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Detecting the Position of Nonlinear Component in Periodic Structures from the System Responses to Dual Sinusoidal Excitations

Z.K. Peng, Z.Q. Lang

Department of Automatic Control and Systems Engineering, University of Sheffield
Mappin Street, Sheffield, S1 3JD, UK
Email: z.peng@sheffield.ac.uk; z.lang@sheffield.ac.uk

Abstract: Based on the Nonlinear Output Frequency Response Functions (NOFRFs), a novel method is developed to detect the position of nonlinear components in periodic structures. The detection procedure requires exciting the nonlinear systems twice using two sinusoidal inputs separately. The frequencies of the two inputs are different; one frequency is twice as high as the other one. The validity of this method is demonstrated by numerical studies. Since the position of a nonlinear component often corresponds to the location of defect in periodic structures, this new method is of great practical significance in fault diagnosis for mechanical and structural systems.

1 Introduction

Periodic structures are defined as structures consisting of identical substructures connected to each other in identical manner. The real life systems which can be modelled as either finite or infinite, one-dimension or multi-dimension periodic structures range from the simple structures like periodically supported beams [1]~[6], plates[5][6] and building block [7]. Analysis of the free and forced vibration and the mode analysis of linear periodic structures are of particular interests [1]-[6]. Mead [8] provides an excellent review about periodic structure studies.

Attentions also have been paid to the study of nonlinear periodic structures [9]-[13]. Chi and Rosenberg [9] have studied the existence of classical normal mode motion in one-dimension non-linear mass-spring-damper systems with many degrees of freedom where all springs and / or all dampers may be strongly non-linear. Using exact and asymptotic techniques, Vakakis *et al* [10] have studied the localized responses of a nonlinear periodic oscillator chain subjected to harmonic excitations with general spatial distributions. Royston and Singh [11] have studied the periodic response of mechanical systems with local nonlinearity using an enhanced Galerkin technique where a new semianalytical framework for the study of mechanical systems with local nonlinearities has been presented. Chakraborty and Mallik [12] have investigated the harmonic vibration propagation in an infinite, non-linear periodic chain using a perturbation approach where the case of cyclic one-dimension nonlinear chain has also been discussed. Marathe and Chatterjee [13] have studied the Wave attenuation in one-dimension nonlinear periodic structures using harmonic balance and multiple scale method. In engineering practices, there are considerable periodic structures that behave nonlinearly just because one or a few components have nonlinear properties, and the nonlinear component is often the component where a fault or abnormal condition occurs. One of the well known examples is beam structures [14] with breathing cracks, the global nonlinear behaviors of which are caused only by a few cracked elements. Therefore it is of great significance to effectively detect the position of the nonlinear component in a periodic structure. The detection of damage in large periodic structures has been studied by Zhu and Wu [15]. In their studies, the periodic structure with damage is still considered to be linear and the locations and magnitude of damage in large mono-coupled periodic systems have been estimated using measured changes in the natural frequencies. Based on a one-dimensional periodic structure model, Sakellariou and Fassois [7][16] have used a stochastic output error vibration-based methodology to detect the damage in structures where the damage elements were modeled as components of cubic stiffness.

The Volterra series approach [17] is a powerful method for the analysis of nonlinear systems, and extends the familiar concept of the convolution integral for linear systems to a series of multi-dimensional convolution integrals. The Fourier transforms of the Volterra kernels are known as the kernel transforms, Higher-order Frequency Response Functions (HFRFs) [18], or more usually Generalised Frequency Response Functions (GFRFs). These provide a convenient concept for analyzing nonlinear systems in the frequency domain. If a differential equation or discrete-time model is available for a nonlinear system, the GFRFs can be determined using the algorithm in [19]~[21]. The GFRFs can be regarded as the extension of the classical frequency response function (FRF) for linear systems to the nonlinear case. The concept of Nonlinear Output Frequency Response Functions (NOFRFs) [22] is an alternative extension of the FRF to the nonlinear case. NOFRFs are one dimensional functions of frequency, which allow the analysis of nonlinear systems to be implemented in a manner similar to the analysis of linear systems, and which provides great insight into the mechanisms which dominate many important nonlinear behaviors.

In this paper, a novel method is derived based on the NOFRF concept to detect the position of the nonlinear component in a periodic structure. The detection procedure requires exciting the nonlinear systems twice using two sinusoidal inputs separately. The frequencies of the two inputs are required to be different; one frequency is twice as high as the other one. Numerical studies verify the effectiveness of the method. The new method is of great practical significance in fault diagnosis for mechanical and structural systems.

The paper is organized as follows. Section 2 gives a brief introduction to the new concept of NOFRFs. Some important properties of the NOFRFs for locally nonlinear MDOF systems, which were first revealed in the authors' recent studies [23], are introduced in Section 3. The novel method for the nonlinear component position detection is presented in Section 4. In Section 5, three numerical case studies are used to verify the effectiveness of the proposed method. Finally conclusions are given in Section 6.

2. Output Frequency Response Functions of Nonlinear Systems

2.1 Output Frequency Response Functions under General Input

The definition of NOFRFs is based on the Volterra series theory of nonlinear systems. The Volterra series extends the well-known convolution integral description for linear systems to a series of multi-dimensional convolution integrals, which can be used to represent a wide class of nonlinear systems [18].

Consider the class of nonlinear systems which are stable at zero equilibrium and which can be described in the neighbourhood of the equilibrium by the Volterra series

$$x(t)=\sum_{n=1}^{N}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}h_n(\tau_1,...,\tau_n)\prod_{i=1}^{n}u(t-\tau_i)d\tau_i \tag{1}$$

where $x(t)$ and $u(t)$ are the output and input of the system, $h_n(\tau_1,...,\tau_n)$ is the nth order Volterra kernel, and N denotes the order of the Volterra series representation. Lang and Billings [18] derived an expression for the output frequency response of this class of nonlinear systems to a general input. The result is

$$\begin{cases} X(j\omega)=\sum_{n=1}^{N}X_n(j\omega) \quad \textbf{\textit{for}}\ \forall\omega \\ X_n(j\omega)=\dfrac{1/\sqrt{n}}{(2\pi)^{n-1}}\displaystyle\int_{\omega_1+,...,+\omega_n=\omega}H_n(j\omega_1,...,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_{n\omega} \end{cases} \tag{2}$$

