptyset$. It implies that $X_O \cap \overline{Y_I} = (w^\flat(X) \cap \overline{Y_I}) \cup (X_I \cap \overline{Y_O}) \cup (X_I \cap w^\flat(Y))$ is empty. Hence, by $w \in Y_O \setminus X_I$, we have $X \sqsubset Y$. Since Y blocks (su, uv), this contradicts to the maximality of X.

(iv) Since $w \in Y_O \setminus X_I$ and $w^\flat(Y) \cap \overline{X_I} = \emptyset$, we have $w \in Y_O \cap \overline{X_I} = Y_I \cap \overline{X_I}$. Hence $X \sqcup Y$ strictly contains X. Since X is non-trivial and by the conditions, it remains to prove that $V \neq X_O \cup Y_O = X_I \cup Y_I$. We have $d_H(s, X_I \cup Y_I) = d_H(s, X_I) + d_H(s, Y_I) - d_H(s, X_I \cap Y_I)$. If Y is critical then, by (5.9) and $d_H(s)$ even, $d_H(s, X_I) + d_H(s, Y_I) \leqslant \frac{1}{2}d_H(s) + \frac{1}{2}d_H(s) - 1 < d_H(s, V)$. If Y is dangerous then $u \in X_I \cap Y_I \cap N_H(s)$, hence by (5.9) and $d_H(s)$ even, $d_H(s, X_I) + d_H(s, Y_I) - d_H(s, X_I \cap Y_I) \leqslant \frac{1}{2}d_H(s) + \frac{1}{2}d_H(s) - 1 < d_H(s, V)$. In both cases $V \setminus (X_I \cup Y_I)$ contains a neighbor of s. □

Claim 5.5. *At least one of the wall of X and Y is non-empty.*

Proof. By contradiction suppose the claim is false. Then, u belongs to $Y_I \cap X_I \cap N_H(s)$ and, by Facts (i), (iii), (ii) and (iv), none of $X \sqcap Y$, $X \sqcap \overline{Y}$, $\overline{X} \sqcap Y$ and $X \sqcup Y$ is a trivial bi-sets of V and $X \sqsubset X \sqcup Y$. Hence by (2.8), $(2k, k)$-connectivity of H and maximality of X, we have

$$\begin{aligned}(2k+1) + (2k+1) &\geqslant f^\flat_H(X) + f^\flat_H(\overline{Y}) \\ &= d^\flat_H(X) + d^\flat_H(\overline{Y}) \\ &\geqslant d^\flat_H(X \sqcap \overline{Y}) + d^\flat_H(\overline{X} \sqcap Y) + 2d_H(s, X_I \cap Y_I) \\ &= f^\flat_H(X \sqcap \overline{Y}) + f^\flat_H(\overline{X} \sqcap Y) + 2d_H(s, X_I \cap Y_I) \\ &\geqslant 2k + 2k + 2\end{aligned}$$

and

$$\begin{aligned}(2k+1) + (2k+1) &\geqslant f^\flat_H(X) + f^\flat_H(Y) \\ &= d^\flat_H(X) + d^\flat_H(Y) \\ &\geqslant d^\flat_H(X \sqcap Y) + d^\flat_H(X \sqcup Y) \\ &= f^\flat_H(X \sqcap Y) + f^\flat_H(X \sqcup Y) \\ &\geqslant 2k + (2k+2).\end{aligned}$$

It follows that equality holds everywhere, in particular, $d^\flat_H(X \sqcap Y) = d^\flat_H(X \sqcap \overline{Y}) = 2k$ are even and $d^\flat_H(X) = 2k+1$ is odd. This contradicts $d^\flat_H(X) = d^\flat_H(X \sqcap Y) + d^\flat_H(X \sqcap \overline{Y}) - 2d_H(X_I \cap Y_I, X_I \cap \overline{Y_I})$. □

Claim 5.6. *None of $w^\flat(X)$ and $w^\flat(Y)$ is empty.*

Proof. Suppose that the wall of X or Y is empty and call this bi-set A. By Claim 5.5, the wall of the other blocking bi-set, say B, is non-empty.

Suppose that $w^\flat(B) \cap A_I = \emptyset$. By Fact 5.2 (i), $A \sqcap B$ is a non-trivial bi-sets of V. If $A = X$ then, by Fact 5.2 (iii), $X \sqcap \overline{Y}$ is non trivial. If $A = Y$ then, by Fact 5.2 (ii), $\overline{X} \sqcap Y$ is non trivial. In both cases $A \sqcap \overline{B}$ is a non-trivial bi-set of V.

Then, by $(2k, k)$-connectivity of H in V, since the edges between $A_I \setminus B_I$ and $A_I \cap B_I$ enters B but not s and by (5.11), we have the following contradiction,

$$\begin{aligned}
2k + 2k &\leqslant f^b_H(A \sqcap B) + f^b_H(A \sqcap \overline{B}) \\
&= d_H(A_I \cap B_I) + d_H(A_I \cap \overline{B_I}) \\
&= d_H(A_I) + 2d_H(A_I \setminus B_I, A_I \cap B_I) \\
&\leqslant d^b_H(A) + 2(d^b_H(B) - d_H(s, B_I)) \\
&= f^b_H(A) + 2(f^b_H(B) - k|w^b(B)| - d_H(s, B_I)) \\
&\leqslant 2k + 1 + 2(k-1).
\end{aligned}$$

Hence, by (5.12) for B, we have $w^b(B) \cap \overline{A_I} = \emptyset$. By Fact 5.2 (iv), $A \sqcup B$ is a non-trivial bi-set of V strictly containing $X \sqsubset A \sqcup B$. If $A = X$ then, by Fact 5.2 (ii), $\overline{X} \sqcap Y$ is non-trivial. If $A = Y$, the by Fact 5.2 (iii), $X \sqcap \overline{Y}$ is non-trivial. In both cases, $\overline{A} \sqcap B$ is a non-trivial bi-set of V. We have also

$$d^b_H(B) - d_H(s, A_I \cap B_I) \leqslant k \tag{5.13}$$

since B is a blocking bi-set with a non-empty wall and, if B is dangerous then $u \in A_I \cap B_I$. By maximality of X, the $(2k, k)$-connectivity of H, since the edges between $\overline{A_I \cup B_I}$ and $B_I \setminus A_I$ enters B but not $A_I \cap B_I$ and A is a blocking bi-set, by (5.11) and (5.13), we have the following contradiction,

$$\begin{aligned}
(2k+2) + 2k &\leqslant f^b_H(A \sqcup B) + f^b_H(\overline{A} \sqcap B) \\
&= d_H(A_I \cup B_I) + d_H(B_I \setminus A_I) \\
&= d_H(\overline{A_I \cup B_I}) + d_H(B_I \setminus A_I) \\
&= d_H(\overline{A_I}) + 2d_H(\overline{A_I \cup B_I}, B_I \setminus A_I) \\
&\leqslant f^b_H(A) + 2(d^b_H(B) - d_H(s, A_I \cap B_I)) \\
&\leqslant (2k+1) + 2k.
\end{aligned}$$

□

Claim 5.7. *Y and X have the same wall.*

Proof. Suppose $w^b(X) \neq w^b(Y)$. By Claim 5.6 and (5.12), both $w^b(X)$ and $w^b(Y)$ are singletons, we have 4 cases.

Case 1 Both $w^b(X) \cap Y_I$ and $w^b(Y) \cap X_I$ are empty. Then $|w^b(X \sqcup Y)| = 2$ and, by Fact (i), $X \sqcap Y$ is a non-trivial bi-set of V. Hence, by (5.10), since X and Y are blocking bi-sets, by (5.11) and the choice of w, we have the following contradiction

$$\begin{aligned}
d^b_H(X \sqcup Y) &\leqslant (f^b_H(X) - 2k) + (f^b_H(Y) - 2k) \\
&\leqslant (d_H(s, X_I) - 1) + d_H(s, Y_I \setminus X_I) \\
&= d_H(s, X_I \cup Y_I) - 1 \\
&\leqslant d^b_H(X \sqcup Y) - 1.
\end{aligned}$$

Case 2 Both $w^b(X) \cap Y_I$ and $w^b(Y) \cap \overline{X_I}$ are empty. Then $|w^b(\overline{X} \sqcup Y)| = 2$ and, by Fact (ii), $\overline{X} \sqcap Y$ is a non-trivial bi-set of V. By symmetry of f^b_H and (5.11), $f^b_H(\overline{X}) - 2k = f^b_H(X) - 2k < d_H(s, X_I) \leqslant d^b_H(\overline{X} \sqcup Y) + d_H(X_I \cap Y_O, \overline{X_O} \cap \overline{Y_I})$. If Y is dangerous, then $u \in X_I \cap Y_I$. Hence $f^b_H(Y) - 2k \leqslant d(s, X_I \cap Y_I) \leqslant d_H(\overline{Y_O} \cap \overline{X_I}, Y_I \cap X_O)$. So we have, $(f^b_H(\overline{X}) - 2k) + (f^b_H(Y) - 2k) < d^b_H(\overline{X} \sqcup Y) + d_H(X_I \cap Y_O, \overline{X_O} \cap \overline{Y_O}) + d_H(\overline{Y_I} \cap \overline{X_I}, Y_I \cap X_O)$ and this contradicts (5.10).

Case 3 Both $w^b(X) \cap \overline{Y_I}$ and $w^b(Y) \cap X_I$ are empty. Then $|w^b(X \sqcup \overline{Y})| = 2$ and, by Fact (iii), $X \sqcap \overline{Y}$ is a non-trivial bi-set of V. By symmetry of f^b_H

and (5.11), $f^\flat_H(\overline{Y}) - 2k = f^\flat_H(Y) - 2k < d_H(s, Y_I) \leqslant d^\flat_H(X \sqcup \overline{Y}) + d_H(Y_I \cap X_O, \overline{Y_O} \cap \overline{X_I})$. Since $w^\flat(Y) \cap X_I = \emptyset$, we have $u \in Y_I \cap X_I$. Hence $f^\flat_H(X) - 2k \leqslant d(s, X_I \cap Y_I) \leqslant d_H(\overline{X_O} \cap \overline{Y_I}, X_I \cap Y_O)$. So we have, $(f^\flat_H(X) - 2k) + (f^\flat_H(\overline{Y}) - 2k) < d^\flat_H(X \sqcup \overline{Y}) + d_H(\overline{X_O} \cap \overline{Y_I}, X_I \cap Y_O) + d_H(Y_I \cap X_O, \overline{Y_O} \cap \overline{X_I})$ and this contradicts (5.10).

Case 4 Both $w^\flat(X) \cap \overline{Y_I}$ and $w^\flat(Y) \cap \overline{X_I}$ are empty. Then $|w^\flat(X \sqcap Y)| = 2$ and, by Fact (iv), $X \sqcup Y$ is a non-trivial bi-set of V and $X \sqsubset X \sqcup Y$. If Y is dangerous, then $u \in X_I \cap Y_I$, thus, since X is a blocking bi-set, $1 + d^\flat_H(X \sqcap Y) \geqslant 1 + d_H(s, X_I \cap Y_I) \geqslant (f^\flat_H(X) - 2k) + (f^\flat_H(Y) - 2k)$. By maximality of X and submodularity of $f^\flat_H$, we have the following contradiction,

$$\begin{aligned}
2k + 2 &\leqslant f^\flat_H(X \sqcup Y) \\
&\leqslant f^\flat_H(X) + f^\flat_H(Y) - f^\flat_H(X \sqcap Y) \\
&\leqslant d^\flat_H(X \sqcap Y) + 1 + 4k - f^\flat_H(X \sqcap Y) \\
&= 1 + 2k.
\end{aligned}$$ □

Claims 5.6 and 5.7 prove Lemma 5.4. □

Proposition 5.1. *Let $H = (V + s, E)$ be a $(2k, k)$-connected graph in V with $d_H(s)$ even and X and Y two critical bi-sets with wall $\{w\}$ such that $d_H(s, w)$ is odd. Then $N_H(s) \setminus (X_O \cup Y_O)$ is non-empty. In particular, $X \sqcup Y$ is a non-trivial bi-set of V.*

Proof. By (5.9), $d_H(s)$ even and $d_H(s, w)$ odd, we have

$$\begin{aligned}
d_H(s, X_O \cup Y_O) &= d_H(s, X_I \cup Y_I) + d_H(s, w) \\
&\leqslant d_H(s, X_I) + d_H(s, Y_I) + d_H(s, w) \\
&< \frac{1}{2}(d_H(s) - d_H(s, w)) + \frac{1}{2}(d_H(s) - d_H(s, w)) \\
&\quad + d_H(s, w) \\
&= d_H(s).
\end{aligned}$$

Hence, there exists a neighbor of s in $V \setminus (X_O \cup Y_O)$. □

Claim 5.8. *Let $H = (V + s, E)$ be a $(2k, k)$-connected graph in V with $d_H(s)$ even. Let X be a maximal blocking bi-set for (su, su) where $u \in V$ such that $d_H(s, u) \geqslant \frac{d_H(s)}{2}$. Then the pair (su, sv) is admissible for all $v \in N_H(s) \setminus X_O$.*

Proof. Since X is obviously dangerous and $v \in N_H(s) \setminus X_O$, none of u and v belongs to $w^\flat(X)$. Suppose that (su, sv) is non-admissible, that is, by Lemma 5.3, there exists a bi-set Y blocking the pair (su, sv). Then, by Lemma 5.4, $w^\flat(X)$ and $w^\flat(Y)$ coincide and are reduced to a singleton. Hence, $v, u \in Y_I$ which gives $d_H(s, Y_I) \geqslant d_H(s, u) + d_H(s, v) \geqslant \frac{d_H(s)}{2} + 1$, contradiction to (5.9). □

5.3.3 *Obstacles*

Let $H = (V + s, E)$ be a $(2k, k)$-connected graph in V such that $d_H(s)$ is even. The following definition of an obstacle extends the one given by Jordán in [41]. The pair $(t, \mathcal{C})$ is called a t-*star obstacle* at s (for short, an *obstacle*) if

$$\begin{aligned}
&t \text{ is a neighbor of } s \text{ with } d_H(s,t) \text{ odd}, && (5.14a)\\
&\mathcal{C} \text{ is a collection of critical bi-sets}, && (5.14b)\\
&\text{each element of } \mathcal{C} \text{ has wall } \{t\}, && (5.14c)\\
&\text{the elements of } \mathcal{C} \text{ are pairwise innerly-disjoint}, && (5.14d)\\
&N_H(s) \setminus \{t\} \subseteq V_I(\mathcal{C}). && (5.14e)
\end{aligned}$$

If $(t, \mathcal{C})$ is an obstacle at s, note that, by Lemma 5.3, no pair (st, su) with $u \in N_H(s) \setminus \{t\}$ is admissible.

