

Introduction to Linear Dynamical Systems and Linear Control Strategies

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Acknowledgements

- ❖ Prof. Li Qiu (HKUST)—Linear control theory, multivariable control theory, robust control theory, sampled-data systems and multirate control theory
- ❖ Dr. Li-Xin Wang (HKUST)—Fuzzy control theory, neural networks and genetic algorithms
- ❖ Prof. Xiren Cao (HKUST)—Stochastic control theory
- ❖ Prof. Zexiang Li (HKUST)—Robotics
- ❖ Dr. Bertram Shi (HKUST)—Nonlinear control theory and cellular neural networks
- ❖ Prof. Guanrong Chen (CityU)—Chaos control theory
- ❖ Prof. Xinghuo Yu (RMIT)—Sliding mode control theory
- ❖ Dr. Peter Kwong-Shun Tam (PolyU)—Adaptive control theory
- ❖ Prof. Kok-Lay Teo (CurtinU)—Optimal control theory and impulsive control theory

Outline

- ❖ Linear Time Invariant Systems
- ❖ Linear Time Invariant Feedback Controls
 - ∞ Pole Placement Approach
 - ∞ State Feedback
 - ∞ Output Feedback
- ❖ Linear Multirate Systems
- ❖ Control of Linear Multirate Systems via Filter Banks Approach
- ❖ Conclusions
- ❖ Questions and Answers

Linear Time Invariant Systems

❖ Definition of linear systems

$$\sum_{i=0}^N a_i y_i(k) = T\left(\sum_{i=0}^N a_i u_i(k)\right)$$

❖ Definition of time invariant systems

$$y(k-1) = T(u(k-1))$$

❖ Definition of linear time invariant systems

∞ A system is both linear and time invariant.

Linear Time Invariant Systems

- ❖ Definition of an impulse response

$$h(k) \equiv T(\delta(k))$$

$$\text{where } \delta(k) \equiv \begin{cases} 1 & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

- ❖ Definition of a frequency response

$$y(k) \equiv T(e^{j\omega k})$$

Linear Time Invariant Systems

❖ Properties of linear time invariant systems

⌘ A system is linear and time invariant if and only if

$$y(n) = \sum_{\forall k \in \mathbb{Z}} h(k)u(n-k)$$

⌘ A system is linear and time invariant if and only if

$$Y(z) = H(z)U(z)$$

where

$$Y(z) = \sum_{\forall n \in \mathbb{Z}} y(n)z^{-n}$$

$$H(z) = \sum_{\forall n \in \mathbb{Z}} h(n)z^{-n}$$

$$U(z) = \sum_{\forall n \in \mathbb{Z}} u(n)z^{-n}$$

Linear Time Invariant Systems

❖ Characterization of linear time invariant systems

∞ Constant linear coefficients difference equations

$$\sum_{i=0}^N a_i y(k-i) = \sum_{j=0}^M b_j u(k-j)$$

∞ Transfer function

$$H(z) = \frac{\sum_{j=0}^M b_j z^{-j}}{\sum_{i=0}^N a_i z^{-i}}$$

∞ State space representation

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}u(k)$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}u(k)$$

Linear Time Invariant Systems

❖ Responses

$$\mathbf{x}(k) = \mathbf{A}^k \mathbf{x}(0) + \sum_{j=0}^{k-1} \mathbf{A}^{k-1-j} \mathbf{B} \mathbf{u}(j) \quad \forall k \geq 1$$

$$\mathbf{y}(k) = \mathbf{C} \mathbf{A}^k \mathbf{x}(0) + \mathbf{C} \sum_{j=0}^{k-1} \mathbf{A}^{k-1-j} \mathbf{B} \mathbf{u}(j) + \mathbf{D} \mathbf{u}(k) \quad \forall k \geq 1$$

zero input response

zero state response

$$y(n) = \sum_{\forall k \in \mathbb{Z}} h(k) u(n-k)$$

Linear Time Invariant Systems

❖ Similarity transforms

∞ Define

$$\tilde{\mathbf{x}}(k) \equiv \mathbf{T}^{-1}\mathbf{x}(k)$$

$$\tilde{\mathbf{A}} \equiv \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$$

$$\tilde{\mathbf{B}} \equiv \mathbf{T}^{-1}\mathbf{B}$$

$$\tilde{\mathbf{C}} \equiv \mathbf{C}\mathbf{T}$$

∞ then

$$\tilde{\mathbf{x}}(k+1) = \tilde{\mathbf{A}}\tilde{\mathbf{x}}(k) + \tilde{\mathbf{B}}\mathbf{u}(k)$$

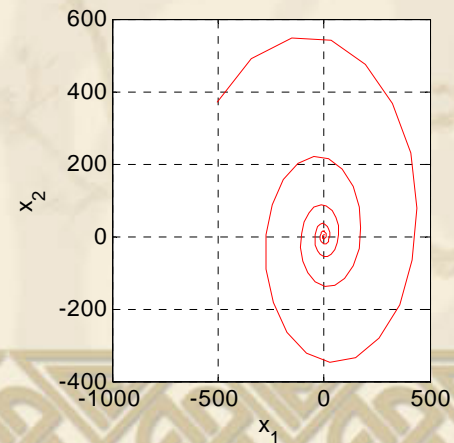
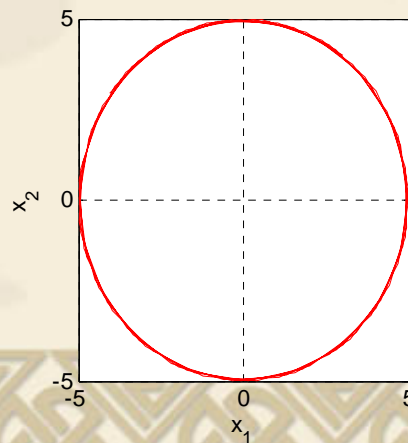
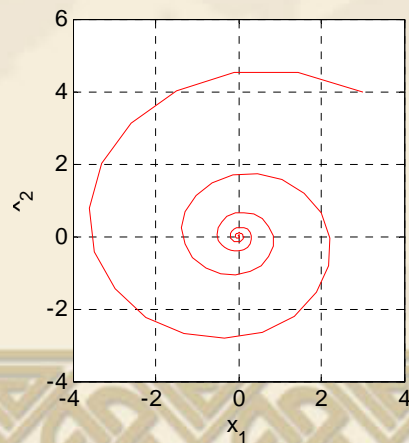
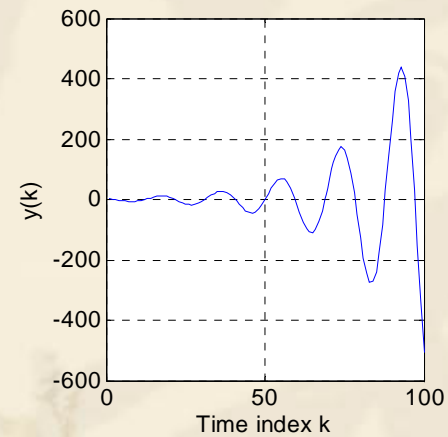
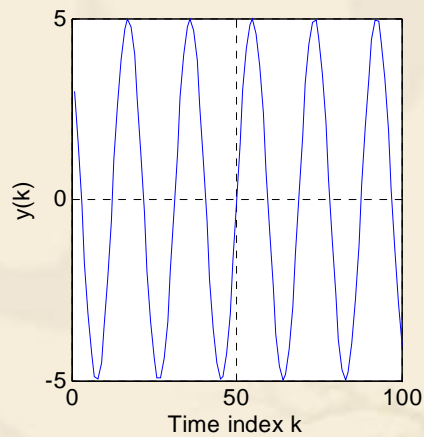
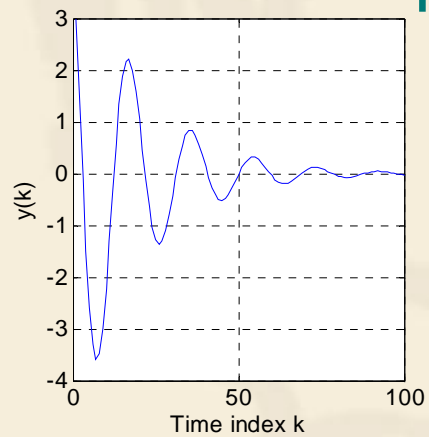
$$\mathbf{y}(k) = \tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + \mathbf{D}\mathbf{u}(k)$$