This expression reveals how nonlinear mechanisms operate on the input spectra to produce the system output frequency response. In (2), $X(j\omega)$ is the spectrum of the system output, $X_n(j\omega)$ represents the *n*th order output frequency response of the system,

$$H_n(j\omega_1,...,j\omega_n) = \int_{-\infty}^{\infty}...\int_{-\infty}^{\infty} h_n(\tau_1,...,\tau_n)e^{-(\omega_1\tau_1+,...,+\omega_n\tau_n)j}d\tau_1...d\tau_n \qquad (3)$$

is the *n*th order Generalised Frequency Response Function (GFRF) [18], and

$$\int_{\omega_1+,...,+\omega_n=\omega} H_n(j\omega_1,...,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_{n\omega}$$

denotes the integration of $H_n(j\omega_1,...,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)$ over the n-dimensional hyper-plane $\omega_1+\cdots+\omega_n=\omega$. Equation (2) is a natural extension of the well-known linear relationship $X(j\omega)=H(j\omega)U(j\omega)$, where $H(j\omega)$ is the frequency response function, to the nonlinear case.

For linear systems, the possible output frequencies are the same as the frequencies in the input. For nonlinear systems described by equation (1), however, the relationship between the input and output frequencies is more complicated. Given the frequency range of an input, the output frequencies of system (1) can be determined using the explicit expression derived by Lang and Billings in [18][24].

Based on the above results for the output frequency response of nonlinear systems, a new concept known as the Nonlinear Output Frequency Response Function (NOFRF) was recently introduced by Lang and Billings [22]. The NOFRF is defined as

$$G_n(j\omega) = \frac{\int_{\omega_1+,...,+\omega_n=\omega} H_n(j\omega_1,...,j\omega_n)\prod_{i=1}^{n}U(j\omega_i)d\sigma_{n\omega}}{\int_{\omega_1+,...,+\omega_n=\omega} \prod_{i=1}^{n}U(j\omega_i)d\sigma_{n\omega}} \qquad (4)$$

under the condition that

$$U_n(j\omega) = \frac{1/\sqrt{n}}{(2\pi)^{n-1}} \int_{\omega_1+,...,+\omega_n=\omega} \prod_{i=1}^{n}U(j\omega_i)d\sigma_{n\omega} \neq 0 \qquad (5)$$

Notice that $G_n(j\omega)$ is valid over the frequency range of $U_n(j\omega)$, which can be determined using the algorithm in [18].

By introducing the NOFRFs $G_n(j\omega)$, $n=1,\cdots N$, equation (2) can be written as

$$X(j\omega) = \sum_{n=1}^{N} X_n(j\omega) = \sum_{n=1}^{N} G_n(j\omega)U_n(j\omega) \qquad (6)$$

which is similar to the description of the output frequency response for linear systems. The NOFRFs reflect a combined contribution of the system and the input to the system output frequency response behaviour. It can be seen from equation (4) that $G_n(j\omega)$ depends not only on H_n (n=1,…,N) but also on the input $U(j\omega)$. For any structure, the dynamical properties are determined by the GFRFs H_n (n= 1,…,N). But, according to equation (4), the NOFRF $G_n(j\omega)$ is a weighted sum of $H_n(j\omega_1,...,j\omega_n)$ over $\omega_1+\cdots+\omega_n=\omega$ with the weights depending on the test input. Therefore $G_n(j\omega)$ can be used as an alternative representation of the dynamical properties described by H_n. The most important property of the NOFRF $G_n(j\omega)$ is that it is one dimensional, and thus allows the analysis of nonlinear systems to be implemented in a convenient manner similar to the analysis of linear systems. Moreover, there is an effective algorithm [22] available which allows the estimation of the NOFRFs to be implemented directly using system input output data.

2.2 Output Frequency Response Functions under Harmonic Inputs

When system (1) is subject to a harmonic input

$$u(t)=A\cos(\omega_F t+\beta) \tag{7}$$

Lang and Billings [18] showed that equation (2) can be expressed as

$$X(j\omega)=\sum_{n=1}^{N}X_n(j\omega)=\sum_{n=1}^{N}\left(\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{kn}=\omega}H_n(j\omega_{k_1},\cdots,j\omega_{k_n})A(j\omega_{k_1})\cdots A(j\omega_{k_n})\right) \tag{8}$$

where

$$A(j\omega_{k_i})=\begin{cases}|A|e^{j\mathrm{sign}(k)\beta} & if \quad \omega_{k_i}\in\{k\omega_F,k=\pm1\}, i=1,\cdots,n\\ 0 & \text{otherwise}\end{cases} \tag{9}$$

Define the frequency components of nth order output of the system as Ω_n. Then according to equation (8), the frequency components in the system output can be expressed as

$$\Omega=\bigcup_{n=1}^{N}\Omega_n \tag{10}$$

and Ω_n is determined by the set of frequencies

$$\{\omega=\omega_{k_1}+\cdots+\omega_{k_n} \mid \omega_{k_i}=\pm\omega_F, i=1,\cdots,n\} \tag{11}$$

From equation (11), it is known that if all $\omega_{k_1},\cdots,\omega_{k_n}$ are taken as $-\omega_F$, then $\omega=-n\omega_F$. If k of these are taken as ω_F, then $\omega=(-n+2k)\omega_F$. The maximal k is n. Therefore the possible frequency components of $X_n(j\omega)$ are

$$\Omega_n=\{(-n+2k)\omega_F, k=0,1,\cdots,n\} \tag{12}$$

Moreover, it is easy to deduce that

$$\Omega=\bigcup_{n=1}^{N}\Omega_n=\{k\omega_F, k=-N,\cdots,-1,0,1,\cdots,N\} \tag{13}$$

Equation (13) explains why superharmonic components are generated when a nonlinear system is subjected to a harmonic excitation. In the following, only those components with positive frequencies will be considered.

