MIDDLE SCHOOL RATIONAL NUMBER KNOWLEDGE

by

SHERRI L. MARTINIE

B.S., University of Kansas, 1991
M.S., University of Kansas, 1998

AN ABSTRACT OF A DISSERTATION

Submitted in partial fulfillment of the requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Curriculum and Instruction
College of Education

KANSAS STATE UNIVERSITY
Manhattan, Kansas

2007

Abstract

This study examined end-of-the-year seventh grade students' rational number knowledge using comparison tasks and rational number subconstruct tasks. Comparison tasks included: comparing two decimals, comparing two fractions and comparing a fraction and a decimal. The subconstructs of rational number addressed in this research include: part-whole, measure, quotient, operator, and ratio. Between eighty-six and one-hundred-one students were assessed using a written instrument divided into three sections. Nine students were interviewed following the written instrument to probe for further understanding. Students were classified by error patterns using decimal comparison tasks. Students were initially to be classified into four groups according to the error pattern: whole number rule (WNR), zero rule (ZR), fraction rule (FR) or apparent expert (AE). However, two new error patterns emerged: ignore zero rule (IZR) and money rule (MR). Students' knowledge of the subconstructs of rational numbers was analyzed for the students as a whole, but also analyzed by classification to look for patterns within small groups of students and by individual students to create a thick, rich description of what students know about rational numbers. Students classified as WNR struggled across almost all of the tasks. ZR students performed in many ways similar to WNR but in other ways performed better. FR and MR students had more success across all tasks compared to WNR and ZR. On average AEs performed significantly better than those students classified by errors. However, further analysis revealed hidden misconceptions and deficiencies for a number of AEs. Results point to the need to make teachers more aware of students' misconceptions and deficiencies because in many ways errors reflect the school experiences of students.

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This is dedicated to the friends and family that supported me throughout this endeavor in so many ways: emotionally, intellectually, and financially.

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CHAPTER 1 - Introduction

Proficiency in mathematics is necessary for a person's success both in school and throughout life. Low mathematics achievement is correlated with a number of social problems including high school dropout rates, delinquency, unemployment and homelessness. Regardless of the importance of learning mathematics or the detriment of not learning it, it has become acceptable in American society to claim to have a lack of ability to learn mathematics and then become excused from learning it. According to Marilyn Burns in her book, Math: Facing an American Phobia, "The negative attitudes and beliefs that people hold about mathematics have seriously limited them, both in their daily lives and in their long-term options" (Burns, 1998, p. ix).

This not only limits what individuals do with their lives but it affects our society as a whole as well. According to government statistics, (http://www.ed.gov/rschstat/research/progs/mathscience/whitehurst.html)

> While levels of achievement in mathematics among U.S. students are low, the demand for a mathematically proficient workforce is increasing. The United States cannot fill all the jobs in mathematically intensive fields with qualified U.S. citizens. As a result, Congress has been forced in recent years to provide an expanded pool of visas for foreign nationals with high-tech skills. At the same time, the number of college degrees awarded in technical areas has dropped sharply for United States citizens.

The mathematics that is needed in daily life has grown in sophistication. Fortunately, we have also learned a great deal recently related to how people learn such complex material (National Research Council (NRC), 2000 & 2001). Recent research and development on cognitive development theory and informational processing models can assist in creating a description of the complex mathematical concepts students study as well as assist in the intervention process, making the much-needed progress toward improvement possible.

One mathematical concept that is frequently encountered in everyday life is that of rational numbers. Decimal fractions, for example, are everywhere – on the labels of bottles, cans, or boxes that we use, on tickets or tags where we shop, in advertisements and newspaper reports that we read, on the library shelves where we retrieve books, on gas pumps, on meters, and on the gages in the cars that we drive. Despite their regular occurrence, the persistence of problems working with decimal fractions has been well-documented (Bell, Swan, & Taylor, 1981; Carpenter, Corbitt, Kepner, Lindquist & Reys, 1981; Post, 1981; Kouba, Carpenter, & Swafford, 1989). On the fourth National Assessment of Educational Progress (NAEP), basic misconceptions about decimal fractions were held by half of the seventh graders (Kouba, Carpenter, & Swafford, 1989). There is evidence that these misconceptions are lasting and continue into adulthood (Putt, 1995; Thipkong & Davis, 1991; Stacey, Helme, Steinle, Baturo, Irwin, & Bana, 2001). "NAEP data has shown reasonably skilled performance with rational numbers is not widely achieved until high school. If competence requires years of learning, studies of older, more advanced students are also needed" (Smith, 1995, p 8).

A major concern lies with those students who progress through instruction of rational numbers, beginning with formal instruction in early elementary school and continuing through middle school, and still do not grasp the concepts (Glasgow et. al., 2000). This happens all too frequently. Given the importance of rational number concepts, intervention is necessary. But before this can happen, we must clearly understand the problem. The importance of considering various interpretations of teaching and learning mathematics is not a new idea. Hiebert (1984) states, "If instruction is going to build upon children's strengths and remediate their weaknesses, we must become aware of how mathematics looks to them." (p. 507) In order to "prescribe" an appropriate program for students we must consider a description of their learning behavior that is dynamic and allows for flexibility when describing change and growth. Students can be in different places in terms of their development and the purpose of this research is to create a thick, rich description of the range of knowledge that students have about rational numbers at the end of seventh grade.

Background

Much research has been done in the field of rational number, but many questions remain. There is a great deal of research that focuses on fractions, decimals and percents in isolation from one another. Researchers have also approached the study of rational number by closely examining pre-rational number concepts such as partitioning, place value, and the concept of a unit. Many studies take an in-depth look at just one of these topics. Other research focused on identifying one or two subconstructs of rational number and analyzed students' understanding of those subconstructs extensively. However, five subconstructs exist across these studies and no research has encompassed all five. Research also has focused on identifying and understanding student misconceptions or deficiencies regarding the learning of rational number such as over-generalizing whole number principles. Research has yet merged these ideas to form complete description of the knowledge that students possess. Creating such a model of this knowledge starts with a description of the range of knowledge and types of skills that students have related to rational numbers. The research reported here aims to provide that necessary first step toward an understanding of 7th grade students' rational number knowledge.

Subconstructs of Rational Numbers

The concept of rational number is complex and sophisticated. It consists of several interpretations and requires understanding many other concepts. Creating a complete, cohesive, and concise picture is difficult. Because the concept of rational number is extensive, studying and then describing it requires breaking it down into subconstructs, identifying what students know about the subconstructs, which subconstructs they understand and then researching the relationships that exist among rational number concepts. Taking these ideas from the isolated cases where they have been previously studied and using what has been learned, a description of how students conceptualize rational numbers can be created. It would be comprehensive because it would do three things. It would merge the subconstructs of rational number. Describe how well students understand the multiple ways in which rational numbers can be represented (fractions, decimals, and percent). Describe how students apply their prior knowledge of whole numbers.

Rational number has been broken down into several subconstructs (Behr, Lesh, Post, & Silver, 1983; Kieren 1980). At least five interpretations have been identified and to deeply understand rational numbers children have to know each subconstruct independently as well as the relations among them. Although the names may vary from one researcher to another, the five constructs are: a part-whole comparison; a measure of continuous or discrete quantities; a quotient (result of division); a ratio; and an operator. Each subconstruct is briefly described here and described in detail in chapter two.

Part-Whole Subconstruct

The part-whole subconstruct, commonly expressed as a fraction, is often considered the foundation of rational number knowledge and basic to the all other interpretations (Behr et al., 1983; Freudenthal, 1983; Kieren, 1988; Pitkethly and Hunting, 1996; Ni & Zhou, 2005). The part-whole subconstruct is based on a student's ability to partition a continuous amount or a set of discrete items into equal-sized groups. For example, four-fifths as parts of a whole is interpreted as four of five equal-size pieces.

Measure

The measure subconstruct involves identifying a fixed unit of measure, a length, and then using that quantity repeatedly as well as dividing that unit into smaller parts. For example, when considering 3/8, one must identify the unit fraction 1/8 and then use that repeatedly to reach 3/8. It can be thought of as a ruler and the fraction or decimal as representing a length. The number connected with an object when "measuring" is the number of units or parts of a unit the object is equal to on the "ruler". A common model used for measure purposes is a number line. The number line is formed from an iteration of a unit and also from the partitioning of each unit and parts of units into smaller, equal-sized parts. This model is different from an area model or a discrete set of objects because it focuses attention on how much rather than how many. In this way, "the fractional measure subconstruct of rational number represents a reconceptualization of the part-whole notion of fractions" (Behr et. al., 1983, p. 99). This encourages students to shift their thinking from seeing a fractional amount as two separate whole numbers to representing a single amount.

Ratio

A ratio is defined as "a statement of the numeric relationship between two entities" (Hart, 1988) and "it is more correctly considered as a comparative index rather than as a number" (Behr et al, 1983). Ratio is based on the ability to coordinate, for example, the number of people sharing and the number of objects shared and to think of this relationship as a composite unit (Lamon, 1994; Streefland, 1991). For example, 3/8 is interpreted as 3 pizzas for every 8 people. It is also commonly thought about as a part-part relationship, which is more obvious when comparing the same units. For example, 4/5 is interpreted, as four red balls for every five white balls where part of the balls is red and part are white.

Quotient

The quotient subconstruct involves thinking of a rational number as an answer to a division problem (an indicated quotient). For example, 4/5 is interpreted as five children sharing four pizzas. This subconstruct is like part-whole and ratio in that the symbols a/b are used in each, however in the quotient subconstruct the meaning for the student is that a/b is a÷b. Behr et al. (1983) state that

> "According to the part-whole interpretation of rational numbers, the symbol a/b usually refers to a fractional part of a single quantity. In the ratio interpretation of rational numbers, they symbol a/b refers to a relationship between two quantities. The symbol a/b may also be used to refer to an operation. That is, a/b is sometimes used as a way of writing a÷b."

Operator

The operator subconstruct is used to refer to a rational number that "represents a multiplicative size transformation where a quantity is reduced to a fraction of its original size by both partitioning the quantity and duplicating various portions of the quantity" (Mack, 2000, p. 309). For example ½ behaves as an operation when taking ½ the length and ½ the width of a rectangle, which reduces the area to ¼ of its original size. It is an operation that can enlarge or "stretch" as well as reduce or "shrink" an object. Behr et al. (1983) suggest that the operator interpretation of fractions helps a student to understand multiplication of fractions, especially in the case where the operator is viewed as "finding" or "taking a part of a part of a whole".

Prior Knowledge

Whole Number Knowledge

Students use their prior knowledge of whole numbers when dealing with rational numbers and this often impedes the development of rational number concepts (Mack, 1995). Students must understand basic numeric properties for working with whole numbers but they must also understand how the properties of rational numbers are different. Order and equivalence comparing whole number properties look different than comparing rational number properties (Smith, 1995). Not recognizing this, students will often manipulate fractions in symbolic notation as if they are two independent whole numbers rather than a single quantity. This will result in addition and subtraction of fraction procedures where students add numerators together and add denominators together rather than obtaining a common denominator (Mack, 1995). Decimals are also viewed as two independent whole numbers, one before and one after the decimal point. This often results in incorrect strategies for writing fractions as decimals. For example, ¼ becomes 1.4. Another example of misapplication of arithmetic properties is the infamous misconception “multiplication makes bigger and division makes smaller”; a misconception that often carries into adulthood (Graeber, Tirosh, & Glover, 1989; Tirosh, Fischbein, Graeber, & Wilson, 1998).

Equivalence/order

Rational number equivalence is a complex mathematical concept (Vance, 1992). It is complex because rational numbers have numerous equivalent representations. Renaming a number using another representation changes the appearance but does not change its properties. In addition, the best choice for representing a number often depends on the situation. Understanding this and identifying the best representation stems from experience with multiple contexts. The development of equivalence with rational numbers progresses slowly over time (Vance, 1992). Research indicates that students construct the concept of equivalence one subconstruct at a time making the development of equivalence a recursive process (Ni, 2001).

Comprehensive Analysis of Rational Number

Research in the field of rational number has focused on identifying ways in which rational numbers can be interpreted based on a given situation. These interpretations form the five subconstructs of rational number. Rational number can also be represented in multiple ways, which gets away from the one-to-one correspondence of symbol to referent that students have grown accustomed to in early years. In addition, research has identified how prior knowledge influences the development of rational number knowledge. What remains to be done is the “packaging” of that knowledge. Liping Ma (1999), in a study comparing teachers’ understanding of fundamental mathematics in the United States and China, found that the Chinese teachers consider the mathematics being taught in terms of a knowledge package. A knowledge package is a network of conceptual and procedural knowledge of connected topics in which some knowledge supports what is being learned presently and other knowledge is supported by it. It is the roadmap used on a journey. As a teacher presents a new topic, he or she must know the role of that knowledge in the knowledge package. Extending this to rational numbers, relationships need to be identified among the subconstructs and mapped in such a way as to illustrate how the development of rational number concepts occurs. This would take all of the pieces of research and unite them in a way that reflects how students learn about rational number. Before a knowledge package can be created, a thorough description of the range of knowledge that students possess is necessary. To proceed without this description would be like identifying towns on the roadmap without identifying the roads and highways connecting them. Those connections would occur later. The research provided here seeks to describe the range of student knowledge of rational number.

Theoretical Perspective and Conceptual Framework

Overview

Some experts contend that understanding develops in stages and the states increase in sophistication and complexity over time (Piaget, 1950; Bruner, 1964; Biggs & Collins, 1991). Instruction of rational numbers primarily occurs during the concrete symbolic stage of development from about age six to sixteen. According to research, there are "learning levels" within each of the stages of development, including the concrete symbolic stage. These levels indicate the progression of a child's ability to move from the basic elements of the stage "to an integrated and sophisticated use of those elements".

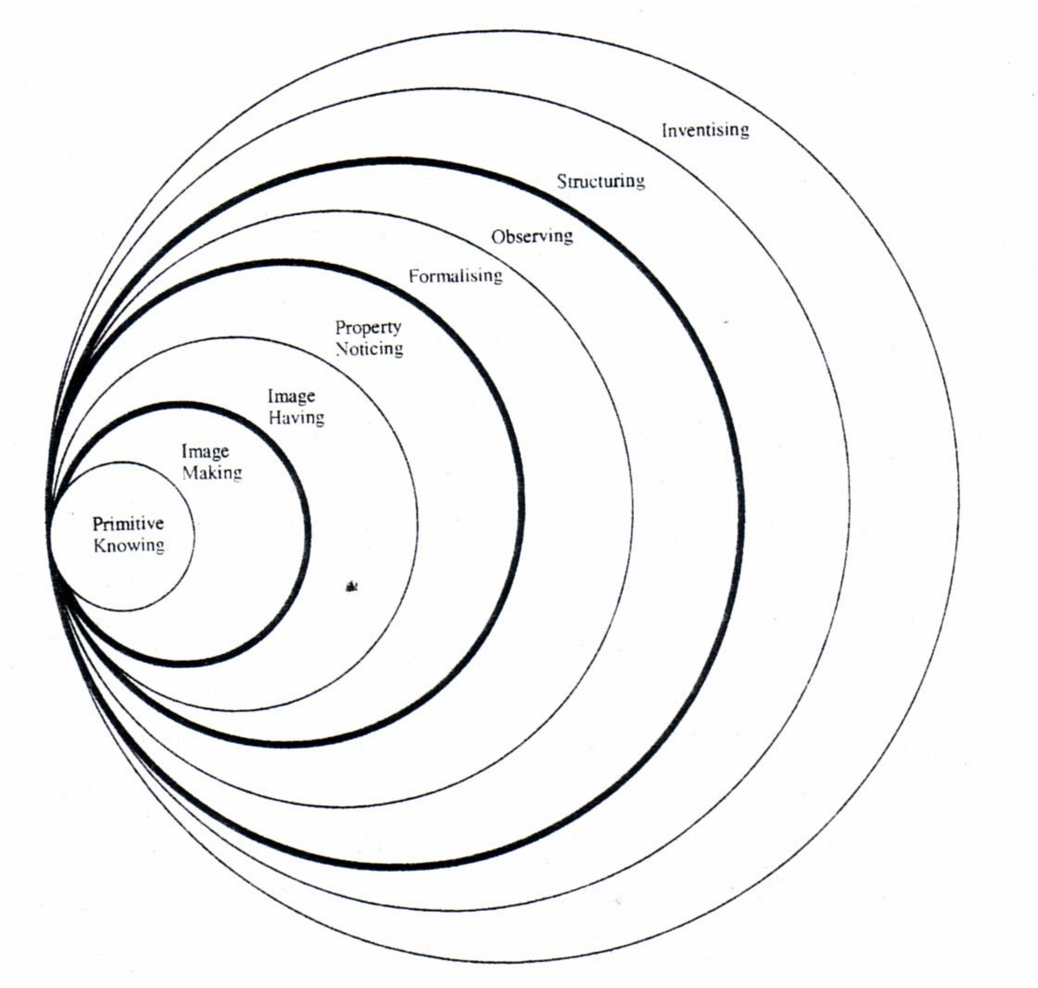
However, others do not view learning as occurring in levels but rather as a dynamic and interactive process. Rather than a linear development of mathematical thinking as being linear, it is viewed as recursive with opportunities for folding back to previous levels or using knowledge at any level as input for another level. Experts such as, "Di Sessa, Hatano, and Pirie and Kieren theorize that understanding develops in a nonlinear manner that is characterized by a need for frequent returns to students' initial understandings" (Mack, 2001, p. 268).

The goal of the research reported here was to describe what students know about rational numbers. Research on rational number indicates that learning does not occur all at once and does not occur systematically by combining partial understandings together. Rather it occurs by a process of reorganization, "disconnecting, connecting, reorganizing appear to be the rule rather than gradual addition to a stable structure" (Hiebert, Wearne, & Tabor, 1991, p. 339). Therefore, descriptions of students in this research will vary. Partial understandings exist and can overlap and students can apply prior knowledge in different ways. The range of knowledge students possess can be explained in a number of ways. One such way is Pirie-Kieren model of growth of mathematical understanding will highlight why describing student's knowledge of rational numbers is a complex endeavor. This framework provides structure for the description of students' knowledge of rational number across subconstructs and representations.

Pirie and Kieren Dynamic Theory

Pirie-Kieren dynamical theory for the growth of mathematical understanding is one model that informs this research. It is dynamic. It involves not only identifying the types of knowledge that students possess but also considers the growth of their understanding. Pirie and Kieren (1994) committed several years to the development of just such a model, which was subsequently revised many times (Pirie & Kieren, 1989; Pirie & Kieren, 1990). The significant piece of their model is that, "It is a theory of growth of mathematical understanding as a whole, dynamic, leveled but non-linear, transcendentally recursive process". Their model involves eight levels labeled within nested circles. The levels are: primitive, image making, image having, property noticing, formalizing, observing, structuring, inventising.

Figure 1-1 Pirie-Kieren model of growth of mathematical understanding (Pirie & Kieren, 1994)



Each level contains and structurally includes all prior levels nested within it. Growth is described as a dynamic organizing process. Extending knowledge involves abstracting to a new outer level as well as folding back to recursively reconstruct the inner levels.

How a concept is learned can influence what the knowledge looks like and therefore drive the description of the knowledge that has been acquired. When describing what students know about rational numbers, this recursive theory will be considered and help frame the description.

Statement of the Problem

Purpose

Students struggle with rational numbers even into adulthood. A growing body of research has identified various pieces of the complex and web-like concept of rational number. What is needed is a way to tie this all together and determine how the subconstructs; part-whole, quotient, measure, ratio, and operator, weave their way through the representations of rational number, which can then be used in later research to determine if there is a developmental process by which students acquire understanding across the five subconstructs. The types of errors students make and misconceptions they have are well documented and were integrated into the description of what students know. My primary objective was to describe what students know about rational number concepts by the end of seventh grade.

Research Questions

How do seventh grade students in a small rural school district conceptualize rational numbers? Specifically:

1. What type of strategies, including error patterns, do students use when comparing within and across decimal and fraction representations?
2. How successful are students at solving rational number tasks for each of the five subconstructs?

Significance of the Study

This study adds to the existing body of knowledge by providing an overview of the extent to which middle school students understand tasks across the five subconstructs of rational number and the frequency of middle school students who apply prior knowledge of whole number or fractions to developing notions of rational number.

Collecting this information produced a much-needed comprehensive picture, a knowledge package, of what students know about rational numbers. This pulls together the great body of knowledge that exists regarding rational numbers to create a thick, rich description of rational number knowledge. Later research will involve finding a relationship between the pieces of knowledge collected here. Finally, if relationships exist, then this study will also inform curriculum design and the development of diagnostic assessments.

Methodology

Subjects

All students in grade 7 of one middle school in a rural setting took two written assessments, a conceptual rational number test and a district minimal competency test. Some of these students were purposefully selected for interviews based on their performance on these two assessments.

Strategy of Inquiry

Students were required to take and pass a minimal competency test to receive their high school diploma. The test measures basic skills and computational fluency. Students begin taking the test at the end of 7th grade. If they do not pass, they continue to receive instruction and take the test at the end of each semester in December and May until they pass. All students participated in a rational number concepts test whose items represented each of the five subconstructs as well as items that assessed understanding of order and equivalence applied to rational numbers. Based on the order and equivalence items, students were classified as successfully applying the following “rules”: whole number rule, fraction rule, zero rule, ignore zero rule, money rule, or expert. Several were considered unclassified. Students from three of the classifications, whole number rule, fraction rule and expert, were selected for an interview. The written assessments and the interview items assessed students understanding of the five subconstructs and were scored and responses analyzed for patterns. These patterns were used to form a description of how students conceptualize rational numbers.

Limitation and/Delimitations of the Study

Limitations

This study was limited in several ways. One, it looked at only one population of students from the same school studying from a nontraditional, standards-based program. This could influence how student organize and think about the different constructs when compared to students using a traditional approach to learning. Second, the students in this study were all at the same grade level, and while this will present a picture of the range of knowledge students possess at this grade, it may not have the range of understanding that would represent all middle school students. Third, this study was limited because it assumes the students are trying to be successful. While the study was designed to give the written conceptual rational number test in small sections over several short testing periods, students were not “graded” on this instrument and may not have felt obligated to do their best work. In addition, testing at the end of the school year was not an ideal time. However, it was necessary in this case so that no additional instruction would occur between the written instruments and the interviews, which occurred over the summer. Finally, the items on the conceptual instrument were taken from previous research. Because the objective of this research was to look at all the subconstructs simultaneously, a small sample of items was selected. This could alter the effectiveness of these items to measure knowledge of the subconstruct that it was intended to measure.

Definition of Terms

Decimal fractions are the result of dividing by ten, one-hundred, one-thousand, and so forth in the base-ten number system. In this case, the denominator is a power of ten but is expressed with out the use of the denominator. The digits 0-9 are used with a decimal point, which separates the whole number digits (integers) from the fractional part. In this paper decimal fractions are simply referred to as decimals if there is a decimal point and a fractional part.

Rational numbers, A rational number is any number that can be written in the form r=a/b, where a and b are integers and b is not zero. Rational numbers include all terminating or repeating decimals. Some examples include: 1/2, 23.45, -1.3, $0.\overline{3}$. Also note that all integers are rational numbers because n=n/1.

Subconstructs, in this paper, are intended to be the various interpretations of uses of rational numbers as discussed throughout. The larger concept of rational number can be broken down into parts.

Summary

This study contributes to the existing literature by compiling current research and creating a comprehensive picture of what students know about rational numbers. This study was based on a need for a system that identifies students' strengths and weaknesses when dealing with rational numbers in order to help jump the hurdles that impede instruction. Needed was a model for describing learning behavior related to rational numbers that is dynamic and allows for continuous growth and change. It provides information regarding the important background knowledge that students bring with them and the prior experiences that influence their level of understanding. It enabled us to assess students' current levels of understanding in order to prescribe the necessary instruction to continue to progress.

Previous research focused on specific components of learning rational numbers. It was used as a basis for this research, which described rational number knowledge broken into five subconstructs and described what students know about comparison, order, and equivalence of rational numbers. Future research will describe how these highly specialized concepts are related and together build a strong foundation for understanding rational numbers. The description created here formed a much-needed "big picture" of the understanding of rational number.

In the story Seven Blind Mice (Young, 1992) seven blind mice encounter something strange by the pond. One by one they go to investigate and each returns with a different theory. One says, "It's a pillar". Another says, "It's a fan". The seventh mouse takes his time investigating. He runs on top of the object and back and forth. He finally returns a more accurate description of the thing. The moral of the story is that "Knowing in part may make a fine tale, but wisdom comes from seeing the whole". In order to really understand rational numbers, educators need to put all of the pieces of research together to form a cohesive, coherent, well-articulated package.

CHAPTER 2 - Review of the Literature

The teaching and learning of rational numbers has been researched extensively, yet middle school students in the U.S. and internationally continue to struggle in this area. “Much has been written about the competencies of experts and the deficiencies of novices. But how does one characterize persons in between? What do partial understandings look like as students are beginning to grapple with difficult mathematical ideas?” (Hiebert, Wearne, & Tabor, 1991 p. 324). These are the question this research proposed to answer. More precisely, the researcher’s purpose with this study was to describe, in a comprehensive and coherent fashion, the way in which students conceptualize rational numbers and to investigate the development and/or interdependent factors in students’ attainment of the subconstructs of rational numbers.

The concept of rational numbers is complex; but as this review of literature will illustrate, it can be broken down into manageable pieces that, like a puzzle, can then be put together. This review is intended to provide a complete picture of rational number from the perspective of the researcher, based on interpretation of research studies and research reviews. The review also provides a framework for the study. This chapter begins with a definition of rational number and an explanation of where it “begins”. Following this, the researcher discusses important “unifying and supporting” elements, addresses multiple representations, describes each of the subconstructs, and presents the theoretical framework on which this study is based.

Rational number refers to a formal system built up by generations of mathematicians. Thompson & Saldanha, (2003) contend “that understanding the rational number system, where ‘rational number’ is used as mathematicians use it, is so far beyond the grasp of school students that curriculum and instruction designers must be clear on what they mean by ‘fractions’ and ‘rational numbers so they avoid designing for incoherent learning goals” (p. 98). “ In elementary and middle school, rational number is interpreted as a more personal and informal understanding of rational number ideas. It is still a challenging concept, but not at the abstract level originally intended by mathematicians. One reason rational numbers are complex is because they have various interpretations based on particular situations, which are referred to as subconstructs.

Analysis of the rational number construct has produced several subconstructs. Separately each provides a different interpretation of rational number. Interwoven they shape the foundation of a meaningful understanding of rational number. Kieren's (1980) analysis produced five subconstructs: part-whole relations; ratios; quotients; measures; and operators. These subconstructs have been confirmed in other research and additional subconstructs have been suggested. Some argue that these additional subconstructs have a place in the original five. Those with a mature, fully developed understanding of rational number differentiate as well as integrate the subconstructs in a meaningful way (Behr, Lesh, Post, & Silver, 1983).

In addition to breaking down rational number knowledge into subconstructs based on the interpretation of the situation in which they are being used, scholars have identified other important ideas related to rational number. Pitkethly and Hunting (1996) identified "unifying elements" and "supporting elements" which appear to be of importance when looking across all of the subconstructs. Unifying and supporting elements will be addressed in this chapter. Many of these elements are extensions of whole number knowledge, which is the natural place to begin.

Whole Numbers

Whole numbers are the foundation upon which students build their rational number knowledge (Hunting & Davis, 1996). Prior knowledge of whole numbers applied to rational numbers both helps and hinders development of rational number knowledge. Constructivist theory contends that learners actively build their knowledge on prior knowledge. Applying acquired notions of whole numbers to developing notions of decimals is a common occurrence among middle school students. The inherent relationships between whole number and decimal knowledge are in some ways similar and some ways different (Resnick, Nesher, Leonard, Magone, Omanson, & Peled, 1989; Sackur-Grisvard & Leonard, 1985; Vance, 1986). For example, the structure of decimals is similar to whole numbers in that the values of digits increase the further they are to the left of the decimal point, but different in that the value of the digits decreases the further they are to the right of the decimal point.

Research indicates that tasks perceived as the same result in retrieval of rules for the task that was learned earliest, as it tends to be strongest (Behr, Wachsmuth, Post, & Lesh, 1984; Hiebert & Wearne, 1985; Resnick et al., 1989; Zazkis & Khoury, 1993). For example, a child's schema for ordering whole numbers is very strong and tends to be over generalized initially, although this tends to diminish with instruction (Behr et al., 1984). This is evident when studying items taken from the second National Assessment of Educational Progress (NAEP) that indicate many 13-year-olds ignore the decimal and treat the number as a whole number (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981).

Whole Number Bias

Ni and Zhou (2005) use the term "whole number bias" when they refer to the use of prior knowledge of whole numbers applied to emerging fraction concepts. A bias is a divergence from the norm, and the norm is considered "an adult model of something in developmental psychology" (Ni & Zhou, 2005 p. 29). Fractions and decimals can appear similar to whole numbers in many ways or can be interpreted as similar when they are not, thus causing a "bias". Initially, whole number knowledge inhibits learning rational numbers because children over generalize their counting principles and because whole numbers have a "next" number where rational numbers do not (Behr et al., 1984; Gelman & Meck, 1992). Over generalizing counting principles explains why students believe that a fraction with a larger denominator is larger than one with a smaller denominator.

Applying whole number properties inappropriately also occurs because students fail to understand the symbolic representation of fractions, which is often prematurely introduced. Research indicates that students base their informal knowledge of fractions on partitioning units and then treat the parts as whole numbers (Ball, 1993; D'Ambrosio & Mewborn, 1994; Mack, 1990, 1995; Streefland, 1991). Without a meaningful understanding for fraction symbols, students acquire misconceptions from attempting to apply rules and operations for whole numbers to fractions. Based on what they know about whole numbers and their limited knowledge of rational numbers, it is natural that students would do this. Mack (1995) indicates that with time and direct effort students can separate whole number from rational number constructs and develop a meaningful understanding of how fractions are represented symbolically.

Pre-rational Number Knowledge

Students possess certain prerequisite concepts that apply to whole number and rational number, and understanding of those has an impact on the subconstructs. Therefore, each of these will be described here. These are the building blocks for understanding rational number.

Partitioning

Partitioning is defined as subdividing a continuous whole into equal parts. Partitioning is fundamental to building initial rational number concepts (Ball, 1993; Mack, 1991, 1993; Kieren et al., 1992; Streefland, 1991, 1993) and is a cognitive predecessor to fractional numbers. Partitioning experiences permit students to discern the compensatory relationship between the size and number of parts and the inverse relationship between the denominator and the value of the fraction even before formal instruction on symbols begins.

In a review of research of initial fraction concepts, Pitkethly and Hunting (1996) identified two perspectives on the early thoughts of children and how these thoughts form a primitive understanding of fractions, one focused on partitioning as the foundation of rational number knowledge and the other on ratio. According to the first perspective, initial fraction concepts appear to develop from experience with partitioning, in continuous or discrete contexts, which leads to unit identification and iteration of units. Children as young as first grade can use their developing notion of partitioning to recognize that a quantity divided more times results in smaller pieces (Empson, 1995). Research indicates that reasoning about fractions emerges around the time that children enter school (Goswami, 1989; Spinillo & Bryant, 1991). Using several informal strategies, young children root their initial fraction knowledge in experiences involving dividing sets and amounts among several recipients (Frydman & Bryant, 1988; Hunting & Sharpley, 1988).

Kieren (1980) suggests that partitioning is fundamental to the meaningful construction of rational number just as counting is the basis to understanding the construction of whole number. In fact, the construction of initial fraction concepts hinges on the coordination of counting and partitioning schemes (Mack, 1993). Fundamental to rational number is the notion that partitioning results in a quantity that is represented by a new number. For many children, coordinating their ideas to reach this level takes time and experience. For example, Mack (1995) found that third and fourth graders use their understanding of partitioning to connect operations on fractions to their prior knowledge of whole numbers. This enables them to solve addition and subtraction problems with a common denominator by separating the numerator and denominator and thinking only about the number of pieces combined or removed. Although this process assists them with the operation, it limits their conceptions about fractions by treating them as whole numbers. The same occurs with decimal numbers where children separate parts from the number and treat them as a whole number.

Research indicates that young children have a rich store of informal knowledge in the context of equal sharing (Davis & Pitkethly, 1990; Hunting & Sharpley, 1991; Pothier & Sawanda, 1983). Streefland (1991, 1993) found that initiating prior knowledge of equal-sharing and using equal-sharing problems assists elementary children as they develop knowledge of fractions. Empson (1999) found that children as young as first grade can understand fractions through equal-sharing tasks that draw on their informal knowledge. It is theorized that partitioning skills develop systematically through a five-level process, which occurs over time as students integrate part-whole relationship with their understanding of area (Pothier & Sawada, 1983). Within the five level theory, “each level is distinguished by certain conceptual characteristics, procedural behaviors, and partitioning capabilities” (Pothier & Sawada, 1983, p. 316). Level one involves sharing. Level two involves schemes that involve halving and lead to iterated halving. Level three is evenness. Level four extends partitioning to include oddness. Research indicates that partitioning that is limited to knowledge of ½ and algorithmic halving from level two hinders the child’s ability to develop partitioning schemes to create fractions with odd number denominators. Level five involves a composition of partitioning strategies.

The significance of partitioning extends beyond the concept of a fraction; it may also provide a basis for multiplication of fractions (Behr, Harel, Post, & Lesh, 1992, 1993, 1994; Empson, 1999; Mack 2000). In their review, Lachance and Confrey (2002) indicated that splitting actions, which include sharing, folding and magnifying, are derived from early experiences and exist intuitively in children. These actions and the resulting thoughts about them enable students to develop important ideas about multiplication, and in tandem, division and ratio. Thus, it follows multiplication, division, and ratio be introduced to children simultaneously to encourage the use of their prior knowledge and develop meaningful constructs. Constructing understanding of multiplication of fractions based on informal knowledge of partitioning has lasting effects (Mack, 2000). First, students used ideas about partitioning to reconceptualize units, partition units in different ways, and solve multiplication of fraction problems in ways that made sense to them. Second, students did not have a strong connection between knowledge of partitioning and the symbolic expression, which took time to rebuild. Third, after being introduced to algorithmic procedures for multiplication of fractions, students' reliance on the informal knowledge of partitioning was transformed. Finally, students used informal knowledge of partitioning to defend their answers when implementing algorithmic procedures.

Unitizing

The identification of a unit as well as the flexible concept of a unit (the unit as it is decomposed, recomposed, and reconstructed) is considered to be "pre-fractional" knowledge. Lamon (2002, p. 80) defines unitizing as "the process of mentally constructing different-sized chunks in terms of which to think about a given commodity". She refers to the ability to reconceptualize quantities into different-sized groups that are easier to think about as "chunking". An example is thinking about a 24-pack of cola as a case of pop, two 12-packs, or four 6-packs. Another example is thinking of five fingers as one hand. A child uses knowledge of whole number counting to form iterable units and then merges this with the act of division. Despite the fact that counting seems to be the basis of unitizing, Pepper (1991, 1993) found no significant relationship between children's counting competence and their ability to equally distribute discrete items and in essence form units.

Unitizing and reunitizing, in conjunction with partitioning, are essential skills for understanding decimal numbers. For example, a hundredths grid has one hundred, small squares that are thought of as counting units. Two different "composite units" are formed. A "tenth" is a new composite unit consisting of a row or column of ten small squares. A "whole" is a new composite unit consisting of one hundred small squares arranged in ten rows of ten. Identifying "tenths" as the square composed of ten rows (or columns) requires the ability to consider units of units (Baturo & Cooper, 1997). "Tenths" are produced from partitioning a whole unit into ten parts. "Tenths" then represents part of a unit as well as a unit itself by re-unitizing to produce hundredths. Relating this back to the original whole, 3 tenths is equivalent to 30 hundredths.

The concept of unit is fundamental to developing operations with whole numbers leading to rational numbers. Multi-digit numbers require students to think about the various sizes of multi-units and how to decompose and recompose multi-digit numbers to perform operations

(Fuson, 1990). When extending this to multiplication of fractions, students must recognize that units can be partitioned in a variety of ways and that the partitioned units can be redefined and reconceptualized as a unit resulting in a unit of units and enable multiplication schemes (Behr et al., 1992; Kieren, 1995; Mack, 2000; Olive, 1999; Steffe, 1988). D'Ambrosio and Mewborn (1994) found "the fundamental hypothesis of the Fractions project staff was that children need to construct a definition of a unit fraction as an iterable unit which can be used to reproduce the whole or to produce any other fraction with that denominator" (p. 151).

Notion of a Quantity

The construction of sound rational number concepts should be based on a strong quantitative notion of rational number (Behr, Wachsmuth, & Post, 1983). An individual must acquire the ability to perceive of the relative size, or "bigness", of rational numbers (Lamon, 1993). Rational numbers are dense, unlike whole numbers; there is always another number between them. They don't form a fixed succession of numbers as counting numbers do, which makes it more difficult to conceptualize the size. When considering where a fraction might be on a number line, students must shift their thinking. For example, consider 4/5, a student must shift from thinking about the numbers four and five as independent numbers one unit apart on a number line to thinking about them as a single quantity that falls between zero and one.

Place Value

Children are often instructed on decimals as an extension of the whole number system without an adequate understanding of place-value concepts that enable them to work with whole numbers (Fuson, 1990). Decimal numbers aside for a moment, multi-digit whole numbers are problematic for children due to the nature of the English language used to name them. English-speaking children struggle to construct meaning based on ten and the multi-unit structure of our place value system (Miura, 1987; Miura, Kim, Chang, & Okamoto, 1988; Miura & Okamoto, 1989). Results of studies suggest a difference in the cognitive representation of number when comparing U.S. first graders to those in Japan that may positively affect Japanese children's and adversely affect U.S. children's understanding of place value and consequent mathematics achievement (Miura & Okamoto, 1989).

English written number words do not correspond with spoken numerals as they do in Japanese, and English number words are irregular in several ways (Fuson, 1990; Miura &

Okamoto, 1989). English number words are missing elements of tens and one and the decades are reversed from their written form, resulting in a similar sound when pronouncing fifteen and fifty. As a result, students have to memorize names of the English numbers rather than construct a number representation that reflects the meaning of individual digits in the base-ten number system. Compounding the problem in the U.S., instruction in place-value is traditionally restricted to the placement of digits in columns and the digits are focused on separately (Sowder, 1997).

Without an understanding of the base ten number system, students often attempt to write too many digits into a column. Extending the place value structure to include digits to the right of the decimal point presents additional challenges. Without understanding the value of the columns, some students think the further away from the decimal point, the larger the value of the digits (0.35 is larger than 0.41 because 53 is larger than 14).