Linear Time Invariant Systems

- ❖ Only three types of behaviors for autonomous response
 - ∞ converge to zero (all system poles are strictly inside the unit circle.)
 - ∞ oscillates (Some system poles are on the unit circle, while all other system poles are strictly inside the unit circle.)
 - ∞ diverge to infinity (Some system poles are outside the unit circle.)

Linear Time Invariant Systems

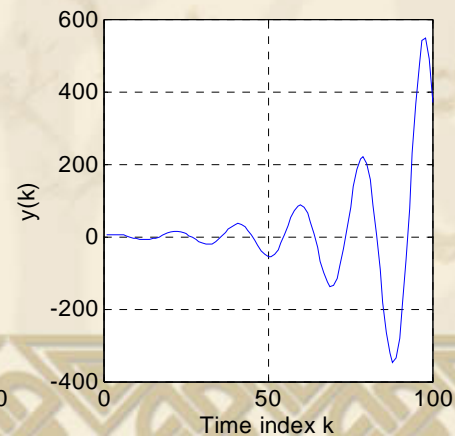
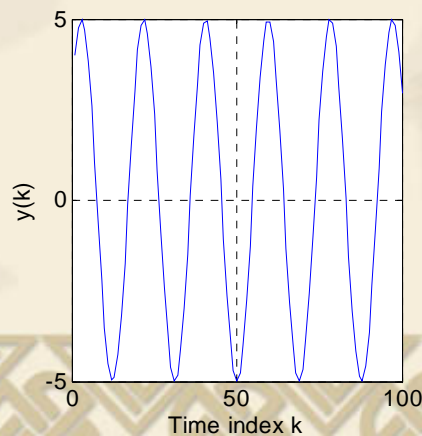
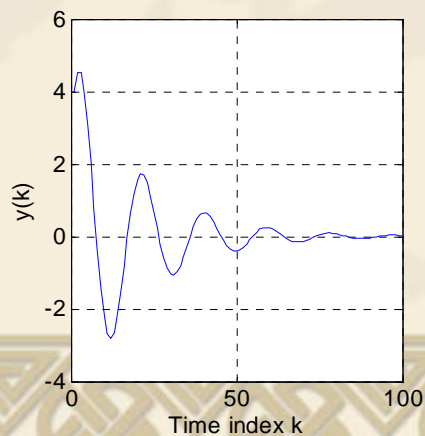
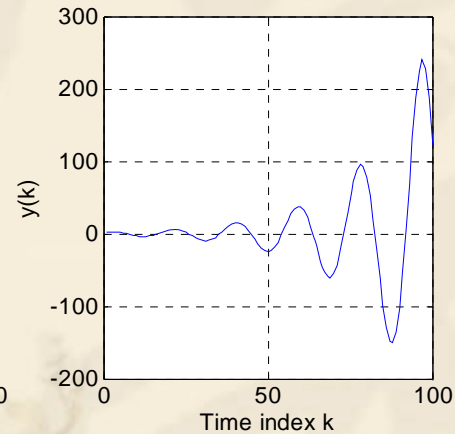
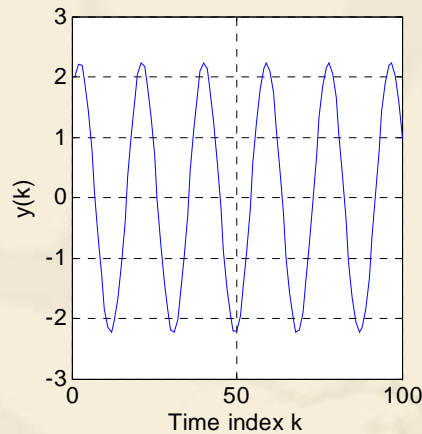
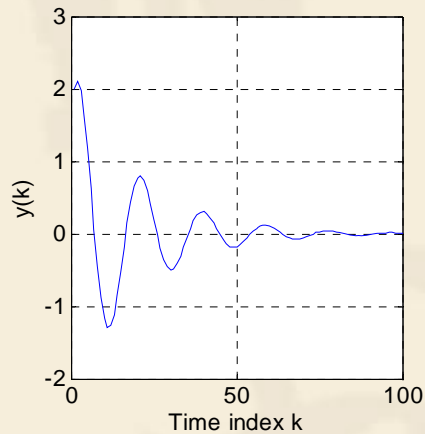
❖ Autonomous responses



Linear Time Invariant Systems

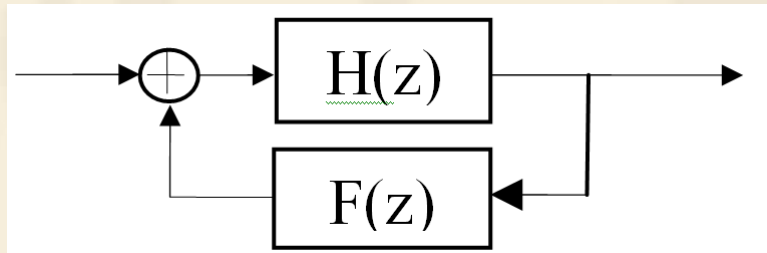
❖ Effects on initial conditions

☞ Behaviors only depend on the system poles, not on initial conditions.



Linear Time Invariant Feedback Controls

❖ Pole placement



Plant transfer function $H(z) = \frac{N_H(z)}{D_H(z)}$

Controller transfer function $F(z) = \frac{N_F(z)}{D_F(z)}$

$$T(z) = \frac{H(z)}{1 + H(z)F(z)} = \frac{N_H(z)D_F(z)}{N_H(z)N_F(z) + D_H(z)D_F(z)}$$

$N_H(z)N_F(z) + D_H(z)D_F(z)$ is stable.

