Approximations and Asymptotic Expansions for Sums of Heavy-tailed Random Variables

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Abstract

This thesis is devoted to the study of sums of independent and identically distributed (i.i.d.) *heavy-tailed* random variables (heavy-tailed sums). To be more precise, we are interested in the behavior of distribution functions of such sums. The problem with heavy-tailed sums is that the classical *central limit theorem* is not applicable in many cases and thus such sums cannot be approximated using the standard normal distribution. Moreover, for some classes even known alternative ways do not provide a good approximation. One of such classes, the class of Pareto-like distributions, was our field of study.

We start our investigation with *random sums* of heavy-tailed random variables, which are often used in applications. "Random sum" means that the number of summands in the sum is not fixed but also is a random variable. We consider different classes of heavy-tailed distributions (subexponential, distributions with regularly varying tails, Pareto-like distributions) and analyze the asymptotic results that are already known. Then we concentrate on the class of *Pareto-like distributions*, which is connected with the class of *stable distributions* by the following asymptotic result. A normalized sum of n i.i.d. Pareto-like distributed random variables converges to some stable distribution as $n \to \infty$. This allows the approximation of the sum by the corresponding stable distribution. The problem is that such approximations are usually very rough, in particular if we deal with Pareto-like random variables with shape parameter $\alpha < 2$. For this case correction terms for the limit distributions are needed. A powerful method is provided by asymptotic expansions of the distribution functions of the sums. The case $\alpha \in (1, 2)$ was already well studied in this regard, and correction terms were obtained using the so-called *pseudomoments*. Unfortunately, these known results do not provide the correction term for the limit distribution in the case of Pareto-like random variables with $\alpha \in (0, 1)$.

Our main result concerns this case. By modifying the concept of pseudomoments we obtain a good approximation of the distribution function of the sum of n i.i.d. Pareto-like random variables with parameter $\alpha \in (0, 1)$. We have also obtained a non-uniform estimate of the remainder.

Using our main theorem for the sums of n random variables with fixed $n \in \mathbb{N}$ we obtain some asymptotic results for random sums. As an application we consider the Cramér-Lundberg model, used in insurance mathematics, and show that for some cases our results help to construct improved estimates of the ruin probability.

Zusammenfassung

In dieser Arbeit untersuchen wir Summen von unabhängigen identisch verteilten (u.i.v.) *heavy-tailed* Zufallsvariablen (heavy-tailed Summen). Genauer sind wir an dem asymptotischen Verhalten der Verteilungsfunktionen solcher Summen interessiert. Das Problem bei heavy-tailed Summen ist, dass der klassische *zentrale Grenzwertsatz* in vielen Fällen nicht anwendbar ist und sich solche Summen somit nicht mit der Standardnormalverteilung approximieren lassen. Des Weiteren liefern auch bekannte alternative Ansätze für einige Klassen keine gute Approximation. Eine solche Klasse, die Klasse der Pareto-like Verteilungen, steht im Mittelpunkt dieser Arbeit.

Wir beginnen die Untersuchungen mit *zufälligen Summen* der heavy-tailed Zufallsvariablen, die in den Anwendungen oft verwendet werden. "Zufällige Summe" bedeutet, dass die Anzahl der Summanden selbst eine Zufallsvariable ist. Wir betrachten verschiedene Klassen der heavy-tailed Verteilungen (subexponentielle Verteilungen, Verteilungen mit regulär variierenden tails, Pareto-like Verteilungen) und analysieren einige bekannte asymptotische Resultate. Danach konzentrieren wir uns auf die Klasse der *Pareto-like Verteilungen*, die mit der Klasse der *stabilen Verteilungen* durch das folgende asymptotische Resultat verbunden ist. Eine normierte Summe von n u.i.v. Pareto-like verteilten Zufallsvariablen konvergiert für $n \to \infty$ gegen eine stabile Verteilung, so dass sich die Summe durch die entsprechende stabile Verteilung approximieren lässt. Solche Approximationen sind aber leider oft sehr ungenau, was insbesondere für Pareto-like Zufallsvariablen mit Parameter $\alpha < 2$ gilt. In diesem Fall werden Korrekturterme für die Grenzverteilungen benötigt. Um diese zu konstruieren, betrachten wir oft asymptotische Entwicklungen der Verteilungsfunktionen der Summen. Der Fall $\alpha \in (1, 2)$ wurde in dieser Hinsicht bereits untersucht, und es wurden Korrekturterme mithilfe der sogenannten *Pseudomomente* konstruiert. Die bekannten Methoden liefern allerdings keine Korrekturterme für die Grenzverteilung im Fall der Pareto-like Zufallsvariablen mit $\alpha \in (0, 1)$.

Unser Hauptresultat bezieht sich auf diesen Fall. Wir konstruieren eine gute Approximation der Verteilungsfunktion der Summe von n u.i.v. Pareto-like Zufallsvariablen mit Parameter $\alpha \in (0, 1)$, indem wir den Begriff eines Pseudomomentes modifizieren. Wir stellen außerdem eine nicht-gleichmäßige Abschätzung des entsprechenden Restgliedes bereit.

Als Folgerung unseres Hauptsatzes für die Summen von n Zufallsvariablen mit festem $n \in \mathbb{N}$ ergeben sich einige asymptotische Resultate für zufällige Summen. Als Anwendung betrachten wir das Cramér-Lundberg-Modell, das in der Versicherungsmathematik verwendet wird. Wir zeigen, dass unsere Ergebnisse in einigen Fällen zu verbesserten Schätzungen der Ruinwahrscheinlichkeit führen.

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Contents

Preface

Sums of independent and identically distributed (i.i.d.) random variables is an important and popular pattern not only in probability theory and statistics but also in many other branches of science. In particular, an important topic is to investigate the sums of so-called *heavy-tailed* random variables, i.e. random variables whose tails are not exponentially bounded. The behavior and possible approximations of the distribution function of such sums is of great interest.

A classical tool when considering sums of i.i.d. random variables is the Central Limit Theorem (CLT), which provides an approximation of sums by the normal distribution. However, the CLT is only a special case of the following more general result which was obtained independently by Lévy [38] and Khintchine [35]. If the distribution of a normalized sum of n i.i.d. random variables converges to some distribution as $n \to \infty$, then the limit distribution must be *stable*. So the normal distribution is one member of the class of stable distributions. Lévy and Khintchine also showed that the normal distribution is not the only one possible limit distribution for sums of i.i.d. random variables.

Why is this fact important for our investigation? There are classes of heavy-tailed distributions for which the CLT is not applicable, but any suitably normalized sum of random variables from these classes converges to some stable distribution. One of these classes is the class of *Pareto-like* distributions with shape parameter $\alpha < 2$. The distribution function of the sum of n i.i.d. Pareto-like distributed random variables has the same behavior as the corresponding stable distribution function as $n \to \infty$. One may ask the following question: Are the approximations by stable distributions good enough and if not, how could we construct better ones? This question is answered in Christoph and Wolf [12] for Pareto-like random variables. They constructed correction terms for the stable limit distribution and obtained good asymptotic results for the case $\alpha \in (1, 2)$ using the concept of *pseudomoments*. Pseudomoments combine the features of moments and metrics. This helps to "reflect the geometry of the distribution more accurately and informatively than with ordinary moments", as Weiner wrote in [50].

Pseudomoment results do not provide good correction terms for the limit distribution in case $\alpha \in (0, 1)$. This was a motivation for investigating this case, to which the thesis is devoted. Modifying the concept of pseudomoments we obtained a good approximation of the distribution function of the sum of n i.i.d. Pareto-like random variables with parameter $\alpha \in (0, 1)$. We also provide a non-uniform bound for the corresponding remainder.

Note that using the modified pseudomoments it should be possible to get better approximations for the sums of Pareto-like random variables with $\alpha \in [1, 2)$. We did not consider this case, since the technical realization of the proof would be much more difficult.

Using our main theorem for sums of n random variables with fixed $n \in \mathbb{N}$ we also obtain asymptotic results for *random sums* of Pareto-like random variables. As an application we consider the Cramér-Lundberg model, used in insurance mathematics, and show that for some cases our results help to construct improved estimates of the ruin probability.

Below we give an overview of the thesis.

In **Chapter 1** we introduce notations and basic concepts. Next, we formulate the problem that we are interested in: how to estimate the asymptotic behavior of a random sum of i.i.d. heavy-tailed (in particular subexponential) random variables. We define these two classes of distributions, formulate some properties of their members and give some examples. We finish this introductory section by describing an application in insurance mathematics.

Chapter 2 is devoted to the analysis of previous research on the asymptotic behavior of random sums. We start with the class of subexponential distributions, for which the first-order behavior of the distribution function of the sum is known. Unfortunately, the second-order result requires stronger assumptions than just subexponentiality. Therefore, we move on from the subexponential class to a subclass: distributions with regularly varying tails. For this subclass we analyze three known second-order asymptotic results. They all require the existence of a density and some additional assumptions on the random variables in the sum. We compare the asymptotic results that these three theorems provide for different examples. We also construct some distributions with regularly varying tails for which all three theorems are not applicable. Next, we restrict ourselves to a subclass of distributions with regularly varying tails, for which some asymptotic results are known without requiring the existence of a density for the random variables in the sum. This subclass is the class of Pareto-like distributions. To obtain these asymptotic results a completely different method, which is connected with limit theorems, is used.

Chapters 3 and **4** are central in this thesis. These chapters are devoted to limit theorems. Here we switch from random sums (considered in Chapters 1 and 2) to sums of n i.i.d. random variables, where $n \in \mathbb{N}$ is fixed. We start Chapter 3 by explaining the connection between stable distributions and Pareto-like distributions. Here we formulate a generalization of the CLT, already obtained around 1930 [35, 38], which allows us to approximate the sums of Pareto-like distributions with parameter $\alpha \in (0, 2]$ by the corresponding stable limit distributions.

Next, we discuss the quality of such approximations and ways of getting better ones. Here we distinguish three cases. The case $\alpha = 2$ was studied a long time ago and good approximations of the distribution function of the sum were obtained using the normal distribution. The research of the case $\alpha < 2$ is relatively young and involves the consideration of pseudomoments. We discuss the notion of pseudomoments and present the known approximation for sums of Pareto-like random variables with $\alpha \in (0, 2)$. Since this approximation consists only of the stable limit distribution in the case $\alpha \in (0, 1)$, we investigate this case separately. Modifying the concept of pseudomoments we present our main result: a better approximation for the distribution function of sums of Pareto-like random variables with $\alpha \in (0, 1)$ and a non-uniform estimate of the corresponding remainder.

Chapter 4 is devoted to the long and technical proof of our result. In the beginning of the chapter we introduce some auxiliary functions and give a concise plan of the proof. In order to prove our main result we have to estimate three terms. The estimation of the second one is the most difficult part. It is carried out in four steps. The proofs of some technical results needed in this chapter are given in Appendix B.

In **Chapter 5** we come back to random sums of random variables and discuss the second-order behavior of the distribution function of such sums. We start with two special cases that are already known and involve the use of pseudomoments and stable distributions. Next, we derive our asymptotic result for random sums of i.i.d. Pareto-like random variables with parameter $\alpha \in (0, 1)$ and give some examples. As an application we obtain the asymptotic result for the Cramér-Lundberg model introduced in Chapter 1.

The thesis contains two Appendices. Appendix A presents some useful mathematics, which we need throughout the work. In Appendix B we collect the proofs of some technical results from Chapter 4 along with some auxiliary lemmata needed for the proofs.

1 Introduction and problem definition

1.1 Notation

In this section we introduce some notational conventions that will be used throughout this thesis.

Let $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{C}$ be sets of natural, real and complex numbers, respectively. We put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+ = (0, +\infty)$ and $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. The integer part of $r \in \mathbb{R}$ is denoted by $[r]$. For $a \in \mathbb{R}$ we define $a^+ = \max\{0, a\}$.

A function $f : \mathbb{R} \to \mathbb{R}$ is *nondecreasing* (respectively, *increasing*) if $f(x) \le f(y)$ (respectively, $f(x) < f(y)$) for all $x, y \in \mathbb{R}$ with $x < y$.

The expression $f(x) = o(g(x))$ as $x \to \infty$ means that $\lim_{x\to\infty} f(x)/g(x) = 0$.
The expression $f(x) = O(g(x))$ as $x \to \infty$ means that the function $|f(x)| \,/\, |g(x)|$ is bounded for sufficiently large x.

Let X be a real random variable defined on the probability space $(\Omega, \mathfrak{A}, P)$. The distribution function F_X of X is given by $F_X(x) = P(X \le x)$ for all $x \in \mathbb{R}$. The *support* of F is the set $\operatorname{supp}(F) = \{x \in \mathbb{R} : 0 < F(x) < 1\}$.
The *expectation* of X is given by $EX = \int_\Omega X dP$, if the integral exists. It can be also written as Riemann-Stieltjes integral:

$$EX = \int_{-\infty}^{+\infty} x \, dF_X(x).$$

Let $p \in \mathbb{N}_0$ and $r \in \mathbb{R}_+$. Then the expectations of random variables X^p and $|X|^r$ are called the *p-th order moment* of X and the *absolute moment of order* r of X, respectively.

Let X be a random variable with distribution function F_X, then $1 - F_X(x) = P(X > x)$ and $F_X(-x) = P(X \le -x)$ for $x \to \infty$ are the *right tail* and the *left tail* of the distribution function F_X, respectively. If $P(X \ge 0) = 1$, we will use "tail" instead of "right tail" and write $\overline{F}_X(x) = 1 - F_X(x)$, $x \ge 0$.

The *characteristic function* of a random variable X with distribution function F_X is given by

$$f_X(t) = E\left(e^{itX}\right) = \int_{-\infty}^{+\infty} e^{itx} dF_X(x), \quad t \in \mathbb{R}.$$

The *moment generating function* of a random variable X with distribution function F_X is given by

$$M_X(t) = E\left(e^{tX}\right) = \int_{-\infty}^{+\infty} e^{tx} dF_X(x), \quad t \in \mathbb{R}.$$

The *gamma function* is denoted by $\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t} dt$ for $x > 0$.

We say that a random variable X has a *normal distribution* $N(\mu, \sigma^2)$ with parameters $\mu \in \mathbb{R}$ and $\sigma \in (0, \infty)$ if the distribution function F_X of X has the following form:

$$F_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(y-\mu)^2/(2\sigma^2)} dy, \quad x \in \mathbb{R}.$$

The distribution function of $N(0,1)$ is denoted by $\Phi(x)$, $x \in \mathbb{R}$.

Throughout the thesis we will often skip the name of a random variable in the notation of distribution function, characteristic function and moment generating function, i.e. we will use $F(x) := F_X(x)$, $f(t) := f_X(t)$ and $M(t) := M_X(t)$.

The *n-fold convolution* of a function $F : \mathbb{R} \to \mathbb{R}$ of bounded variation is denoted by F^{n*} for all $n \in \mathbb{N}_0$ with $F^{0*}(x) = \mathbb{1}_{[0,+\infty)}(x)$, $x \in \mathbb{R}$ (distribution function of the unit measure at zero) and $F^{n*} = F^{(n-1)*} * F$ for $n \geq 1$. Recall that the convolution $F * G$ of two functions F and G of bounded variation is defined as

$$(F * G)(x) = \int_{-\infty}^{+\infty} F(x - y)\, dG(y), \quad x \in \mathbb{R}.$$

Let X and Y be two random variables defined on some probability spaces (not necessarily on a common one). We write $X \stackrel{d}{=} Y$ if X and Y have the same distribution function.

Let $F, F_1, F_2, \dots$ be bounded nondecreasing real functions on $\mathbb{R}$. The sequence $(F_n)_{n\in\mathbb{N}}$ converges weakly to F if $F_n(x) \to F(x)$ as $n \to \infty$ at every point x of continuity of F. Throughout this thesis we write $F_n \to F$ as $n \to \infty$ and mean weak convergence.

For integers u and v such that $u < v$ we set by convention: $\sum_{i=v}^{u} a_i = 0$ and $\prod_{i=v}^{u} a_i = 1$ for any $a_i \in \mathbb{C}$.

Unless otherwise specified the symbols $C, C_1, C_2, \dots$ denote positive constants. One and the same letter used in different parts of the thesis may stand for different values.

We use the standard abbreviation "i.i.d." for "*independent and identically distributed*".

1.2 Basic definitions and problem formulation

The aim of this section is to introduce the problem under consideration and to give some elementary definitions and properties of objects, which we use in what follows.

Let $X, X_1, X_2, ...$ be i.i.d. nonnegative random variables with common distribution function F and let ν be a nonnegative integer-valued counting random variable, which is supposed to be independent of $X_1, X_2, ...$. Consider a compound sum $S_\nu = X_1 + \cdots + X_\nu$, where $S_0 = 0$. Let $p_n = P(\nu = n)$ with $\sum_{n=0}^{\infty} p_n = 1$. Then the distribution function of the sum S_ν can be written in the form

$$K_\nu(x) = P(S_\nu \leq x) = \sum_{n=0}^{\infty} P(\nu = n) F^{n*}(x) = \sum_{n=0}^{\infty} p_n F^{n*}(x), \ x \in \mathbb{R}, \tag{1}$$

where F^{n*} denotes the n-fold convolution of F, i.e. $F^{n*}(x) = P(X_1 + \cdots + X_n \leq x)$ for $n \geq 1$ and F^{0*} is the distribution function of the unit measure at zero.

We are interested in the behavior of the ratio

$$\frac{1 - K_\nu(x)}{1 - F(x)} \tag{2}$$

for large x. We restrict ourselves to the case of *heavy-tailed* distribution function F, and specifically to the case of *subexponential* F. The importance of ratio (2) in applications for such F is explained at the end of Section 1.3.

Often one uses heavy-tailedness and subexponentiality as synonyms. Actually, the class $\mathcal{S}$ of subexponential distributions is only a subclass (but a very important one) of heavy-tailed distributions. For the latter no definition is universally accepted. We will use the following one.

Definition 1.1 (Heavy-tailed distribution, [2, Appendix 5], [46, Section 2.5])**.**
A random variable X (or its distribution function F) is said to be heavy-tailed on the right (or to have heavy right tail) if

$$E[\, e^{tX} \mathbb{1}_{\{X>0\}}] = \int_0^{+\infty} e^{tx} dF(x) = \infty \quad \text{for all} \quad t > 0, \tag{3}$$

i.e. if the moment generating function of $X \cdot \mathbb{1}_{\{X>0\}}$ is infinite for all $t > 0$.

Remark 1.1. If a random variable X is heavy-tailed on the right, then for all $\lambda > 0$ we have (see [30])

$$\lim_{x \to \infty} e^{\lambda x} P(X > x) = \infty. \tag{4}$$

Condition (4) means that the right tail of F decreases to 0 as $x \to \infty$ more slowly than any exponential function $e^{-\lambda x}$ with $\lambda > 0$.

Remark 1.2. For a random variable X with heavy left tail equality (3) holds with $e^{-tX} \mathbb{1}_{\{X<0\}}$ instead of $e^{tX} \mathbb{1}_{\{X>0\}}$. For commonly considered applications the right tail of a distribution is of interest, but a distribution may have heavy left tail, or both tails may be heavy.

There are two important subclasses of heavy-tailed distributions, namely, long-tailed distributions and subexponential distributions. In applications all commonly used heavy-tailed distributions belong to the subexponential class, which is actually defined only for positive random variables.

Definition 1.2 (Subexponential distribution, [30])**.**
A distribution function F of a positive random variable with $F(x) < 1$ for all $x > 0$ is called subexponential (we write $F \in \mathcal{S}$) if for all $n \geq 2$ the following condition holds:

$$\lim_{x \to +\infty} \frac{1 - F^{n*}(x)}{1 - F(x)} = n. \tag{5}$$

Remark 1.3. The class $\mathcal{S}$ of subexponential distributions was first invented and examined by Chistyakov [13]. He proved that (5) holds for all $n \geq 2$ if and only if it holds for $n = 2$. Embrechts and Goldie [17] showed that (5) holds for $n = 2$ if it holds for some $n > 2$.

Remark 1.4. The assumption $F(x) < 1$ for all $x > 0$ means that the support of F is unbounded from above.

Remark 1.5. Definition 1.2 may be extended to any distribution on the real line. A distribution function G will be called *subexponential on* $\mathbb{R}$ if there exists a subexponential distribution function F such that $\lim_{x\to+\infty}(1-G(x))/(1-F(x)) = 1$, [18, Appendix 3.2].

Some properties of subexponential distributions that will be used in this thesis are given below.

Lemma 1.3. *Let $F \in \mathcal{S}$. Then the following properties hold:*

(i) $\displaystyle\int_0^{+\infty} e^{tx} dF(x) = \infty$ *for all $t > 0$, i.e. F is heavy-tailed.*

(ii) $\displaystyle\lim_{x\to+\infty} \frac{1-F(x-y)}{1-F(x)} = 1$ *for all $y > 0$.*

(iii) For each $\varepsilon > 0$ there exists a finite constant $K = K(\varepsilon)$ such that for all $x \geq 0$ and $n \in \mathbb{N}_0$:

$$\frac{1-F^{n*}(x)}{1-F(x)} \leq K(1+\varepsilon)^n. \tag{6}$$

Proof. Both properties (i) and (ii) were proved by Chistyakov in [13]. Property (iii) is due to Kesten (for a proof see [3]). □

Remark 1.6. Property (i) demonstrates that the tail of $F \in \mathcal{S}$ is not exponentially bounded. The latter in turn accounts for the name "subexponential". Also we see from (i) that class $\mathcal{S}$ is a subclass of heavy-tailed distributions.

