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An implicit high order discontinuous Galerkin level set method for two-phase flow problems

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Abstract

An implicit high order time (BDF) and polynomial degree discontinuous Galerkin (DG) level set method is presented in this talk. The major advantage of this new approach is an accurate mass conservation during the convection of the level set function, thanks to the implicit method. Numerical experiments are presented for the Zalesak and the Leveque test cases. The convergence rates versus time and space are investigated for both BDF and DG high orders. The capture of the zero level set interface is then improved by using an auto-adaptive mesh procedure. The problem is approximated by using the discontinuous Galerkin method for both the level set function, the velocity and the pressure fields.

Keywords: Level set method, discontinuous Galerkin FEM, high order methods

1. Introduction

Let us denote, at any time $t \geq 0$ by $\Omega(t) \subset \mathbb{R}^d$, $d = 2, 3$, the bounded moving domain and $\Gamma(t) = \partial \Omega(t)$ its boundary. A level set function $\phi$ is defined for any time in a bounded computational domain denoted by $\Lambda \subset \mathbb{R}^d$, and containing $\Omega(t)$ at any time, such that $\Gamma(t) = \{ x \in \Lambda; \phi(t, x) = 0 \}$. Since every point belonging to the boundary $\Gamma(t)$ will still continue to belong to it for any time, we have:

$$\frac{d}{dt}(\phi(t, x)) = 0 \iff \frac{\partial \phi}{\partial t} + u \cdot \nabla \phi = 0 \text{ in } [0, +\infty[ \times \Lambda$$

(1)

where $u = \dot{x}$ denotes the velocity field. The normal to the boundary $\Gamma(t)$ writes $\nu = \nabla \phi/||\nabla \phi||$ and then deformations of $\Gamma(t)$ are only due to the normal component $u \cdot \nu$ of the velocity on $\Gamma(t)$. For a given velocity field $u$, the problem is to find $\phi$ defined in $[0, +\infty[ \times \Lambda$ and satisfying (1) together with an initial condition $\phi(t=0) = \phi_0$ where $\phi_0$ is given. This is a linear hyperbolic problem. The level set method was introduced in 1988 by Osher and Sethian [8] (see also [12]). Notice that several choices are possible for the function $\phi$: the only requirement being that a fixed isocountour of $\phi$ coincides with the front at each time $t$. A common choice is the signed distance from the front: e.g. $\Omega(t)$ is the part where $\phi(t, x)$ is negative and $\phi(t, x) = -\text{dist}(\Gamma(t), x)$ for all $x \in \Omega(t)$.

Observe also that, assuming a divergence free velocity field $u$, the volume of $\Omega(t)$ remains constant at any time. This property is not strictly maintained after discretization by most numerical methods suitable for hyperbolic problems and this problem is often referred as a mass loss, as the mass conservation of the fluid is violated. Several cures to this problem have already proposed. In 1999, Sussman and Fatemi [14] proposed, in the context of a finite difference method, to add a constraint in order to improve the mass conservation. This idea was extended in 2010 by Laadhari et al [4] to the finite element context (see also [5]). In 2006, Di Pietro et al. [2] and, independently Marchandise et al. [7] proposed to discretize the level set equation (1) by using a discontinuous Galerkin finite element method. Note that the discontinuous Galerkin method applies also to a closely related fluid interface problem, the Cahn-Hilliard equations (see e.g. [9]).

In this paper, we revisit this second approach and propose some improvements that dramatically decrease the mass error.

2. Numerical method

2.1. Discontinuous space approximation

Let $\mathcal{T}_h$ be a triangulation of the computational domain $\Lambda$. We introduce the following finite dimensional space:

$$X_h = \{ \phi_h \in L^2(\Lambda); \phi_{h|K} \in P_k, \forall K \in \mathcal{T}_h \}$$

where $k \geq 0$ is the polynomial degree. The variational formulation of the semi-discretized problem writes (see e.g. [11, p. 8] or [1, p. 57]):

$$\int_{\Lambda} \frac{\partial \phi_h}{\partial t} \phi_h \, dx + \int_{\partial \Lambda} \max(0, -u \cdot n) \phi_h \, ds$$

$$+ \sum_{K \in \mathcal{T}_h} \int_{K} u \cdot \nabla \phi_h \, dx$$

$$+ \sum_{S \in \partial \mathcal{T}_h} \int_{S} [\phi_h] \left( \frac{|u|_S}{2} [\phi_h] - u \cdot n [\phi_h] \right) \, ds = 0,$$

$$\forall \phi_h \in X_h$$

(2)

Figure 1: Notations for the discontinuous Galerkin method.