Linear Time Invariant Feedback Controls

❖ State feedback

∞ Plant state space matrices $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$

∞ Controller state space matrices $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}, \tilde{\mathbf{D}})$

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}(\mathbf{u}(k) - \tilde{\mathbf{y}}(k))$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}(\mathbf{u}(k) - \tilde{\mathbf{y}}(k))$$

$$\tilde{\mathbf{x}}(k+1) = \tilde{\mathbf{A}}\tilde{\mathbf{x}}(k) + \tilde{\mathbf{B}}\mathbf{x}(k)$$

$$\tilde{\mathbf{y}}(k) = \tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + \tilde{\mathbf{D}}\mathbf{x}(k)$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}(\mathbf{u}(k) - (\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + \tilde{\mathbf{D}}\mathbf{x}(k)))$$

$$= (\mathbf{C} - \mathbf{D}\tilde{\mathbf{D}})\mathbf{x}(k) - \mathbf{D}\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + \mathbf{D}\mathbf{u}(k)$$

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}(\mathbf{u}(k) - (\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + \tilde{\mathbf{D}}\mathbf{x}(k)))$$

$$= (\mathbf{A} - \mathbf{B}\tilde{\mathbf{D}})\mathbf{x}(k) - \mathbf{B}\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + \mathbf{B}\mathbf{u}(k)$$

Linear Time Invariant Feedback Controls

❖ State feedback

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \tilde{\mathbf{x}}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\tilde{\mathbf{D}} & -\mathbf{B}\tilde{\mathbf{C}} \\ \tilde{\mathbf{B}} & \tilde{\mathbf{A}} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \tilde{\mathbf{x}}(k) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{0} \end{bmatrix} \mathbf{u}(k)$$

$$\mathbf{y}(k) = \begin{bmatrix} \mathbf{C} - \mathbf{D}\tilde{\mathbf{D}} & -\mathbf{D}\tilde{\mathbf{C}} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \tilde{\mathbf{x}}(k) \end{bmatrix} + \mathbf{D}\mathbf{u}(k)$$

$$\begin{bmatrix} (\mathbf{A} - \mathbf{B}\tilde{\mathbf{D}}) & -\mathbf{B}\tilde{\mathbf{C}} \\ \tilde{\mathbf{B}} & \tilde{\mathbf{A}} \end{bmatrix} \text{ is stable.}$$

Linear Time Invariant Feedback Controls

❖ Output feedback

∞ Plant state space matrices $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$

∞ Controller state space matrices $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}, \tilde{\mathbf{D}})$

$$\mathbf{x}(k+1) = \mathbf{A}\mathbf{x}(k) + \mathbf{B}(\mathbf{u}(k) - \tilde{\mathbf{y}}(k))$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}(\mathbf{u}(k) - \tilde{\mathbf{y}}(k))$$

$$\tilde{\mathbf{x}}(k+1) = \tilde{\mathbf{A}}\tilde{\mathbf{x}}(k) + \tilde{\mathbf{B}}\mathbf{y}(k)$$

$$\tilde{\mathbf{y}}(k) = \tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + \tilde{\mathbf{D}}\mathbf{y}(k)$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) + \mathbf{D}(\mathbf{u}(k) - (\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + \tilde{\mathbf{D}}\mathbf{y}(k)))$$

$$= \mathbf{C}\mathbf{x}(k) - \mathbf{D}\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + \mathbf{D}\mathbf{u}(k) - \mathbf{D}\tilde{\mathbf{D}}\mathbf{y}(k)$$

$$\mathbf{y}(k) = (\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}})^{-1} (\mathbf{C}\mathbf{x}(k) - \mathbf{D}\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + \mathbf{D}\mathbf{u}(k))$$

$$= (\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}})^{-1} \mathbf{C}\mathbf{x}(k) - (\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}})^{-1} \mathbf{D}\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + (\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}})^{-1} \mathbf{D}\mathbf{u}(k)$$

Linear Time Invariant Feedback Controls

❖ Output feedback

$$\begin{aligned}
 \tilde{\mathbf{y}}(k) &= \tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + \tilde{\mathbf{D}}\left(\left(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{C}\mathbf{x}(k) - \left(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + \left(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\mathbf{u}(k)\right) \\
 &= \tilde{\mathbf{D}}\left(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{C}\mathbf{x}(k) + \left(\mathbf{I} - \tilde{\mathbf{D}}\left(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\right)\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + \tilde{\mathbf{D}}\left(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\mathbf{u}(k) \\
 \mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}\left(\mathbf{u}(k) - \left(\tilde{\mathbf{D}}\left(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{C}\mathbf{x}(k) + \left(\mathbf{I} - \tilde{\mathbf{D}}\left(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\right)\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + \tilde{\mathbf{D}}\left(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\mathbf{u}(k)\right)\right) \\
 &= \left(\mathbf{A} - \mathbf{B}\tilde{\mathbf{D}}\left(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{C}\right)\mathbf{x}(k) - \mathbf{B}\left(\mathbf{I} - \tilde{\mathbf{D}}\left(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\right)\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + \left(\mathbf{B} - \mathbf{B}\tilde{\mathbf{D}}\left(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\right)\mathbf{u}(k) \\
 \tilde{\mathbf{x}}(k+1) &= \tilde{\mathbf{A}}\tilde{\mathbf{x}}(k) + \tilde{\mathbf{B}}\left(\left(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{C}\mathbf{x}(k) - \left(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\tilde{\mathbf{C}}\tilde{\mathbf{x}}(k) + \left(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\mathbf{u}(k)\right) \\
 &= \tilde{\mathbf{B}}\left(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{C}\mathbf{x}(k) + \left(\tilde{\mathbf{A}} - \tilde{\mathbf{B}}\left(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\tilde{\mathbf{C}}\right)\tilde{\mathbf{x}}(k) + \tilde{\mathbf{B}}\left(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}}\right)^{-1}\mathbf{D}\mathbf{u}(k)
 \end{aligned}$$

Linear Time Invariant Feedback Controls

❖ Output feedback

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \tilde{\mathbf{x}}(k+1) \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\tilde{\mathbf{D}}(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}})^{-1}\mathbf{C} & -\mathbf{B}(\mathbf{I} - \tilde{\mathbf{D}}(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}})^{-1}\mathbf{D})\tilde{\mathbf{C}} \\ \tilde{\mathbf{B}}(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}})^{-1}\mathbf{C} & (\tilde{\mathbf{A}} - \tilde{\mathbf{B}}(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}})^{-1}\mathbf{D}\tilde{\mathbf{C}}) \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \tilde{\mathbf{x}}(k) \end{bmatrix} + \begin{bmatrix} (\mathbf{B} - \mathbf{B}\tilde{\mathbf{D}}(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}})^{-1}\mathbf{D}) \\ \tilde{\mathbf{B}}(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}})^{-1}\mathbf{D} \end{bmatrix} \mathbf{u}(k)$$

$$\tilde{\mathbf{y}}(k) = \begin{bmatrix} \tilde{\mathbf{D}}(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}})^{-1}\mathbf{C} & (\mathbf{I} - \tilde{\mathbf{D}}(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}})^{-1}\mathbf{D})\tilde{\mathbf{C}} \end{bmatrix} \begin{bmatrix} \mathbf{x}(k) \\ \tilde{\mathbf{x}}(k) \end{bmatrix} + \tilde{\mathbf{D}}(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}})^{-1}\mathbf{D}\mathbf{u}(k)$$

$$\begin{bmatrix} \mathbf{A} - \mathbf{B}\tilde{\mathbf{D}}(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}})^{-1}\mathbf{C} & -\mathbf{B}(\mathbf{I} - \tilde{\mathbf{D}}(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}})^{-1}\mathbf{D})\tilde{\mathbf{C}} \\ \tilde{\mathbf{B}}(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}})^{-1}\mathbf{C} & (\tilde{\mathbf{A}} - \tilde{\mathbf{B}}(\mathbf{I} + \mathbf{D}\tilde{\mathbf{D}})^{-1}\mathbf{D}\tilde{\mathbf{C}}) \end{bmatrix} \text{ is stable.}$$