Remark 1.7. A distribution function F with property (ii) is often referred to as *long-tailed* , [2, Appendix 5]. From Lemma 1.3 it follows that the class $\mathcal{S}$ is a subset of the class of long-tailed distributions.

We will give some examples of heavy-tailed and subexponential distributions.

Example 1.1 ([18, Section 1.4, Ex. 1.4.2])**.**
Consider a game where the first player (Peter) tosses a fair coin until it falls head for the first time, receiving from the second player (Paul) 2^k roubles, if this happens at trial k. The distribution function of Peter's gain is

$$F(x) = \sum_{k\in\mathbb{N}\,:\,2^k\leq x} 2^{-k}, \quad x \geq 0.$$

The problem underlying this game is the famous St. Petersburg paradox (see [22, Section X.4]). Note that for any fixed $\ell \in \mathbb{N}$ we have

$$\frac{1-F(2^\ell-1)}{1-F(2^\ell)} = \frac{1-\sum\limits_{k=1}^{\ell-1} 2^{-k}}{1-\sum\limits_{k=1}^{\ell} 2^{-k}} = 2$$

so that property (ii) from Lemma 1.3 is not satisfied. Therefore, $F \notin \mathcal{S}$. On the other hand, F is heavy-tailed:

$$\int_0^{+\infty} e^{tx} dF(x) = \sum_{k=1}^{\infty} e^{t \cdot 2^k}\, 2^{-k} = \infty \quad \text{for all} \quad t > 0.$$

In fact, according to the Cauchy convergence test, the latter infinite series diverges for all $t > 0$, since $\limsup_{n\to\infty} \sqrt[n]{a_n} = \limsup_{n\to\infty} \exp\{t\, 2^n/n\}/2 = \infty > 1$.
This example shows that the class $\mathcal{S}$ does not coincide with the class of heavy-tailed distributions. Some other examples of heavy-tailed but not subexponential distributions can be found in [16], [45]. □

Example 1.2. Consider a (μ, λ)-*Cauchy distributed* random variable X with density function p_X, distribution function F_X and characteristic function f_X, given by

$$p_X(x) = \frac{1}{\pi}\, \frac{\lambda}{\lambda^2 + (x-\mu)^2}, \qquad F(x) := F_X(x) = \frac{1}{\pi} \arctan\left(\frac{x-\mu}{\lambda}\right) + \frac{1}{2}, \quad x \in \mathbb{R},$$

$$f_X(t) = \exp\{i\, \mu\, t - \lambda\, |t|\}, \quad t \in \mathbb{R}.$$

Let us consider the case $\mu = 0$ and $\lambda = 1$, i.e. the standard Cauchy distribution. It is easy to show that the condition $\int_0^{+\infty} e^{tx}\, dF(x) = \int_{-\infty}^0 e^{-tx}\, dF(x) = \infty$ holds for all $t > 0$. Hence, both tails of F are heavy. Moreover, the distribution function F is subexponential. In order to prove this fact note that the characteristic function of the sum of two standard Cauchy distributed independent random variables X_1, X_2 has the form

$$f_{X_1+X_2}(t) = (f_{X_1}(t))^2 = \left(e^{-|t|}\right)^2 = e^{-2|t|}.$$

Hence, $X_1 + X_2$ is also a Cauchy distributed random variable but with parameters $(0, 2)$ instead of $(0, 1)$. This implies that $F_{X_1+X_2}(x) = (1/\pi) \arctan(x/2) + 1/2$ and consequently

$$\lim_{x\to+\infty} \frac{1 - F^{2*}(x)}{1 - F(x)} = \lim_{x\to+\infty} \frac{1/2 - (1/\pi) \arctan(x/2)}{1/2 - (1/\pi) \arctan x} = 2.$$

□

Example 1.3. Let X be a *Pareto-distributed* random variable with distribution function F, given by

$$1 - F(x) = \begin{cases} (\kappa/x)^\alpha & \text{for} \quad x \geq \kappa, \\ 1 & \text{for} \quad x < \kappa, \end{cases} \tag{7}$$

where $\kappa > 0$ and $\alpha > 0$ are scale and shape parameters, respectively. For $x > 2\kappa$ we have

$$\begin{aligned} \frac{1 - F^{2*}(x)}{1 - F(x)} &= \int_\kappa^\infty \frac{1 - F(x-y)}{1 - F(x)}\, dF(y) = \left(\int_\kappa^{x-\kappa} + \int_{x-\kappa}^\infty\right) \frac{1 - F(x-y)}{1 - F(x)}\, dF(y) \\ &= \int_\kappa^{x-\kappa} \frac{(x-y)^{-\alpha}}{x^{-\alpha}}\, dF(y) + \int_{x-\kappa}^\infty \frac{1}{1 - F(x)}\, dF(y) \\ &= \int_\kappa^\infty \left(1 - \frac{y}{x}\right)^{-\alpha} \frac{\alpha\, \kappa^\alpha}{y^{1+\alpha}}\, \mathbb{1}_{[\kappa, x-\kappa]}(y)\, dy + \frac{1 - F(x-\kappa)}{1 - F(x)}. \end{aligned}$$

Using Corollary 1 from [26, Chapter XIV, § 3-518], we can show that the first term converges to 1 as $x \to \infty$. From this it follows that

$$\lim_{x\to+\infty} \frac{1 - F^{2*}(x)}{1 - F(x)} = 2.$$

Therefore, the Pareto distribution is subexponential, and, consequently, heavy-tailed. □

Remark 1.8. We proved that in all three examples we deal with heavy-tailed distributions. Moreover, for the Cauchy distribution, Peter's gain distribution, and the Pareto distribution with $\alpha \in (0, 1)$ even the expectation is infinite or does not exist. It is typical of heavy-tailed distributions to have infinite moments of high orders. This makes investigating the models with such distributions very difficult, since a lot of commonly used methods fail to work.

Other examples of subexponential distributions are Burr, log-gamma, lognormal, Weibull with shape parameter $\tau \in (0, 1)$, "almost" exponential etc. (see [30]).

As we already noted, subexponential distributions are widely used in applications. This fact can be explained very well by the following equivalent description of the class $\mathcal{S}$. It gives a physical interpretation of subexponentiality.

Lemma 1.4. *A distribution function F on $(0, +\infty)$ such that $F(x) < 1$ for all $x > 0$ is subexponential ($F \in \mathcal{S}$) if and only if for all $n \geq 2$ the following condition holds:*

$$\lim_{x\to+\infty} \frac{P(X_1 + \cdots + X_n > x)}{P(\max(X_1, \ldots, X_n) > x)} = 1. \tag{8}$$

Proof. See Embrechts and Goldie [16]. □

Remark 1.9. If (8) holds for some $n \geq 2$, then it holds for all $n \geq 2$, see Remark 1.3.

Remark 1.10. Condition (8) means that the sum of n i.i.d. subexponential random variables and their maximum are comparable quantities, if they are sufficiently large. Or in other words, the sum is large if and only if the maximum is large. This makes it possible to use subexponential distributions for modeling events, that occur rarely, but have a considerable influence on the situation. Such events are typical for catastrophe insurance and for finance.

1.3 Motivation and possible applications

In this section we will give a detailed description of basic insurance models, where the ratio (2) with subexponential distribution functions F occurs. Our investigation of (2) may be useful for the following applications.

Definition 1.5 (The Cramér-Lundberg model, [18, Section 1.1]).
The Cramér-Lundberg model is given by conditions (a)-(e):

(a) The claim size process:
the claim sizes $(X_k)_{k\in\mathbb{N}}$ are positive i.i.d. random variables having common non-lattice distribution function F, finite mean $\mu = EX_1 > 0$, and variance $\sigma^2 = \mathrm{Var}(X_1) \leq \infty$.

(b) *The claim times (point process):*
the claims occur at the random instances of time

$$0 < T_1 < T_2 < \dots .$$

(c) *The claim arrival process (counting process):*
the number of claims in the interval $[0, t]$ *is denoted by*

$$N(t) = \sup\{n \geq 1 : T_n \leq t\}, \quad t \geq 0,$$

where, by convention, $\sup \emptyset = 0$.

(d) *The inter-arrival times*

$$Y_1 = T_1, \quad Y_k = T_k - T_{k-1}, \quad k = 2, 3, \dots, \tag{9}$$

are i.i.d. exponentially distributed with finite mean $EY_1 = 1/\lambda$, $\lambda > 0$.

(e) *The sequences* (X_k) *and* (Y_k) *are independent of each other.*

Remark 1.11. The claim arrival process $(N(t))$ is a homogeneous Poisson process with intensity $\lambda > 0$, i.e.

$$P(N(t) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \quad k \in \mathbb{N}_0. \tag{10}$$

Remark 1.12. In the literature the Cramér-Lundberg model is also referred to as the classical risk model or the basic insurance risk model.

Definition 1.6 (The renewal model, [18, Section 1.1])**.**
The renewal model is given by conditions (a)-(c), (e) and

(d′) *the inter-arrival times* Y_k *given in* (9) *are i.i.d. with finite mean* $EY_1 = 1/\lambda$.

Remark 1.13. The only difference between the Cramér-Lundberg and the renewal model is that the process $(N(t))$ for the claim arrivals of the latter does not have to be a homogeneous Poisson process. It can be an arbitrary renewal counting process. This means that the Cramér-Lundberg model is a special case of the renewal model.

For the renewal model in general and for the Cramér-Lundberg model in particular, the *risk process* $U(t)$ and the *ruin probability* $\psi(u)$ are defined by

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \quad t \geq 0, \tag{11}$$

$$\psi(u) = P\Big(U(t) < 0 \quad \text{for some} \quad t \geq 0\Big), \quad u \geq 0, \tag{12}$$

where $u \geq 0$ is the *initial capital* and $c > 0$ is the *premium rate*. Note that the set $\{U(t) < 0 \text{ for some } t \geq 0\}$ is a measurable set. It follows from its alternative representation (1.8) in [18, Section 1.1]. In the literature one distinguishes the ruin probability in finite time (or with finite horizon) and the ruin probability in infinite time (or with infinite horizon). The former is denoted by $\psi(u, T)$ with

$\psi(u,T) = P\Big(U(t) < 0 \text{ for some } 0 \le t \le T\Big)$, $0 < T < \infty$, $u \ge 0$. In the thesis we deal with the ruin probability in infinite time, which is defined by (12).

To avoid ruin with probability 1 in renewal models the natural condition of solvency is always supposed: $c - \lambda\mu > 0$. This condition is usually known in applications as the basic *net profit condition* and given in the form: $\rho = \frac{c}{\lambda\mu} - 1 > 0$.

An interesting question is:

How to estimate the ruin probability?

This question is tightly bound with another one: How large does the income premium rate c have to be? The first guess is provided by the net-profit condition, but this is a rather coarse estimate. Since premiums have to be specified before any claims occur, there are some difficulties in finding a more accurate estimate for c. Furthermore, any insurance company can be ruined at any claim time. That is why it seems reasonable to take the ruin probability as a measure of validity of the value c. To be more precise, the premium rate c should be chosen so that the value of $\psi(u)$ is small for given u.

Then the next question appears: how can we speak about the "smallness" of the function defined by (12). The definition tells us nothing about the behavior of the function $\psi(u)$. Luckily, it was shown (see [18, Section 1.1, (1.10)]) that the non-ruin probability $1 - \psi(u)$ can be expressed as follows:

$$1 - \psi(u) = (1-\alpha)\sum_{n=0}^{\infty} \alpha^n H^{n*}(u), \tag{13}$$

with some constant $\alpha \in (0,1)$ and some distribution function H. How to find these H and α is described in [24, Sections XII.3 and XVIII.3]. From (13) it follows that $1 - \psi$ can be interpreted as the distribution function of a random sum (for more details see considerations after Theorem 1.8). Representation (13) for the non-ruin probability holds for all renewal models. Moreover, for the Cramér-Lundberg model the function H and the constant α were explicitly found (see [18, Section 1.2]):

$$1 - \psi(u) = \frac{\rho}{1+\rho}\sum_{n=0}^{\infty}(1+\rho)^{-n} F_I^{n*}(u), \tag{14}$$

where

$$F_I(x) = \frac{1}{\mu}\int_0^x (1 - F(y))dy, \quad x \ge 0 \tag{15}$$

denotes the *integrated tail distribution*. This representation is known as Pollaczek-Khinchin formula.

If the *Cramér-Lundberg condition* holds, i.e. if there exists some $v > 0$ such that

$$\int_0^{+\infty} e^{vx} dF_I(x) = \frac{c}{\lambda\mu} = \rho + 1, \quad \rho > 0, \tag{16}$$

then the ruin probability $\psi(u)$ in the Cramér-Lundberg model can be estimated as follows (for a proof see [18, Theorem 1.2.2]):

$$\psi(u) \le e^{-vu} \quad \text{for all} \quad u \ge 0. \tag{17}$$

Note that if v in (16) exists, then it is uniquely determined.

A result similar to (17) was also obtained for the renewal model (see [20]). Inequality (17) gives a good estimate of $\psi(u)$ even for relatively small u. But for subexponential integrated tails F_I, which fit real insurance data very well, it is easy to see from Lemma 1.3 (i) that the Cramér-Lundberg condition (16) does not hold.

If the distribution function F of claim sizes X_i satisfies the Cramér-Lundberg condition, then the corresponding risk processes are called risk processes with "small claims". Risk processes with F such that F_I is subexponential are referred to as risk processes with "large claims". Figure 1 demonstrates the validity of such names.

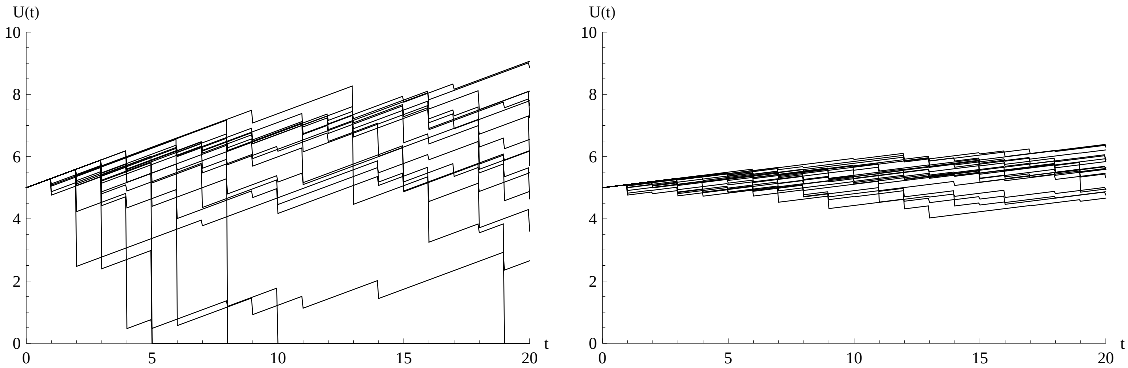

Figure 1: Some realizations of risk processes $U(t)$ for large (Pareto, left) and small (exponential, right) claim sizes.

For risk processes with "large claims" inequality (17) for the ruin probability does not hold. For such occasions the following theorem was obtained.

Theorem 1.7 (The Cramér-Lundberg theorem, [18, Theorem 1.3.8]).
Consider the Cramér-Lundberg model with net profit condition $\rho > 0$. Then the following statements are equivalent:

(i) $F_I \in \mathcal{S}$,

(ii) $1 - \psi \in \mathcal{S}$,

(iii) $\lim_{u\to+\infty} \dfrac{\psi(u)}{1 - F_I(u)} = \rho^{-1}$.

Remark 1.14. The idea of the proof is the following (for a detailed proof see [18, Section 1.3]). Representation (14) implies

$$\frac{\psi(u)}{1 - F_I(u)} = \frac{\rho}{1+\rho} \sum_{n=0}^{\infty} (1+\rho)^{-n} \frac{1 - F_I^{n*}(u)}{1 - F_I(u)}. \tag{18}$$

Then using property (5) of subexponential distributions, Lemma 1.3 (iii) and Lebesgue's dominated convergence theorem (to interchange the limit and the infinite sum in (18)) we get equality (iii) of Theorem 1.7.

The following result is a generalization of Theorem 1.7 for the renewal risk model.

Theorem 1.8 ([30])**.** *Consider the renewal risk model with net profit condition $\rho > 0$. Then the following two statements are equivalent:*

(i) $F_I \in \mathcal{S}$,

(ii) $1 - \psi \in \mathcal{S}$.

If (i) or (ii) holds, then $\lim\limits_{u\to+\infty} \dfrac{\psi(u)}{1-F_I(u)} = \rho^{-1}$.

Theorem 1.7 and Theorem 1.8 give us only the first-order asymptotic result for the ratio $\psi(u)/(1-F_I(u))$. The approximation $\psi(u) \approx \rho^{-1}(1-F_I(u))$ is acceptable only for very large u, which is hardly possible in practice. As Embrechts et al. [18, Section 1.4] write, such estimate for ψ is "however mainly of theoretical value" in comparison with estimate (17). Therefore, it would be useful for insurance companies to get more precise asymptotic behavior of the ratio $\psi(u)/(1 - F_I(u))$. This task is connected to our problem under consideration, namely, to the behavior of ratio (2).

Indeed, using (14) and (15) we can interpret the non-ruin probability $1 - \psi(u)$ as the distribution function of the sum $S^*_{\nu^*} = X^*_1 + X^*_2 + \cdots + X^*_{\nu^*}$, $S^*_0 = 0$, where $X^*_1, X^*_2, \ldots$ are i.i.d. random variables with common distribution function $F_I(u)$ and ν^* is a counting random variable with distribution $P(\nu^* = n) = \rho\,(1+\rho)^{-(n+1)}$ for $n \in \mathbb{N}_0$, which is independent of $X_1, X_2, \ldots$. In other words, according to (1) we have $1-\psi(u) = K^*_{\nu^*}(u)$. Then $\psi(u)/(1-F_I(u))$ from Theorem 1.7 and Theorem 1.8 can be written as

$$\frac{\psi(u)}{1-F_I(u)} = \frac{1-K^*_{\nu^*}(u)}{1-F_I(u)},$$

which is a particular case of expression (2).

Thus, it is no coincidence that such an "unnaturally-looking" ratio (2) was chosen for our investigation. The method of analyzing the data in insurance is quite different compared to the analysis of usual statistical data. In the latter case the possibility of very large events, which can be found on the fast decreasing tail of K_ν, is considered as negligible. In the insurance models with heavy-tailed data this is not allowed, since the tails contain rare but very influential events. This makes the influence of the tail of the distribution much more significant. The fact that the ruin probability itself is the tail of the random sum defined above corroborates this significance. Thus, the tail of a random sum, such as $1 - K_\nu$ with K_ν from (1), is of interest in insurance. But we investigate not the tail itself but the quotient of it to the tail of one random variable. Such consideration of the problem gives the possibility to use known properties of subexponentiality. In addition, such a pattern as ratio (2) is often seen in some already obtained results in insurance (for example, Theorems 1.7 and Theorem 1.8). This means that the research of the general problem about the behavior of (2) provides the ability to obtain better estimates for the ruin probability in the basic insurance models.

We shall return to random sums and the Cramér-Lundberg model as an application of our main theorem in Chapter 5.

2 Analysis of previous research

2.1 Subexponential and regularly varying distributions

In this section we will give a short review of the results that were obtained concerning our problem under consideration.

Recall that we consider a compound sum $S_\nu = X_1 + X_2 + \cdots + X_\nu$, $S_0 = 0$, of i.i.d. nonnegative random variables $X, X_1, X_2, \ldots$ with common *subexponential* distribution function F (i.e. $F \in \mathcal{S}$) and a nonnegative integer-valued counting random variable ν, which is supposed to be independent of $X_1, X_2, \ldots$. Let $p_n = P(\nu = n)$ with $\sum_{n=0}^{\infty} p_n = 1$. We already know from (1) that the distribution function K_ν of the sum S_ν can be expressed as follows:

$$K_\nu(x) = P(S_\nu \le x) = \sum_{n=0}^{\infty} p_n F^{n*}(x), \; x \in \mathbb{R}. \tag{19}$$

We are interested in the behavior of the quotient

$$\frac{1 - K_\nu(x)}{1 - F(x)} \quad \text{as} \quad x \to \infty. \tag{20}$$

The following theorem gives the first-order result for the problem under consideration.

Theorem 2.1. *Let $(p_n)_{n \in \mathbb{N}_0}$ be a distribution of a random variable ν and let ν be independent of $X_1, X_2, \ldots$. Suppose that for some $\varepsilon > 0$ we have*

$$\sum_{n=0}^{\infty} p_n (1 + \varepsilon)^n < \infty. \tag{21}$$

If $F \in \mathcal{S}$, then

$$\lim_{x \to +\infty} \frac{1 - K_\nu(x)}{1 - F(x)} = E\nu. \tag{22}$$

Remark 2.1. This result was obtained by Chover et al. [8, Theorem 4] in a more general setting. The proof is similar to the proof of Theorem 1.7. Condition (21), Lemma 1.3 (*iii*), property (5) of subexponential distributions and Lebesgue's dominated convergence theorem provide (22).

Remark 2.2. Condition (21) is equivalent to the condition that the moment generating function of ν is finite in a neighborhood of the origin, i.e. $Ee^{t\nu} < \infty$ for $|\, t| < \varepsilon$ for some $\varepsilon > 0$.