Multiplicative Reasoning

Children initially abandon relational schemes in favor of strategies based on addition, and not until about age nine do children begin to apply multiplicative quantity relationships systematically (Streefland, 1993). For example, in a study by Tournaire (1986), students between grades 4 and 5 made significant gains in successfully using proportional reasoning methods. The results could not be explained by the curriculum because the students did not receive instruction in proportional reasoning. It appears that multiplicative reasoning has everything to do with students' ability to reason with two ratios. The study also reported that the greatest improvement in multiplication and division did not occur between grades 4 and 5, but rather between grades 3 and 4. It was hypothesized that the success rates probably occurred due to a deeper understanding of multiplication reasoning. Thus, the critical component of proportional reasoning appears to be the multiplicative relationship. Multiplicative situations provide informal work in preparing students for proportional reasoning.

Ratio and proportion are a natural part of an individual's multiplicative conceptual field (Lo & Watanabe, 1997). Often students are exposed to situations that require multiplicative thinking. When students deal with "for every" type problems they are tapping into an area ripe with potential to use multiplicative thinking (Clark & Kamii, 1996). Research indicates that multiplicative thinking appears early but develops very slowly. In one study, 45% of second graders use multiplicative thinking but only 48% of fifth graders were solid multiplicative thinkers (Clark & Kamii, 1996). It is hypothesized that, although students may learn their times tables, they may not learn the conceptual importance of multiplication, which results in a lack of meaning for multiplication and division computation (Lo & Watanabe, 1997).

Research by Clark & Kamii (1996) supports the belief that multiplication involves higher-order thinking because it involves simultaneous relations on two levels. It is not just repeated addition, which is one level of abstraction. Higher-order multiplicative thinking develops systematically through five levels of reasoning. Students at level one are nonnumeric, but not additive. Levels two and three involve progress in additive thinking toward more sophisticated adding techniques at level three. Levels four and five are multiplicative thinking with level five characterizing a student working with a command of multiplicative thinking.

Additive reasoning, which occurs in early stages of developing multiplicative thinking is an inhibitor to reasoning with ratio and proportion. Children using additive reasoning find the difference in the ratio and apply the difference to find the missing value. The data are treated in an absolute way. Students are familiar with concept of addition and they learn to rely on it early and often. Therefore, its use is often resistant to change. This is supported by additive reasoning which is used in a qualitative, intuitive way, not just by seven-year-olds, but by ages eleven-through sixteen-year-olds who have been taught something about proportions (Hart, 1988). Furthermore, research shows that children do not "grow out" of erroneous addition methods (Thornton & Fuller, 1981). Moreover this additive error may cause a delay in the development of multiplicative thinking (Markovits & Hershokowitz, 1997) which can disrupt work with rational number. For example, students inappropriately apply additive thinking when judging the equivalency of two fractions with different numerators and denominators (Behr et al., 1984).

Proportional Reasoning

Proportional reasoning tasks can basically be separated into two types: missing value and numerical comparison problems (Karplus, Pulos, & Stage, 1983). One of the best-known missing value problems is Mr. Tall and Mr. Short (Karplus & Peterson, 1970). Presented with a short and tall stick person measured using big paperclips and the short person also measured using small paperclips, students were asked to predict the height of Mr. Tall in small paperclips. Several other well-known missing value problems include: Fish and Food (Clark & Kamii, 1996), the shadow puzzle (Inhelder & Piaget, 1958; Thorton & Fuller, 1981), and the recipe puzzle (Thorton & Fuller, 1981).

One well-known numerical comparison task is Noelting's (1980) orange juice problem which involves comparing the orange taste of two drink mixtures of varying numbers of glasses of orange juice and water. Two trays, each with a large glass accompanied by several smaller glasses of water and orange juice, were used for the experiment. One tray was called A and the other B. The experimenter pretended to pour the smaller glasses into a larger glass and asked the child to determine which will have a stronger orange juice taste. Another well-known numerical comparison task is Karplus, Pulos and Stage's (1983) lemonade problem.

Problems types can also be categorized as one of four "semantic type". They are: well-chunked measures, part-part-whole, unrelated sets or associated sets, and stretcher/shrinker or scaling problems (Lamon, 1993). Students were found to use different methods depending on the context of the situation (Hart, 1988; Lamon, 1993). Part-part-whole problems, such as comparing the boys to girls in a class of students, did not elicit proportional reasoning in students because they could solve the problems using more primitive strategies, such as the "building-up" strategy or by using a table and chart to create and extend a pattern. Stretcher/shrinker problems, which involve scaling up or scaling down, failed to elicit much proportional reasoning due to the difficulty of this problem type (Cramer, Post, & Currier, 1993; Lamon, 1993). Students failed to see the multiplicative nature required of the task. The most sophisticated reasoning came from associated sets, which is pairing two unrelated sets such as people and pizza, because of the concrete pictorial mode in which they were presented. The discrete quantities in associated sets made them the least abstract.

Two basic types of strategies that can potentially produce successful results with proportional reasoning have been identified in the literature: Multiplicative and building-up strategies (Tourniaire & Pulos, 1985). Two multiplicative strategies are the unit rate and factor of change methods. The most popular (intuitive) strategy selected by students involves finding the amount per one or finding the unit rate (Cramer et al., 1993; Kennedy et al., 2002; Post et al., 1988). It is popular because it brings out children's natural thought process. Unit rate relies on a student finding the multiplicative relationship between the ratio pairs. Another strategy is the factor of change method, which requires finding the multiplicative relationship in one rate pair and applying it to the other in it equivalent rate pair (Cramer et al., 1993; Post et al., 1988). This involves the “times as many” or “how many times greater” type thinking.

Building-up strategies are considered more elementary (Tourniaire, 1986; Tourniaire & Pulos, 1985). Students who have not had formal instruction on proportions typically use building-up strategies (Kennedy et al., 2002). It involves finding a relationship and extending it using addition. Building-up is a repeated addition model that produces the correct solution without using the operation of multiplication. It can be a natural transition to a multiplicative strategy, but this does not always occur. Lo and Watanabe (1997) studied a child who used a ratio-unit/build-up method to solve proportional reasoning problems but did so in a way that was not originally considered deliberate or reflective and used a teaching method to try to get him to schematize his strategy.

Students may also use an equivalent fraction strategy or the traditional cross-product algorithm (Cramer et al., 1993) but Karplus and his colleagues (1983) found that only a very small number of subjects used this strategy.

Fallback strategies are used when students revert to more simplistic methods in problem situations that are unfamiliar or numerically complex (Tourniaire, 1983). Additive methods replaced proportional reasoning to a very large degree on more difficult problems (Karplus, Pulos, & Stage, 1983). Sometimes this is because the task is more complex semantically or numerically. The difficulty level of the problem situation seems to play a significant role in the use of fallback strategies (Markovits & Hershkowitz, 1997). The attempt to escape working with noninteger ratios is referred to as “fraction avoidance syndrome” and is frequently observed (Karplus, Pulos, & Stage, 1983; Lo & Watanabe, 1997; Tourniaire & Pulos, 1985).

The ability to reason proportionally requires the ability to reason formally in a quantitative sense and often comes later developmentally because of the second-order reasoning involved (Piaget & Inhelder, 1975; Karplus & Peterson, 1980). To be able to reason logically with rational numbers, students need to have this second-order reasoning ability. For example, they would use it to compare and order rational numbers, which are discussed in the following section.

Equivalence/order

Rational number equivalence is a complex mathematical concept. It is complex because rational numbers have numerous equivalent representations (Vance, 1992). Also, renaming a number using another representation changes (e.g. from decimal to a fraction) the appearance but not its properties. In addition, the best choice for representing a number often depends on the situation. Understanding this and identifying the best representation stems from experience with multiple contexts. The development of equivalence with rational numbers progresses slowly over time (Vance, 1992). Research indicates that students construct the concept of equivalence one subconstruct at a time, making the development of equivalence a recursive process (Ni, 2001). Research also suggests that the various graphical representations used to depict equivalence present different challenges. According to Ni (2001), "understanding and representing simple and equivalent fractions did not develop in a parallel fashion with respect to different subconstructs embodied in graphical representations" (p. 411).

As stated earlier, young children's early partitioning strategies involve knowledge of ½. Spinillo and Bryant (1991) reported on three experiments, which validate the crucial importance of the "half" boundary. When judging equivalence of two fractions, young children were more successful comparing fractional amounts that were cross-half comparisons, where one fraction is less than ½ and one greater than ½ (i.e. 3/8 vs. 5/8), rather than within-half comparisons, where both fractions were less than ½ or both greater than ½ (1/8 vs. 3/8). They also found that students were more successful using an actual half (1/4 vs. 2/4) than they were compared to within-half comparisons.

In comparison of fractions tasks, the pairs of fractions can be categorized into three types of tasks: same numerators; same denominators; and different numerators and denominators. Strategies for comparing the three types of fractions pairs have been identified (Behr et al., 1984). Students used one of five strategies for comparing fractions with the same numerators. Four of them were valid and one was not. The valid strategies included responses where students referred to the numerator and denominator, the denominator only, a reference point, or manipulatives and pictures. The invalid strategy focused on the denominator using rules consistent with whole numbers but did not recognize the inverse relation between the numerator and denominator.

Five strategies were also used for comparing fractions with same denominators, some of them the same as those described above. Again, four were valid and one was not. The four valid strategies included responses that referred to the numerator and denominator, a reference point, manipulatives and pictures, and the numerator only (number of parts). The invalid strategy involved an incorrect comparison of the size of the parts caused by inverting the relationship between the numerator and denominator.

Finally, six strategies were used for comparing fractions with different numerators and denominators. Three of them are valid and three are invalid. The valid strategies include references to application of ratios, a reference pint, or manipulatives and/or pictures. The invalid strategies made reference to addition (compared fractions by adding to the numerator and denominator), an incomplete proportion, and whole number dominance described above. The reference point strategy was used across all three types of fractions. The use of a reference point for comparing fractions signals that conceptions are generalized. There is a positive relationship between reasoning based on reference point and quantitative understanding of rational number (Behr et al., 1983)

To compare and order decimals students use specific strategies. Research documents three common errors that students make as they learn decimals (Moloney & Stacey, 1997; Nesher & Peled, 1986; Resnick et al., 1989; Sackur-Grisvard and Leonard, 1985). When determining the larger of two decimal numbers, students tend to progress through a series of stages. Students at first select the number that has more digits following the decimal point. This is often referred to as the "longer is larger" rule or more commonly the "whole number rule". Another error sometimes occurs when students select as smaller a number that has a zero following the decimal point, believing that a zero indicates a small number, but then after that they apply the whole number rule. The zero rule is considered a slight improvement over the whole number rule. A third error, that typically follows the "whole number rule", is to select as larger a number that has fewer digits following the decimal point, drawing on knowledge of place value, and believing that extending a number more places after the decimal results in smaller portions. This is commonly referred to as the "fraction rule". This predictable pattern of errors (whole number rule, zero rule, fraction rule) is consistent with a constructivist model of learning, which predicts that students use their most accessible mental model to apply to a new area of knowledge in order to understand it. For them, this is whole number followed by fraction knowledge.

Some studies report that students outgrow these rules over time (Nesher & Peled, 1986) as they become more competent with rational number concepts in high school (Smith, 1995), but other studies indicate that these errors persist into adulthood (Grossman, 1983; Putt, 1995; Stacey, Helme, Steinle, Baturo, Irwin, & Bana, 2001; Thipkong & Davis, 1991). For example, Grossman (1983) found that preservice elementary teachers struggled to identify the smallest number from a set of five decimal numbers. More preservice teachers used the "whole number" rule and selected the "longest" number as larger more frequently than the correct answer. In addition, research indicates that practicing teachers appear to be deficient in knowledge of decimals (Post, Behr, Lesh, & Wachsmuth, 1985; Post, Harel, Behr, & Lesh, 1991).

In their review of textbooks from the fourth and fifth grades, Lachance and Confrey (2002) found separate treatment of decimal and fraction concepts. The referents used in the chapters to address the concepts were different. Glasgow et al. (2000) report that 90% of teachers indicated that they use money as a model for teaching decimals while only 50% use equivalence to fractions and pictures to teach the same concept. This puts students at a disadvantage. In particular, academically low-achieving students rely heavily on visual representation of decimal concepts in order to make sense of them (Woodward, Baxter, & Robinson, 1999). This may well result in a missing link between fractions and decimals and explain why some students do not outgrow the "whole number rule." Students need an understanding of place value and fractions in order to understand decimals; otherwise, they are left unable to make sense out of the decimal symbols (Bell, Swan, & Taylor, 1981) and will continue to struggle to understand equivalent representations of fractions and decimals which are addressed in the next section.

Representations of Rational Number

Knowledge of rational number is built upon a foundation of prerequisite knowledge, equivalence being one piece. Equivalent representations of rational numbers can look very different in form yet represent the same concept. For example, rational numbers can be represented as fractions, decimals, or percents. Each of those representations will be discussed below.

Fraction

Early notions of fractions stem from thinking of them as "parts of things." This begins with partitioning and equal sharing tasks and moves towards more abstract ideas such as operating with fractions as numbers. To work with fractions in a more abstract symbolic manner, students must have a meaningful understanding of fractions symbolically. Fractions are represented symbolically with a numerator and denominator. When beginning to use symbols, students must be required to construct concrete representations to parallel the symbolic notation.

A rate is often connected with a fraction representation. A rate source "defines a new quantity as a relationship between two other quantities. For example, speed is defined as a relationship between two other quantities." It is distinguished from a simple fraction by the fact that rate is presented in a context, meaning it has a label such as miles/hour and a fraction is not (Heller et al., 1990). Students struggle to understand the composition of rate questions. In particular, they struggle to understand the structure of inverse relationships, such as exchange between currencies (Onslow, 1988). It is also difficult for them to conceive of the relationship between two discrete quantities, which does not represent a distinct quantity, but describes an inter-relationship (Onslow, 1988). Two problem situations in which students apply rates and fractions are numerical comparison problems, involving the equality or inequality of two fractions, and missing value problems, where student are given three components of two equal fractions or rates and they are asked to find the "missing value", the fourth component (Heller et al., 1990).

Decimal

The decimal representation of rational number involves the merging of whole number knowledge and common fractions with very specific kinds of units. In addition, decimals can be viewed as both continuous and discrete. Research indicates that students struggle with rational numbers, in general, but decimals are apparently the greatest challenge. Recent results from the Third International Mathematics and Science Study (TIMSS) indicate that students do not perform as well on questions involving decimals compared to those involving fractions (Glasgow, Ragan, Fields, Reys, & Wasman, 2000). Based on interviews with students and surveys of teachers, Glasgow et al. (2000) found that less emphasis is placed on decimals during classroom instruction than fraction instruction.

Another source of poor performance may be the sophistication of the concept. Decimals are more complex than fractions due to the fact that decimals rely on an understanding of both place value and fractions (Hiebert, 1984; Watson, Collis, & Campbell, 1995). In addition, the partitioning procedure that motivates the expression of a rational number as a decimal is complicated by the fact that a whole unit must be partitioned into precisely ten equal parts. Each part is simultaneously a tenth of a whole and a unit, which itself may be divided into ten parts (Hiebert, 1984). Common fractions provide more explicit information than do decimal numbers. In decimal notation the denominator is hidden, just as the place value of the columns in whole number numeration is hidden. This complication causes errors in thinking. Interpreting the portion following the decimal point as the denominator of a fraction is referred to as reciprocal thinking. Students without this understanding think that 0.8 is equivalent to one-eighth. The belief that in a decimal number the decimal point separates the number into two distinct whole numbers persists into adulthood and demonstrates the lack of understanding of the structure of our decimal place value number system (Bell, Swan, & Taylor, 1981; Zazkis & Khoury, 1993). Students with this belief write 1.4 as the decimal for ¼ (Hiebert, 1985).

The impact of whole number knowledge on students' development of decimal fraction knowledge may also relate to the nature of the knowledge. Students' knowledge of whole numbers includes procedural knowledge (form) and conceptual knowledge (understanding) (Hiebert, 1984). Students' prior knowledge may be predominately procedural, which may account for misconceptions applied to decimal fractions (Hiebert & Wearne, 1985). Often students can inappropriately apply rules, which can result in the right answer for the wrong reason. This reinforces the inappropriate use of the rule, and the error rules discussed above persist and procedural flaws are not corrected (Hiebert & Wearne, 1985). In addition, students who are instructed using only procedural methods tend to regress in performance over time (Woodward, Howard & Battle, 1997). To be competent with decimals, students must develop conceptual knowledge along with procedural knowledge (Hiebert & Wearne, 1985, 1986, 1988; Hiebert et al, 1991; Resnick et al. 1989)

Percent

Percent is a particular way to quantify multiplicative relationships. According to Parker and Leinhardt (1995), percent is "a comparative index number, an intensive quantity, a fraction or ratio, a statistic or a function" (p. 444). Throughout all of these interpretations, it is "an alternative language used to describe a proportional relationship" (Parker & Leinhardt, 1995, p. 445). Together these interpretations create the full concept of percent and are essential understandings in order to solve a wide variety of problems involving percent (Risacher, 1992).

As with the other interpretations of rational number, students use incorrect rules and procedures related to percent with confidence (Gay, 1997). This may be a direct consequence of students studying from a curriculum that emphasizes rules and procedures (Gay, 1997; Hiebert, 1984; Rittle-Johnson, Siegler & Alibali, 2001). When they are not sure what to do, students will revert to rules and procedures from concepts that are more familiar, more intuitive, and/or resistant to change, such as whole number (Risacher, 1992).

Common errors working with percent have been identified in research (Parker & Leinhardt, 1995; Risacher, 1992). First, students tend to ignore the percent label and treat the percent as a whole number. Second, they follow what is referred to as the "numerator rule" where they exchange the percent sign on the right with a decimal on the left. Third, they implement the "times table", also known as a "random algorithm" (Parker & Leinhardt, 1995: Payne & Allinger, 1984; Risacher, 1992). In addition to problems that arise from over generalizing whole number rules and procedures, misconceptions also result from students' limited instruction of percent as part of a whole. Researchers regard the interpretation of a percent as a fractional part of a whole to be of primary importance before working percent problems (Allinger & Payne, 1986). Yet, students who are over reliant on part-whole notions find percent greater than one hundred problematic since in their minds the part cannot exceed the whole (Parker & Leinhardt, 1995).

Focusing on this controversy, Moss and Case (1999) report on an experimental curriculum that introduced the rational number subconstructs of fraction, decimal, percent in reverse. The curriculum begins rational number instruction with percent in a linear measurement context. It then extends that with instruction on decimals to two places then to three and one places. Finally, instruction leads to fraction notation. Results indicate that students using this curriculum model have a deeper understanding of rational number, less reliance on whole number knowledge, and make more frequent references to proportional reasoning concepts.

Subconstructs of Rational Number

Much of the research on rational number has focused on the subconstructs, where different studies concentrate on different subconstructs. This section focuses on the five subconstructs that are common across research on rational number. Whereas others exist, these are subconstructs that are reoccurring and well-established in the research. They are: part-whole, measure, ratio, quotient, and operator.

Part-whole Relations

According to LeFevre (1986) there is a relationship between students' achievement on working with fractions and their understanding of part-whole relationships. The part-whole subconstruct, commonly expressed as a fraction, is often considered the basis of rational number knowledge and fundamental to the other interpretations (Behr et al., 1983; Freudenthal, 1983; Kieren, 1988; Pitkethly & Hunting, 1996; Ni & Zhou, 2005). The part-whole subconstruct is based on a students' ability to partition a continuous amount or a set of discrete items into equal-sized groups.

Spinillo and Bryant (1991) indicate that children acquire notions of part-part relations before that of part-whole. Likely this stems from their application of whole number knowledge. Using a part-part relation allows young children to treat the fractional parts as whole numbers. Sophian and Wood (1997) elaborated on work done by Spinillo and Bryant to include a three-part stimuli rather than two-part. Their research documents a developmental shift in preference from part-part toward part-whole relations and confirms earlier work by Noelting (1980a, 1980b) who found that young children compared the number of glasses of concentrate to the number of glasses of water in two drinks to determine which glass was more concentrated. Older students compared the part that was orange juice to the total amount of liquid, indicating a shift toward part-whole reasoning in terms of a relationship between both parts of the fraction simultaneously. Research indicates that young children who receive instruction in number concepts using a part-part-whole curriculum develop a deeper understanding of number concepts and a greater understanding of place value (Fischer, 1990). The curriculum stresses set-subset relationships and enables the child to explore various combinations of subsets that create a whole set. Students learn to decompose and recompose numbers while considering the original whole set.

The part-whole subconstruct appears to have its roots in partitioning, which is discussed in more detail in the next section. Partitioning is fundamental in developing early equal-sharing strategies. Partitioning leads to equal-sharing, which in turn given rise to the part-whole concept where it becomes necessary for students to understand the compensatory relation between the number of pieces and their size. Behr et al. (1984) found variation in students' ability to acquire an understanding this relationship between the size and number of equal parts in a partitioned unit.

Armstrong and Larson (1995) wanted to know more about how students' thoughts about fractions developed with time and experience at different grade levels, as well as how the properties of tasks assist or impede problem solving. To do this they analyzed responses students gave on comparison-of-area tasks (where rectangles were partitioned with part of the area shaded), identified common strategies and classified strategies into categories. In addition, the tasks were introduced both with and without fraction symbols to study the affect of the symbols on the selection of a strategy. Strategies were grouped into three categories: Part-Whole (PW), Direct Comparison (DC), and a combination Part-Whole/Direct Comparison. Prior to the introduction of fractional terms and symbols, the use of PW strategies was observed in only about one fourth of the responses. Following the introduction of terms and symbols, the application of PW strategies increased and those of DC decreased. Younger students continued to use a DC strategy even after fractional terms and symbols were introduced with the models, indicating they cannot coordinate all of the part-whole relationship conditions. As previous research indicates, they prefer to consider the parts independently from the whole and compare pieces directly. With a jump in use of PW strategies from 30% to 73%, eighth graders showed the most dramatic increase in the use of PW strategy with the introduction of fractional terms and symbols.

Results of their study also indicate that students, when making comparisons, often do not remain attentive to the size of the wholes from where the parts come (Armstrong & Larson, 1995). This indicates that they do not apply rational number reasoning without a suggestion to do so (given terms or symbols). They also only deal well with the obvious. Students do not perform well if the fraction symbol is connected with a visual model where the unit is not divided into the exact number of equal parts as indicated by the denominator (Behr et al, 1983). One way to address these issues in middle school is to include comparison-of-area tasks in which the size of the wholes or units differ and where the parts to be compared are not congruent or similar in shape. This is confirmed by D'Ambrosio and Mewborn (1994) who studied how students interpreted "fair" in "fair shares". They found that students define "equal" as not just having parts that are the same size, but also parts that look alike. Their research supports the variation in visual representation of component parts so that students are exposed to "nonstandard" illustrations of part-whole phenomena.

In addition to the fraction representation, research on part-whole subconstruct has focused on percent. Research indicates that when learners select a meaningful referent for percent, that the part-whole interpretation dominated their thinking (Risacher, 1992). For example, the part-whole conception of 4% was represented two ways in their study: using a graph paper diagram with four squares out of one hundred squares shaded and using the statement "4 out of 100". While students benefit from instruction on part-whole interpretation, instruction is often limited to part-whole contexts, and there often is an over reliance on the area model as a representation of the part-whole subconstruct (Kerslake, 1986). Students have less difficulty seeing a rectangular region as a whole compared to a discrete set of circles (Gay, 1997). In particular, when using an area model, the partitioning is initially limited to units with a "measure of one," meaning that the geometric shape being partitioned represents one unit (Mack, 2001). This limited view promotes the stereotype that a fraction is always a part less than a whole. To be able to perceive of improper fractions, students have to reorganize their conception of a fraction as part of a whole. They can do this by abstracting the unit fraction, iterating it a number of times to form a whole, and to continue iterating it to form a fraction that exceeds the reference whole (Tzur, 1999). In addition the area model also enables the students to treat each part as a whole rather than a fraction (Mack, 1990, 1993), which restricts the view of fractional amounts as independent whole numbers or as counting units (Mack, 2001).

Treating the rational number as counting units creates problems within percent representation as well. In their review, Parker and Leinhardt (1995) reported on studies that revealed students' struggles with percent greater than one-hundred because of initial instruction focusing on percent as part of a whole. The whole is implied with percent, which makes attending to the whole even more difficult. It is a challenge for students to expand the concept of percent to situations that are not part-whole in nature which sometimes produce percents greater than one-hundred. On a final note, probability is commonly viewed as a part-whole representation. It is formed by the number of favorable outcomes compared to the total number of possible outcomes.

Research indicates that the part-whole subconstruct is the basis of rational number knowledge. It appears to be then entry point to rational number and fundamental to the other subconstructs.

Ratio

Ratio is defined as “a statement of the numeric relationship between two entities” (Hart, 1988) and “it is more correctly considered as a comparative index rather than as a number” (Behr et al, 1983). Ratio is based on the ability to coordinate, for example, the number of people sharing and the number of objects shared and to think of this relationship as a composite unit (Lamon, 1994; Streefland, 1991). For example, the fraction 3/5 would be interpreted as three pizzas for every five people.

In a review of research of initial fraction concepts, Pitkethly and Hunting (1996) identified two interpretations on the early thoughts of children and how these thoughts form a primitive understanding of fractions. One of those perspectives indicates that initial fraction concepts emerge from early thoughts about ratio and proportion. Young children use ratios in informal ways in a framework of natural standards to make proportional comparisons. In one study, a child stated that the whale in the movie poster was too large based upon his memory of a whale he had seen the year before. He used this memory of the whale and the size of the man (the trainer working with the whale) and applied this ratio to the whale and man in the poster (Van den Brink & Streefland, 1978). Young children seem able to use their knowledge and judgment to determine if something looks natural or distorted. It appears that instruction hinders this progression of the intuitively expressed “ratio” notion of fractions that is possessed by young children. The introduction of symbols and symbolic manipulations shifts thinking about ratio away from personal and concrete experiences, toward formal and abstract thinking. Abstract representations often are introduced too early without a thorough understanding of the concept of ratio and its underlying features.

Regarding formal instruction, Karpus et al. (1983) argue that the prevailing equivalent fraction method to teaching ratio and proportion in the U.S. fails to meaningfully instruct more than a small fraction of students. The unit ratio method is more effective, but still limited.

Students need to have an understanding of the concept of a unit before using procedures that build on it. Unitizing (an important element which will be addressed more specifically in a later section) appears to be a process from which more advanced reasoning about ratio evolves (Lamon, 1993,1994, 2002).

Fractions can be viewed as a subset of ratio and are used in this way to express quantities and measurements. Decimals, as a base ten representation of fractions, are also used to express quantities and measurements. Therefore, it is believed that fractions and decimals are derived from ratio (Lachance & Confrey, 2002). As a result, decimal instruction that is grounded in the context of ratio assists students' understanding of decimal notation, as well as fractions and percent, which are more conceptual in nature (Lachance & Confrey, 2002)

According to Parker and Leinhardt (1995), percent can be viewed as a relationship or comparison. When conceived in this manner, percent is considered a ratio comparison. There are three ways percent can be perceived of as ratio comparisons. They are ratios when used to compare: different sets (number of boys to number of girls), different characteristics within a set (arm span to height), or the change in a set over time (current population to past population). For example, this would involve finding percent increase or percent decrease. Like all ratio situations, percent as a ratio involves multiplicative structures. Research indicates that young students, grades 4 and 5, as well as many older students, grade 8, have not developed a sense of multiplicative or ratio relationships and instead rely on incorrect additive reasoning when solving percent problems (Risacher, 1992).

Ratio is based on the ability to coordinate parts and that comparison is multiplicative in nature. Research indicates that young children are able to use ratios in informal ways but the formal methods for teaching ratio and proportion to older students tend to be problematic.

Measure

The measure subconstruct involves using a unit of measure, a length, and then dividing that unit into smaller and smaller subunits. It can be thought of as a ruler and the fraction as representing a length. The number connected with an object when "measuring" is the number of units or parts of a unit the object is equal to on the "ruler". The number line is a common model used for measure purposes. The number line is formed from an iteration of a unit and also from the partitioning of each unit into smaller equal parts. Then each of those parts are again partitioned into smaller, equal-sized parts. This model is different from an area model or a discrete set of objects because it focuses attention on how much rather than how many. In this way, “the fractional measure subconstruct of rational number represents a reconceptualization of the part-whole notion of fractions” (Behr et. al., 1983). This encourages students to shift their thinking from seeing a fractional amount as two separate whole numbers to representing a single amount. D’Ambrosio and Mewborn (1994) also argue “linear models help children construct rich definitions of fractions without becoming dependent on the ‘number of parts shaded out of total number of parts’ view of fractions typically afforded by area model.”

The linear coordinate subconstruct of rational number is related to the measure subconstruct and also emphasizes the notion of rational numbers representing a single amount. It is different in that rational numbers are viewed as specific and unique points on a number line. It highlights important properties of the rational number lines such as betweenness, density, distance, and completeness (Behr et al, 1983).

Research indicates that the measure subconstruct, when assessed using number line tasks, is challenging for students (Larson, 1980; Behr et. al., 1983). Children as late as 7th grade are unable to conceptualize a fraction as a point on a number line. Challenges include interpreting the unit on the number line and unit subdivisions that are not equal in number to the denominator (Larson, 1980). Number lines that are one unit in length tend to enable students to use the part-whole model because they view the number line as a “whole” unit. Researchers began to use number lines greater than one. When using number lines greater than one, the number line model is difficult because it is completely continuous in nature. For example geometric regions and sets provide visual discreteness, which makes the part-whole model more accessible.

Despite the fact that number lines have been used extensively to assess the measure subconstruct, questions have been raised as to whether the number line model fairly assesses students’ understanding of the measure subconstruct. Ni (2000) used comparing fraction tasks along with number line tasks to assess the measure subconstruct and found almost no correlation between the two tasks despite the hypothesis that they measure the same concept. Fraction-size comparisons were able to predict measurement knowledge, while number lines were not. However, there has been no other research on this topic and the number line is accepted in general as a way to assess the measurement subconstruct.

In general the measure subconstruct can be visualized as a ruler or as representing a length or amount of something. A common model is the number line, which research indicates is difficult for students to grasp. This may mean that learners have an incomplete understanding of the measure subconstruct or may struggle to represent that understanding with a number line.

Quotient

As discussed earlier, the part-whole subconstruct tends to be dominant initially. Research indicates that children progress from perceiving fractions as solely a part-whole relationship toward regarding fractions as a division operation (Middleton et al., 2001). One way to make a distinction is to consider the symbolism a/b and how it is interpreted according to a part-whole, ratio, and quotient perspective. Behr et al. (1983) state that, "according to the part-whole interpretation of rational numbers, the symbol a/b usually refers to a fractional part of a single quantity. In the ratio interpretation of rational numbers, they symbol a/b refers to a relationship between two quantities. The symbol a/b may also be used to refer to an operation. That is, a/b is sometimes used as a way of writing a ÷ b" (p. 95).

Graeber and Tanenhaus (1993) expand the definition of division of whole numbers and identify both the measurement and partitive model of division. The partitive model can be thought of as "sharing" or "dealing". Using the above symbolism, partitive division occurs when "a" is split into "b" equal parts resulting in "q" in each part. For example 8 ÷ 4 is interpreted as 8 candies are shared with 4 people means each person gets 2 candies. The size of the groups (number of candies) is unknown. The measurement model is also referred to as the quotitive model of division. It can be thought of as a repeated subtraction. It is interpreted as "a" split with "b" in each part resulting in "q" groups. For example, 8 ÷ 4 is interpreted as 8 candies are split into groups of 4 candies with 2 people getting candy. The number of groups is unknown. Students are often introduced to division using the measurement model. However, once the partitive model is introduced it tends to dominate the way most people, including adults, think about division. It is the measurement model, though, that enables a student to interpret division by a rational number and to allow quotients that are less than one.

Research indicates that initially students' prior knowledge of whole numbers hinders students as they work with division problems in which the quotient is less than one. They are often reluctant to denote quotients less than one as division problems because in the beginning work done with division always produces a whole number quotient. To deal with this, students will inappropriately apply the commutative property and try to switch the position of the numbers in order to produce a quotient that is greater than one. Most textbooks treat whole number division and fractions as separate topics. Therefore, students do not see them as related making the quotient subconstruct more difficult to grasp. In addition, textbooks often rely heavily on the part-whole construct of fractions and the partitive model of division making it more difficult for students to conceptualize fractions less than one and represent them using division number sentences.

Behr et al (1983) stated that formal understanding of rational numbers are beyond the grasp of elementary and middle school students. The level of abstract thinking necessary for interpreting rational numbers as indicated quotients is the reason for that statement and highlight two levels of sophistication.

> On the one hand, 8/4 or 2/3 interpreted as an indicated division results in establishing the equivalence of 8/4 and 2, or 2/3 and .666. But rational numbers can also be considered as elements of a quotient field, and, as such, can be used to define equivalence, addition, multiplication, and other properties from a purely deductive perspective, all algorithms are derivable from equations via the field properties. This level of sophistication generally requires intellectual structures not available to middle school children because it relates rational numbers to abstract algebraic systems (Behr, 1983, 96).

It seems that the link between fractions and division, for example, should be obvious, because a fraction is often interpreted as a whole that has been "divided" into equal-sized pieces. However, the quotient subconstruct tends to be an interpretation that is often overlooked or taken for granted.

Operator

The operator subconstruct involves viewing rational number as a function or a transformation. The operator subconstruct can be further divided into two strategies: Duplicator/partition- reducer (DPR) and stretcher/shrinker (SS). Behr et al. (1997) supplied preservice teachers with a pile of sticks each. The pile consisted of eight bundles of four sticks. Students were further instructed to find ¾ of the pile of sticks. Students using DPR found ¾ of the number of bundles while SS students found ¾ the size of each bundle. In the case of SS, rational number was considered as a rate. DPR strategy was used more often than the SS strategy. The DPR strategy uses partitive division while the SS relies on quotitive division. Given that the partitive model of division tends to dominate the way most people think about division, the choice of operator strategy appears to confirm such dominance.

When working with percent, students were found to do the worst on interpreting a quantity expressed as a percent of a number (Gay, 1997). Parker and Leinhardt (1995) refer to percent as a functional operator and consider it one of the most important meanings of percent. This is the interpretation that enables us to compute taxes, interest and discounts. Despite its practical importance, in a study of both younger students (grades 4 and 5) and older students (grade 8), both groups lacked an understanding of the subconstruct of percent as a multiplicative operator. Even when solving percent problems in which the operator interpretation is most appropriate, students frequently applied other interpretations (Risacher, 1992). In particular, students were found to be less familiar and have more difficulty working with percent greater than 100.

Some researchers contend that partitioning schemes lead students to an understanding of the subconstruct of operator (Behr et al., 1992, 1993; Kieren, 1988). Furthermore, the operator subconstruct lends to an understanding of multiplication of fractions, particularly when students are asked to find ¼ of ½ or are asked to take ¼ of ½ (Behr et al., 1992, 1993). In general, it appears that there is certain prerequisite knowledge that weaves its way through the subconstructs, that there are multiple ways to represent rational numbers and that the subconstructs need to be connected with each other. This is the case with the operator subconstruct.

There is a great deal of evidence about how rational number is broken down trying to analyze what students know about fractions across the subconstructs, but the question remains as to whether one concept is a prerequisite understanding in order to have access to another. A few studies have hinted at that. This study presented tasks that span the five subconstructs to examine how students perform.

Theoretical Framework

Pirie-Kieren Dynamical Theory for Growth of Mathematical Understanding

A model that informs this research involves not only the identifying the types of knowledge that students possess but also considers the growth of their understanding. Pirie and Kieren (1994) committed several years to the development of just such a model, which was subsequently revised many times (Pirie & Kieren, 1989; Pirie & Kieren, 1990). The significant piece of their model is that, "It is a theory of growth of mathematical understanding as a whole, dynamic, leveled but non-linear, transcendentally recursive process" (p. 62). Their model involves eight levels labeled within nested circles. Each circle contains all prior circles nested within it and is rooted in all subsequent circles.

Starting with the inner most circle the levels are: primitive knowing; image making; image having; property noticing; formalizing; observing; structuring; and inventising. Primitive knowledge is the beginning of growth of understanding and represents what the learner can do initially. Image making involves using prior knowledge in new ways. Image having is when the learner is freed from the need to use physical objects to solve the problem and instead uses a mental construct. At the level of property noticing, one combines aspects of the developing images to build context specific, relevant properties. The fifth level, formalizing, involves abstracting or generalizing a method, definition or algorithm that would work in any case. Observing, the sixth level, occurs when the learning coordinates the formal activities to construct theorems. The seventh level, structuring, then involves further coordinating the theorems to see them as inter-related and to use them to justify or verify. The final and outermost level is referred to as inventising. This level represents a completely structured understanding, which allows the learner to generate new questions with the potential to grow into new

Summary

In summary, developing rational number knowledge is a complex endeavor. Development does not occur in systematic way, but rather in a forward and back, zigzagging fashion. This is due to the influence of several factors in acquiring a mature understanding of rational number. Students who are the same age with similar school experiences may not be at the same place developmentally. The goal of this study is to describe the range of how students conceptualize rational numbers based on the subconstructs of rational numbers as well as several other unifying elements.

CHAPTER 3 - DESIGN AND METHODOLOGY

Introduction

"A myth about students' learning of mathematics is that understanding comes all at once" (Hiebert, Wearne, & Tabor, 1991, p. 321). The concept of rational number is complex and sophisticated and learning about it takes time. Because the learning is gradual, students can be in a number of different places in terms of their knowledge of rational numbers, despite their similarity in age and experience. The objective of this research was to describe how students conceptualize rational numbers by the end of 7th grade. To do so required breaking down the concept of rational number into subconstructs, identifying what students know about the subconstructs and determining if relationships exist among them. Mapping this information with how students applied their prior knowledge of whole numbers provided a powerful picture of the developing notion of rational numbers. The application of prior knowledge of whole numbers was assessed using equivalence exercises and analyzing response patterns.

My research question, therefore, was: How do 7th grade students in a small rural school district conceptualize rational numbers? Specifically this research addresses:

1. What type of strategies, including error patterns, do students use when comparing within and across decimal and fraction representations?
2. How successful are students at solving rational number tasks for each of the five subconstructs?

Overview

This study was a step toward a grounded theory. "The intent of a grounded theory study is to generate or discover a theory, an abstract analytical schema of a phenomenon, that relates to a particular situation" (Creswell, 1998, p. 56). The design of this study was qualitative: exploratory and interpretive in nature. To describe how students conceptualize rational numbers, solutions to tasks involving rational numbers were used, responses to interview questions were analyzed, and observations of students' behavior as they work were documented. All students participated in a written assessment that focused on rational number concepts. The instrument used to do this is thoroughly described in this chapter.