The NOFRFs defined in equation (4) can be extended to the case of harmonic inputs [25] as

$$G_n^H(j\omega)=\frac{\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{kn}=\omega}H_n(j\omega_{k_1},\cdots,j\omega_{k_n})A(j\omega_{k_1})\cdots A(j\omega_{k_n})}{\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{kn}=\omega}A(j\omega_{k_1})\cdots A(j\omega_{k_n})}\quad (n=1,\ldots,N)\tag{14}$$

under the condition that

$$A_n(j\omega)=\frac{1}{2^n}\sum_{\omega_{k_1}+\cdots+\omega_{kn}=\omega}A(j\omega_{k_1})\cdots A(j\omega_{k_n})\neq 0\tag{15}$$

Obviously, $G_n^H(j\omega)$ is only valid over Ω_n defined by equation (12). Consequently, the output spectrum $X(j\omega)$ of nonlinear systems under a harmonic input can be expressed as

$$X(j\omega)=\sum_{n=1}^{N}X_n(j\omega)=\sum_{n=1}^{N}G_n^H(j\omega)\,A_n(j\omega)\tag{16}$$

When k of the n frequencies of $\omega_{k_1},\cdots,\omega_{k_n}$ are taken as ω_F and the remainders are as $-\omega_F$, substituting equation (9) into equation (15) yields,

$$A_n(j(-n+2k)\omega_F)=\frac{1}{2^n}|A|^n\,C_n^k e^{j(-n+2k)\beta}\tag{17}$$

Moreover

$$G_n^H(j(-n+2k)\omega_F)=\frac{\frac{1}{2^n}H_n(\overbrace{j\omega_F,\cdots,j\omega_F}^{k},\overbrace{-j\omega_F,\cdots,-j\omega_F}^{n-k})|A|^n\,C_n^k e^{j(-n+2k)\beta}}{\frac{1}{2^n}|A|^n\,C_n^k e^{j(-n+2k)\beta}}$$

$$=H_n(\overbrace{j\omega_F,\cdots,j\omega_F}^{k},\overbrace{-j\omega_F,\cdots,-j\omega_F}^{n-k})\tag{18}$$

where $H_n(j\omega_1,\ldots,j\omega_n)$ is assumed to be a symmetric function. Therefore, in this case, $G_n^H(j\omega)$ over the nth order output frequency range $\Omega_n=\{(-n+2k)\omega_F, k=0,1,\cdots,n\}$ is equal to the GFRF $H_n(j\omega_1,\ldots,j\omega_n)$ evaluated at $\omega_1=\cdots=\omega_k=\omega_F$, $\omega_{k+1}=\cdots=\omega_n=-\omega_F$, $k=0,\cdots,n$.

Substituting equations (17) and (18) into (16), it can be derived that the output spectrum $Y(j\omega)$ of nonlinear systems subjected to a harmonic input can be expressed as

$$X(jk\omega_F)=\sum_{n=1}^{[(N-k+1)/2]}G_{k+2(n-1)}^H(jk\omega_F)\,A_{k+2(n-1)}(jk\omega_F)\qquad (k=0,1,\cdots,N)\tag{19}$$

where $[\cdot]$ denotes the operator of taking the integral.

3. NOFRFs of Nonlinear Periodic Structures

Consider the one-dimension nonlinear periodic structures where the Lth component is nonlinear, which have be used in [7], [12], [13] and [15], shown in Figure 1.

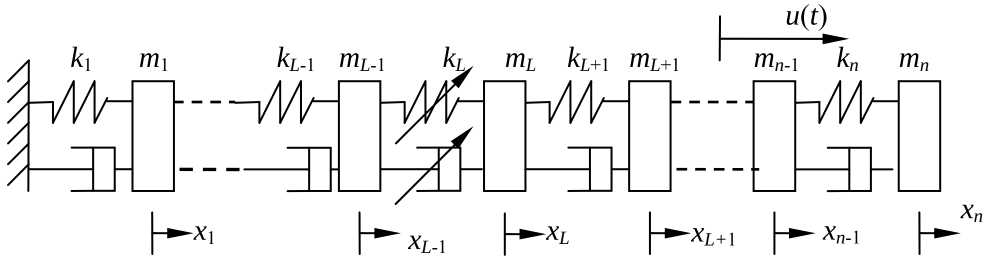

Figure 1, a locally nonlinear multi-degree freedom oscillator

Assume the restoring forces $S_{LS}(\Delta)$ and $S_{LD}(\dot{\Delta})$ of the Lth spring and damper are the polynomial functions of the deformation Δ and $\dot{\Delta}$ respectively, *e.g.*,

$$S_{LS}(\Delta)=\sum_{i=1}^{P} r_i \Delta^i\ ,\ \ S_{LD}(\dot{\Delta})=\sum_{i=1}^{P} w_i \dot{\Delta}^i \tag{20}$$

where P is the degree of the polynomial. Without loss of generality, further assume $L \neq 1, n$. Denote

$$NonF=\sum_{i=2}^{P} w_i(\dot{x}_{L-1}-\dot{x}_L)^i+\sum_{i=2}^{P} r_i(x_{L-1}-x_L)^i \tag{21}$$

$$NF=\begin{pmatrix} \overbrace{0\cdots0}^{L-2} & NonF & -NonF & \overbrace{0\cdots0}^{n-L} \end{pmatrix}' \tag{22}$$

Then the motion of the nonlinear oscillator in Figure 1 is described in a matrix form as.

$$M\ddot{x}+C\dot{x}+Kx=-NF+F(t) \tag{23}$$

where M is the system mass matrix,

$$M=\begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix}$$

and

$$C=\begin{bmatrix} c_1+c_2 & -c_2 & 0 & \cdots & 0 \\ -c_2 & c_2+c_3 & -c_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -c_{n-1} & c_{n-1}+c_n & -c_n \\ 0 & \cdots & 0 & -c_n & c_n \end{bmatrix} K=\begin{bmatrix} k_1+k_2 & -k_2 & 0 & \cdots & 0 \\ -k_2 & k_2+k_3 & -k_3 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -k_{n-1} & k_{n-1}+k_n & -k_n \\ 0 & \cdots & 0 & -k_n & k_n \end{bmatrix}$$

are the system mass, damping and stiffness matrix respectively. $x=(x_1,\cdots,x_n)'$ is the displacement vector, and

$$F(t) = (\overbrace{0,\cdots,0}^{J-1}, u(t), \overbrace{0,\cdots,0}^{n-J})' \tag{24}$$

is the external force vector acting on the J^{th} mass of the oscillator.

The system described by equation (23) is a typical locally nonlinear periodic structure. The *L*th nonlinear component can lead the whole system to behave nonlinearly. In this case, the Volterra series can be used to describe the relationships between the displacements $x_i(t)$ ($i = 1,\cdots,n$) and the input force $u(t)$ as below

$$x_i(t) = \sum_{j=1}^{N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_{(i,j)}(\tau_1,...,\tau_j) \prod_{Z=1}^{j} u(t-\tau_Z) d\tau_Z \tag{25}$$

where $h_{(i,j)}(\tau_1,...,\tau_j)$ is the *j*th order Volterra kernel associated to the *i*th mass. In the frequency domain, the relationship (16) can be expressed as

$$X_i(j\omega) = \sum_{l=1}^{N} X_{(i,l)}(j\omega) = \sum_{l=1}^{N} G_{(i,l)}(j\omega) U_l(j\omega) \qquad (i = 1,\cdots,n) \tag{26}$$

where $G_{(i,l)}(j\omega)$ is the *l*th order NOFRF associated to the *i*th mass.