Below we give some examples of the distributions, for which condition (21) holds.

Example 2.1. Let us consider a Poisson-distributed random variable ν with parameter $\lambda \in \mathbb{R}_+$, i.e. $p_n = P(\nu = n) = e^{-\lambda} \lambda^n / n!$ for all $n \in \mathbb{N}_0$. Then for any fixed $\varepsilon > 0$ we have

$$\sum_{n=0}^{\infty} p_n (1 + \varepsilon)^n = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda\,(1 + \varepsilon))^n}{n!} = e^{-\lambda}\, e^{\lambda(1+\varepsilon)} = e^{\lambda\,\varepsilon} < \infty.$$

This means that condition (21) holds for the Poisson distribution. □

Example 2.2. For a random variable ν^* considered at the end of Section 1.3 with $p_n = P(\nu^* = n) = \rho(1+\rho)^{-(n+1)}$, $n \in \mathbb{N}_0$, $\rho \in \mathbb{R}_+$, condition (21) is also satisfied. Indeed, for all $0 < \varepsilon < \rho$ we obtain

$$\sum_{n=0}^{\infty} p_n (1+\varepsilon)^n = \frac{\rho}{1+\rho} \sum_{n=0}^{\infty} \left(\frac{1+\varepsilon}{1+\rho}\right)^n = \frac{\rho}{\rho - \varepsilon} < \infty.$$

□

Relation (22) gives us a first-order result for $\big(1-K_\nu(x)\big) \big/ \big(1-F(x)\big)$ as $x \to \infty$. This means that $1 - K_\nu(x)$ can be estimated by $E\nu \big(1 - F(x)\big)$ for large x. The question is: How good is this estimate? The quality of such an approximation is characterized by the rate of convergence of

$$\Delta(x) := \frac{1 - K_\nu(x)}{1 - F(x)} - E\nu \quad \text{to} \quad 0 \quad \text{as} \quad x \to \infty\,. \tag{23}$$

In general, without some additional conditions on $F \in \mathcal{S}$ we cannot predict the behavior of $\Delta(x)$, except that $\Delta(x) \to 0$ as $x \to \infty$, which is provided by Theorem 2.1. The next step in the investigation is an attempt to obtain some estimations of $\Delta(x)$ considering not all subexponential distributions but only a subclass of them, namely, distributions with regularly varying tails. This subclass is rather popular for modeling heavy-tailed phenomena. The idea of regular variation was introduced by Karamata in 1930, [34]. For an encyclopedic treatment of regular variation see Bingham et al. [7], de Haan [14], Feller [24] or Seneta [48].

Definition 2.2 (Regular variation, [18, Appendix 3.1])**.**
A positive measurable function h on $(0,\infty)$ is regularly varying at infinity of index $\alpha \in \mathbb{R}$ (we write $h \in \mathcal{RV}_\alpha$) if

$$\lim_{x \to +\infty} \frac{h(tx)}{h(x)} = t^\alpha, \quad \textit{for all} \quad t > 0. \tag{24}$$

Remark 2.3. If a function L is regularly varying of index $\alpha = 0$, i.e. $L \in \mathcal{RV}_0$, then we say that L is a *slowly varying* function.

Remark 2.4. Regular variation defined above is called regular variation in Karamata's sense.

Remark 2.5. If (24) holds, then it holds uniformly on each compact subset of $(0,\infty)$.

The following lemma gives an equivalent description of regular variation.

Lemma 2.3. *A positive measurable function h on $(0,\infty)$ is regularly varying at infinity of index $\alpha \in \mathbb{R}$ if and only if*

$$h(x) = x^\alpha\, L(x), \quad x > 0, \tag{25}$$

where L is a slowly varying function.

Proof. See [48, Section 1.1]. □

Remark 2.6. Regular variation may be defined not only at infinity, but at any $a \in \mathbb{R}$. In the thesis we use regular variation only at infinity. Therefore in what follows we will say "regularly varying" and mean "regularly varying at infinity".

Remark 2.7. The property of regular variation depends only on the behavior at infinity and it is therefore not necessary for $h(x)$ to be positive, or even defined, for all $x > 0$, [24, Section VIII.8].

Example 2.3 ([18, Appendix 3.1])**.**
Positive constants, functions converging to a positive constant, logarithms and iterated logarithms are slowly varying functions. Typical examples of regularly varying functions of index α are the following:

$$x^\alpha, \qquad x^\alpha \log(\log(e+x)), \qquad x^\alpha(\log(1+x))^\gamma,\ \gamma \in \mathbb{R}.$$

□

Definition 2.4 (Distributions with regularly varying tails)**.**
We say that the distribution function F has regularly varying (right) tail of index $-\alpha$ if $1 - F \in \mathcal{RV}_{-\alpha}$, $\alpha > 0$, i.e. if

$$1 - F(x) = x^{-\alpha} L(x), \quad x > 0, \tag{26}$$

where L is a slowly varying function.

Remark 2.8. If the distribution function F has regularly varying *left* tail of index $-\alpha$, $\alpha > 0$, then instead of (26) we have

$$F(-x) = x^{-\alpha} L(x), \quad x > 0, \tag{27}$$

where L is a slowly varying function. Generally we consider nonnegative random variables. Their distribution functions only have nontrivial right tails. Therefore, as it was said in Section 1.1, we write "tail" and mean "right tail" unless otherwise specified.

Lemma 2.5. *Each distribution with a regularly varying tail is subexponential and thereby heavy-tailed.*

Proof. Subexponentiality is proved in [30], and heavy-tailedness follows from subexponentiality and from Lemma 1.3 (i). □

Example 2.4. The standard Cauchy distribution function

$$F(x) = \frac{1}{\pi} \arctan x + \frac{1}{2}, \quad x \in \mathbb{R},$$

from Example 1.2 has regularly varying tail of index -1. This follows from the following representation as $x \to +\infty$:

$$1 - F(x) = \frac{1}{2} - \frac{1}{\pi} \arctan x = x^{-1} L(x) = \frac{1}{\pi x} + O\left(\frac{1}{x^3}\right), \tag{28}$$

since $L(x) = x\,(1/2 - 1/\pi \arctan x)$ is a slowly varying function. Since the standard Cauchy distribution is symmetric, then it is clear that the left tail of its distribution function is also regularly varying with the same index. □

Example 2.5. Pareto distribution function F with

$$1 - F(x) = \begin{cases} (\kappa/x)^\alpha & \text{for} \quad x \geq \kappa, \\ 1 & \text{for} \quad x < \kappa, \end{cases} \qquad \kappa > 0,\ \alpha > 0$$

has regularly varying tail of index $-\alpha$. A slowly varying function L from representation (26) can be chosen as $L(x) = \kappa^\alpha$, $x > 0$. □

Example 2.6. Consider the *Lévy distribution* with density function p:

$$p(x) = \frac{1}{2\sqrt{\pi}}\, e^{-\frac{1}{4x}}\, x^{-\frac{3}{2}}, \quad x > 0.$$

The corresponding distribution function F can be represented in the following way:

$$1 - F(x) = \frac{1}{\sqrt{\pi}\,\sqrt{x}} + O(x^{-3/2}) \quad \text{as} \quad x \to \infty. \tag{29}$$

According to Definition 2.4 the latter means that the Lévy distribution has regularly varying tail with of index $-1/2$. □

Remark 2.9. Notice that Pareto, Lévy and Cauchy distributions not only have regularly varying tails, but also regularly varying densities. This is easy to show using Definition 2.2.

Many papers have been devoted to the study of the behavior of the difference $R(x) = 1 - K_\nu(x) - E\nu\big(1 - F(x)\big)$ in the case of regularly varying F. For example, see Baltrūnas, Omey [4], [5]; Geluk [28]; Omey, Willekens [41], [42], [43].

Below we give some results due to Omey and Willekens, which can be useful for the investigation of the behavior of $\Delta(x)$. They concern *nonnegative* absolutely continuous random variables with regularly varying density p. Recall from (19) that

$$K_\nu(x) = P(S_\nu \leq x) = \sum_{n=0}^{\infty} p_n F^{n*}(x),\ x \in \mathbb{R},$$

where $S_\nu = X_1 + X_2 + \cdots + X_\nu$, $S_0 = 0$, with i.i.d. nonnegative random variables $X, X_1, ..., X_\nu$ with common distribution function F and a nonnegative integer-valued counting random variable ν with distribution $p_n = P(\nu = n)$.

Theorem 2.6 (Omey, Willekens, [43])**.**
Assume that $\sum_{n=0}^{\infty} p_n(1+\varepsilon)^n < \infty$ *for some* $\varepsilon > 0$ *and* $\mu = EX < \infty$. *If* F *has a continuous density* $p \in \mathcal{RV}_{-\beta}$ *with* $\beta > 1$, *then*

$$\lim_{x\to\infty} \frac{1 - K_\nu(x) - E\nu\big(1 - F(x)\big)}{p(x)} = \mu\, E\big(\nu(\nu-1)\big). \tag{30}$$

Remark 2.10. In the paper [43] Omey and Willekens require the condition of analyticity of the function $P_\nu(z) := \sum_{n=0}^{\infty} p_n z^n$ at $z = 1$ instead of $\sum_{n=0}^{\infty} p_n(1+\varepsilon)^n < \infty$ for some $\varepsilon > 0$. These two conditions are equivalent (see [18, Remark on p. 45]).

Remark 2.11. If the conditions $p \in \mathcal{RV}_{-\beta}$ and $\mu < \infty$ in Theorem 2.6 are satisfied, then the condition $\beta \geq 2$ holds true (this follows from [18, Proposition A.3.8]).

The following theorem is an analogue of Theorem 2.6, for the case when the expectation of a random variable X is infinite. The asymptotic result below is quite different from the finite mean case, although the techniques of the proof are similar to those used in proving Theorem 2.6.

Theorem 2.7 (Omey, Willekens, [41]).
Assume that $\sum_{n=0}^{\infty} p_n(1+\varepsilon)^n < \infty$ *for some* $\varepsilon > 0$. *If* F *has a density* $p \in \mathcal{RV}_{-\beta}$ *with* $1 < \beta < 2$, *then*

$$\lim_{x\to\infty} \frac{1 - K_\nu(x) - E\nu\left(1 - F(x)\right)}{p(x)\int_0^x (1 - F(y))dy} = c(\beta) E\Big(\nu(\nu - 1)\Big) \tag{31}$$

with

$$c(\beta) = -\frac{(2-\beta)\,(3-2\beta)\,(\Gamma(2-\beta))^2}{(\beta-1)\,\Gamma(4-2\beta)}, \qquad \beta \in (1,2). \tag{32}$$

Remark 2.12. Theorem 2.6 and Theorem 2.7 provide the first-order results for $\Delta(x)$, and, consequently, the second-order results for $(1 - K_\nu(x))/(1 - F(x))$. Namely, if $\mu < \infty$, then for large x

$$\Delta(x) = \frac{1 - K_\nu(x)}{1 - F(x)} - E\nu \;=\; \mu\, E\Big(\nu(\nu-1)\Big)\frac{p(x)}{1 - F(x)}\;(1 + o(1)), \tag{33}$$

and in the case $\mu = \infty$ we have

$$\Delta(x) \;=\; c(\beta)\, E\Big(\nu(\nu-1)\Big)\frac{p(x)\int_0^x (1 - F(y))dy}{1 - F(x)}\;(1 + o(1)) \tag{34}$$

with $c(\beta)$ from (32).

Remark 2.13. Much less is known about the case $\mu = \infty$ in comparison with the case $\mu < \infty$. Therefore, asymptotic equalities like (31) play a very important role.

Remark 2.14. If $\beta = 3/2$ in Theorem 2.7, then $c(\beta) = 0$ and (31) does not yield the exact asymptotic behavior of $1 - K_\nu(x) - E\nu\left(1 - F(x)\right)$. Omey and Willekens [41] improved this result in the case of a stable distribution function F (for a definition see Section 3.1).

Since the theorems of Omey and Willekens assume regularly varying densities, the natural question arises whether all distribution functions with regularly varying tails have regularly varying densities (in the case when densities exist). Unfortunately, the answer is No.

Example 2.7. Let us consider a nonnegative random variable X with density function p:

$$p(x) = \frac{\alpha D(1 - \cos x)}{x^{1+\alpha}}, \quad x \geq 0, \qquad \alpha \in (0,1), \tag{35}$$

where

$$D = \left(\int_0^{+\infty} \frac{\alpha(1 - \cos x)}{x^{1+\alpha}}\, dx\right)^{-1} = \frac{1}{\Gamma(1-\alpha)\,\cos(\pi\alpha/2)}.$$

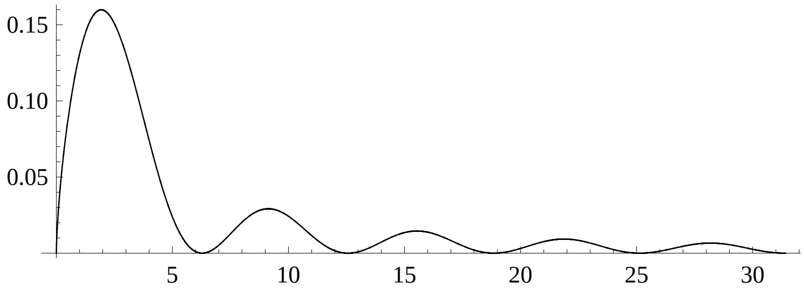

Figure 2: Density function $p(x)$ for $\alpha = 1/3$.

Unfortunately, Theorems 2.6 and 2.7 cannot be applied for random variable X, since $p(x) = 0$ at each point $x = 2\pi n$, $n \in \mathbb{N}_0$ and therefore p is not regularly varying. This fact is also illustrated by Figure 2, where the density function p for $\alpha = 1/3$ is represented.

Note that the distribution function F of X can be written in the form:

$$1 - F(x) = \frac{D}{x^{\alpha}} + \frac{\alpha D \sin x}{x^{1+\alpha}} + O\left(\frac{1}{x^{2+\alpha}}\right), \quad x \to \infty.$$

The latter means that the distribution function of X has a regularly varying tail although the density function is not regularly varying. This fact indicates that the assumptions of Theorems 2.6 and 2.7 might be too restrictive. □

We will give one more example in order to show that there are random variables with discontinuous densities, whose distribution functions are regularly varying although their density functions are not.

Example 2.8. Let us consider a nonnegative random variable X with density function p:

$$p(x) = \begin{cases} 0, & \text{for} \quad x < 0, \\ 1 - A, & \text{for} \quad 0 \le x < 1, \\ \dfrac{1}{2\sqrt{\pi}} x^{-3/2} \left(1 - \sum\limits_{k=2}^{\infty} \mathbb{1}_{\left[k - \frac{1}{k^2}; k + \frac{1}{k^2}\right)}(x)\right), & \text{for} \quad x \ge 1. \end{cases} \tag{36}$$

where $A\,(\approx 0.492)$ is chosen such that $\int_{-\infty}^{+\infty} p(x)dx = 1$. The graph of this function is presented in Figure 3.

The density function of X is equal to 0 at each point $x = n$, $n \in \mathbb{N} \setminus \{1\}$ and is therefore not regularly varying. But the corresponding distribution function allows the following representation for large x:

$$\begin{aligned} 1 - F(x) &= \int_x^{\infty} p(y)dy = \frac{1}{2\sqrt{\pi}} \left(\int_x^{\infty} \frac{1}{y^{3/2}}\, dy - \sum_{k=2}^{\infty} \int_x^{\infty} \frac{1}{y^{3/2}} \mathbb{1}_{\left[k - \frac{1}{k^2}; k + \frac{1}{k^2}\right)}(y)\, dy \right) \\ &= \frac{1}{\sqrt{\pi}\sqrt{x}} - \frac{1}{\sqrt{\pi}} \sum_{k=k^*}^{\infty} \frac{2k}{\sqrt{k^6 - 1}\left(\sqrt{k^3 + 1} + \sqrt{k^3 - 1}\right)} \\ &= \frac{1}{\sqrt{\pi}\,\sqrt{x}} + O(x^{-5/2}), \quad x \to \infty, \qquad k^* = \min\left\{k \in \mathbb{N} : k - \frac{1}{k^2} \ge x\right\}. \end{aligned}$$

This means that the distribution function of X has a regularly varying right tail. □

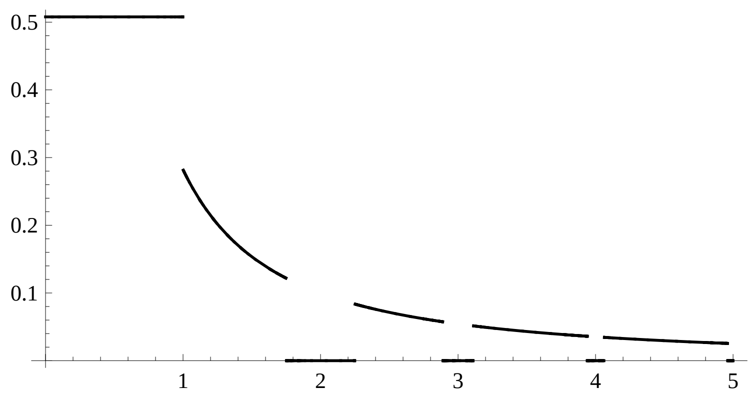

Figure 3: Density function $p(x)$.

Next, the other (even more general) question arises whether an analogue of Theorem 2.7 holds for a distribution function without density or for densities that are not regularly varying. As Omey and Willekens wrote in [41]: "in view of the dependence of the limit on α this question seems to be nontrivial".

Further investigation of this problem is due to Mikosch and Nagaev [40]. They showed that for random variables without a regularly varying density "uncritical" use of the approximation of $1 - K_\nu(x)$ by $E\nu\big(1 - F(x)\big)$ provided by Theorem 2.1 even in the case of a regularly varying distribution function F can be problematic.

In [40] an example of a distribution function F with regularly varying tails is presented, for which the convergence rate in (23) is arbitrarily slow. This means that in order to get more information about the behavior of $\Delta(x)$ some additional conditions (besides regular variation) are required. As an example consider the following theorem by Mikosch and Nagaev (for a proof see [40]).

Theorem 2.8 ([40]). *Assume that a distribution function F of a positive random variable X with finite mean $\mu < \infty$ satisfies the following conditions:*

(i) $\displaystyle\limsup_{x\to\infty} \frac{1 - F(cx)}{1 - F(x)} < \infty$ *for all* $c \in (0,1)$*;*

(ii) $\displaystyle\liminf_{x\to\infty} x^\alpha(1 - F(x)) > 0$ *for some* $\alpha > 1$*;*

(iii) F *has density* p *which is non-increasing on* $[x_0, \infty)$, $0 < x_0 < \infty$.

If for random variable ν *with* $p_n = P(\nu = n)$ *condition* (21) *holds, then*

$$\Delta(x) = \frac{1 - K_\nu(x)}{1 - F(x)} - E\nu = O(x^{-1}), \quad x \to \infty.$$

Remark 2.15. Distribution functions that satisfy condition (i) from Theorem 2.8 are said to be of dominated variation. Actually, it is enough to check the limit inequality from (i) only for $c = 0.5$. If it holds for $c = 0.5$, then it holds for all $0 < c < 1$ (for a proof see [7] or [23, Corollary 2.0.6]).

Remark 2.16. If the distribution function F has a regularly varying tail of non-positive index, i.e. $1 - F \in \mathcal{RV}_{-\alpha}$, $\alpha \geq 0$, then condition (i) from Theorem 2.8 is satisfied (for a proof see [19] or [37]).

Remark 2.17. Theorem 2.8 is not applicable for the Lévy distribution from Example 2.6 and for the Pareto distribution with $\alpha \in (0, 1]$ from Example 1.3, since the expectations of the corresponding random variables are infinite. The same holds for the random variables with infinite expectations from Examples 2.7 and 2.8. Note that for both of them condition (iii) is also not satisfied.

Theorem 2.8 provides a rather good rate of convergence for $\Delta(x)$ in (23). But at the same time it requires some strong conditions. This makes the theorem applicable only for relatively "smooth" functions. Furthermore, as before the problem of obtaining estimates of $\Delta(x)$ for random variables without density is not solved.

In the next section we will give some results, which do not require existence of the density of random variables. These results are motivated by results of Christoph from [9], [10] and concern a popular subclass of regularly varying functions, namely, the class of Pareto-like distributions.

2.2 Pareto-like distributions

In this section we give a definition of a Pareto-like distribution, explain why it is chosen for further investigation and formulate some results concerning the asymptotic behavior of $\Delta(x)$ in the case if $F(x)$ is Pareto-like.

As was already said, if we consider a regularly varying function F, which is not necessarily continuous (or whose density is not necessarily regularly varying), the rate of convergence of $\Delta(x)$ to 0 can be arbitrarily slow. Such examples may be constructed by choosing a slowly varying function $L(x)$ from representation (26) in a special way (for details see [40]). In order to exclude such cases some conditions must be imposed on the slowly varying function. One class of distribution functions with a special form of $L(x)$ has been popular recently, namely, the class of *Pareto-like* distributions.

Definition 2.9 (Pareto-like distribution).
We say that a nonnegative random variable is Pareto-like distributed with parameter $\alpha > 0$ *if its distribution function* F *can be represented in the following form:*

$$1 - F(x) = C(\alpha)\, x^{-\alpha} + O(x^{-r}), \quad \text{as} \quad x \to \infty \tag{37}$$

for some $r > \alpha$ *and some* $C(\alpha) > 0$.