This formulation introduces some notations that are usual in
the context of the discontinuous Galerkin method. The second
term involves the outer unit normal \( n \) on the boundary \( \partial \Omega \)
of the computational domain. The last term involves a sum over
\( \mathcal{S}_{h}^{(i)} \), the set of internal sides of the mesh \( \mathcal{T}_{h} \). Each internal side
\( S \in \mathcal{S}_{h}^{(i)} \) has two possible orientations: one is chosen defini-
tively. On this internal side, \( n \) denotes the normal to the orien-
ted side \( S \): as \( S \) is an internal side, there exists two elements
\( K_{-} \) and \( K_{+} \) such that \( S = \partial K_{-} \cap \partial K_{+} \) and \( n \) is the outward
unit normal of \( K_{-} \) on \( \partial K_{-} \cap S \) and the inward unit normal of \( K_{+} \)
on \( \partial K_{+} \cap S \), as shown on Fig. 1. For each \( \phi_{h} \in \mathbb{X}_{h} \), recall that \( \phi_{0} \)
is in general discontinuous across the internal side \( S \). We define on
\( S \) the inner value \( \phi_{h} = \phi_{h|\partial K_{-}} \) of \( \phi_{h} \) as the restriction \( \phi_{h|K_{-}} \)
of \( \phi_{h} \) in \( K_{-} \) along \( \partial K_{-} \cap S \). Conversely, we define the outer value
\( \phi_{h} = \phi_{h|\partial K_{+}} \). We also denote on \( S \) the jump \( [\phi_{h}] = \phi_{h} - \phi_{h}^{+} \)
and the average \( \| \phi_{h} \| = (\phi_{h}^{+} + \phi_{h}^{-})/2 \). The case \( k = 0 \), i.e.,
a piecewise constant approximation, coincides with the popular
upwinding finite volume scheme.

### 2.2. Time discretization by an implicit scheme

The final discrete problem is obtained from the semi-discrete
one (2) by using a time discretization scheme. For instance, Di
Pietro et al. [3] and Marchandise et al. [7] used an explicit Runge-
Kutta (RK) method [3]. Here, we propose to use a high order
implicit BDF (k + 1) scheme. The mesh is the same for all these
computations (5878 elements, 3040 vertices). The comparison
between the initial disk position and the final one after one period
shows a dramatic improvement when using high order polynomi-
als and schemes: for \( k \geq 3 \), the changes are no more perceptible
and the disk after one period fits the initial shape. Fig. 3 shows
the effect of an uniform mesh refinement when \( k = 2 \).

#### 3. Tests and discussion

##### 3.1. The Zalesak rotating disk test

The Zalesak slotted disk in rotation [15] is a widely used
test for comparing the performances of interface capturing
methods. In this example, the slotted disk is rotated around
the center of the computational domain \( \Lambda = [0, H] \times [0, H] \)
with \( H = 4 \) and a constant angular velocity 0.5, such that
\( u(x, y) = (0.5(y - 2), -0.5(x - 2)) \). The disk of radius 0.5
is initially centered at \((2, 2.75)\). The width of the slot is 0.12 and
the maximum width of the upper bridge, that connects two parts
of the disk, is 0.4. Notice that the slotted disk returns to its initial
position after a period \( T = 4\pi \).

Numerical computations are performed on the time interval
\([0, T] \) with a time step \( \Delta t = T/N \) where \( N \) is the number of
time steps. Fig. 2 shows the effect of increasing the polynomial
degree \( k \) of the discontinuous Galerkin method while using an
implicit BDF(k + 1) scheme. The mesh is the same for all these
computations (5878 elements, 3040 vertices). The comparison
between the initial disk position and the final one after one period
shows a dramatic improvement when using high order polynomi-
als and schemes: for \( k \geq 3 \), the changes are no more perceptible
and the disk after one period fits the initial shape. Fig. 3 shows
the effect of an uniform mesh refinement when \( k = 2 \).