Some basic properties of obstacles are proven in the following proposition.

Proposition 5.2. *Let* $H = (V+s, E)$ *be a* $(2k,k)$*-connected graph in* V *with* $d_H(s)$ *even and* $(t, \mathcal{C})$ *an obstacle at* s*. Then*

$$|\mathcal{C}| \geqslant 3, \tag{5.15}$$

$$H - st \text{ is } (2k,k)\text{-connected in } V. \tag{5.16}$$

Proof. (5.15): By (5.14e), (5.14a) and $d_H(s)$ even, $|\mathcal{C}| \geqslant 1$. Let X and Y be two (not necessarily distinct) elements of $\mathcal{C}$. By (5.14b), (5.14c), (5.14a) and Proposition 5.1, $N_H(s) \setminus (X_O \cup Y_O)$ is non-empty. Thus, by (5.14e), there exists an element in $\mathcal{C} \setminus \{X, Y\}$.

(5.16): Suppose that $H - st$ is not $(2k,k)$-connected in V, that is, by $(2k,k)$-connectivity of H, there exists in H a non-trivial tight bi-set X of V such that $t \in X_I$. Note that, by (5.14a), $|w^\flat(X)| \leqslant 1$.

Since H is $(2k,k)$-connected in V and by (5.14c), for every $Y \in \mathcal{C}$, $d_H(t, Y_I) = d_H(Y_I) - (f^\flat_H(Y) - k|w^\flat(Y)|) \geqslant 2k - (2k-k) = k$. If $X_I = \{t\}$ then, by tightness of X, (5.14d), (5.15), (5.14a) and $|w^\flat(X)| \leqslant 1$, we have the following contradiction $2k = f^\flat_H(X) \geqslant d^\flat_H(X) = d_H(X_I) - d_H(X_I, w^\flat(X)) = d_H(t) - d_H(t, w^\flat(X)) \geqslant d_H(t,s) + \sum_{Y \in \mathcal{C}, w^\flat(X) \notin Y_I} d_H(t, Y_I) \geqslant 1 + 2k$. So $X_I \neq \{t\}$.

Suppose that there exists $Y \in \mathcal{C}$ such that $\overline{X} \sqcap Y$ and $X \sqcap \overline{Y}$ are both non-trivial bi-sets of V. Then, since X is tight, Y is critical, by symmetry of $f^\flat_H$, $(2k,k)$-connectivity of H in V and Fact 2.3, we have the following contradiction, $0 = d_H(X_I \cap Y_O, \overline{X_O} \cap \overline{Y_I}) \geqslant d_H(s,t) \geqslant 1$. Hence, for all $Y \in \mathcal{C}$, $\overline{X} \sqcap Y$ or $X \sqcap \overline{Y}$ is trivial, that is, since X and Y are non-trivial, $Y_I \subseteq X_O$ or $X_I \subseteq Y_O$.

If, for all $Y \in \mathcal{C}$, $Y_I \subseteq X_O$ then, by $t \in X_I$ and (5.14e), $N_H(s) \subseteq X_O$ which, by the tightness of X, contradicts (5.8). So there exists an element Y of $\mathcal{C}$ such that $X_I \subseteq Y_O$. By $X_I \neq \{t\}$ and (5.14d), this element is unique. Then, by (5.15), (5.14d) and (5.14c), there are at least two distinct elements $A, B \in \mathcal{C}$ such that $A_I, B_I \subseteq (X_O \setminus \{t\}) \cap \overline{Y_I} = (X_I \cup w^\flat(X)) \cap \overline{Y_O} \subseteq w^\flat(X)$, a contradiction to $|w^\flat(X)| \leqslant 1$. □

The following lemma shows that to find an obstacle one does not have to focus on the disjointness of the inner-sets. Its proof relies on the classic uncrossing technique.

Lemma 5.5. *Let* $H = (V+s, E)$ *be a* $(2k,k)$*-connected graph in* V *with* $d_H(s)$ *even. If there exists a pair* $(t, \mathcal{F})$ *satisfying* (5.14a), (5.14b), (5.14c) *and* (5.14e) *then there exists a* t*-star obstacle at* s*.*

Proof. Choose a pair $(t, \mathcal{C})$ satisfying (5.14a), (5.14b), (5.14c) and (5.14e) such that $\sum_{X \in \mathcal{C}} |X_I|$ is minimal. Suppose there exist two distinct elements X and Y in $\mathcal{C}$ such that $X_I \cap Y_I \neq \emptyset$ that is $X \sqcap Y$ is a non-trivial bi-set of V. By choice of $\mathcal{C}$, $X \sqsubseteq Y$ or $Y \sqsubseteq X$ is not possible. Hence, by (5.14c), $X \sqcap \overline{Y}$ and $\overline{X} \sqcap Y$ are non-trivial bi-sets of V. By Proposition 5.1, $X \sqcup Y$ is a non-trivial bi-set of V. Hence, by Fact 2.3, $X \sqcap Y$, $X \sqcap \overline{Y}$ and $\overline{X} \sqcap Y$ are tight. The bi-sets among them which contain a neighbor of s are critical bi-sets with wall t. Hence replacing X and Y by these critical bi-sets in $\mathcal{C}$ contradicts the minimality of $\sum_{X \in \mathcal{C}} |X_I|$. □

5.3.4 *A New Splitting-off Theorem*

The aim of this section is to prove the following result.

Theorem 5.7. *Let $H = (V+s, E)$ be a $(2k,k)$-connected graph in V with $k \geqslant 2$ such that $d_H(s) \geqslant 4$ is even. There is a complete admissible splitting-off at s if and only if there exists no obstacle at s.*

As a first result we prove the existence of an obstacle when there exists no admissible pair at all.

Theorem 5.8. *Let $H = (V+s, E)$ be a graph that is $(2k,k)$-connected in V with $d_H(s)$ even and $k \geqslant 2$. If there exists no admissible splitting-off at s then $d_H(s) = 4$ and there exists an obstacle at s.*

Proof. Suppose that there exists no admissible splitting-off at s, that is, by Lemma 5.3, for each pair of edges incident to s, there exists a bi-set that blocks it.

Let X be a maximal blocking bi-set for a pair (su, sv) with $u \in X_I$. By (5.8), there exists a neighbor w of s in $\overline{X_O} \subseteq \overline{X_I}$. Let Y be a maximal blocking bi-set for the pair (su, sw). By Lemma 5.4, the wall of X and the wall of Y coincide and are reduced to a singleton, say $\{t\}$. By choice of $u \in X_I$ and $w \in \overline{X_O}$, t is different from u or w thus Y is a dangerous blocking bi-set.

For the same reasons, every maximal blocking bi-set for a pair (sa, sb) with $a \in Y_I$ and $b \in N_H(s) \cap \overline{Y_O} \neq \emptyset$ is a dangerous bi-set with wall $\{t\}$. By repeating this argument once more, we have that every pair (sa, sb) with $a, b \notin \{t\}$ is blocked by a dangerous bi-set with wall $\{t\}$. Hence, there exists a family $\mathcal{F}$ of (maximal) dangerous bi-sets such that (5.14c) holds for $\mathcal{F}$ and every pair of edges adjacent to s but not t is blocked by an element of $\mathcal{F}$.

Now consider the graph $H - t$ which is, by $(2k,k)$-connectivity of H in V, k-edge-connected in $V - t$. If (su', sv') is a pair of edges in $H - t$ then, by the definition of $\mathcal{F}$, there exists a dangerous bi-set $Z \in \mathcal{F}$ such that $u', v' \in Z_I$ and $w^\flat(Z) = \{t\}$. Hence $d_{(H-t)_{u',v'}}(Z_I) = d^\flat_{H_{u',v'}}(Z) = d^\flat_H(Z) - 2 \leqslant f^\flat_H(Z) - k|w^\flat(Z)| - 2 \leqslant k - 1$, that is splitting-off the pair (su', sv') destroys the k-edge-connectivity of $H - t$ in $V - t$.

Hence, since $k \geqslant 2$, by the theorem of Mader (Theorem 3.18), $d_{H-t}(s) = 3$. So, by $d_H(s)$ even and Claim 5.8, $d_H(s,t)$ is odd and smaller than $\frac{d_H(s)}{2}$. So $d_H(s,t) = 1$ and $d_H(s) = 4$. Hence, by (5.8), the inner-set of each element of $\mathcal{F}$ contains exactly two neighbors of s and $|\mathcal{F}| = 3$. So, for $X \in \mathcal{F}$, $X' = (\overline{X_I} - s, \overline{X_O} - s)$ is a non-trivial bi-set of V and X'_I contains exactly one neighbor of s, say x. We have $f^\flat_H(X') = f^\flat_H(X) - d_H(s, X_I) + d_H(s, V \setminus X_O) \leqslant 2k + 1 - 2 + 1 = 2k$ thus X' is a critical bi-set blocking (st, sx). So $(t, \mathcal{F}') = (t, \{X' : X \in \mathcal{F}\})$ satisfies (5.14a), (5.14b), (5.14c) and (5.14e). The obstacle at s is obtained by applying Lemma 5.5 on $(t, \mathcal{F}')$. $\square$

To prove the main result of this section we need to consider obstacles arising after an admissible splitting-off. The following lemma describes the two interesting situations that may occur.

Lemma 5.6. *Let $H = (V+s, E)$ be a $(2k,k)$-connected graph in V with $d_H(s) \geqslant 6$ even, (su, sv) an admissible pair in H and $(t, \mathcal{C})$ an obstacle at s in $H_{u,v}$.*

(a) If $t \in \{u, v\}$ then $d_H(s,t) \geqslant 2$ and (st, st) is admissible in H.

(b) If $t \notin \{u, v\}$ then either there exists a t-star obstacle at s in H or there exists no obstacle at s in $H_{t,w}$ for some admissible pair (st, sw) in H.

Proof. (a) Suppose $t = u$. By (5.14a) in $H_{u,v}$, $d_H(s,t) = d_{H_{u,v}}(s,t) + 1 \geqslant 2$.

Suppose now that (st, st) is non-admissible in H. Then, by Lemma 5.3, there exists in H a maximal blocking bi-set X for this pair. If v belongs to the inner-set of an element of $\mathcal{C}$ denote by Y this element and let $Y = (\emptyset, \emptyset)$

otherwise. Since $u \in X_I$, X is a blocking bi-set, Y is a critical or empty bi-set, by (5.9) and $d_H(s)$ even, we have,

$$\begin{aligned} d_{H_{u,v}}(s, X_I \cup Y_I) &\leqslant d_{H_{u,v}}(s, X_I) + d_{H_{u,v}}(s, Y_I) \\ &\leqslant (d_H(s, X_I) - 1) + d_{H_{u,v}}(s, Y_I) \\ &\leqslant (\frac{1}{2}d_H(s) - 1) + (\frac{1}{2}d_{H_{u,v}}(s) - 1) \\ &= d_{H_{u,v}}(s) - 1. \end{aligned}$$

So, by (5.14e), there exists $Z \in \mathcal{C} \setminus Y$ and $w \in N_{H_{u,v}}(s) \setminus (X_I \cup \{v\})$ such that $w \in Z_I$ and $v \notin Z_I$. Hence, Z is also a blocking bi-set for (st, sw) in H. Then, by Lemma 5.4 applied in H, $w^\flat(X) = w^\flat(Z) = \{t\}$ which contradicts the fact that X blocks the splitting (st, st) in H.

(b)

Claim 5.9. *If* st *belongs to no admissible pair in* H *then there exists a* t*-star obstacle in* H.

Proof. By $t \notin \{u, v\}$ and (5.14a), $d_H(s, t) = d_{H_{u,v}}(s, t)$ is odd thus it remains to construct a collection of critical bi-sets in H satisfying (5.14c), (5.14d) and (5.14e). By Lemma 5.5, it suffices to find one satisfying (5.14c) and (5.14e).