Without loss of generality, assume *L*<*J*, as revealed in [23], for any two consecutive masses, the NOFRFs of system (23) satisfy following relationships

$$\lambda_2^{i,i+1}(j\omega) = \frac{G_{(i,2)}(j\omega)}{G_{(i+1,2)}(j\omega)} = \cdots = \lambda_N^{i,i+1}(j\omega) = \frac{G_{(i,N)}(j\omega)}{G_{(i+1,N)}(j\omega)} \qquad (1 \le i \le n-1) \tag{27}$$

$$\lambda_1^{i,i+1}(j\omega) = \frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} = \frac{G_{(i,Z)}(j\omega)}{G_{(i+1,Z)}(j\omega)} = \lambda_Z^{i,i+1}(j\omega) \qquad (1 \le i \le L-2 \text{ or } J \le i \le n-1;\ 2 \le Z \le N) \tag{28}$$

$$\lambda_1^{i,i+1}(j\omega) = \frac{G_{(i,1)}(j\omega)}{G_{(i+1,1)}(j\omega)} \ne \frac{G_{(i,Z)}(j\omega)}{G_{(i+1,Z)}(j\omega)} = \lambda_Z^{i,i+1}(j\omega) \qquad (L-1 \le i \le J-1, 2 \le Z \le N) \tag{29}$$

Based on these relationships of the NOFRFs, a novel method can be developed to determine the position of the nonlinear element in system (23).

As equations (27)~(29) are the basis of this study, but the derivation and justification of these equations can take quite a lot of contents and, therefore, the elaboration of them are not given in this paper. However, for the sake of completion, a numerical case is given here to justify equations (27)~(29). The numerical case study is conducted on a damped 8-DOF oscillator whose fourth spring (L = 4) was nonlinear. As widely used in modal analysis, the damping was assumed to be proportional to the stiffness, e.g., $C = \mu K$. The values of the system parameters are taken as

$$m_1 = \cdots = m_8 = 1,\ r_1 = k_1 = \cdots = k_8 = 3.5531 \times 10^4,\ \mu = 0.01$$
$$r_2 = 0.8 \times r_1^2,\ r_3 = 0.4 \times r_1^3,\ w_1 = \mu r_1,\ w_2 = 0.1\mu^2 k_2, w_3 = 0$$

and the input was a harmonic force acting on the 6th mass ($J = 6$), $u(t) = A\sin(2\pi \times 20t)$.

If only the NOFRFs up to the 4th order is considered, according to equations (16) and (17), the frequency components of the outputs of the 8 masses can be written as

$$\begin{aligned} X_i(j\omega_F) &= G^H_{(i,1)}(j\omega_F)U_1(j\omega_F) + G^H_{(i,3)}(j\omega_F)U_3(j\omega_F) \\ X_i(j2\omega_F) &= G^H_{(i,2)}(j2\omega_F)U_2(j2\omega_F) + G^H_{(i,4)}(j2\omega_F)U_4(j2\omega_F) \\ X_i(j3\omega_F) &= G^H_{(i,3)}(j3\omega_F)U_3(j3\omega_F) \\ X_i(j4\omega_F) &= G^H_{(i,4)}(j4\omega_F)U_4(j4\omega_F) \qquad (i = 1,\cdots,8) \end{aligned} \tag{30}$$

From equation (30), it can be seen that, using the method in [22], two different inputs with the same waveform but different strengths are sufficient to estimate the NOFRFs up to 4th order. Therefore, in this numerical study, two different inputs are used with *A*=0.8 and *A*=1.0 respectively. The simulation studies were conducted using a fourth-order *Runge-Kutta* method to obtain the forced response of the system.

The evaluated results of $G^H_1(j\omega_F)$, $G^H_3(j\omega_F)$, $G^H_2(j2\omega_F)$ and $G^H_4(j2\omega_F)$ for all masses are given in Table 1. According to relationships (27)~(29), it is known that the following relationships should be tenable.

$$\begin{aligned} \lambda_1^{i,i+1}(j\omega_F) &= \frac{G^H_{(i,1)}(j\omega_F)}{G^H_{(i+1,1)}(j\omega_F)} = \frac{G^H_{(i,3)}(j\omega_F)}{G^H_{(i+1,3)}(j\omega_F)} = \lambda_3^{i,i+1}(j\omega_F) \qquad \text{for } i = 1,2,6,7 \\ \lambda_1^{i,i+1}(j\omega_F) &= \frac{G^H_{(i,1)}(j\omega_F)}{G^H_{(i+1,1)}(j\omega_F)} \neq \frac{G^H_{(i,3)}(j\omega_F)}{G^H_{(i+1,3)}(j\omega_F)} = \lambda_3^{i,i+1}(j\omega_F) \qquad \text{for } i = 3,4,5 \\ \lambda_2^{i,i+1}(j2\omega_F) &= \frac{G^H_{(i,2)}(j2\omega_F)}{G^H_{(i+1,2)}(j2\omega_F)} = \frac{G^H_{(i,4)}(j2\omega_F)}{G^H_{(i+1,4)}(j2\omega_F)} = \lambda_4^{i,i+1}(j2\omega_F) \qquad \text{for } i = 1,\cdots,7 \end{aligned} \tag{31}$$

Table 1. The evaluated results of $G^H_1(j\omega_F)$, $G^H_3(j\omega_F)$ $G^H_2(j2\omega_F)$ and $G^H_4(j2\omega_F)$