### Table 1: Coefficients of the BDF(k) schemes, \( 1 \leq p \leq 6 \).

<table>
<thead>
<tr>
<th>( p )</th>
<th>( \alpha_0 )</th>
<th>( \alpha_1 )</th>
<th>( \alpha_2 )</th>
<th>( \alpha_3 )</th>
<th>( \alpha_4 )</th>
<th>( \alpha_5 )</th>
<th>( \alpha_6 )</th>
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<td>1</td>
<td>1</td>
<td>(-1)</td>
<td>(-1)</td>
<td>(-1)</td>
<td>(-1)</td>
<td>(-1)</td>
</tr>
<tr>
<td>2</td>
<td>3/2</td>
<td>-2</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
</tr>
<tr>
<td>3</td>
<td>11/6</td>
<td>-3</td>
<td>3/2</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
<td>1/3</td>
</tr>
<tr>
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<td>25/12</td>
<td>-4</td>
<td>3</td>
<td>-4/3</td>
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<td>1/4</td>
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<tr>
<td>6</td>
<td>191/700</td>
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<td>-20/3</td>
<td>15/4</td>
<td>-6/5</td>
<td>-6/5</td>
</tr>
</tbody>
</table>

Let us introduce the following bilinear form, defined for all
\( \phi, \varphi \in \mathbb{X}_{h} \) by

\[
a(\phi, \varphi) = \frac{\alpha_0}{\Delta t} \int_{\Lambda} \phi \varphi \, dx + \int_{\partial \Lambda} \max(0, -u \cdot n) \phi \, ds
+ \sum_{K \in \mathcal{T}_{h}} \int_{K} u \cdot \nabla \phi \varphi \, dx
+ \sum_{S \in \mathcal{S}_{h}^{(i)}} \int_{S} \left| \phi \right| \frac{1}{2} \left( |\varphi| - u \cdot n \|\varphi\| \right) \, ds
\]

At a time step \( t_n = n \Delta t \), when \( n \geq p \), the fully discrete problem
writes

\[
(P)_{h}: \text{find } \phi_{h}^{(n)} \text{ such that } a(\phi_{h}^{(n)}, \varphi_{h}) = -\frac{\alpha_0}{\Delta t} \int_{\Lambda} \phi_{h}^{(n-k)} \varphi_{h} \, dx
\]

In order to quantify the difference between the approximate
solution and the exact one, let us introduce the following \( L^{1} \) error
between the two shapes:

\[
\| \phi - \phi_{h} \| = \max_{0 \leq t \leq N} \frac{1}{L} \int_{\Lambda} \| H(\phi(t)) - H(\phi_{h}(t)) \|_{L^{1}(\Lambda)}
\]

where \( H \) denotes the usual Heaviside function and \( L \) is the
perimeter size of the initial interface. Fig. 4 shows, for this error
measurement, the convergence of the approximate solution to
the exact one when the time step tends to zero. Remark that, for
each \( h \), the error tends to a constant that is independent of the
time step. We are looking for a representation of the error as
the sum of two terms, one that depends only upon \( h \) and the other
only upon \( \Delta t \) as \( \| \phi - \phi_{h} \| = O(h^{n} + \Delta t^{p}) \). Since the solution
is regular versus \( t \), the error is expected to depend optimally upon
\( \Delta t \), asymptotically as \( \Delta t^{p} \). Conversely, the shape is poorly regu-
lar in space, due to sharp corners, and the convergence properties
will be investigated numerically by looking for the $\alpha \leq k + 1$ power index. Fig. 5 shows the error versus $h$ for various mesh $h$ and polynomial degrees $k$. The time step $\Delta t$ has been chosen sufficiently small for the error to depend only upon $h$ and $k$ and not upon $\Delta t$ (see also Fig. 4). Observe that the approximate solution converges to the exact one with mesh refinement with a power index $\alpha \approx 2$ that appears to be independent upon $k \geq 1$. Table 2 provides the error data for the purpose of comparison: observe that the $L_1$ error is of about one order of magnitude lower than those in [7]. Let us choose for $\Omega$ a full disk, which is regular, instead of a slotted one. Fig. 6 shows the error versus $h$ after one complete revolution of the full disk. It is estimated that $\alpha \approx k/2 + 1$. Thus, the power index $\alpha$ is strongly dependent upon $k$ when the shape $\Omega$ is regular and increasing $k$ decreases dramatically the error.
for $0 \leq t \leq T = 8$. The initial disk is compressed and become very stretched: its shape tends to become very thin. The shape reaches its maximum deformation at time $t = T/2$, and has returned to its initial state at time T. This test is also investigated with different polynomial degrees $k$ and with the BDF($k$+1) scheme.