Remark 2.18. A Pareto-like distribution is a distribution with a regularly varying tail of index $-\alpha$ with a slowly varying function $L(x) = C(\alpha) + O(x^{-(r-\alpha)})$.

Example 2.9. Typical examples of Pareto-like distributions with parameter α are Pareto distributions themselves (defined in Example 1.3) with the same parameter. According to representation (29) the Lévy distribution from Example 2.6 is also Pareto-like with parameter $\alpha = 1/2$. □

For a Pareto-like distribution function F with parameter $\alpha > 0$ define

$$u_\alpha(x) := 1 - F(x) - C(\alpha)\, x^{-\alpha}, \quad C(\alpha) > 0.$$

Christoph has obtained the following results in terms of the function $u_\alpha(x)$.

Theorem 2.10 ([10]). *Suppose* $1 < \alpha < 2$,

$$\int_z^\infty x^{[r]}\,|du_\alpha(x)| \;=\; O(z^{[r]-r}) \quad as \quad z \to \infty \tag{38}$$

for some $r \in (1+\alpha,\, 2\alpha]$, *and additionally, in case* $r \in \mathbb{N}$,

$$\left|\int_0^z x^r\, du_\alpha(x)\right| < \infty \quad for\ all \quad z > 0. \tag{39}$$

If $E\nu^3 < \infty$ *and* $\mu = EX$, *then*

$$\Delta(x) = \frac{P\left(S_\nu > x\right)}{P(X > x)} - E\nu = \frac{\alpha\mu\left(E\nu^2 - E\nu\right)}{x} + O\left(x^{-(r-\alpha)}\right) \ \text{as } x \to \infty. \tag{40}$$

Theorem 2.11 ([9]). *Suppose* $0 < \alpha < 2$, $\alpha \neq 1$. *Let* (38) *hold for some* $r \in (\alpha, \min\{2\alpha, 1+\alpha\}]$ *and additionally let* (39) *hold in case* $r \in \mathbb{N}$. *If* $E\nu^3 < \infty$, *then*

$$\Delta(x) = \frac{P\left(S_\nu > x\right)}{P(X > x)} - E\nu = O\left(x^{-(r-\alpha)}\right) \ \text{as } x \to \infty. \tag{41}$$

Remark 2.19. In case $\alpha \in (1, 2)$ relation (40) gives the exact first-order result for $\Delta(x)$, since the coefficient $\alpha\mu\left(E\nu^2 - E\nu\right)$ at x^{-1} vanishes only in the trivial cases $P(X = 0) = 1$ or $P(\nu = 1) = 1$.

Remark 2.20. For $\alpha \in (0, 1)$ and $\alpha < r \leq 2\alpha$ Theorem 2.11 can provide only the $O\left(x^{-(r-\alpha)}\right)$-behavior of $\Delta(x)$ as $x \to \infty$, where $0 < r - \alpha < 1$. Such a deterioration of quality is connected with the infiniteness of the expectation of X for $\alpha \in (0, 1)$. In general, infinite expectation of X is the reason, why some methods do not give any estimates of $\Delta(x)$ at all.

Remark 2.21. In the special case of $\alpha = 1/2$ Christoph [9] improved the asymptotic result (41) from Theorem 2.11. This improvement will be discussed in Section 5.1.

Below we consider some examples for which the asymptotic results are provided by Theorems 2.10 and 2.11. We compare these results to the asymptotics provided by Theorems 2.6 and 2.7 due to Omey and Willekens [41, 43] and Theorem 2.8 due to Mikosch and Nagaev [40].

Example 2.10. First, let us consider the most popular Pareto-like distribution, namely, the Pareto distribution with parameters $\alpha > 0$ and $\kappa > 0$ (see Example 1.3). If $\alpha \in (1, 2)$, then the expectation of a Pareto-distributed random variable X is finite and Theorems 2.6 and 2.10 give the same first-order result for $\Delta(x)$ as $x \to \infty$:

$$\Delta(x) = \frac{\alpha\,\mu\,E\Big(\nu(\nu-1)\Big)}{x} + O(x^{-\alpha}), \quad \text{where} \quad \mu = EX = \frac{\kappa\,\alpha}{\alpha - 1}.$$

In this case Theorem 2.8 provides less information, namely: $\Delta(x) = O(x^{-1})$.
In the case of infinite expectation of X, i.e. if $\alpha \in (0, 1)$, we have

$$\Delta(x) = \frac{c(\alpha,\, \kappa)}{x^\alpha} + o(x^{-\alpha}), \quad x \to \infty,$$

with

$$c(\alpha,\,\kappa) = \begin{cases} \dfrac{-\kappa^{\alpha}\,\Gamma^2(1-\alpha)\,E\Big(\nu(\nu-1)\Big)}{2\,\Gamma(1-2\alpha)} & \text{for} \quad \alpha \neq 1/2, \\ 0 & \text{for} \quad \alpha = 1/2, \end{cases} \tag{42}$$

according to Theorem 2.7, and only

$$\Delta(x) = O(x^{-\alpha}), \quad x \to \infty,$$

according to Theorem 2.11 with $r = 2\alpha$. □

Example 2.11. We consider a nonnegative random variable X from Example 2.8 with the density function p defined by (36). Note that p is not regularly varying. As was already shown, the distribution function F of X can be represented as follows:

$$1 - F(x) = \frac{1}{\sqrt{\pi}\,\sqrt{x}} + O(x^{-5/2}), \quad x \to \infty.$$

This means that X is Pareto-like distributed with $\alpha = 1/2$ and the conditions of Theorem 2.11 are satisfied with $r = 1$. Therefore, we have

$$\Delta(x) = \frac{P\left(S_\nu > x\right)}{P(X > x)} - E\nu = O(x^{-1/2}) \quad \text{as} \quad x \to \infty. \tag{43}$$

Theorems 2.6 and 2.7 are not applicable for this example, since the condition of regular variation of the density is not satisfied, and Theorems 2.8 and 2.10 are not applicable, since the expectation of X is infinite. □

Example 2.12 ([10])**.** Now let us consider an example of a random variable X without density. Let the distribution function F of X have the following form:

$$F(x) = \frac{1}{2\sqrt{\pi}}\,F_{3/2}(x) + \left(1 - \frac{1}{2\sqrt{\pi}}\right)\Pi(x), \quad x \geq 0,$$

where $F_{3/2}(x) = 1 - x^{-3/2}$ for $x > 1$ is the Pareto distribution function ($\alpha = 3/2$, $\kappa = 1$ in (7)) and $\Pi(x)$ is the standard Poisson distribution function with intensity 1. The function F has jumps at every integer $k \geq 0$. Among all theorems considered above only Theorems 2.10 and 2.11 are applicable, but the first one (with $\alpha = 3/2$, $r = 3$ and $\mu = 1 + 1/\sqrt{\pi}$) gives a more precise approximation of $\Delta(x)$, namely:

$$\Delta(x) = \frac{3\left(1 + \pi^{-1/2}\right)\,E\left(\nu(\nu-1)\right)}{2x} + O(x^{-3/2}) \quad \text{as} \quad x \to \infty.$$

□

Remark 2.22. The examples considered above show that Theorems 2.10 and 2.11 provide quite good results for random variables with regularly varying densities. More importantly, these theorems are also applicable in case of random variables with non-regularly-varying densities or even without densities.

Remark 2.23. Note that there are examples of distribution functions, for which Theorem 2.8 gives better results than Theorems 2.10 and 2.11, see [10].

As we can see, in different situations each of the theorems considered above (concerning the behavior of $\Delta(x)$) can give the best as well as the worst asymptotic result in comparison with the results obtained from the other considered theorems. Such a difference between the quality of asymptotic estimates can be explained by the difference of methods, which provided the corresponding results. In order to obtain Theorem 2.8 Mikosch and Nagaev approximated the n-fold convolution of F with the function F itself, using property (5) of subexponentiality of F:

$$1 - F^{n*}(x) \sim n(1 - F(x)) \quad \text{as} \quad x \to \infty,$$

whereas in order to obtain Theorems 2.10 and 2.11 Christoph used the approximation of the n-fold convolution of F with some stable distribution G (for details see the next section):

$$1 - F^{n*}(x) \sim n(1 - G(x)) \quad \text{as} \quad x \to \infty.$$

Though the asymptotic results for $\Delta(x)$ considered above have different quality in different situations, they still have something in common: the smaller the parameter α of Pareto-like (or regularly varying) distribution, the worse the asymptotic estimate of $\Delta(x)$. As we already noted, one reason for this is the infiniteness of the expectation of Pareto-like and regularly varying distributions with parameter $\alpha \in (0, 1]$. This makes some theorems non-applicable at all. But the theorems from above that can be applied for such α give us not much information about the behavior of $\Delta(x)$ either. They provide only first-order results, which usually are of the order $x^{-\alpha}$. If α is very small, then the convergence of $\Delta(x)$ to 0 is slow and the approximation of $1 - K_\nu(x)$ by $E\nu\,(1 - F(x))$ in applications is not very useful. This is another reason for the decrease of the quality of asymptotic results with the decrease of α.

For example, even for very "smooth" Pareto distribution with $\alpha \in (0, 1)$, $\alpha \neq \frac{1}{2}$, the best that we are able to obtain is the following (see Example 2.10):

$$\Delta(x) = \frac{c(\alpha,\, \kappa)}{x^\alpha} + o(x^{-\alpha}), \quad x \to \infty,$$

where $c(\alpha,\, \kappa)$ is given by (42). Roughly speaking, this information is "nothing" for small α. In Example 2.11 for Pareto-like distribution with $\alpha = 1/2$ the asymptotic result is even worse: we obtained only the O-estimate (43) for $\Delta(x)$. That is why Omey and Willekens [41] pointed out the importance of finding second order results for $\Delta(x)$. Unfortunately, they "have not been able to obtain second order results for arbitrary regular varying densities p", [41]. Nevertheless, they proved some theorems about the second order behavior for stable densities (for details see [41]). In the next section we will give more general results concerning stable distributions.

3 Limit theorems

3.1 Stable distributions. Connection with Pareto-like distributions

The aim of this section is to introduce the concept of stable distributions and explain, how it can help to approximate $\Delta(x)$.

We already mentioned in Section 2.2 that in order to obtain Theorems 2.10 and 2.11 Christoph used the approximation of the n-fold convolution F^{n*} of F with some stable distribution G as follows: $1 - F^{n*}(x) \sim n(1 - G(x))$ as $x \to \infty$. In this connection the following two questions naturally arise:

1) Why do we have to approximate F^{n*}?
2) Why are we able to approximate F^{n*} with some stable distribution?

We begin with the first question. Recall that (see (19))

$$\Delta(x) = \frac{1 - K_\nu(x)}{1 - F(x)} - E\nu = \sum_{n=0}^{\infty} p_n \frac{1 - F^{n*}(x)}{1 - F(x)} - E\nu.$$

This representation shows that the quality of approximation of $\Delta(x)$ depends on how well F^{n*} is approximated. In general, it is not easy to deal with convolutions of distribution functions, since they can not be expressed explicitly. Therefore, one tries to find the best approximation of F^{n*} in each particular situation. When dealing with subexponential distributions, it is natural to use property (5) for estimation of F^{n*}, i.e. $1 - F^{n*}(x) \sim n(1 - F(x))$ for large x. But for some kinds of distribution functions it is possible to obtain a better approximation of $F^{n*}(x)$ and, therefore, a better approximation of $\Delta(x)$. It can be done with the help of stable distributions.

At this point we are moving to the second question. How do stable distributions arise in this context? In order to understand this let us forget for a moment that we consider subexponential distributions. Recall that $F^{n*}(x)$ is the distribution function of the sum $X_1 + \cdots + X_n$ of i.i.d random variables with distribution function F. The first result that comes to mind of every probabilist when considering such sums is the *central limit theorem*. This theorem tells us that for i.i.d. random variables X_i with finite expectation μ and finite variance σ^2 we have

$$P\left(\frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \le x\right) \to \Phi(x) \quad \text{as} \quad n \to \infty,$$

where $\Phi(x)$ is the distribution function of the standard normal distribution $N(0, 1)$. This means that for large enough n we can approximate $F^{n*}(\sigma\sqrt{n}\,x + n\mu)$ with $\Phi(x)$ for any $x \in \mathbb{R}$. At this point we recall that we consider subexponential (in particular, Pareto-like) random variables. As the examples from the previous chapter show, variances (and expectations) of such random variables are often infinite. Therefore, the central limit theorem is not applicable. Does this mean that distribution functions of (suitably normalized) sums of i.i.d. random variables with infinite variance (or infinite expectation) could not converge to any distribution function? Luckily it does not. It turns out that the limit distribution can be not only normal, but any *stable distribution*. This fact and this class of distributions was discovered by Paul Lévy. Below we give a formal definition.

Definition 3.1 (Stable distribution, [31, Section 9.1]).
The distribution of a random variable X is stable if $X, X_1, X_2, \ldots$ are independent, identically distributed random variables, and there exist constants, $c_n > 0$, and $d_n \in \mathbb{R}$, $n \geq 1$, such that

$$X_1 + \cdots + X_n \stackrel{d}{=} c_n X + d_n \quad \textit{for all} \quad n. \tag{44}$$

Remark 3.1. The name "stable" of the class accounts for the fact that a sum of i.i.d. random variables has the same distribution as a linearly transformed summand, [31, Section 9.1].

Example 3.1. It is easy to see that the normal distribution $N(\mu, \sigma^2)$ is stable with $c_n = \sqrt{n}$ and $d_n = \mu(n - \sqrt{n})$ in (44). □

Let $S_n = X_1 + \cdots + X_n$. We consider the following sums:

$$\frac{S_n - a_n}{b_n}, \tag{45}$$

where (a_n) and (b_n) are some normalizing sequences such that $a_n, b_n \in \mathbb{R}$, $b_n > 0$, for all $n \in \mathbb{N}$. The following theorem describes the class of all possible limit distributions of normalized sums (45).

Theorem 3.2 ([44, Section IV.3, Theorem 10]).
The set of distributions that are limits of distributions of sums (45) *of i.i.d. random variables $X_1, ..., X_n$ coincides with the set of stable distributions.*

Remark 3.2. This theorem was proved independently by Lévy [38] and Khintchine [35].

Theorem 3.2 is mainly of theoretical value in comparison with the central limit theorem. In order to use this result in practice, first of all, we need an analytic representation of stable distributions. Second of all, we want to have a rule to make a decision with which concrete stable distribution the distribution function of sums (45) can be approximated in each particular situation. The solution of the first problem is given by the following theorem.

Theorem 3.3 (Canonical representation of stable distributions I, [18, § 2.2]).
The distribution function $G(x)$ is stable if and only if its characteristic function can be represented by the formula

$$g(t) = \exp\left\{i\,\gamma^*\,t - \lambda^*\,|t|^\alpha\,\left(1 - i\,\beta^*\,\omega^*(t,\alpha)\,\operatorname{sign} t\right)\right\}, \tag{46}$$

where $\alpha, \beta^*, \gamma^*, \lambda^*$ are parameters such that $\alpha \in (0, 2]$, $\beta^* \in [-1, 1]$, $\lambda^* \in [0, \infty)$, $\gamma^* \in \mathbb{R}$ and

$$\omega^*(t, \alpha) = \begin{cases} \tan(\pi\alpha/2) & \textit{if} \quad \alpha \neq 1, \\ -\dfrac{2}{\pi}\,\ln|t| & \textit{if} \quad \alpha = 1. \end{cases} \tag{47}$$

Remark 3.3. This result was obtained by Khintchine and Lévy, [36].

Remark 3.4. The value $\lambda^* = 0$ corresponds to the degenerate distribution. Formally it must be included in the theorem, since every sequence (S_n) can be normalized and centered in such a way that it converges to a constant in probability, [18, § 2.2]. However, this trivial case is not of interest for us and will therefore be *excluded* from our consideration hereafter.

Remark 3.5. As we can see, stable distributions form a four-parametric family of functions. There exist some other commonly used parameterizations of the characteristic function of a stable distribution beside (46). One of them will be given below. The choice of the representation depends on the application, where stable distributions are needed. By changing the parametrization we can make our research easier or more difficult.

Remark 3.6. Any nondegenerate stable distribution has four parameters: the characteristic exponent α, the skewness parameter β^*, the scale parameter $\lambda^* > 0$, and the shift parameter γ^*. The most important parameter is the parameter α, since it determines the basic properties of distributions such as finiteness of moments, behavior of tails, the sequences (c_n) and (d_n) from (44).

Example 3.2. 1) Since the characteristic function of the normal distribution $N(\mu, \sigma^2)$ has the form $f(t) = \exp\{it\mu - t^2\sigma^2/2\}$, then from Theorem 3.3 it follows that $N(\mu, \sigma^2)$ is stable with parameters $(\alpha, \beta^*, \lambda^*, \gamma^*) = (2, 0, \sigma^2/2, \mu)$. The value $\alpha = 2$ always corresponds to the normal distribution. In this case $\omega^*(t, \alpha) = 0$ and the parameter β^* can be chosen arbitrarily (it is generally accepted to put $\beta^* = 0$). In other words, the class of normal distributions is a subclass of the stable distributions which depends only on two parameters (instead of four).
2) The standard Cauchy distribution (see Example 1.2) with characteristic function $f(t) = \exp\{-|t|\}$ is stable with parameters $(\alpha, \beta^*, \lambda^*, \gamma^*) = (1, 0, 1, 0)$.
3) According to formula (46) the Lévy distribution defined in Example 2.6 with characteristic function $f(t) = \exp\{-(\sqrt{2}/2)\, |t|^{1/2}\big(1 - i\ \mathrm{sign}(t)\big)\}$ is also stable with parameters $(\alpha, \beta^*, \lambda^*, \gamma^*) = (1/2, 1, \sqrt{2}/2, 0)$. □

For the investigation of some analytic properties of stable distributions it is more useful to consider another parametrization.

Theorem 3.4 (Canonical representation of stable distributions II, [12, § 1.1])**.** *The distribution function $G(x)$ is stable if and only if its characteristic function can be represented by the formula*

$$g(t) = \exp\left\{i\,\gamma\, t - \lambda\, |t|^\alpha\, \omega(t, \alpha, \beta)\right\}, \tag{48}$$

where $\alpha, \beta, \gamma, \lambda$ are parameters such that $\alpha \in (0, 2]$, $\beta \in [-1, 1]$, $\lambda \in [0, \infty)$, $\gamma \in \mathbb{R}$ and

$$\omega(\alpha) = \begin{cases} \exp\left(-i\,\dfrac{\pi}{2}\,\beta\, K(\alpha)\ \mathrm{sign}\, t\right) & \text{if} \quad \alpha \neq 1, \\ \pi/2 + i\beta \ln|t|\, \mathrm{sign}\, t & \text{if} \quad \alpha = 1. \end{cases} \tag{49}$$

with $K(\alpha) = \alpha - 1 + \mathrm{sign}(1 - \alpha)$.

Remark 3.7. For a proof and for a connection between the parameters $\beta^*, \lambda^*, \gamma^*$ from Theorem 3.3 and the parameters β, λ, γ from Theorem 3.4 see [12, § 1.1] or [51, Theorem C.3]. The parameter α is the same in both theorems.

Remark 3.8. All nondegenerate stable distributions are absolutely continuous (for a proof see Lemma 3.13).

Note that though an explicit form of the characteristic function of a stable distribution is found, explicit expressions for the stable densities in terms of elementary functions are known only in a few cases. These are the normal distribution, the Cauchy distribution, and the Lévy distribution (see Example 3.2).

In order to distinguish between stable distributions we will index distribution functions and characteristic functions as $G_{\alpha,\beta}(x;\lambda,\gamma)$ and $g_{\alpha,\beta}(t;\lambda,\gamma)$, respectively. In what follows we use parametrization (48).

For each nondegenerate stable distribution we define its domain of attraction as follows.

Definition 3.5 (Domain of attraction, [29, § 35])**.**
Let $X, X_1, X_2, \ldots$ be independent, identically distributed random variables with common distribution function F and partial sums S_n, $n \geq 1$. If for suitably chosen normalizing sequences (a_n) and (b_n) the distribution functions of sums (45) *converge as $n \to \infty$ to a distribution function $G_{\alpha,\beta}(x;\lambda,\gamma)$, i.e.*

$$P\left(\frac{S_n - a_n}{b_n} \leq x\right) \to G_{\alpha,\beta}(x;\lambda,\gamma) \quad \textit{as} \quad n \to \infty, \tag{50}$$

then we say that $F(x)$ is attracted to $G_{\alpha,\beta}(x;\lambda,\gamma)$. The set of distribution functions attracted to $G_{\alpha,\beta}(x;\lambda,\gamma)$ is called the domain of attraction of $G_{\alpha,\beta}(x;\lambda,\gamma)$ and is denoted by $DA(G_{\alpha,\beta}(\,.\,;\lambda,\gamma))$.

Remark 3.9. In this thesis we use the concepts "domain of attraction of distribution" and "domain of attraction of distribution function" as synonyms. Moreover, if we say that the random variable X is attracted to some distribution, than we mean that the distribution function of X is attracted to the considered distribution.

Remark 3.10. In general, for any distribution function we can define its domain of attraction in the same way as in Definition 3.5. But it is empty if this distribution function is not stable. From Theorem 3.2 it follows that only the stable distributions possess (non-empty) domains of attraction.

Remark 3.11. If convergence (50) takes place, then the sequence (b_n) must have the form $b_n = n^{1/\alpha}\, h(n)$, where $h(n)$ is a slowly varying function in the sense of Karamata (for a proof see [33, § 2.2, p. 46]).