Students do not always learn what the teacher intends, nor do right answers always indicate understanding. Like a fingerprint, no two are alike; acquisition of knowledge is an individual process, in the sense that we organize our experiential work based on our individual interpretation of the experiences that we encounter. According to constructivist beliefs, each student creates meaning in his/her own mind based on his/her own experiences and understandings (Cobb & Steffe, 1983, Steffe & Kieren, 1994). As a result, students will have varying levels of understanding, no two people have exactly the same knowledge, that knowledge is not defined or represented in exactly the same way, and each will arrive at that understanding differently. However, students will share some common knowledge as a result of "taken as shared" understandings (Cobb, Yackel, & Wood, 1992). Full understanding is achieved when a student has connected knowledge of the representations and subcontructs of rational knowledge. "Taken as shared" imagery may exist in both understanding and misconceptions. If teachers are alerted to the misconceptions and deficiencies of students they can use this information to drive instruction. Assuming a constructivist perspective, the researcher focused on how students applied or misapplied prior knowledge of whole numbers, to developing notions of fractions, and finally to a mature (as defined in chapter two) understanding of rational numbers. To assess this "mature" understanding, the selection of tasks and the analysis of the data focused on how well students perform with each of the subconstructs.

Using the information about students' prior knowledge gathered through the written assessment, students were classified based on how they performed on order and equivalence exercises. A smaller sample population of students, some from each classification, was selected to participate in an interview protocol where they performed tasks drawing on knowledge of the rational number subconstructs. Student work, verbal responses, and actions were recorded and analyzed in order to identify trends or patterns.

Subjects/Setting

Sample

This sample was a convenience sample. Participants were an accessible population of 7th grade students all from one middle school in a rural district in Kansas. By the time they graduate high school, these students are expected to pass a minimal competency test (high school graduation requirement), which focuses mainly on computational procedures involving whole numbers and rational numbers. They begin taking this test in May at the end of seventh grade. If they are not successful, they will have the opportunity to take the test during a set testing period in December and May of each school year following instructional intervention. The results of this research would be very beneficial to the middle and high school teachers working with them as these students transition to the high school. Therefore, they were selected as the sample for this study.

Setting

The setting of this study was a rural district in Northeast Kansas. The district has approximately 1380 students. Approximately 4% of the population is minority, 54% is male, 46% is female, 28% is economically disadvantaged. The district consists of one K-2 elementary school, one 3-5 elementary school, one middle school, and one high school. The middle school has approximately 332 students, with 3% of the population being minority. Approximately 41% of the middle school students are economically disadvantaged while 59% are not. In addition, 52% of the middle school students are male and 48% are female.

Instruments for Data Collection

Three instruments were used to collect data. At the end of the school year, students took a written conceptual test and the district's minimal competency test, which focuses on computation and procedural skill. All seventh grade students also participated in a written conceptual test. The data collected was used to classify students according to the type of strategy they used and the type of errors they made. Finally, after classifying the students by patterns in their performance on these assessments, they were sorted into classifications and students from each classification were interviewed. The data collected was work samples, student responses, and observations of students as they worked.

The three instruments (minimal competency test, rational number conceptual test, and an interview) enable triangulation of the data and provide sufficient evidence to describe the range of knowledge students possess with regard to rational number understanding.

District Minimal Competency Test

The District Minimal Competency Test (DMCT) measures students' basic mathematics skills. The skills measured are largely procedural or factual in nature. The DMCT is based on basic skills and procedures identified in the Kansas Mathematics Standards. The individual skills assessed are spread throughout the standards, covering a range of elementary and middle school grades, up to and including the seventh grade. The test consists of eighty-eight questions that cover the following content: integers, rational number knowledge, place value, rounding, estimation, computation (whole numbers, fractions, and decimals), percent of a number, measurement conversions, exponents, and problem solving. Fifty-two items touch on some form of rational number knowledge and computation. The data collected was work samples and an overall score.

The DMCT was administered in two parts, spread over two days at the end of the school year in May. Students were not permitted to use a calculator, but they could take as much time as they need to complete the test.

Written Conceptual Instrument

The written conceptual instrument assessed the students' knowledge of rational numbers. The researcher used data from this instrument to break the concept down into manageable pieces in order to study them separately as well as combined.

The instrument was framed based on the five subconstructs of rational number. They are: part-whole, ratio, measure, quotient, and operator. It was also designed to identify how a student uses prior knowledge of whole numbers with respect to rational numbers. It was also framed based on research on order and equivalence. Research indicates that students appear to move through developmental stages as they learn to compare and order fractions and decimals.

The items on this instrument were a collection of tasks identified in research as tasks that assess knowledge of rational numbers and equivalence. They are described below:

Order and equivalence of decimals. Students were asked to compare pairs of decimal numbers. These number pairs and the analysis of the answers used to classify students were taken from research done by Moloney and Stacey (1997). Students were asked to circle the number with the largest value or if they are equal in value to circle both. The decimal pairs were designed to identify developmentally where students are in terms of working with decimals: whole number dominant, zero-rule (slight improvement of whole-number dominance), fraction dominant, and expert. Follow-up occurred during the interview, which enabled the researcher to analyze the responses. Students' errors were classified based on how they were using their prior knowledge. Responses identified in previous research were applied to this research study (Moloney & Stacy, 1997; Nesher & Peled, 1986, Resnick et al., 1989; Sackur-Grisvard & Leonard, 1985; Stacey et al, 2001).

Order and equivalence of fractions. Students were asked to compare fraction pairs of three types: same numerator; same denominator; and different numerator and denominator (Behr et al, 1984; Markovits & Sowder, 1994). Students were directed to circle the number with the largest value or if equal in value to circle both. These items assessed a student's ability to compare fractions represented by symbolic notation. This data was analyzed to determine students' level of skill comparing fractions. In addition, analysis of errors determined how they used their prior knowledge.

The interview task was important because it enabled the researcher to confirm and to analyze their solution strategy. Response strategies for each class of fractions were identified in previous research and used these to classify students in terms of valid and invalid solution strategies and how students were using their prior knowledge (Behr et al., 1983; Smith, 1995).

Order and equivalence between representations. Students were asked to compare a pair of numbers composed of a decimal number and a fractional number. They were asked to circle the number with the largest value or if equal in value to circle both. These items assessed students' ability to compare between representations.

Part-whole subconstruct. To study students' knowledge of part-whole subconstruct, which research indicates students develop first and the equivalence between representations, students were given a chart to complete. The first column included an area model with a shaded region. For each area model students were asked to write the fraction, decimal, and percent for the shaded region. This assessed their ability to write equivalent forms of rational numbers. Tasks such as these were expected to highlight misconceptions. Student responses were also expected to provide additional evidence as to how students used their prior knowledge. For example, this was where students wrote the fraction ¼ but the decimal 1.4 indicating that they saw the fraction and/or decimal as two whole numbers separated by a bar or a decimal point (Nesher & Peled, 1986). Tasks challenged students to look closely at visual representations.

Ratio Subconstruct. Items in this section drew on students' knowledge of ratio and the multiplicative relationship that exists between ratios. Items used to assess ratio sense were taken from Lamon (1995). These statements were used in a true/false format to be followed up by interviews. Having this on the written assessment rather than the interview enabled me to use "ratio sense" as a factor for determining how well a student understands the ratio subconstruct. Students were also asked to interpret and write ratios.

Measure Subconstruct. The measure subconstruct involved interpreting a rational number as representing a length. A common model used for measure purposes is a number line. Modified from Thipkong and Davis (1991) and Vance (1992), questions in this section involved the use of a number line. This model is different form an area model or a discrete set of objects because it focuses attention on how much rather than how many. It is also completely continuous. Students were asked to conceptualize a fraction or decimal as a point on a line. They were asked to interpret the unit and unit subdivisions on a number line. This subconstruct draws heavily upon the notion of density of rational numbers. Tasks in this section were taken from Markovits and Sowder (1994) and Vance (1992). Students answered questions such as, "Are there decimals between 0.74 and 0.75? If so, how many?" A small sample of struggling high school students were asked these questions and NO ONE answered infinitely many. They answered with a finite number or with 0. The way in which they understand the density of numbers could be a useful factor in describing their understanding of the measure subconstruct.

Quotient Subconstruct. Research indicates that initially students' prior knowledge of whole numbers hinders them as they work with division problems in which the quotient is less than one. They are often reluctant to denote quotients less than one as division problems because initial work with division always resulted in a whole number quotient. To deal with this they will inappropriately apply the commutative property and try to switch the position of the numbers in order to produce a quotient more than one. Questions in this section involved interpreting a fraction as a division problem. Tasks included asking students to represent a division situation as many ways as they could and then looking for this notation when analyzing results. Tasks also included division situations in context. Students were asked to divide cake or a candy bar among several people and tell how much cake or candy bar each would get.

Operator Subconstruct. The items that fall under the operator subconstruct were items where an operator results in a transformation of the other factor. It is the "rational number as a function concept". Students were asked to find the length of a rectangle if the dimensions are ¼ the dimensions of another rectangle. They were also asked to find a fraction of a set of items to determine how many of an item would be present. Students were also asked to find the fraction of a number out of context, for example find 1/6 of 18.

Procedure

Written Conceptual Instrument

Written directions accompanied the instrument, which were discussed with the teacher prior to administration. Students were given the assessment in three sessions of approximately thirty minutes. Calculators were permitted however, no student used one. Students were asked to show their work in the space provided.

The Interview

The goal of the interview was to provide clarification of tasks on the written conceptual items if necessary. Students were asked to clarify or give an explanation for their written answer or lack of a written answer. Students were asked about items that they didn't answer. They were asked about items in which directions were not followed correctly. They were asked about items that were illegible. In addition, the goal of the interview was to provide a deeper understanding of what students were thinking when they answered questions on the written assessment when they described their thinking. Students were asked about ideas that are not expressed in writing. Students were asked to tell what they were thinking when answering a particular question. When asked to do this, it may have actually stimulated thought processes that were not stimulated when answering the question in isolation. Finally, students were asked to perform various tasks that required them to demonstrate their understanding of the subconstructs of rational numbers. The interview also helped describe how students in each category perceive rational numbers and perform based on the subconstructs.

In order to develop an interview protocol and to facilitate the analysis of the data, the framework for this study was based on prior research of the rational number subconstructs and the existing knowledge gained from that research. The five subconstructs of rational number are: part-whole, ratio, measure, quotient, and operator. In addition to these various interpretations of rational numbers, they can also be represented in other ways as well, such as fractions, decimals, and percentages. They can also be modeled using discrete sets, area models, or number lines. Each model features something different about rational numbers. In addition, it comes with its own set of challenges.

Order and Equivalence. Items on this part of the instrument were intended to add clarification and depth to the answers on the written conceptual instrument. The researcher may be able to understand the students' thinking by analyzing and interpreting their answers, but the best way to gain access to the students' thinking is for them to tell someone what they are thinking. Students were asked to explain their reasoning as they answered comparison problems involving decimal and fractions. They were asked to order a list of numbers from smallest to largest. In addition they were asked about an item with blanks spaces rather than numbers (Irwin, 2001).

Part-whole Subconstruct. Because students appeared to have a stronger understanding of the part-whole subconstruct, interview items focused on the other subconstructs in an effort to control the number of tasks and amount of time students were engaged in the interview.

Ratio Subconstruct. Students were shown a group of X's and O'x and asked what the fraction represented in this situation. They were also given a situation involving golf balls and baseballs and were asked to give the ratio of golf balls to baseballs as many ways as you can. This was similar to the ratio items on the written assessment. The goal of these tasks was to add clarification and depth to the data collected during the written section.

Measure Subconstruct. The measure subconstruct involved interpreting a rational number as representing a length. A common model used for measure purposes is a number line. Questions in the written section involved the use of a number line. Students were also given a decimal number and asked to select from a set of five numbers, that which is closest. Students were also asked questions focusing on the density of fractions and decimals, similar to those on the written assessment. The goal of these tasks was to add clarification and depth to the data collected during the written section.

Operator Subconstruct. Using a pile of sticks that were in eight bundles of four sticks, the interviewer assessed students' understanding of the operator construct and identify their use of two strategic approaches to the operator subconstruct: duplicator/partition-reducer (DPR) (3/4 of the number of bundles) and stretcher/shrinker (SS). Students were given three situations, which highlighted these approaches. Some students were able to use both approaches, some only one and some struggled to use either approach. Students used toothpicks as boards to model their approach.

Quotient Subconstruct. The interviewer presented tasks that require students to use "divided" quantities. These tasks assessed students' willingness to write division number sentences for fractions less than one. There were also tasks asking student to divide cakes and candy bars equally among people. These items were similar to items on the written assessment and allowed the interviewer to probe students for more understanding.

Subjects

The decimal and fraction comparison items were used to classify students into the following categories: whole number rule, fraction rule, zero rule, ignore zero rule, money rule and expert. The subjects for the interview were volunteers. The volunteers were randomly drawn from a stratified sample pool of all 7th graders. They were stratified by the classifications listed above. Selected students were then asked to volunteer for the study. The parents of those that volunteered signed permission slips. Members of the volunteer group were paid $10 per hour for their participation. Most interviews lasted approximately one hour. All the students that volunteered attended the same middle school during their 7th grade year and will attend the same high school in a rural school district in Midwestern United States.

Procedure

Interviews were conducted in a semi-structured manner, collecting the intended core of data while respecting the freedom to pursue emerging issues as they arose. The interviews took place over the summer, which alleviated some of the fatigue factor of a lengthy interview process. They were conducted in a quiet location where the student was comfortable such as the middle school, library, or the student's home. These options were given to the child and his/her parents.

The interview tasks were read aloud to the subject followed by time to work, using paper and pencil or other materials if needed, and time to think through their response. They often explained what they were doing as they worked, or sometimes they worked first and explained their thinking when finished. The researcher listened while they explained their reasoning and asked them to record their work on paper if they didn't already. In addition, the researcher took notes regarding the student's reasoning. Requests for clarification were made when necessary.

Rephrasing what was heard enabled clarification. If students were in some way inconsistent in their responses as they answer the questions, they were asked to clarify their response, even if it meant going back to a previous question. Prompts, such as, "Is there another way you can think about that?" were used to elicit responses and to probe for depth of understanding. If a student was unable to reach a solution, they were asked to explain what they do and do not understand about the task. In some cases the researcher moved on without an answer if the student indicated that they didn't know and seemed uncomfortable being probed.

The data collected was in words and was audiotaped to capture them and be sure none were missed. The researcher used field notes whenever possible to record responses, thoughts regarding those responses, and initial inferences to avoid forgetting later what those thoughts were. In addition, the researcher observed the student work on the instrument and during the interview. The data collected included expressions and actions. Again, field notes were kept whenever possible and appropriate to document impressions and thoughts as students worked.

Data analysis

Pattern Analysis

The researcher used pattern or trend analysis when analyzing the data collected. Data was collected from written conceptual test for all students in the accessible population in order to create classifications. Using decimal comparison tasks students were sorted into categories based on their response pattern or the "rule" they appeared to follow: whole number rule, fraction rule, zero rule, a unique category of students that struggled when the digits in the ones, tenths, and hundredths were the same, and the expert group. These groups were then interviewed to probe for the range of understanding that students possessed and to create a description of what students initially appear to know about the five subconstructs and the equivalent representations of rational numbers.

The researcher analyzed the data collected on fraction comparison tasks and decimal-fraction comparison tasks. They were analyzed by the type of task and by their relationship to the benchmarks ½ and one.

The researcher analyzed the responses to the questions "what is a fraction" to identify common response patterns to such an open ended question in terms of the type of subconstruct students accessed to describe a fraction. The researcher then analyzed questions specific to each subconstruct and identified patterns in responses. The data was then disaggregated to look across all the students to see if there are any common features in student responses and create classification based on these patterns. Responses were often sorted into three categories: responses that indicated an understanding of the subconstruct, responses that were correct but it was unclear as to whether they had an understanding of the subconstruct, and responses that indicated and misunderstanding of the subconstruct.

Confirming the categories

The interview responses were used to confirm the classifications created from the data collected in the written instrument and confirm that the students in certain categories share some common knowledge as well as common limitations. The interview data was used to further describe what students know about the subconstructs and to get an explanation for some of the common responses that were given. Two classifications were not interviewed and therefore the classification was not confirmed because the patterns in their answers were not discovered until after the interviews. In the case of the money rule, one fraction rule student offered insight into the money rule because she used the rule on one occasion.

Summary

In summary, the concept of rational number is multi-faceted. There are many pieces, and learning about each of the subconstruct and making connections among them takes time. Unlike rational numbers themselves, acquiring notions of rational numbers does not occur in a linear or continuous manner. Students that have completed the same grade with the same basic exposure to rational number concepts are hypothesized to perform in a range of ways. The objective of this research is to describe the range of how students conceptualize rational numbers by the end of 7th grade.

The first step was to collect all of the “pieces.” This was done by breaking down the concept of rational number into subconstructs, and assessing what students know about these subconstructs. They were intended to supply the pieces to the puzzle.

Upon capturing the thoughts and ideas that have been constructed by students, an attempt was made to classify them based on the data. This involved studying the pieces to find a pattern and determine which ones appear to “fit” together. Assuming a constructivist perspective, the researcher focused on how to describe students understanding of rational number knowledge as they transition from whole number dominance, to developing notions of fractions, and finally to a mature understanding of rational numbers and described students' knowledge.

Research Issues

This study was a naturalistic study that involved both qualitative and quantitative data collection and analysis strategies, however, the primary focus was on qualitative analysis with the purpose of describing what students know. The analysis of data determined initial categories to which students were assigned.

Trustworthiness

To ensure the trustworthiness of the study, several criteria were considered (Lincoln & Guba, 1985). In the next section, the term used in qualitative research is given followed by the term used in quantitative research for comparison sake. The strategies used to meet the expectations for each criterion are given.

Credibility

Credibility establishes that the results of qualitative research are credible or believable from the perspective of the participant in the research. Because the purpose of qualitative research is to describe the phenomena from the participant's perspective, the participants are the only ones who can legitimately judge the credibility of the results.

Qualitative research is interpretive in nature and to increase credibility, the research must make known the perspectives and assumptions that may influence the study. In addition, personal connections with the topic or participants should be acknowledged. Triangulation enhances credibility. In this study the three instruments used enables triangulation of data. Another strategy is to triangulate the data analysis. This involves people other than the primary researcher to analyze and confirm data. This was accomplished in this study by using peer debriefers or member checks in the data analysis phase of research. Two mathematics teachers, one middle school teacher and one high school teacher, were asked to score the written assessments. These scores were then compared to the scores obtained by the researcher and any discrepancies were discussed and agreed upon. To validate the interview results a modified version of member check was used. Rather than ask the student to confirm what they said during the interview, the researcher asked the classroom teacher to give background information about the student after the interview took place. The researcher shared the results of the interview with the teacher to confirm that the description matched the student. This can be viewed as another form of triangulation through the use of several analysts.

Transferability

Transferability is the extent to which the research findings can be transferred to another setting. The researcher needs to provide enough information so that the reader can determine whether the findings can be transferred to a new situation (Lincoln & Guba, 1985). A thick, rich description makes it easier for the reader to determine if the study is transferable to their site. Therefore, in this research pattern analysis was used to group students by common knowledge that they share. What they share was described as thoroughly as possible to aide in transferability. In addition, an interview protocol was used to confirm what the student knows but also to provide an even better description. Purposeful sampling was also a strategy for increasing transferability because it provides information-rich cases studied in depth.

Dependability

Dependability is the extent to which the research findings are judged to be reliable or repeatable across time. The context in which the research takes place is constantly changing. Dependability is based on the ability to identify and describe changes that occur and how they impact the approach to the study. To increase dependability, the researcher maintained prolonged engagement in the field to identify changes. The researcher also used numerous subjects and triangulated data.

Confirmability

Confirmability is the degree to which the results can be confirmed or corroborated by others. It is a search for the potential for bias or distortion. Qualitative research is highly dependent on interpretation and is acknowledged as value-bound which contradicts the idea of objectivity in the quantitative sense. Admittedly, values will have an impact and should be identified and taken into account when interpreting and reporting findings. Lincoln and Guba (1985) refer to confirmability as the extent to which the researcher can be neutral or nonjudgemental when interpreting and reporting the data. They recommend a “confirmability audit” which is again an audit trail which included the following: raw data, analysis notes, reconstruction and synthesis products, process notes, personal notes, preliminary developmental information (p. 320-321). In this research an “iquuiry audit” was used. Documentation included: originaldata, early data interpretations or analysis, research reports, communication with others that analyzed the data.

Summary

In summary, the concept of rational number is multi-faceted. There are many pieces, and learning about each of the interpretations and making connections among them takes time. Unlike rational numbers themselves, acquiring notions of rational numbers does not occur in a linear or continuous manner. Students that have completed the same grade with the same basic exposure to rational number concepts were hypothesized to be on a continuum in terms of development. The objective of this research was to describe the range of how students conceptualize rational numbers by the end of 7th grade.

The first step was to collect all of the “pieces”. This was done by breaking down the concept of rational number into subconstructs, and assessing students using these subconstructs. That was the purpose of the three instruments used in this study. They were intended to supply the pieces to the puzzle.

Upon capturing the thoughts and ideas that have been constructed by students’, an attempt was made to classify them based on the data. This involved studying the pieces to find a pattern and determine which ones appear to “fit” together. Assuming a constructivist perspective, I focused on the rational number knowledge that students possess as they transition from whole number dominance, to developing notions of fractions, and finally to a mature understanding of rational numbers and will describe students’ knowledge at various points along the continuum.

CHAPTER 4 - RESULTS

The purpose of this research was to describe what 7th grade students know about rational numbers. Previous research on rational number knowledge indicates that learning occurs by combining partial understandings in a reorganization process. The data collected in this study identified ways in which students apply or misapply their partial understandings and the way partial understandings influenced the way in which they “know” rational numbers. This research studied how students applied knowledge of whole numbers, order, and equivalence. Rational numbers can be represented in multiple ways: fractions, decimals, and percents. The ways in which students represent rational numbers was considered in this research as well. In addition, this research focused on five well-known interpretations of rational number referred to here as the subconstructs of rational number: part-whole, measure, ratio, quotient, operator. Describing the understandings that students possess helped to form a clearer picture of how students conceptualize rational numbers.

The range of knowledge that students possess regardless of the fact that they are similar in age and have had similar school experiences highlights the complexity of the concept. The range of knowledge can be explained many ways, however, one way was used in this research: Pirie-Kieren model of growth of mathematical understanding. The ideas presented in this model provided the structure for forming a description of what students know about rational number. Data collection involved a written test, which was taken by all available 7th graders at the end of the school year. The written tests were used to classify students based on how they used prior knowledge to compare decimals and fractions. The purpose of the classification was to provide guidance in terms of selecting students to interview that would represent a range of success working with rational numbers and therefore, a range of knowledge that students would possess.

Nine students were selected for interviews based on the data collected on the written test. Two students were classified as whole number dominant thinkers, two students were classified as fraction dominant thinkers, and five were classified as “experts” because they correctly answered all of the decimal comparison tasks. More students were interviewed from the “expert” category because the majority of the students fell into this category. The expert classifications were confirmed using the scores on a District Minimal Competency Test (DMCT) as a measure of success working with rational numbers. Experts all scored above 80% with three scoring above 90%. The whole number dominant and fraction dominant thinkers all scored below 80% with two scoring below 70% on the DMCT.

The results have been divided into two parts. The first part of this chapter reports data from the comparison tasks and answers the first research question: What type of strategies, including error patterns, do students use when comparing within and across decimal and fraction representations? First, results of the decimal comparison tasks from the written test are reported. The results were used to classify students into the following groups based on the "rule" they apply to comparing tasks: Whole Number Rule (WNR), Fraction Rule (FR), Zero rule (ZR), Ignore Zero Rule (IZR), Money Rule (MR), and "apparent" experts (AE). This is followed by the results of the decimal comparison tasks from the interview. Second, results of the fraction comparison tasks from the written test are reported and followed by the results of the fraction comparison tasks from the interview. Finally, results of the fraction-decimal comparison tasks from the written test are reported, followed by the results from the interview. The three sections of the written test were given on different days; therefore the number of students responding varies from tasks to task. In some cases one hundred one students responded, other times ninety-nine, and in some cases only eighty-six were able to respond. This is discussed further in the limitations of the study.

The second part of this chapter reports on data collected about the subconstructs of rational numbers and answers the second research question: How successful are students at solving rational number tasks for each of the five subconstructs? Results are reported about the part-whole subconstruct from the written test. There were no interview items to follow up this subconstruct because research reports that the part-whole subconstruct is the dominant interpretation of rational number and the one that students appear to grasp first. With so many questions to ask, it was decided that further probing on the part-whole subconstruct was not as important at this time. Second, results are reported about the quotient subconstruct. The results of the written test are shared, followed by the results of follow-up tasks from the interview. Reporting of the results alternated between written test results and interview results, which provided deeper insight into the written test results. The remaining three subconstructs are reported in a parallel fashion.

All of the data is then collapsed into a summary of the description of how 7th grade students in a small rural school district in the central Midwest conceptualize rational numbers.

Decimal Comparison Tasks

Written Test Results: Decimal Comparison Tasks

Students were given a list of decimal pairs and were asked to circle the number with the largest value or, if they are equal in value, to circle both. The purpose of this task was to measure students' success with comparing decimals and to analyze the errors that were made looking for patterns in order to build a description of what 7th graders know about rational numbers. These number pairs and the error patterns that emerged were taken from research done by Moloney & Stacey (1997). Sixty-eight students (68%) out of the one hundred one that completed the decimal comparison tasks correctly answered all of the decimal comparison tasks. Thirty-two students (32%) missed one or more comparing problems. Table 4.1 lists the seventeen comparison items from the most missed item to least.

Table 4-1 Decimal Comparison Tasks

"Which is larger?"	Number of Students that answered incorrectly (n=101)	Incorrect Answer Selected (Number of Students)
4.4502 or 4.45	15	4.45 (10); Indicated equal (5)
17.35 or 17.353	12	17.35 (4); Indicated equal (8)
0.36 or 0.5	10	0.36 (10)
4.63 or 4.8	10	4.63 (10)
8.24563 or 8.245	9	8.245 (5); Indicated equal (4)
2.621 or 2.0687986	9	2.0687986 (9)
0.100 or 0.25	9	0.100 (9)
0.4 or 0.04	9	0.04 (7); Indicated equal (2)
3.72 or 3.073	9	3.073 (9)
0.37 or 0.216	8	0.216 (8)
8.514 or 8.0525738	7	8.0525738 (7)
1.27 or 1.270	6	1.270 (4); 1.27 (2)

4.7 or 4.08	5	4.08 (5)
0.3 or 0.30	5	0.30 (4): 0.3 (1)
0.4 or 0.457	4	0.4 (3), Indicated equal (1)
0.08 or 0.75	2	0.08 (2)
5.62 or 5.736	2	5.62 (2)

A third column in table 4-1 indicates the incorrect answer that was selected. From this table, it became evident that students were using more than the length of the number to determine the size. In several cases, numbers that were unequal in length were both circled, indicating that the student believed they were equal in value. Individual student tests were analyzed to search for error patterns. A new classification emerged and was labeled “money rule” (MR). The boxes in table 4-2 mark items missed by the thirty-two students that missed one or more tasks. Students are sorted by type of response.

Table 4-2 Individual Decimal Comparison Results

	4.8 or 4.63	0.5 or 0.36	0.25 or 0.100	0.37 or 0.216	4.7 or 4.08	2.621 or 2.0687986	3.72 or 3.073	0.04 or 0.4	8.514 or 8.0525738	4.4502 or 4.45	0.457 or 0.4	17.353 or 17.35	8.24563 or 8.245	5.736 or 5.62	Match pattern	Classifiction	DMCT score
Type of items	W	W	W	W	Z	Z	Z	Z	Z	F	F	F	F	F			
WNR																	
Paige	X	X	X	X	X	X	X	X	X						100%	WNR	51
Tiff	X	X	X	X	X	X	X	X	X			=			93%	WNR	
Kate	X	X		X		X	X	X							79%	WNR	57
Wendy	X	X	X	X		X	X		X						86%	WNR	73
Wade	X	X		X	X	X		X							79%	WNR	60
Ty	X	X	X	X	X	X	X	X	X						100%	WNR	51
Jami	X	X	X	X	X	X	X	X	X						100%	WNR	40
																Avg	55.3%
ZR																	
Mark	X	X													86%	ZR	56
Derek	X	X		X		X									86%	ZR	60
Mike	X	X			X										71%	ZR	59
																Avg	58.3%
IZR																	
Chris							X	X							79%	IZR	
Steph							X	X	X						86%	IZR	67
FR																	
Faith										X	X	X	X	X	100%	FR	74
Brad										X	X	X	X		93%	FR	86
Fran											X	=	X		86%	FR	47
Ashley										X		X	X		86%	FR	80
																Avg	71.8%
FR/IZR																	
Chase	X				X	X	X	X	X	X	X	X	X	X	93%	Fr/IZR	47%
MR										=	=	=	=				
Charlie										=		=			85%	MR	88
Taylor			X							=		=			79%	MR	55
KC										=		=	=		93%	MR	86
Jeff										=	=	=	=		100%	MR	49
Shelly	X											=	=		79%	MR	
																Avg	69.5%
UC																	
Josh			X	X						X		X				U	60
Jake										X						U	95
Lacy																U	76
Shay										X						U	82
Kay													X			U	77
Mel										X						U	75

Students constructed “rules” based on their experiences. Often the rules they’ve constructed contain a misconception that lead to predictable mistakes in solving comparison tasks. Three of these have been established in literature and are discussed in chapter 2 they are: Whole Number Rule (WNR); Fraction Rule (FR), Zero Rule (ZR). Two additional error rules emerged through on-going analysis of the data that were not found in previous research, they were the Money Rule (MR) and the Ignore Zero Rule (IZR). In this section, each of the error rules is briefly reviewed and linked to the written survey. Then each classification of thinking pattern is discussed in more detail. In the tables that follow, the errors are broken down into patterns and students are classified based on the type of “rule” they appeared to use.

Students that used the “whole number rule” (WNR) treated the portion after the decimal point as a whole number and chose the “longer” as bigger. They were expected to miss items 1, 2, 4, 5, 6, 8, 10, 11, 13, 15, and 16.

Students that used the “fraction rule” (FR) would select the shorter number as larger and were expected to miss items 3, 4, 7, 9, 11, 14, and 17. These students recognize that the columns decrease in value as you move from left to right and because of this will select shorter decimal numbers as bigger.

Zero Rule (ZR) students selected as smaller the number with zero immediately after the decimal but otherwise followed the whole number rule. Some would consider this an improved version of the WNR and students were expected to miss only items 1, 5, 8, and 15.

Students that used the Ignore Zero Rule (IZR) would remove the zero and compare the numbers without the zero (because zero means nothing) and they would miss items 2, 6, 12, 13, and 16. Without careful analysis, these students might be classified as using the whole number rule, however they do not miss all of the WNR items. They sometimes miss the items that the ZR students answer correctly and they correctly answer the items that the ZR students would miss. The difference is in the treatment of zero.

A fifth rule pattern was identified which will be titled the “Money Rule”. These students specifically missed tasks where the ones, tenths, and hundredths columns were equal but columns beyond that were not and would therefore miss items 3, 9, and 14. These students correctly compared numbers unless the digits in the columns up to and including the hundredths place were the same. In this case, they ignored the columns beyond the hundredths place. Not only did they answer these items incorrectly, they did so by indicating that the two numbers are equal. Students circling just one of the numbers would be using a whole number rule or fraction rule because students applying these rules first and foremost consider the length of the number. Fraction rule students, for example, recognize that columns decrease in value as they move from left to right. The “Money Rule” applies a more sophisticated comparison because students don’t just look at the length of the number; they consider the value of the digits in the tenths and hundredths columns. For example, when comparing 5.62 and 5.735 a fraction rule student would select 5.62 as the bigger number because it is shorter (i.e. hundredths are bigger than thousandths) but MR students would recognize that 73 cents is more than 62 cents and selects 5.735 as bigger. The length of the number is not the determining factor anymore. However, their understanding of the values appear limited to the hundredths place and therefore will ignore the digits after the hundredths column. For example they would consider 17.35 and 17.353 equal. There were fraction rule students that missed only these items but weren’t included in this group because they did not indicate the numbers to be equal. These items were the most commonly missed items. A group of “unclassified” students (students that missed only one item) tended to miss one of these tasks as well. Whole number rule students were the most successful with these items because they could use their rule to answer correctly for the wrong reason.

Each of these rules will be discussed in more detail in the data that follows. In tables 4-3-4-4, the four errors are broken down into four patterns, based on the type of “rule” a student appeared to use.

Interview Results: Decimal Comparison Tasks

Analysis of the written test was used to select students for interviews. Five students who missed one or zero on the written test were interviewed; two WNR and two FR students were selected. The interviews were used to further probe into student thinking regarding how tasks were solved. The findings of each interview are reported within the error pattern classifications in the next section. Only three students were classified as zero rule students and weren’t available or willing to do formal interviews. MR and IZR classifications weren’t discovered until after the interviews. However, one of the students interviewed was able to provide some insight into the new classifications.

Classifications: Whole Number Rule

Students following the whole number rule would select the longer decimal number as the larger. Students following this rule were expected to miss the eleven comparing decimal tasks listed in table 4-3. Included in the table is the number of WNR students that missed the task. Seven students were classified as WNR students because they answered the tasks following the expected pattern with 79% or more accuracy. Often students were found to have formed coping strategies so that misconceptions would not be apparent unless the student perceived the task as difficult, then they would regress to the use of their “rule”. For this reason, not all students reach 100% accuracy.

Table 4-3 Whole Number Rule Students

Decimal comparing task	Number of students that answered incorrectly n=7
0.36 or 0.5	7
2.621 or 2.0687986	7
4.63 or 4.8	7
0.4 or 0.04	6
3.72 or 3.073	6
0.37 or 0.216	6
0.100 or 0.25	5
8.514 or 8.0525738	5
4.7 or 4.08	5
*0.3 or 0.30	3

The task comparing 0.3 or 0.30 was not used to classify WNR students because FR students would be expected to miss this item as well and the results reported in this table indicate that it is not frequently missed by students that would otherwise use the WNR. Based on interviews this is because the features of the number encourage an annex zero coping strategy. Of the nine remaining items, three students missed all of the items and one was selected for the interview. One student missed six items. Two students missed five items and one of those was selected for an interview.

Whole Number Rule Interview Results

Nine students were interviewed throughout this study. Two WNR students were interviewed. Wendy and Wade were classified as WNR students because they made an error by over-generalizing their knowledge of whole numbers. The tasks used in the interview and the answers selected by the two WNR students that were interviewed are in table 4-4.

Table 4-4 Decimal Comparison Interview Tasks

	0.64 or 0.7	0.6 or 0.281	2.41 or 2.043	0.317 or 0.2	1.38 or 1.38209	6.396 or 6.39653	0.2 or 0.3	0.41 or 0.67	1.7 or 1.6	0 or 0.8
Type of items	W	W	W/Z	F	F	F	E	E	E	
WNR										
Wendy	X									
Wade		X	X	X						

When the decimals were of equal length, as the exercise marked "E" in the above table would indicate, Wendy's strategy was to look at the digits after the decimal point and pick the "bigger" number. When choosing between 0.2 and 0.3 she selected 0.3 and stated that "three is more than two". When comparing 1.7 and 1.6, she selected 1.7 and stated, "1.7 is one more number bigger". When comparing 0.41 and 0.67, she chose 0.67 and explained that sixty-seven is more than forty-one. When the decimals were of unequal length, as the exercises above marked W and F indicate, she would pick the number with more digits as larger. When comparing 0.64 and 0.7, she selected 0.64 because "it has two numbers and the other only has one number" (in reference to the digits after the decimal point). When comparing 0.317 to 0.2, she selected 0.317 justifying her answer saying that "one has 3 numbers in it and the other has only one". When comparing 2.41 to 2.043 she at first said she didn't know. After going on to the next question where she compared 6.396 or 6.39653 and she annexed zeroes to correctly select 6.39653, she went back to the previous question and selected 2.41 as bigger and first stated that "the zero in the other number makes is smaller" and when probed a little more said "if I lined them up then four is greater than zero". She used the annex zeroes strategy when comparing

1.38 or 1.38209 to correctly select 1.38209. The only time she used the annex zero strategy were with the two tasks where the digits were the same up to and including the hundredths place. When answering the second to the last problem was comparing 0.6 or 0.281 where she selected 0.6 but when asked to explain stated “I don’t know” and couldn’t elaborate. By the end she had used had used the whole number thinking through the majority of the tasks but used annexing zeroes twice and considered “lining up” the numbers to compare the digits in columns on one problem. It is interesting to note that she correctly selected 0.6 and 0.281 but couldn’t explain her thinking – perhaps she was beginning to think beyond whole numbers.

During the interview, Wade (although classified as WNR) was predominately WNR thinking and on one occasion used the FR. On four tasks he picked the longer as bigger. For example, when comparing 0.6 to 0.281 he selected 0.281 as bigger “because it is thousandths.” When comparing 2.41 or 2.043 he selected 2.043 stating that “Forty-three thousandths is bigger than forty-one hundredths.” When comparing 6.396 or 6.39653 and when comparing 1.38 or 1.38209 he picked the longer as bigger “because they are in the hundred-thousandths.” In the case of numbers of equal length (0.2 or 0.3 and 1.7 or 1.6 and 0.41 or 0.67) he selected the larger number explained, “they have the same amount of numbers and 3 is bigger than 2.” In two cases he selected shorter as bigger, but for only one he used a fraction rule explanation. For example, when comparing 0.317 to 0.2 he picked 0.2 as bigger because “the shorter the number the bigger the value.” He also picked shorter as bigger in some cases, but for a different reason. When comparing 0.64 to 0.7, he selected 0.7 as bigger “because 7 is bigger than 6 or 4.” This still appears to be whole number dominant thinking. He is comparing the whole number 7 to the whole numbers 6 and 4. Although he often correctly selected the larger decimal number, there were clearly errors in his thinking. Wade referred to the column names and often seemed to use those names to justify his selection but seems to have some idea that a fraction rule exists.

To confirm the strategy students used to compare decimals, in the interview they were asked, “Which is bigger 0._ _ or 0._ _ _.” Both Wendy and Wade answered, “You can’t tell. You need to have numbers.” It was expected that they would select the “longer” decimal number but they did not.

To increase the level of difficulty with the comparing task, students were asked to order six different numbers from smallest to biggest. The numbers were all written on index cards, one number per index card: 0.6, 0.060, 6.006, 0.66, 0.0666, 0.606. This enabled the researcher to see if the student’s strategy would hold up when comparing six numbers to each other.