![Figure 6: Test of circle: error vs mesh refinement $h$ and polynomial degree $k$.](image6.png)

![Figure 8: Leveque test: error vs uniform mesh refinement $h$ and polynomial degree $k$.](image8.png)

### Table 2: Zalesak test: data and comparison with [7] for the error vs mesh refinement $h$ and polynomial degree $k$.  

<table>
<thead>
<tr>
<th>order</th>
<th>mesh $h/H$ from [7]</th>
<th>present $h$</th>
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<td>0.02429</td>
</tr>
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<tr>
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<td>0.04 0.04</td>
<td>0.00148</td>
</tr>
<tr>
<td>1</td>
<td>0.02</td>
<td>0.01125</td>
</tr>
<tr>
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<td>0.02</td>
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<td>0.00017</td>
</tr>
<tr>
<td>5</td>
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<td>9.729696e-05</td>
</tr>
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</table>

Notice that the mass of the slotted disk at time $t_n$ expresses:

$$m^{(n)} = \int_\Lambda \left(1 - H\left(\phi_h^{(n)}\right)\right) \, dx$$  

Remark that the mass error

$$m_e = \max_{0 \leq n \leq N} \left|m(\phi(t_n)) - m_h^{(n)}\right|$$

is related to the previous error in $L^1$ norm $m_e \leq L\|\phi - \phi_h\|$. The evolution of the relative mass error is reported on Fig. 7. Observe that when $k = 1$, the mass error on the coarsest mesh is of about 5% of the total mass during the whole period $t = 4\pi$. Conversely, on the finest mesh, it is dramatically decreased and remains bounded. As in the previous computation, the time step $\Delta t$ has been chosen sufficiently small for the error to depend only upon $h$ and $k$. While increasing the polynomial degree $k$, these fluctuations strongly decreases in amplitude and for $k = 4$ the relative error is bounded on all meshes.

#### 3.2. The vortex-in-box Leveque test

The second test, proposed in [6], considers a disk of radius 0.15 at $(0.5, 0.75)$ in a unit square domain and the velocity field:

$$u(t, x, y) = \left(\frac{-\sin^2(\pi x)\sin(2\pi y)\cos(\pi t/\pi)}{\sin^2(\pi y)\sin(2\pi x)\cos(\pi t/\pi)}\right)$$
4. Conclusion

In this contribution, a level set transport is investigated with an implicit high order time (BDF) and polynomial degree discontinuous Galerkin finite element method. Using three well-known cases, i.e. a rotation circle, and the Zalesak and Leveque tests, we establish that our method present a nice mesh convergence. For the Zalesak case, the numerical solution converges toward the exact solution in $h^2$ whatever the polynomial degree mainly due to the non-regularity of the exact solution. For the Leveque test, the mesh convergence is better but stays limited at high polynomial degree due to the sharp interface observed during the advection process. For the rotating circle, up too our tests, there is no a priori limit for the convergence rate versus $h$ and we can conclude that the limitation of the convergence rate appears when the region $\Omega$ has poor regularity.

It is noteworthy that the mass loss becomes very small when the polynomial degree increases. This means that it is not necessary to introduce artificially the mass conservation. This is an important feature for the future work when the transport scheme will be coupled with the Navier-Stokes equations.
References