We initialize $\mathcal{F}$ as $\{X \in \mathcal{C}, |X_I \cap \{u, v\}| < 2\}$. Clearly $\mathcal{F}$ is a collection of critical bi-sets satisfying (5.14c). Suppose $\mathcal{F}$ does not satisfy (5.14e), that is, there exists $w \in N_H(s) \setminus (V_I(\mathcal{F}) \cup \{t\})$. Since st belongs to no admissible pair, by Lemma 5.3, there exists a maximal blocking bi-set X for the pair (st, sw). Now we prove that $w^\flat(X) = \{t\}$ that is X can be added into the collection $\mathcal{F}$ constructed so far.

Assume, by contradiction, that $t \in X_I$. We have $N_H(s) \cap V_I(\mathcal{F}) \subseteq X_I$ otherwise, there exists $Z \in \mathcal{F}$ such that $(N_H(s) \cap Z_I) \setminus X_I \neq \emptyset$, thus by Lemma 5.4, $w^\flat(X) = w^\flat(Z) = \{t\}$, a contradiction. Thus, by $t \in X_I$, (5.9), $d_H(s)$ even and $d_H(s) \geqslant 6$, we have,

$$\begin{aligned} d_{H_{u,v}}(s) - d_{H_{u,v}}(s, V_I(\mathcal{F}) \cup \{t\}) &\geqslant d_{H_{u,v}}(s) - d_{H_{u,v}}(s, X_I) \\ &\geqslant d_H(s) - 2 - d_H(s, X_I) \\ &\geqslant \frac{1}{2}d_H(s) - 2 \geqslant 1. \end{aligned}$$

Hence, by (5.14e) in $H_{u,v}$, there exists a unique $Y \in \mathcal{C} \setminus \mathcal{F}$ such that $N_{H_{u,v}}(s) \setminus \{t\} \subseteq V_I(\mathcal{F}) \cup Y_I$. Since $\{u, v\} \subseteq Y_I$, we have $N_H(s) \setminus \{t\} \subseteq V_I(\mathcal{F}) \cup Y_I$ and, in particular, $w \in Y_I$. If X is dangerous, $w \in X_I \cap Y_I$ and, by (5.9), we have $d_H(s, X_I) - d_H(s, X_I \cap Y_I) \leqslant \frac{1}{2}d_H(s) - 1$. If X is critical, by (5.9), we have $d_H(s, X_I) \leqslant \frac{1}{2}d_H(s) - 1$. Hence, by $N_H(s) \subseteq X_I \cup Y_I$, and by (5.9), we have the contradiction,

$$\begin{aligned} d_H(s) &= d_H(s, Y_I) + d_H(s, X_I) - d_H(s, X_I \cap Y_I) \\ &= d_{H_{u,v}}(s, Y_I) + 2 + d_H(s, X_I) - d_H(s, X_I \cap Y_I) \\ &\leqslant (\frac{1}{2}(d_H(s) - 2) - 1) + 2 + (\frac{1}{2}d_H(s) - 1) \\ &= d_H(s) - 1. \end{aligned}$$

□

Claim 5.10. *If* (st, sw) *is an admissible pair in* H *and there exists an obstacle* $(t', \mathcal{C}')$ *in* $H_{t,w}$ *then* $t = t'$.

Proof. Suppose $t \neq t'$.

Proposition 5.3. *Let* $X \in \mathcal{C}$ *and* $X' \in \mathcal{C}'$ *such that* $X \sqcap X'$ *is a non-trivial bi-set of* V. *Then* $t \in X'_I$ *or* $t' \in X_I$.

Proof. By contradiction assume that $t \notin X'_I$ and $t' \notin X_I$. Thus, by $t \neq t'$ and X' critical in $H_{t,w}$, we have $t \notin X'_O$ and $f^{\flat}_H(X') = f^{\flat}_{H_{t,w}}(X') = 2k$. Since X is critical in $H_{u,v}$ and by (5.11), we have, $f^{\flat}_H(X) - d^{\flat}_H(X \sqcup X') \leqslant f^{\flat}_H(X) - d_H(s, X_I) \leqslant f^{\flat}_{H_{u,v}}(X) - d_{H_{u,v}}(s, X_I) \leqslant 2k - 1$. Hence, since $|w^{\flat}(X \sqcup X')| = |\{t, t'\}| = 2$ and $X \sqcap X'$ is non-trivial, by (5.10), $0 \leqslant (f^{\flat}_H(X) - 2k) + (f^{\flat}_H(X') - 2k) - d^{\flat}_H(X \sqcup X') \leqslant -1$, a contradiction. □

Proposition 5.4. *There exists* $X \in \mathcal{C}$ *such that* $t' \in X_I$.

Proof. Suppose for a contradiction that $t' \notin V_I(\mathcal{C})$. Then by $t \neq t'$ and (5.14e) for $(t, \mathcal{C})$ in $H_{u,v}$, t' is not a neighbor of s in $H_{u,v}$ but a neighbor of s in $H_{t,w}$ (by (5.14a) for $(t', \mathcal{C}')$) so t' coincides with u or v, say u. By (5.15) and (5.14d) for $(t', \mathcal{C}')$ in $H_{t,w}$, there exists an element $X' \in \mathcal{C}'$ containing neither t nor v. Hence, there exists a vertex in $(N_H(s) \cap X'_I) \setminus \{t', u, v, t\}$ which, by (5.14e) for $(t, \mathcal{C})$ in $H_{u,v}$, is contained in the inner-set of an element $X \in \mathcal{C}$. Thus, $X' \sqcap X$ is a non-trivial bi-set of V such that $t \notin X'_I$ and $t' \notin X_I$, a contradiction to Proposition 5.3. □

By Proposition 5.4, there exists $X \in \mathcal{C}$ such that $t' \in X_I$. By (5.15) and (5.14d) for $(t, \mathcal{C})$ in $H_{u,v}$, there exists an element $Y \in \mathcal{C} \setminus X$ not containing w. Hence, there exists a vertex in $(N_H(s) \cap Y_I) \setminus \{t', w, t\}$ which, by (5.14e) in $H_{t,w}$, is contained in the inner-set of an element $X' \in \mathcal{C}'$. Thus $Y \sqcap X'$ is non-trivial, so by Proposition 5.3 and $t' \notin Y_I$, we have $t \in X'_I$.

Suppose that there exists a neighbor z of s in $H' = H - \{su, sv, sw\}$ that belongs to none of X_I and X'_I. Then, by (5.15), (5.14d), $t' \in X_I$ and $t \in X'_I$, there exists $Z \in \mathcal{C} \setminus X$ and $Z' \in \mathcal{C}' \setminus X'$ such that $z \in Z_I \cap Z'_I$. By $t' \in X_I$, $t \in X'_I$ and (5.14d), this contradicts Proposition 5.3 for Z and Z'. Hence, by (5.9), we have the following contradiction

$$\begin{aligned} d_H(s) - 3 &= d_{H'}(s) \\ &\leqslant d_{H'}(s, X_I) + d_{H'}(s, Y_I) \\ &\leqslant d_{H_{u,v}}(s, X_I) + d_{H_{t,w}}(s, Y_I) \\ &\leqslant \Big(\frac{d_{H_{u,v}}(s)}{2} - 1\Big) + \Big(\frac{d_{H_{t,w}}(s)}{2} - 1\Big) \\ &= d_H(s) - 4. \end{aligned}$$

□

Suppose there exists no t-star obstacle at s in H. Hence, by Claim 5.9, there exists an admissible pair (st, sw) in H. By Claim 5.10, if there exists an obstacle in $H_{t,w}$ then it is a t-star obstacle $(t, \mathcal{C}')$. By $t \notin \{u, v\}$ and (5.14a) in $H_{u,v}$, $d_H(s, t)$ is odd. Hence, by (5.14a) in $H_{t,w}$, $w = t$. Thus $(t, \mathcal{C}')$ is a t-star obstacle in H, a contradiction. □

Now we are in the position to prove our main result that characterizes the existence of a complete admissible splitting-off.

Proof of Theorem 5.7. Suppose there exists an obstacle $(t, \mathcal{C})$ at s. By (5.14a), every sequence of $\frac{1}{2}d_H(s)$ splitting-off of disjoint pairs at s contains a pair (st, su) with $u \in N_H(s) \setminus \{t\}$. As we noticed after the definition of an obstacle, such a pair is not admissible. Hence there exists no admissible complete splitting-off at s.

Now, we prove, by induction on $d_H(s)$, that if there exists no obstacle at s, then there exists an admissible complete splitting-off at s. For $d_H(s) = 2$, the only splitting-off is obviously admissible. Suppose $d_H(s) = 4$ and there exists no obstacle at s. By Theorem 5.8, there exists an admissible splitting-off (su, sv) at s. Since the only possible splitting-off in $H_{u,v}$ is admissible, there exists an admissible complete splitting-off at s in H.

Now suppose that the theorem is true for $d_{H'}(s) = 2\ell$ and $\ell \geqslant 2$. Let $H = (V+s, E)$ be a $(2k, k)$-connected graph in V such that $d_H(s) = 2\ell + 2 \geqslant 6$ and there exists no obstacle at s. By Theorem 5.8, there exists an admissible splitting-off (su, sv) at s. If there exists no obstacle at s in $H_{u,v}$, then, by induction, there exists an admissible complete splitting-off at s and we are done. So we may assume that there exists a t-star obstacle at s in $H_{u,v}$. Since there exists no obstacle at s in H, if case (b) of Lemma 5.6 occurs then there exists some admissible pair (st, sw) in H such that there exists no obstacle at s in $H_{t,w}$. Thus, by induction, there exists a complete splitting at s in H and we are done. So we may assume that case (a) of Lemma 5.6 occurs and we consider $H_{t,t}$ that is $(2k, k)$-connected in V. If there exists an obstacle $(t', \mathcal{C}')$ at s in $H_{t,t}$, for the same reason as above, case (a) of Lemma 5.6 occurs. Hence $t = t'$ and $(t, \mathcal{C}')$ is an obstacle in H, a contradiction. □

As we shall see in Subsection 5.3.6, having a polynomial time algorithm that provides either an obstacle or a complete admissible splitting-off at s would have an interesting application. However deriving such an algorithm from the proof of Theorem 5.7 does not seem straightforward.

Testing whether a given splitting-off is admissible can be done in polynomial time since this reduces to checking the $(2k, k)$-connectivity of a graph. The difficulty arises from the fact that there may exist an obstacle in the graph resulting from an admissible splitting-off (see Lemma 5.6). Hence a sequence of consecutive admissible splitting-off may result in a graph with no admissible splitting-off at s. Finding an algorithm that bounds (with a polynomial upper bound) the number of such sequences that are explored during the search for a complete splitting-off is not obvious.

5.3.5 *Construction of* $(2k, k)$*-Connected Graphs*

In this section we provide a construction of the family of $(2k, k)$-connected graphs for k even. The special case $k = 2$ has been previously proved by Jordán [41].

We need the following extension of Lemma 5.1 of [41] for k even. Let $G = (V, E)$ be a $(2k, k)$-connected graph, s a vertex of degree even, $(t, \mathcal{C})$ and $(t, \mathcal{C}')$ two obstacles at s. We say that $(t, \mathcal{C})$ is a *refinement* of $(t, \mathcal{C}')$ if there exists $X' \in \mathcal{C}'$ such that $X \sqsubseteq X'$ for all $X \in \mathcal{C}$. An obstacle that has no proper refinement is called *finest*.

Lemma 5.7. *Let $G = (V, E)$ be a $(2k, k)$-connected graph with k even. Let s be a vertex of degree $2k$ and $(t, \mathcal{C})$ a finest obstacle at s. Let $X \in \mathcal{C}$, s' a vertex in X_I of degree $2k$ and $(t', \mathcal{C}')$ an obstacle at s'. Then there exists $X' \in \mathcal{C}'$ such that ${X'}_I \subseteq X_I$.*

Proof. By contradiction assume that the lemma is false.

Suppose $t' \in X_I$. By assumption there exists $Y' \in \mathcal{C}'$ such that $Y'_I \setminus X_I \neq \emptyset$. Suppose that $t \notin Y'_I$, then $\overline{X} \sqcap Y'$ is non-trivial and $|w^\flat(\overline{X} \sqcup Y')| = |\{t, t'\}| = 2$. Hence, by (5.10) and since $\overline{X}$ and Y' are tight, we have $0 + 0 \geqslant d^\flat_G(\overline{X} \sqcup Y') \geqslant d_G(s', Y'_I) \geqslant 1$, a contradiction. Hence, $t \in Y'_I$. Now suppose that $X_O \cup Y'_O \neq V$ that is $\overline{X} \sqcap \overline{Y'}$ is non-trivial. We have $w^\flat(\overline{X} \sqcup \overline{Y'}) = |\{t, t'\}| = 2$, then, by (5.10) and since $\overline{X}$ and $\overline{Y'}$ are tight, we have $0 + 0 \geqslant d_G(Y'_I \cap \overline{X_I}, \overline{Y'_O} \cap$

$X_O) + d^\flat_G(\overline{X} \sqcup \overline{Y'}) \geqslant d_G(s', Y'_I) \geqslant 1$, a contradiction. Hence, $Y'_O \cup X_O = V$, and, for all $X' \in \mathcal{C}' - Y'$, $X'_I \subseteq X_I$, a contradiction. Hence we proved $t' \notin X_I$.