Both students lined up the cards from shortest to longest (except for 6.006 which they both put last in their list), indicating that they were using the longer is bigger strategy. When asked about the strategy, Wendy noticed the zeroes in 0.060 and 0.0666 and then used a “line up” strategy to compare the columns. She then rearranged the cards in the right order. She even went back to the previous comparing task and corrected her answer to comparing 0.64 and 0.7 (however I continued to use her original answer in the analysis above). She originally circled 0.64 but changed her answer to 0.7. Something about the ordering task stimulated her to think about the comparing task differently and she changed strategies, which when asked about it she couldn’t explain.

Wade ordered his numbers from shortest to longest and when asked to explain, pointed to the first number and said, “this has one number (0.6), the next has two numbers (0.66), these two have three numbers (0.606 & 0.060) and the one with the zero goes on the end, the next one has four numbers and it looks like it is repeating (0.0666), and the last one has a whole number so it is the biggest (6.006)."

The WNR student interviewed were dominated by whole number thinking when comparing decimals. However, they did hint at the use of other ideas. They were beginning to use other strategies such as annexing zeroes or lining up the digits to compare columns, however they were doing it without the ability to explain the value of each column. They were still very much focused on length of the numbers.

Classification: Fraction Rule

Students using the "Fraction Rule" would select the shorter decimal number as the larger. It was predicted that they would miss the eight comparing tasks listed in table 4-5. Five students were classified as fraction rule students because they answered in the expected pattern with 86% or more accuracy. One student answered in a combination of both IZR and FR with 93% accuracy. He is included in the table below but was separated from the other fraction rule students during future analysis and is in a group of his own.

Table 4-5 Fraction Rule Students

Comparing decimals tasks:	Number of student that answered incorrectly n=5
17.35 or 17.353	5
4.4502 or 4.45	4
8.24563 or 8.245	5
0.4 or 0.457	4
5.62 or 5.736	2
*1.27 or 1.270	2
*0.3 or 0.30	1

* The tasks comparing 0.3 or 0.30 and 1.27 and 1.270 were not used to classify WNR students because FR students would be expected to miss this item as well and the results reported

in this table indicate that it is not frequently missed by students that would otherwise use the WNR. Of the five FR tasks, two students missed all five tasks and one was interviewed. One student missed four tasks. Two missed three tasks and one was selected for an interview. This student also answered one of the tasks indicating they were equal when the items were unequal in length and provided insight into the money rule discovered later.

Fraction Rule Interview Results

Nine students were interviewed throughout this study. Two FR students were interviewed. Faith and Fran were classified as WNR students because they made an error by over-generalizing their knowledge of whole numbers. The tasks used in the interview and the answers selected by the two WNR students that were interviewed are in table 4-6.

Table 4-6 Fraction Rule Interview Results

	0.64 or 0.7	0.6 or 0.281	2.41 or 2.043	0.317 or 0.2	1.38 or 1.38209	6.396 or 6.39653	0.2 or 0.3	0.41 or 0.67	1.7 or 1.6	0 or 0.8
Type of items	W	W	W/Z	F	F	F	E	E	E	
FR										
Fran					=	=				X
Faith				x	x	x				

Fran and Faith were classified as users of the Fraction Rule and selected for interviews. When comparing two decimals of equal length, both students were able to correctly choose the larger number. When unequal in length, for example comparing 0.6 and 0.281, Fran annexed zeroes and correctly selected the larger number. She did this on four tasks. However, when the digits in the tenths and hundredths were the same (6.396 or 6.39652 and 1.38 or 1.38209), she did not annex zeroes. She said they were equal because “you just take off the last digits because they don’t matter”. This was consistent with her written assessment where she chose the shorter decimal numbers as the larger in the case of comparing 4.4502 and 4.45 and comparing 8.24563 and 8.245. She also incorrectly stated during the interview that 0 was greater than 0.8 because “one is a whole number and the other was a decimal”. This is an example of negative thinking, where students see numbers less than one as negative numbers or below zero. She showed no indication of annexing zeroes on her written assessment as she did during the interview.

Faith’s thinking was clearly dominated by the Fraction Rule. She correctly selected the largest number when the decimals were equal in length. When unequal in length she selected the number that had fewer digits after the decimal point as the larger number. She stated the larger number has “fewer numbers after the decimal point”.

To confirm the strategy students are using to compare decimals, in the interview they were asked, “Which is bigger 0._ _ or 0._ _ _.” Both Fran and Faith selected 0. _ _ . Faith stated this was “because 0. _ _ has only two numbers after the decimal point” confirming her original strategy. Fran however, first selected 0._ _ as larger but changed her mind after being asked to explain herself saying that you can’t tell “it could be bigger or it could be the same for example 0.62 and 0.620.” Again, using the annex zero strategy but she did suggest that 0._ _ _ could be bigger. She was beginning to question her own thinking but seemed to get hung up on the annexing zero strategy.

On the ordering task (0.6, 0.060, 6.006, 0.66, 0.0666, 0.606), FR students would likely order the sequence from smallest to greatest as 0.0666, 0.060, 0.606, 0.66, 0.6, 6.006. Faith continued to use the fraction rule and line up the numbers from longest to shortest as was expected using the number of digits after he decimal point to justify herself. Fran struggled with this task for sometime before coming up with this list from least to greatest: 0.060, 0.0666, 0.606, 0.6, 0.66, and 6.006. To explain herself she “put the last one on the end because it had a whole number.” She “put the first two at the front because they both had a zero after the decimal point and then more numbers, and if you put zeroes on the end of .060 then you can see it is smaller than 0.0666. Next in order she put 0.606 and put it before 0.6 “because it is longer and 0.6 is shorter.” Then she put 0.66 after 0.6 “because if you put a zero on the end of 0.6 it is smaller than 0.66.” Her logic reveals that she understands the value of “whole numbers” and that she uses the “zero rule” to help select numbers that are smaller based on a zero in the tenths place. She is also very dependent on an annex zero strategy to equalize the length of the numbers but still will use the fraction rule as well.

Faith’s thinking was dominated by the fraction rule, while Fran’s thinking during the interview indicated that she has other ideas as well. Fran’s explanation to comparing decimals with the same digits in the tenths and hundredths and just ignoring all the digits after that were the same (1.38 and 1.38209) and calling them equal led to the identification of a new rule, the Money Rule. However, this was not established until after students were selected for interviews and the interviews were complete.

Classification: Zero Rule

Just as WNR students used prior knowledge of whole numbers to make decisions and FR students used knowledge of fractional parts related to the decimal columns, Zero Rule (ZR) and Ignore Zero Rule (IZR) students apply prior knowledge of zero, but in different ways. Specifically, the zero is considered by these students to mean “nothing” or signals a very small amount. The three students classified as IZR students often missed the questions with the zero in the tenths place. They would compare the numbers ignoring the zero, which is verified by the fact that two of the three IZR students claimed that 0.4 and 0.04 were equal. Conversely, the ZR students would consistently answer these correctly. They would recognize the zero in the tenths place as a tool for making the number small. Three students were classified as ZR students. The breakdown of the five zero tasks listed in table 4-7. One of the IZR students was also classified as a fraction rule and is included in both categories. None of these students were interviewed.

Table 4-7 Zero Rule and Ignore Zero Rule

Decimal Comparison tasks	Number of ZR students that answered correctly N=3	Number of IZR students that answered correctly N=3
3.72 or 3.073	3	0
0.4 or 0.04	3	0 (2 said they were =)
8.514 or 8.0525738	3	1
0.08 or 0.75	3	2
2.621 or 2.0687986	2	2
4.7 or 4.08	2	2
.25 or .100	3	3
.3 or .30	2	3
.27 or .270	2	2

IZR students would correctly answer comparison questions such as 0.3 and 0.30 because ignoring the zero results in the same number. In the case of the ZR students, a common coping strategy is to annex zeroes to make the decimals the same length. Doing so would produce the correct answer. However, one ZR student missed the last two decimal comparison tasks in the table. He selected 0.30 and 0.270 as larger indicating that WNR thinking coincides with ZR thinking. None of the ZR students missed the task of comparing 0.25 or 0.100 as originally expected. However, if a zero indicates a small number, perhaps the appearance of two zeroes enabled ZR students to identify it as the smaller of the two numbers.

Classification: Money Rule

Five students answered in another pattern that was different than any of the patterns identified thus far. These students consistently missed only the problems that had the same digits in the whole number columns, tenths and hundredths places, with the exception of the last two items in table 4-8 and did so indicating that the two numbers with equal. They missed two or more of the tasks listed in table 4-8.

Table 4-8 Money Rule Students on Comparing Decimal Tasks

Decimal comparing tasks	Number of students that answered incorrectly N=5
17.35 or 17.353	5
4.4502 or 4.45	4
8.24563 or 8.245	3
0.4 or 0.457	1
1.27 or 1.270	0

Of the thirteen errors that were made among these five students in the above table, all of them were errors indicating that the two decimals were equal. Students in this group circled both decimal numbers. Two of the people interviewed (one fraction dominant and one whole number dominant) also indicated some use of the money rule. In the interviews they both stated that what happened after the hundredths column “didn’t matter”. They were able to explain that the value of ones was more than the value of the tenths and tenths more than hundredths. According to research, most of the study of decimal places focuses on the tenths and hundredths columns. Students also seemed to understand that the columns decreased in value and that “thousandths” were small pieces – so small that they didn’t matter, were ignored and treated like zero.

Other Errors

Ten of the thirty-two students that incorrectly answered the comparing decimal tasks above only missed one problem (except one student who missed four apparently random tasks) and were not included in any of the classifications. The error they made may have been a mental "slip" or they may have been using some sort of "coping strategy", such as annexing zeroes to equalize the places after a decimal that either didn't work for this one item or wasn't used in this case. However, as mentioned earlier, tasks where the ones, tenths, and hundredths columns (and thousandths in some cases) were equal were the most frequently missed, which is the case with these students as well. There were mainly two tasks that were most frequently missed by these ten unclassified students. They are listed here with the single mistake they made:

- Three students selected 0.100 as the larger number compared to 0.25.
- Six students selected 4.45 as the larger number compared to 4.4502.

Comparing Decimals Tasks "Apparent" Experts

Five of the nine interviewees were classified as "apparent" experts (AE) on the written test and were able to correctly answer the comparing decimal tasks during the interview as well. The main strategy that enabled all of these students to be successful was comparing columns. The strategy involved comparing columns, one at a time beginning with the column with the largest value, looking for a place where the digits were not the same and selecting as larger the number with a larger digit in the comparison columns. For example, when comparing 0.64 and 0.7 Greg selected 0.7 as larger explaining that "in the first place (referring to the tenths column) after the decimal has a larger number in it than the other". Katie stated she knew the answer to 0.64 or 0.7 because "she looked at the tenths place and seven-tenths is more than six-tenths". When comparing 0.41 or 0.67, all of the AE selected 0.67 and stated "the tenths place is higher" none of them stated that 67 was greater than 41 as the WNR and FR students did. If the digits in the tenths column were the same, for example, when comparing 1.38 and 1.38209, Greg would go to the first column where the digits weren't the same, in this case he pointed to the two and stated that "two is greater than zero". Chad did the same and when comparing 6.396 or 6.39653 annexed zeroes and stated that "fifty-three is more than those zeroes". Two others (John and Elane) shared this thinking and indicated that "one has more digits and the other would have zeroes in those places" and "more digits on the end makes it bigger". Katie, the fifth AE, besides referring to the place value columns also was successful using a strategy of annexing zeroes to make the numbers equal in length. She used this strategy whenever the numbers did not have the same number of digits except when comparing 0.6 and 0.281. When comparing this pair, she said 0.6 was bigger "because it is more than one half and 0.281 was less than one half", in other words she used the benchmark of ½. She struggled with the one pair of numbers, 0 and 0.8, and when asked to explain she said she wasn't sure.

To better identify the strategy students are using to compare decimals, in the interview they were asked, "Which is bigger 0._ _ or 0._ _ _?" These same five students all gave similar correct answers. They answered, "You can't tell, you need numbers… It depends on where the digits are… The digits could all be zeroes or they could all be the same."

On the ordering task (0.6, 0.060, 6.006, 0.66, 0.0666, 0.606), AE students ordered the sequence from smallest to greatest correctly (0.060, 0.0666, 0.6, 0.606, 0.66, 6.006). These students were successful and all implemented the same strategy of comparing columns one at a time to look for the digit that was the smallest (or biggest). One student lined the cards up vertically to "line up" the columns. The others lined the cards up horizontally and referred to the columns by name. Chad explained his ordering selection by saying, "I put this one (6.006) at the end because it was the only one with a whole number. Then I put these two (0.060 and 0.0666) at the front because they were the only ones with a zero in the tenths place and the others had six. I put 0.060 first because they both had six in the hundredths place but it had a zero in the thousandths and the other had a six. Then I put this one (0.6) next because it had 6 in the tenths place and nothing more. The next one (0.606) had six in the tenths place but also a six in the thousandths place which would make it a little bit bigger. Then I put 0.66 next but before this one (6.006) because it had a six in the tenth and a six in the hundredths column and the others had a zero." The other AE's had explanations that were amazingly similar and they all referred to columns, first tenths, then hundredths, then thousandths.

Summary of Decimal Comparison Tasks

Across the WNR, FR, ZR, IZR, MR error types, students' misuse of whole number knowledge and limited understanding of the depth of rational numbers, especially beyond "hundredths" resulted in errors. In the interviews the strategy used by WNR and FR students often avoided thinking about rational numbers and the actual value of the digits that make up a rational number, instead they focused on the length of the numbers or on trying to equalize the length of the numbers by adding on zeroes or in some cases even dropping off digits at the end of the number, as a result errors were not obvious to them and these students tended to be less successful on comparing tasks.

It appears that "expert" students referred to columns and recognized the values of the columns when comparing and ordering decimals while WNR and FR students tended to be limited to an "annex zeroes" strategy and treating the decimal portion of the number as a whole number. Two new "rules" for comparing pairs of decimal numbers emerged: Money Rule (MR) and Ignore Zero Rule (IZR).

Certain tasks seemed to bring misconceptions to the surface, such as comparing decimals of unequal length highlights the use of WNR or FR, while others tasks encouraged students to look more closely at the features of the numbers or prompted strategic thinking of some form, for example, when the digits were the same in the tenths and hundredths columns one WNR student interviewed use an annex zero strategy that was not used on any other task and one FR student eliminated numbers at the end of the decimals and called them equal which she did not do on any other task.

Fraction Comparison Tasks

Written Test Results: Fraction Comparison Tasks

Students were given a list of fraction pairs and were asked to circle the number with the largest value or if they are equal in value to circle both. The purpose of this task was to measure students' success with comparing fractions but also to analyze the strategies used and errors made. These number pairs were created and analyzed based on research done by Behr et al (1984). Three groups of fractions were created: same numerator, same denominator and different numerator and denominator. Of the eighty-six students that completed the fraction comparison tasks, thirty-three students (39%) correctly answered all of the tasks. This is significantly less than the 68% of students that correctly answered all of the decimal comparison tasks (68/101). The results to the fraction comparisons items are provided in table 4-9, including a brief description of the type of task and the relation to ½.

Table 4-9 Fraction Comparison Item Analysis

Fraction comparison item	Type of task	Relation to ½	Percent correct
1/7 or 2/7	Common denominator	Both < ½	96%
1/8 or 7/8	Common denominator	Cross ½	93%
3/5 or 4/5	Common denominator	Both > ½	94%
4/3 or 2/3	Common denominator	Cross 1	94%
½ or 1/5	Same numerators	Both < ½	100%
2/3 or 2/9	Same numerators	Cross ½	96%
9/100 or 9/10	Same numerators	Cross ½	85%
3/11 or 3/14	Same numerators	Both < ½	92%
4/3 or 4/5	Same numerator	Cross 1	89%
¾ or 6/8	Equivalent	Both > ½	80%
2/3 or 6/9	Equivalent	Both > ½	84%
1/5 or 7/9	Neither the same	Cross ½	93%
5/8 or 6/5	Neither	Cross 1	93%
6/12 or 2/5	Neither	½ vs. < ½	89%
4/10 or 1/7	Neither	Both < ½	88%
43/100 or 6/10	Neither	Cross ½ (but close!)	88%
2/5 or 5/9	Neither	Cross ½ (but close!)	74%
6/7 or 8/9	Neither	Both > ½	66%

Results are broken down by type of fraction pairs. Of all the students that completed the tasks 93% or more of these students correctly answered fraction comparison tasks with common denominators. Same numerator type tasks has more variability with as low as 85% and as high as 100% answering correctly on these tasks as you can see in the table above. Students had less success comparing equivalent fractions with 80% and 84% of the students answering these two tasks correctly. Fractions with different numerators and denominators were the most challenging for students. The success on these tasks varied greatly from as low as 66% correct on one tasks to as high of 93% correct on two tasks. The variability in these tasks required further analysis. Across each type of task further analysis was done to study the relation to one-half. When comparing fractions that are both less than 1/2, common dominator tasks resulted in 96% accuracy while same numerator resulted in 92% accuracy and different numerator and denominator comparison tasks resulted in 88% accuracy. When comparing fractions that are both greater than ½, the common denominator task yielded a 94% correct while different numerator and denominator yielded a 66% (there were no same numerator tasks comparing two fractions both greater than ½). When comparing fractions across ½, 93% of the students correctly answered the common denominator tasks while an average of 90.5 (two tasks) and different numerator and denominator tasks averaged 85% (three tasks). Finally, when comparing fractions across 1, 94% of the students correctly answered the common denominator task, 89% correctly answered the same numerator task, and 93% correctly answered the different numerator and denominator task.

Across these tasks, students were most successful with common denominator tasks, followed by same numerator tasks and with different numerator and denominator tasks resulting in the least amount of success with only comparing fractions across one as an exception. When comparing fractions with different numerators and denominators, the closer the two fractions are in value, for example 2/5 and 5/9 or 6/7 and 8/9, the less frequently they are compared correctly. The fraction comparison tasks were further analyzed by the classification (WNR, FR, ZR, IZR, MR) and can be found in table 4-10.

Table 4-10 Fraction Comparison Tasks by Classification

	1/8 or 7/8	1/7 or 2/7	3/5 or 4/5	4/3 or 2/3	½ or 1/5	2/3 or 2/9	9/100 or 9/10	3/11 or 3/14	4/3 or 4/5	1/5 or 7/9	4/10 or 1/7	6/12 or 2/5	5/8 or 6/5	2/5 or 5/9	43/100 or 6/10	6/7 or 8/9	2/3 or 6/9	¾ or 6/8
Item Type	C D	C D	C D	C D	S N	S N	S N	S N	S N	D	D	D	D	D	D	D	D E	D E
WNR																		
Paige	X	X	X	X		X			X	X	X		X	X		=	X	X
Tiff						X	X	X	X						X			X
Kate	X						=								=	X	X	
Wendy	X						=	X	X	X	X			X		=	X	X
Wade														X		=		
Ty								X	X		X							
Jami	X	X	=	X				X	X	X	X			X		X	X	X
ZR																		
Mark	X						=							X	X	=	X	
Derek									X					=		=	X	X
Zack			X	X					X	X	X	X	X	X		=		
IZR																		
Steph							=	X										
Chris												X		X		=		X
FR																		
Faith																		
Brad																X		
Fran				X			=				X			X				X
Ashley														X				
FR/ IZR																		
Chase	X	X	X	X						X	X	X	X	X		X	X	X
MR																		
Charlie									X								X	X
Taylor							=											
KC																	X	
Jeff															=	=		
Shelly								X			X	X	X		X			
UC																		
Josh									X								X	X
Jake																		
Lacey																		
Shay							X	X							X			
Kay						X										X	X	X
Mel														X				
CJ																X		X
Sam												X			X			
Emma																X		
Matt																X		

- WNR students (n=7) correctly answered common denominator comparison tasks with 64.3% accuracy, same numerator tasks with 60% accuracy, different numerator and denominator tasks with 61.2% accuracy, and different but equivalent with 42.9% accuracy.
- ZR students (n=3) correctly answered common denominator comparison tasks with 75% accuracy, same numerator with 80% accuracy, different numerator and denominator tasks with 47.6% accuracy, and different but equivalent with 50% accuracy.
- IZR students (n=2, not including student classified in two categories) correctly answered common denominator comparison tasks with 100% accuracy, same numerator tasks with 80% accuracy, different numerator and denominator tasks with 78.5% accuracy, and different but equivalent with 75% accuracy.
- FR students (n=4, not including student classified in two categories) correctly answered common denominator comparison tasks with 93.8% accuracy, same numerator tasks with 95% accuracy, different numerator and denominator tasks with 85.7% accuracy, and different but equivalent with 87.5% accuracy.
- MR students (n=5) correctly answered common denominator comparison tasks with 100% accuracy, same numerator tasks with 88% accuracy, different numerator and denominator tasks with 82.9% accuracy, and different but equivalent tasks with 70% accuracy.
- The unclassified group (n=10) correctly answered the common denominator comparison tasks with 100% accuracy, same numerator tasks with 92% accuracy, different numerator and denominator tasks with 87.1% accuracy and different but equivalent tasks with 75% accuracy.
- AE students (n=53) correctly answered common denominator comparison tasks with 99% accuracy, the same numerator tasks with 97.7% accuracy, different numerator and denominator

tasks with 90.3% accuracy, and different but equivalent tasks with 91.5% accuracy. Twenty-four of the apparent experts missed at least one item. A break-down of the number of items missed among those twenty-four students can be found in table 4-11. A break-down of the actual item missed can be found in table 4-12.

Table 4-11 Apparent Experts and Frequency of Number of Fraction Comparison Items missed

Number of Items Missed	Frequency (n=24)
1	13
2	5
3	1
4	1
5	3
9	1

Table 4-12 Apparent Experts and Frequency of Comparison Task Missed

Comparison Item	Frequency of Errors N=54
1/8 or 7/8	1
1/7 or 2/7	0
3/5 or 4/5	1
4/3 or 2/3	0
½ or 1/5	0
2/3 or 2/9	0
9/100 or 9/10	5
3/11 or 3/14	1
4/3 or 4/5	1
1/5 or 7/9	2
4/10 or 1/7	3
6/12 or 2/5	3
5/8 or 6/5	1
2/5 or 5/9	10
43/100 or 6/10	2
6/7 or 8/9	15
2/3 or 6/9	4
¾ or 6/8	5

The AE group performed better than any other classification across all tasks with average scores in the 90% range. Across all classifications, including apparent experts, students were less successful comparing fractions with different numerators and denominators, specifically when the fractions are close in value such as 6/7 and 8/9 or 2/5 and 5/9. WNR and ZR students were less successful than the other classifications when comparing fractions with common denominators and same numerators. However, ZR students performed better than WNR students comparing fractions with the same numerator (60% compared to 80% however, the number of students in these groups are small). FR students were the most consistent across tasks. MR, and the unclassified group performed similarly with the most success comparing fractions with common denominators and the least success comparing equivalent fractions.

Interview Results: Fraction Comparison Tasks

Interview of WNR students

The interviews included twelve comparing fractions tasks listed below in table 4-13. What was surprising about the interview results was the variation in the strategies the students used.

Table 4-13 WNR Interview Results

	½ or 1/5	3/5 or 4/5	4/3 or 2/3	9/100 or 9/10	3/11 or 3/14	1/8 or 7/9	4/10 or 1/7	2/5 or 5/9	5/8 or 6/5	2/3 or 6/9
Item Type	CD	CD	CD	SN	SN	D	D	D	D	DE
WNR										
Wade							X	X		
Wendy										

Wendy, a WNR student, relied on the cross multiplication technique resulting in two whole numbers and then compared those numbers and correctly answered all of the fraction comparison tasks in the interview, which was surprising because she had missed nine out of eighteen on the written test. There was no evidence that she used the cross multiplication technique on the written test. There were occasional mental slips with this technique, not being sure which resulting number corresponds with which fraction, The interview exchange regarding strategies is provided here:

Researcher: Can you explain why the technique worked?

Wendy: I don't know

Researcher: Are there other ways to determine which is bigger besides cross multiplication?

Wendy: I don't know

Researcher: What about the fractions 2/3 and 6/9 is there another way to compare them?

[pause]

Wendy: You could reduce 6/9 to 2/3 and then they would be the same.

Wade, the other WNR student, tried to use the benchmark of one but had a misconception when doing this. He would compare the numerator and denominator of each fraction to see "how far from one" they were. For example, 9/100 has 91 more to go to get to one and 9/10 only has one to go, which was a correct answer with inaccurate reasoning. Another example would be 4/10 and 1/7 both are 6 away from one so they are equal which is incorrect. Another example was when Wade thought 2/5 was greater than 5/9 because 2/5 is three away from one and 5/9 is four away from one. He did recognize that 4/3 was "past a whole number" and selected it as bigger when compared to 2/3. The reliance on whole numbers is evident as he attempts to determine "how far apart" the numerator and denominator are and then treat that "missing part" to make whole number comparisons. These two tasks in the interview were the same (2/5 or 5/9) or similar to the two tasks he missed on the written test. Both task were different numerator and denominator tasks. On the written test he circled both 6/7 and 8/9 as equivalent and if you follow the strategy he explained in the interview, they are both "one away" from a whole number and would be equivalent.

Table 4-14 Interview of FR Students

	½ or 1/5	3/5 or 4/5	4/3 or 2/3	9/100 or 9/10	3/11 or 3/14	1/8 or 7/9	4/10 or 1/7	2/5 or 5/9	5/8 or 6/5	2/3 or 6/9
Item Type	CD	CD	CD	SN	SN	D	D	D	D	DE
Fran				X	X				X	
Faith										

The dominant strategy for Fran, a FR student, was to rewrite fractions to get common denominators. She even did this when she already had common denominators, for example she changed 3/5 to 15/10 and 4/5 to 20/10. In this case, her procedure for common denominators was to multiply the numerator by the denominator and then multiply both denominators by two. When she struggled trying to get common denominators and resorted to trying other strategies. She drew circles for ½ and 1/5 and circled ½ and said, "This would be a bigger piece."

When comparing 4/10 to 1/7, she selected 4/10 as larger because "the numbers four and ten are bigger than one and seven." When probed, she then tried to change the fractions to decimals because she was successful with comparing decimals using an annexing zero strategy but struggled to do this as well because she was troubled by 1/7.

When comparing 3/11 and 3/14, she tried to multiply the denominator by the numerator and put in a decimal point in the front to get .33 for 3/11 and .42 for 3/14. When probed for other ideas she compared the denominator and suggested that "fourteen is more than eleven."

I sensed that she recognized something was wrong with her decimal technique so she seemed to make up new strategies. For example when comparing 5/8 to 6/5 she just took out the fraction bar and wrote 58 and 65 and told me "58 is less than 65". When I probed her for other strategies, she tried to divide the two digits. She divided eight by five and got 1.3 and divided 6 by 5 and got 1.1 and still ended up selecting 5/8.

She maintained a positive attitude throughout and seemed to enjoy the attention and the challenge. She was persistent and appeared to enjoy being creative. On her written test, Fran missed five of the eighteen items. She missed four items on the written test, but she answered correctly the same item or a parallel item on her interview (4/10 or 1/7, 4/3 or 2/3, 2/5 or 5/9, ¾ or 6/8). She missed two items during the interview that she did not miss on her written test (3/11 or 3/14 and 5/8 or 6/5). She did miss the comparison task 9/100 or 9/10 on both the written and interview setting. Her written test shows no indication that she tried to get common denominators or tried any other strategy. The interview may have encouraged her to try harder, show more work and be more creative than she did on her written test.

Faith, the other FR student, relied mostly on cross multiplication even when comparing 2/3 for 6/9. However, she was able to suggest other strategies when probed but they were not her first choice.

Researcher: Are there other strategies that you could use to compare these fractions?

Faith: Well, to compare 9/100 and 9/10 you could get a common denominator and then you've got 9/100 and 90/100.

Researcher: Any other ideas?

Faith: If you compare 3/5 and 4/5 since they have the same denominator you can see that four is bigger than three.

Researcher: Interesting, what else?

Faith: When you look at 4/3 you can has a whole number in it so it is bigger than 2/3 [pause].

. Researcher: What about 3/11 and 3/14? Do you know anything about those fractions that would help you compare them?"

Faith: Well, 3/11 would have bigger size pieces than 3/14 and it would be bigger.

Researcher: What about 1/8 or 7/9?

[pause, 1 minute]

Faith: I don't know how to explain it another way.

Researcher: What about 1/5 or ½?

Faith: Well, they both have a one. I guess you could get a common denominator.

Interview of the apparent experts

All of the five AE students in the interview were successful with the comparing fractions tasks and correctly compared all of them. Four of the five also correctly compared all the fractions on the written test as well. One student, Katie, incorrectly compared 3/11 or 3/14 and 2/5 or 5/9 on the written test but answered correctly during the interview.

The students with the most success used several strategies and seemed to select a strategy based on the type of problem. The following strategies were used by four of the most successful students (John, Elane, Greg, and Chad) on the written tasks and in the interviews: Use benchmarks of ½ and 1, find a common denominator (sometimes by simplifying) and compare numerators, compare denominators when the numerators are the same (bigger denominator means smaller pieces). Each student was interviewed separately, however the examples that follow group the responses by question for comparison sake.

Comparing same denominators 3/5 or 4/5:

Researcher: Why did you select 4/5 as the larger number?

John: They have the same denominator and four is more than three in the numerator.

Chad: The numerator is bigger and the denominators are the same.

Katie: The numerator is bigger.

Elane: Four is closer to five and 5/5 is a whole number.

Greg: They are the same size pieces so just compare the numerators that tell you the number of pieces.

Comparing same numerators ½ to 1/5:

Researcher: Why did you select ½ as the larger number?

John: I made the denominator the same and 2.5/ is more than 1/5.

Chad: If you change ½ to a decimal you get 0.5 and if you change 1/5 to a decimal you get 0.2 and two-tenths are more than five-tenths.

Researcher: Chad, do you have any other ideas?

Chad: Well, you could think about them as percents also and 50% is more than 20%.

Katie: This one because fifty out of one-hundred is more than twenty. (She manipulated ½ and 1/5 to get a common denominators which resulted in fractions 50/100 and 20/100 and circled ½).

Elane: If you picture a pie or a pizza ½ would be more than 1/5.

Greg: The pieces are larger.

Comparing equivalent fractions 2/3 or 6/9:

When comparing 2/3 to 6/9, John multiplied 2 by 3 and 3 by 3 to convert 2/3 to 6/9 and circled both fractions. When probed for other strategies suggested simplifying 6l/9 to 2/3.

Researcher: Why did you circle both fractions?

John: I multiplied two by three and three by three to change 2/3 to 6/9.

Researcher: Is there another way to compare them?

John: You could simplify 6/9 by dividing six by three and nine by three.

Researcher: Why did you circle both fractions?

Chad: You can just simplify 6/9 by divided six and nine both by three and you get 2/3.

Katie: Well, I multiplied the denominators three and nine and then multiplied the numerator by the other denominator. (She manipulated the fractions to get a common denominator for 2/3 and 6/9 by changing 2/3 to 18/27 and 6/9 to 18/27.)

Researcher: How did you get the denominators here?

Katie: I just used cross multiplication technique and got these fractions (pointing to 18/27 and 18/27).

Reseracher: Why did you circle both fractions?

Elane: They are equal. I just got a common denominator by changing 2/3 to 6/9.

Researcher: Could you do this another way?

Elane: You could simplify 6/9 to 2/3.

Greg: I simplified 6/9 to 2/3. You can see then that they are the same.

Comparing different numerators and denominators 1/8 to 7/9:

Researcher: Why did you choose 7/9?

John: 7/9 is closer to a whole and is over ½ and 1/8 is less than ½ and closer to zero.

Katie: This one has more if you get a common denominator. (She manipulated the fractions 1/8 and 7/9 to get 9/72 and 63/72 again multiplying the denominators to get the common denominator.)

Researcher: Is there another way to compare them?

Katie: If you think about ½ then 1/8 is less than ½ and 7/9 is more.

Researcher: Why did you select 7/9?

Elane: 7/9 is more than ½ and 1/8 isn't.

Researcher: Is there another way to compare them?

Elane: Yes, there are other ways. You could get a common denominator and look at the numerator for example.

Reearcher: Why did you select 7/9?

Greg: Well, they are similar size pieces and there are more of the ninths.

Researcher: Is there another way to compare them?

Greg: 7/9 is greater than ½ and 1/8 isn't.

It is important to note that none of these students used a cross-multiplication technique that was popular with WNR and FR students.

Katie, classified as an "expert", appeared to be more limited in her strategies for comparing fractions compared to the other experts. She rewrote fractions to have common denominators on all the tasks. She was successful on the interview tasks, but she missed two items on the written test. She selected 3/14 as bigger than 3/11. With bigger denominators it was likely that she made a mistake finding a common denominator. With probing, Katie was able to determine that she could use a benchmark of ½ on one problem comparing 1/8 to 7/9.

Summary of Fraction Comparison Tasks

Students used a variety of strategies. WNR and FR students in the interview were for the most part limited to a single strategy, such as cross multiplication or "how far from one" for example, which at times they struggled to implement. One fraction rule student used multiple approaches, however they were ones created by the student and incorporated limited application of the meaning of numerators and/or denominators. In general, strategies used by WNR and FR students do not draw on the meaning of the numerator and denominator of the fraction but rather on comparing one whole number to another. On other hand, the AE students that were interviewed were more flexible in the use of strategies. Depending on the features of the numbers on the comparing tasks, students would use techniques such as using ½ or 1 as a reference, common denominators including simplifying fractions, and reasoning about the size of the piece(s) "missing." All of these strategies indicate that the structure of the fraction has meaning to the students.

WNR students that were interviewed relied on whole numbers when comparing fractions. They used techniques, in particular cross-multiplication, to transform the fractions into some form of whole number for comparison. There is also evidence on the written tests of other WNR students that indicates the use of cross multiplication.

FR students that were interviewed often relied on one strategy. In the case of these two students, one relied heavily on common denominators. The second FR student interviewed suggested the use of other strategies, some accurate and most inaccurate. There is also evidence on the written tests of other FR students that indicate the use of common denominator or cross multiplication technique.

Fraction-Decimal Comparison Tasks

Written Test Results: Fraction-Decimal Comparison Tasks

Students were given a list of eighteen number pairs. One was a fraction and the other a decimal. They were asked to circle the number with the largest value or if they are equal in value to circle both. The purpose of this task was to measure students' success with comparing fractions to decimals and to analyze strategies used and the errors that made looking for patterns. These items were at the end of the third section of the written test given over two days. Due to scheduling difficulties several students were not able to complete the test. Of the eighty-six students that attempted to compare the fraction to the decimal in the tasks listed in the table 4-15, eighteen students (21%) answered all the questions correctly. This is significantly less than the decimal comparison tasks (68%) and less than the fraction comparison tasks (39%).

Table 4-15 Fraction-Decimal Comparison Tasks

Comparison Item	Percent Correct	Type of task	Relation to ½
1/3 or 0.013	91%	Remove / zero in tenth place (Z)	Both < ½
3/10 or 0.3	89%	Equivalent (tenths) (E)	Both < ½
7/10 or 0.07	87%	Remove / zero in tenth place (Z)	Cross ½
2/3 or 0.25	85%	Drop / put decimal in front (DF)	Cross ½
3/7 or 3.7	84%	Replace / with decimal (R)	Cross 1
¾ or 0.750	84%	Equivalent (E)	Both > ½
4/8 or 0.5	81%	Equivalent (E)	½
4/7 or 4.7	80%	Replace / with decimal (R)	Cross 1
1/10 or 1.10	80%	Replace / with decimal (R)	Cross 1
1/6 or 0.6	78%	Denominator becomes decimal (DD)	Cross ½
1/8 or 0.8	76%	Denominator becomes decimal (DD)	Cross ½
5/6 or 0.59	76%	Drop / put decimal in front (DF)	Cross ½
3/6 or 0.36	75%	Drop / put decimal in front (DF)	½ and < ½
9/10 or 0.910	74%	Drop / put decimal in front (DF)	Both > ½
3/5 or 0.45	73%	Drop / put decimal in front (DF)	Both < ½
1/5 or 0.15	68%	Drop / put decimal in front (DF)	Both < ½
2/5 or 0.4	66%	Equivalent (E)	Both < ½
3/5 or 0.6	59%	Equivalent (E)	Both > ½

Comparing tasks were designed to identify strategies that students use to compare a fraction to a decimal and to determine what "procedures" students may be using to compare a fraction to a decimal. One procedure was to drop the fraction bar and put a decimal in the front of the digits (DF), for example 3/6 or 0.36. A second procedure involved replacing the fraction bar with a decimal point (R), for example 3/7 or 3.7. These items were the most frequently missed. Further analysis of table 4-15 indicated that students that miss these items do so by indicating that they are equal in value. A third procedure involved removing the fraction bar, putting a zero in the tenth places and a decimal in the front (Z), for example 1/3 or 0.013. Students compared these tasks with the most accuracy with 9% of the students answering correctly on one of the tasks and 87% answering correctly on the second task. A fourth procedure involved dropping the decimal point and using only the denominator as the decimal portion (denominator becomes decimal), for example 1/6 and 0.6. Students correctly answered these type comparing tasks 78% and 76% of the time. Finally students were given an equivalent fraction and decimal pair. The more familiar the fraction, such as 4/8 and 0.5, or the more "alike" the pair look, for example 3/10 or 0.3, the more likely students were to correctly compare these equivalent tasks. Of all the items compared, two of the equivalent fraction pairs, found at the bottom of the table above, were the most frequently missed items with 66% and 59% of the students answering them correctly.

The relation to ½ doesn't seem to have much influence as item type on successful comparison. When ordering tasks based on percent correct from highest to lowest, tasks types are fairly well clumped together while relation to ½ is fairly evenly distributed from high to low percent correct. The results were then broken down across classification using the item types describe above and can be found in table 4-16.

Table 4-16 Comparing Fraction/Decimal Pairs by Classification.