	$G^H_1(j\omega_F)$ ($\times 10^{-6}$)	$G^H_3(j\omega_F)$ ($\times 10^{-9}$)	$G^H_2(j2\omega_F)$ ($\times 10^{-9}$)	$G^H_4(j2\omega_F)$ ($\times 10^{-10}$)
Mass 1	-1.9442+2.8776i	5.4586-7.3663i	6.0215-12.9855i	-1.9521-3.4108i
Mass 2	-4.1766+4.8383i	11.5721-12.2812i	18.5089-19.1412i	-1.3474-7.1855i
Mass 3	-6.7369+5.061i	18.3492-12.5736i	38.1986-9.3255i	3.9767-10.0350i
Mass 4	-9.2319+2.952i	-12.7969+5.4557i	-38.0890+6.2165i	-4.6556+9.5167i
Mass 5	-10.7758-1.6643i	-5.4352+7.5922i	-16.5271+16.8545i	1.1500+6.3785i
Mass 6	-10.1014-8.3275i	1.2207+7.2432i	-1.2526+13.2872i	2.7770+2.3907i
Mass 7	-15.1122-0.8377i	6.0974+5.9104i	6.2132+5.7292i	2.2699-0.4817i
Mass 8	-17.3365+3.5237i	8.6436+4.8795i	8.6698+0.5735i	1.5053-1.8507i

Table 2, the evaluated values of $\lambda_1^{i,i+1}(j\omega_F)$, $\lambda_3^{i,i+1}(j\omega_F)$ $\lambda_2^{i,i+1}(j2\omega_F)$ and $\lambda_4^{i,i+1}(j2\omega_F)$

	$\lambda_1^{i,i+1}(j\omega_F)$	$\lambda_3^{i,i+1}(j\omega_F)$	$\lambda_2^{i,i+1}(j2\omega_F)$	$\lambda_4^{i,i+1}(j2\omega_F)$
i=1	0.5396-0.0639i	0.5396-0.0639i	0.5078-0.1764i	0.5078-0.1765i
i=2	0.7412-0.1614i	0.7412-0.1614i	0.5727-0.3613i	0.5729-0.3613i
i=3	0.8211-0.2856i	-1.5678 +0.3142i	-1.0158+0.0791i	-1.0158+0.0791i
i=4	0.7955-0.3968i	1.2730+0.7743i	1.3178+0.9677i	1.3176+0.9674i
i=5	0.7160-0.4255i	0.8963+0.9014i	1.3735+1.1144i	1.3735+1.1145i
i=6	0.6969+0.5124i	0.6969+0.5124i	0.9568+1.2563i	0.9568+1.2562i
i=7	0.8277+0.2166i	0.8277+0.2166i	0.7570+0.6108i	0.7570+0.6108i

From the NOFRFs in Table 1, $\lambda_1^{i,i+1}(j\omega_F)$, $\lambda_3^{i,i+1}(j\omega_F)$, $\lambda_2^{i,i+1}(j2\omega_F)$ and $\lambda_4^{i,i+1}(j2\omega_F)$ ($i=1,\cdots,7$) can be evaluated, and the results are given in Tables 2. The results shown in Tables 2 have a strict accordance with the relationships in (31). Therefore, the numerical study verifies relationships (27)~(29).

4. The Nonlinear Component Position Detection Method

When the input $u(t)$ is a sinusoidal type force of frequency ω_F, according to the definition of NOFRFs under a harmonic input in Section 2.2, it is known from equations (27)~(29), that

$$\frac{G_{(i,1)}(j\omega_F)}{G_{(i+1,1)}(j\omega_F)} = \frac{G_{(i,3)}(j\omega_F)}{G_{(i+1,3)}(j\omega_F)} = \cdots = \frac{G_{(i,2D+1)}(j\omega_F)}{G_{(i+1,2D+1)}(j\omega_F)} = \cdots = \lambda^{i,i+1}(j\omega_F)$$
$$(D=1,2,\cdots)\ (1\le i\le L-2 \text{ or } J\le i\le n-1) \quad (32)$$

$$\frac{G_{(i,1)}(j\omega_F)}{G_{(i+1,1)}(j\omega_F)} \ne \frac{G_{(i,3)}(j\omega_F)}{G_{(i+1,3)}(j\omega_F)} = \cdots = \frac{G_{(i,2D+1)}(j\omega_F)}{G_{(i+1,2D+1)}(j\omega_F)} = \cdots = \lambda^{i,i+1}(j\omega_F)$$
$$(D=1,2,\cdots)\ (L-1\le i\le J-1) \quad (33)$$

$$\frac{G_{(i,2)}(j2\omega_F)}{G_{(i+1,2)}(j2\omega_F)} = \frac{G_{(i,4)}(j2\omega_F)}{G_{(i+1,4)}(j2\omega_F)} = \cdots = \frac{G_{(i,2D)}(j2\omega)}{G_{(i+1,2D)}(j2\omega)} = \cdots = \lambda^{i,i+1}(j2\omega_F)$$
$$\frac{G_{(i,3)}(j3\omega_F)}{G_{(i+1,3)}(j3\omega_F)} = \frac{G_{(i,5)}(j3\omega_F)}{G_{(i+1,5)}(j3\omega_F)} = \cdots = \frac{G_{(i,2D+1)}(j3\omega_F)}{G_{(i+1,2D+1)}(j3\omega_F)} = \cdots = \lambda^{i,i+1}(j3\omega_F)$$
$$\vdots \qquad (D=1,2,\cdots)\ (i=1,\cdots,n-1) \quad (34)$$

According to equation (19), the first harmonic components of $x_i(t)$ ($i=1,\cdots,n$) can be written as

$$X_i(j\omega_F) = \sum_{k=1}^{[N/2]} G_{(i,2k-1)}^H(j\omega_F)\, A_{2k-1}(j\omega_F) \qquad (i=1,\cdots,n) \quad (35)$$

For the masses which are on the left of the nonlinear spring or on the right of the input force, substituting equation (32) into (35) yields

$$X_i(j\omega_F) = \lambda^{i,i+1}(j\omega_F)\sum_{k=1}^{[N/2]} G^H_{(i+1,2k-1)}(j\omega_F)\, A_{2k-1}(j\omega_F) = \lambda^{i,i+1}(j\omega_F) X_{i+1}(j\omega_F)$$

$$(1 \le i \le L-2 \text{ or } J \le i \le n-1) \quad (36)$$

Therefore,

$$\lambda^{i,i+1}(j\omega_F) = \frac{X_i(j\omega_F)}{X_{i+1}(j\omega_F)} \qquad (1 \le i \le L-2 \text{ or } J \le i \le n-1) \quad (37)$$