Example 3.3. The central limit theorem states that each random variable X with finite mean μ and finite variance σ^2 is attracted to the standard normal distribution with $a_n = n\mu$ and $b_n = \sigma\sqrt{n}$. □

The classical limit theorems of probability (de Moivre-Laplace, Lévy) show that for the convergence to the normal distribution (which is a stable distribution with $\alpha = 2$) the most interesting case is the one with $b_n = an^{1/2}$ with some constant $a > 0$, [33, p. 91]. This fact caused the following definition.

Definition 3.6 (Domain of normal attraction).
We say that the distribution function $F(x)$ belongs to the domain of normal attraction of the distribution function $G_{\alpha,\beta}(x;\lambda,\gamma)$ (and write $F \in DNA(G_{\alpha,\beta}(\,.\,;\lambda,\gamma))$) if $F(x)$ is attracted to $G_{\alpha,\beta}(x;\lambda,\gamma)$ with a normalizing sequence (b_n) in (50) *such that $b_n = an^{1/\alpha}$ for some $a > 0$.*

Remark 3.12. The adjective "normal" in the definition above is equivalent to the adjective "natural" in some sense. To consider "parts" of domains of attraction with $b_n = an^{1/\alpha}$ is natural enough, since only for this choice of b_n any stable distribution function $G_{\alpha,\beta}(x;\lambda,\gamma)$ is attracted to itself.

Now we move to the second problem formulated after Theorem 3.2: For any given distribution function F to be able to decide whether it is attracted to some stable distribution and if it is, to which one. This problem is equivalent to determining the domains of attraction for each stable distribution. To determine the domain of attraction of stable $G_{\alpha,\beta}(x;\lambda,\gamma)$ means to find the necessary and sufficient conditions, which must be imposed on the function F, in order for convergence (50) to take place. This problem was solved completely in the 1930's. Below we give the results only for *domains of normal attraction*, since we use only them in what follows.

We distinguish two cases: the case of stable distribution with parameter $\alpha = 2$ and the case of stable distribution with $\alpha \in (0,2)$. The first case concerns the normal distribution as limit distribution.

Theorem 3.7 (Normal limit distribution, [33, Theorem 2.6.6]).
The random variable X with distribution function $F(x)$ belongs to the domain of normal attraction of the normal distribution $N(0,1)$ if and only if it has finite variance σ^2. In this case we can put $b_n = \sigma\sqrt{n}$.

If we get some stable distribution with $\alpha \in (0,2)$ as limit distribution, then we speak about non-normal limit distribution.

Theorem 3.8 (Non-normal limit distribution, [33, Theorem 2.6.7]).
The distribution function $F(x)$ belongs to the domain of normal attraction of the stable distribution $G_{\alpha,\beta}(x;\lambda,\gamma)$ with $b_n = an^{1/\alpha}$, $a > 0$ if and only if

$$\begin{cases} 1 - F(x) = \dfrac{c_1 a^\alpha}{x^\alpha} + o(x^{-\alpha}), \\ F(-x) = \dfrac{c_2 a^\alpha}{x^\alpha} + o(x^{-\alpha}), \end{cases} \qquad x > 0, \quad x \to \infty, \tag{51}$$

where c_1 and c_2 are nonnegative constants determined by the parameters α, β, λ such that $c_1 + c_2 > 0$.

Remark 3.13. Because of cumbersome expressions we do not give c_1 and c_2 from Theorem 3.8 explicitly. This information can be found in [33, § 2.6].

Remark 3.14. Similar results have also been obtained for domains of attraction in general (for a proof see [29, § 35], [33, § 2.6]).

Remark 3.15. From Theorems 3.7 and 3.8 we see the following. While the normal distribution attracts a very wide class of distributions, the domains of attraction of the other stable distributions consist only of those distribution functions whose behavior is similar to the behavior of the attracting distribution function, [29, § 35].

Finally we can see how Pareto-like distributions are connected with stable distributions. From Definition 2.9 and Theorem 3.8 it follows that each Pareto-like distribution function F with parameter $\alpha \in (0, 2)$ belongs to the domain of normal attraction of some $G_{\alpha,\beta}(x; \lambda, \gamma)$, i.e. $F \in \mathrm{DNA}(G_{\alpha,\beta}(\,.\,; \lambda, \gamma))$. If we consider Pareto-like F with $\alpha > 2$, then Theorem 3.7 states that $F \in \mathrm{DNA}(\Phi)$.

This means that for the n-fold convolution F^{n*} of Pareto-like F with $\alpha \in (0, 2)$ we have

$$F^{n*}(b_n x + a_n) \to G_{\alpha,\beta}(x; \lambda, \gamma), \quad \text{as} \quad n \to \infty,$$

where $b_n = an^{1/\alpha}$ with some constant $a > 0$.

At this point the same question as before arises: how good is the approximation

$$F^{n*}(b_n x + a_n) \approx G_{\alpha,\beta}(x; \lambda, \gamma)? \tag{52}$$

In order to answer this question we should provide estimates of remainder terms by such approximation. The next sections are devoted to this problem.

3.2 Remainder term estimates. Case $\alpha = 2$

In this section we discuss the quality of approximation of the distribution functions of normalized sums (45) with the normal limit distribution function. We are interested mostly in the case of non-normal limit distribution, i.e. in remainder term estimates of approximation (52) with $\alpha \in (0, 2)$. But let us start with the classical case which corresponds to the central limit theorem. Since this theorem is known for a long time, it is natural to find out what was done relative to the problem formulated above in this case and to ponder, whether the same methods are applicable in our case of consideration.

Let us consider a distribution function F from the domain of normal attraction of some stable distribution. Then we put $F_n(x) := F^{n*}(b_n x + a_n)$ with normalizing constants a_n and b_n, which depend on the stable distribution mentioned above. In the case of normal limit distribution $F_n(x) = F^{n*}(\sigma\sqrt{n}x + n\mu)$.

Theorem 3.9 (The Berry-Esseen Theorem, [31, Chapter 7, §6.1])**.**
Consider a random variable X with distribution function F, finite mean μ and finite positive variance σ^2. If $E|X|^3 < \infty$, then

$$\sup_x |F_n(x) - \Phi(x)| \le C\, \frac{E|X-\mu|^3}{\sigma^3\sqrt{n}}, \tag{53}$$

where C is a purely numerical constant which does not depend on F.

Remark 3.16. This result was obtained independently by Berry and Esseen. They also showed that the order relative to n in (53) can not be improved.

Remark 3.17. An exact value of the absolute constant C is still unknown. Esseen [21] proved that $C \ge 0.4097$. The upper bound is being constantly updated. The latest estimate is $C < 0.4748$, [49].

Under the same conditions as in Theorem 3.9 the following non-uniform bound for the difference $F_n - \Phi$ was obtained:

$$|F_n(x) - \Phi(x)| \le \widetilde{C}\ \frac{E|X-\mu|^3}{\sigma^3\sqrt{n}\ (1+|x|^3)},$$

where $\widetilde{C} = C + 8\,(1+e)$ with constant C from Theorem 3.9 (for details see [39] and [44, Section V.4]).

Since the central limit theorem is used mostly for approximation for finite n in applications, both mentioned results are needed for the justification of such use. But in some situations if n is not large enough, the error by such approximation can be significantly large. Therefore, it becomes necessary to consider some corrections to the limit distribution function. The most powerful and general method of finding such corrections is to consider the various asymptotic expansions for the distribution function $F_n(x)$, [29, Section 8]. The first such asymptotic expansion was suggested by Chebyshev. His idea was developed and led to the following result.

Theorem 3.10 ([44, Section VI.3]).
If $E|X|^r < \infty$ for some $r \ge 3$ and $\limsup_{|t|\to\infty} |f(t)| < 1$, then

$$(1+|x|)^r \left| F_n(x) - \Phi(x) - \sum_{k=3}^{[r]} Q_k(x) n^{-(k-2)/2} \right| = O(n^{-(r-2)/2}), \quad n \to \infty, \qquad (54)$$

where for each integer $k \ge 3$ the function Q_k depends on the moments EX^m with $1 \le m \le k$.

Remark 3.18. The explicit form of Q_k is known (see [29, Section 8] or [44, Section VI.1]). But because of its cumbersome expression we do not give it here.

Remark 3.19. Cramér's condition $\limsup_{|t|\to\infty} |f(t)| < 1$ from Theorem 3.10 means that the distribution function F of a random variable X with characteristic function f has a non-zero absolutely continuous component (for a proof see [33, § 1.4]).

Considering Theorem 3.10 it is natural to ask if it is possible to use the same methods to construct similar functions Q_k and to obtain estimates similar to (54) in the case of non-normal limit distribution? Unfortunately, the methods used to prove Theorem 3.10 cannot be applied in that case. As usual, the reason is the infiniteness of moments of higher orders for the random variables attracted to $G_{\alpha,\beta}(\,.\,;\lambda,\gamma)$ with $0 < \alpha < 2$.

Lemma 3.11 ([29, § 35]). *If $F \in DNA(G_{\alpha,\beta}(\,.\,;\lambda,\gamma))$ with $\alpha \in (0,2)$, then*

$$\int_{-\infty}^{+\infty} |x|^{\delta} dF(x) = \infty \quad \textit{for} \quad \delta \ge \alpha.$$

From Lemma 3.11 it follows that a random variable X with $F \in \mathrm{DNA}(G_{\alpha,\beta}(\,.\,;\lambda,\gamma))$ has an infinite variance for $\alpha \in (0,2)$ and even an infinite expectation for $\alpha \in (0,1]$, not to mention the third moment. Therefore, the asymptotic expansion of $F_n(x)$, which depends on the moments, can not be constructed in the non-normal case. However, if instead of the moments we consider a more general concept of *pseudo-moments* (see Section 3.4 for details), then it becomes possible to obtain results similar to (54). Sections 3.4 and 3.5 are devoted to the solution of this problem.

3.3 Properties of stable distributions

This section is rather technical. Here we will give some properties of stable distributions and some results that will be needed in what follows. Many of them are formulated only for $\alpha \in (0,1)$ or for $\alpha \neq 1$. The reason is that each result looks differently depending on the parameter area. In order not to overload the section with many cases we consider only the most necessary facts. The omitted information can be found in [29], [33] and [51].

Recall that we are interested in nonnegative random variables X_i, $i \geq 1$. In this connection the following fact is useful: if $X \geq 0$ is attracted to stable $G_{\alpha,\beta}(\,.\,;\lambda,\gamma)$, then $\beta = 1$, [51, p. 18]. Therefore, in what follows we set $\beta = 1$.

Without loss of generality we can consider $\lambda = 1$ and $\gamma = 0$ because of

$$\begin{aligned} G_{\alpha,1}(x;\lambda,\gamma) &= G_{\alpha,1}(\lambda^{-1/\alpha}(x-\gamma);1,0) \quad \text{for} \quad \alpha \neq 1, \\ G_{1,1}(x;\lambda,\gamma) &= G_{1,1}(\lambda^{-1}(x-\gamma-\lambda\,\ln\lambda);1,0). \end{aligned} \tag{55}$$

For the sake of brevity we will write $G_{\alpha,1}(x,\lambda) := G_{\alpha,1}(x\,;\lambda,0)$ and $G_{\alpha,1}(x) := G_{\alpha,1}(x\,;1,0)$. The same holds for characteristic functions: $g_{\alpha,1}(t,\lambda) := g_{\alpha,1}(t\,;\lambda,0)$ and $g_{\alpha,1}(t) := g_{\alpha,1}(t\,;1,0)$.

If $\alpha < 1$ and $\gamma = 0$, then the corresponding stable random variable is nonnegative, i.e. $G_{\alpha,1}(x,\lambda) = 0$ for $x < 0$.

The next property of stable distributions accounts for their name. For $x > 0$ we have

$$G_{\alpha,1}^{n*}(x) = G_{\alpha,1}(n^{-1/\alpha}x) \quad \text{for} \quad \alpha \neq 1 \tag{56}$$

and

$$G_{1,1}^{n*}(x) = G_{1,1}(n^{-1}(x - n\ln n)).$$

This follows from representation (48) and is called the *stability property*.

Since an explicit form of stable distribution functions is known only in a few cases, the study of the properties of stable random variables is rather difficult. One way to make it easier is to consider asymptotic expansions of stable distribution functions. For $\alpha \in (0,2)$, $\alpha \neq 1$, we have (see [51, Section 2.4])

$$1 - G_{\alpha,1}(x,\lambda) = C_1(\alpha)\,\lambda\,x^{-\alpha} + C_2(\alpha)\,\lambda^2\,x^{-2\alpha} + \cdots + C_j(\alpha)\,\lambda^j\,x^{-j\alpha} + O(x^{-(j+1)\alpha}) \tag{57}$$

as $x \to \infty$, where

$$C_j(\alpha) = \frac{1}{\pi j!}\,(-1)^{j+1}\,\Gamma(j\alpha)\,\sin(j\alpha\pi)\,, \quad j \in \mathbb{N}. \tag{58}$$

Note that in this case c_1 from (51) is equal to $C_1(\alpha)$ defined in (58).

Further properties will be formulated as a sequence of lemmata.

Lemma 3.12 ([29, § 35])**.** *If $F \in DNA(G_{\alpha,\beta}(\,.\,;\lambda,\gamma))$ with $\alpha \in (0,2)$, then*

$$\int_{-\infty}^{+\infty} |x|^\delta dF(x) < \infty \quad \textit{for} \quad 0 \leq \delta < \alpha, \quad \delta \in \mathbb{R}.$$

Remark 3.20. Since every stable distribution with characteristic exponent α belongs to its own domain of normal attraction, it has finite absolute moments of order $\delta < \alpha$.

Lemma 3.13 ([29, § 36])**.** *All nondegenerate stable distributions are absolutely continuous and their distribution functions have derivatives of all orders at every point.*

Proof. Notice that for a nondegenerate stable distribution we have (see parametrization (46))

$$|g_{\alpha,\beta^*}(t;\lambda^*,\gamma^*)| \leq \exp\left(-\lambda^*|t|^\alpha\right), \quad \lambda^* > 0.$$

It follows from this and from the inversion theorem that nondegenerate stable distribution are absolutely continuous. The corresponding density function $p_{\alpha,\beta^*}(x;\lambda^*,\gamma^*)$ can be written in the form:

$$p_{\alpha,\beta^*}(x;\lambda^*,\gamma^*) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-itx}\, g_{\alpha,\beta^*}(t;\lambda^*,\gamma^*)dt.$$

Differentiating this formula formally n times with respect to x, we obtain

$$\frac{d^n}{dx^n}\, p_{\alpha,\beta^*}(x;\lambda^*,\gamma^*) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (-it)^n\, e^{-itx}\, g_{\alpha,\beta^*}(t;\lambda^*,\gamma^*)dt.$$

The latter integral converges absolutely, which completes the proof. □

The next lemma provides some estimates for derivatives of a stable distribution function $G_{\alpha,\beta}(x,\lambda)$ with respect to x and λ.

Lemma 3.14 ([12, Theorem 1.5])**.**
Let $\alpha < 1$. For any integers k, $j \geq 0$, $k,j \in \mathbb{N}_0$ with $k+j>0$ there exist positive constants $A_{k,j}, B_{k,j}$, and $D_{k,j}$, depending only on k and j, such that

$$\left|\frac{\partial^{k+j}}{\partial x^k \partial\lambda^j} G_{\alpha,\beta}(x,\lambda)\right| \leq D_{k,j}\, \lambda^{-j-k/\alpha} \tag{59}$$

for all x and $0 < \lambda \leq 2$ and

$$\left|\frac{\partial^{k+j}}{\partial x^k \partial\lambda^j} G_{\alpha,\beta}(x,\lambda)\right| \leq B_{k,j}\, |x|^{-k-\alpha J} \tag{60}$$

for $|x| > A_{k,j} > 0$ and $0 < \lambda \leq 2$, where

$$J = \begin{cases} j+1 & \textit{if} \quad j(\alpha + K(\alpha)\beta) \quad \textit{is an even integer}, \\ j & \textit{otherwise}, \end{cases}$$

and $K(\alpha) = \alpha - 1 + \operatorname{sign}(1-\alpha)$. Moreover, the constants $A_{k,j}, B_{k,j}$, and $D_{k,j}$ can be chosen as follows:

$$\begin{aligned} D_{k,j} &= \Gamma(j+k/\alpha)(\pi\alpha)^{-1}\Big(\cos(K(\alpha)\beta\pi/2)\Big)^{-j-k/\alpha}, \\ A_{k,j} &= (\alpha J + k)^{1/\alpha}, \qquad B_{k,j} = 4\Gamma(\alpha J + k)/\pi. \end{aligned}$$

Remark 3.21. Lemma 3.14 holds also for $1 \leq \alpha < 2$. But in this case the constants $A_{k,j}, B_{k,j}$, and $D_{k,j}$ have another form (for details and for a proof of the lemma see [12]).

3.4 Remainder term estimate: $0 < \alpha < 2$

The goal of this section is to formulate an asymptotic result for the case $\alpha < 2$ similar to Theorem 3.10. For this purpose we first have to define the pseudomoments.

3.4.1 Pseudomoments

Pseudomoments are constructed to replace ordinary moments of random variables from the domain of normal attraction of a stable distribution. Pseudomoments combine the features of moments with the features of metrics. This helps to "reflect the geometry of the distribution more accurately and informatively then with ordinary moments", as Weiner wrote in [50].

We start with a motivating example. Let us consider a nonnegative Pareto-like distributed random variable X with distribution function F, i.e.

$$1 - F(x) = C(\alpha)\, x^{-\alpha} + O(x^{-r}) \quad \text{as} \quad x \to \infty \tag{61}$$

for some $r > \alpha$ and $C(\alpha) > 0$. From Theorem 3.8 it follows that $F \in \mathrm{DNA}(G_{\alpha,1}(\,.\,;\lambda,\gamma))$. Therefore, according to Lemma 3.11

$$E|X|^\delta = \int_{-\infty}^{+\infty} |x|^\delta dF(x) = \infty \quad \text{for} \quad \delta \geq \alpha.$$

Without loss of generality we can put the normalizing constant $a = 1$ in (51), the scale parameter $\lambda = 1$ in (57), and $C(\alpha) = C_1(\alpha)$ in (61). Then we find

$$F(x) - G_{\alpha,1}(x) = O(x^{-\min\{r, 2\alpha\}}) \quad \text{for} \quad r > \alpha \quad \text{from (61)}, \quad x \to \infty.$$

Hence, $F(x) - G_{\alpha,1}(x)$ has better tail-behavior than $1 - F(x)$ and $1 - G_{\alpha,1}(x)$, which permits to use the concept of pseudomoments. It was first introduced by Bergström in [6]. In this thesis we are interested in nonnegative Pareto-like distributions, which very well illustrate the reasonability to introduce the pseudomoments.

But in general, pseudomoments can be defined for any random variable with distribution function $F \in \mathrm{DNA}(G_{\alpha,\beta}(\,.\,;\lambda,\gamma))$.

Definition 3.15 (Pseudomoments)**.**
Consider a random variable X with distribution function $F \in DNA(G_{\alpha,\beta})$. Put $H(x) := F(x) - G_{\alpha,\beta}(x)$. If the corresponding integrals exist, we define the $k-th$ order pseudomoment

$$\mu_k = \mu_k(H) = \int_{-\infty}^{+\infty} x^k\, dH(x), \quad k \in \mathbb{N}_0, \tag{62}$$

the $r-th$ order absolute pseudomoment

$$\nu_r = \nu_r(H) = \int_{-\infty}^{+\infty} |x|^r\; |dH(x)|\,, \quad r \geq 0, \quad r \in \mathbb{R}, \tag{63}$$

and the $r-th$ order truncated pseudomoment

$$\gamma_r = \gamma_r(H) = \begin{cases} \sup\limits_{z>0} z^{r-[r]} \displaystyle\int_{|x|>z} |x|^{[r]}\, |dH(x)|\,, & r \neq [r],\ r > 0, \\ \sup\limits_{z>0} \left(\left| \displaystyle\int_{-z}^{z} x^r dH(x) \right| + z \displaystyle\int_{|x|>z} |x|^{\,r-1}\, |dH(x)| \right), & r = [r],\ r > 0. \end{cases}$$

Remark 3.22. Note that (62) is the improper Riemann-Stieltjes integral of x^k with respect to $H(x)$. Let us briefly recall what this means.