	3/6 or 0.36	9/10 or 0.910	1/5 or 0.15	5/6 or 0.59	3/5 or 0.45	2/3 or 0.25	3/7 or 3.7	4/7 or 4.7	1/10 or 1.10	1/3 or 0.013	7/10 or 0.07	1/6 or 0.6	1/8 or 0.8	3/10 or 0.3	¾ or 0.750	4/8 or 0.5	2/5 or 0.4	3/5 or 0.6
Item type	D F	D F	D F	D F	D F	D F	R	R	R	Z	Z	D D	D D	E	E	E	E	E
WNR																		
Paige		=	=				=		=		X	X	X		X	X		X
Tiff	X	X	=				X		=		=	=	X	X		X		X
Katie	X				X	X	=		=	X	X		=		X		X	X
Wendy		=			X		=	=	=			X	X		X	X	X	X
Wade	X		X	X						X	=		=		X		X	X
Ty	=	X					=	=							X	X	X	X
Jami	=	X	X		X	X	=	=	=	=	Z	=		X	X	X	X	X
ZR																		
Mark	X			X	X				=	X			X	X		X	X	X
Derek		X	=	X	=	X		=	X		X			X	X	X	X	
IZR																		
Steph			X	X		X	=		=	X		X					X	X
Chris		X					=	=	=						X		X	X
FR																		
Faith	=	X										=	=				X	X
Brad	X	X			=	X			=			X	=	X		X	X	X
Fran	=	X	X	X		X	X	X	X				=			X		
Ashley	X	X																
FR/ IZR																		
Chase			X	X	X	X			X			X			X	X	X	X
MR																		
Charlie																		
Taylor	=	X	X		X		=	=	=			=	=				X	X
KC			X		X							=						
Jeff		X	X			X		X			=	X	=			X		X
Shelly	=		=		X	X		X				X	=					X
UC																		
Josh	=	=	X		X	X	=	=		X		X	X	X	X		X	X
Jake																		
Shay			X					X						X			X	X
Kay			=	X	=							X	=				X	X
Mel				X													X	X
CJ	X																	X
Sam				=									=					X
Emma											=			X				

The data on individual students was further analyzed by the type of fraction-decimal comparison task. The percent correct for each classification can be found in table 4-17. The percent correct was calculated by counting the number of questions answered correctly in each task type by all the students in each classification and dividing by the total number of questions in each task type.

Table 4-17 Average Percent Correct by Rule Type and Type of Relationship.

	DF	R	Z	DD	E
WNR	52%	33%	43%	36%	29%
ZR	33%	50%	50%	75%	20%
IZR	67%	17%	75%	75%	50%
FR	46%	67%	100%	38%	65%
FR/IZR	33%	33%	100%	50%	20%
MR	57%	67%	90%	30%	80%
UC	75%	88%	88%	69%	65%
AE (n=53)	88%	92%	97%	91%	86%

The data indicated that WNR students performed poorly across all item types, but struggled the most with equivalent fraction pairs where only 29% of the time they indicated the two fractions were equal in value. They also struggled with tasks where the fraction bar is replaced with a decimal point (33%). Of the fourteen combined errors made among these students, thirteen were errors indicating the two items were equivalent. Clearly they were looking for common features between the two numbers and using a procedure that appeared to make them equivalent which was trading one symbol for another. This appeared to be a trend across all the classifications.

In general, students performed lowest on equivalent fraction tasks (including the apparent experts). Of all the errors made across the classifications (not including AE) 19/74 of the errors made (26% of the errors) on type tasks where students drop the fraction bar and place a decimal point in front of the digits such as (DF) were errors indicating the two items were equivalent when they were not. But looking at these tasks more closely, only three of the six tasks would actually be equal if this procedure was followed. They are 3/6 or 0.36 and 9/10 or 0.910 and 1/5 or 0.15. Specifically looking at these three tasks, forty-two combined errors were made across classifications and 14/42 (33%) were errors indicating the two items were equivalent when they were not which is slightly higher than the figure that includes all of the tasks types. Of the fourteen errors made on tasks where the procedure was to remove the fraction bar, place a zero in the tenths place and decimal point in front (Z), 5/14 (33%) were errors indicating the two items were equivalent. Denominator becomes decimal (DD) type items resulted in a combined twenty-nine error, with 15/29 (50%) being errors indicating the two items were equivalent.

The most overwhelming error involved the procedure of replacing the fraction bar with a decimal point where the error is in indicting the two numbers are now equivalent. There were 26/35 errors (74%) on tasks where the procedure is used. Across all classifications there were students that indicated that removing the fraction bar and replacing it with a decimal point would make the two items equivalent. Rather than consider the value of the numbers, such as more or less than 1/2, students tended to look for numbers that “looked” equivalent or were fractions and decimals with which they were already familiar such as 0.3 and 3/10 or 4/8 and 0.5. Students struggled the most with equivalent items that did not look alike, such as 2/5 and 0.4 or 3/5 and 0.6.

In general, AE students performed fairly consistently across all types of items and had more success with R item types that any of the other classifications. Thirty-nine of the fifty-three apparent experts that completed these tasks missed at least one item. A break-down of the number of items missed among those thirty-nine students can be found in table 4-18. A break-down of the frequency of the item missed by the apparent experts can be found in table 4-19.

Table 4-18 Apparent Experts and Frequency of Number of Fraction - Decimal Comparison Items missed

Number of Items Missed	Frequency (n=39)
1	13
2	**10**
3	**5**
4	**2**
5	**2**
6	**4**
8	**2**
11	**1**

Table 4-19 Apparent Expert Frequency of Incorrect Responses by Task

Task	Task type	Frequency Incorrect of an incorrect choice	Frequency of incorrectly stating items are equivalenct
5/6 or 0.59	DF	11	
3/6 or 0.36	DF	7	
9/10 or 0.910	DF	6	3
3/5 or 0.45	DF	10	2
1/5 or 0.15	DF	11	1
2/3 or 0.25	DF	2	
3/7 or 3.7	R	1	2
4/7 or 4.7	R	2	3
1/10 or 1.10	R	0	4
1/3 or 0.013	Z	1	
7/10 or 0.07	Z	1	1
1/6 or 0.6	DD	6	
1/8 or 0.8	DD	4	1
3/10 or 0.3	E	1	
¾ or 0.750	E	4	
4/8 or 0.5	E	6	
2/5 or 0.4	E	12	
3/5 or0.6	E	15	

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Interview Results: Fraction-Decimal Comparison Tasks

Interview with WNR students

In the interview, students were asked to circle the number with the largest value and if they were equal circle both, just as they were asked to do on the written test, however now they were also asked to explain their thinking. The items used in the interview are in table 4-20.

Table 4-20 WNR Interview Results.

	9/10 or 0.910	2/5 or 0.25	5/6 or 0.59	7/9 or 0.45	2/3 or 0.25	4/7 or 4.7	1/10 or 1.10	1/6 or 0.6	1/7 or 0.7	¾ or 0.750	3/6 or 0.5	4/5 or 0.8
Item type	DF	DF	DF	DF	DF	R	R	DD	DD	E	E	E
Wade	X	X								X		
Wendy	X				X	X	X			X		

Wendy, a WNR student, used one strategy, which was to change the decimal to a fraction and cross multiply. This procedure allowed her to use the cross-multiply technique she relied heavily on when comparing fractions. For example, when comparing 1/7 to 0.7 she converted 0.7 to 7/10 and they used cross multiplication to compare two whole numbers. She explained that "seventy is more than ten." This was the technique and explanation that she used for all the decimals she knew how to write as a fraction. She did not use this technique to compare ¾ to 0.750 because she said, "that's a lot of numbers after the decimal and I'm not sure what that means." Her ability to write decimals as fractions appears limited to tenths and hundredths. When probed for another way to think about it she said, "I don't know. I really don't know."

Wendy answered, "I don't know" to the following: 9/10 or 0.910, ¾ or 0.750, 4/7 or 4.7, and 1/10 or 1.10. She struggled with the first two because they had digits into the thousandths place and she wasn't sure how to convert that to a fraction. When asked about the last two comparison tasks she said, “I’m not sure, there are numbers in front of the decimal and don’t know what to do with it or what it means.” On her written test she missed these items as well but indicated they were all equal.

Wade, the other WNR student, also relied on a dominant strategy similar to his strategy for comparing two fractions. He changed all the decimals to fractions and then compared the differences between the numerator and denominator to see how close to a whole number the fractions were. A smaller difference between the numerator and denominator meant the fraction was bigger than a fraction with a larger difference. This was consistent with the strategy he used in the comparing fractions task.

When comparing 1/7 to 0.7, he converted 0.7 to 7/10 and then stated, “Seven is closer to ten than one is to seven so it is closer to a whole number.” In this case it resulted in a correct answer, however when comparing 9/10 to 0.910, he converted 0.910 to 910/1000 and then explained, “Nine is only one away from ten and 910 has 80 left so 9/10 is bigger.” In this case the same thinking resulted in the wrong answer. He continued with this “how far away” strategy that he used when comparing fraction in the previous task as well. When probed for other strategies he was able to come up with some. For example, when comparing 4/7 to 4.7 rather than convert to a fraction he pointed out that “4.7 has a whole number in it.” He recognized the same thing when comparing 1/10 and 1.10. When comparing 3/6 to 0.5 he converted 0.5 to 5/10 and then saw that 3/6 and 5/10 were both equal to ½. Finally, when comparing 1/6 and 0.6 he was able to suggest that 0.6 was bigger because when he converted 0.6 to 6/10 and simplified it to 3/5 he stated “it was over half way to five”. An interesting thing to note is that he missed items on the interview that he did not miss on the written test and he missed items on the written test that he did not miss in the interview. Apparently he approached the task differently when he was taking the written test.

The interesting thing about both whole number dominant thinkers is that they opted to change the decimal to a fraction. It was expected that these students would prefer to work with decimal numbers, which more closely resemble whole numbers. However, this was not the case due to a dominant strategy that they had for working with fractions that resulted in correct answers more consistently.

Interview with FR students

In the interview, students were asked to circle the number with the largest value and if they were equal circle both, just as they were asked to do on the written test, however now they were also asked to explain their thinking. The items used in the interview are in table 4-21.

Table 4-21 FR Interview Results. Legend of abbreviations needed

	9/10 or 0.910	2/5 or 0.25	5/6 or 0.59	7/9 or 0.45	2/3 or 0.25	4/7 or 4.7	1/10 or 1.10	1/6 or 0.6	1/7 or 0.7	¾ or 0.750	3/6 or 0.5	4/5 or 0.8
Item type	DF	DF	DF	DF	DF	R	R	DD	DD	E	E	E
Fran	=	=				=	X	=		=	X	X
Faith												

Fran, a FR student, relied on a dominant strategy which involved converting the fraction to a decimal. She did this several ways. One way was to take out the fraction bar and put a decimal in the front, recognizing that the fraction was less than one so didn’t want to replace the fraction bar with a decimal point. She started out using this technique with for example 9/10 compared to 0.910 and for 2/5 compared to 0.25, which she said were equal. She then switched and used division to change the fraction to a decimal but divided the denominator by the numerator and put a decimal point in the front of her answer. For example, when comparing 3/6 to 0.5 she divided 6 by 3 and got 2 so she wrote the decimal for 3/6 as 0.2 and compared it to 0.5. She also used this strategy to compare 2/3 and 0.25. Three divided by two resulted in 1.1 which was bigger than 0.25. She did this when she compared 4/5 to 0.8. She divided 5 by 4 and got 1.1 which she compared to 0.8. She used this division strategy on six of the twelve tasks. She changed strategies when comparing 1/6 to 0.6 where she ignored the numerator completely and put a decimal point in front of the denominator so that 1/6 became 0.6 and circled both 1/6 and 0.6 as equivalent. When comparing 1/10 to 1.10 however, she dropped the fraction bar and ignored the numerator to convert 1/10 to 10 and circled 1/10 as larger than 1.10. This has become a pattern with Fran to try various techniques to compare decimals, fractions or fraction/decimal pairs. She tends to follow various procedures and can explain what she is doing but can't explain the meaning behind why she is doing it.

Researcher: Why are you using division?

Fran: To change the fraction to a decimal.

Researcher: Why do you use division to change a fraction to a decimal?

Fran: I don't know that is just what we always do.

Researcher: How do you know if you decimal makes sense with a fraction?

Fran: Well, you really don't know, but you don't have to you just change them.

Faith, the other FR student, answered very differently from Fran. Faith relied on a dominant strategy which involved converting a decimal to a fraction (rather than fraction to a decimal) and then using the cross multiplication technique that she used in the comparing fraction tasks. All of her tasks involved the same explanation of describing the procedure for changing a decimal to a fraction and multiplying. She used this same cross multiplication technique when comparing fractions in the previous task with 100% accuracy and did not have the same success comparing decimals so perhaps she too has picked up on the fact that she has more success with fractions than decimals. There were a couple of exceptions to using the cross multiplication technique. She at first suggested converting the decimal to a fraction for 0.750 which she recognized equaled ¾ and circled both. The same was true when she converted 0.5 to 5/10, which equaled ½ and circled both 3/6 and 0.5. She was also able to recognize 4.7 was greater than 4/7 and 1.10 was greater than 1/10 because it has "whole numbers in it" and in these cases didn't use cross multiplication.

Most of the students in the interview were fairly successful with the comparing decimal to a fraction tasks. The WNR and FR students interviewed did better on the interview tasks than they did on the written test.

Interview with the Apparent Experts

The five AE students interviewed were able to complete the fraction-decimal comparison tasks with 100% accuracy. The answers came much easier for several of the students and the flexible use of strategies by some students were more evident with this type of comparison task. The students with the most success used several strategies and seemed to select a strategy based on the type of problem. The following strategies were used by four of the most successful students (John, Elaine, Greg, and Chad) that were classified as experts: Use benchmarks of ½ and 1, convert decimal to a fraction and compare two fractions, convert fraction to a decimal and compare two decimals, convert one or both to a percent and compare (i.e., when comparing 4/5 to 0.8, both were determined to be equal to 80%).

When comparing 1/6 to 0.6 Greg and John both circled 0.6 and explained that "six-tenths is more than one-half and one-sixth is less than one-half." Elaine circled 0.6 and stated, "six-tenths is closer to one." Chad did something a little different that the others, he changed 1/6 to the decimal 0.16 (repeating) and reported that 0.6 would be bigger. When comparing 3/6 to 0.5, Greg and John both converted both to the fraction ½ and circled both indicating they were equivalent while Elane and Chad converted 3/6 to the decimal 0.5 and circled both.

When comparing 5/6 to 0.59, Chad explained that 5/6 was bigger than 0.59 because "0.59 is barely over one-half while 5/6 is about 16% away from being one so it is closer to one." John took a slightly different approach and selected 5/6 saying, "This one (.59) is almost 0.4 away from one and 5/6 is definitely less than 0.4 away from one." Greg simplified it a bit saying, "5/6 is almost one and 0.59 is nearly ½ so 5/6 is larger." Elane wasn't quite as specific, she selected 5/6 because "it is closer to one."

Katie, as with the fraction tasks, struggled more than the other "experts". She had a single dominant strategy, which was to change the decimal to a fraction and get a common denominator. She performed better on the interview, answering questions correctly that she missed on the written test. She struggled to use her strategy during the interview. For example, comparing 2/5 to 0.25 she converted 0.25 to ¼ and then converted 2/5 to 8/20 and ¼ to 5/20 resulting in a correct answer after a lot of work. When comparing 5/6 to 0.59, she converted 0.59 to 59/100 and recognizing she could get a common denominator she changed her strategy and tried to change 5/6 to a decimal, she got as far as 0.8 and was able to determine that 5/6 was bigger. She struggled as well with 7/9 compared to 0.45. It is possible that she was more persistent and more resourceful during the interview with someone watching and waiting for an explanation, resulting in a better performance. She did use other strategies such as changing 2/3 to 0.6666. She recognized that 3/6 and 0.5 were both ½. She took the zero off of 0.750 to make

0.75 and recognized that as equal to ¾. The last example being an item she missed on the written assessment indicating then that 0.750 was greater than ¾.

Summary of Fraction-Decimal Comparison Tasks

The WNR and FR students interviewed performed better on the interview tasks than they did on the written test. To compare fraction-decimal pairs, they used a strategy similar to what they used to compare fractions. It appeared that the focus was on converting the decimal to a fraction and using common denominators or cross multiplication. This worked for tasks in which they knew how to convert to fractions. More complex decimals such as 0.750 and 1.10 created some problems.

AE students performed better on the interview tasks and the written test than did the WNR and FR students. They showed greater flexibility in selecting and implementing a strategy for comparing fraction and decimal tasks.

Student Thinking Across Comparison Tasks

Across all three comparison type tasks, there appear to be some patterns emerging that answer research question #1: What type of strategies, including error patterns, do students use when comparing within and across decimal and fraction representations? Students that were classified as WNR students in decimal comparison tasks continue to rely on the use of whole numbers on fraction comparison tasks and fraction/decimal comparison tasks. They are often limited to the use of one strategy, for example during the interview when comparing decimal they relied on the length of the number, for comparing fraction tasks they would use cross-multiplication and then compare two whole numbers, and when comparing fractions to decimals they would drop the fraction bar and put a decimal point in its place or they would convert the decimal to a fraction and use fraction comparison strategy. One student classified as FR students in the decimal comparison tasks attempted to use to more than one strategy. However, as with the other WNR and FR students she followed procedures which had little connection to the value of the numbers but rather involved converting one number into a form that looked more like the other. In decimal comparison tasks they relied on the length of the number, but when probed further would consider the value of the digits in the column and use an annex zero strategy.

In fraction comparison tasks, the FR students interviewed relied on cross multiplication so they could compare two numbers or finding common denominators so that the fractions looked almost alike. For example, in the interview one student was also very dependent on the cross multiplication technique even when probed for other ways to think about the comparison tasks. The second FR student frequently used common denominators but would consider other strategies when probed such as drawing pictures or annexing zeroes. However, some of these strategies didn't really make sense, for example when comparing 5/8 and 6/5 she first dropped the fraction bars and compared 58 to 65. When probed further, she divided eight by five resulting in 1.3 and divided six by five resulting in 1.1 and now selected 5/8 as bigger. WNR and FR students in the interview did not even suggest the use of benchmarks to compare the fractions. Students that were classified as apparent experts on decimal comparison tasks tend to have and use various strategies depending on the type of task are frequently used benchmarks. However, the interviews showed that even experts can be limited in the strategies they use which can cause errors when that strategy is inappropriate or inefficient.

Across the types of classifications, students were more successful with decimal comparison tasks. This is probably because the "annex zeroes" coping strategy is much easier to use than strategies necessary for comparing fractions or comparing fractions to decimals. Students were less successful with comparing fraction tasks. Students used a variety of strategies such as cross-multiplication, finding common denominators, or using benchmarks. Students were the least successful when comparing a decimal to a fraction. This involved an extra step of attempting to make the two representations “look alike”. Despite the success with comparing decimals, converting fractions to decimals was not a common occurrence. It did occur on occasion with the “experts”, however they used a variety of strategies with converting to decimals being just one. The students with the less of success with decimal comparison problems, whole number dominant thinkers, tended to rely on using fractions and a cross multiplication technique when comparing fraction or comparing fraction to a decimal. Perhaps they recognize the complexity of decimal numbers and have found that they have more success with the cross multiply technique than the rule of picking longer as bigger. The students that struggled with comparison tasks often avoided working with rational numbers, as a result errors were not obvious to them and these students tended to be less successful on comparing tasks.

Across all classifications, student struggled comparing equivalent fractions (an average of 82% on two tasks) and comparing an equivalent fraction and decimal (an average of 72.5% on four tasks). Of the twenty-nine errors made comparing equivalent fractions only nine of those errors were made by the apparent experts, which made up approximately half the total number of students in the study. Across all classifications, including AE, equivalent pairs that didn’t “look alike” (2/5 or 0.4 and 3/5 or 0.6) resulted in the most errors across all the tasks. It would appear that a common strategy for students is to look for common features when comparing fractions or fraction/decimal pairs.

Subconstructs of Rational Number Tasks

The second part of this chapter reports on data collected about the subconstructs of rational numbers and answers the second research question: How successful are students at solving rational number tasks for each of the five subconstructs? The tasks are reported by subconstruct alternating between written test results and follow-up interview responses. The test and the interview are the same one reported above. This section divides the items into subconstructs, rather than types of rational numbers.

What is a fraction?

Students were asked to answer the question, "What is a fraction? Explain all the ways you can think of to describe the meaning or meanings of the fraction ¾." The purpose of this task was to see which subconstructs students would access from their knowledge base and use to explain a fraction. The remainder of the written tests items specifically targeted the subconstructs, but this was asked to see how they would initially respond. Eighty-five students attempted to answer this question. The responses given are listed in table 4-22 with the frequency of the students that gave the response. The percent of students with these responses adds up to more than 100% because a student could give more than one explanation.

Table 4-22 Responses to "What is a Fraction?"

Responses by subcontract	% of students N=85	WNR N=7	ZR N=3	IZR N=2	FR N=4	FR/ IZR N=1	MR N=5	UC N=10	AE N=53
Subconstruct responses:									
Fraction is a part out of a whole – three out of four, drew 4 parts and shaded 3, 3 pennies out of 4 coins, "one number out of another number", "It's not a whole"	N=48 (56.5%)	3	1	1	2		2	6	33
Quotient: a division problem, 3÷4, "something that goes into something else", "cut up into equal parts"	N=8 (9.4%)			1		1			6
Operator: "Part of a number", ¾ of a whole, ¼ x 3	N=7 (8.2%)								7
Ratio: For every three you get four, 3:4, 3 to 4	N=3 (3.5%)			1					2
Measure: ¾ of the way to a whole, a number between 0 and 1, a way to be more specific	N=4 (4.7%)								4
No response:	N=13 (15.3%)	2	2	1			1	2	5
Other responses:									
75%	N=18 (21.1%)	1			2		1	4	10
Decimal: 0.75	N=15 (17.6%)				2		3	3	7
75 out of 100, 75/100, "They are always over a hundred", "It's always out of one hundred", 100÷3, 100-25=75	N=7 (8.2%)				1		1	1	4
Two numbers: "It's a numerator and denominator with a line separating them", "numerator of 3 and denominator of 4", "two numbers put together"	N=7 (8.2%)	1					1		5

A part-whole definition was the most frequent response occurring in 56.5% of the responses, followed by quotient (9.4%), operator (8.2%), ratio (3.5%), and measure (4.7%). In response to the meaning of ¾, 21.1% gave the percent equivalent and 17.6% gave the decimal equivalent. A surprising 15.3% did not respond at all to this question, although they responded to the question before and after it. Of those thirteen students, eight came from the thirty-two students (25%) that were not classified as experts.

WNR responses were limited to the part whole responses and gave answers such as "a part of a number", "3 of the blocks are filled out of 4", and "It means that, lets say you had a pie and you cut it into 4 pieces but then one person took a piece so there is 3 pieces left so 3 out of 4 pieces are left”. Two students didn’t respond (or responded with ?) and one gave the percent for ¾. One focused on the fraction as if it were two numbers, stating “two numbers put together, same as a decimal but a fraction”. ZR students responded similarly to WNR. One gave a part-whole response simply stating that the meaning of ¾ is “3 out of 4”. The other two failed to answer. Of the two IZR students, one didn’t respond and the other responded touching on three subconstructs.

FR students again were limited to the part-whole interpretation of fractions. Two of the four FR students only gave the decimal and percent equivalent of the fraction ¾ and didn’t explain the meaning of the fraction. One student fell into two classifications and his response stated, “It is smaller than a number, and with a / you can divide it”. His response indicates that fractions are less than the whole number one and the reference to division indicates that the student believes division makes a number smaller.

Two MR students gave part-whole responses that were more detailed compared to the previous groups. One responded, “A fraction is numbers over each other to explain how much you have or don’t have of something.” The other answered, “A fraction is a way to describe something that is not whole. Three out of four. Someone ate three of the four sections.”

Responses of students in the UC category were limited to the part-whole subconstruct as well, answers included: “Three fourths means that out of four socks three are blue or out of four cars three are red”, “A fraction is a number that’s not complete,” and “A fraction is one number out of a different number (I think).” Of the 32 students that were not classified as apparent experts, only one explained the meaning of a fraction using more than one subconstruct.

AE students also tended to give explanations focusing on the part-whole subconstruct. Of the fifty-three AE students, twenty-nine gave responses limited to one subconstruct and ten gave responses that focused on two subconstructs. One student explained the meaning of a fraction using three subconstructs. No student explained the meaning with more than three. Generally, students that used more than one subconstruct gave a part-whole explanation as one response. Some sample responses from AE students by subconstruct are listed below:

- Part-whole
    - ¾ means there are four things and you have three"
    - "A fraction is part of a whole"
    - "The amount of parts over the total"
    - "A fraction is a way of saying you have so many things out of a total of that thing"
- Quotient
    - "3÷4"
    - "If you were to divide something, the fraction is a way of saying how much of whatever you divide"
    - "A fraction is something that goes into something else"
- Ratio
    - "For every three you get four"
    - "3 to 4"
    - "3:4"
- Measure
    - "A fraction is a number between 0 and 1"
    - "Three quarters of the way to a whole"

  - “A fraction is a number that can be expressed as a certain number. ¾ is more exact that round up to one. So with fractions, you can be more specific”

- Operator
  - “A fraction is part of a number. A number broken down as part of a number. ¾ of the number 1 is the fraction ¾.”
  - “It can also be used to describe how to split a number, like they get a certain amount of a number. You would get most of it but not all of it for ¾”
  - “It is a fragment of a number”

These answers provided an overview in terms of how students interpreted fractions, however because of the written format they are limited because there was no one to encourage or probe them to think about the concept more deeply. Because of this, tasks were designed to try to bring out other important ideas related to the concepts.

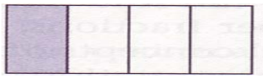
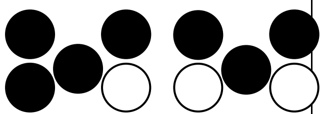
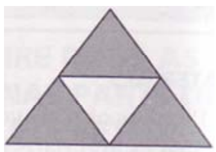
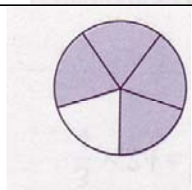
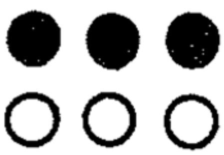
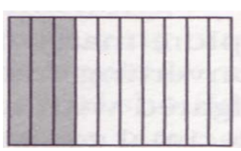
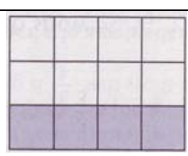
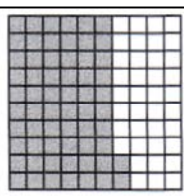
Part-whole Subconstruct

The part-whole relation is often the first interpretation that students experience and forms the basis for the other interpretations (citation). Consider the fraction 3/5. Using a part-whole relation, 3/5 describes three of five equal-size parts. Research indicates that there is an over-reliance on part-whole contexts and instruction is often limited to the area model. The research here focuses on area models, discrete sets, and number line (continuous model).

Written Test Results: Identifying Shaded Portion

Ninety-nine students were asked to give the fraction, decimal, and percent for the shaded part of a shape, the shaded part of a set of shapes, or the shaded part of a line. The purpose of the tasks was to assess students’ ability to use symbolic notation for part-whole subconstruct. Table 4-23 reports the percent of correct responses given by students.

Table 4-23 Fraction, Decimal, Percent Name for Part-Whole Models.

Figure		Fraction	Decimal	Percent
	Correct responses: Percent of students correct	$\frac{1}{4}$ 97	.25 87	25% 91
	Correct responses: Percent of students correct	$\frac{7}{10}$ 97	0.7 or 0.70 94	70% 93
	Correct responses: Percent of students correct	$\frac{3}{4}$ 99	0.75 89	75% 96
	Correct responses: Percent of students correct	$\frac{4}{5}$ 98	0.8 or 0.80 84	80% 86
	Correct responses: Percent of students correct	$\frac{3}{6}$ or $\frac{1}{2}$ 98	0.5 or 0.50 94	50% 96
	Correct responses: Percent of students correct	$\frac{3}{10}$ 97	0.3 or0.30 93	30% 94
 0 1	Correct responses: Percent of students correct	½, $\frac{0.5}{1}$, 5/10 78, 11, 5 Total: 93%	0.5 or 0.50 92	50% 91
	Correct responses: Percent of students correct	$\frac{4}{12}$ or $\frac{1}{3}$ 97	$0.\overline{3}$ or $0.3\overline{3}$ 52	$33\frac{1}{3}\%$ or $33.\overline{3}\%$ 58
	Correct responses: Percent of students correct	$\frac{62}{100}$ *or* $\frac{31}{50}$ 86	0.62 77	62% 78

The percent of students that gave an appropriate fraction name for the shaded portion ranged from 86% to 99%. The percent of students that gave an appropriate decimal name for the shaded portion ranged from 52% to 94%. The percent of students that gave an appropriate percent for the shaded portion ranged from 58% to 96%. Not surprisingly, the decimal and percent for 1/3 was the most difficult for students, with 52% and 58% of the students answering correctly respectively. The decimal and percent for 1/3 is non-terminating, making it more challenging for students. The fraction, decimal and percent for the 10 by 10 grid with 62/100 shaded was also challenging, with 86%, 77% and 78% of the students answering correctly respectively. Students also struggled with the decimal and percent for 4/5, with 84% and 86% of the students answering correctly respectively, even though 98% of the students were able to give the fraction name. The results for number line tasks indicate that students perform about the same writing fractions, decimals, and percents for this task. It is interesting to note that in addition to the fraction ½, students also used the fractions   and 5/10 to represent the shaded amount. While these are equivalent to ½ it is uncertain as to whether these students recognized that ½ of the number line was shaded. In general, students were better able to write the part out of the whole amount shaded in terms of a fraction rather than a decimal or even a percent. This is not surprising because the "whole" in the decimal and percent is not as obvious as it is in the fraction and with decimals and percent the "whole" has to be converted to a base-ten number system. To better analyze how students performed on the above part-whole tasks, students were sorted by the number of items they missed. Results are found in tables 4-24 through 4-27.

Table 4-24 Number of Students with Correct Responses Broken Down by Representation.

Number of part-whole items missed (N=27)	Fraction (n=99)	Decimal (n=99)	Percent (n=99)
0	75	33	40
1	20	41	38
2	3	8	7
3	1	7	7
4		4	4
5		3	2
6		1	
7		1	1
8		1	

The total number of students was then broken down into three groups. Those that were classified by a rule (WNR, FR, ZR, IZR, MR), those that were unclassified (UC) and those that were classified as apparent experts (AE)

Table 4-25 Frequency of Part-whole Representation Missed by Students "Classified" by a Rule (n=22).

	Fraction	Decimal	Percent
0	11	2	4
1	9	8	7
2	1	3	3
3	1	2	2
4		2	3
5		2	2
6		1	
7		1	1
8		1	

Table 4-26 Frequency of Part-whole Representations Missed by Unclassified Students (n= 10)

	Fraction	Decimal	Percent
0	8	4	4
1	2	3	4
2		2	2
3		1	

Table 4-27 Frequency of Part-whole Representations Missed by Apparent Expert Students (n= 67)

	Fraction	Decimal	Percent
0	56	27	32
1	9	30	27
2	**2**	**3**	**2**
3		**4**	**5**
4		**2**	**1**
5		**1**	

Overall, 75% of the students correctly use a fraction to represent a part-whole task while only 33% were able to correctly use a decimal to represent part-whole and 40% were able to correctly use a percent to represent a part-whole situation. While only one student missed three fraction tasks, seventeen students missed three or more decimal tasks and fourteen missed three or more percent task. This lends more support to the statement that students have a more difficult time using decimals and percents in part-whole situations but are successful using fractions across all classifications.

Quotient Subconstruct

A quotient interpretation implies division. Here a fraction is perceived as a division operation and the symbol a/b is used as a way of writing a ÷ b. This research focuses on the partitioning aspect, or “sharing”, of equal parts, which involved dividing an object or objects among several people. It also focuses on the meaning of the symbolic form a/b. The quotient interpretation makes a shift from how many to how much.

Written Test Results: Sharing Task #1

One hundred one students were asked the following question: Three cakes were divided equally among four people. Shade in the amount each person would get. How much cake does each person get? The purpose of this task was to find how successful students were with partitioning exercises and with using fraction notation to represent the "how much" a person would get. Sixty-seven students responded indicating some level of understanding. However, twenty-six of these students gave no numerical answer: They only drew an accurate picture and shaded or labeled the appropriate amount for each person. Because they didn't give a numerical answer, the researcher is uncertain as to whether they do or do not know how to represent the fractional amount and therefore were not counted as students with responses that demonstrate knowledge of the quotient subconstruct because the symbolic notation was central to this interpretation. Other students gave answers that did not use rational numbers. Because the purpose was to measure success with quotient subcontruct data was put in 3 categories: Demonstrated knowledge of quotient subconstuct using rational number answers, Correct answers but not clear quotient understanding (because rational numbers were not used), and incorrect answers.

Only twenty-nine students demonstrated knowledge of the quotient subconstruct by using a fraction or percent to represent the amount of cake each person would get. Responses are listed in table 4-28.

Table 4-28 Responses that Demonstrate Knowledge of Quotient Subconstruct

Response	Overall Frequency N=29/101	WNR N=0/7	ZR N=1/3	IZR N=0/2	FR N=1/4	FR/IZR N=1/1	MR N=0/5	UC N=2/10	AE N=24/69
¾	25	0	1	0	1	0	0	2	21
9/12	1	0	0	0	0	0	0	0	1
75%	3	0	0	0	0	1	0	0	2

Thirty-three students gave responses that, while may be correct, do not demonstrate clear quotient understanding because students failed to represent the amount each person would get as a rational number. These responses are listed in table 4-29.

Table 4-29 Correct but Unclear Knowledge of Quotient Subconstruct.

Responses	Frequency N=33/101	WNR N=5/7	ZR N=0/3	IZ N=0/2	FR N=2/4	MR N=2/5	UC N=6/10	AE N=18/69
Accurate picture/ no answer	20	1	0	0	1	1	6	11
3 pieces each	9	2	0	0	1	0	0	6
1 piece from each cake	2	2	0	0	0	0	0	0
1/3 +1/3 = 2/3 + ¼ of a third	1	0	0	0	0	1	0	0
¼ of each cake	1	0	0	0	0	0	0	1

A larger number of students (approximately 25%) were able to draw an accurate picture and label the parts for each person indicating that they understood how to “share” equally. They did not give a rational number answer for the amount each person would get and therefore it is unclear as to whether they have knowledge of the quotient subconstruct of rational numbers. Answers given by thirty-four students were incorrect. The responses of these students are in table 4-30.

Table 4-30 Incorrect responses.

Answer given	Frequency N=39/101	WNR N=2/7	ZR N=2/3	IZR N=2/2	FR N=1/4	MR N=3/5	UC N=2/10	AE N=27/69
3/12 = ¼	12	0	1	0	0	1	0	10
Inaccurate picture & no numerical answer	19	2	1	1	1	1	0	13
Other incorrect responses	8	0	0	1	0	1	2	4

A common inaccurate response was 3/12 = ¼ of a cake. Students with this answer were most likely treating all of the cakes as one whole rather than each cake as a whole. After cutting the cakes into fourths they treated the pieces as twelfths. Ten students gave an incorrect fraction for the amount of cake, each of these ten students gave a different fraction! A list of all the incorrect fractions is in the appendices. Ten students attempted to draw the parts but didn’t shade any of the parts, shaded incorrectly or didn't finish their drawing. None of these students that attempted drawings gave any numerical answer to the task.

Written Test Results: Sharing Task #2

Students were asked a similar question to the one above, but this one using a denominator that could be perceived as more challenging. One hundred one students answered the following question: Three cakes are divided equally among five people. How much cake does each person get? Forty-one students answered indicating some level of understanding. However, ten of those students gave no numerical answer but drew fifths and marked the amount for each person appropriately. It is uncertain if they recognize that the amount shaded is a "fraction" of a cake. Other students gave an answer in pieces but not as a fraction of a cake. Because the purpose was to measure success with quotient subcontruct data was put in 3 categories: demonstrated knowledge of quotient subconstuct using rational numbers, correct answers but not clear quotient understanding, incorrect answers. Twenty-seven students demonstrated knowledge of the quotient subconstruct. The responses are listed in table 4-31 with the frequency of the response.

Table 4-31 Responses that Demonstrate Knowledge of Quotient Subconstruct

Responses	Overall Frequency N=27/101	WNR N=0/7	ZR N=0/3	IZR N=0/2	FR N=0/4	FR/IZR N=0/1	MR N=0/5	UC N=2/10	AE N=25/69
3/5 (or 6/10)	25							2	23
1/5 of each cake	2								2

Fifteen students fell into the category of "correct answers but not clear quotient understanding". Their responses are in table 4-32.

Table 4-32 Correct but Unclear Knowledge of Quotient Subconstruct.

Responses	Overall Frequency N=15/101	WNR N=2/7	ZR N=0/3	IZR N=0/2	FR N=0/4	FR/IZR N=0/1	MR N=0/5	UC N=4/10	AE N=9/69
Accurate picture/ No numerical answer	10	1						4	5
3 pieces	5	1							4

Five students answered “3 pieces,” and while this may be correct; it does not demonstrate clear quotient understanding because students failed to represent the amount each person would get as a rational number. It is unclear as to whether they are able to do so or not. As mentioned earlier, ten students drew fifths and correctly indicated the amount for each person but gave no numerical answer and would fall into this category. This is significantly less than the thirty-eight students that fell into this category on the previous question.

The answers given by sixty students were not correct (60%), compared to 34% on previous item. Common incorrect responses and the frequency of the response are in table 4-33. Twelve students answered incorrectly and their individual answers are not included in this table because they were the only student making that error. Incorrect responses that were given two or more times are included in table 4-34.

Table 4-33 Types of incorrect responses by students using each error pattern.

Answer given	Overall Frequency N=59/101	WNR N=5/7	ZR N=3/3	IZR N=2/2	FR N=4/4	FR/IZR N=1/1	MR N=3/5	UC N=6/10	AE N=35/69
Inaccurate picture & no numerical answer	32	3	2		1	1	2	5	18
3/15 = 1/5	10				1			1	8
½	3	2							1
1/6	3		1	1					1
5/15= 1/5	1			1					
Other incorrect responses	10				2		1		7

Again, the common error was to treat all the cakes as one whole, cut each cake into fifths but then treat the pieces as fifteenths. A surprising thirty students gave no numerical answer but attempted to divide the circles and were unsuccessful. Two of those students answered, “I don’t know”.

Interview Results: Follow-up to Sharing Task #2

Nine students were interviewed and asked, “If three candy bars are divided equally among five people, how much candy bar does each person get?” Students were generally successful with this task. Six students, (Faith, John, Elaine, Greg, Chad and Katie) answered correctly with 3/5. They all divided the candy bars into five equal parts and shaded a three parts for each person. Successful students gave the following explanations.

Katie: 1/5 +1/5+1/5 = 3/5.

Elaine: Each would get three pieces with would be 3/15 of all the pieces or 3/5 of one candy bar.

Greg: Less than a whole candy bar… 3/5 to be exact.

Ryan: Fifteen is a multiple of five and it divides by three.. (He then divided each bar into five and shaded three for one person.)

The remaining three students struggled with this task. Wendy used a guess and check type strategy. She first drew lines dividing each rectangle into three, then four, and finally five parts. Each time she divided the rectangles she counted out equal parts for each person. She used this process until it worked. She numbered the parts 1,2,3,4,5 and then started over again numbering parts 1,2,3,4,5 and continued until she ran out of pieces and they were all numbered. Then she answered, "Each person would get three pieces." After probing her for another way to say this and then asking for a fraction amount she answered 3/5.