For the masses located between the nonlinear spring and the input force, substituting equation (33) into (35) yields,

$$\begin{aligned} X_i(j\omega_F) &= \lambda_1^{i,i+1}(j\omega_F) G^H_{(i+1,1)}(j\omega_F)\, A_1(j\omega_F) + \lambda^{i,i+1}(j\omega_F)\sum_{k=1}^{[N/2]} G^H_{(i+1,2k+1)}(j\omega_F)\, A_{2k+1}(j\omega_F) \\ &\ne \lambda^{i,i+1}(j\omega_F) G^H_{(i+1,1)}(j\omega_F)\, A_1(j\omega_F) + \lambda^{i,i+1}(j\omega_F)\sum_{k=1}^{[N/2]} G^H_{(i+1,2k+1)}(j\omega_F)\, A_{2k+1}(j\omega_F) \\ &= \lambda^{i,i+1}(j\omega_F) X_{i+1}(j\omega_F) \end{aligned}$$

$$(L-1 \le i \le J-1) \quad (38)$$

Obviously,

$$\lambda^{i,i+1}(j\omega_F) \ne \frac{X_i(j\omega_F)}{X_{i+1}(j\omega_F)} \qquad (L-1 \le i \le J-1) \quad (39)$$

According to equation (19), the second harmonic components of $x_i(t)$ ($i = 1,\cdots,n$) can be written as

$$X_i(j2\omega_F) = \sum_{k=1}^{[(N-1)/2]} G^H_{(i,2k)}(j2\omega_F)\, A_{2k}(j2\omega_F) \qquad (i = 1,\cdots,n) \quad (40)$$

Substituting equation (34) into (40) yields

$$\begin{aligned} X_i(j2\omega_F) &= \sum_{k=1}^{[(N-1)/2]} G^H_{(i,2k)}(j2\omega_F)\, A_{2k}(j2\omega_F) \\ &= \lambda^{i,i+1}(j2\omega_F)\sum_{k=1}^{[(N-1)/2]} G^H_{(i+1,2k)}(j2\omega_F)\, A_{2k}(j2\omega_F) = \lambda^{i,i+1}(j2\omega_F) X_{i+1}(j2\omega_F) \end{aligned}$$

$$(i = 1,\cdots,n-1) \quad (41)$$

Consequently

$$\lambda^{i,i+1}(j2\omega_F) = \frac{X_i(j2\omega_F)}{X_{i+1}(j2\omega_F)} \qquad (i = 1,\cdots,n-1) \quad (42)$$

Similarly, it can be deduced that

$$\lambda^{i,i+1}(jD\omega_F) = \frac{X_i(jD\omega_F)}{X_{i+1}(jD\omega_F)} \qquad (D \ge 2,\, i = 1,\cdots,n-1) \quad (43)$$

When using two sinusoidal inputs, the frequencies of which are ω_{F1} and ω_{F2}, to excite the nonlinear system under study, it is easy to extract the harmonic components from the

FFT spectra of the system responses, denote them as $X_{(F1,i)}(jD\omega_{F1})$ and $X_{(F2,i)}(jD\omega_{F2})$ ($i=1,\cdots,n$; $D=1,2,\cdots$) respectively. It can be known from (37), (39) and (43) that

$$\lambda^{i,i+1}(j\omega_{F2}) = \frac{X_{(F2,i)}(j\omega_{F2})}{X_{(F2,i+1)}(j\omega_{F2})} \qquad (1 \le i \le L-2 \text{ or } J \le i \le n-1) \quad (44)$$

$$\lambda^{i,i+1}(j\omega_{F2}) \neq \frac{X_{(F2,i)}(j\omega_{F2})}{X_{(F2,i+1)}(j\omega_{F2})} \qquad (L-1 \le i \le J-1) \quad (45)$$

$$\lambda^{i,i+1}(jD\omega_{F1}) = \frac{X_{(F1,i)}(jD\omega_{F1})}{X_{(F1,i+1)}(jD\omega_{F1})} \qquad (i=1,\cdots,n-1) \quad (46)$$

If $\omega_{F1} = \omega_{F2}/P$ and P is an integral, then for the P^{th} harmonic components of $X_{(F1,i)}$ $(i=1,\cdots,n)$, obviously, equation (46) can be rewritten as

$$\lambda^{i,i+1}(jP\omega_{F1}) = \frac{X_{(F1,i)}(jP\omega_{F1})}{X_{(F1,i+1)}(jP\omega_{F1})} = \lambda^{i,i+1}(j\omega_{F2}) \qquad (i=1,\cdots,n-1) \quad (47)$$

From (44), (45) and (47) it can be known that

$$\frac{X_{(F1,i)}(jP\omega_{F1})}{X_{(F1,i+1)}(jP\omega_{F1})} = \frac{X_{(F2,i)}(j\omega_{F2})}{X_{(F2,i+1)}(j\omega_{F2})} \qquad (1 \le i \le L-2 \text{ or } J \le i \le n-1) \quad (48)$$

and

$$\frac{X_{(F1,i)}(jP\omega_{F1})}{X_{(F1,i+1)}(jP\omega_{F1})} \neq \frac{X_{(F2,i)}(j\omega_{F2})}{X_{(F2,i+1)}(j\omega_{F2})} \qquad (L-1 \le i \le J-1) \quad (49)$$

The relationships given in (48) and (49) provide a simple way to detect the position of nonlinear components in the MDOF nonlinear systems using dual sinusoidal excitations. Usually, for simplicity and convenience, ω_{F1} can be chosen as $\omega_{F1} = (1/2)\omega_{F2}$. The detection procedure is summarized as:

1) Excite the nonlinear system by two sinusoidal inputs separately whose frequencies are ω_{F1} and ω_{F2} respectively, and $\omega_{F1} = 1/2\omega_{F2}$.
2) Calculate the FFT spectra of the two sets of system responses, denote them as $X_{(F1,i)}$ and $X_{(F2,i)}$, $(i=1,\cdots,n)$.
3) Extract the first harmonics $X_{(F2,i)}(j\omega_{F2})$ from $X_{(F2,i)}$, and the second harmonics $X_{(F1,i)}(j2\omega_{F1})$ from $X_{(F1,i)}$, $(i=1,\cdots,n)$.
4) Calculate $X_{(F2,i)}(j\omega_{F2})/X_{(F2,i+1)}(j\omega_{F2})$ and $X_{(F1,i)}(j2\omega_{F1})/X_{(F1,i+1)}(j2\omega_{F1})$, denote them as $R_{F2}^{i,i+1}(j\omega_{F2})$ and $R_{F1}^{i,i+1}(j2\omega_{F1})$, $(i=1,\cdots,n-1)$.
5) Find out the masses where $R_{F2}^{i,i+1}(j\omega_{F2}) \neq R_{F1}^{i,i+1}(j2\omega_{F1})$, then the component on the right side of the furthest left mass is the nonlinear one.