Linear Multirate Systems

❖ Definition

$$y(k) = \sum_{\forall l \in \mathbb{Z}} g(k, l)u(l) \quad \forall k \in \mathbb{Z}$$

where

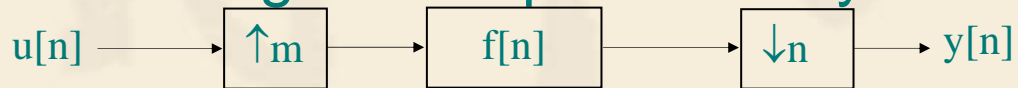
$$g(k, l) = g(k + m, l + n) \quad \forall k, l \in \mathbb{Z}$$

∞ Input shifts by n samples, output shifts by m samples.

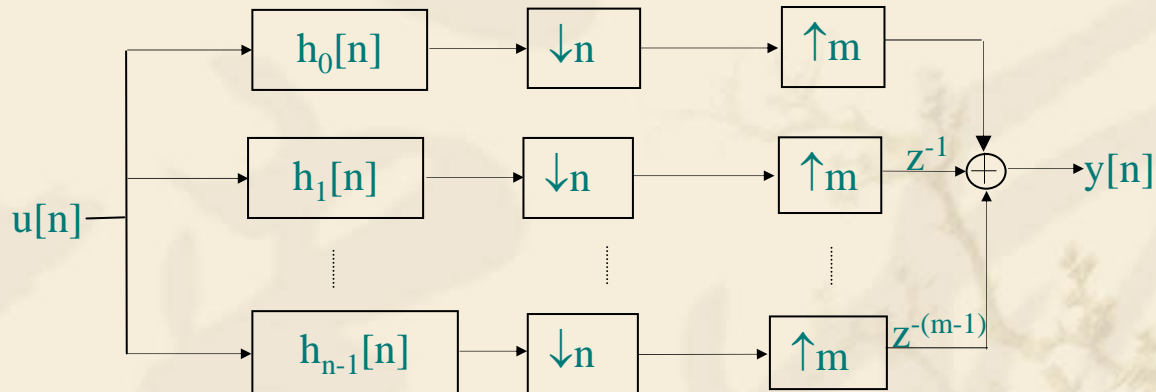
Linear Multirate Systems

❖ Examples:

⌘ Rate changers/sampled data systems



⌘ Filter banks



Linear Multirate Systems

❖ Realization

⌘ A linear multirate system can be realized by a filter bank system.

⌘ Define a blocked input signal as

$$\mathbf{u}(k) \equiv [u(nk) \quad \cdots \quad u(nk + n - 1)]^T$$

⌘ Define a block output signal as

$$\mathbf{y}(k) \equiv [y(mk) \quad \cdots \quad y(mk + m - 1)]^T$$

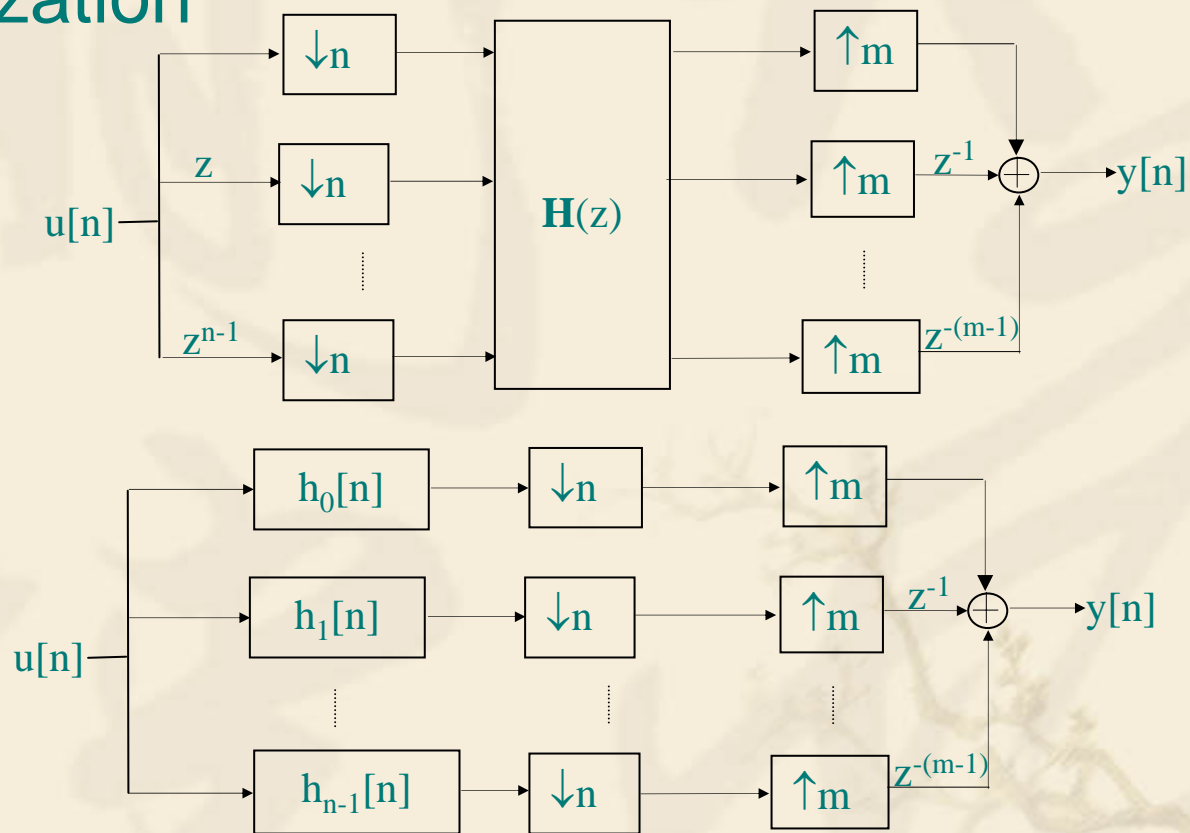
⌘ Input shifts by n samples, the blocked input signal shifts by 1 sample. Output shifts by m samples, the blocked output signal shifts by 1 sample.

⌘ Hence, there exists an $m \times n$ transfer matrix $\mathbf{H}(z)$ such that

$$\mathbf{Y}(z) = \mathbf{H}(z)\mathbf{X}(z)$$

Linear Multirate Systems

❖ Realization



Linear Multirate Systems

❖ Realization

∞ Denote $f[kn - lm] = g[k, l] \quad \forall k, l \in \mathbb{Z}$

∞ Define the map $I : \{0, 1, \dots, m-1\} \times \mathbb{Z} \rightarrow \mathbb{Z}$ such that

$$I(k, l) = kn - lm$$

∞ I is bijective if and only if m and n is co-prime. Or in other words, I is bijective if and only if the highest common factor of m and n is 1.