The function $H(x) = F(x) - G_{\alpha,1}(x)$ is a function of bounded variation on every finite interval $[a,b]$ (as a difference of two nondecreasing functions, see [15, Theorem 7.2.4]) and x^k is continuous for all $k \in \mathbb{N}_0$. Therefore, the definite Riemann-Stieltjes integral $\int_a^b x^k dH(x)$ exists for any $a, b \in \mathbb{R}$ and is given by the following formula:

$$\int_a^b x^k dH(x) := \lim_{\substack{n\to\infty \\ \max\limits_i (x_{i+1}-x_i)\to 0}} \sum_{i=0}^{n-1} t_i^k \left(H(x_{i+1}) - H(x_i)\right),$$

where $a = x_0 < x_1 < \cdots < x_n = b$ is a partition of $[a,b]$ and $t_i \in [x_i, x_{i+1}]$, $i = 0, 1, \ldots, n-1$. If there exists $c \in \mathbb{R}$ with $H(c+) - H(c-) = 0$ such that

$$\lim_{a\to-\infty} \int_a^c x^k dH(x) \in \overline{\mathbb{R}} \qquad \text{and} \qquad \lim_{b\to+\infty} \int_c^b x^k dH(x) \in \overline{\mathbb{R}},$$

and the sum of both limits is not of indeterminate form $\infty - \infty$, then we say that the integral $\int_{-\infty}^{+\infty} x^k dH(x)$ exists. In this case,

$$\begin{aligned}
\int_{-\infty}^{+\infty} x^k dH(x) &:= \int_{-\infty}^{c} x^k dH(x) + \int_c^{+\infty} x^k dH(x) \\
&:= \lim_{a\to-\infty} \int_a^c x^k dH(x) + \lim_{b\to+\infty} \int_c^b x^k dH(x).
\end{aligned}$$

The integral in (63) is defined similarly to that from above:

$$\int_a^b |x|^r \, |dH(x)| := \lim_{\substack{n\to\infty \\ \max\limits_i (x_{i+1}-x_i)\to 0}} \sum_{i=0}^{n-1} |t_i|^k \, |H(x_{i+1}) - H(x_i)|,$$

and

$$\int_{-\infty}^{+\infty} |x|^r \, |dH(x)| := \lim_{a\to-\infty} \int_a^c |x|^r \, |dH(x)| + \lim_{b\to+\infty} \int_c^b |x|^r \, |dH(x)|$$

if both limits exist for suitably chosen $c \in \mathbb{R}$. Also note that the definition does not depend on the choice of c.

Remark 3.23. The pseudomoments μ_0 and ν_0 are finite for any random variable X with distribution function $F \in \mathrm{DNA}(G_{\alpha,\beta})$. Moreover, $\mu_0 = 0$ and $\nu_0 \leq 2$. Indeed,

$$\begin{aligned}
\mu_0 &= \int_{-\infty}^{+\infty} dH(x) = \int_{-\infty}^{+\infty} dF(x) - \int_{-\infty}^{+\infty} dG_{\alpha,\beta}(x) = 0; \\
\nu_0 &= \int_{-\infty}^{+\infty} |dH(x)| \leq \int_{-\infty}^{+\infty} dF(x) + \int_{-\infty}^{+\infty} dG_{\alpha,\beta}(x) = 2.
\end{aligned}$$

Note that for a discrete random variable X we always have $\nu_0 = 2$.

The following three lemmata give some properties of pseudomoments, which we need in what follows.

Lemma 3.16. *Let p, $q \in \mathbb{R}$ and $p \geq 0$, $q \geq 0$.*

(a) If $\nu_p < \infty$, then $\nu_q < \infty$ for all $q \leq p$. Moreover, $\nu_q \leq \nu_0^{(p-q)/p}\, \nu_p^{q/p}$.

(b) If $\nu_p = \infty$, then $\nu_q = \infty$ for all $q \geq p$.

Proof. *(a)* follows from Hölder's inequality (for a detailed proof see [12, Lemma 2.2]). *(b)* For $q \geq p \geq 0$ we have

$$\begin{aligned}
\nu_q &= \int_{-\infty}^{+\infty} |x|^q\, |dH(x)| = \int_{-1}^{1} |x|^q\, |dH(x)| + \int_{|x|\geq 1} |x|^{q-p}\, |x|^p\, |dH(x)| \\
&\geq \int_{-1}^{1} |x|^q\, |dH(x)| + \int_{|x|\geq 1} |x|^p\, |dH(x)| \\
&= \int_{-1}^{1} |x|^q\, |dH(x)| - \int_{-1}^{1} |x|^p\, |dH(x)| + \nu_p = \infty,
\end{aligned}$$

since the first two integrals in the last equality are finite. Indeed, for any $l \geq 0$ we find:

$$\int_{-1}^{1} |x|^l\, |dH(x)| \leq \int_{-1}^{1} |dH(x)| \leq \int_{-\infty}^{+\infty} |dH(x)| = \nu_0 \leq 2.$$

Therefore, $\nu_q = \infty$ for all $q \geq p$. □

Lemma 3.17. *If $\nu_r < \infty$, then $\mu_{[r]}$ exists and $|\mu_k| < \infty$ for $k \in \{0, 1, \ldots, [r]\}$, where $[r]$ is an integer part of real $r \geq 0$. Moreover, $|\mu_k| \leq \nu_k$ for any $k \in \{0, 1, \ldots, [r]\}$.*

Proof. The assertion follows from Definition 3.15, Lemma 3.16 and Remark 3.22. □

Lemma 3.18. *With the notation of Definition 3.15 we have*

$$\gamma_r \leq \nu_r, \qquad \forall\, r \geq 0,\ r \in \mathbb{R}.$$

Proof. We consider two cases. First let $r \neq [r]$. Then we have

$$\begin{aligned}
\gamma_r &= \sup_{z>0} z^{r-[r]} \int_{|x|>z} |x|^{[r]}\, |dH(x)| \leq \sup_{z>0} \int_{|x|>z} |x|^{r-[r]}\, |x|^{[r]}\, |dH(x)| \\
&\leq \sup_{z>0} \int_{|x|>z} |x|^r\, |dH(x)| = \int_{-\infty}^{+\infty} |x|^r\, |dH(x)| = \nu_r.
\end{aligned}$$

In case $r = [r]$ we find

$$\begin{aligned}
\gamma_r &= \sup_{z>0} \left(\left| \int_{-z}^{z} x^r dH(x) \right| + z \int_{|x|>z} |x|^{-1}\, |x|^{\,r}\, |dH(x)| \right) \\
&\leq \sup_{z>0} \left(\left| \int_{-z}^{z} x^r dH(x) \right| + \int_{|x|>z} |x|^{\,r}\, |dH(x)| \right) \\
&\leq \int_{-\infty}^{+\infty} |x|^{\,r}\, |dH(x)| = \nu_r.
\end{aligned}$$

□

Remark 3.24. Lemma 3.18 shows that the truncated pseudomoment γ_r is a modification of ν_r. In the examples below we will see that γ_r can be finite even if the corresponding absolute pseudomoment is infinite.

Example 3.4. We consider a random variable X with distribution function F and density function

$$p(x) = \frac{1}{\pi}\frac{1}{x^2}\,\mathbb{1}_{\{|x|\geq 2/\pi\}}(x), \quad x \in \mathbb{R}.$$

From Theorem 3.8 it follows that $F \in \mathrm{DNA}(G_{1,0})$, where $G_{1,0}$ is the Cauchy distribution with density function $p_{1,0}(x) = 1/(\pi(1+x^2))$, $x \in \mathbb{R}$. From

$$\begin{aligned} \nu_r &= \int_{-\infty}^{+\infty} |x|^r\,|dH(x)| = 2\int_0^{+\infty} x^r\,|p(x) - p_{1,0}(x)|\,dx \\ &= \frac{2}{\pi}\left(\int_{2/\pi}^{+\infty} \frac{x^r}{x^2(1+x^2)}\,dx + \int_0^{2/\pi} \frac{x^r}{1+x^2} dx\right) \end{aligned} \tag{64}$$

we have that $\nu_r < \infty$ for $r < 3$. From Lemma 3.17 it follows that $|\mu_i| < \infty$ with $i = 0, 1, 2$. Easy calculations using Definition 3.15 yield

$$\mu_0 = \mu_1 = 0, \qquad \mu_2 = 1 - \frac{4}{\pi^2}.$$

Similarly we can show that $\gamma_3 < \infty$. Moreover, it is possible to obtain the exact value of γ_3:

$$\begin{aligned} \gamma_3 &= \sup_{z>0}\left(\left|\int_{-z}^{z} x^3\,(p(x) - p_{1,0}(x))\,dx\right| + z\int_{|x|>z} x^2\,|p(x) - p_{1,0}(x)|\,dx\right) \\ &= \sup_{z>0}\underbrace{\left(2z\int_z^{+\infty} x^2\,|p(x) - p_{1,0}(x)|\,dx\right)}_{:=f(z)}. \end{aligned}$$

Note that $f(z)$ is a piecewise function that can be computed explicitely. Using basic curve tracing techniques it can be shown that $f(z)$ is increasing on $\mathbb{R}_+$. Due to cumbersome expressions we skip these steps for the sake of readability and continue

$$\gamma_3 = \sup_{z>0}\left(2z\int_z^{+\infty} x^2 \cdot \frac{1}{\pi\,x^2(1+x^2)}\,dx\right) = \lim_{z\to\infty}\left(z - \frac{2z\,\arctan z}{\pi}\right) = \frac{2}{\pi}.$$

□

Sometimes it is not easy to calculate pseudomoments directly using only Definition 3.15. In such situations the next two lemmata are useful.

Lemma 3.19 ([12, Lemma 2.5])**.**
Suppose $\nu_r < \infty$ for some $r \geq 0$. Then

$$f(t) - g_{\alpha,\beta}(t) = \sum_{k=0}^{[r]} \frac{(it)^k}{k!}\,\mu_k + o(|t|^r) \quad \textit{as} \quad t \to 0$$

and

$$f(t) - g_{\alpha,\beta}(t) = \sum_{k=0}^{R} \frac{(it)^k}{k!}\,\mu_k + \theta\,|t|^r\,\nu_r \quad \textit{if} \quad |t| < 1,$$

where

$$\begin{cases} R = [r] & \textit{and} \quad |\theta| \leq 2\,\Gamma(r - [r] + 1)/\Gamma(r+1) \qquad \textit{if} \quad r \neq [r], \\ R = r - 1 & \textit{and} \quad |\theta| \leq 1/r! \qquad \textit{if} \quad r = [r]. \end{cases}$$

Lemma 3.20 ([12, Lemma 2.6]).
If $\nu_k < \infty$ for some integer $k \geq 1$, then $f(t) - g_{\alpha,\beta}(t)$ is k times differentiable and, moreover, $\left(f(t) - g_{\alpha,\beta}(t)\right)^{(m)}\Big|_{t=0} = i^m \mu_m$ for $m \in \{0, 1, \ldots, k\}$.

Remark 3.25. The two lemmata given above illustrate the endless consistency and harmony of mathematics. We see that the connection between pseudomoments and the difference $f(t)-g_{\alpha,\beta}(t)$ is of the same nature as the connection between moments of a random variable and its characteristic function.

Example 3.5 (Pareto with $\alpha = 1/2$).
Consider the Pareto distribution defined in (7) with $\alpha = 1/2$ and $\kappa = 1/\pi$. Its distribution function F is attracted to the Lévy distribution, i.e. $F \in \mathrm{DNA}(G_{1/2,1})$. Denote the Pareto density function with $p(x)$ and the Levý density function with $p_{1/2,1}(x)$. Then we have

$$\begin{aligned} p(x) - p_{1/2,1}(x) &= \frac{1}{2\sqrt{\pi}} x^{-3/2} - \frac{1}{2\sqrt{\pi}} x^{-3/2} e^{-1/(4x)} \\ &= \frac{1}{8\sqrt{\pi}} x^{-5/2} + O(x^{-7/2}), \quad x \to \infty. \end{aligned}$$

Therefore $\nu_r = \int_0^\infty x^r \left| p(x) - p_{1/2,1}(x) \right| dx < \infty$ for any $r < 3/2$. From Lemma 3.17 it follows that $|\mu_1| < \infty$. Moreover, integrating by parts and using the substitution $t := 1/\left(2\sqrt{x}\right)$ we obtain

$$\begin{aligned} \mu_1 &= \int_{-\infty}^{+\infty} x\left(p(x) - p_{1/2,1}(x)\right) dx = -\int_0^{1/\pi} x \cdot \frac{1}{2\sqrt{\pi}} x^{-3/2} e^{-1/(4x)}\, dx \\ &+ \int_{1/\pi}^{+\infty} x \cdot \frac{1}{2\sqrt{\pi}} x^{-3/2}\left(1 - e^{-1/(4x)}\right) dx = -\frac{1}{\sqrt{\pi}} \int_0^{1/\pi} e^{-1/(4x)}\, d\left(\sqrt{x}\right) \\ &+ \frac{1}{\sqrt{\pi}} \int_{1/\pi}^{+\infty} \left(1 - e^{-1/(4x)}\right) d\left(\sqrt{x}\right) = \frac{1}{\sqrt{\pi}}\left(-\frac{1}{\sqrt{\pi}} e^{-\pi/4} + \int_{\sqrt{\pi}/2}^{+\infty} e^{-t^2} dt \right. \\ &\left. + \frac{1}{\sqrt{\pi}}\left(e^{-\pi/4} - 1\right) + \int_0^{\sqrt{\pi}/2} e^{-t^2} dt \right) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} e^{-t^2} dt - \frac{1}{\pi} = \frac{1}{2} - \frac{1}{\pi}. \end{aligned}$$

Note that $\nu_{3/2} = \infty$, but $\gamma_{3/2} < \infty$. Indeed,

$$\begin{aligned} \gamma_{3/2} &= \sup_{z>0}\left(z^{1/2} \int_z^{+\infty} x \left| p(x) - p_{1/2,1}(x) \right| dx\right) \\ &\leq \frac{1}{8\sqrt{\pi}} \sup_{z>0}\left(z^{1/2} \int_z^{\infty} x \cdot x^{-5/2}\, dx\right) = \frac{1}{4\sqrt{\pi}} < \infty. \end{aligned}$$

□

Example 3.6 (Pareto with $\alpha \in (0,1)$, $\alpha \neq 1/2$).
Let X be a Pareto-distributed random variable with $\alpha \in (0,1), \alpha \neq 1/2$. We know that $F \in \mathrm{DNA}(G_{\alpha,1})$. Without loss of generality we can put $\kappa = (C_1(\alpha))^{1/\alpha}$, where $C_1(\alpha)$ is defined in (58). Using representation (57) for $G_{\alpha,1}(x,\lambda)$ with $\lambda = 1$ we find

$$F(x) - G_{\alpha,1}(x) = C_2(\alpha) x^{-2\alpha} + O(x^{-3\alpha}), \quad x \to \infty,$$

where $C_2(\alpha) \neq 0$. From this it follows that $\nu_r < \infty$ for $r < 2\alpha$, $\nu_{2\alpha} = \infty$, but $\gamma_{2\alpha} < \infty$. Proofs are similar to that of Example 3.5. □

3.4.2 A remainder term estimate

A result similar to Theorem 3.10 will be formulated here for the case $\alpha < 2$. Pseudomoments defined in the previous subsection made it all possible.

Recall that we consider a sequence of i.i.d. random variables $X, X_1, X_2, \dots$ with distribution function $F \in \mathrm{DNA}(G_{\alpha,\beta})$ for $\alpha < 2$. Our goal is to build an approximation of the distribution function $F_n(x)$ defined by

$$F_n(x) := P\left(\frac{S_n - a_n}{b_n} \le x\right),$$

where $S_n = X_1 + X_2 + \dots + X_n$ and (a_n), (b_n) are some suitably chosen normalizing sequences such that $F_n(x) \to G_{\alpha,\beta}(x)$ as $n \to \infty$.

Under some conditions on pseudomoments Christoph [12] suggested to approximate $F_n(x)$ by the sum of the corresponding stable distribution $G_{\alpha,\beta}(x)$ and some correction term $W_{r,n}(x)$. He also obtained a non-uniform bound of the remainder term for such an approximation. His result is an analogue of Theorem 3.10. However in this case the correction term depends on the pseudomoments, and not on the moments of X.

Theorem 3.21 ([12, Theorem 5.2]).
Let the case $\alpha = 1$, $\beta \neq 0$ be excluded. Suppose $\gamma_r < \infty$ for some $r > \alpha$, $\nu_0 < 1$ and

$$\max\{\gamma_r^{1/r}, \gamma_r, \gamma_r\, \nu_\alpha^{r/(r-\alpha)}\} n^{-(r-\alpha)/\alpha} \le 1/8,$$

then for all $n \ge R + 2$

$$|F_n(x) - G_{\alpha,\beta}(x) - W_{r,n}(x)| \le C n^{-(r-\alpha)/\alpha} (1 + |x|)^{-r}, \tag{65}$$

where constant C does not depend on n, x and

$$W_{r,n}(x) = \sum_{u=1}^{U} \sum_{j=0}^{J} \sum_{k=1}^{K} \binom{n}{k} a_{kuj} \frac{(-1)^u}{u!\, j!} \left(\frac{\partial^{u+j}}{\partial x^u\, \partial \lambda^j} G_{\alpha,\beta}(x, \lambda)\bigg|_{\lambda=1} \right) n^{-j-u/\alpha}$$

with $U = [2(r-\alpha)/(2-\alpha)]$, $J = [u + (r - u - \alpha)/\alpha]$, $K = [\min\{u, (u+j)/2\}]$,

$$a_{ukj} = \sum_{\substack{u_1+\ldots+u_k=u \\ j_1+\ldots+j_k=j}} \prod_{i=1}^{k} \frac{1}{u_i!\, j_i!}\, d_{u_i\, j_i} \quad \textit{with} \quad 1 \le u_i \le R,\ j_i \ge 0,$$

$$d_{u0} = (-1)^u\, \mu_1^u + d_u, \qquad d_{uj} = (-1)^j\, d_u \qquad \textit{for} \quad j > 0,$$

and

$$d_u = \sum_{v=\max\{0,u-R\}}^{u-1} \binom{u}{v} (-\mu_1)^v\, \mu_{u-v}, \qquad R = \begin{cases} [r], & \textit{if} \quad [r] \neq r, \\ r-1, & \textit{if} \quad [r] = r. \end{cases}$$

Remark 3.26. Note that the centering sequence (a_n) can be chosen as $a_n = n\,\mu_1$ if $r > 1$ and $a_n = 0$ if $r \le 1$ (see [12, Chapter 5.2]).

Remark 3.27. Roughly speaking, the correction term $W_{r,n}(x)$ is a sum of derivatives of $G_{\alpha,\beta}(x)$ with coefficients depending on the pseudomoments. Similar correction term will be constructed (in detail) in Section 3.5 for the case $\alpha \in (0,1)$. For more details and for the proof of Theorem 3.21 see [12, Theorem 5.2].

Remark 3.28. Theorem 3.21 (as well as Theorem 3.10) holds only for distribution functions F, that have a non-zero absolutely continuous component. For such an F Cramér's condition $\limsup_{|t|\to\infty} |f(t)| < 1$ holds true. This condition is not mentioned explicitly in the theorem, since it follows from the condition $\nu_0 < 1$, [12, Section 5.2].

Remark 3.29. The approximation of $F_n(x)$ and the estimates of the remainder term given in (65) were used in order to obtain Theorems 2.10 and 2.11. Conditions (38) and (39) from these theorems provide $\gamma_r < \infty$ for the specified $r > \alpha$.

Example 3.7 ([12, Example 5.3])**.** Let us consider a random variable X from Example 3.4. We know that $\gamma_3 < \infty$. Also, from (64) with $r = 0$ we have

$$\nu_0 = \frac{2}{\pi}\left(\int_{2/\pi}^{+\infty} \frac{1}{x^2(1+x^2)}\,dx + \int_0^{2/\pi} \frac{1}{1+x^2}dx\right) = \frac{4}{\pi}\arctan\left(\frac{2}{\pi}\right) \approx 0.72 < 1.$$

Therefore, Theorem 3.21 yields

$$\left|F_n(x) - G_{1,0}(x) - \frac{1}{2n}\,\mu_2\,\frac{d^2}{dx^2}\,G_{1,0}(x)\right| \le Cn^{-2}(1+|x|)^{-3},$$

where the constant C does not depend on n or x. □

Remark 3.30. Theorem 3.21 gives good results if α is not very small. Namely, if α is small then r can also be small and $U = [2(r-\alpha)/(2-\alpha)] = 0$ in Theorem 3.21. This means that the term $W_{r,n}$ in (65) is equal to 0.

Example 3.8. Let us consider a Pareto-distributed random variable X with $\alpha = 1/3$ and $\kappa = (C_1(1/3))^3$, where $C_1(1/3) \approx 0.74$ is defined in (58). The distribution function F of X has the form:

$$1 - F(x) = \frac{C_1(1/3)}{x^{1/3}}, \quad x \ge (C_1(1/3))^3.$$

Therefore, $F \in \mathrm{DNA}(G_{1/3,1})$. From Example 3.6 it follows that $\gamma_{2/3} < \infty$, but $\nu_{2/3} = \infty$ and $\mu_1 = \infty$. Then Theorem 3.21 provides the following remainder term estimates for the approximation of $F_n(x)$ by $G_{1/3,1}(x)$:

$$\left|F_n(x) - G_{1/3,1}(x)\right| \le Cn^{-1}(1+|x|)^{-2/3}.$$

As we can see, the correction term $W_{r,n}$ vanishes in this case and the quality of approximation is much worse than in Example 3.7. □

Naturally the following question arises: is it possible to improve the approximation of F_n in the case of $F \in \mathrm{DNA}(G_{\alpha,1})$ with small values of α. This problem will be discussed in the next section.

3.5 Remainder term estimate: $0 < \alpha < 1$

This section is devoted to the case $F \in \mathrm{DNA}(G_{\alpha,1})$ with $\alpha \in (0,1)$. Our goal is to improve Theorem 3.21 for small α. On the whole, the case $\alpha \in (0,1)$ is considerably less studied than the case $\alpha > 1$. Remark 3.30 and Example 3.8 make clear that for small α we can not obtain a very good approximation of F^{n*} using Theorem 3.21. This can be explained by the infiniteness not only of the moments of order ≥ 1 but also of the pseudomoments of higher orders. That is why in this section we also discuss a possible modification of pseudomoments.