Suppose $t' \neq t$. If t belongs to an element $Z' \in \mathcal{C}'$ then, by (5.9), $d_G(s') - d_G(s', Z'_I) > 2k - k = d^\flat_G(X)$. Hence there exists $Y' \in \mathcal{C}'$ with $Y'_I \cap X_I \neq \emptyset$ and $t \notin Y'_I$. Thus $X \sqcap Y'$ is non-trivial and $|w^\flat(X \sqcup Y')| = |\{t, t'\}| = 2$. Since X and Y' are both tight, by (5.10) and (5.14a), $0 + 0 \geqslant d_G(\overline{X_O} \cap Y'_O, X_I \cap \overline{Y'_I}) \geqslant d_G(t', s') \geqslant 1$, a contradiction. Hence we proved that $t = t'$.

By $(2k,k)$-connectivity of G and $d_G(s') = 2k$, $d_G(s', t) \leqslant k$. Thus, by (5.14a) and k even, $d_G(s', t) < k$. Hence $d_G(s') - d_G(s', t) > 2k - k = d^\flat_G(X)$ and there exists $Y' \in \mathcal{C}'$ with $Y'_I \cap X_I \neq \emptyset$. By $|\mathcal{C}'| \geqslant 3$ and assumption, $X \sqcup Y', \overline{X} \sqcap Y'$ and $X \sqcap \overline{Y'} = \overline{\overline{X} \sqcup Y'}$ are non-trivial, thus, by Fact 2.3, $X \sqcap Y'$ and $X \sqcap \overline{Y'}$ are tight bi-sets with wall t. Thus, in $\mathcal{C}$, X can be replaced by the bi-sets among $X \sqcap Y'$ and $X \sqcap \overline{Y'}$ which contain at least one neighbor of s in their inner-set. Hence, $(t, \mathcal{C})$ is not a finest obstacle at s, a contradiction. □

We can now describe and prove the construction of the family of $(2k,k)$-connected graphs. We denote by kK_3 the graph on 3 vertices where each pair of vertices is connected by k parallel edges. Note that kK_3 is $(2k,k)$-connected and it is the only minimally $(2k,k)$-connected graph on 3 vertices.

Theorem 5.9. *A graph G is $(2k,k)$-connected with k even if and only if G can be obtained from kK_3 by a sequence of the following operations:*

(a) adding a new edge,

(b) pinching a set F of k edges such that, for all vertices v, $d_F(v) \leqslant k$.

Proof. First we prove the sufficiency, that is these operations preserve $(2k,k)$-connectivity. It is clearly true for (a). Let G' be a graph obtained from a $(2k,k)$-connected graph $G = (V, E)$ by the operation (b) and call s the new vertex. We must show that for every non-trivial bi-set X of $V + s$, we have $f^\flat_{G'}(X) \geqslant 2k$. If X is a non-trivial bi-set of V then $s \notin X_O$ and, by $(2k,k)$-connectivity of G, $f^\flat_{G'}(X) = d^\flat_{G'}(X) + k|w^\flat(X)| \geqslant d^\flat_G(X) + k|w^\flat(X)| = f^\flat_G(X) \geqslant 2k$. So, by symmetry of $f^\flat_{G'}$, we may assume that $X_I = \{s\}$ or $w^\flat(X) = \{s\}$. If $X_I = \{s\}$ then, by $d_{G'}(s) = 2k$ and $d_F(w^\flat(X)) \leqslant k$, we have $f^\flat_{G'}(X) = d^\flat_{G'}(X) + k|w^\flat(X)| = d_{G'}(s) - d_{G'}(s, w^\flat(X)) + k|w^\flat(X)| = d_{G'}(s) - d_F(w^\flat(X)) + k|w^\flat(X)| \geqslant 2k$. If $w^\flat(X) = \{s\}$ then $\emptyset \neq X_I \neq V$. Hence, by $(2k,k)$-connectivity of G and $|F| = k$, we have $f^\flat_{G'}(X) = d^\flat_{G'}(X) + k|w^\flat(X)| = d_G(X_I) - d_F(X_I) + k \geqslant d_G(X_I) - |F| + k \geqslant 2k$.

To see the necessity, let G be a $(2k,k)$-connected graph with at least 4 vertices. Note that the inverse operation of (a) is deleting an edge and that of (b) is a complete splitting-off at a vertex s of degree $2k$ such that $d_G(s, v) \leqslant k$ for all $v \in V$. Note also that these inverse operations must preserve $(2k,k)$-connectivity. Thus we may assume that, on the one hand, G is minimally $(2k,k)$-connected and hence, by Lemma 5.2, G contains a vertex of degree $2k$, and, on the other hand, for every such vertex u, there exists no admissible complete splitting-off at u, that is, by Theorem 5.7, there exists an obstacle at u.

We choose in $\{(u, (t, \mathcal{C}), X) : d_G(u) = 2k, (t, \mathcal{C}) \text{ a finest obstacle at } u, X \in \mathcal{C}\}$ a triple $(u^*, (t^*, \mathcal{C}^*), X^*)$ with X^* minimal for inclusion. By Lemma 5.2, there exists a vertex u' of degree $2k$ in X^*_I. Then, as we have seen, there exists a finest obstacle $(t', \mathcal{C}')$ at u'. By Lemma 5.7, there exists $X' \in \mathcal{C}'$ such that $X'_I \subseteq X^*_I$. Since $X'_I \cup u \subseteq X^*_I$, the triple $(u', (t', \mathcal{C}'), X')$ contradicts the choice of $(u^*, (t^*, \mathcal{C}^*), X^*)$. □

We mention that the condition k even is necessary in Lemma 5.7 and Theorem 5.9. Consider the graph obtained from K_4 by adding a new vertex t and 3 edges between t and each vertex of K_4 (see Figure 16). This graph is minimally $(6,3)$-connected but there exists no complete admissible splitting-off at any of the 4 vertices of degree 6. Indeed, if s, a, b, c denote the vertices of degree 6, then $\{(\{a,t\},\{a\}), (\{b,t\},\{b\}), (\{c,t\},\{c\})\}$, is a t-star obstacle at s.

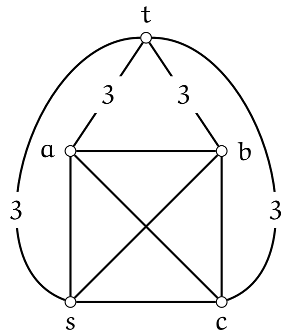


Figure 16: A minimally $(6,3)$-connected graph that has no admissible complete splitting at any vertex of degree 6.

5.3.6 *Augmentation*

Augmenting the connectivity of a graph means adding a minimum number of new edges such that the resulting graph satisfies a given connectivity requirement. Regarding global edge-connectivity, Watanabe and Nakamura [71] gave a polynomial time algorithm that solves the problem and then Cai and Sun [4] gave a min-max formulation. Frank [23] developed a new approach based on the spitting-off operation to address this problem as well as the augmentation problem for local edge-connectivity.

For local vertex-connectivity the problem turns to be NP-complete [59] and augmenting global vertex-connectivity remains an open problem. However, the particular case of 2-vertex-connectivity is solved independently by Plesnik [63] and Eswaran and Tarjan [17]. Furthermore Végh [70] showed how to augment the vertex-connectivity by one by adding a minimum number of edges.

In this section, we answer the following question for $k \geqslant 2$: given a graph what is the minimum number of edges to be added to make it $(2k,k)$-connected. For $k = 1$, the problem reduces to the 2-vertex-connectivity. However, we have no polynomial time algorithm to find this minimum set of edges since our approach is based on finding a complete admissible splitting-off (a problem for which we did not develop a polynomial time algorithm).

We shall need the following definitions. Let $G = (V, E)$ be a graph and k an integer. An *s-extension* of G is a graph $H = (V + s, E \cup F)$ where F is a set of edges between V and the new vertex s. The *size* of an s-extension of G is defined by $|F|$.

We mimic the approach of Frank [24] for the augmentation problem: first we prove a result on minimal extensions and then, by applying our splitting-off theorem, we get a result on minimal augmentation.

Lemma 5.8. *Let $G = (V, E)$ be a graph and k an integer. The minimal size of an s-extension of G that is $(2k,k)$-connected in V is equal to* $\max\left\{\sum_{X\in\mathcal{X}}(2k - f^{b}_{G}(X))\right\}$ *where $\mathcal{X}$ is a family of non-trivial pairwise innerly-disjoint bi-sets of V.*

Proof. If $H' = (V+s, E\cup F')$ is an s-extension of G that is $(2k,k)$-connected in V and $\mathcal{X}'$ is an arbitrary family of non-trivial pairwise innerly-disjoint bi-sets of V then $\sum_{X'\in\mathcal{X}'}(2k - f^{\flat}_G(X')) \leqslant \sum_{X'\in\mathcal{X}'}(f^{\flat}_{H'}(X') - f^{\flat}_G(X')) = \sum_{X'\in\mathcal{X}'} d^{\flat}_{(V+s,F')}(X') \leqslant |F'|$. This shows that max $\leqslant$ min.

To prove that equality holds, we provide a family $\mathcal{X}$ of non-trivial pairwise innerly-disjoint bi-sets of V and an s-extension of G that is $(2k,k)$-connected in V of size $\sum_{X\in\mathcal{X}}(2k - f^{\flat}_G(X))$. We consider the s-extension of G whose set of new edges consists of $\max_{\mathcal{X}}(2k - f^{\flat}_G(X))$ parallel edges sv, for each $v\in V$. This extension is obviously $(2k,k)$-connected in V. Then we remove as many new edges as possible without destroying the $(2k,k)$-connectivity in V. Let us denote by F the set of remaining edges and $H = (V+s, E\cup F)$. In H, by minimality of F, for each $e\in F$, there exists a tight bi-set of V entered by e. Let $\mathcal{X}$ be a family of non-trivial tight bi-sets of V such that

$$\text{each edge of } F \text{ enters at least one element of } \mathcal{X} \text{ and} \tag{5.17}$$

$$\textstyle\sum_{X\in\mathcal{X}} |X_I| \text{ is minimal.} \tag{5.18}$$

Claim 5.11. *The elements of $\mathcal{X}$ are pairwise innerly-disjoint.*

Proof. Note that, the degree of each tight bi-set X in $\mathcal{X}$ is at least one thus $|w^{\flat}(X)| \leqslant 1$. Suppose there exist two distinct elements X and Y in $\mathcal{X}$ such $X_I\cap Y_I \neq \emptyset$, that is $X\sqcap Y$ is a non-trivial bi-set of V.

If $X\sqcup Y$ is a non-trivial bi-set of V then, by $(2k,k)$-connectivity in V of H, tightness of X and Y and Fact 2.3, $X\sqcup Y$ is tight. Since all the edges of F entering X_I or Y_I enters $(X\sqcup Y)_I$, the family obtained from $\mathcal{X}$ by substituting $X\sqcup Y$ for X and Y satisfies (5.17) and, by $X_I\cap Y_I\neq\emptyset$, contradicts (5.18). So $X_O\cup Y_O = V$.

If $X\sqcap\overline{Y}$ and $\overline{X}\sqcap Y$ are non-trivial bi-sets of V then, by $(2k,k)$-connectivity in V of H, tightness of X and Y and Fact 2.3, both $X\sqcap\overline{Y}$ and $\overline{X}\sqcap Y$ are tight and $d_H(\overline{X_O}\cap\overline{Y_I}, X_I\cap Y_O) = d_H(Y_I\cap X_O, \overline{Y_O}\cap\overline{X_I}) = 0$. Hence all the edges of F entering X_I or Y_I enters $(X\sqcap\overline{Y})_I$ or $(\overline{X}\sqcap Y)_I$. Thus the family obtained from $\mathcal{X}$ by substituting $X\sqcap\overline{Y}$ and $\overline{X}\sqcap Y$ for X and Y satisfies (5.17) and, by $X_I\cap Y_I\neq\emptyset$, contradicts (5.18). So, by symmetry, we may assume that $X_I\subseteq Y_O$.

We have $N_H(s)\cap X_I \not\subseteq Y_I$ otherwise $\mathcal{X} - X$ satisfies (5.17) and contradicts the minimality of $\mathcal{X}$. Thus, by $X_I\subseteq Y_O$, $d_H(s, w^{\flat}(Y)) \geqslant 1$ and, since $X_O\cup Y_O = V$ and Y is non-trivial, $w^{\flat}(X)\setminus Y_O = X_O\setminus Y_O = (X_O\cup Y_O)\setminus Y_O = V\setminus Y_O$ is non-empty. So $|w^{\flat}(\overline{X}\sqcup Y)| \geqslant 2$.