Linear Multirate Systems

❖ Realization

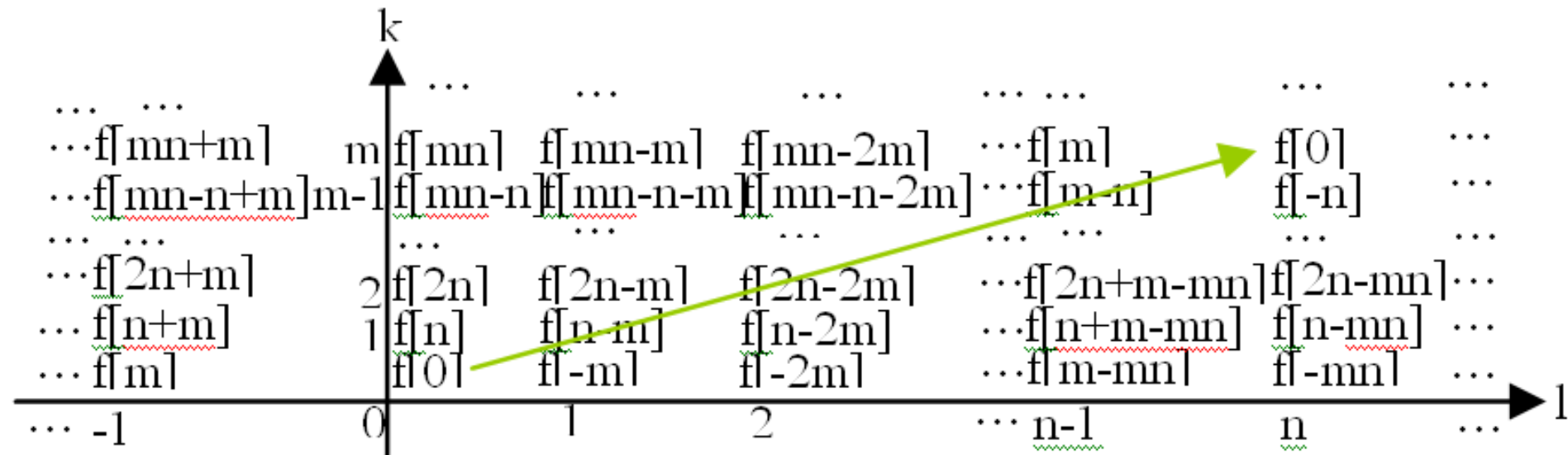


Figure 1a. Mapping from $g[n,k]$ to $f[k]$ when m and n are co-prime.

Linear Multirate Systems

❖ Realization

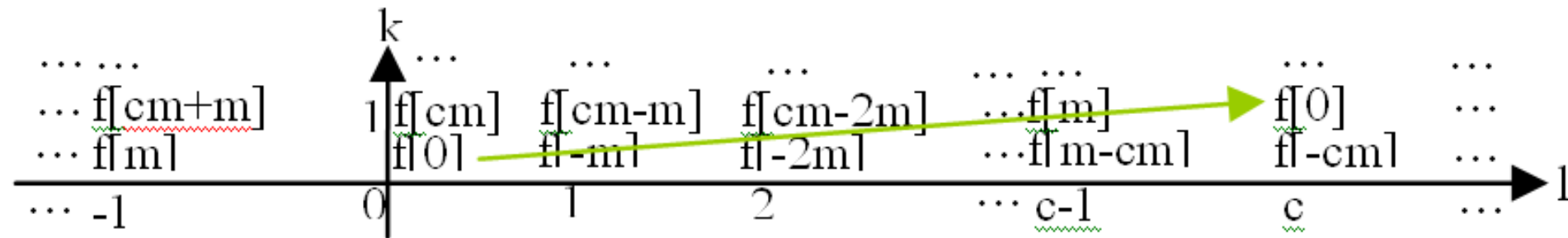
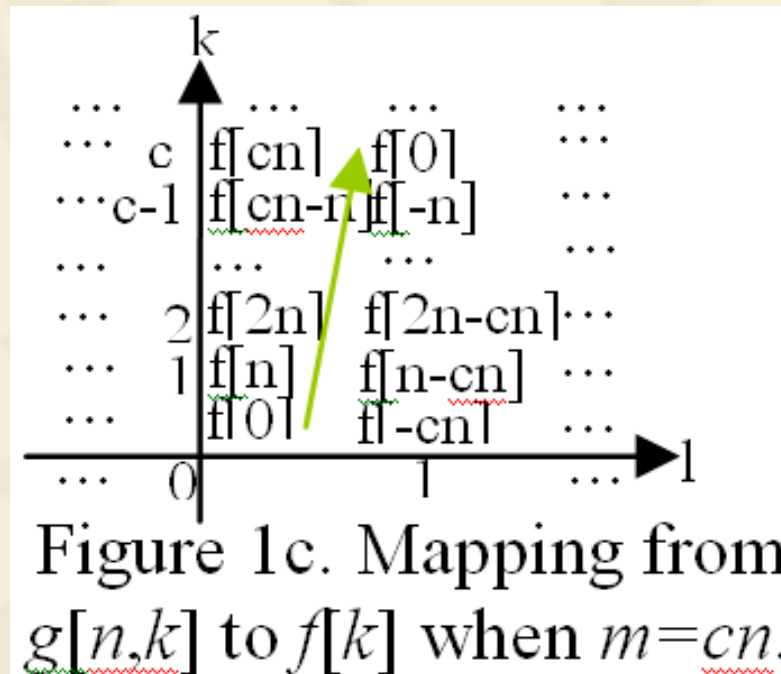


Figure 1b. Mapping from $g[n, k]$ to $f[k]$ when $n = cm$.