3.5.1 Function $\widetilde{G}_\alpha$

The goal of this subsection is to construct a function whose behavior is more similar to the behavior of $F(x)$ as $x \to \infty$ than that of the attracting distribution function $G_{\alpha,1}(x)$.

We consider a nonnegative random variable X with distribution function $F \in \mathrm{DNA}(G_{\alpha,1})$, where $\alpha \in (0,1)$. It follows from Theorem 3.8 that

$$1 - F(x) = \frac{c_1 a^\alpha}{x^\alpha} + o(x^{-\alpha}), \quad x > 0, \quad x \to \infty, \tag{66}$$

where c_1 is the first coefficient in the expansion of $G_{\alpha,1}$ (see (57)):

$$1 - G_{\alpha,1}(x,\lambda) = \frac{c_1\,\lambda}{x^\alpha} + \frac{c_2\,\lambda^2}{x^{2\alpha}} + \cdots + \frac{c_j\,\lambda^j}{x^{j\alpha}} + O(x^{-(j+1)\alpha}), \quad x \to \infty, \tag{67}$$

with $c_k := C_k(\alpha)$, $k \in \mathbb{N}$, from (58). Without loss of generality we can put the normalizing constant $a = 1$.

Usually the distribution function F is given (known). This means that we can consider more terms in the asymptotic expansion of F in comparison with (66). Therefore, in this section we suppose that F can be represented in the following form:

$$1 - F(x) = \frac{c_1}{x^\alpha} + \frac{c_2\,d_2}{x^{2\alpha}} + \cdots + \frac{c_s\,d_s}{x^{s\alpha}} + u(x), \quad x \to \infty, \tag{68}$$

where $s \in \mathbb{N}$ and $u(x)$ are such that

$$s\alpha \geq 1 + \alpha, \qquad \int_0^{+\infty} x^q |du(x)| < \infty \quad \text{for some } q > s\alpha,$$

$c_i := C_i(\alpha)$, $i = 1, ..., s$, from (58) and d_i are suitable constants for $i = 2, \ldots, s$.

Remark 3.31. If some of the coefficients c_i in (68) are equal to 0, then the corresponding d_i can be chosen arbitrarily. It is important to fix all d_i, $i = 2, ..., s$, before we go to the next step.

Remark 3.32. Note that the coefficients in (68) have the unusual form $c_k \cdot d_k$ with suitable $d_k \in \mathbb{R}$. This can be explained by the requirement that if the coefficient c_k from (67) is equal to 0, then the corresponding coefficient in representation (68) must also be equal to 0.

The introduction of pseudomoments defined in Section 3.4 is based on the fact that $F(x) - G_{\alpha,1}(x)$ has a better tail-behavior than $1 - F(x)$ and $1 - G_{\alpha,1}(x)$. Note that if $c_2 \neq 0$ and $d_2 \neq 1$ in (68), then $F(x) - G_{\alpha,1}(x) = O(x^{-2\alpha})$ as $x \to \infty$. This means that only the first terms in both expansions are equal and vanish in the difference. Our goal now is to construct another function $\widetilde{G}_\alpha$ (instead of $G_{\alpha,1}$) that can eliminate all known terms in expansion (68) of $1 - F(x)$. Namely, we can define $\widetilde{G}_\alpha$ as follows:

$$\widetilde{G}_\alpha(x) = G_{\alpha,1}(x) + \sum_{j=2}^{s} A_j\, G^{(0,j)}(x,1), \quad x \in \mathbb{R}, \tag{69}$$

with the coefficients

$$A_2 = \frac{d_2 - 1}{2}, \quad A_k = \frac{d_k - 1}{k!} - \sum_{u=2}^{k-1} A_u \frac{1}{(k-u)!}, \quad k = 3, \ldots, s, \tag{70}$$

which are chosen in such a way that

$$F(x) - \widetilde{G}_\alpha(x) = -u(x) + O\left(\frac{1}{x^{(s+1)\alpha}}\right) \quad \text{as} \quad x \to \infty.$$

By $G^{(k,j)}(x,1)$ we denote the derivatives of $G_{\alpha,1}(x,\lambda)$ with respect to x and λ:

$$G^{(k,j)}(x,1) := \left.\frac{\partial^{k+j}}{\partial x^k \partial \lambda^j} G_{\alpha,1}(x,\lambda)\right|_{\lambda=1}, \quad k,j \in \mathbb{N}_0. \tag{71}$$

They exist and are bounded according to Lemma 3.14. Note that $F(x) - \widetilde{G}_\alpha(x) = 0$ for $x < 0$, since $F(x) = G_{\alpha,1}(x) = 0$ for $x < 0$.

Lemma 3.22. *For the function $\widetilde{G}_\alpha$ defined by* (69) *the following properties hold true.*

(i) $\widetilde{G}_\alpha(x)$ *is absolutely continuous and differentiable on* $\mathbb{R}$ *with* $\widetilde{G}_\alpha(x) = 0$ *for* $x < 0$.

(ii) There exists a constant $\widetilde{G} > 0$ *such that* $\left|\widetilde{G}_\alpha(x)\right| \leq \widetilde{G}$ *for all* $x \in \mathbb{R}$.

(iii) $\lim_{x \to +\infty} \widetilde{G}_\alpha(x) = 1$.

Proof. (i) According to the definition, $\widetilde{G}_\alpha$ is a linear combination of an absolutely continuous and infinitely differentiable (with respect to x and λ) function $G_{\alpha,1}(x,\lambda)$ and derivatives of $G_{\alpha,1}(x,\lambda)$, which are also absolutely continuous and differentiable (see Lemma 3.14). This fact, the definition of $\widetilde{G}_\alpha(x)$ and the fact that $G_{\alpha,1}(x,\lambda) = 0$ for $x < 0$ and $\alpha < 1$ (see Section 3.3) give us statement (i).
(ii) Boundedness of the function $\widetilde{G}_\alpha(x)$ follows from Lemma 3.14. We have the following estimate:

$$\begin{aligned}\left|\widetilde{G}_\alpha(x)\right| &\leq |G_{\alpha,1}(x)| + \sum_{j=2}^{s} |A_j| \left|G^{(0,j)}(x,1)\right| \\ &\leq 1 + \sum_{j=2}^{s} |A_j| \cdot D_{0,j} =: \widetilde{G} < \infty \qquad \forall x \in \mathbb{R},\end{aligned} \tag{72}$$

where $D_{0,j}$ are defined in Lemma 3.14.
(iii) The third property follows from (60):

$$\lim_{x\to+\infty} \widetilde{G}_\alpha(x) = \lim_{x\to+\infty}\left(G_{\alpha,1}(x)+\sum_{j=2}^{s} A_j\, G^{(0,j)}(x,1)\right)$$
$$= 1+\sum_{j=2}^{s} A_j \lim_{x\to+\infty} G^{(0,j)}(x,1) = 1. \tag{73}$$

□

The following lemma illustrates the "goodness" of the function $\widetilde{G}_\alpha(x)$ for our purposes. Namely, the n-fold convolution of $\widetilde{G}_\alpha$ can be expressed explicitly. We use the following notation:

$$\varphi_{\alpha,1}(t,\lambda) := \ln\left(g_{\alpha,1}(t,\lambda)\right), \quad \text{i.e.} \quad \varphi_{\alpha,1}(t,\lambda) = -\lambda|\,t|^{\alpha} e^{-i\,\alpha\,(\pi/2)\,\mathrm{sign}\,t}, \tag{74}$$

where $g_{\alpha,1}(t,\lambda)$ is a characteristic function of $G_{\alpha,1}(x,\lambda)$. For the sake of brevity we will also write $\varphi_{\alpha,1}(t) := \varphi_{\alpha,1}(t,1)$.

Lemma 3.23. *For all $\rho = 1,2,...,n$, $n\in\mathbb{N}$, and $x\in\mathbb{R}$ we have*

$$\widetilde{G}_\alpha^{\rho *}\left(n^{1/\alpha}x\right) = G_{\alpha,1}\left(x,\frac{\rho}{n}\right) + \sum_{k=2}^{s\rho}\frac{c_{k,\rho}}{n^k}\,G^{(0,k)}\left(x,\frac{\rho}{n}\right) \tag{75}$$

with

$$c_{k,\,\rho} = \sum_{\substack{k_0+k_2+\cdots+k_s=\rho\\ k=2k_2+\cdots+sk_s}} \frac{\rho!}{k_0!k_2!\cdots k_s!}A_2^{k_2}\cdots A_s^{k_s}, \tag{76}$$

where the summation is carried out over all non-negative integer solutions $k_0, k_2, \ldots, k_s$ of the equation $k_0+k_2+\cdots+k_s=\rho$ and $k = 2k_2+\cdots+sk_s$.

Proof. We consider the inverse Fourier transform $\widetilde{g}_\alpha$ of $\widetilde{G}_\alpha$. We obtain

$$\widetilde{g}_\alpha(t) = \int_{-\infty}^{+\infty} e^{itx}d\widetilde{G}_\alpha(x) = \int_{-\infty}^{+\infty} e^{itx}dG_{\alpha,1}(x) + \sum_{j=2}^{s} A_j \int_{-\infty}^{+\infty} e^{itx}dG^{(0,j)}(x,1)$$
$$= g_{\alpha,1}(t) + \sum_{j=2}^{s} A_j\left(\frac{d^j}{d\lambda^j}g_{\alpha,1}(t,\lambda)\right)\Bigg|_{\lambda=1} = g_{\alpha,1}(t) + \sum_{j=2}^{s} A_j\, g_{\alpha,1}(t)\,\varphi_{\alpha,1}^{j}(t). \tag{77}$$

Note that the inverse Fourier transform of $\widetilde{G}_\alpha\left(n^{1/\alpha}x\right)$ is $\widetilde{g}_\alpha\left(tn^{-1/\alpha}\right)$:

$$\widetilde{g}_\alpha\left(tn^{-1/\alpha}\right) = \int_{-\infty}^{+\infty} e^{itn^{-1/\alpha}x}d\widetilde{G}_\alpha(x) = \int_{-\infty}^{+\infty} e^{ity}d\widetilde{G}_\alpha\left(n^{1/\alpha}y\right).$$

It is known that the inverse Fourier transform of $\widetilde{G}_\alpha^{\rho *}$ is equal to the ρ-th power of $\widetilde{g}_\alpha$. Our task now is to expand and transform $\widetilde{g}_\alpha^{\rho}$ into a suitable form. Using formula (147) from Lemma A.5 and the definition of $g_{\alpha,1}(t)$ we write:

$$\begin{aligned}
\widetilde{g}_\alpha^{\rho}\Big(tn^{-1/\alpha}\Big) &= \left(g_{\alpha,1}\Big(tn^{-1/\alpha}\Big) + \sum_{j=2}^{s} A_j\, g_{\alpha,1}\Big(tn^{-1/\alpha}\Big)\, \varphi_{\alpha,1}^{j}\Big(tn^{-1/\alpha}\Big)\right)^{\rho} \\
&= g_{\alpha,1}^{\rho}\Big(tn^{-1/\alpha}\Big) \sum_{k_0+k_2+\dots+k_s=\rho} \frac{\rho!}{k_0!k_2!\cdots k_s!} 1^{k_0} A_2^{k_2} \varphi_{\alpha,1}^{2k_2}\Big(tn^{-1/\alpha}\Big)\cdots A_s^{k_s}\varphi_{\alpha,1}^{sk_s}\Big(tn^{-1/\alpha}\Big) \\
&= g_{\alpha,1}\left(t\left(\frac{\rho}{n}\right)^{1/\alpha}\right) \sum_{k_0+k_2+\dots+k_s=\rho} \frac{\rho!}{k_0!k_2!\cdots k_s!} A_2^{k_2}\cdots A_s^{k_s} \left(\varphi_{\alpha,1}(tn^{-1/\alpha})\right)^{2k_2+\ldots+sk_s} \\
&= g_{\alpha,1}\left(t\left(\frac{\rho}{n}\right)^{1/\alpha}\right) \sum_{k=0}^{s\rho} \varphi_{\alpha,1}^{k}(tn^{-1/\alpha}) \underbrace{\sum_{\substack{k_0+k_2+\dots+k_s=\rho \\ 2k_2+\dots+sk_s=k}} \frac{\rho!}{k_0!k_2!\cdots k_s!} A_2^{k_2}\cdots A_s^{k_s}}_{c_{k,\rho}} = \text{Ⓢ}.
\end{aligned}$$

Now, taking into account that $c_{0,\rho} = 1$ and $c_{1,\rho} = 0$ for all $\rho \in \mathbb{N}$ and using the definition of $\varphi_{\alpha,1}$ we continue

$$\begin{aligned}
\text{Ⓢ} &= g_{\alpha,1}\left(t\left(\frac{\rho}{n}\right)^{1/\alpha}\right) + \sum_{k=2}^{s\rho} c_{k,\rho}\, \varphi_{\alpha,1}^{k}(tn^{-1/\alpha})\, g_{\alpha,1}\left(t\left(\frac{\rho}{n}\right)^{1/\alpha}\right) \\
&= g_{\alpha,1}\left(t\left(\frac{\rho}{n}\right)^{1/\alpha}\right) + \sum_{k=2}^{s\rho} \frac{c_{k,\rho}}{\rho^k}\, \varphi_{\alpha,1}^{k}\left(t\left(\frac{\rho}{n}\right)^{1/\alpha}\right) g_{\alpha,1}\left(t\left(\frac{\rho}{n}\right)^{1/\alpha}\right).
\end{aligned} \tag{78}$$

The inverse Fourier transform of $G^{(0,k)}$ is equal to $g_{\alpha,1}\varphi_{\alpha,1}^{k}$. Indeed,

$$\begin{aligned}
\int_{-\infty}^{+\infty} e^{itx}\, d\,G^{(0,k)}\left(x\left(\frac{n}{\rho}\right)^{1/\alpha}, 1\right) &= \frac{d^k}{d\lambda^k}\left(\int_{-\infty}^{+\infty} e^{itx} d\,G_{\alpha,1}\left(x\left(\frac{n}{\rho}\right)^{1/\alpha}, \lambda\right)\right)\Bigg|_{\lambda=1} \\
&= \frac{d^k}{d\lambda^k}\, g_{\alpha,1}\left(t\left(\frac{\rho}{n}\right)^{1/\alpha}, \lambda\right)\Bigg|_{\lambda=1} = \frac{d^k}{d\lambda^k}\, e^{\lambda\varphi_{\alpha,1}\left(t\left(\frac{\rho}{n}\right)^{1/\alpha},1\right)}\Bigg|_{\lambda=1} \\
&= \varphi_{\alpha,1}^{k}\left(t\left(\frac{\rho}{n}\right)^{1/\alpha}, 1\right) g_{\alpha,1}\left(t\left(\frac{\rho}{n}\right)^{1/\alpha}, 1\right),
\end{aligned}$$

which leads us to

$$\widetilde{G}_\alpha^{\rho *}\left(xn^{1/\alpha}\right) = G_{\alpha,1}\left(x\left(\frac{n}{\rho}\right)^{1/\alpha}\right) + \sum_{k=2}^{s\rho} \frac{c_{k,\rho}}{\rho^k}\cdot G^{(0,k)}\left(x\left(\frac{n}{\rho}\right)^{1/\alpha}, 1\right).$$

Finally, using property (55) for $G_{\alpha,1}\Big(x\left(\frac{n}{\rho}\right)^{1/\alpha}\Big) = G_{\alpha,1}\left(x, \frac{\rho}{n}\right)$ and

$$\begin{aligned}
G^{(0,k)}\left(x\left(\frac{n}{\rho}\right)^{1/\alpha}, 1\right) &= \frac{d^k}{d\lambda^k} G_{\alpha,1}\left(x\left(\frac{n}{\rho}\right)^{1/\alpha}, \lambda\right)\Bigg|_{\lambda=1} = \frac{d^k}{d\lambda^k} G_{\alpha,1}\left(x, \lambda\frac{\rho}{n}\right)\Bigg|_{\lambda=1} \\
&= \left(\frac{\rho}{n}\right)^k \cdot \frac{d^k}{dy^k} G_{\alpha,1}(x,y)\Bigg|_{y=\frac{\rho}{n}} = \left(\frac{\rho}{n}\right)^k G_{\alpha,1}^{(0,k)}\left(x, \frac{\rho}{n}\right)
\end{aligned}$$

we obtain the assertion of the lemma.

□

3.5.2 Pseudomoments: new approach

The construction of the function $\widetilde{G}_\alpha$ allows us to use modified pseudomoments.

We put $\widetilde{H}(x) := F(x) - \widetilde{G}_\alpha(x)$ and consider *"new pseudomoments"* $\mu(\widetilde{H})$, $\nu(\widetilde{H})$ and $\gamma(\widetilde{H})$: the k-th order pseudomoment

$$\mu_k^* = \mu_k(\widetilde{H}) = \int_0^{+\infty} x^k \, d\widetilde{H}(x), \quad k \geq 0, \quad k \in \mathbb{N}_0, \tag{79}$$

the r-th order absolute pseudomoment

$$\nu_r^* = \nu_r(\widetilde{H}) = \int_0^{+\infty} x^r \left| d\widetilde{H}(x) \right|, \quad r \geq 0, \quad r \in \mathbb{R}, \tag{80}$$

and the r-th order truncated pseudomoment

$$\gamma_r^* = \gamma_r(\widetilde{H}) = \begin{cases} \sup\limits_{z>0} z^{r-[r]} \displaystyle\int_z^{+\infty} x^{[r]} \left| d\widetilde{H}(x) \right|, & r \neq [r], \ r > 0, \\ \sup\limits_{z>0} \left(\left| \displaystyle\int_0^z x^r d\widetilde{H}(x) \right| + z \displaystyle\int_z^{+\infty} x^{r-1} \left| d\widetilde{H}(x) \right| \right), & r = [r], \ r > 0, \end{cases}$$

if the corresponding integrals exist.

Remark 3.33. All new pseudomoments are well-defined. It follows from Remark 3.22 and from the fact that $\widetilde{H}(x)$ is a function of bounded variation, same as $H(x)$. Indeed,

$$\widetilde{H}(x) = F(x) - \widetilde{G}_\alpha(x) = H(x) - \sum_{j=2}^{s} A_j \, G^{(0,j)}(x, 1).$$

According to Lemma 3.14 each of the functions $G^{(0,j)}(x, 1)$, $j = 2, \ldots, s$, has a bounded derivative with respect to x, which makes its variation also bounded (see [27, Kapitel XV/568, Satz 3]). From [27, Kapitel XV/569, Satz 2] it follows that $\widetilde{H}(x)$ is also of bounded variation as a difference of functions of bounded variation.

New pseudomoments modify the old ones. In the example below we will see that ν_r^* can be finite even if ν_r defined by (63) is infinite.

Example 3.9. Let us consider a Pareto-distributed random variable X from (7) with $\alpha = 1/3$ and $\kappa = (C_1(1/3))^3$, where $C_1(1/3) \approx 0.74$ is defined in (58). The distribution function F of X has the form

$$1 - F(x) = \frac{C_1(1/3)}{x^{1/3}}, \quad x \geq (C_1(1/3))^3.$$

Therefore, $F \in \mathrm{DNA}(G_{1/3,1})$. According to representation (68) of $F(x)$ we can put $d_2 = \cdots = d_s = 0$ for all integer $s \geq 2$ and $u(x) = 0$. Let us consider the case $s = 4$ and construct the function $\widetilde{G}_{1/3}$ using formula (69) with $A_2 = -1/2$, $A_3 = 1/3$, $A_4 = -1/8$ defined by (70):

$$\widetilde{G}_{1/3}(x) = G_{1/3,1}(x) - \frac{1}{2} G^{(0,2)}(x, 1) + \frac{1}{3} G^{(0,3)}(x, 1) - \frac{1}{8} G^{(0,4)}(x, 1).$$

Then we have

$$\widetilde{H}(x) = F(x) - \widetilde{G}_{1/3}(x) = O\left(\frac{1}{x^{5/3}} \right) \quad \text{as} \quad x \to \infty.$$

Therefore, $\nu_r^* < \infty$ for $r < 5/3$, whereas from Example 3.6 it follows that $\nu_r = \infty$ already for $r \geq 2/3$. Note also that $\gamma_{5/3}^* < \infty$. □

New pseudomoments have partly the same properties as the old ones. Lemmata 3.16, 3.17 and 3.18 hold true with μ_k^*, ν_r^* and γ_r^* instead of μ_k, ν_r and γ_r. We also have the following result.

Lemma 3.24. *For the pseudomoments μ_0^* and ν_0^* the following statements hold true:*

$$\textit{(i)}\quad \mu_0^* = 0; \qquad\qquad \textit{(ii)}\quad \nu_0^* \le 2 + \sum_{j=2}^{s} |A_j| \Big(D_{1,j}\, A_{1,j} + B_{1,j}/\alpha j \Big),$$

where $A_{1,j}$, $B_{1,j}$ and $D_{1,j}$ are constants from Lemma 3.14.