Wade (WNR) cut each bar into five and shaded on piece of each bar. Then he explained, "1/5 of each bar so 3/15 of a candy bar."

Researcher: What fraction of a single candy bar each would get?

Wade: 1/5.

Researcher: What would you get if you put all three pieces for one person in one candy bar how much it would be?

Wade: 3/15

Fran (FR) divided the bar into five parts and then counted by twenties so that a whole candy bar equaled one-hundred. Then she answered that each person would get "a twentieth of a candy bar." When I asked her why she numbered by twenty she said that a whole candy bar had to be out of one-hundred. It was apparent that she was thinking in terms of percent of candy bar but confusing the fraction and percent. She answered 1/20 but her explanation indicates she meant 20%.

All students were able to answer. With probing, all students were able to give a rational number answer. For example, for Wendy (WNR), it was not immediately apparent how to make the cuts. This may have been the case for many of the student that didn't completely answer this question. The interview environment may have encouraged the students to be more persistent.

Written Test Results: Sharing Task #3

Another question similar to the two previous questions was asked of the one hundred one students. The question was: Five candy bars are divided equally among three children. How much candy bar does each child get? This question resulted in more than one candy bar per student rather than less than one in the previous questions. Forty-six of the one hundred one answered indicated some level of understanding. The answers from thirty-two students (32%) were considered "acceptable", meaning that it was correct and applied the quotient subconstruct. They are listed in table 4-34 below with their frequency.

Table 4-34 Responses that Demonstrate Knowledge of Quotient Subconstruct

Answer given	Frequency N=32/101	WNR N=0/7	ZR N=0/3	IZR N=1/2	FR N=0/4	FR/IZR N=1/1	MR N=0/5	UC N=2/10	AE N=28/69
5/3 or 1 2/3 or 1 4/6	29			1		1		2	25
1/3 of each bar	2								2
1 6.6/10	1								1

The answers given by sixteen students were correct, but it is unclear whether they have knowledge of the quotient subconstruct because there was no use of rational numbers. Thirteen students did not give a numerical answer but drew thirds and marked or labeled those parts correctly indicating how much each person would get. The responses given by these students are in table 4-35.

Table 4-35 Correct but Unclear Knowledge of Quotient Subconstruct.

Responses	Frequency N=16/101	WNR N=3/7	ZR N=0/3	IZR N=0/2	FR N=0/4	FR/IZR N=0/1	MR N=2/5	UC N=3/10	AE N=8/69
Accurate picture/ no answer	12	1					1	3	7
5 pieces	4	2					1		1

The answers from the remaining fifty-three students were not acceptable. The answers and the frequency are listed in table 4-36.

Table 4-36 Incorrect answers to sharing task

Answer given	Frequency N=53/101	WNR N=4/7	ZR N=3/3	IZR N=1/2	FR N=4/4	FR/IZR N=0/1	MR N=3/5	UC N=5/10	AE N=33/69
Inaccurate picture & no numerical answer	21	2	1		2		1	3	12
1 2/6 or 1 1/3	10		1		1			1	8
5/15 or 1/3	6								6
3/5	2				1				1
1 ¾ or 1.75	2								2
1.5 or 1 ½	3	1	1						1
Other incorrect responses	8	1		1			2	1	3

Eight students gave incorrect fraction or decimal answers not included in the table because each answer was given only once. Twenty-one students gave no numerical answer and attempted the drawing but did so incorrectly.

Interview Results: Follow-up to Sharing Task #3

Students were interviewed and asked, “If six candy bars are divided equally among four people. How much does each get?” Seven students correctly answered the question with one of several equivalent responses: 1 ½, 1 2/4, 3/2, and 6/4. The division techniques for this task varied more than the previous task. One strategy was to leave four whole bars and cut the remaining two in half, then shade one whole bar and ½ of another for each person which makes 1 ½ . A second strategy was to cut all six rectangles into four parts making twenty-four pieces and then sharing those with four people makes six pieces a person. If each piece is a fourth of a candy bar this makes 6/4. Another student explained it as "four would make one whole with 2/4 left" making 1 2/4. A third strategy was to cut each candy bar into two parts making twelve pieces, then sharing with four people makes three pieces per person. If each piece is half of a candy bar this makes 3/2.

Wendy (WNR) was able to answer correctly with the answer 6/4 using the same guess and check strategy she used earlier. The first tried dividing the rectangle into thirds, counted out the pieces and when that wouldn't work tried fourths, which worked. When she came up with the answer 6/4 she asked what the question meant. I reread the question but given the answer, she didn't understand what it meant. In previous interview questions it was clear that she struggled to understand rational numbers greater than one. Although she gave the correct answer, she could not explain it.

Two students answered incorrectly. Fran (FR) divided each bar into four parts. She then shaded one piece for each person in each candy bar. She recognized that four pieces would make one whole. The remaining amount she represented with the decimal 0.25 resulting in the answer 1.25. She could see ¼ of the remaining two candy bars for each person and used this for the decimal.

Katie (AE) divided each bar into four parts and shaded the parts of the bars differently for each person, counted the number of pieces total and the number for each person and came up with 6/24. After probing her she compared this to the previous question that she answered correctly and she was able to revise her answer to come up with 6/4 or 1 ½.

Summary of Sharing Tasks

Students struggle to accurately partition an object into equal parts. Approximately 20-30% of the students did not accurately partition the picture (percent depending on the task). Partitioning is a building block to the quotient subconstruct.

The use of a fraction less familiar or friendly to the students (in particular an odd denominator) led to fewer students that could (1) partition the objects correctly and (2) give an answer that indicates what each persons "part" would look like. Students with unclear knowledge of the quotient subconstruct were those with correct illustrations and an answer that indicated the right number of "parts" but the parts were written in a rational number form, for example, "three pieces" rather than 3/5. Fewer students used a rational number to represent the part when the fraction was less friendly.

When students were able to accurately partition a picture, they still struggled to use a rational number for the "part" each person would get. This would indicate that students see division as a whole number operation and not one involving fractions.

Written Test Results: Division Problem in Context #1

For the next quotient subconstruct task, students were given a division problem in context. The purpose is to compare how well student perform division resulting in a whole number versus division resulting in a fraction. Division resulting in a fraction is at the heart of quotient subconstruct of rational numbers. However, if students struggle with the quotient subconstruct is it because they struggle with the concept of division or because they struggle specifically when the quotient is less than one? One hundred one students were asked the following question: Five friends shared 15 pounds of trail mix. How much did each person get? The purpose of this task was to see if students would recognize this sharing problem as division and to see if they could answer it correctly. Ninety-five students (95%) gave a correct response. Those responses can be found in table 4-37.

Table 4-37 Five friends share Fifteen Pounds of Trail Mix.

Answers given	Frequency N=95/101	WNR N=6/7	ZR N=3/3	IZR N=2/2	FR N=4/4	FR/IZR N=0/1	MR N=4/5	UC N=10/10	AE N=66/69
3 pounds	94	6	3	2	4		3	10	66
3/15	1						1		

Only six students gave answers that were considered incorrect. Those answers and their frequency are in table 4-38.

Table 4-38 Incorrect Responses for Five Friends Sharing Task.

Answers given	Frequency N=6/101	WNR N=1/7	ZR N=0/3	IZR N=0/2	FR N=0/4	FR/IZR N=1/1	MR N=1/5	UC N=0/10	AE N=3/69
5	4	1				1	1		1
15/75 = 1/5	1								1
No answer	1								1

The results indicate that students were generally successful, with the exception of six students, in recognizing this as a division problem and correctly find the quotient by calculating the amount each would get. This indicates that generally students are comfortable with the conception of division when the quotient is a whole number.

Written Test Results: Division Problem in Context #2

Students were asked another division problem in context. This one resulted in a "fraction" of an amount which means applying a fraction as a quotient. One hundred one students were asked the following question: There was 10 feet of licorice rope. If 20 people shared the licorice rope, how much did each person get? The purpose of this task was to see if students still recognized division when the divisor was greater than the dividend resulting in an answer that is a fraction less than one. Sixty-three students (63%) gave a correct response. This is significantly less than the 95% that correctly answered the previous question. Correct answers and their frequency are found in table 4-39.

Table 4-39 Correct Responses to Licorice Division Task.

Answers given	Frequency N=63/101	WNR N=1/7	ZR N=0/3	IZR N=0/2	FR N=2/4	FR/IZR N=1/1	MR N=2/5	UC N=7/10	AE N=50/69
.5 or ½ foot	32				1		1	5	25
6 inches	25	1					1	1	22
2/4 or ½ (no label)	6				1	1		1	3

Thirty-eight answers were considered incorrect. Those answers and their frequency are given in table 4-40.

Table 4-40 Incorrect Responses to Licorice Division Task.

Incorrect Responses	Frequency N=38/101	WNR N=6/7	ZR N=3/3	IZR N=2/2	FR N=2/4	FR/IZR N=0/1	MR N=3/5	UC N=3/10	AE N=19/69
2 feet	18	4	1				2	3	8
2 inches	4		1	1					2
1/20	4								4
No answer	6	1	1				1		3
Other	6	1		1	2				2

Results indicate that students are not as successful at finding the quotient by calculating the amount each person would get when the resulting quotient is less than one. Fewer students answered this division problem correctly (n=63) with this task compared to the number that answered the division problem correctly (n=94) on the previous task. Students were more likely to change the order of the digits in the division problem resulting in a whole number answer of two. Students were also more likely to not answer this task (n=5) compared to the previous task (n=1) and indicates that they are less comfortable with a rational number response to a division problem or the quotient interpretation of rational number.

Written Test Results: Division Notation Task #1

One hundred one students were asked to complete the following task. To determine if students recognize the fraction notation as a way of writing a division problem, students were asked to write the division problem twelve divided by four as many different ways as they could. Because students were asked to write the notation as many was as they could, students often gave more than one answer and the total frequency in the table will be more than one hundred one. The frequency of responses is broken down by classification in table 4-41.

Table 4-41 Frequency of Correct Responses.

Number of Correct responses	Total N=101	WNR N=7	ZR N=3	IZR N=2	FR N=4	FR/IZR N=1	MR N=5	UC N=10	AE N=69
0	6	2					1	1	2
1	17		1	1	1		1	3	10
2	33	4	1		2	1	2	3	20
3	45	1	1	1	1		1	3	37

Accurate responses can be found in table 4-42.

Table 4-42 Notation for twelve divided by four.

Answer given	Frequency N=101	WNR N=7	ZR N=3	IZR N=2	FR N=4	FR/IZR N=1	MR N=5	UC N=10	AE N=69
$4\overline{)12}$	86	5	2	2	3	1	3	9	61
12 ÷ 4	76	4	3	1	3	1	3	5	56
12/4	54	1	1	1	2		1	4	44
12 x 1/4	2	1					1		

Of the answers that were given between 94% of the students were able to use one form or another. Long division notation was the most frequently occurring form of division notation while only 54% gave the “fraction” notation. This is not surprising given the fact that when they were asked in a previous task “what is a fraction” only eight students out of eighty-five students (9%) indicated that fractions are a quotient. However 54% is significantly, likely the attention drawn to division stimulated more quotient knowledge than simply asking students to come up with this on their own. Some students struggled with the division notation. The number of incorrect responses is given in table 4-43. Incorrect responses and the frequency are given in table 4-44.

Table 4-43 Frequency of Incorrect Responses

Number of Incorrect responses	Total N=101	WNR N=7	ZR N=3	IZR N=2	FR N=4	FR/IZR N=1	MR N=5	UC N=10	AE N=69
0		4	1	1	3		2	6	52
1		1	1	1	1	1	2	4	15
2		2					1		2
3			1						

Table 4-44 Incorrect responses to twelve divided by four.

Answer given	Frequency	WNR	ZR	IZR	FR	FR/IZR	MR	UC	AE	
4 ÷ 12	11	2	2		1		1		5	
$12\overline{)4}$	7	2	1				1	1	4	
12:4	5	1							4	
4/12	6		1			1	1	1	2	
Other	4	1					1		4	

Written Test Results: Division Notation Task #2

The previous question resulted in a whole number quotient and involved “a larger number divided by a smaller number”. The following question was asked to see how students comparatively would deal with a division problem with a fractional quotient and that involved “a smaller number divided by a larger one”. Students were asked to write the division problem 3 divided by 9 as many different ways as they can. Correct responses can be found in table 4-45.

Table 4-45 Correct responses to Three divided by Nine

Answer given	Frequency	WNR N=7	ZR N=3	IZR N=2	FR N=4	FR/IZR N=1	MR N=5	UC N=10	AE N=69
$3 \div 9$	72	4	3		3	1	1	6	54
$9\overline{)3}$	64	4	2	1	4	1	2	6	44
3/9 or 1/3	49	1	1		1	1	1	5	39
3 out of 9	2								2
Picture form 3/9 shaded	1							1	

Of the answers that were given 90% of the students were able to use one form or another. This is fewer than the previous division problem resulting in a whole number. This confirms the previous findings that students are less successful with division problems with a quotient less than one. However, 49% gave the “fraction” notation, which is close to the 51% in the previous question. In general, about half the students applied fraction notation to division. Incorrect responses can be found in table 4-46.

Table 4-46 Incorrect Responses to Three Divided by Nine.

Answer given	Frequency	WNR N=7	ZR N=3	IZR N=2	FR N=4	FR/IZR N=1	MR N=5	UC N=10	AE N=69
$9 \div 3$	16	1	2	1			1		11
$3\overline{)9}$	30	3	1	1	1		2	4	18
9/3	11		1	1			1		8
3:9	5	1							4
9:3	1			1					
Other	6	2						1	3

There was an increase in the number of students that wrote the traditional division problem “backwards.” Again, indicating that students struggle with division notation for quotients less than one. Answers were included in the table only if three or more students gave that answer.

The purpose of the division tasks was to explore the frequency at which students would give the fraction notation for a division problem, thus getting as the quotient interpretation. In the first division notation task (twelve divided by four) 54% of the students used a fraction notation correctly and 6% gave the fraction notation incorrectly and the remaining percent did not use fraction notation. In the second division notation task (three divided by nine), 49% used fraction notation correctly and 11% used fraction notation incorrectly and the remaining percent did not use fraction notation. Between the two tasks, it appears that approximately half the students were stimulated by the task to use fraction notation correctly and apply the quotient subconstruct.

Summary of Quotient Subconstruct

Across the quotient subconstruct tasks, (the sharing tasks, the division problems in context, and in the division notation tasks) results were analyzed for evidence that students perceived the fraction notation as a way of reporting the quotients in these tasks. When asked how much, the focus was on whether students made the shift from "how many" type responses (such as 3 pieces) to "how much" type responses which involve the fraction notation. Students' success on tasks varied greatly with the nature of the task. Responses from the three sharing tasks indicate that between 29% and 32% of the students on the written test responded with a rational number when asked "how much" would each person get when the item are "divided" equally. Students classified by "error rules" were less likely to report their quotient as a fraction compared to "apparent expert" students. The division problems in context and the division notation task were more than just division tasks. The first division problem in context illustrated that 95% of the students can interpret whole number division in context. This indicates that problems with partitioning with these tasks do not come from a lack of understanding about division itself. The second division problem in context illustrated that 63% of the students can correctly interpret a division problem with a rational number quotient. This indicated that division problems that do stem from a larger number divided by a smaller one are more problematic. However, when looking back at the partitioning tasks, fewer than 63% of the students performed successfully. Results indicate that if a quotient resulted in a whole number, such as a fifteen divided by three, students were more successful than if it resulted in a fraction, such as six divided by twelve, even when the quotient was a familiar fraction such as ½.   By looking at the frequency of the fraction notation in the fraction notation tasks, there is evidence that students apply fraction notation to quotients. Students did in fact use fraction notation for a division problem that resulted in a whole number (12/4) 54% of the time and used the fraction notation for a division problem resulting in a fraction (3/9) 49% of the time. Interviews suggest that with probing, students may be able use the fraction notation, but just didn't think of it on their own.

Ratio Subconstruct

A ratio is used to represent the numeric relationship between to objects and is often used as a comparative index. A ratio is a part-part relationship rather than a part whole relationship. For example, 4/5 as a ratio represents four white balls for every five black balls.

Written Test Results: Ratio of Boys and Girls Task

To see if students were able to connect the fraction notation to ratios, they were asked: "Here is a picture of a group of boys and girls. What does 3/5 represent in this situation?" Of the ninety-nine students that were asked this question, sixty-four students (64%) (41 AE, 4 WNR, 3 ZR, 2 IZR, 4 FR, 2 MR, 1 FR/IZR,7 UC) gave an accurate answer to this question, by answering something similar to “for every three boys there are five girls”. Thirty-seven students gave inaccurate answers. Students’ incorrect responses and their frequency are in table 4-47.

Table 4-47 Incorrect Responses to Boys and Girls Task

Incorrect Responses	Frequency N=35/99	WNR N=3/7	ZR N=0/3	IZR N=0/2	FR N=0/4	FR/IZR N=0/1	MR N=3/5	UC N=3/10	AE N=26/67
No answer or “nothing”	21	3					1	2	15
Boys/amount of boys	4								4
3 boys out of 5 girls	2						1		1
4.8	2								2
3 girls with hand-up	2								2
Other answers	4						1	1	2

The most common incorrect response was actually no response at all. If the part-whole interpretation of fractions was the dominant interpretation then using the fraction in part-part terms may not make sense to them and thus, 3/5 would represent “nothing” to them. Two students responded with three boys and five girls but used the words “out of” rather than “for every”.

Written Test Results: Ratio of Circles Task

Again, to see if students were able to connect the fraction notation to ratios, they were asked: “Here is a picture of some circles. What does ½ represent in this situation?” Out of the ninety-nine that were asked this question, thirty-four students (34%) gave an acceptable answer to this question (2 WNR, 1 ZR, 1IZR, 2 FR, 5 UC, 23 AE). A few examples are:

- “For every one white circle there are two dark circles.”
- “It is the ratio of white to black circles.”
- “Ratio for unshaded to shaded.”

• “The black circles happen twice the amount the white ones do.”

In a few cases students stated the ratio a little differently, such as “1 dark, ½ light” or they circled ½ of a white circle and a whole black circle.

This percent correct is less than the previous question. It seemed that their part-whole understanding of ½ interfered with their ability to express ½ as a ratio. Many of the students that gave no response answered the previous question. In fact, twenty-one students that gave no response or answered with “nothing” or “I don’t know” gave a correct response to the previous question. Sixty-one students gave unacceptable answers. Those answers and their frequency are in table 4-48.

Table 4-48 Incorrect response to Ratio of Circles Task

Incorrect Responses	Frequency N=65	WNR N=5/7	ZR N=2/3	IZR N=1/2	FR N=2/4	FR/IZR N=1/1	MR N=5/5	UC N=5/10	AE N=44/67
No answer, “nothing” or I don’t know	43	5	1	1	2	1	3	2	28
1 ½ circles	7						1		6
½ the balls aren’t shaded	2							1	1
Other responses	13		1				1	2	9

There were nine students that answered incorrectly and each gave a different answer and are listed as “other responses” in the table. Only those responses that occurred more than once are in the table. There were many more students that did not answer this task compared to the previous task (43 vs. 21). There were also more students that answered the question “What does ½ represent in this situation” with “nothing” or “I don’t know”. It is possible that those that gave no answer did so because they couldn’t find a relationship between the picture and the ratio.

Written Test Results: Write a Ratio for Socks Task

In the next question, students were asked to give a ratio rather than interpret one. The intent was also to see if how often students used the fraction notation for the ratio. Students were asked, “In her sock drawer, Jan has 10 pairs of socks. Seven pairs are white and the rest are black. What is the ratio of white socks to black socks?” Of the ninety-nine students that were asked this question, the answers given by eighty-four students (84%) were accurate.

Table 4-49 Ratio of White Socks to Black Socks

Correct Responses	Frequency N=86	WNR N=7	ZR N=2/3	IZR N=2/2	FR N=2/4	FR/IZR N=1/1	MR N=4/5	UC N=8/10	AE N=64
7:3	45	2	2	2	1	1	2	1	34
7 to 3	33				1		2	5	24
7/3	6	1						1	4
There are 7 white and 3 black	2							1	1
14/6 or 14 to 6	2								1

The answers given by fifteen students were incorrect. The responses are in table 4-50.

Table 4-50 Incorrect responses to ratio of White Socks to Black Socks.

Incorrect Responses	Frequency N=15	WNR N=3/7	ZR N=1/3	IZR N=0/2	FR N=2/4	FR/IZR N=0/1	MR N=1/5	UC N=2/10	AE N=67
No answer	6	2			1			1	2
4:7	2							1	
7/10 7 to 10	2	1					1		
3/10	1		1						
7:2	1								
3:7	1				1				
5	1								1

In general students were relatively successful writing a ratio with 84% of the students correctly writing the ratio. However, only 7% of the students used fraction notation when writing a ratio. However, an open-ended task like this may not have stimulated the students to think about writing the ratio as a part-part fraction.

Interview Results: Ratio of Golf Balls to Baseballs Task

Nine students were interviewed and asked a similar type ratio question. They were asked, “Bob has 8 balls. Five are golf balls and the rest are baseballs. What is the ratio of golf balls to baseballs? Write the ratio as many ways as you can.” The purpose of this task during the interview was to probe for all possible notations, fraction notation in particular. Generally, the students were successful with this task. For comparison with the item on the written test the results of these nine students are in table 4-51.

Table 4-51 Correct Responses Ratio of Balls Task

Correct Responses	Frequency N=9
5:3	6
5 to 3	6
5/3	6
5-3 (correct?)	2

The students using the notation 5/3 were all experts and one FR student (Faith). Wendy (WNR) and Fran (FR) did not include the fraction notation. Wendy (WNR) gave only two notations: 5 to 3 and 5:3. On her written test she gave one form (7:3). During the interview, Fran (FR) gave two notations as well with neither being correct: 3:5 and (3,5). She did not respond to this question on the written assessment. Wade (WNR) used fraction notation but incorrectly wrote it as 3/5. His other forms of a ratio were 3 ÷ 5 written two different ways. He had no correct notations during the interview and also did not respond to this item on the written test. The other six students had three correct notations each. The fraction notation occurred as often as the ratios 5:3 and 5 to 3, which is inconsistent with the written test. On the written test, these six students had only one notation each and none of them used fraction notation. The interview setting encouraged students to be more persistent and offer as many ideas as possible and they performed better.

Summary of Ratio Subconstruct

When asked to write a ratio for a situation, approximately 84% of the students were able to do so in one form or another, however, only 7% used the fraction notation for the ratio. When given the fraction notation and asked to interpret it in a given situation, 62% of the students were able to tell what the fraction 3/5 represented in a ratio context, while only 38% were able to tell what the fraction ½ represented. Perhaps this is because the fraction ½ is so familiar to students in a part-whole context. It appears that students are more successful with the fraction notation in a ratio context compared to a quotient context

Operator Subconstruct

The operator interpretation involves a rational number that behaves as a multiplicative size transformation. The rational number behaves as an operation that reduces or enlarges an object. Tasks in this subconstruct involve enlarging and shrinking.

Written Test Results: Dimensions of a Rectangle Task

The first operator item students were asked was, “The dimensions of rectangle A are ¼ of dimensions of rectangle B. What is the length of rectangle A?” (Rectangle B has a width of 12 and length of 20. Rectangle A has a width of 3.) This would be considered a “stretcher-shrinker” type problem. The purpose of this question was to measure student success with the concept of the fraction as an operation that reduces the dimensions of a rectangle. Seventy-eight students out of the ninety-nine (78%) students answered correctly with the response of 5. Twenty-three students answered incorrectly. Those answers and their frequency are in table 4-52.

Table 4-52 Incorrect Responses to Dimensions of Rectangle Task.

Incorrect Responses	Frequency N=21	WNR N=5/7	ZR N=3/3	IZR N=0/2	FR N=1/4	FR/IZR N=1/1	MR N=0/5	UC N=2/10	AE N=67
No answer	9	3	1			1		1	3
.75	2		1					1	
20	2		1						1
10	2	1			1				
Other	6	1							5

Students classified as WNR and ZR struggled with this task with only 20% of the students in these two groups answering correctly while approximately 87% of the AE responded correctly. In particular, of the nine students that did not respond to the question, 3 of those were WNR students and one was a ZR student accounting for a large part of the non-responders.

Written Test Results: Pencil Task

To see how well student could interpret an operator task in a different context, students were asked to answer the following question: “Susan bought ten pencils to school today. One-fifth of the pencils were yellow. How many of the pencils were yellow?” Seventy-eight students out of the ninety-nine (78%) that were asked this question answered this question correctly with the answer of two. This number of students was consistent to the number of students in the previous question. Twenty students answered incorrectly and their answers are in table 4-53.

Table 4-53 Incorrect Responses to Pencil Task

Incorrect Responses	Frequency N=21	WNR N=7	ZR N=2/3	IZR N=1/2	FR N=1/4	FR/IZR N=0/1	MR N=1/5	UC N=2/10	AE N=67
No answer	5	1	1	1					2
5	4	1						1	2
1/5	2							1	1
8	2		1						1
Other	8	2			1		1		4

The answer of five is interesting because subtracting ten and five (1/5) results in five. From the comparing task in part one, a strategy of whole number dominant students was to find the difference between the numerator and denominator of the fraction to see which fraction is larger. It is possible that this same strategy was being implemented here.

Written Test Results: Cupcake Task

Another similar question was asked: "Tom made 12 cupcakes. That afternoon, his friend ate one-third of the cupcakes. How many cupcakes did he eat that afternoon?" Eighty-seven students out of the ninety-nine that were asked this question (87%) answered correctly with the answer of 4. This is more than the previous two questions. The fraction 1/3 may have been easier to work with in this context than 1/5. Twelve students gave inaccurate answers, which can be found in table 4-54.

Table 4-54 Incorrect Responses to Cupcake Operator Task

Incorrect Responses	Frequency N=12	WNR N=4/7	ZR N=2/3	IZR N=0/2	FR N=1/4	FR/IZR N=0/1	MR N=0/5	UC N=1/10	AE N=67
No answer	5	2	1						2
3	4	1	1						2
9	2	1						1	
40%	1				1				

Approximately the same number of students gave no answer compared to the pencil task. The answer of nine is interesting because subtracting 12 and 3 (1/3) gives nine. As stated earlier, a common strategy of whole number dominant student is to find the difference between the numerator and denominator to see which is larger. This is seen again here, but by fewer students.

Written Test Results: Tomato Plant Task

This question was similar to the previous ones only it did not use a unit fraction, making it potentially more challenging. "Joe planted 24 pots of tomato plants. He gave three-eighths of the pots to his mother. How many pots of tomato plants did his mother get?" Sixty-six of the ninety-nine students that were asked this question answered correctly (66%). This was less than the previous tasks as expected. Thirty-three did not. Their answers are in table 4-55.

Table 4-55 Incorrect Responses to Tomato Plant Task

Incorrect Responses	Frequency N=33	WNR N=4/7	ZR N=3/3	IZR N=2/2	FR N=1/4	FR/IZR N=1/1	MR N=1/5	UC N=10	AE N=67
No answer	10	1	3	1		1	1		3
8	6	2						2	2
3	3								3
½	3								3
24	1								1
3/8	3	1						1	2
Other	7			1	1			2	3

There were seven students that answered incorrectly and gave a different answer and were recorded as "other" in the table. Responses that occurred more than once are in the table. Compared to the number of students that answered correctly in the previous question (n=87), this was significantly less. It was also the fewest number of students answering correctly of all the questions of this type.

Written Test Results: Operator Task out of Context #1

The next two questions involved a unit fraction and an operator type question without a context. The purpose of this task was to see if the context influenced the success of students on operator tasks. Students were asked: "What is 1/6 of 18?" Eighty-four students answered correctly (84%) Fifteen students responded incorrectly. The responses are in table 4-56.

Table 4-56 Incorrect responses to 1/6 of 18

Incorrect Responses	Frequency N=15	WNR N=2/7	ZR N=2/3	IZR N=0/2	FR N=3/4	FR/IZR N=1/1	MR N=3/5	UC N=3/10	AE N=67
No answer	7	1	2			1			3
6	3							2	1
Other	5	1			1		2	1	

There were five students that answered incorrectly and each gave a different answer and are counted as "other" in the table. Other answers included: 1.66%, .16, 2.88, 14, and 50%. This was closely related to the question involving 1/3 of 12 cupcakes and had the same frequency of students answering correctly (approximately 85%).

Written Test Results: Operator Task Out of Context #2

The purpose of the next tasks was to see how successful students were at performing operator tasks out of context using a fraction other than a unit fraction. Students were asked, “What is 2/3 of 15?” Seventy-nine students (79%) answered correctly with the answer of ten. Twenty answered incorrectly and their responses are in table 4-57.

Table 4-57 Incorrect Answers to 2/3 of 15

Incorrect Responses	Frequency N=20	WNR N=4/7	ZR N=3/3	IZR N=1/2	FR N=1/4	FR/IZR N=1/1	MR N=2/5	UC N=2/10	AE N=69
No Answer	10	2	3			1			4
5	3			1				1	1
Other	7	2			1		2	1	1

There were nine students that answered incorrectly and each gave a different answer and are counted as “other” in the table. Other answers include: 13, 12, 30, 9.9, 0.66, 0.66%, and 50%. This was designed to be similar to the 3/8 of the 24 potted plants because it is not a unit fraction, however, students performed better on this task compared to the potted plants task (65%). This is also similar to the pencil task in that the pencil task involved the fraction 1/3 and this task involved the fraction 2/3. While 79% of the students answered correctly with the pencil task, 80% answered correctly in this task. This may indicate that the type of fraction influences success on the task more than the task being in context or not.

Interview Results: Follow-up to Operator Task out of Context

Nine students were interviewed. To follow up with operator tasks out of context, students were asked, “What is 2/3 of 18?” Seven students correctly answered with twelve. All of them divided 18 by 3 (because of thirds) and got 6 and them multiplied 6 by 2 (because of numerator) and got 12. All of these students also correctly answered similar items on the written test.

Two students were incorrect. Wendy (WNR) subtracted two from eighteen and got sixteen for her answer. She skipped all the operator tasks on written test. The other was Fran (FR) divided 3 by 2 and got 1.1 and then divided 18 by 1.1 and got 0.177. She also missed these items on the written assessment. When asked, "What is 1/6 of 18?" and "What is 2/3 of 15?" she gave the same answer 50% for both. She missed all the operator tasks.

Summary of Operator Subconstruct

Student success on operator tasks varied depending on the fraction used. There were six operator tasks. The frequency of students correctly responding to those tasks is in table 4-58.

Table 4-58 Number of Correct Responses to Operator Tasks

Number of Correct responses N = 6	Total N=99	WNR N=7	ZR N=3	IZR N=2	FR N=4	FR/IZR N=1	MR N=5	UC N=10	AE N=67
6	50	1			3		2	3	41
5	18	1		1			1	1	14
4	12	1					2	3	6
3	7			1				3	3
2	2	1							1
1	6	1	3			1			1
0	4	2			1				1

Whether or not the task was designed with a context did not appear to affect the success of the students on operator tasks. Students had more success with unit fractions such as ¼, 1/5, 1/3 and 1/6 (78%, 78%, 87%, 84% correct respectively). They had less success with non-unit fractions such as 3/8 and 2/3 (66% and 79% respectively). Approximately 79% answered correctly on the non-unit fraction 2/3 which is similar to results of unit fractions ¼ and 1/5 (both 78%). However, it is less that the success of the unit fraction 1/3 where students were correct

87% of the time. In general, students answered operator tasks with 78% or better results. The students that struggled the most were also classified as WNR and ZR.

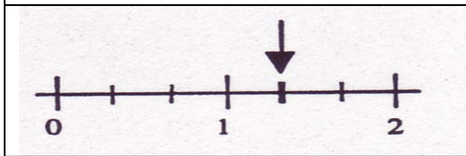
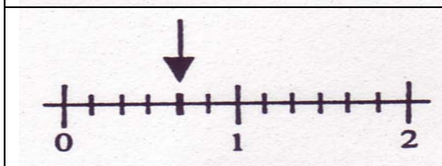
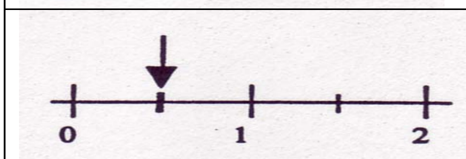
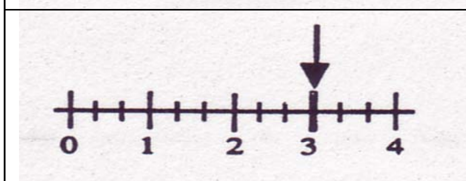
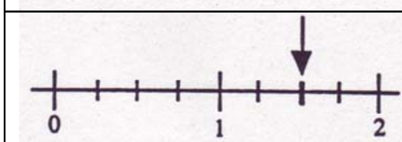
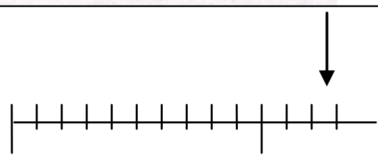
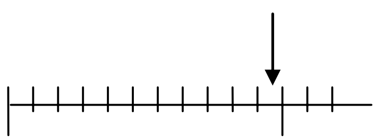
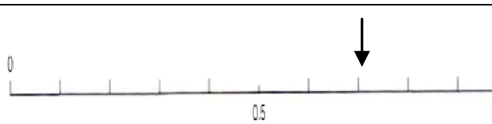
Measure Subconstruct

The measure subconstruct is frequently accompanied by a number line and involves identifying a fixed unit of measure.

Written Test Results: Number Line Tasks

Students were asked to label the length of the number line at the point marked with the arrow. The purpose of this task was to see if students could identify a rational number on a number line. A fraction, decimal or percent would be considered acceptable and students were not told to use one form of number. The eight number line tasks, the percent of students answering correctly and the incorrect responses given are in table 4-59.

Table 4-59 Number line Tasks and Responses

Number line tasks	Percent Correct	Incorrect responses (frequency)
	58%	No answer (10), 1.25 or 1 ¼ (11), 1.50 or 1 1/2 (4), 1.1 (3), 1.4(2), 1.35, 1.3, 0.60, ½, 1 2/3, 2/4
	52%	No answer (14), 4/5 (4), 0.4 (4), 0.75 (3), 0.80(2), 0.60(2), 0.2(2), ¼, 5/7, 0.18, 1/8, 0/3, 4, 1/5, 0.65, 2, -3
	88%	No answer (6), 0.25 (2), 50, .1
	93%	No answer (4), 8/11, 0.30
	89%	No answer (4), 7/9, 0.70, 12/4, 0.5,1.2
	54%	No answer (10), 1 2/6 or 1 1/3 (14), 1 2/5 (2), 1 2/9(2), 1.25 (2), 1 1/6 (2), 1.3 (2), 1/6, 2/10, 1/5, 1.10, 1 ½
 0 1.1	80%	No answer (11), 0.9 (3), 0.99, -1, 10/11
	78%	No answer (11), 0.75 (3), ¾, 1.5, 1, 0.07,0.10

It was more common for students to give an answer as a fraction. For example, in the first number line task 45/49 wrote as fraction and 4/49 wrote as decimal. Students were the most successful identifying whole numbers on the numbers line, however it is interesting to note that even though the answer was labeled on number line #4, there were still four students that answered incorrectly. Four students didn't answer the number line task at all, one student answered 8/11 which perhaps indicates that the student considered the whole number line as "1" and found the fractional amount for the part indicated (which would actually be 9/12). One other student gave an answer of 0.3, again maybe thinking of the number line as a whole and finding the "part" that was marked.

Students had similar success with number lines marked by some form of 1/2. For example, line #3 students labeled with 88% accuracy. Incorrect answers included 0.25, 50, and .1. A student looking at the whole number line as "1" would label the mark 0.25 which two students did. Students had similar success with line #5 and labeled it with 89% accuracy. Although there were more marks on this number line and the arrow was moved between one and two rather than zero and one students still performed the same. There was more variation in the incorrect responses to this number line than number line #3. If a student counted each mark (including zero), there are nine marks on this number line and the arrow is pointing to the seventh mark resulting in the incorrect response 7/9. One student did answer 0.5 rather than 1.5.

Students were also able to label line #7 with 80% accuracy, which is less than the other number line labeled with a whole number but there were also less clues. In addition, students had to use the number 1.1 for help rather than whole number labels. Eleven students did not label this number line which is more than most of the other tasks. Eleven students also did not label line #8 and 78% were able to label it correctly. Students needed to recognize that the number line was divided into tenths using only 0.5 and not a whole number. A common incorrect answer was ¾ or 0.75. Students may be more familiar with the fractions divided into "fourths" rather than "tenths" and estimated that 0.75 would be after five-tenths and before one.

Students struggled to accurately label several of these lines. Only 57% of the students correctly labeled line #1. The majority of these students were not able to identify that the number line was divided into thirds. Instead they used fourths, halves, tenths, and even hundredths. The most frequent incorrect response was 1.25 or 1 ¼, with eleven students labeling the line with one of those responses. If students count the marks starting with one and ending with two there are four marks and the arrow would be pointing to one of them.

Students also struggled with line #6. This number line showed the numbers zero and one and was marked with tenths, however it was marked beyond the one but the number line did not show the number two. If students used the marks between zero and one to determine the size of the "cuts" then they could determine the size of the marks after one as well. A common incorrect response was 1 2/6 or 1 1/3. If the student counted the number of marks after one there were six and the arrow was pointing to the second mark.

Finally students performed lowest on line #2. This number line was marked with sixths. There was more variety of incorrect responses that corresponded to this number line than any other. The frequency of incorrect response can be found in table 4-60. There were also more people that didn't answer it compared to any other task (fourteen). It seemed that students just didn't know what to do with sixths. It is a fraction less familiar in a measure context. Halves, fourths, and eights are measures used in the standard measurement system, and tenths and hundreds are used in metrics.

Table 4-60 Number of Correct Responses to Number Line Tasks

Number of Correct responses N = 8	Total N=86	WNR N=7	ZR N=3	IZR N=2	FR N=4	FR/IZR N=1	MR N=5	UC N=10	AE N=54
8	20				1			1	18
7	21				1		1	5	14
6	14	1					2		11
5	12	1	1	1	2	1	1		5
4	8	1	1					2	4
3	3	1		1					1
2	4	1					1	1	1
1	0								
0	4	2	1					1	

WNR and ZR students answered correctly with less frequency than the other classifications. There were four students that responded incorrectly to all of the tasks and two were WNR students, one was a FR student and one was unclassified. AE students performed better than the students classified by some “error rule.” Approximately 80% of the apparent experts missed two or fewer number line tasks. Sixty percent of the UC and MR students missed two or fewer. Fifty percent of the FR students and none of the WNR, ZR, IZR students missed two or fewer items.