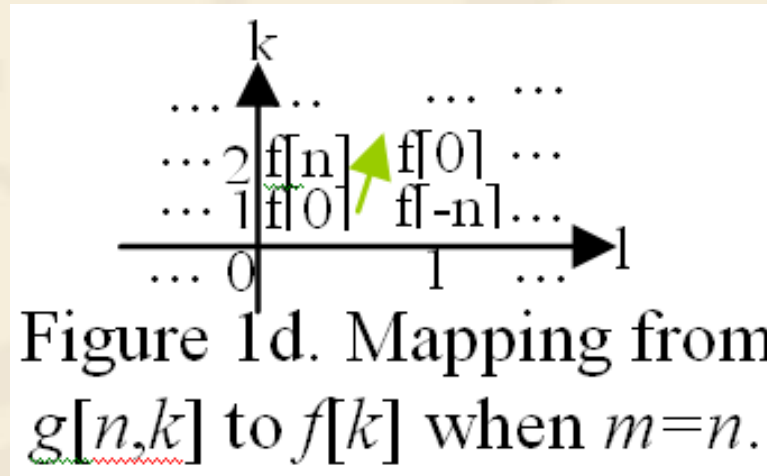
Linear Multirate Systems

❖ Realization



Linear Multirate Systems

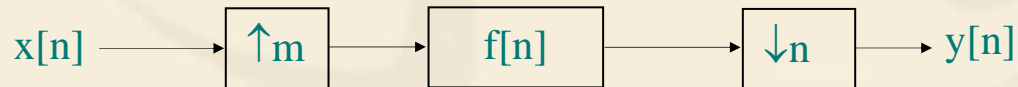
❖ Realization



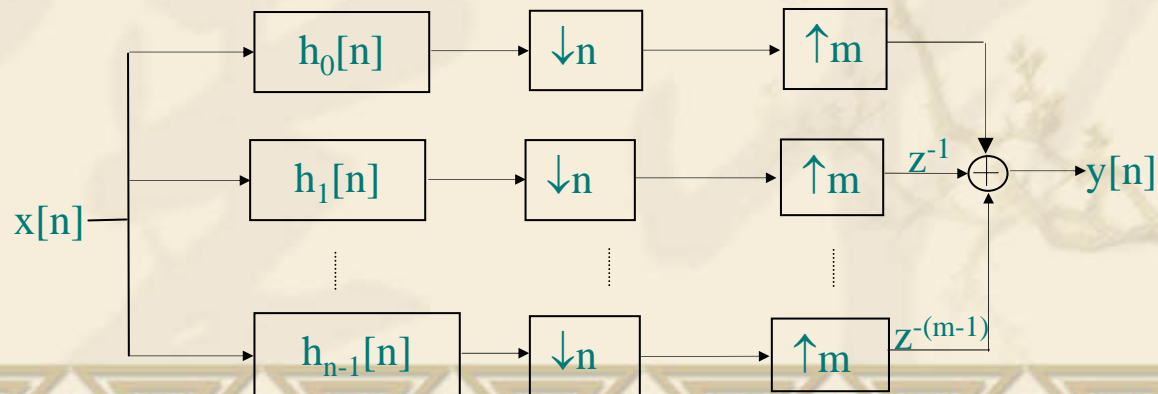
Linear Multirate Systems

❖ Realization

∞ A linear multirate system is equivalent to a rate changer if and only if m and n is co-prime. That is:



is equivalent to



Linear Multirate Systems

❖ Properties

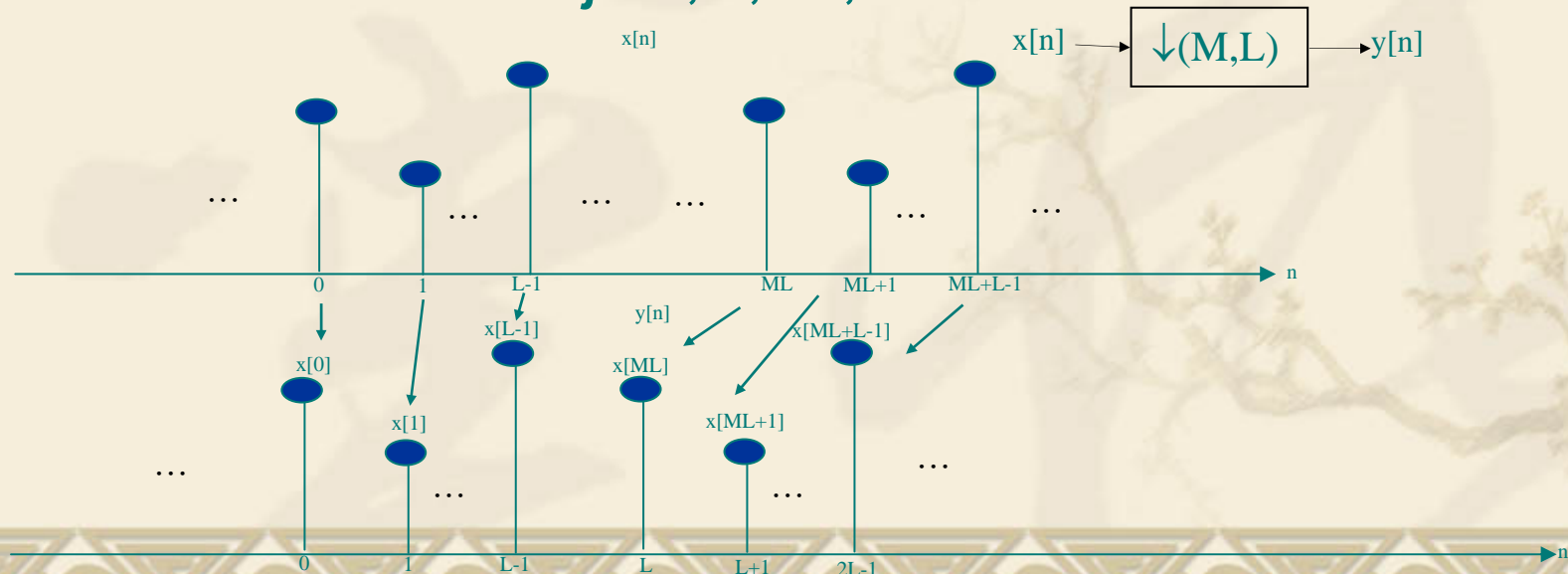
- ⌘ A linear multirate system is stable if and only if $h_i[n]$ for $i=0,1,\dots,n-1$ are all stable.
- ⌘ A linear multirate system is finite impulse response if and only if $h_i[n]$ for $i=0,1,\dots,n-1$ are all finite impulse response.

Linear Multirate Systems

❖ Realization

⌘ Block decimators (decimation ratio M and block length L)

$$y[Lk + j] = x[kML + j] \text{ for } j=0,1,\dots,L-1 \text{ and } k \in \mathbb{Z}.$$

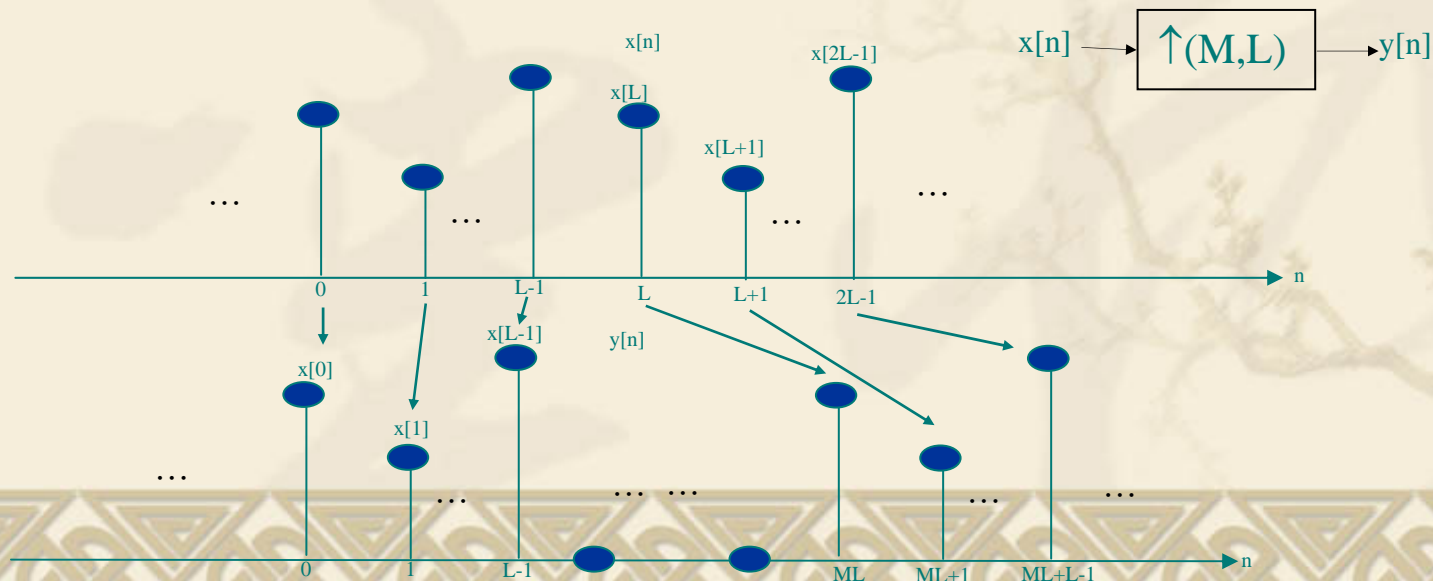


Linear Multirate Systems

❖ Realization

⌘ Block expanders (expansion ratio M and block length L)

$$y[k] = \begin{cases} x\left[\frac{k - \text{mod}(k, ML)}{M} + \text{mod}(k, ML)\right] & k - \text{mod}(k, ML) \leq k < k - \text{mod}(k, ML) + L \\ 0 & k - \text{mod}(k, ML) + L \leq k < k + ML - \text{mod}(k, ML) \end{cases}$$