It is worth noting here that, if the force position is located on the left side of the nonlinear component, that is, $J \le L$, then, at the 5^{th} step of the above detection procedure, the nonlinear component is the one on the right side of the furthest right mass which satisfies the relationship $R_{F2}^{i,i+1}(j\omega_{F2}) \neq R_{F1}^{i,i+1}(j2\omega_{F1})$. In addition, if there are no nonlinear components in the system, then there is no superharmonic component in the system response spectrum. Therefore there is no need to use the above detection procedure.

The novel nonlinear component position detection method requires only two tests where the MDOF system excited by two different sinusoidal forces. This is obviously very easy to carry out in practices. In the following section, the effectiveness of this method will be demonstrated using numerical studies.

5 Numerical Studies

In order to verify the nonlinear component position detection method, the same oscillator used in Section 3 was adopted.

Case Study 1 (*L* < *J*):

Table 3, $X_{(F1,i)}(j2\omega_{F1})$ and $R_{F1}^{i,i+1}(j2\omega_{F1})$ (*i*=1,…,8)

	$X_{(F1,i)}(j2\omega_{F1})\,(\times 10^{-8})$	$R_{F1}^{i,i+1}(j2\omega_{F1})$
i=1	-0.4384 + 0.5793i	0.5078 - 0.1764i
i=2	-1.1240 + 0.7503i	0.5727 - 0.3613i
i=3	-1.9951 + 0.0516i	-1.0158 + 0.0791i
i=4	1.9562 + 0.1014i	1.3177 + 0.9677i
i=5	1.0011 - 0.6582i	1.3735 + 1.1144i
i=6	0.2051 - 0.6456i	0.9568+ 1.2563i
i=7	-0.2466 - 0.3510i	0.7570 + 0.6108i
i=8	-0.4239- 0.1217i	

In this case study, the 4th spring are nonlinear, that is *L* = 4. The two sinusoidal forces used are $u_1(t) = \sin(2\pi \times 20t)$ and $u_2(t) = \sin(2\pi \times 40t)$ respectively, and were imposed on the 6th mass of this system, that is *J* = 6. The responses of the system were obtained using a fourth-order *Runge–Kutta* method to integrate equation (31). The second super-harmonics were used to calculate $R_{F1}^{i,i+1}(j2\omega_{F1})$, which were extracted from the FFT spectra of the responses of the system subjected to $u_1(t)$. The results of the second super-harmonics are given in Table 3 together with the calculated values of $R_{F1}^{i,i+1}(j2\omega_{F1})$ (i=1,…,7). Table 4 gives the first harmonics of the FFT spectra of the responses of the system subjected to $u_2(t)$, together with the values of $R_{F2}^{i,i+1}(j\omega_{F2})$ (i=1,…,7). The moduli of $R_{F1}^{i,i+1}(j2\omega_{F1})$ and $R_{F2}^{i,i+1}(j\omega_{F2})$ (i=1,…,7) are given in Table 5.

Table 4, $X_{(F2,i)}(j\omega_{F2})$ and $R_{F2}^{i,i+1}(j\omega_{F2})$ (i=1,…,8)

	$X_{(F2,i)}(j\omega_{F2})$ ($\times 10^{-6}$)	$R_{F2}^{i,i+1}(j\omega_{F2})$
i=1	0.4874-0.2344i	0.5078-0.1765i
i=2	0.9996-0.1142i	0.5730-0.3611i
i=3	1.3385+0.6442i	0.5045-0.4136i
i=4	0.9605+2.0646i	0.4783-0.3916i
i=5	-0.9134+3.5687i	0.4844-0.3797i
i=6	-4.7447+3.6478i	0.9568+1.2562i
i=7	0.0172+3.7901i	0.7570+0.6107i
i=8	2.4604+3.0218i	

Table 5, Moduli $R_{F1}^{i,i+1}(j2\omega_{F1})$ and $R_{F2}^{i,i+1}(j\omega_{F2})$ (i=1,…,7)

	$\left\|R_{F1}^{i,i+1}(j2\omega_{F1})\right\|$	$\left\|R_{F2}^{i,i+1}(j\omega_{F2})\right\|$
i=1	0.5376	0.5376
i=2	0.6772	0.6773
i=3	1.0189	0.6524
i=4	1.6349	0.6181
i=5	1.7687	0.6155
i=6	1.5791	1.5791
i=7	0.9727	0.9726

The results given in Table 6 clearly show that $R_{F1}^{i,i+1}(j2\omega_{F1}) \neq R_{F2}^{i,i+1}(j\omega_{F2})$ at i = 3,4,5. According to the detection method in Section 4, it can be known that the component on the right side of the 3rd mass is the nonlinear one, that is, the 4th spring component.

Case Study 2 (*L* = *J*):

In this case study, the 4th spring are nonlinear, that is L=4. The sinusoidal forces used in this case are the same as the ones used in Case 1, but were imposed on the 4th mass of this system, that is J=4, so L=J. The results of $X_{(F1,i)}(j2\omega_{F1})$ and $R_{F1}^{i,i+1}(j2\omega_{F1})$ (i=1,…,8) are given in Table 6, and the results of $X_{(F2,i)}(j\omega_{F2})$ and $R_{F2}^{i,i+1}(j\omega_{F2})$ (i=1,…,8) are given in Table 7. Table 8 gives the moduli of $R_{F1}^{i,i+1}(j2\omega_{F1})$ and $R_{F2}^{i,i+1}(j\omega_{F2})$ (i=1,…,7).