For the same reason as above, $N_H(s)\cap Y_I\not\subseteq X_I$. Thus, by $|w^{\flat}(X)|\leqslant 1$ and $w^{\flat}(X)\setminus Y_O\neq\emptyset$, the set $Y_I\setminus X_O = Y_I\setminus X_I$ contains a neighbor of s, that is $\overline{X}\sqcap Y$ is non-trivial. Thus, by symmetry of $f^{\flat}_H$, tightness of X and Y and (5.10), we have the following contradiction $0+0 = (f^{\flat}_H(\overline{X}) - 2k) + (f^{\flat}_H(Y) - 2k) \geqslant d_H(X_I\cap Y_O, \overline{X_O}\cap\overline{Y_I}) \geqslant d_H(s, w^{\flat}(Y)) \geqslant 1$. □

By Claim 5.11, (5.17) and by tightness of the elements of $\mathcal{X}$, we have $|F| = \sum_{X\in\mathcal{X}} d^{\flat}_{(V+s,F)}(X) = \sum_{X\in\mathcal{X}}(f^{\flat}_H(X) - f^{\flat}_G(X)) = \sum_{X\in\mathcal{X}}(2k - f^{\flat}_G(X))$. □

The augmentation theorem goes as follows.

Theorem 5.10. *Let $G = (V,E)$ be a graph and $k\geqslant 2$ an integer. The minimum cardinality γ of a set F of edges such that $(V, E\cup F)$ is $(2k,k)$-connected is equal to*

$$\alpha = \left\lceil \frac{1}{2}\max\left\{\sum_{X\in\mathcal{X}}(2k - f^{\flat}_G(X))\right\}\right\rceil,$$

where $\mathcal{X}$ is a family of non-trivial pairwise innerly-disjoint bi-sets of V.

Proof. We first prove $\gamma \geqslant \alpha$. Let $\mathcal{X}$ be a family of non-trivial bi-sets of V such that the elements of $\mathcal{X}$ are pairwise innerly-disjoint. For each $X \in \mathcal{X}$, we must add at least $2k - f^{\flat}_G(X)$ new edges entering the bi-set X when this quantity is positive. Since the elements of $\mathcal{X}$ are pairwise innerly-disjoint, a new edge may enter at most 2 elements of $\mathcal{X}$. Hence $2\gamma \geqslant \sum_{X\in\mathcal{X}}(2k - f^{\flat}_G(X))$.

We now prove $\gamma \leqslant \alpha$. By Lemma 5.8, there exists an s-extension $H = (V+s, E\cup F)$ of G that is $(2k,k)$-connected in V and a family $\mathcal{X}$ of non-trivial pairwise innerly-disjoint bi-sets of V such that

$$|F| = \sum_{X\in\mathcal{X}} (2k - f^{\flat}_G(X)).$$

If $|F|$ is odd, then there exists a vertex $u \in V$ such that $d_H(s,u)$ is odd, in this case, let $F' = F \cup su$ otherwise let $F' = F$. So, in the graph $H' = (V\cup s, E\cup F')$, $d_{H'}(s)$ is even. Suppose there exists an obstacle $(t, \mathcal{C})$ at s. By Claim 5.16, $H' - st$ is $(2k,k)$-connected in V. If $H = H'$ this contradicts the minimality of $|F|$. Then $d_H(s)$ is odd and $F' = F + su$ for some vertex $u \in V$ such that $d_H(s,u)$ is odd. If $u \in X_I$ for some $X \in \mathcal{C}$, then we have $f_H(X) = f_{H'}(X) - 1 = 2k - 1$, a contradiction to the $(2k,k)$-connectivity of H. Thus, by (5.14e), $u = t$ and hence $d_{H'}(s,t) = d_H(s,t) + 1$ is even which contradicts (5.14a). Hence, by Theorem 5.7, there exists an admissible complete splitting-off at s in H'. Let us denote by F'' the set of edges obtained by this complete splitting-off. Then $(V, E\cup F'')$ is $(2k,k)$-connected and

$$|F''| = \frac{1}{2}|F'| = \left\lceil \frac{1}{2}|F| \right\rceil = \left\lceil \frac{1}{2} \sum_{X\in\mathcal{X}} (2k - f^{\flat}_G(X)) \right\rceil .$$

This proves $\gamma \leqslant \alpha$. □

Following the above proof, having a polynomial time algorithm that finds a complete admissible splitting-off when it exists would Developping an algorithm that finds a set F of edges such that $(V, E\cup F)$ is $(2k,k)$-connected in polynomial time

6 ON FRANK'S CONJECTURES

In this chapter we disprove Conjectures 5.2 and 5.3 of Frank. Moreover, we show that, given $k \geqslant 3$, the problem of deciding whether a graph has a k-vertex-connected orientation and the problem of deciding whether a graph has a rooted k-vertex-connected orientation at a given vertex are both NP-complete. This work has been conducted when I was visiting University of Waterloo. The material of this section is from [6].

6.1 PRELIMINARIES

In this section we provide tools that are used to prove connectivity properties. In all this section $G = (V, E)$ is a graph, $D = (V, A)$ is a digraph and k is a positive integer.

First, we extend the definitions of fans and difans introduced in Section 2.2. For $U \subset V$, a pair of paths of G (resp. dipaths of D) is called U-disjoint if each vertex of U is contained in at most one path (resp. dipath). Let X and Y be two disjoint vertex sets. A *k-fan joining X and Y* (resp. a *k-difan from X to Y*) is a set of k pairwise U-disjoint paths joining X and Y (resp. dipaths from X to Y) where U is defined by

$$U = \begin{cases} V \setminus (X \cup Y) & \text{if } |X| = |Y| = 1, \\ V \setminus X & \text{if } |X| = 1 \text{ and } |Y| > 1, \\ V \setminus Y & \text{if } |Y| = 1 \text{ and } |X| > 1, \\ V & \text{if } |X| > 1 \text{ and } |Y| > 1. \end{cases}$$

We recall, from Subsection 2.2.4, that the weak $2k$-connectivity, the mixed $(2k, 2)$-connectivity and the g-bounded $2k$-connectivity where $g = 2$ coincide and, two vertices are weakly $2k$-connected if there exists 2 edge-disjoint k-fans joining them. In G, a set of vertices is called *weakly $2k$-connected* if every pair of vertices contained in this set is weakly $2k$-connected. We do not prove the following statements that derive from Theorem 2.2. If X is a weakly $2k$-connected set of at least k vertices and v is a vertex in $V \setminus X$ such that there exist 2 edge-disjoint k-fans joining v and X then $X \cup v$ is weakly $2k$-connected. If X and Y are two disjoint weakly $2k$-connected sets each of at least k vertices such that there exist 2 edge-disjoint k-fans joining X and Y then $X \cup Y$ is weakly $2k$-connected.

As for graphs, in digraphs the existence of k-difans proves k-vertex-connectivity. In D, a set of vertices is called *k-vertex-connected* if for every pair of vertices u, v of this set there exist a k-difan from u to v and a k-difan from v to u. Again the following statements are direct corollaries of of Theorem 2.3. If X is a k-vertex-connected set of at least k vertices and v is a vertex in $V \setminus X$ such that there exist a k-difan from X to u and a k-difan from u to X then $X \cup v$ is k-vertex-connected. If X and Y are two disjoint k-vertex-connected sets each of at least k vertices such that there exist a k-difan from X to Y and a k-difan from Y to X then $X \cup Y$ is k-vertex-connected.

The constructions in this chapter are based on the following easy observation deriving from Theorem 2.3.

Fact 6.1. *Let $G = (V, E)$ be a graph and S, T two disjoint sets of vertices such that $|T| = k - 1$ and $S \cup T \neq V$. Then for any k-vertex-connected orientation D of G, in $D - T$, at least one arc enters S and at least one arc leaves S.*

Fact 6.2. *Let $G = (V, E)$ be a graph and P a path of simple edges such that each inner vertex v of P satisfies*

$$d_G(v) = 2k \text{ and} \tag{6.1}$$

$$v \text{ has exactly } k+1 \text{ neighbors.} \tag{6.2}$$

Then in any k-vertex-connected orientation of G the orientation of the edges of P results in a dipath.

Note that this fact holds also for cycles considered as closed paths where every vertex is an inner-vertex. The next results is well known [1] and can be proved by induction.

Proposition 6.1. *For every positive integer k the complete graph on $2k + 1$ vertices K_{2k+1} has a k-vertex-connected orientation.*

6.2 GLOBAL VERTEX-CONNECTIVITY

6.2.1 *Counterexamples*

First, we disprove Conjecture 5.2 for $k = 3$, and then, we extend the construction to higher connectivity. For $k = 3$, the graph G_3 defined in Figure 17 is claimed to be a counterexample.

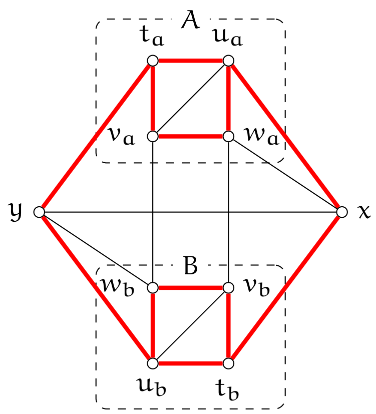


Figure 17: The graph G_3. Every thick and red edge represents a pair of parallel edges and black edges represent simple edges.

Proposition 6.2. *The graph G_3 is weakly 6-connected and has no 3-vertex-connected orientation.*

Proof. First we show that G_3 is weakly 6-connected. Observe that there exist 2 edge-disjoint 3-fans joining any pair of vertices in $A \setminus w_a$. Then, note that there exist 2 edge-disjoint 3-fans joining w_a and $A \setminus w_a$. Hence A is weakly 6-connected. Symmetrically, B is also weakly 6-connected. There exist 2 edge-disjoint 3-fans joining A and B so $A \cup B$ is weakly 6-connected. There exists 2 edge-disjoint 3-fans joining x (resp. y) and $A \cup B$. It follows that G_3 is weakly 6-connected.

Suppose for a contradiction that G_3 has a 3-vertex-connected orientation D. Since each vertex of the path $P = v_a w_b y x w_a v_b$ satisfies (6.1) and (6.2) for $k = 3$ and, by Fact 6.2, the orientation in D of the edges of P results in a directed path from v_a to v_b or from v_b to v_a. In particular, both $v_a w_b$ and $v_b w_a$ are directed from A to B or from B to A. In both cases $D - \{x, y\}$ is not strongly connected, a contradiction. □

Szigeti observed that G_3 is not a minimal counterexample. Indeed, the graph H_3 obtained from G_3 by deleting the two vertices t_a and t_b and adding the new edges $u_a v_a$, $v_a y$, $y u_a$, $u_b v_b$, $v_b x$ and $x u_b$ is weakly 6-connected but has no 3-vertex-connected orientation (see Figure 18). (Suppose that H_3 has a 3-vertex-connected orientation D. Then, by Fact 6.2, in D the orientation of the edges of the two triangles $v_a y w_b$ and $v_b x w_a$ results in circuits. Considering the cut $\{x, y\}$, we see that these circuits must be either both clockwise or both counterclockwise, say clockwise. Hence, by Fact 6.2, in D the orientation of the path $u_a y x u_b$ results in a dipath from u_a to u_b or from u_b to u_a. In the first case $D - \{y, v_b\}$ is not strongly connected, in the other case $D - \{x, v_a\}$ is not strongly connected.)

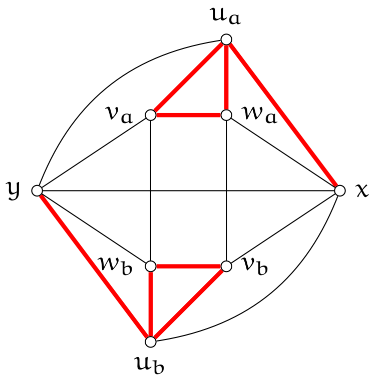


Figure 18: The graph H_3. A weakly 6-connected graph having no 3-vertex-connected orientation.

We now extend the construction of counterexamples to higher connectivity. Let $k \geqslant 4$ be an integer. We define the graph $G_k = (V, E)$ as follows (see Figure 19). Let $n \geqslant \max\{2k+1, 3k-4\}$ be an odd integer. Let A and B be two sets of n vertices such that $|A \cap B| = k - 3$. The vertex set V is the union of A, B and 4 new vertices w, x, y and z. Now we add edges such that each of A and B induces a complete simple graph and we add pairs of parallel edges between vertices in $(A \cup B) \setminus (A \cap B)$ and $\{w, x, y, z\}$ such that

$$
\begin{aligned}
&\text{each vertex of} \quad A \cup B \quad \text{is incident to at most one pair of parallel edges,} \\
&\qquad d_{G_k}(w, A) \quad = \quad d_{G_k}(z, B) = 2k - 2, \\
&\qquad d_{G_k}(y, A) \quad = \quad d_{G_k}(x, B) = 2 \text{ and} \\
&\qquad d_{G_k}(x, A) \quad = \quad d_{G_k}(y, B) = 2k - 4.
\end{aligned}
$$

Since $n \geqslant 3k - 4$, this is possible and we can also choose $a \in A \setminus B$ and $b \in B \setminus A$ such that neither a nor b is a neighbor of $\{w, x, y, z\}$. We conclude the construction by adding the path $P = azyxwb$.