Linear Multirate Systems

❖ Realization

∞ $\forall m, n \in \mathbb{Z}^+$ (no matter m and n are co-prime or not), all linear multirate systems (shifting input by n samples resulting to shifting an output by m samples) can be represented via a series cascade of $\uparrow m$, followed by an LTI filter with an impulse response $f[k]$, and then followed by $\downarrow(n, m)$.

Linear Multirate Systems

❖ Realization

∞ The input output relationship of all linear multirate rate systems is $y[km+i] = \sum_{l \rightarrow -\infty}^{+\infty} g[i, l-kn] u[l]$, $\forall k, l \in \mathbb{Z}$, $\forall m, n \in \mathbb{Z}^+$ and $i=0, 1, \dots, m-1$.

∞ The input output relationship of the system with block sampler is $y[km+i] = \sum_{\forall l} f[kmn - ml + i] u[l]$, $\forall k, l \in \mathbb{Z}$, $\forall m, n \in \mathbb{Z}^+$ and $i=0, 1, \dots, m-1$.

∞ $\forall k, l \in \mathbb{Z}$, $\forall m, n \in \mathbb{Z}^+$ and $i=0, 1, \dots, m-1$, the mapping from $\{0, 1, \dots, m-1\} \times \mathbb{Z}$ to \mathbb{Z} , where $[i, l-kn] \in \{0, 1, \dots, m-1\} \times \mathbb{Z}$ and $kmn - ml + i \in \mathbb{Z}$ is bijective.

Linear Multirate Systems

❖ Realization

- ∞ Hence, $\forall k, l \in \mathbb{Z}$, $\forall m, n \in \mathbb{Z}^+$ and $i = 0, 1, \dots, m-1$, there exists a unique time index $k m n - m l + i$ corresponding to the time index $[i, l - k n]$.
- ∞ As a result, there exists an LTI filter with an impulse response $f[k]$ satisfying $f[k m n - m l + i] = g[i, l - k n]$, $\forall k, l \in \mathbb{Z}$, $\forall m, n \in \mathbb{Z}^+$ and $i = 0, 1, \dots, m-1$, that the linear multirate rate systems and the system with block sampler are input output equivalent.

Linear Multirate Systems

❖ Realization

∞ $\forall m, n \in \mathbb{Z}^+$ (no matter m and n are co-prime or not), all linear multirate rate systems (with shifting input by n samples resulting to shifting an output by m samples) can be represented via a series cascade of $\uparrow(m, n)$, followed by an LTI filter with an impulse response $f[k]$, and then followed by $\downarrow n$.

Linear Multirate Systems

❖ Realization

∞ The input output relationship of all linear multirate rate systems is $y[k] = \sum_{l \rightarrow -\infty}^{+\infty} \sum_{i=0}^{n-1} g[k, nl+i] u[nl+i]$, $\forall k, l \in \mathbb{Z}$, $\forall m, n \in \mathbb{Z}^+$ and $i=0, 1, \dots, n-1$.

∞ The input output relationship of the system with block sampler is $y[k] = \sum_{l \rightarrow -\infty}^{+\infty} \sum_{i=0}^{n-1} f[kn - mnl - i] u[nl+i]$, $\forall k, l \in \mathbb{Z}$, $\forall m, n \in \mathbb{Z}^+$ and $i=0, 1, \dots, n-1$.

∞ $\forall l \in \mathbb{Z}$, $\forall m, n \in \mathbb{Z}^+$, $k \in \{0, 1, \dots, m-1\}$ and $i \in \{0, 1, \dots, n-1\}$, the mapping from $\{0, 1, \dots, m-1\} \times \mathbb{Z}$ to \mathbb{Z} , where $[k, nl+i] \in \{0, 1, \dots, m-1\} \times \mathbb{Z}$ and $kn - mnl - i \in \mathbb{Z}$ is bijective.

Linear Multirate Systems

❖ Realization

- ∞ Hence, $\forall l \in \mathbb{Z}$, $\forall m, n \in \mathbb{Z}^+$, $k \in \{0, 1, \dots, m-1\}$ and $i \in \{0, 1, \dots, n-1\}$, there exists a unique time index $kn - mnl - i$ corresponding to the time index $[k, nl+i]$.
- ∞ As a result, there exists an LTI filter with an impulse response $f[k]$ satisfying $f[kn - mnl - i] = g[k, nl+i]$, $\forall k, l \in \mathbb{Z}$, $\forall m, n \in \mathbb{Z}^+$ and $i = 0, 1, \dots, n-1$, that the linear multirate rate systems and the system with block sampler are input output equivalent.

Linear Multirate Systems

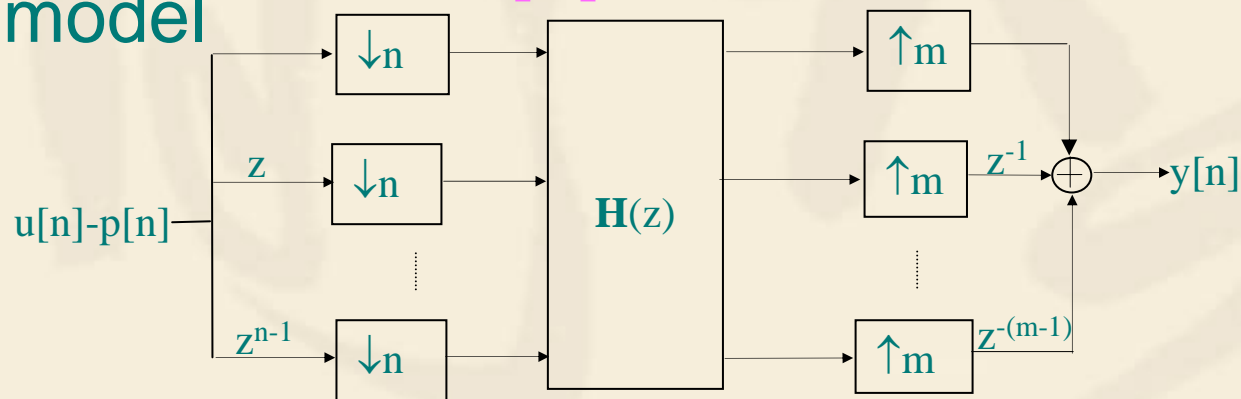
❖ Properties

- ⌘ A linear multirate system is stable if and only if $f[n]$ is stable.
- ⌘ A linear multirate system is finite impulse response if and only if $f[n]$ is finite impulse response.