Interview Results: Follow-up to Number Line Tasks

Nine students were interviewed. There was a number line on each of three index cards. Students were shown the number lines one at a time and asked to give the number for the point marked by the arrow. The first card had an arrow pointing to 1 1/3. Six students were able to correctly identify this point and labeled it as a mixed number. Three labeled it incorrectly. Wendy (WNR) labeled it 1/16, Fran (FR) labeled is 1.25, and Katie (AE) labeled it 1.25 or 1 1/3. Because Katie (AE) seemed to think the two numbers represented the same point on the line she was considered “incorrect.”

The next card had an arrow pointing to 4/6 so the answers 4/6 or 2/3 were considered correct. Five students were correct. Four were incorrect. Wendy gave the answer 4/16, Fran and Faith both gave the answer 0.4, and Katie gave the answer 0.65 or 4/6.

The third card had an arrow pointing to 1 4/5. Eight students labeled this point correctly. All except one gave the mixed number 1 4/5. Fran was the only one who didn't and she answered 0.8. Again, Wendy answered 4/16.

Written Test Results: Placing Points on a Number Line

Eighty-five students were asked to mark the following points on the number line to represent the following numbers: 0.5, 1.05, 4, 2.6, 3.25, 1.9. The results of the markings are in table 4-61.

Table 4-61 Correct Marking of Points on a Number Line

Number to mark On number line	Number of students With correct marking	WNR N=7	ZR N=3	FR N=4	FR/IZR N=1	IZR N=2	MR N=5	UC N=10	AE N=54
0.5	82 (95%)	6	1	3	1	2	5	10	54
4	80 (93%)	5	1	4	1	2	5	10	52
2.6	78 (93%)	5	1	4	1	2	3	9	53
3.25	76 (91%)	4	1	3	1	0	5	9	53
1.9	66 (80%)	3	1	3	1	0	3	6	49
1.05	57 (67%)	1	0	3	0	0	2	5	46

The results indicate that 1.05 was the most difficult. Student often placed this where 1.5 would be on the number line. Of the eighteen students classified WNR, ZR, FR/IZR, IZR, MR only three of them (16%) correctly place this number on the number line (two of them were MR students). Only 50% of the unclassified students were successful with this task. FR students had more success than the other classifications with three out of four students labeling the number accurately on the number line. Apparent experts accurately put 1.05 on a number line 85% of the time. It appears that the ranking from least difficult to more difficult to place as indicated in the above table is fairly consistent across classifications.

Interview Results: Which Number is Closest Task

To better assess how well students conceptualize how close numbers are to each other, they were asked the following question during the interview, "The number 0.28 is closest to which number? 0.2, 0.3, 27, 0.25 or 3" Seven of the nine students correctly answered 0.3. The common strategy, used by four students was annexing zeroes so that all the numbers had digits in the hundredths place. This strategy was used by Fran and Faith, both FR students, and used by Katie, a struggling AE and John an AE. The other three AE students, Elaine, Chad and Greg, all crossed out 27 and 3 and then calculated how many "hundredths away" each of the numbers were from 0.28. These three students were successful on almost all the tasks on the written test and interview. Their strategy appears to put more thought it how the numbers are related to each other while annexing zeroes "equalizes" them but makes them more like whole numbers.
Wade (WNR) selected 0.25.

Researcher: Explain your thinking here.

Wade: Twenty-seven is a long way away because it is a whole number, and a big one. Then three is too high too because 0.25 has a zero and isn't a whole number. Then 0.2 and0.3 are way lower than 0.28 because two and three are way smaller than twenty-eight so I picked twenty-five. It is apparent that when comparing the decimal numbers Wade is ignoring the decimal point and comparing the numbers as if they are whole numbers.

Wendy (WNR) was the other student that was incorrect and selected 0.25.

Researcher: Explain your thinking here.

Wendy: Three is pretty far away and twenty-seven is the farthest away. I picked this one (0.25) because it is only a couple away.

Researcher: What about the other two?

Wendy: I don't know. They just look too small.

The interview lends some insight into why WNR and ZR students would struggle so much. They continue to use the length of numbers to order numbers on a number line.

Written Test Results: Density of Fraction Task

Density is important to understanding the measure subconstruct. To further assess students' knowledge of density of fractions additional tasks were assigned. Students were asked if there are fractions between 3/5 and 4/5 and if so, how many? Of the eighty-five students that were asked this question, twenty-seven (31.4%) said no and eight (9.3%) gave no answer. Fifty-one students correctly answered yes (59.3%), however only fourteen students (16%) indicated that there were infinitely many fractions between the two they were given. Table 4-62 shows the responses to the question "how many" and gives the frequency of the response.

Table 4-62 Responses to Density of Fraction Task

Response to "how many"	**Frequency N=86**	**Percent of students**	WNR N=7	ZR N=3	FR N=4	FR/IZR N=1	IZR N=2	MR N=5	UC N=10	AE N=54
Endless or infinitely many	14	16%						1	1	12
1	13	15.1%	3		1		1	1		7
10	5	5.8%	1			1		1		2
2	5	5.8%		1				1	2	1
"Not sure" , ?, or no response to "how many"	6	7%			1					5
9	3	3.5%	1							2
1/5	2	2.3%								2
Other	3	3.5%							1	2
Responded "No or None"	27	31.4%	2	2	1		1	1	4	16
No answer	8	9.3%			1				2	5

Only fourteen of the eight-six students were able to communicate that there are infinitely many fractions between any given fractions. One student wrote, "Yes, there are some if you don't use the same denominator. If you don't there is an unlimited number." Some students may not have considered the many other equivalent forms of 3/5 and 4/5. Another student wrote that there was one fraction between the two and stated that it would be 7/10. It appears this student was able to use an equivalent representation for 3/5 and 4/5 converting them to 6/10 and 8/10, however it appears this student's use of equivalent representation is limited to tenths. The results indicate that students lack an understanding of the density of fractional numbers, which would impact their ability to understand the "measure" subconstruct of rational numbers.

Interview Results: Follow-up to Density of Fractions Task

To probe deeper into students' knowledge of density of fractions, nine students were interviewed and asked, "Are there fractions between 4/7 and 5/7? And if so, how many?" Seven of the nine students answered "Yes". Five of these students were all classified as "experts" in the decimal comparison tasks. Three of these students indicated that there were infinitely many numbers in between these two fractions. Two of the other four students said that there were nine other numbers, for example 4.5/7, 4.1/7, etc.. The other two students said that there was one fraction between the two at the half-way mark. One of these two drew a picture of sevenths and tried to cut it into more parts and then answered, "4/7 and a half."

Two students answered "no", there are no fractions between 4/7 and 5/7. One was a WNR student that answered, "I don't get how there could be". The other was a FR student that said, "There's no whole number between 4 and 5." When asked about this it became clear that she was focusing on the numerator and did not find it appropriate to use 4.1 or 4.5 for the numerator to list more fractions.

Written Test Results: Density of Decimals Task #1

Students were asked if there are fractions between 0.3 and 0.4 and if so, how many? Of the eighty-six students that were asked this question, 12 (14%) said no and 4 gave no answer. Sixty-four students correctly answered yes (74%), however only eighteen (20.9%) students indicated there would be infinitely many numbers between the two decimal numbers given. This is more than the fifty-three (62%) that answered yes and fourteen (16%) that answered infinitely many to the previous fraction question. Table 4-63 shows the responses to the question "how many" and gives the frequency of the response.

Table 4-63 Responses to Density of Decimals Task

Response to "how many"	**Frequency N=86**	**Percent of students**	WNR N=7	ZR N=3	FR N=4	FR/IZR N=1	IZR N=2	MR N=5	UC N=10	AE N=54
Endless or infinitely many	18	20.9%						1	1	16
1	13	15.1%	1	1	2		1	1	3	4
10	10	11.6%						2	2	6
9	10	11.6%			1					9
Yes, but no indication of how many	3	3.5%								3
2	2	2.3%	1							1
99	2	2.3%								2
Other	6	7%	1							5
"No, None, or 0"	12	14%	4						2	6
No answer	10	11.6%		2	1	1	1	1	2	2

Only nineteen of the eighty-five students were able to communicate that there are infinitely many decimal numbers between any given decimal numbers. This is slightly better than the 17% in the previous fraction question. Results indicate that students lack an understanding of the density of decimal numbers which would impact their ability to understand the "measure" subconstruct of rational numbers. However, students grasp this concept with decimals better than they do with fractions. One student (UC) that answered "1" to the question how many gave the number 0.35 (that same student said there were no numbers between the decimal numbers given in the following question). Another student (MR) answered "1 away" to all three of the density questions indicating that she was finding the difference between the numbers.

Interview Results: Follow-up to Density of Decimals Task #1

To probe deeper into students' knowledge of density of decimals, nine students were interviewed and asked, "Are there fractions between 0.6 and 0.7? And if so, how many?" Seven students answered "yes." To answer the question "how many" four indicated infinitely many with comments such as "too many to count". Three students did not indicate an infinite amount. One students answered one to the question "how many". When asked more about this, it became clear that he was finding the difference between the numbers. When asked to name a number between the two he said that he couldn't. Then after thinking about it he appeared to comprehend the task and started naming different number, and then said that there was "a bunch of different numbers". Another student answered, "Nine, they are 0.60, 0.61… until you get to 0.70." The third student answered, "Eight, between 0.61 through 0.69" (although this is nine numbers).

Two students answered "No, I don't know how there could be" to the question. They were both WNR students.

Written Test Results: Density of Decimals Task #2

Students were asked if there are fractions between 0.74 and 0.75 and if so, how many? Of the eighty-five students that were asked this question, 27 (32%) said no and 4 gave no answer. Fifty-three students answered yes (61.6%) which is fewer the 64 students (74%) that answered yes to the previous decimal question. However, nineteen students indicated that there were infinitely many decimal numbers between any two numbers. This is one more student than the previous decimal question. Table 4-64 shows the responses to the question "how many" and gives the frequency of the response.

Table 4-64 Responses to Density of Decimal Task #2

Response to "how many"	**Frequency N=86**	WNR N=7	ZR N=3	FR N=4	FR/IZR N=1	IZR N=2	MR N=5	UC N=10	AE N=54
Endless or infinitely many	19						1	1	17
1	14	1	1	1		1	1	2	7
10	7						2		5
9	4			1					3
Yes, but no indication of how many	3								3
0.010	2								2
Other	4	1							3
"No, None, or 0"	28	5	2	1	1	1	1	5	12
No answer	5			1				2	2

Results indicate that students recognize that there are decimals numbers in between two tenths but that they don't recognize this with hundredths. Often instruction is limited to hundredths and students don't get much experience with numbers smaller than that. One student that responded "1" to "how many" also gave the decimal 0.5 indicating there is a decimal number half-way between 0.74 and 0.75. Another student that responded "1" gave the decimal 0.745 as the number. Of the five WNR students that answered "no" to the decimal question four said "yes" to the fraction question. The reverse is true for other classifications. One of the two IZR students two of the three ZR students, and the IZR/FR student was a consistent "no" across all three questions. Three of the five MR students that said "no" to this question answered "Yes, 1" to the previous decimal question.

Interview Results: Follow-up to Density of Decimals Task #2

To probe deeper into students' knowledge of density of decimals by exploring decimal numbers extended from tenths in the previous task to hundredth in this task. Nine students were interviewed and asked, "Are there decimals between 0.46 and 0.47? And if so, how many?" Six students answered “yes.” Of these, five answered indicating there were infinitely many numbers. One answered, “Nine, they are 0.460 through 0.470.”

Three students answered “no.” Two of them were classified as whole number dominant thinking and when asked to explain said, “I don’t get how there could be.” The third student, Fran, answered no even though they answered yes to the previous decimal task. She was the same student that said the columns past the hundredths were small and “didn’t matter.” It appears that her knowledge of decimals ends with the hundredths place.

Summary of Measure Subconstruct

Success with labeling a point on a number line depended on the type of part used. Students performed better with whole numbers and parts in halves, fourths, and tenths. Students struggled working with thirds and sixths. There were a small number of students that treated the number line as one-whole and labeled the marks accordingly. Students also struggled when the “clues” given on a number line were numbers other than whole numbers. For example, a number line marked by tenths, labeled with 0.5 with several equal increments after the 0.5 but one is not on the number line. When given a list of decimals to place on a number line, again, students had greater success with whole number and familiar parts such as halves and fourth but struggled more with tenths and hundredths. Many students misplaced 1.05 and put it where 1.5 would be.

When asked if there were numbers in between two fractions students answered yes 62% of the time and 18% knew that an infinite number of fractions existed between any two fractions. When asked if there were numbers in between 0.3 and 0.4 students answered yes 74% of the time and 22% knew that there were infinitely many numbers in between two decimals. When asked if there were numbers in between 0.74 and 0.75 students answered yes 64% of the time and 20% knew there were infinitely many numbers in between these two decimals. Based on responses to other tasks and based on interviews, it appears that students' knowledge of decimals drops off after the hundredths place, which may explain why fewer students would say there are numbers in between 0.74 and 0.75. When working with the tenths, students seemed more comfortable with 0.3 being equivalent to 0.30 using the annex zeroes strategy. Doing the same thing for 0.4 writing it as 0.40 makes some of the numbers in between easier to find. Students seemed to have less success with fractions because they didn't have a strategy for rewriting the fractions and finding fractions in between. Those that were successful used fractions such as 3.5/5.

Summary of Description of Student Knowledge of Rational Number

Question 1

What type of comparing and ordering strategies do students use with fractions and decimal comparison tasks?

There were five error patterns identified in this research: Whole Number Rule (WNR), Zero Rule (ZR), Ignore Zero Rule (IZR), Fraction Rule (FR), and Money Rule (MR). Of these, three; WNR, ZR, and FR, are consistent with prior research. Two error patterns, IZR and MR, emerged in this research and were not identified in previous research. Results indicate that students experience a range of success with comparison tasks. Students were most successful with decimal comparison tasks with 68% of the students correctly answered all of the tasks and 7% of the students only missed one task. Students were less successful with comparing fraction tasks with 39% of the students completing all the tasks correctly. Students were the least successful when comparing a decimal to a fraction with 21% of the students correctly answering all the tasks.

Success on decimal comparison tasks was a fairly good predictor of success on fraction comparison and fraction-decimal comparison tasks. The average percent correct for each classification can be found in table 4-65.

Table 4-65 Average Percent Correct per Classification

	DMCT	Decimal Comparison	Fraction Comparison	Decimal-Fraction Comparison
FR/IZR	47%	21.4%	33%	44%
WNR	55.3%	43%	59.5%	39.7%
ZR	58.3%	78.5%	63%	39%
IZR	67%	82%	83%	55%
MR	69.5%	78.6%	87%	65.5%
FR	71.8%	73%	90%	59.7%
UC	77%	91%	90.5%	76.6%
AE	81.7%	100%	94.3%	85%

ZR students performed similar to the WNR students on the DMCT and across comparison tasks, except they were significantly more successful on decimal comparison tasks than WNR students. MR and FR students had a significantly higher average percent correct than WNR students across all comparison tasks. They also performed similar to each other. The apparent experts had the highest average percent correct on all comparison tasks and on the DMCT.

Question 2

How successful are students at solving rational number tasks for each of the five subconstructs?

There was a similar pattern of success on the subconstruct tasks when broken down by classification. Table 4-66 shows the average percent correct on eight measure tasks (number line tasks) and the six operator tasks by classification.

Table 4-66 Average Percent Correct on Measure and Operator Tasks per Classification

	Measure tasks	Operator tasks
WNR	35.7	43
ZR	36	16.6
IZR	50	66.7
MR	65	83
FR	78	75
UC	71	73
AE	82	87.5

WNR students had the lowest average percent correct, with the exception of the three zero rule students that struggled with the operator tasks. When ranking the classifications by success on measure and operator tasks, the results are very similar to those on comparison tasks. The average percent correct on measure tasks is similar to the percent correct on fraction-decimal comparison tasks, except fraction rule students were more successful on measure tasks than any other classification other than the apparent experts. Apparent Experts had the highest average percent correct.

Some of the tasks used in this research attempted to get at the meaning of the fraction notation in various subconstructs. Students were very comfortable with fraction notation used in the part-whole tasks. In quotient situations, students were less likely to use fraction notation to describe “how much” with fraction notation used between 29-32% of the time on three sharing tasks and fraction notation used between 49-54% of the time as a means of representing a division problem. Students were even less likely to use fraction notation in part-part ratio situations. When asked to write a ratio as many ways as they can only 7% of the students used fraction notation. However, 62% of the students could interpret a ratio in a part-part manner when given a context and a fraction and asked to explain what it means. The exception appears to be when interpreting the fraction ½. Only 38% of the students could explain that one dot was shaded for every two that is not. Earlier it was stated that familiar fractions resulted in more success, however this is a case where it did not because the part-whole interpretation of fractions appears to dominate.

In addition, the fact that the task was presented in context or not in a context or the type of model used didn't appear to be a factor. For the most part students performed the same on part-whole tasks whether the figure was an area model, discrete set of objects or a simple number line. However there was a slight difference between a set of six circles where three were shaded (98% correct) and a number line marked zero to one with up to one-half shaded (93% correct). Both resulted in the "part shaded" as one-half. While the number line was "shaded" and intended to be a part-whole task, it still was a task involving "length" and therefore a "measure" task as well. The number line task was an attempt to "relate" part-whole and measure subconstructs. Further research into the measure subconstruct indicates that students struggle to apply the concept of density to fractions and decimals making the measure concept harder to grasp. Again, on measure tasks students performed better on labeling number line tasks that involved numbers with which they were more familiar such as 0.5 (95%) and whole numbers (93%) and were less successful with slightly more complex decimal numbers such as 1.9 (80%) and 1.05 (67%) and fractions that are less familiar such as sixths (52%).

Students were in many ways successful with operator tasks. From interviews it appears that part-whole interpretation and division play a part in the success with operator tasks. For example, when finding 1/6 of 18 (84% correct) students would describe dividing eighteen by six. The part-whole interpretation of 1/6 involves six parts in the whole. A student that knows this and knows that division involves equal groups would then divide eighteen into six equal groups (one group for each part) and take one of them. This is connected to the quotient subconstruct because 1/6 means to divide by six. Students have less success with a non-unit fraction such as

3/8 of 24. A common incorrect response was eight, which is the result of dividing twenty-four by three. The second common incorrect response was three, which is the result of dividing twenty for by eight but not multiplying by three (taking three groups of eight). Another common response was 3/8, which is the result of twenty-four divided by eight (three) multiplied by three (nine) rewritten as a fraction 9/24 and simplified to get 3/8. Despite the understanding of the division inherent in operator task problems and the fact that students were generally successful on operator tasks (in general 78% correct on these tasks with the exception of the above task 3/8 of 24), students didn't use fraction notation to denote division or an answer to a division problem with the same frequency that they appear to use it correctly in operator tasks. As stated earlier, the fraction notation was used given in quotient tasks between 49-54% of the time. For example, students gave the answer to three divided by nine as 1/3 or 3/9 only 49% of the time.

CHAPTER 5 - DISCUSSSION, IMPLICATONS, & RECOMMENDATIONS

Students' understanding of rational number is the focus of this study. In particular, the work, summarized in this chapter, identifies the conceptions and related strategies that students employ when solving tasks with rational numbers. In addition, the study analyzes the implementation of such strategies for (1) looking across decimals, fractions, and percents across different representations and (2) looking at fraction, decimal, and percent tasks across subconstructs, or conceptual categories, of rational number. This chapter includes a discussion of (1) the summary of the problem, (2) an overview of the methodology, (3) a summary of the results, (4) discussion of the results, (5) recommendations for practice, and (6) recommendations for further research.

Summary of the Problem

Background

Rational numbers are used consistently in everyday situations. Instruction on rational numbers begins in elementary school and continues through middle and even into high school. Despite these formal and informal experiences with rational numbers, students' struggles with rational number knowledge have been well-documented.

Representations of Rational Numbers

Research has focused on errors students make working with rational numbers and on the underlying misconceptions that lead to those errors. Research on rational numbers has focused on three representations of rational numbers: fractions, decimals and percent. In addition, research has focused on identifying student misconceptions and deficiencies, such as over-reliance on whole number principles. When comparing decimal numbers, error patterns have been identified and their rationale explained. For example, it is well established in the research that when determining the larger of two decimal numbers, students tend to progress through a series of stages. Students at first select the number that has more digits following the decimal point. This is often referred to as the "longer is larger" rule or more commonly the "whole number rule" (Moloney & Stacey, 1997; Nesher & Peled, 1986; Resnick et al., 1989; Sackur-Grisvard & Leonard, 1985). They then progress through a series of other "error patterns" before developing a meaningful understanding of decimal comparison. Often research in this area, however, has focused exclusively on one representation without determining how such a misconception in fractions, for example, might play out with decimals or percents. What remained to be done was to merge the research on error patterns in each of the representations and analyze students' thinking across representations of rational numbers.

Subconstructs of Rational Numbers

The concept of rational number consists of several interpretations and requires layers of understanding. Studying rational number knowledge requires breaking it down into manageable pieces. Across numerous studies, different researchers have identified subconstructs, or ways in which rational numbers can be conceptualized. Rational number has been broken down into several subconstructs (Behr, Lesh, Post, & Silver, 1983; Kieren 1981). At least five interpretations have been identified and to deeply understand rational numbers children have to know each subconstruct independently as well as the relations among them. Although the names may vary slightly from one researcher to another, the five constructs are: a part-whole comparison; a measure of continuous or discrete quantities; a quotient (result of division); a ratio; and an operator. The Rational Number Project (RNP)—a research project sponsored by the National Science Foundation since 1979—has done extensive research on children's learning of rational numbers and also investigated how prior knowledge facilitates or impedes children's progress toward understanding rational number concepts. Many research articles have been written by the principal investigators (Behr, Cramer, Harel, Lesh, & Post) as a result of the project, which focus on the subconstructs as well as other rational number related concepts. Each subconstruct is briefly described in chapter one and in detail in chapter two. What remained to be done was to study the subconstructs as a unit rather than in isolation. Future research would then identify relationships among them.

Need for the Study

A growing body of research identified various pieces of the sophisticated and web-like concept of rational number. What was needed was a way to bring it all together. Needed was a description of how the subconstructs weave their way through the representations of rational number and vice versa. The misconceptions and deficiencies identified in one set of research needed to be connected to the other pieces of research. The purpose of my research was to describe what students know about rational number concepts by the end of seventh grade and to answer the question "How do seventh grade students in a small rural school district conceptualize rational numbers?" Specifically:

1. What type of comparing and ordering strategies do students use with fractions and decimal comparison tasks?
2. How successful are students at solving rational number tasks for each of the five subconstructs?

This study adds to the existing body of research by providing an overarching description of the rational number knowledge of one group of students. It provides a comprehensive picture of what students know that provides a basis for future research that will involve finding a relationship among the pieces presented here.

Methodology

Data was collected on all 7th grade students (n = 101) in one small mid-western district. These students were required to take and pass a district minimal competency test (DMCT) to receive their high school diploma. Students first took the test at the end of 7th grade and if they do not pass, each subsequent year until they do pass. The scores on the DMCT were collected from the district and provided insight into students' computational proficiency with whole numbers, fractions, decimals and percent. In addition, all students participated in a rational number conceptual test whose items assessed order and equivalence as applied to decimals and fraction comparison items and included items that represented each of the five subconstructs of rational numbers. Based on the pattern of correct and incorrect response to comparison tasks, students were classified as using certain "rules": Whole Number Rule (WNR), Zero Rule (ZR), Ignore Zero Rule (IZR), Fraction Rule (FR), Money Rule (MR) or Apparent Experts (AE). Several students were considered unclassified because they were inconsistent in the strategies they applied to the given tasks.

Nine students from WNR, FR, and AE classifications were interviewed. These classifications were selected because these rules had already been established in previous research making them easy to identify. The other classifications weren't discovered until later at which time it was too late to interview them. The written assessments and interview items were analyzed for patterns and relationships across representations, across subconstructs, and across types of thinkers to provide a thick, rich description of what students know about rational numbers.

Summary of the Results

The scope and design of this study resulted in some interesting findings that begin to address what conceptual relationships might exist for students across rational number representations and subconstructs. In this section, nine important findings are briefly described, followed by a more detailed description organized by research question.

Key Findings

1. Based on comparison of decimal tasks, five error patterns were identified. Three of which are consistent with prior research and two error patterns were novel to this study. The two new error patterns were the Ignore Zero Rule (IZR) and the Money Rule (MR). IZR students, like ZR students, took notice of the zero in the tenths place. However, ZR students used this zero as a clue that the number would be smaller than one with a digit other than zero in the tenths place. On the other hand, IZR students basically ignored the zero because zero means nothing and compared the numbers as if the zero wasn't there. Money rule students also ignored digits, but in this case they would ignore anything after the hundredths place. Once the pattern was identified they were easy to locate because they would circle two numbers that were not equal but had the same digits in the ones, tenths, and hundredths places.
2. In comparison tasks, the representation of the numbers had an impact on students' ability to solve the task correctly. Specifically, students correctly compared two decimal numbers more frequently than they correctly compared two fractions. Comparing a decimal to a fraction resulted in the least accuracy.
3. Student success on the comparison tasks was related to their "rule" classification. The most primitive error rule in terms of application of prior knowledge is the "whole number

rule" (WNR). In this study, 7% of the seventh graders were classified as WNR, which is consistent with most research. It is important to note that they were tested at the end of the year so may have performed more like eighth graders. Sackur-Grisvard and Leonard (1985) reported that the WNR was used by 40% of fourth graders and 25% of fifth graders. Resnick et al (1989) found that the WNR was used by 35% of the fifth graders. Zucker (1985) reported that 18% of the 7th graders and 5% of ninth graders used WNR. Similar results were collected by Steinle and Stacey (1998) with 32% of fifth graders down to 5% of the 10th graders. It is important to note the decline of WNR with the increase in grade that is found consistently across the research. Students classified as WNR students in this study were the least successful on comparison tasks with the lowest average percent correct across all tasks. This is consistent with findings that WNR users have "very underdeveloped concepts of place value and fractions, and the decimal point emerged as nothing more than a separator of whole numbers" (Moloney & Stacy, 1997, p.35).

4. Zero rule (ZR) students and "Ignore Zero Rule" (IZR) students had slightly more success on tasks compared to WNR. This is consistent with previous research that indicated the ZR was an improvement over the WNR. Resnick et al. (1989) documented a shift from the whole number rule toward the ZR between fourth and fifth grades. Sackur-Grisvard Leonard (1985) reported that ZR was used by 8% of the fourth graders and 14% of the 5th graders. In this study 3% of the 7th graders used the ZR. Moloney & Stacey (1997) and suggest that the ZR students are only a slight improvement over the WNR students. They also found little difference in the performance of ZR and WNR students.

5. Fraction Rule (FR) and Money Rule (MR) students had similar percent correct on the written and interview items; however when comparing directly the average percent correct, MR students performed slightly better than FR students on two of the three comparison tasks. Sackur-Grisvard and Leonard (1985) reported that FR was used by 6% of their students. Zuker (1985) found that the FR was used by 23% of the seventh and ninth graders. Moloney and Stacey (1997) found that the FR is apparent across all levels with an astounding 20% of tenth graders using the rule. This is inconsistent with the findings of this study where the FR was used by only 5% of the seventh graders (one of those being in combination with IZR). FR students are more successful than WNR because they "integrate knowledge about fractional parts and fraction notation with their place value knowledge" (Resnick et al., 1989, p.21). However this success can be detrimental to further progress. Research indicates the FR misconception "has the potential to remain with students until adulthood unless it is challenged" (Moloney & Stacey, 1997 p. 36).
6. Ten percent of the students were "unclassified" because they missed one or two comparison tasks and therefore were not considered experts and their errors did not fit a pattern. This parallels other research where 88% of the children could be classified consistently using three rules: WNR, FR, and ZR (Resnick et al., 1989). Using the same instrument Moloney & Stacey (1997) were able to classify 80% of their students. They then modified this instrument to include more questions and found that they could classify students with 85-95% consistency. A criterion of four out of five items was set as an indicator of a given rule. In this study, fourteen items were used and students needed to match expected answers with 70% accuracy or better to be classified (all but one of the

thirty-two students matched with 79% accuracy or better). In this study unclassified students were those that generally only missed a single question. They most likely would qualify as experts. In other research students were classified as experts if they had no more than one error in the comparison tasks (Roche & Clarke, 2004). This is supported by the fact that they performed significantly better than the error rule students on comparison tasks. If this were the case, there would be only one unclassified student in this study. The fact that approximately 90% of learners apply a predictable error pattern underscores the need to apply a more diagnostic approach to instruction of rational numbers in order to identify student misconceptions and plan instruction accordingly.

7. Students used a variety of strategies to make comparisons. Interview results illustrated that students that were more successful on comparison tasks used a wider variety of strategies that were more often conceptual in nature compared to the WNR and FR students in the interview that frequently used a single strategy that was procedural in nature.
8. The type of fraction used appeared to be a factor in the success of students on subconstruct tasks. Students performed better with even-number denominators compared to odd-number denominators. For example, on the partitioning task they were more successful with the quotient ¾ than 3/5. They also had more success with unit fractions. For example, on operator tasks they performed better with 1/3 and 1/5 than 3/8.
9. Equivalency was a stumbling block for many students. WNR students for example, were able to compare fractions with 60% accuracy or better on all tasks other than equivalent fraction tasks where they only did so with 43% accuracy. This is consistent with research on the problems inherent with equivalent fractions (Behr et al., 1983; Behr et al., 1984;

Kamii & Clark, 1995; Post et al., 1985). Issues with equivalency are even more evident on tasks comparing a fraction to a decimal where the two lowest scoring items were equivalent fraction-decimal pairs.

10. Students used fraction notation for part-whole relationships with ease. For example, when asked to give the fraction for the shaded part of a figure, students were successful 95% of the time. They struggled the most with the fraction notation for a hundred grid with sixty-two squares shaded. Despite the success with fraction notation, they were less likely to use fraction notation with the other subconstructs and, in particular, were more reluctant to use it with the ratio subconstruct.
11. Results indicate that the part-whole interpretation of fractions tends to dominate student thinking even in part-part situations. This was particularly evident in the measure subconstuct tasks where students would interpret the number line as one whole regardless of the labels and markings on the line. In addition, students struggled to interpret fraction notation for the part-part ratio ½ because of the dominant use of ½ in part-whole contexts.
12. There are several concepts that form the foundation of rational number knowledge. This research suggests that students have misconceptions and/or deficiencies with regard to these topics. In particular, there was evidence that students were lacking skill with partitioning. It was also apparent that they had not fully developed the concept of density in terms of understanding that in between any two numbers there are infinitely many others. Finally, they have a limited understanding of place value in terms of decimal fractions.

13. Students struggled with the concept of equivalence. Students expected that equivalent representations would "look alike". When fractions and decimals used the same digits they would select them as equivalent. Students performed lowest on fraction-decimal comparison tasks where the numbers were equivalent.
14. Students vary in their conceptual understanding and procedural skill with rational numbers, but few seemed to have it completely developed. In some cases students may have attained the status of Apparent Expert by using the annex zero rule to equalize the length, which would enable them to compare the numbers as if they are two whole numbers. In this case, they would correctly answer the decimal comparison tasks, however they could still be whole-number-dominant or fraction-dominant in their thinking but not be classified that way. In this case they would have procedural skill but not conceptual understanding.

Research Question #1

The first part of chapter four reported data from the comparison tasks and answered the first research question: What type of strategies, including error patterns, do students use when comparing within and across decimal and fraction representations? Several conclusions are summarized and discussed here.

There were five error patterns identified in this research: Whole Number Rule (WNR), Zero Rule (ZR), Ignore Zero Rule (IZR), Fraction Rule (FR), and Money Rule (MR). Of these, three WNR, ZR, and FR, are consistent with prior research (Moloney & Stacey, 1997; Nesher & Peled, 1986; Resnick et al., 1989; Sackur-Grisvard & Leonard, 1985). Two new error patterns, IZR and MR, were identified in this research.

The results documented a range of success with comparison tasks. Students were most successful with decimal comparison tasks, with 68% of the students correctly answering all of the tasks. Some students provided written evidence of the strategy they used to compare decimals in the form of “annexing zeroes.” This is a coping strategy that is much easier to implement than strategies used when comparing fractions or comparing fractions to decimals, which may explain why students were less successful with comparing fraction tasks with 39% of the students completing all the tasks correctly. Students were most successful comparing fractions with common denominators. They were also successful comparing fractions with the same numerators. Fractions with neither the same numerator nor the same denominator presented students with the biggest problem. They struggled the most with fractions that were close in size, for example 6/7 and 8/9. Students used a variety of strategies such as cross-multiplication, finding common denominators, or using benchmarks.

Students struggle the most to compare a decimal to a fraction. Only 21% of the students correctly answered all the fraction/decimal comparison tasks. A common strategy involved attempting to make the two representations “look alike.” For example, students would remove the fraction bar from the fraction ¼ and replace it with a decimal point resulting in 1.4. Despite the success with comparing decimals, converting fractions to decimals was not a common occurrence during interviews. Instead students would change the decimal to a fraction. The students with the least success with decimal comparison problems, WNR students, tended to rely on a cross multiplication technique when comparing fraction or when comparing fraction to a decimal. Apparent experts (AE) used a variety of strategies with converting the fraction to a decimals being just one.

Across all three comparison tasks, student success appeared dependent upon their classification. Students classified as WNR students were less successful than other classifications across all three comparison tasks. They also had the lowest average minimal competency test score compared to the other classifications. The success experienced by WNR students was followed by ZR, IZR, MR, FR and then the UC groups in terms of percent correct on comparison tasks (with some exception) and average minimal competency test scores. ZR and IZR students scored lower than the other classification (other than WNR) on the comparison tasks, except for the decimal comparison tasks where they were more successful than all classifications (other than the UC group). This is consistent with research that reports the ZR is only a slight improvement of the WNR (Nesher & Peled, 1986).

There was variability when analyzing MR and FR students' success on comparison tasks. MR students were slightly more accurate on decimal comparison and decimal-fraction comparison tasks and FR students were more accurate on fraction comparison tasks and had a slightly higher average minimal competency test score. This is consistent with other research that found the FR students recognized that the digits after the decimal point represented a fractional amount; however they did not connect the size of the parts and number of parts (Nesher & Peled, 1986). In fraction notation, the size of the parts is indicated by the denominator. In decimal notation, the size of the parts is not explicitly noted but is implied by the location of the digits. When comparing ¾ to a decimal form, FR students would most likely select 3.4 or 0.3 or 0.34. This provides evidence that a student can show a high level of success on in a particular area of rational number when they are applying a misconception. It also indicates that as students applied more knowledge to their understanding of rational numbers, they had more success.

WNR students relied on the whole number knowledge as they used the length of the number to determine the value and used techniques to transform the rational number into a whole number “substitute”. ZR and IZR students are slightly more advanced because they have added in their knowledge of zero. MR students are further advanced because they apply their knowledge of money to help them consider the value of the digits rather than the length, leading to greater success on decimal tasks. However, this knowledge was not very useful on fraction comparison tasks. FR students are dependent on the length of the decimal number when determining its value, which may initially make us want to put them on the same level as WNR students. However, FR students appear to have a deeper knowledge of fractions; and while they do not apply it in meaningful ways to help them understand decimal numbers they are able to use their knowledge of fractions to perform successfully on other tasks. Unlike the WNR student, FR students are not looking at rational numbers as if they are whole numbers.

Across all three comparison tasks, patterns emerged in the type of strategies students used. Students classified as WNR or FR students, based on the interviews and evidence of written work on their tests, appear to rely more on procedural knowledge of rational numbers than on conceptual understanding. In general WNR and FR students use techniques that produce numbers that can be used as whole numbers and AE students provide more evidence of conceptual understanding as they describe the size and number of pieces involved in their rational number explanations and use benchmarks and estimation strategies that consider the value of the numbers with which they are working. WNR students in decimal comparison tasks often used a single strategy during the interview that converted rational numbers to whole numbers on fraction comparison tasks and fraction/decimal comparison tasks. Students classified as FR students in the decimal comparison tasks had knowledge of more strategies when probed during the interview but were often over-reliant on a single strategy. Students classified as experts on decimal comparison tasks used a variety of strategies that they selected based on the type of task and were more successful. This is consistent with previous research that there is a positive relationship between reasoning based on reference points (benchmarks) and quantitative understanding of rational number (Behr et al., 1983).

Research Question #2

The second part of chapter four reported on data collected about the subconstructs of rational numbers and answers the second research question: How successful are students at solving rational number tasks for each of the five subconstructs?

Across all subconstructs, students performed better on tasks that involved fractions that were more familiar such as ½ or ¼. The exception is the part-part interpretation of ratio applied to the fraction ½. Students struggled to interpret this ratio, most likely because of the domination of part-whole understanding of ½ on which so much of their prior knowledge is built. Fractions with odd number denominators such as 1/3 or 1/5 used on tasks resulted in less success. Students performed better with unit fractions than they did with non-unit fractions. While students performed better on tasks comparing decimals than they did comparing fractions, they gave correct fraction responses to part-whole tasks more often than they gave correct decimal responses. This is most likely because the part-whole nature of a decimal number is embedded in the value of the columns.

Some of the tasks used in this research attempted to get at the meaning of the fraction notation in various subconstructs. When asked to describe the use of a fraction as many ways as they can the part-whole interpretations dominated. Students were very comfortable using fraction notation in part-whole tasks. In quotient situations, students used fraction notation about half the time to describe “how much” or as a means of representing a division problem. Again, students’ prior knowledge of whole numbers hinders students as they work with division problems in which the quotient is less than one. Students were more likely to incorrectly use fraction notation for three divided by nine compared to fraction notation used to write twelve divided by four. This is consistent with research. Students used fraction notation less consistently in part-part ratio situations. When asked to write a ratio as many ways as they can only 7% of the students used fraction notation. However, 62% of the students could interpret a ratio in a part-part manner when given a context and a fraction and asked to explain what it means. The exception appears to be when interpreting the fraction ½. Only 38% of the students could explain that one dot was shaded for every two that is not. Earlier it was stated that familiar fractions resulted in more success, however this is a case where it did not because the part-whole interpretation of fractions appears to dominate.

From interviews it appears that part-whole interpretation and division play a part in the success with operator tasks. For example, when finding 1/6 of 18 students would describe dividing eighteen by six. The part-whole interpretation of 1/6 involves six parts in the whole. A student that knows this and knows that division involves equal groups would then divide eighteen into six equal groups (one group for each part) and take one of them. This is connected to the quotient subconstruct because 1/6 means to divide by six. Students have less success with a non-unit fraction such as 3/8 of 24. Despite the application of the division inherent in operator task problems and the fact that students were generally successful on operator tasks, students didn’t use fraction notation to denote division with the same frequency that they appear to use division correctly in operator tasks.