Proposition 6.3. *Let $k \geqslant 4$ be an integer. The graph $G_k = (V, E)$ is weakly $2k$-connected and there exists $F \subset E$ such that doubling each edge in F results in an Eulerian graph that has no k-vertex-connected orientation.*

Proof. Since $n \geqslant 2k + 1$ both of the complete graphs induced by A and B are weakly $2k$-connected. Note that there exist 2 edge-disjoint k-fans

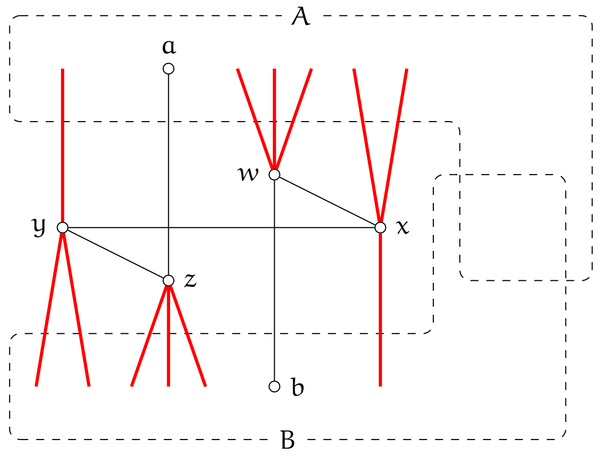


Figure 19: G_4 every thick and red edge represents a pair of parallel edges and black edges represent simple edges.

joining A and $B \setminus A$ (one uses $(A \cap B) \cup \{w, x, y\}$ the other one uses $(A \cap B) \cup \{x, y, z\}$), thus $A \cup B$ is weakly $2k$-connected. Note also that, for any vertex $v \in \{w, x, y, z\}$, there exist 2 edge-disjoint k-fans joining v and $A \cup B$. Hence, G_k is weakly $2k$-connected.

Since n is odd each vertex in $A \setminus (B \cup a)$ or $B \setminus (A \cup b)$ or $\{w, x, y, z\}$ has even degree. Pick arbitrarily a vertex $c \in (A \cap B)$. We define F as the union of $\{ac, cb\}$ and, if the degree of the vertices in $A \cap B$ is odd, a perfect matching of $A \cap B$.

Suppose for a contradiction that $G_k + F$ has a k-vertex-connected orientation D. By Fact 6.2, since each inner vertex of P satisfies (6.1) and (6.2), the orientation of the edges P results in the dipath $azyxwb$ or the dipath $bwxyza$. In both cases, $D - ((A \cap B) \cup \{x, y\})$ is not strongly connected, a contradiction. □

6.2.2 *NP-completeness*

In this section we prove the following result.

Theorem 6.1. *Let $k \geqslant 3$ be an integer. Deciding whether a graph has a k-vertex-connected orientation is NP-complete. This holds also for Eulerian graphs.*

A *reorientation* of a digraph D is a digraph obtained from D by reversing a subset of arcs. Obviously, the problem of finding a k-vertex-connected orientation of a graph and the problem of finding a k-vertex-connected reorientation of a digraph are equivalent. For convenience we prove the NP-completeness of the second problem by giving a reduction from the problem of NOT-ALL-EQUAL 3-SAT which is known to be NP-complete [66].

Actually in this Chapter we consider a slight variation of the classic NOT-ALL-EQUAL 3-SAT defined in Subsection 4.1.3. In this classic version each clause consists of three non-negated variables and an assignment true-false of the variables satisfies the instance if and only if each clause contains both a true and a false value. Here we consider that the clauses may contain negated variables. Obviously this preserves the NP-completeness of the problem and this will enables us to use the same construction for the rooted vertex-connected orientation problem.

Let Π be an instance of NOT-ALL-EQUAL 3-SAT and let $k \geqslant 3$ be an integer. We define a directed graph $D_k = D_k(\Pi) = (V, A)$ satisfying the following fact.

Claim 6.1. *There exists a* k*-vertex-connected reorientation of* D_k *if an only if there is an assignment of the variables which satisfies* Π.

The construction of D_k associates to each variable x a circuit Δ_x and to each pair (C, x), where x is a variable that appears in the clause C, a special arc e_x^C (see Figure 20). A reorientation of D_k is called *consistent* if, for each variable x, the orientations of the special arcs of type e_x^C and the circuit Δ_x are either all preserved or all reversed. A consistent reorientation of D_k defines a *natural assignment* of the variables in which a variable x receives value true if Δ_x is preserved and false if Δ_x is reversed.

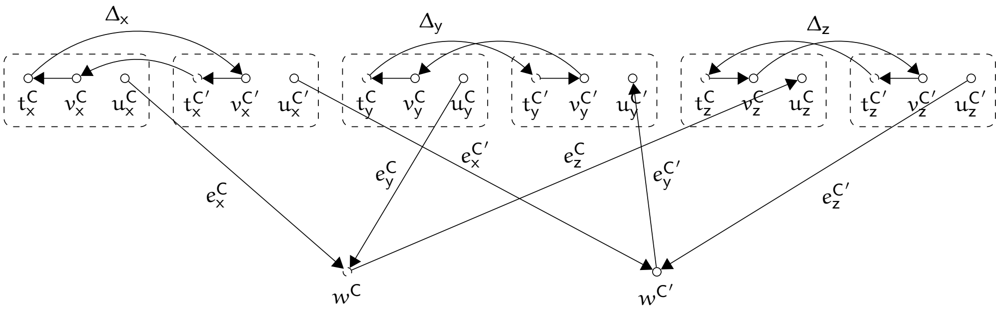


Figure 20: Representation of the circuits and the special arcs of $D_3(\Pi)$ where Π is composed of the clauses $C = (x, y, \bar{z})$ and $C' = (x, \bar{y}, z)$. The dashed boxes represent the clause-variable gadgets.

For each clause C we construct a C-gadget (see Figure 21 (a)) that uses the special arcs associated to C and a vertex w^C. The following property will result from the C-gadgets.

Proposition 6.4. *Let* D' *be a consistent reorientation of* D_k *and* Ω *the natural assignment defined by* D'*. Then* Ω *satisfies* Π *if and only if, for each clause* C*, there exist at least one special arc entering* w^C *and at least one special arc leaving* w^C.

For each pair (C, x) where C is a clause and x is a variable that appears in C we define a (C, x)-gadget (see Figure 21 (b)) which links the orientation of Δ_x to the orientation of e_x^C. We will prove the following fact.

Proposition 6.5. *Every* k*-vertex-connected reorientation of* D_k *is consistent.*

Let L be a set of $k-1$ vertices. We construct a clause gadget as follows. For a clause C composed of the variables x, y and z we add the vertices w^C, u_x^C, u_y^C, u_z^C. We add arcs such that $L \cup w^C$ induces a complete digraph. We add the special arc $u_x^C w^C$ if $x \in C$ and the special arc $w^C u_x^C$ if $\bar{x} \in C$. This special arc is denoted by e_x^C. We define similarly the special arcs e_y^C and e_z^C. This ends the construction of the C-gadget. Let W denote the set of all vertices of type w^C.

Let M be a set of $k-2$ new vertices and choose arbitrarily one vertex $m \in M$. For each pair (C, x) where C is a clause and x is a variable that appears in C we add the new vertices $t_x^C, t_x'^C, v_x^C$ and denote U_x^C a set containing u_x^C and $2k$ new vertices. In $U_x^C \setminus u_x^C$ we choose two vertices $u_x'^C$ and $u_x''^C$ and a set A_x^C of $k-1$ vertices. We add arcs such that U_x^C induces a k-vertex-connected digraph (Proposition 6.1), $k-2$ vertex-disjoint arcs from M to A_x^C and $k-2$ vertex-disjoint arcs from A_x^C to M. We add pairs of opposed arcs between the pairs of vertices $(t_x^C, t_x'^C)$, $(t_x^C, u_x''^C)$, $(v_x^C, u_x'^C)$, $(v_x^C, u_x''^C)$ and the pairs of type (t_x^C, m') and (v_x^C, m') for each $m' \in M \setminus m$. Note that, so far, the undirected degree of t_x^C and v_x^C is $2k-2$. We add an arc $v_x^C t_x^C$ if $x \in C$ and an arc $t_x^C v_x^C$ if

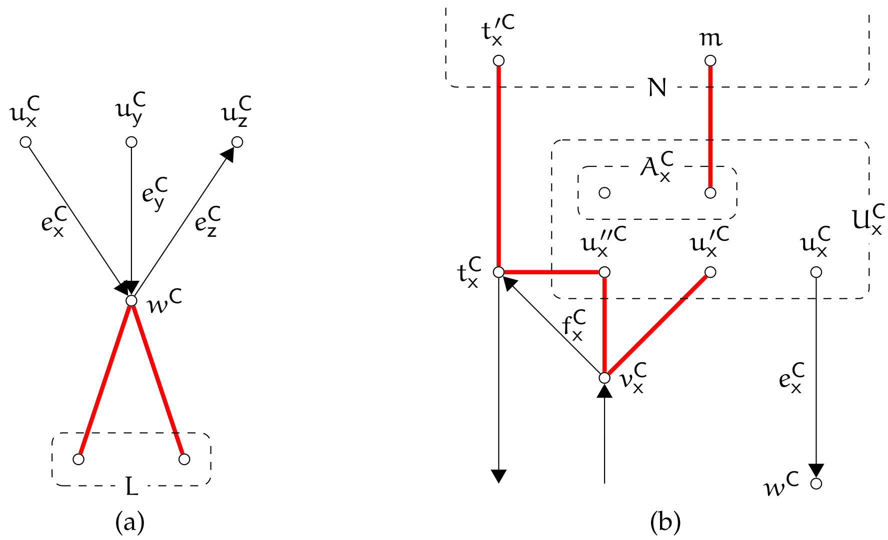


Figure 21: (a) A clause gadget for $k = 3$ and $C = (x, y, \bar{z})$. (b) A (C, x)-gadget for $k = 3$. In both pictures, each red and thick edge represents a pair of opposed arcs.

$\bar{x} \in C$. Call this arc f_x^C. The definition of the (C, x)-gadget is concluded by the following definition of the circuit Δ_x.

For each variable x add arcs such that the set of vertices of type t_x^C and v_x^C induce a circuit Δ_x that traverses (in arbitrary order) all the (C, x)-gadgets where C is a clause containing x. In this circuit connect a (C, x)-gadget to the next (C', x)-gadget by adding an arc leaving the head of f_x^C and entering the tail of $f_x^{C'}$ (see Figure 20). Note that now the undirected degree of t_x^C and v_x is $2k$.

We denote by N the union of L, M and all the vertices of type $t_x'^C$. By adding new vertices in N if necessary, we may assume that $|N|$ is odd and larger that $2k$. To conclude the definition of D_k we add arcs such that N induces a k-vertex-connected digraph (Proposition 6.1).

The proof of Proposition 6.4 follows from the definition of the C-gadgets.

Proof of Proposition 6.4. Let e_x^C be a special arc associated to a clause C and a variable x. In D', the arc e_x^C enters w_C if and only if either $x \in C$ and e_x^C had been preserved that is $x =$ true or $\bar{x} \in C$ and e_x^C has been reversed that is $x =$ false. Hence e_x^C enters w^C if and only if the variable x brings to C a value true. Thus C is satisfied if and only if there exist a special arc entering w^C and a special arc leaving w^C. □

The proof of Proposition 6.5 follows from the definition of the (C, x)-gadgets.

Proof of Proposition 6.5. Let D' be a k-vertex-connected reorientation of D_k and let x be a variable. Observe that each vertex incident to Δ_x satisfies (6.1) and (6.2) in the underlying undirected graph. Hence, by Fact 6.2, Δ_x is either preserved or reversed. Let C be a clause in which x appears. In $D' - (M \cup t_x^C)$ exactly one arc enters $U_x^C \cup v_x^C$ and exactly one arc leaves $U_x^C \cup v_x^C$ (see Figure 21 (b)). One of these arcs belongs to Δ_x and the other is the special arc e_x^C. Hence, by k-vertex-connectivity of D', e_x^C is reversed if and only if Δ_x is reversed. □

The following fact derives easily from the definition of D_k. We recall that W is the set of vertices of type w^C.

Proposition 6.6. *Let D' be a consistent reorientation of D_k such that the orientation of each non special arc that belongs to no circuit of type Δ_x is preserved. Then the set $V \setminus W$ is k-vertex-connected in D'.*

Proof. Let C be a clause and x be a variable that appears in C. Without loss of generality we may assume that x appears in an other clause C'. The circuit Δ_x contains a dipath from (resp. to) t_x^C to (resp. from) $t_x'^{C'}$ that is disjoint from $M \cup t_x'^C$. Hence $N \cup t_x^C$ is k-vertex-connected.

Since D' is consistent, we may assume without loss of generality that e_x^C leaves u_x^C and f_x^C leaves v_x^C (see figure 21 (b)). Observe that there is a k-difan from $M \cup t_x'^{C'} \cup t_x^C$ to U_x^C (the dipath from $t_x'^{C'}$ uses arcs of Δ_x and v_x^C) and a k-difan from U_x^C to $M \cup t_x^C \cup L$ (the dipath to L uses the arc e_x^C). Hence, since M and L are subsets of N, $N \cup t_x^C \cup U_x^C$ is k-vertex-connected. The circuit Δ_x contains a dipath from (resp. to) v_x^C to (resp. from) t_x^C that is disjoint from $M \cup U_x^C$. Hence $N \cup t_x^C \cup U_x^C \cup v_x^C$ is k-vertex-connected and the proposition follows. □

We can now prove Claim 6.1.