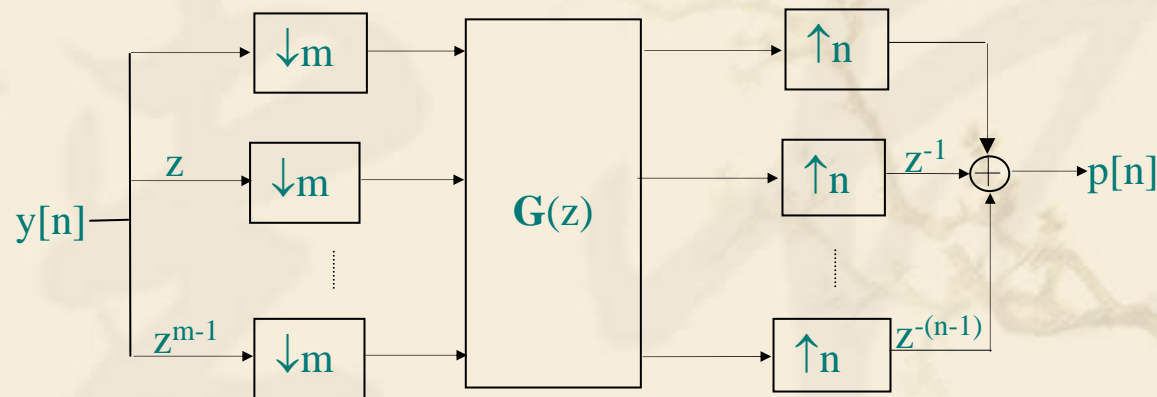
Control of Linear Multirate Systems via Filter Banks Approach

Approach

❖ Plant model



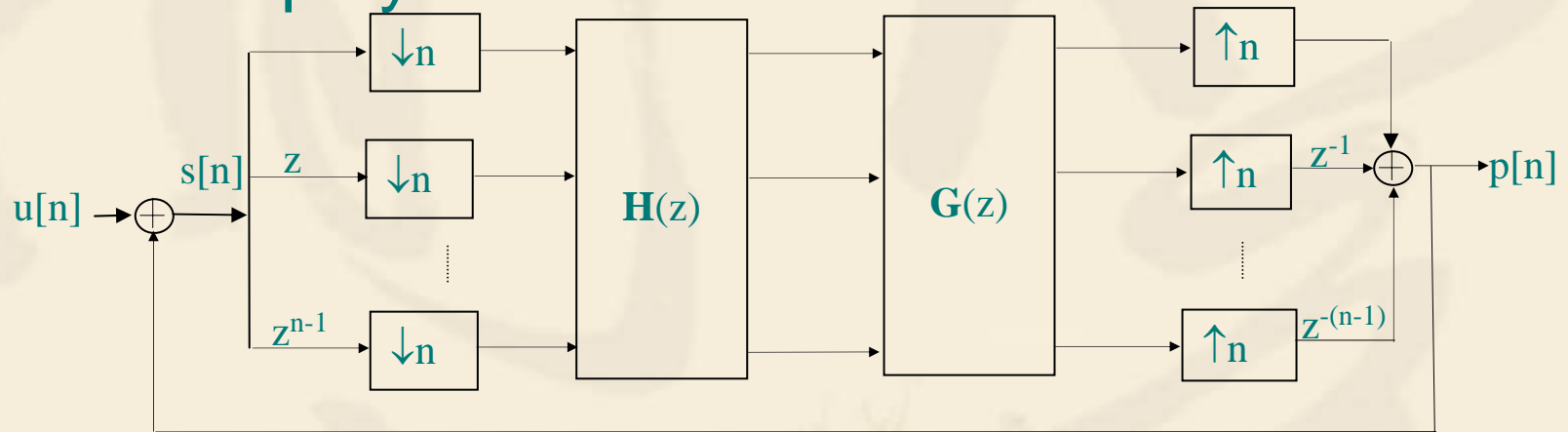
❖ Controller model



Control of Linear Multirate Systems via Filter Banks

Approach

❖ Closed loop system model



$$U(z) \equiv \sum_{\forall k} u(k)z^{-k}$$

$$P(z) \equiv \sum_{\forall k} p(k)z^{-k}$$

$$U_i(z) \equiv \sum_{\forall k} u(nk + i)z^{-k}$$

$$P_i(z) \equiv \sum_{\forall k} p(nk + i)z^{-k}$$

$$U(z) = \sum_{i=0}^{n-1} z^{-i} U_i(z^n)$$

$$P(z) = \sum_{i=0}^{n-1} z^{-i} P_i(z^n)$$

$$\mathbf{U}_p(z) \equiv [U_0(z) \quad \cdots \quad U_{n-1}(z)]^T$$

$$\mathbf{P}_p(z) \equiv [P_0(z) \quad \cdots \quad P_{n-1}(z)]^T$$

Control of Linear Multirate Systems via Filter Banks Approach

- ❖ Closed loop system model

$$Y(z) \equiv \sum_{\forall k} y(k)z^{-k}$$

$$Y_i(z) \equiv \sum_{\forall k} y(mk + i)z^{-k}$$

$$Y(z) = \sum_{i=0}^{m-1} z^{-i} Y_i(z^m)$$

$$\mathbf{Y}_p(z) \equiv [Y_0(z) \quad \cdots \quad Y_{m-1}(z)]^T$$

$$\mathbf{G}(z)\mathbf{H}(z)(\mathbf{U}_p(z) - \mathbf{P}_p(z)) = \mathbf{P}_p(z)$$

$$\mathbf{P}_p(z) = (\mathbf{I} + \mathbf{G}(z)\mathbf{H}(z))^{-1} \mathbf{G}(z)\mathbf{H}(z)\mathbf{U}_p(z)$$

Control of Linear Multirate Systems via Filter Banks Approach

❖ Closed loop system model

$$\begin{aligned} \mathbf{Y}_p(z) &= \mathbf{H}(z)(\mathbf{U}_p(z) - \mathbf{P}_p(z)) \\ &= \mathbf{H}(z)\left(\mathbf{I} - (\mathbf{I} + \mathbf{G}(z)\mathbf{H}(z))^{-1}\mathbf{G}(z)\mathbf{H}(z)\right)\mathbf{U}_p(z) \\ &\quad \mathbf{H}(z)\left(\mathbf{I} - (\mathbf{I} + \mathbf{G}(z)\mathbf{H}(z))^{-1}\mathbf{G}(z)\mathbf{H}(z)\right) \text{ is stable.} \end{aligned}$$

Conclusions

- ❖ Only three types of behaviors for autonomous response of linear time invariant systems.
- ❖ Behaviors of linear time invariant systems only depend on the system poles, not on initial conditions.
- ❖ Stability conditions based on pole placement, state feedback and output feedback of linear time invariant systems are derived.
- ❖ Linear multirate systems can be realized via a filter bank.
- ❖ When the input rate and the output rate is co-prime, then linear multirate systems can be realized via linear rate changers. Otherwise, they can be realized via block samplers.
- ❖ Stability conditions for linear multirate feedback systems are derived based on filter bank approach.

Questions and Answers