The results of this research are consistent with previous findings that students are more familiar with the part-whole interpretation of rational numbers. They are dependent on this interpretation to the point that it can interfere with the development of other subconstructs. Evidence of this surfaced when students labeled a number line interpreting the whole number line as "one" rather than using the labels given to them as clues for identifying additional marks. Part-whole interpretation also interfered on partitioning tasks where students redefined a unit and used all the cookies and cakes as one whole group rather than finding the fractional amount of a single cake. The part-whole interpretation interfered when students tried to interpret a part-part ratio of ½ as described above.

However, the knowledge of part-whole interpretation is also a useful application, for example as applied to operator tasks. According to research, the part-whole interpretation appears to develop first and is basic to all other interpretations (Behr et al., 1983; Freudenthal, 1983; Kieren, 1988; Pitkethly & Huntinig, 1996; Ni & Zhou, 2005). Students appear to apply it in meaningful ways to the operator subconstruct. For example, when finding 1/3 of eighteen pencils they use their part-whole knowledge to divide the eighteen pencils into three "whole" groups and then take one "part" of them. This then lends itself to a better understanding of the quotient subconstruct, applying the idea that the denominator is both our divisor and the total number of parts in the whole after dividing. Part-whole relationships encourage a "double-count" schema and as a result students often treat fractions as two separate whole numbers. However, students must move beyond treating rational numbers as two whole numbers separated by some symbol. The measure subconstruct is a more complex combination of ideas. To have a meaningful interpretation of the measure subconstruct, students must shift from seeing a fraction as two separate whole numbers to a single amount. They have to be able to look at the line and see it differently with each new fraction. For example, they've got to be able to see "1/2 of the line". Then they've got to be able to see "1/3 of the line", "1/4 of the line" and so on. Each time they partition the line they should see how density applies. In this way, operator and quotient interpretations build up to the measure subconstruct. Ratio seems the least related to the other subconstruct in the way students talk about it. However, the interpretation of ratio has its foundation in very early number sense where students experience part-part-whole relationships. Research indicates that they just need to become more familiar with the notation but have a grasp on the concept.

The type of model used in the tasks didn't appear to be a factor. For the most part students performed the same on part-whole tasks whether the figure was an area model, discrete set of objects or a simple number line. However there was a slight difference between a set of six circles where three were shaded and a number line marked zero to one with up to one-half shaded. While the number line was "shaded" and intended to be a part-whole task, it was still a task involving "length" and therefore a "measure" task as well. This particular number line task was an attempt to "relate" part-whole and measure subconstructs. Further research into the measure subconstruct using additional number lines and density tasks indicated that students struggle to apply the concept of density to fractions and decimals making the measure concept harder to grasp. Students performed better on labeling number line tasks that involved numbers with which they were more familiar such as 0.5 (95%) and whole numbers (93%) and they were less successful with slightly more complex decimal numbers such as 1.9 (80%) and 1.05 (67%) and fractions that are less familiar such as sixths (52%).

Error Patterns across Subconstructs

Across all subconstructs, WNR and ZR students performed with significantly less success and the apparent experts were significantly more successful when looking at the average percent correct. For example, on the eight number line tasks used to assess understanding of the measure subcontract, WNR and ZR students correctly labeled the number lines approximately 36% of the time. The other groups scored an average of 65% or higher with apparent experts correct an average of 82% of the time. The average percent correct on the operator tasks were similar in that the WNR and ZR students performed the lowest and the apparent experts performed the highest.

Discussion

This section highlights issues that surfaced within this research. Issues include those that are developmental in nature, those that focus on expanding students' knowledge of rational number to more difficult tasks, problems with equivalency, the variation in strategies used and the hidden misconceptions of experts.

Developmental Nature of Rational Numbers

Results indicate that there are building blocks inherent to rational numbers which successful students are able to apply to the various subconstructs that others are not. Within the subconstruct tasks, there were fundamental concepts with which students struggled. One issue was partitioning. When asked to partition pictures in sharing tasks, students struggled to do this accurately and fairly. Partitioning experiences enable students to develop an understanding of the relationship between the number of parts and the size of the pieces. Fundamental to this understanding is the notion that partitioning results in a new number.

Density of Numbers

A second issue revolved around the concept of density, which is essential to the development of a meaningful interpretation of the measure subconstruct. Less than 25% of the students recognized that there were infinitely many fractions between two fractions or infinitely many decimal numbers between any two decimal numbers. This is likely the result of the interference of whole number knowledge. Because it is so strong, a child's schema for ordering whole numbers is often inappropriately applied. As a result, notions of whole numbers inhibit acquiring the concept of density because whole numbers have a "next" number where rational numbers do not (Behr et al., 1984 & Gelman & Meck, 1992). The concept of density shifts thinking from "how many" to "how much".

Partitioning appears to be fundamental to the concept of density. Partitioning experiences encourage students to think about how more cuts can always be made resulting in smaller size pieces. However, even if students have a solid base of partitioning experiences they still struggle to recognize density of rational numbers because they have not developed a meaningful understanding of how the partitioning is represented symbolically with decimals and fractions. Which brings us full circle, without meaning behind the symbols, students retreat to knowledge that was learned the earliest and therefore is the strongest – whole number knowledge.

Expanding Knowledge to More Difficult Tasks

A third issue is the limited knowledge of the value of decimals beyond hundredths. Students were able to use the relationship between tenths and hundredths to identify that there would be numbers between 0.4 and 0.5 for example, but they were limited to nine or ten numbers that would be the result of adding hundredths to tenths 0.41, 0.42, etc. They were not able to extend this beyond the hundredths place to recognize the "infinitely many" concept. The limited understanding of decimals beyond hundredths place was more extreme in some cases. Identified in this research was a group of students classified as the "Money Rule" students. If two different decimal numbers had the same digits in the whole number, tenths, and hundredths columns, MR students (several others students did this as well but not as consistently as those identified as money rule students) would circle both numbers indicating that they are equal. When asked about this, one student explained that what happened after the hundredth place "didn't matter". Research indicates that 90% of teachers use money as a model for teaching decimals while only 50% use equivalence of fractions and pictures to teach the same concept (Glasgow et al, 2000). The money model for decimals is limited in its usefulness and as this research has found creates misconceptions and leave deficiencies in the understanding students have of decimal numbers.

Equivalency

Students struggled with the concept of equivalency. This was consistent with research. Equivalence is a complex concept because rational numbers have infinitely many equivalent representations (Vance, 1992). Renaming a number using another representation, for example rewriting a decimal as a fraction, often changes the appearance of the number but doesn't affect is properties. Students had less success comparing equivalent fractions and decimals indicting that it develops later than other comparison ideas. The multiplicative nature of equivalent fractions is often sited as the source of the problem (Kamii & Clark, 199). A WNR student for example would not find 2/3 and 6/9 to be equivalent because 2/3 is one away from a whole number and 6/9 is three away, which is additive rather than multiplicative thinking. This type of thinking was shared in interviews. There is evidence that students that have moved beyond WNR can still possess these thoughts. According to research, "A substantial number of students "back slide" into a whole-number-dominance strategy when confronted with problem-solving situations where they must apply their knowledge of the order and equivalence of fractions" (Behr et al., 1984, p333). Comparing a fraction to a decimal is perceived as more difficult and "children whose knowledge of rational numbers is insecure can regress to more primitive strategies in the face of cognitive disequilibrium" (Behr et al., 1984, p. 337). Apparent experts were more successful with equivalency tasks. This is consistent with research that indicates the development of equivalence occurs slowly over time (Vance, 1992) and in a subconstruct-by-subconstruct course as the concept of equivalence evolves (Ni, 2001). Ni (2001) found that students performed better with equivalent fractions applied to a part-whole interpretation than those applied to the measure interpretation.

Use of Multiple Strategies

During the interviews, it became apparent that students accessed and used different types of strategies. Some of those strategies were based more on conceptual knowledge and some of those strategies were based more on procedural knowledge. Students classified as WNR and FR were more likely to use procedural based strategies before and after being probed during the interview. Students classified as apparent experts were more likely to use a variety of strategies and many of those being conceptual in nature. Apparent experts appeared to select strategies based on the features of the numbers used in the task and would not approach every problem the same way. WNR and FR students used the same technique throughout. For example, when comparing fractions they would use the cross multiplication technique for every task, even when comparing 2/3 and 6/9. The type of thinking applied to comparison tasks may be a good predictor of success on other rational number tasks as well

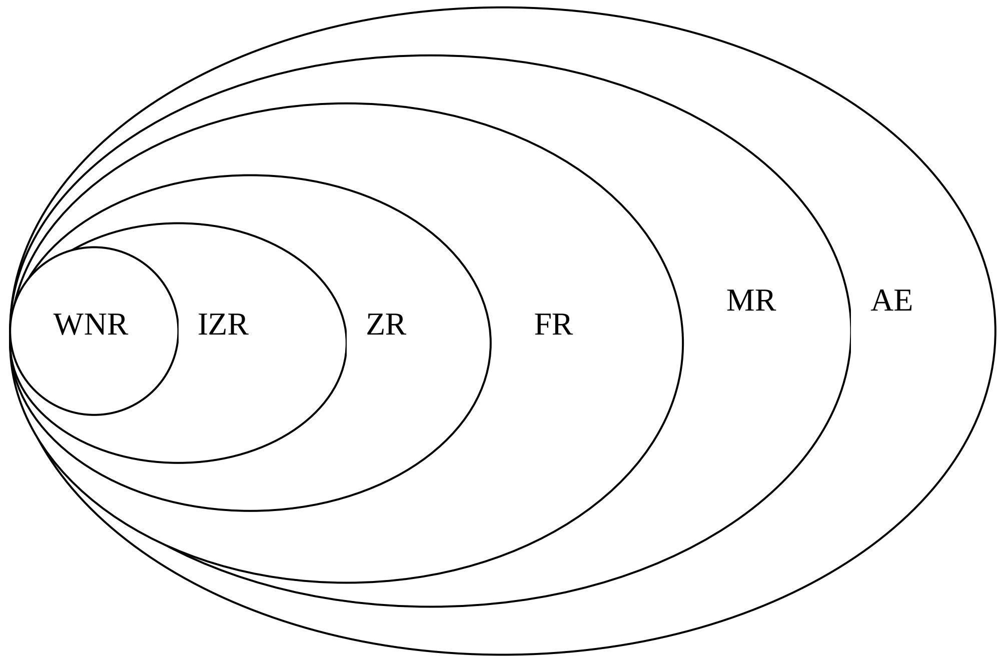
Apparent Experts with Hidden Misconceptions

A large number of students were classified as “apparent experts”. They successfully compared decimals with 100% accuracy. They also completed a minimal competency test with 80% accuracy or better. However, these raw scores on these assessments don’t indicate mastery. Looking more closely at the subconstruct tasks indicates that many students still have fundamental deficiencies and misconceptions. . Not all students classified as apparent experts on decimal comparison tasks would perform as apparent experts across all rational number tasks. Research indicates that students use an annex zero strategy to equalize the length of the decimal numbers so that they can treat them as whole numbers (Roche & Clarke, 2004). While these students perform well on decimal comparison tasks they do not demonstrate conceptual knowledge of rational numbers. When comparing fractions the apparent experts did not do so with 100% accuracy as they did when comparing decimals. Eleven of the fifty-four apparent experts missed two or more comparing fraction tasks and one missed nine. Fifteen of these students incorrectly selected the larger fraction when comparing 6/7 or 8/9. With conceptual understanding about the size of the “missing piece” this question would be quite simple to answer. However, even those students that were successful are still under question because they may still have been able to use a cross-multiplication technique resulting again in two whole numbers which they could compare without a conceptual understanding of fractions. Comparing a fraction to a decimal requires more of an understanding about the relationship between fractions and decimals; and therefore, more of a need for a conceptual understanding of rational numbers. Twenty-six of the fifty-four apparent experts (nearly half) missed two or more comparing fraction-decimal tasks and one missed eleven. The most frequently missed item was comparing 3/5 to 0.6. This is consistent with the research on the slow development of equivalency and a good indicator of an “unstable” understanding of rational number.

Knowledge is good when it works at when we can achieve our goal. Knowledge is not good when it hinders our ability to reach our goal because of some misconception that we have, unless it enables a student to “fallback” on previous understandings and build on them to form new ideas that are better and stronger. This knowledge can be reshaped to be useful if the individual that holds this view recognizes this misconception. Knowledge is gained from experiences where the learner has a purpose. The purpose is not to memorize right answers or regurgitate someone else’s meaning. Students will successfully remember this information but that does not ensure that it means anything to them. Understanding rational numbers is a complex process. It is more than just getting the correct answer. Often this is done with a procedure or method that is not meaningful to the student. For the knowledge to be useful, the subcontructs and representations must be interconnected. The growth of understanding occurs as the relationship between the levels and among the concepts are strengthened. Constructivist theory contends that present understanding is dependent on previous understandings. Pirie and Kieren’s theory provided a foundation for hypothesizing how the subconstructs and representations may be interconnected.

In my version of the theory, I’ve placed each of the “error rules” in nested circles, indicating that as students grow in their understanding of rational numbers they move through a series of misconceptions before constructing a solid understanding. Figure 5.1 illustrates how this growth could look using the nested circles.

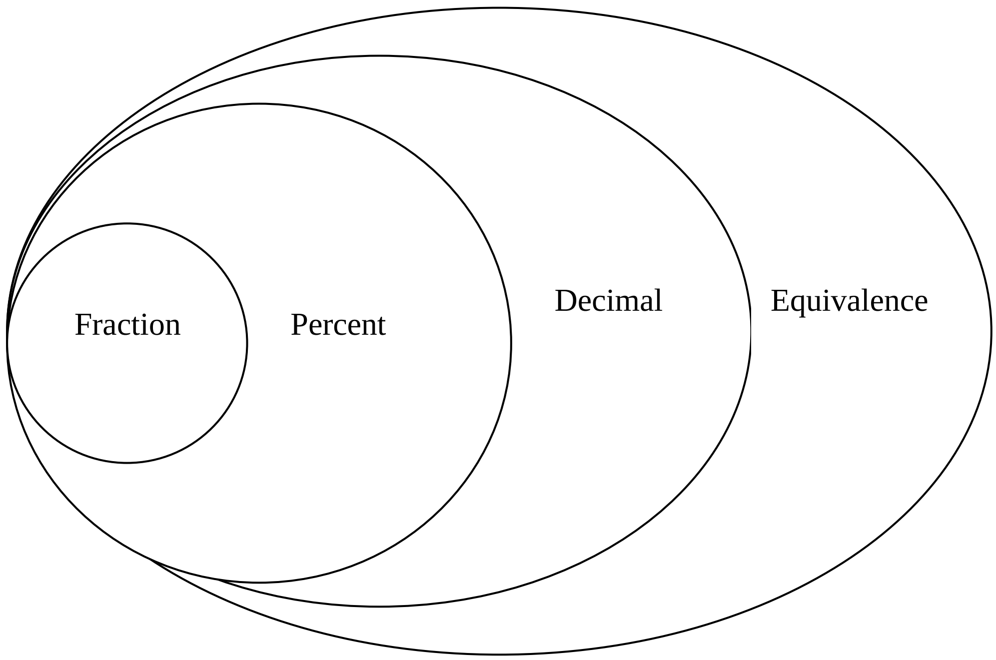
Figure 5-1 Modified Pirie & Kieren nested circles applied to error rules



WNR would be the most primitive form of knowledge and is where most students are in terms of understanding numbers when learning begins. Students begin to apply knowledge of zero which builds on their knowledge of whole numbers and therefore is “outside” of the WNR. As students begin to develop the “concept” of fraction and come to understand the portion after the decimal point is the “fraction” portion of the number they misapply their knowledge of fractions and place value. However, this is an indication of growth. MR students have been placed outside of the FR because students have images (money) related to the values that they are applying to their understanding of decimals, however they are restricted to the model. Finally, students become “apparent experts” in terms of analyzing their knowledge using comparison tasks.

As stated earlier, students can provide the right answers without real understanding of the concept. Therefore, the above figure must be layered with other figures. One of those involves the representations of rational number. Students appear to identify first with the fraction representation, followed by percent, and finally decimal representation. The figure below maps the growth of the representation with equivalence being the final level.

Figure 5-2 Modified Pirie & Kieren nested circles to applied to representations of rational number.



As students develop a complete understanding of rational number representation they will need to move back and forth among these circles.

When students have an understanding of multiple representation (moved beyond WNR) they will begin to recognize that they encounter various ways that rational numbers are used. Again, this must be layered on top of the previous models to illustrate the complexity of the concept applied to the description of the growth of understanding.

Future work and Recommendations

Implications for Instruction

The key instructional recommendations from this study lie mainly with concerns about a lack of flexibility in interpreting rational numbers and an over-reliance on some features of rational number throughout instruction. First, on the part of the students there still seems to be an over-reliance on whole numbers. Mistakes can reveal student misconceptions or over-generalizations and provide valuable opportunities to learn if teachers can use the mistakes to drive instruction. Teachers need to be aware of the fact that students struggle to move beyond what is “safe” and familiar in particular when they are struggling with a new idea. Students need opportunities to expand their knowledge of number beyond whole numbers. They need to do so in a way that expands their number knowledge so that rational numbers are not taught separately and disconnected from whole numbers.

Second, on the part of the teacher there is an over-reliance on certain types of instructional models. Student knowledge of rational number appears limited to hundredths because the models that make sense to them are limited to money or base-ten blocks with pieces that are no smaller than hundredths. Just as we teach students to comprehend “how much is a million” by providing them with number sense activities involving large numbers, we need to teach students to comprehend “how much is a millionth” in the same manner.

A third concern is an over-reliance on procedural knowledge. Because the concept of rational number is complex, teachers often try to simplify the concept by focusing on a series of “steps”. For example, teaching students to cross multiply uses skills for which they are more likely to be proficient. In addition, it transforms the fraction into what appears to be two whole numbers for comparison purposes. In attempt to make computation “easier”, teachers take away the thought that should accompany the rational numbers. For example, research finds that "teaching students to annex zeroes before comparing decimals may be to the detriment of their conceptual understanding" (Roche & Clarke, 2004, p. 7).

There is also an over-reliance on the part-whole subconstruct of rational numbers. Textbooks often rely heavily on the part-whole subconstruct and the partitive model of division making it more difficult for students to conceptualize fractions less than one and represent them using division. More attention needs to focus on the different interpretations of rational numbers and the fact that the use of them depends on the context in the same way that a word can have different meanings depending on the context in which it is used. In general, instruction needs to focus on developing a flexible interpretation of fractions. Students need a wide variety of rational number experiences in multiple contexts.

Implications for Future Research

Future research should focus on the new error patterns identified in this research. WNR, ZR, and FR were the only classifications identified at the time of the interviews. After further analysis IZR and MR emerged from the "unclassified" group where they were initially placed but at a time that was too late for interviews. In particular, the money rule group of students is an interesting phenomenon that merit further investigation.

Future research should also focus on the use of conceptual knowledge versus procedural knowledge applied to strategies or techniques. More students should be interviewed in the future to determine how the use of knowledge type impacts success with rational number tasks. It appears, from the limited number of interviews performed in this study, that there is a relationship between success on comparison tasks and conceptual versus procedural knowledge.

Finally, the subconstructs provide separate interpretations of rational number but when interwoven create the very fiber of a meaningful understanding of rational number. Future research, therefore, needs to focus on how the subconstructs are related to one another. This research barely scratched the surface but it did lay a foundation for describing the web-like connections among the subconstructs. Future research should build on those connections and seek to find additional relationships.

Transferability Issues

The limitations of the study were identified before the study began, however others emerged as the study unfolded. One limitation identified early in the study was the use of one population of students from the same school studying from the same curricula, which was a nontraditional, standards-based program. However, prior to their middle school experience most of these students were taught from a traditional curricula. This could influence how they organize and think about the different constructs compared to students in other districts. It can create more variability in the way students think about rational number and create a range of knowledge that may not represent all middle school students.

A second limitation identified early in the study was that it assumes the students are trying to be successful. While the study was designed to give the written conceptual test in small sections over several short testing periods, students were not graded on this instrument and may not have felt obligated to do their best work. In addition, testing at the end of the school year as the study was designed to do was not an ideal time frame. However, it was necessary so that limited additional instruction would take place between the written test and the interviews that would take place in the summer. After the research took place, this limitation became more of a concern. The test took place only a week or two before the end of the school year. Initially, the three sections were planned to be given over three different days. At the last minute, a school event was planned and two sections of the test had to be given in one day. Several students were not able to complete the entire third section. This resulted in different numbers of students completing the three sections, with 101 students completing the first section, 99 completing the second section, and only 86 completing the third section. Almost all of the students that didn't complete section three (and therefore weren't included in the analysis) were classified as apparent experts (AE) in the first section. During the analysis stage, it was observed that there were questions that students left blank. Often the question before and after would be answered indicating that they didn't stop working but rather they skipped over a question. Again, this highlighted the initial limitation that this study assumes the students are trying to be successful. It was interesting to note that students in the interview performed better on the same tasks in the interview than they did on the written test.

A third limitation is the fact that the items on the conceptual instrument were taken from previous research. Because the objective of this research was to look at a range of representations and subconstructs simultaneously only a small sample of items were selected from a much larger set of tasks in other research studies. This could alter the effectiveness of those items to measure the knowledge that they were intended to measure.

The final limitation lies with the theoretical framework. Pirie and Kieren theory is about *growth* of understanding. This research was done at a particular time not over a span of time. However, because students no two people come to the same understanding at the same time studying a group of students allows for the study of growth of understanding. Longitudinal research needs to be done to track growth of students over time to compare with this study. This would confirm or deny the results and the interpretations of this study.

Summary

This study contributes to current research and literature by describing as thoroughly as possible what students know about rational number. While there is a danger in scraping a mile wide and an inch deep, there is value in finding out what lies below the surface and taking it slowly allows the layers to expose themselves a little at a time. This study was based on a need to identify students' strengths and weaknesses; schemas that are embedded into though processes; and deficiencies that create chasms in learning. It enabled us to assess students' current levels of understanding in order to determine the next steps of instruction. Previous research on error patterns identified in comparison tasks and student knowledge of the subconstructs of rational were used as a basis for this research. Developing rational number knowledge is a complex endeavor. It doesn't occur in a linear fashion, but rather in a forward and back, spiraling manner. This research shows that students are not in the same place in terms of the development of rational number knowledge and that identifying their misconceptions requires careful analysis beyond looking at percentage correct on a written test and looking at tasks in isolation. Misconceptions can be identified when students are asked to apply their knowledge across types of tasks and when the observer is able to notice the patterns of errors created by the student.

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Appendix A - Rational Number Concept TEST

Section 1

The purpose of these questions is to find out how much you and other 7th graders know about fractions, decimals and percent. How you score on these questions will NOT affect your math grade. However, what we learn from how you answer these questions, we will use to teach you more about fractions, decimal and percents next year so you will want to do your best! There are 3 parts to this assessment and you will take one part each day during homebase. If you need more time to answer the questions, please let your teacher know.

Name: ____

Section 1

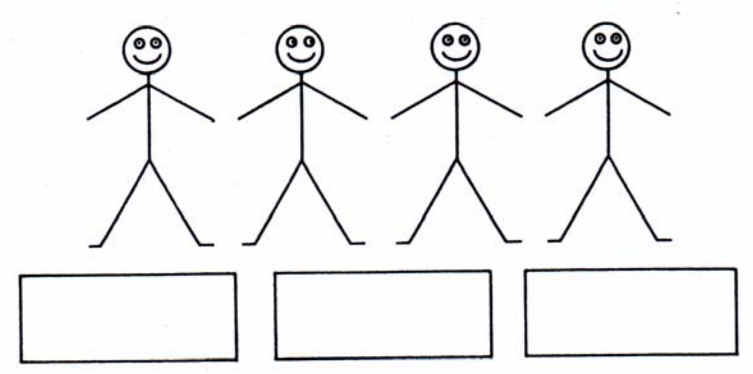
1. Circle the number with the largest value. If they are equal in value circle both.

4.63	or	4.8
4.7	or	4.08
4.4502	or	4.45
0.3	or	0.30
0.36	or	0.5
2.621	or	2.0687986
0.4	or	0.457
0.100	or	0.25
17.35	or	17.353
3.72	or	3.073
1.27	or	1.270
0.08	or	0.75
0.4	or	0.04
8.24563	or	8.245
0.37	or	0.216
8.514	or	8.0525738
5.62	or	5.736

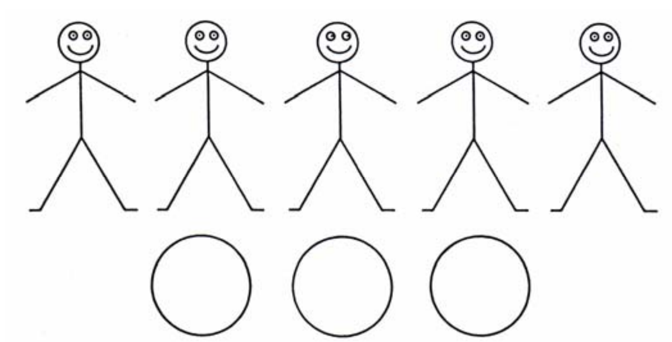
2. Order the set of three numbers from smallest to largest value.

3.682	3.2	3.84	__________	__________	__________
7.651	7.8	7.08	__________	__________	__________
6.796	6.4	6.07	__________	__________	__________

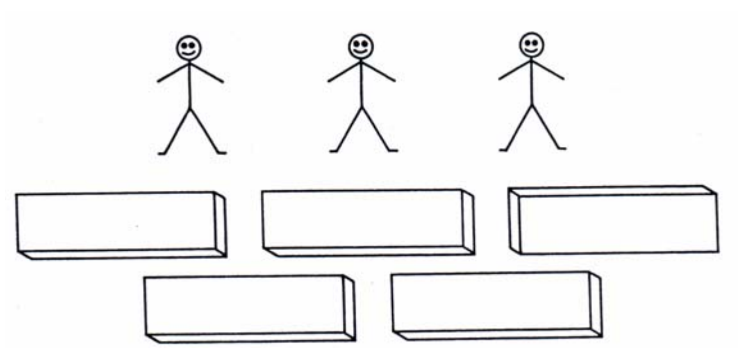
3. Three cakes are divided equally between four people. Shade in the amount each person would get. How much cake does each person get?



4. Three cakes are divided equally between 5 people. How much cake does each person get?



5. Five candy bars are divided equally between three children. How much candy bar does each child get?



6. Five friends shared 15 pounds of trail mix. How many pounds did each person get?

7. There was 10 feet of licorice rope. If 20 people shared the licorice rope, how many feet of licorice rope did each person get?

8. Write the division problem twelve divided by four as many different ways as you can.

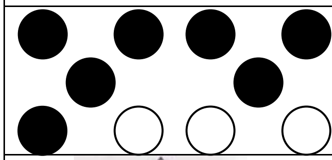
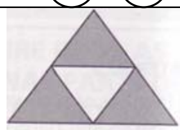
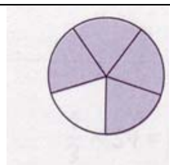
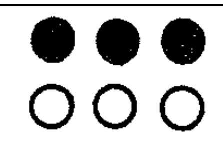
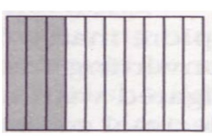
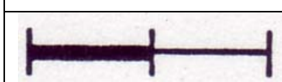
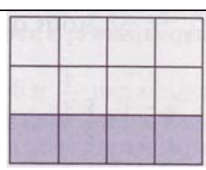
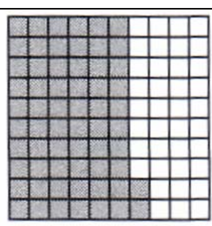
9. Write the division problem three divided by nine as many different ways as you can.

Section 2

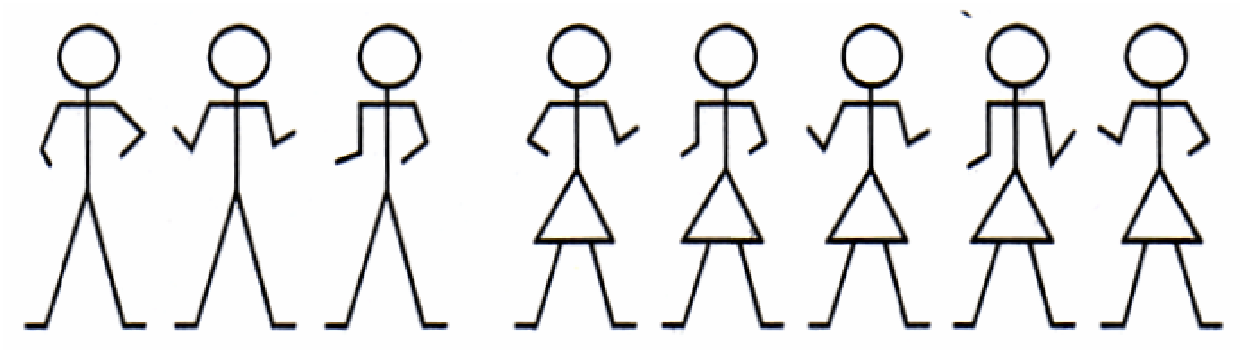
The purpose of these questions is to find out how much you and other 7th graders know about fractions, decimals and percent. How you score on these questions will NOT affect your math grade. However, what we learn from how you answer these questions, we will use to teach you more about fractions, decimal and percents next year so you will want to do your best! There are 3 parts to this assessment and you will take one part each day during homebase. If you need more time to answer the questions, please let your teacher know.

Name: ____

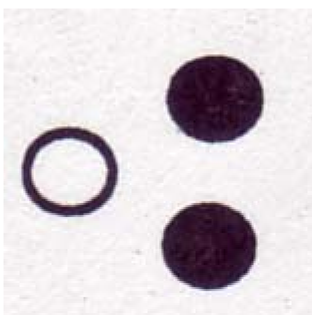
Write the fraction, decimal, and percent for the shaded part.

Figure	Fraction	Decimal	Percent
			
			
			
			
			
			
 0 1			
			
			

1. Here is a picture of a group of boys and girls. What does 3/5 represent in this situation?

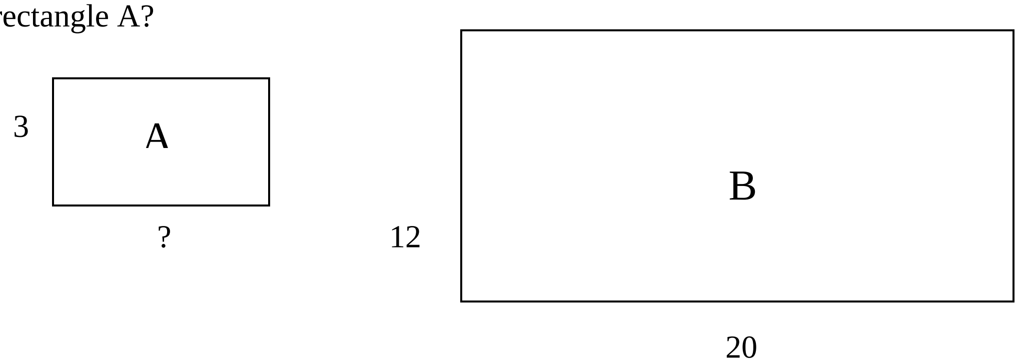


2. Here is a picture of some circles. What does ½ represent in this situation?



3. In her sock drawer, Jan has ten pairs of socks. Seven pairs are white and the rest are black. What is the ratio of white socks to black socks?

4. The dimensions of rectangle A are ¼ of the dimensions of rectangle B. What is the length of rectangle A?



5. Susan brought ten pencils to school today. One-fifth of the pencils were yellow. How many of the pencils were yellow?

6. In class, Tom made twelve cupcakes. That afternoon, his friend ate one-third of the cupcakes Tom had made. How many cupcakes did he eat?

7. Joe planted twenty-four pots of tomato plants. He gave three-eighths of the pots to his mother. How many pots of tomato plants did his mother get?

8. What is 1/6 of 18?

9. What is 2/3 of 15?

Section 3

The purpose of these questions is to find out how much you and other 7th graders know about fractions, decimals and percent. How you score on these questions will NOT affect your math grade. However, what we learn from how you answer these questions, we will use to teach you more about fractions, decimal and percents next year so you will want to do your best! There are 3 parts to this assessment and you will take one part each day during homebase. If you need more time to answer the questions, please let your teacher know.

Name: ____

Name: ____________________________

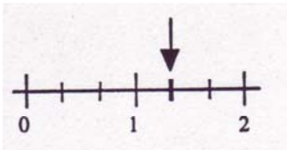
Section 3

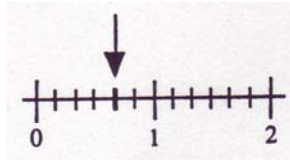
1. Circle the number with the largest value. If they are equal in value circle both.

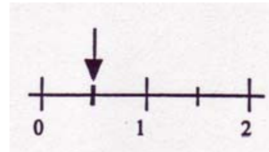
$\frac{1}{2}$ or $\frac{1}{5}$		$\frac{1}{8}$ or $\frac{7}{8}$
$\frac{2}{3}$ or $\frac{6}{9}$		$\frac{1}{5}$ or $\frac{7}{9}$
$\frac{2}{3}$ or $\frac{2}{9}$		$\frac{3}{5}$ or $\frac{4}{5}$
$\frac{6}{7}$ or $\frac{8}{9}$		$\frac{4}{10}$ or $\frac{1}{7}$
$\frac{3}{4}$ or $\frac{6}{8}$		$\frac{6}{12}$ or $\frac{2}{5}$
$\frac{9}{100}$ or $\frac{9}{10}$		$\frac{4}{3}$ or $\frac{2}{3}$
$\frac{3}{11}$ or $\frac{3}{14}$		$\frac{5}{8}$ or $\frac{6}{5}$
$\frac{4}{3}$ or $\frac{4}{5}$		$\frac{2}{5}$ or $\frac{5}{9}$
$\frac{1}{7}$ or $\frac{2}{7}$		$\frac{43}{100}$ or $\frac{6}{10}$

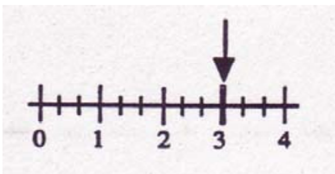
2. **What is a fraction? Explain all the ways you can think to describe the meanin or meanings of the fraction ¾.**

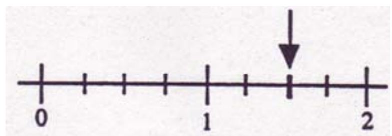
3. **For each number line below, label the length of the number line at the point marked with the arrow.**

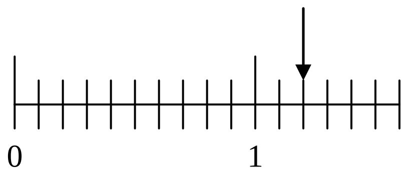


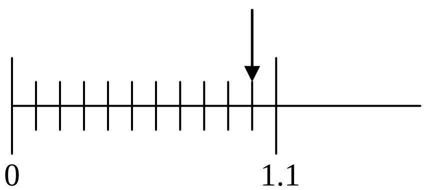


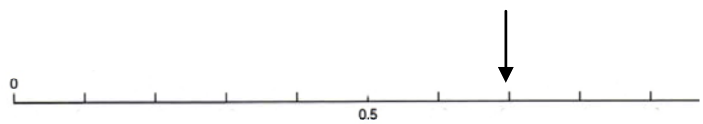




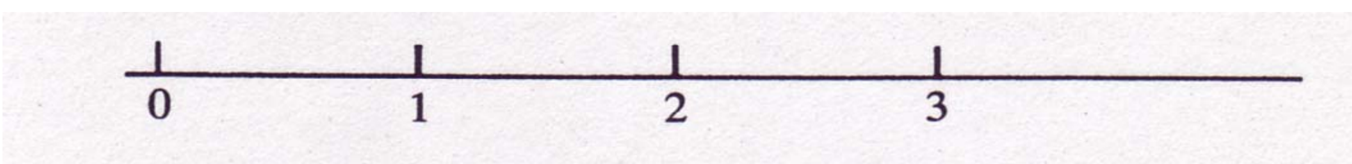








4. **As best you can mark a point on each number line to represent the following decimal numbers: 0.5, 1.05, 4, 2.6, 3.25, 1.9**



5. **Are there fractions between 3/5 and 4/5? How many?**

6. **Are there decimals between 0.3 and 0.4? How many?**

7. **Are there decimals between 0.74 and 0.75? How many?**

1. **Circle the number with the largest value. If they are equal in value circle both.**

1/8	or	0.8
2/3	or	0.25
9/10	or	0.910
5/6	or	0.59
1/10	or	1.10
3/5	or	0.6
3/10	or	0.3
1/5	or	0.15
4/8	or	0.5
1/3	or	0.013
3/7	or	3.7
7/10	or	0.07
2/5	or	0.4
3/6	or	0.36
¾	or	0.750
3/5	or	0.45
1/6	or	0.6
4/7	or	4.7

Appendix B - Interview

Interview

1. Circle the number with the largest value. If they are equal in value circle both.

0.64 or 0.7		0.317 or 0.2
2.41 or 2.043		6.396 or 6.39653
0.2 or 0.3		0.41 or 0.67
0 or 0.8		1.38 or 1.38209
0.6 or 0.281		1.7 or 1.6

2. Which is larger? Can you tell?

0._ _ or 0. _ _ _

3. The number 0.28 is closest to which number?

0.2 0.3 27 0.25 3

4. Order the following numbers from smallest to largest.

5. Circle the number with the largest value. If they are equal in value circle both.

½ or 1/5	1/8 or 7/9
2/3 or 6/9	3/5 or 4/5
9/100	9/10
2/5 or 5/9	3/11 or 3/14
4/3 or 2/3	5/8 or 6/5

6. For each number line on the card shown, give the number for the point marked by the arrow.

7. Are there fractions between 4/7 and 5/7? If so, how many?

8. Are there decimal numbers between 0.6 and 0.7? If so, how many?

9. Are there decimal numbers between 0.46 and 0.47? If so, how many?

10. Circle the number with the largest value. If they are equal in value circle both.

1/7 or 0.7	2/3 or 0.25
9/10 or 0.910	4/5 or 0.8
2/5 or 0.25	4/7 or 4.7
3/6 or 0.5	1/6 or 0.6
7/9 or 0.45	¾ or 0.750
5/6 pr 0.59	1/10 or 1.10

11. Three candy bars are divided equally among five people. How much candy bar does each person get? Show how to divide the candy bars.

12. Six candy bars are divided equally among four people. How much candy bar does each person get? Show how you would divide the candy bars.

13. How many different ways can you write four divided by fifteen?

14. Below is a group of X's and O's. What does 3/5 represent in this situation?

XXXOOOOO

15. In his closet, Bob has eight balls. Five balls are golf balls and the rest are baseballs. What is the ratio of golf balls to baseballs? Write this ratio as many ways as you can.