Proof of Claim 6.1. Let D' be a k-vertex-connected reorientation of D_k. By Proposition 6.5, D' is consistent and defines a natural assignment Ω. By Fact 6.1 in $D' - L$ one special arc enters w^C and one special arc leaves w^C for each clause C. Thus, by Proposition 6.4, Ω is satisfies Π.

Let Ω be an assignment of the variables that satisfies Π. Call D' the reorientation of D_k obtained by reversing the circuit Δ_x and the special arcs e_x^C for each variable x assigned false. By Proposition 6.6, $V \setminus W$ is k-vertex-connected in D'. By Proposition 6.4, for each clause C, there are a special arc entering w^C and a special arc leaving w^C, hence D' is k-vertex-connected. □

Denote by $G'_k = G'_k(\Pi)$ the underlying undirected graph of $D_k(\Pi)$. We can now prove the main theorem of this section.

Proof of Theorem 6.1. By Claim 6.1, $G'_k(\Pi)$ has a k-vertex-connected orientation if and only if there exists an assignment satisfying Π. Since the order of $G'(\Pi)$ is a linear function of the size of Π and NOT-ALL-EQUAL 3-SAT is NP-complete (Theorem 4.6) this proves the first part of Theorem 6.1.

Observe that in G'_k the only vertices of odd degree are of type u_x^C and w^C. Let l be an arbitrary vertex of L. We can add a set F of edges of type $u_x^C m, ml, lw^C$ such that $G'_k + F$ is Eulerian. Observe that for any orientation of F, Claim 6.1 still hold for $D_k + F$. This proves the second part of Theorem 6.1. □

The following fact shows that $G'_k(\Pi)$ is a counterexample to Conjecture 5.2 if Π is not satisfiable.

Proposition 6.7. *The graph $G'_k(\Pi)$ is weakly $2k$-vertex-connected.*

Proof. By Proposition 6.6, $V \setminus W$ is k-vertex-connected in D_k, thus $V \setminus W$ is weakly $2k$-connected in G'_k. Since there exist 2 edge-disjoint k-fans from w^C to $V \setminus W$ for every clause C, G'_k is weakly $2k$-connected. □

We now construct an Eulerian counterexample to Conjecture 5.2 for $k = 3$. Let x be a variable and $C = (x, x)$ be a clause. Let H'_3 be the Eulerian graph obtained from $G'_3(\{C\})$ by adding an edge $u_x^C m$ in each of the two copies of the (C, x)-gadget. The next result follows from the discussion above.

Proposition 6.8. *H'_3 is an Eulerian weakly 6-connected graph that has no 3-vertex-connected orientation.*

6.3 ROOTED VERTEX-CONNECTIVITY

In this section we mimic the previous section to prove the following result and derive a counterexample to Conjecture 5.3.

Theorem 6.2. *Let $k \geqslant 3$ be an integer. Deciding whether a rooted graph has rooted k-vertex-connected orientation at given vertex is NP-complete.*

The proof of Theorem 6.2 follows the proof of Theorem 6.1 and relies on reducing the problem of 3-SAT to the problem of finding a rooted k-vertex-connected reorientation of a digraph. Let Π be an instance of 3-SAT and $k \geqslant 3$ be an integer. Recall the digraph $D_k(\Pi)$ defined in subsection 6.2.2. We define $D_k^r(\Pi) = (V, A)$ as the digraph obtained from $D_k(\Pi)$ by adding the root vertex r and an arc rv for each $v \in N$. In the rest of this chapter all the rooted reorientation of D_k^r are considered rooted at r. A reorientation of $D_k^r(\Pi)$ is called *consistent* if it induces a consistent reorientation of $D_k(\Pi)$. The proof of the following result is very similar to the proof of Claim 6.1.

Claim 6.2. *There exists a rooted k-vertex-connected reorientation of D_k^r if and only if there is an assignment of the variables which satisfies Π.*

The proof of the following fact is very similar to the proof of Proposition 6.4 and is skipped.

Proposition 6.9. *Let D' be a consistent reorientation of D_k^r and Ω the natural assignment defined by D'. Then Ω satisfies Π if and only if, for each clause C, there exists at least one special arc entering w^C.*

The following result is a direct corollary of Proposition 6.6.

Proposition 6.10. *Let D' be a consistent reorientation of D_k^r such that the orientation of each non special arcs that belongs to no circuit of type Δ_x is preserved. Then the set $V \setminus W$ is rooted k-vertex-connected in D'.*

Proposition 6.5 does not hold anymore and is replaced by a weaker result.

Proposition 6.11. *Let D' be a rooted k-vertex-connected reorientation of D_k^r. Then, for each clause C, Δ_C is either preserved or reversed. And there exists a consistent reorientation of D' such that, for each clause C, the number of special arcs entering w^C does not decrease.*

Proof. Let D' be a rooted k-vertex-connected reorientation of D_k^r and let x be a variable and v a vertex incident to Δ_x. Observe that if we remove the $k-1$ neighbors of v that are not incident to Δ_x then only the two arcs of Δ_x are incident to v. Hence by rooted k-vertex-connectivity at least one of these arcs enters v. Thus, since Δ_x is a circuit, each vertex of Δ_x is entered by exactly one arc of Δ_x in D', that is Δ_x is either preserved or reversed.

If D' is not consistent choose a variable x and a clause C such that exactly one of Δ_x and e_x^C is preserved. In $D' - (M \cup t_x^C)$ exactly two arcs are incident to $U_x^C \cup v_x^C$. One of them is the special arc e_x^C and the other one e belongs to Δ_x. By rooted k-vertex-connectivity and choice of x and C both e and e_x^C enter U_x^C. We reverse the special arc e_x^C. Repeat this operation to obtain a consistent reorientation as claimed. □

We can now prove Claim 6.2.

Proof of Claim 6.2. Let D' be a rooted k-vertex-connected reorientation of D_k. Denote D'' a consistent reorientation of D' as in Proposition 6.11 and Ω the natural assignment associated to D''. Let C be clause, by rooted k-vertex-connectivity of D' there exists at least one special arc entering w^C in $D' - L$. This property is preserved in D'' hence, by Proposition 6.9, Ω satisfies C.

Let Ω be an assignment of the variables that satisfies Π. Call D' the consistent reorientation of D_k obtained by reversing the circuit Δ_x and the special arcs e_x^C for each variable x assigned false. By Proposition 6.10, $V - W$ is rooted k-vertex-connected in D'. By Proposition 6.9, for each clause C, there is a special arc entering w^C. Hence D' is rooted k-vertex-connected. □

Since the order $D_k^r(\Pi)$ is a linear function of the size of Π, Theorem 6.2 directly derives from the NP-completeness 3-Sat and Claim 6.2.

The construction of counterexamples to Conjecture 5.3 is based on the following observation.

Proposition 6.12. *Let* G *be weakly* $2k$*-connected graph and* r *any vertex of* G*. Then* (5.1) *is satisfied.*

Proof. By weak $2k$-connectivity of G, $d_G^b(X) \geqslant 2(k - |w^b(X)|)$, for any bi-set X. Thus for any family $\mathcal{F}$ of pairwise innerly-disjoint bi-sets $e_G^b(\mathcal{F}) \geqslant \frac{1}{2}\sum_{X\in\mathcal{F}} d_G^b(X) \geqslant \sum_{X\in\mathcal{F}}(k - |w^b(X)|)$. □

Denote by $G_k^r(\Pi)$ the underlying undirected graph of $D_k^r(\Pi)$. By Proposition 6.7 and $|N| \geqslant 2k$, $G_k^r(\Pi)$ is weakly $2k$-connected and, by Proposition 6.12, (5.1) is satisfied. Hence if Π is not satisfiable then $G_k^r(\Pi)$ is a counterexample to Conjecture 5.3. For instance, the rooted graph $G_3^r(\{(x, x), (\bar{x}, \bar{x})\})$ given in Figure 22 is a counterexample to Conjecture 5.3 for $k = 3$.

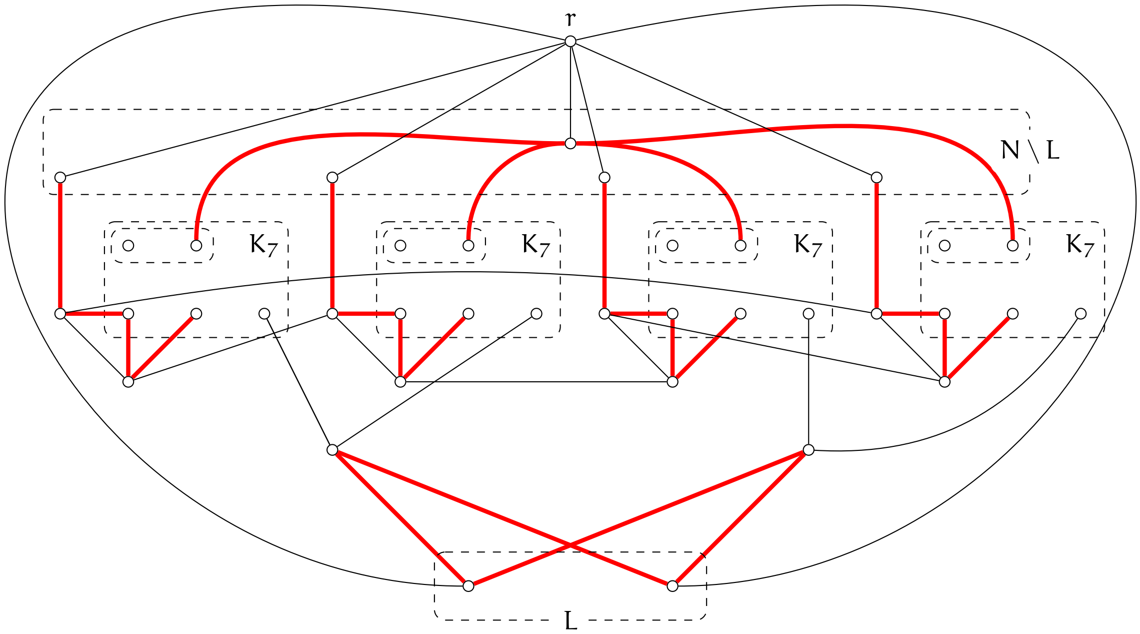


Figure 22: A graph that satisfies (5.1) for $k = 3$ but has no rooted 3-connected orientation. The thick and red edges represent pairs of parallel edges and the vertex set $(N \setminus L) \cup L$ induces a complete graph.

6.4 REMARK ON SIMPLE GRAPHS

In Sections 6.2 and 6.3 the proofs strongly rely on the existence of vertices satisfying (6.1) and (6.2). Such vertices do not exist in a simple graph hence the reader may wonder whether the results seen before still hold in the particular case of simple graphs. In this section we answer this question positively by explaining how the constructions seen so far can be adapted.

First we introduce the gadgets used to remove parallel edges. Given two disjoint vertex sets A and B such that $|A| > |B| \geqslant 1$ an A-B-*directed-connection* is a set of $|B|$ vertex-disjoint arcs from A to B and of $|B|$ vertex-disjoint arcs from B to A that induces no parallel edges. Given a vertex v we define a v-B-directed-connection as the following construction. Create a set S of $2k+1$ new vertices that induces a k-vertex-connected digraph and add an S-$(B \cup v)$-directed-connection. An A-B-connection or v-B-connection is defined as the underlying undirected graph of a directed-connection. By Fact 6.1, in every k-vertex-connected orientation of a graph that contains a v-B-connection where $|B| = k-1$ the two arcs between S and v have opposite directions.

6.4.1 *Undirected Graphs*

We use the definition of the graph family G_k given in Subsection 6.2.1. The pairs of parallel edges incident to w are replaced by a w-T_w-connection where $T_w \subset A \setminus (B \cup a)$. Then the two edges $w'w$, $w''w$ from the connection are removed and the edges wb and wx are replaced by the two edges $w'b$ and $w'x$. Finally, the vertex w is removed. Symmetrically, the edges incident to z are replaced and z is removed. Each pair of parallel edges of type yu such that $u \in A$ (resp. $u \in B$) is replace by a y-T_y-connection where $T_y \subset A \setminus (B \cup a)$ (resp. $T_y \subset B \setminus (A \cup b)$). Symmetrically, each pair of parallel edges incident to x is removed. We call G_k^s the simple graph obtained (see Figure 23).

The proof of the weak $2k$-connectivity of G_k^s is straightforward and is skipped. Now suppose that G_k^s has a k-vertex-connected orientation D. By Fact 6.1, the edges $w'b$ and $w'x$ have opposite directions and the edges $z'a$ and $z'y$ have opposite directions. Each pair of edges incident to y and issued from a y-T_y-connection have opposite directions, hence by $d_{G_k^s}(y) = 2k$, the edges yz' and yx have opposite directions. Similarly, the edges xw' and xy have opposite directions. Hence the orientation of the edges induced by the path $az'yxw'b$ results in a dipath and $D - ((A \cap B) \cup \{x, y\})$ is not strongly connected, a contradiction.

6.4.2 *Directed Graphs*

Recall the definition of D_k given in subsection 6.2.2. For each clause C, the C-gadget is modified the following way (see Figure 24). The pairs of parallel edges incident to w^C are replaced by a w^C-L-directed-connection. Then the two arcs $w^C w'^C$, $w''^C w^C$ from the directed-connection are removed and w^C is replace by w'^C in the three special arcs incident to w^C. Finally, the vertex w^C is removed.