Table 6, $X_{(F1,i)}(j2\omega_{F1})$ and $R_{F1}^{i,i+1}(j2\omega_{F1})$ (i=1,…,8)

	$X_{(F1,i)}(j2\omega_{F1})\,(\times 10^{-8})$	$R_{F1}^{i,i+1}(j2\omega_{F1})$
i=1	0.3146 – 2.7024i	0.5078 - 0.1764i
i=2	2.2027 – 4.5564i	0.5727 - 0.3613i
i=3	6.3414 – 3.9554i	-1.01578+ 0.0791i
i=4	-6.5065+ 3.3876i	1.3177 + 0.9678i
i=5	-1.9809 + 4.0258i	1.3736 + 1.11433i
i=6	0.5642 + 2.4732i	0.9568 + 1.2563i
i=7	1.4624 + 0.6647i	0.7571 + 0.6108i
i=8	0.1599 - 0.0412i	

Table 7, $X_{(F2,i)}(j\omega_{F2})$ and $R_{F2}^{i,i+1}(j\omega_{F2})$ (i=1,…,8)

	$X_{(F2,i)}(j\omega_{F2})\ (\times 10^{-5})$	$R_{F2}^{i,i+1}(j\omega_{F2})$
i=1	0.1007 + 0.1188i	0.5078-0.1765i
i=2	0.1044 + 0.2702i	0.5730-0.3611i
i=3	-0.0823 + 0.4198i	

		0.5030-0.4096i
i=4	-0.5070 + 0.4216i	1.3180+0.9670i
i=5	-0.0975 + 0.3914i	1.3733+1.1145i
i=6	0.0966+ 0.2066i	0.9568+1.2562i
i=7	0.1412+ 0.0306i	0.7570+0.6107i
i=8	0.1327-0.0667i	

Table 8, Moduli $R_{F1}^{i,i+1}(j2\omega_{F1})$ and $R_{F2}^{i,i+1}(j\omega_{F2})$ (*i*=1,…,7)

	$\left\|R_{F1}^{i,i+1}(j2\omega_{F1})\right\|$	$\left\|R_{F2}^{i,i+1}(j\omega_{F2})\right\|$
i=1	0.5376	0.5376
i=2	0.6772	0.6773
i=3	1.0189	0.6487
i=4	1.6350	1.6346
i=5	1.7687	1.7686
i=6	1.5791	1.5791
i=7	0.9727	0.9726

The results given in Table 8 clearly show that $R_{F1}^{i,i+1}(j2\omega_{F1}) \neq R_{F2}^{i,i+1}(j\omega_{F2})$ only at *i* = 3. According to the detection method in Section 4, it can be known that the component on the right side of the 3rd mass is the nonlinear one, that is, the 4th spring component.

Case Study 3 (*L*>*J*):

In this case study, the 7th spring are nonlinear, that is *L*=7. The sinusoidal forces used in this case are the same as the ones used above cases, but were imposed at the 4th mass of this system, that is *J*=4, so *L*>*J*. The results of $X_{(F1,i)}(j2\omega_{F1})$, $R_{F1}^{i,i+1}(j2\omega_{F1})$, $X_{(F2,i)}(j\omega_{F2})$ and $R_{F2}^{i,i+1}(j\omega_{F2})$ (*i*=1,…,8) are given in Table 9 and Table 10 respectively. Table 11 gives the moduli of $R_{F1}^{i,i+1}(j2\omega_{F1})$ and $R_{F2}^{i,i+1}(j\omega_{F2})$ (*i*=1,…,7).

Table 9, $X_{(F1,i)}(j2\omega_{F1})$ and $R_{F1}^{i,i+1}(j2\omega_{F1})$ (*i*=1,…,8)

	$X_{(F1,i)}(j2\omega_{F1})\,(\times 10^{-7})$	$R_{F1}^{i,i+1}(j2\omega_{F1})$

i=1	-0.0691 + 0.0115i	0.5078 - 0.1765i
i=2	-0.1284 - 0.0220i	0.5730 - 0.3611i
i=3	-0.1430 - 0.1286i	0.5046 - 0.4142i
i=4	-0.0444 - 0.2913i	0.4783 - 0.3913i
i=5	0.2429 - 0.4103i	0.4842 - 0.3798i
i=6	0.7221 - 0.2811i	-1.1682 - 0.3649i
i=7	-0.4947 + 0.3951i	0.7568 + 0.6112i
i=8	-0.1405 + 0.6356i	

Table 10, $X_{(F2,i)}(j\omega_{F2})$ and $R_{F2}^{i,i+1}(j\omega_{F2})$ (i=1,…,8)

	$X_{(F2,i)}(j\omega_{F2})$ $(\times 10^{-5})$	$R_{F2}^{i,i+1}(j\omega_{F2})$
i=1	0.1013 + 0.1193i	0.5078 - 0.1765i
i=2	0.1052 + 0.2714i	0.5730 - 0.3611i
i=3	-0.0822 + 0.4218i	0.5047 - 0.4142i
i=4	-0.5073 + 0.41953i	1.3192 + 0.9672i
i=5	-0.0985 + 0.3902i	1.3736 + 1.1183i
i=6	0.0960 + 0.2059i	0.9580 + 1.2488i
i=7	0.1409 + 0.0313i	0.7570 + 0.6107i
i=8	0.1330 - 0.0660i	

Table 11, Moduli $R_{F1}^{i,i+1}(j2\omega_{F1})$ and $R_{F2}^{i,i+1}(j\omega_{F2})$ (i=1,…,7)

	$\left\|R_{F1}^{i,i+1}(j2\omega_{F1})\right\|$	$\left\|R_{F2}^{i,i+1}(j\omega_{F2})\right\|$
i=1	0.5376	0.5376

i=2	0.6772	0.6773
i=3	0.6528	0.6529
i=4	0.6179	1.6357
i=5	0.6154	1.7712
i=6	1.2238	1.5740
i=7	0.9727	0.9726

The results given in Table 11 clearly show that $R_{F1}^{i,i+1}(j2\omega_{F1}) \neq R_{F2}^{i,i+1}(j\omega_{F2})$ at i = 4, 5, 6. According to the detection method in Section 4, it can be known that the component on the right side of the 6th mass is the nonlinear one, that is, the 7th spring component.

6 Conclusions and Remarks

Based on the properties of NOFRFs, a novel method is developed to detect the position of the nonlinear component in a periodic structure. The detection procedure requires exciting the nonlinear system under study twice using two sinusoidal inputs separately. The frequencies of the two inputs are different; one frequency is twice as high as the other one. Three numerical studies have been used to demonstrate the effectiveness of this method. The distinct advantage of this method is that it only needs the test data under two sinusoidal input forces, which can be readily carried out in practices. Since the positions of the nonlinear components in periodic structures often correspond to the locations of faults, the nonlinear component position detection method is of practical significance in the fault diagnosis for mechanical and structural systems. It is worthy to note here that in real periodic structures there are always minor non-linearities between each component and, such structures therefore should be better to model as weakly nonlinear chains which have been by Chakraborty and Mallik [12]. For such structures, the problem is then changed to detect the components with strong nonlinear property in the weakly nonlinear chains. It is a more complicated problem we are planning to investigate in future studies.

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