Let C be a clause and x a variable that appears in C. The (C, x)-gadget is modified the following way (see Figure 24). Since $|A_x^C| = k-1 \geqslant 2$ we may assume that there is no parallel arcs between A_x^C and M. Let B_x^C be a subset of U_x^C disjoint from A_x^C and u_x^C of size $k-1$. The parallel arcs incident to v_x^C are replaced by a v_x^C-B_x^C-directed-connection. Then the two arcs $v_x^C v_x'^C$, $v_x''^C v_x^C$ from the directed-connection are removed and v_x^C is replaced by $v_x'^C$ in the two arcs of Δ_x incident to v_x^C. Finally, the vertex v_x^C is removed. The parallel arcs incident to t_x^C are removed and we add $k-2$ t_x^C-$(M \cup t_x'^C)$-directed-connections and one t_x^C-B_x^C-directed-connections.

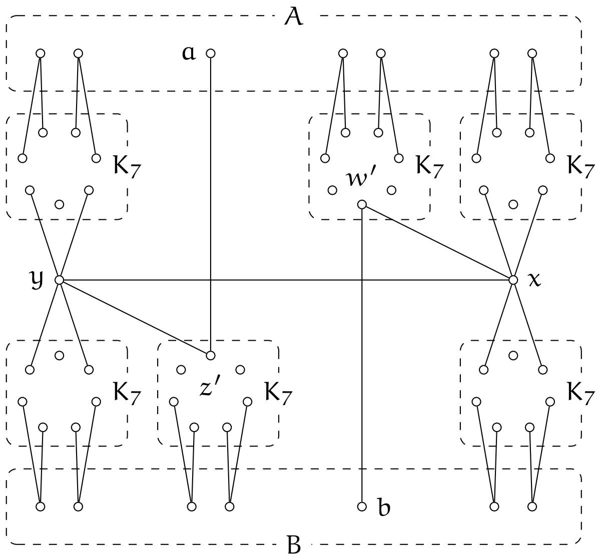


Figure 23: G_3^s: A simple weakly 6-connected graph that admits no 3-connected orientation. The vertex sets A and B induce complete graphs.

The interested reader may check that Propositions 6.4, 6.5 and 6.6 and Claim 6.1 hold for the digraph obtained D_k by the above alterations.

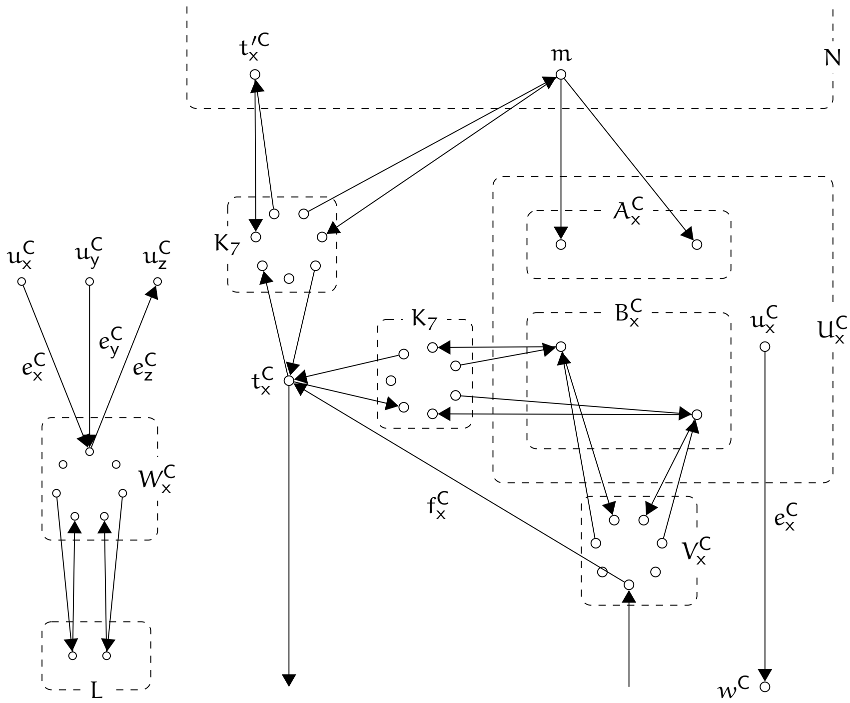


Figure 24: On the left, a clause gadget for $k = 3$ and $C = (x, y, \bar{z})$. On the right, a (C, x)-gadget for $k = 3$ with no parallel arcs.

CONCLUSION

Throughout this document we addressed several problems regarding the orientation of graphs with connectivity constraints. As long as only the global or rooted arc-connectivity is requested, there exist satisfactory methods and algorithms to solve the problems and some variations of it. Some of these approaches are restricted to Graph Theory but others, which use polyhedral and submodular optimization, get into even more abstract issues.

Actually, deep results in the field of orientations of graphs enable us to derive packing theorems in undirected graphs from their directed counterpart. For instance, we deduced a recent theorem on packing matroid constrained trees from a general orientation theorem and a result on packing matroid constrained arborescences that we directly proved. However, we showed that combining a packing problem and an orientation problem that are both deeply understood may drastically rise the difficulty. Indeed, we proved that the problem of Recski that assembles the spanning trees packing problem and the indegree constrained orientation problem is NP-complete. The fact that a slight alteration of the requirements may turn a simple problem into a hard one is not surprising in Combinatorics. A classic example in the area of packing is the problem of finding edge-disjoint Steiner trees for which Kriesell [47] conjectured a sufficient condition: if every pair of vertices of U is $2k$-edge-connected then there exist k edge-disjoint trees, each tree spanning U. Using the strong orientation theorem of Nash-Williams this conjecture could be deduced from its directed counterpart that we propose.

Conjecture 6.1. *Let $D = (V, A)$ be a digraph and let $U \subseteq V$ such that every pair of vertices of U is k-arc-connected. Then there exist $r \in U$ and k arc-disjoint r-arborescences, each arborescence spanning U.*

Concerning local arc-connectivity, the main orientation problem is algorithmically solved efficiently but some variations of this problem turn out to be NP-complete. No successful approach very different from the original odd-pairing technique has been found and we proved that there is no hope to extend this method to a more abstract framework.

In terms of vertex-connectivity, the problems concerning the orientation of graphs are also challenging. There exists a characterization of graphs having an Eulerian and 2-vertex-connected orientation that proves a very special case of a conjecture of Frank. This characterization uses a construction of weakly 4-connected graphs that we generalized to $(2k, k)$-connected graphs for k even. For higher connectivity we gave counterexamples to the conjecture of Frank and proved that the problem of finding a k-vertex-connected orientation is NP-complete for $k \geqslant 3$. For rooted vertex-connectivity, we proved the same statement and disproved another conjecture of Frank.

In this view, the conjecture of Thomassen stating that sufficiently highly vertex-connected graphs have a k-vertex-connected orientation is even more interesting. This conjecture has been settled for $k = 2$ by an approach based on packing rigid spanning subgraphs and the characterization of graphs having an Eulerian and 2-vertex-connected orientation. We improved this result by proving a theorem on packing spanning rigid subgraphs and trees that eventually appears to hold in the more general context of count matroids.

So far, the conjectures of Frank for the case $k = 2$ and Thomassen's conjecture for $k \geqslant 3$ are still open. Having a better insight into the structure of

k-vertex-connected digraphs may be a profitable approach to these problems. In this area a classic problem is to decide the existence of tight vertices in graphs and digraphs that are minimal with respect to a given connectivity. We proved the existence of at least one tight vertex in the undirected case for the g-bounded connectivity but previous works suggest a better lower bound (Question 5.1). In the directed case, we gave a result for the g-bounded 2-connectivity that generalizes results on 2-arc-connectivity and 2-vertex-connectivity. For higher arc-connectivity the problem is solved while it remains open for k-vertex-connectivity if $k \geqslant 3$.

Hopefully, the works we conducted during the past three years will motivate others to challenge the unsolved questions that concern the orientation and the connectivity of graphs. In particular, the g-bounded k-connectivity may be a suitable environment for proofs by induction since it provides a very fine notion in-between vertex-connectivity, for which many essential problems remain open, and edge/arc-connectivity, for which the equivalent problems are solved. We tried to promote the use of bi-sets functions which naturally arise with this connectivity notion and now we hope the reader is convinced of their usefulness.

À travers ce document nous avons abordé plusieurs problèmes traitant l'orientation des graphes avec des contraintes de connexité. Tant que seul l'arc-connexité est en jeu, de façon globale ou enracinée, il existe des méthodes et des algorithmes satisfaisants pour résoudre les problèmes qui se posent. Certaines de ces approches sont contenues dans le cadre de la Théorie des Graphes mais d'autres, qui utilisent des outils d'optimisation polyédrale ou sous-modulaire, ont une portée plus grande.

En effet, certains resultats profonds dans le domaine de l'orientation nous permettent de déduire des théorèmes de packing dans les graphes non-orientés depuis leurs analogues dans les graphes orientés. Par exemple, nous avons déduit un théorème récent sur le packing d'arbres avec des contraintes de matroïde depuis un théorème générale sur l'orientation et un résultat sur le packing d'arborescences avec des contraintes de matroïdes dont nous donnons une preuve. Nous avons également montré que combiner un problème de packing et un problème d'orientation qui, pris séparément, sont polynomiaux peut engendrer un problème considérablement plus difficile. C'est le cas du problème de Recski, regroupant le problème de packing des arbres couvrants et celui de l'orientation avec degrés entrants prescrits, qui s'avère être NP-complet. Le fait qu'une légère modification des contraintes rende un problème simple très difficile n'est pas surprenant en Combinatoire. Un exemple classique est le problème de trouver un packing de Steiner trees pour lequel Kriesell [47] conjecture une condition suffisante : si toute paire de sommet de U est $2k$-arête-connexe alors il existe k arbres arête-disjoints, chacun couvrant U. En utilisant la version forte du théorème d'orientation de Nash-Williams cette conjecture pourraît être déduite de son analogue orienté que nous proposons (Conjecture 6.1).

En ce qui concerne l'orientation avec des contraintes d'arc-connexité locale, le problème principal est théoriquement résolu et une solution peut être calculée de façon efficace. Cependant certaines variations de ce problème se révèlent être NP-complètes. À ce jour, aucune méthode vraiment différente de l'approche originelle basée sur le couplage des sommets impairs n'a pas encore été apportée et nous avons prouvé qu'on ne peut pas espérer étendre cette méthode dans un cadre plus abstrait.

En termes de sommet-connexité, les problèmes qui concernent l'orientation des graphes sont aussi stimulants. Il existe une caractérisation des graphes ayant une orientation eulérienne et 2-sommet-connexe qui établit un cas particulier d'une conjecture de Frank. Cette caractérisation utilise la construction des graphes faiblement 4-connexes que nous avons généralisée aux graphes $(2k, k)$-connexes pour k pair. Pour chaque $k \geqslant 3$, nous avons donné un contre-exemples à la conjecture de Frank et montré que décider si un graphe admet une orientation k-sommet-connexe est NP-complet. Nous avons également prouvé la même assertion concernant la racine sommet-connexité et infirmé la conjecture de Frank correspondante.

De ce fait, la conjecture de Thomassen qui stipule que tout graphe dont la sommet-connexité est suffisament grande a une orientation k-sommet-connexe devient encore plus intéressante. Cette conjecture a été vérifiée pour $k = 2$ en suivant une approche basée sur le packing de sous-graphes rigides couvrants ainsi que la caractérisation des graphes ayant une orientation eulérienne et 2-sommet-connexe. Nous avons améliorer ce résultat en démontrant un théorème de packing de sous-graphes rigides couvrants et d'arbres couvrants qui s'étend dans le cadre plus général des count matroïdes.

Les conjectures de Frank pour $k = 2$ et la conjecture de Thomassen pour $k \geqslant 3$ restent ouvertes. Une meilleure compréhension de la stucture des graphes orientés k-sommet-connexes apporterait probablement de nouveaux éléments de réponse. Dans ce domaine un problème classique est de savoir si un graphe ou un graphe orienté contient des sommets serrés dont le degré est minimal au regard d'une certaine connexité. Dans le cas non orienté, nous avons montré l'existence d'au moins un sommet serré pour la connexité g-bornée mais certains travaux antérieurs suggèrent que cette borne inférieur peut-être améliorée (Question 5.1). Dans le cas orienté, nous avons donné un résultat concernant la 2-connexité g-bornée qui généralise certains résultats sur la 2-arc-connexité et la 2-sommet-connexité. Le problème est résolu pour toute k-arc-connexité mais reste ouvert dans le cas de la k-sommet-connexité dès lors que $k \geqslant 3$.

J'espère que le travail que nous avons effectué pendant ces trois dernières années amèneront d'autres personnes à relever les défis que représentent les nombreuses questions ouvertes que posent l'orientation et la connexité des graphes. En particulier, la k-connexité g-bornée peut être un environnement adapté aux preuves par récurrence puisque qu'elle donne une notion très fine située entre la sommet-connexité, pour laquelle de nombreux problèmes restent ouverts, et l'arête/arc-connexité, pour laquelle les problèmes correspondant sont résolus. Nous avons tenté de promouvoir l'utilisation des bi-ensembles qui accompagnent naturellement cette connexité et espérons maintenant que le lecteur est convaincu de leur utilité.

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