Proof. (i) For μ_0^* we have

$$\begin{aligned} \mu_0^* &= \int_0^{+\infty} d\widetilde{H}(x) = \int_0^{+\infty} dF(x) - \int_0^{+\infty} d\widetilde{G}_\alpha(x) \\ &= 1 - \int_0^{+\infty} d\left(G_{\alpha,1}(x) + \sum_{j=2}^{s} A_j\, G^{(0,j)}(x,1) \right) \\ &= 1 - \int_0^{+\infty} dG_{\alpha,1}(x) - \sum_{j=2}^{s} A_j \frac{d^j}{d\lambda^j} \left(\underbrace{\int_0^{+\infty} dG_{\alpha,1}(x,\lambda)}_{=1} \right) \Bigg|_{\lambda=1} = 0. \end{aligned}$$

The interchange of integral and differentiation in the last step is justified by the Leibniz integral rule [26, Kapitel XIV/520, Satz 3].
(ii) Using inequalities from Lemma 3.14 with $A_{1,j} \ge 1$ we obtain for $j \in \{2, \dots, s\}$

$$\begin{aligned} &\int_0^{+\infty} \left| G^{(1,j)}(x,1) \right| dx = \int_0^{A_{1,j}} \left| G^{(1,j)}(x,1) \right| dx + \int_{A_{1,j}}^{+\infty} \left| G^{(1,j)}(x,1) \right| dx \\ \le\ & A_{1,j}\, D_{1,j} + B_{1,j} \int_{A_{1,j}}^{+\infty} x^{-1-\alpha J} dx \le A_{1,j}\, D_{1,j} + B_{1,j} \int_{A_{1,j}}^{+\infty} x^{-1-\alpha j} dx \\ \le\ & A_{1,j}\, D_{1,j} + B_{1,j}\, A_{1,j}^{-\alpha j}/(\alpha j) \le A_{1,j}\, D_{1,j} + B_{1,j}/(\alpha j). \end{aligned}$$

The latter estimate and the fact that $\nu_0 \le 2$ (see Remark 3.23) yield

$$\begin{aligned} \nu_0^* &= \int_0^{+\infty} \left| d\widetilde{H}(x) \right| \le \int_0^{+\infty} \left| d\left(F - G_{\alpha,1}\right)(x) \right| + \int_0^{+\infty} \left| d\left(\sum_{j=2}^{s} A_j\, G^{(0,j)}(x,1) \right) \right| \\ &\le \nu_0 + \sum_{j=2}^{s} |A_j| \int_0^{+\infty} \left| G^{(1,j)}(x,1) \right| dx \le 2 + \sum_{j=2}^{s} |A_j| \Big(D_{1,j}\, A_{1,j} + B_{1,j}/\alpha j \Big). \end{aligned}$$

This completes the proof of Lemma 3.24. □

To calculate new pseudomoments directly is even more complicated than to compute the old ones. The following lemma, which is an analogue of Lemma 3.19, makes this easier.

Recall that

$$\varphi_{\alpha,1}(t,\lambda) = \ln\left(g_{\alpha,1}(t,\lambda)\right), \quad \text{i.e.} \quad \varphi_{\alpha,1}(t,\lambda) = -\lambda |\, t|^{\alpha} e^{-i\,\alpha\,(\pi/2)\,\mathrm{sign}\, t},$$

where $g_{\alpha,1}(t,\lambda)$ is a characteristic function, which corresponds to $G_{\alpha,1}(x,\lambda)$. Recall also that for the sake of brevity we write $\varphi_{\alpha,1}(t) := \varphi_{\alpha,1}(t,1)$.

Lemma 3.25. *Suppose* $\nu_r^* < \infty$ *for some* $r \geq 0$. *Then*

$$f(t) - \widetilde{g}_\alpha(t) = \sum_{k=0}^{[r]} \frac{(it)^k}{k!} \mu_k^* + o(|t|^r) \quad as \quad t \to 0, \tag{81}$$

where

$$\widetilde{g}_\alpha(t) = g_{\alpha,1}(t) \left(1 + \sum_{j=2}^{s} A_j \, \varphi_{\alpha,1}^j(t)\right) \tag{82}$$

with A_j, $j = 2, \ldots, s$, *defined in* (70).

Proof. In the same way as in the proof of Lemma 3.19 we expand e^{itx} from the integral $\int_{-\infty}^{+\infty} e^{itx} d(F - \widetilde{G}_\alpha)(x)$ into series and obtain the right side of (81). The left side follows from

$$\int_{-\infty}^{+\infty} e^{itx} d(F - \widetilde{G}_\alpha)(x) = f(t) - \int_{-\infty}^{+\infty} e^{itx} d\widetilde{G}_\alpha(x)$$

and from

$$\begin{aligned} \int_{-\infty}^{+\infty} e^{itx} d\widetilde{G}_\alpha(x) &= \int_{-\infty}^{+\infty} e^{itx} dG_{\alpha,1}(x) + \sum_{j=2}^{s} A_j \int_{-\infty}^{+\infty} e^{itx} dG^{(0,j)}(x, 1) \\ = \quad & g_{\alpha,1}(t) + \sum_{j=2}^{s} A_j \left(\frac{d^j}{d\lambda^j} \int_{-\infty}^{+\infty} e^{itx} dG_{\alpha,1}(x) \right)\Bigg|_{\lambda=1} \\ = \quad & g_{\alpha,1}(t) + \sum_{j=2}^{s} A_j \frac{d^j}{d\lambda^j} g_{\alpha,1}(t, \lambda) = g_{\alpha,1}(t) \left(1 + \sum_{j=2}^{s} A_j \, \varphi_{\alpha,1}^j(t)\right) = \widetilde{g}_\alpha(t). \end{aligned}$$

The interchange of integral and differentiation in the previous formula is justified by the Leibniz integral rule [26, Kapitel XIV/520, Satz 3]. ☐

Let us see, how Lemma 3.25 works in special cases.

Example 3.10. Let $\alpha \in (0, 1)$. We consider a Pareto-distributed random variable X with distribution function F such that

$$1 - F(x) = \begin{cases} \dfrac{c_1}{x^\alpha} & \text{for } x \geq c_1^{1/\alpha}, \\ 1 & \text{for } x < c_1^{1/\alpha}, \end{cases}$$

where $c_1 = C_1(\alpha) = (\Gamma(\alpha) \sin \alpha\pi) / \pi$. According to representation (68) of $F(x)$ we can put $d_2 = \cdots = d_s = 0$ for all $s \geq 2$ and $u(x) = 0$. For our convenience let us fix such $s \in \mathbb{N}$ that $(1 + s)\alpha \notin \mathbb{N}$ and construct the function $\widetilde{G}_\alpha$ using formulas (69) and (70). Then we have

$$F(x) - \widetilde{G}_\alpha(x) = O\left(\frac{1}{x^{(s+1)\alpha}}\right) \quad \text{as} \quad x \to \infty.$$

Therefore, $\nu_r^* < \infty$ for $r < (s + 1)\alpha$. Let us calculate pseudomoments μ_j^* for $j = 1, 2, \ldots, [(s + 1)\alpha]$.

Our objective is to obtain the following expansion of $f(t)-\widetilde{g}_\alpha(t)$:

$$f(t)-\widetilde{g}_\alpha(t)=a_0+a_1 it+a_2\,\frac{(it)^2}{2!}+\cdots \quad \text{as} \quad t\to 0 \tag{83}$$

and than use Lemma 3.25, which states that the coefficient a_1 from (83) is the first pseudomoment.

Using integration by parts $f(t)$ can be represented as follows:

$$\begin{aligned} f(t) &= -\int_{-\infty}^{+\infty} e^{itx}d(1-F(x)) = -\int_{c_1^{1/\alpha}}^{+\infty} e^{itx}d\left(\frac{c_1}{x^\alpha}\right) \\ &= -\frac{c_1}{x^\alpha}\,e^{itx}\Big|_{x=c_1^{1/\alpha}}^{x=+\infty} + c_1 it\int_{c_1^{1/\alpha}}^{+\infty} e^{itx}x^{-\alpha}dx \\ &= e^{itc_1^{1/\alpha}} + c_1 it\int_0^{+\infty} e^{itx}x^{-\alpha}dx - c_1 it\int_0^{c_1^{1/\alpha}} e^{itx}x^{-\alpha}dx. \end{aligned} \tag{84}$$

Note that the second term in (84) is equal to $\varphi_{\alpha,1}(t)$. Indeed, using the substitution $y=|t|\,x$, Lemma A.2 and formulas from (145) we have

$$\begin{aligned} c_1 it &\int_0^{+\infty} e^{itx}x^{-\alpha}dx = c_1 it\int_0^{+\infty}(\cos(tx)+i\sin(tx))x^{-\alpha}dx \\ &= c_1 i\ \operatorname{sign} t\,|t|^\alpha\int_0^{+\infty}\Big(\cos(\operatorname{sign} t\,|t|\,x)+i\ \operatorname{sign} t\sin(|t|\,x)\Big)(|t|\,x)^{-\alpha}d(|t|\,x) \\ &= c_1 i\ \operatorname{sign} t\,|t|^\alpha\int_0^{+\infty}\Big(\cos y+i\ \operatorname{sign} t\sin y\Big)y^{-\alpha}dy \\ &= c_1 i\ \operatorname{sign} t\,|t|^\alpha\,\Gamma(1-\alpha)\left(\sin\frac{\alpha\pi}{2}+i\ \operatorname{sign} t\cos\frac{\alpha\pi}{2}\right) \\ &= -|t|^\alpha\left(\cos\frac{\alpha\pi}{2}-i\ \operatorname{sign} t\sin\frac{\alpha\pi}{2}\right) = -|t|^\alpha\,e^{-i\alpha(\pi/2)\operatorname{sign} t} = \varphi_{\alpha,1}(t). \end{aligned}$$

Expanding the first term and e^{itx} from the last integral in (84) into series for small t we obtain

$$\begin{aligned} f(t) &= e^{itc_1^{1/\alpha}} + c_1 it\int_0^{+\infty} e^{itx}x^{-\alpha}dx - c_1 it\int_0^{c_1^{1/\alpha}} e^{itx}x^{-\alpha}dx \\ &= 1+\varphi_{\alpha,1}(t)+c_1^{1/\alpha}\frac{\alpha}{\alpha-1}it+c_1^{2/\alpha}\frac{\alpha}{2(\alpha-2)}(it)^2+c_1^{3/\alpha}\frac{\alpha}{6(\alpha-3)}(it)^3+\cdots. \end{aligned}$$

Now we consider $\widetilde{g}_\alpha(t)$. First of all note that since we put $d_2=\cdots=d_s=0$, equalities (70) for the coefficients A_k take the form

$$\sum_{u=2}^{k}\frac{A_u}{(k-u)!}+\frac{1}{k!}=0, \quad k=2,\ldots,s.$$

Using this representation and expanding $g_{\alpha,1}(t)=e^{\varphi_{\alpha,1}(t)}$ into series with respect to $\varphi_{\alpha,1}(t)$ we obtain

$$\widetilde{g}_\alpha(t) = g_{\alpha,1}(t)\left(1+\sum_{j=2}^{s}A_j\varphi_{\alpha,1}^j(t)\right)$$

$$\begin{aligned}
&= \left(1+\varphi_{\alpha,1}(t)+\frac{\varphi^2_{\alpha,1}(t)}{2!}+\frac{\varphi^3_{\alpha,1}(t)}{3!}+\dots\right)\left(1+\sum_{j=2}^{s}A_j\varphi^j_{\alpha,1}(t)\right)\\
&= 1+\varphi_{\alpha,1}(t)+\left(\frac{1}{2}+A_2\right)\varphi^2_{\alpha,1}(t)+\left(\frac{1}{6}+A_2+A_3\right)\varphi^3_{\alpha,1}(t)+\dots\\
&\quad+\left(\frac{1}{m!}+\sum_{u=2}^{m}\frac{A_u}{(m-u)!}\right)\varphi^m_{\alpha,1}(t)+\dots+\left(\frac{1}{s!}+\sum_{u=2}^{s}\frac{A_u}{(s-u)!}\right)\varphi^s_{\alpha,1}(t)\\
&\quad+\sum_{k=s+1}^{\infty}\varphi^k_{\alpha,1}(t)\left(\frac{1}{k!}+\sum_{u=2}^{s}\frac{A_u}{(k-u)!}\right)=1+\varphi_{\alpha,1}(t)\\
&\quad+\sum_{k=2}^{s}\varphi^k_{\alpha,1}(t)\left(\frac{1}{k!}+\sum_{u=2}^{k}\frac{A_u}{(k-u)!}\right)+\sum_{k=s+1}^{\infty}\varphi^k_{\alpha,1}(t)\left(\frac{1}{k!}+\sum_{u=2}^{s}\frac{A_u}{(k-u)!}\right)\\
&= 1+\varphi_{\alpha,1}(t)+\sum_{k=s+1}^{\infty}\varphi^k_{\alpha,1}(t)\left(\frac{1}{k!}+\sum_{u=2}^{s}\frac{A_u}{(k-u)!}\right).
\end{aligned}$$

Thus,

$$f(t)-\widetilde{g}(t)=\sum_{n=1}^{\infty}c_1^{n/\alpha}\frac{\alpha}{\alpha-n}\frac{(it)^n}{n!}-\sum_{k=s+1}^{\infty}\varphi^k_{\alpha,1}(t)\left(\frac{1}{k!}+\sum_{u=2}^{s}\frac{A_u}{(k-u)!}\right).$$

From this expansion for fixed s we can obtain formulas for pseudomoments μ_j^*. Indeed, we have

$$f(t)-\widetilde{g}(t)=\sum_{n=1}^{[(1+s)\alpha]}c_1^{n/\alpha}\frac{\alpha}{\alpha-n}\frac{(it)^n}{n!}+o\left(|t|^{\alpha(s+1)}\right)\quad\text{as}\quad t\to 0,$$

whence from Lemma 3.25 it follows that

$$\mu_j^*=c_1^{j/\alpha}\frac{\alpha}{\alpha-j}\qquad\text{for}\quad j=1,\dots,[(1+s)\alpha].$$

It is important to note that s can be chosen *arbitrarily*. This means that for the Pareto distribution we can always construct such $\widetilde{H}$ that we have as many finite pseudomoments $\mu_j^*:=\mu_j(\widetilde{H})$ as we want. □

3.5.3 Main result

In this subsection we just formulate our main result concerning the asymptotic expansion of $F_n(x)$ and the remainder term estimate for the case $0<\alpha<1$.

Recall that we consider a sequence of i.i.d. random variables $X, X_1, X_2, \dots$ with distribution function $F\in\mathrm{DNA}(G_{\alpha,1})$ for $0<\alpha<1$. We suppose that F can be represented in the following form:

$$1-F(x)=\frac{c_1}{x^\alpha}+\frac{c_2\,d_2}{x^{2\alpha}}+\dots+\frac{c_s\,d_s}{x^{s\alpha}}+u(x),\quad x\to\infty,\tag{85}$$

where $s\in\mathbb{N}$ and $u(x)$ are such that

$$s\alpha\geq 1+\alpha,\qquad \int_0^{+\infty}x^q|du(x)|<\infty\quad\text{for some } q>s\alpha,\tag{86}$$

$c_i := C_i(\alpha)$, $i = 1, ..., s$, from (58) and d_i are suitable constants for $i = 2, \ldots, s$.

Our goal is to build an approximation of the distribution function F_n defined by

$$F_n(x) := P\left(\frac{S_n - a_n}{b_n} \le x\right),$$

where $S_n = X_1 + X_2 + \cdots + X_n$ and (a_n), (b_n) are some suitably chosen normalizing sequences such that $F_n \to G_{\alpha,1}$ as $n \to \infty$. From the definition of $\mathrm{DNA}(G_{\alpha,1})$ (see Definition 3.5) it follows that $b_n = a\,n^{1/\alpha}$ with $a > 0$. Without loss of generality we put $a = 1$, and also for $\alpha \in (0,1)$ we can take $a_n = 0$, $n \in \mathbb{N}$. This means, we have

$$F_n(x) = P\left(X_1 + \cdots + X_n \le xn^{1/\alpha}\right) = F^{n*}(xn^{1/\alpha}).$$

It is known that $F_n \to G_{\alpha,1}$, but we want to construct a correction term for $G_{\alpha,1}(x)$ in order to get a better approximation. Just as in Theorem 3.10 or Theorem 3.21 this correction function is a linear combination of derivatives of the corresponding limit distribution. The only difference is that the coefficients of this linear combination depend on new pseudomoments $\mu_k^* = \mu_k(F - \widetilde{G}_\alpha)$ defined by (79).

For a given distribution function F we construct a function $\widetilde{G}_\alpha$ using formula (69) and fix it. In terms of new pseudomoments μ_k^* for some $r \in \mathbb{R}_+$, $r > \alpha$ and $n \in \mathbb{N}$ we construct the function $\widetilde{W}_{r,n}(x)$ for all $x \in \mathbb{R}$ as follows:

$$\begin{aligned}\widetilde{W}_{r,n}(x) &= \sum_{k=2}^{\rho} \frac{c_{k,n}}{n^k} G^{(0,k)}(x,1) \\ &\quad + \sum_{k=0}^{p} \sum_{\ell=1}^{m_k} \binom{n}{\ell} \frac{c_{k,n-\ell}}{n^k} \sum_{u=\ell}^{m_{\ell,k}} \sum_{v=0}^{p_{u,\ell,k}} G^{(u,k+v)}(x,1) \frac{(-\ell/n)^v\,(-1)^u}{v!}\, n^{-u/\alpha}\, \widetilde{C}_{u,\ell},\end{aligned} \tag{87}$$

where $\rho = [2(R+1)/\alpha]$, $p = [2R/\alpha]$, $m_{\ell,k} = [R + 1 + \alpha(\ell - 1 - k/2)]$, $m_k = 1 + [(R - \alpha k/2)/(1-\alpha)]$, $p_{u,\ell,k} = \max\{0, [(R+1-u)/\alpha + \ell - 1 - k/2]\}$ with

$$R = \begin{cases} [r], & \text{if } [r] \ne r \\ r - 1, & \text{if } [r] = r \end{cases}, \tag{88}$$

$$c_{r,\rho} = \sum_{\substack{k_0+k_2+\cdots+k_s=\rho \\ 2k_2+\cdots+sk_s=r}} \frac{\rho!}{k_0!k_2!\cdots k_s!} A_2^{k_2}\cdots A_s^{k_s}, \qquad r,\rho \in \mathbb{N}_0, \tag{89}$$

s is from (85), A_j, $j = 2, ..., s$, are from (70), $G^{(u,k)}(x,1)$ are defined by (71) and

$$\widetilde{C}_{u,\ell} = \sum_{\substack{k_1+2k_2+\ldots+Rk_R=u \\ k_1+k_2+\ldots+k_R=\ell}} \frac{\ell!}{k_1!\ldots k_R!} \left(\frac{\mu_1^*}{1!}\right)^{k_1} \ldots \left(\frac{\mu_R^*}{R!}\right)^{k_R}, \qquad u = \ell, ..., m_{\ell,k}. \tag{90}$$

Then we have the following analogue of Theorem 3.21.

Theorem 3.26. *If for $r > 1$ we have $0 < \gamma_r^* < \infty$ and $\nu_0^* < 1$, then for all $x \in \mathbb{R}$ and all integers $n \ge 2$ the following inequality holds:*

$$\left|F_n(x) - G_{\alpha,1}(x) - \widetilde{W}_{r,n}(x)\right| \le C\,(1+|x|)^{-r}\, n^{-\frac{r-\alpha}{\alpha}} \left(1 + n^{\frac{r}{\alpha}}\, Q_n\right),$$

where constant C does not depend on x and n, $\widetilde{W}_{r,n}(x)$ is defined by (87) *and* $Q_n = \nu_0^{*\,n-1} + \left(\sup_{|t|>\widetilde{\varepsilon}}|f(t)| + 2\,\gamma_r^* n^{-r/\alpha}\right)^{n-1}$ *with $\widetilde{\varepsilon}$ defined as follows:*

$$\widetilde{\varepsilon} = \min\left\{1,\ \frac{1}{c_0},\ \frac{1}{(2D)^{1/\alpha}},\ \frac{1}{D^{1/\alpha}}\left(\frac{\cos\left(\frac{\alpha\pi}{2}\right)}{8D}\right)^{\frac{2+\rho/2}{\alpha}},\ \left(\frac{\cos\left(\frac{\alpha\pi}{2}\right)}{16\,e\,c_0}\right)^{1/(1-\alpha)}\right\}, \tag{91}$$

where $c_0 = (\nu_0^* + 1)\gamma_r^{*\,1/r}$, $D = \max\limits_{2\le j\le s}\{2\,|A_j|^{1/j}\}$, $\rho = [2(R+1)/\alpha]$ *with A_j defined by* (70) *and R defined by* (88)*.*

Proof. Section 4 will be devoted to the proof of this result. □

Remark 3.34. If $\nu_0^* < 1$, then for each $\varepsilon > 0$ we have $\sup_{|t|>\varepsilon}|f(t)| < 1$. Indeed, from formula (77) for $\widetilde{g}_\alpha(t)$ and from definition (80) of ν_0^* we have

$$|f(t) - \widetilde{g}_\alpha(t)| = \left|\int_{-\infty}^{+\infty} e^{itx} d\left(F - \widetilde{G}_\alpha\right)(x)\right| \le \int_0^{+\infty}\left|d\left(F - \widetilde{G}_\alpha\right)(x)\right| = \nu_0^*.$$

Then, using formulas (77), (74) for $\widetilde{g}_\alpha(t)$ and the fact that $\nu_0^* < 1$ we obtain:

$$\begin{aligned}|f(t)| &\le \nu_0^* + |\widetilde{g}_\alpha(t)| \le \nu_0^* + |g_{\alpha,1}(t)|\left(1 + \sum_{j=2}^{s}|A_j|\,|\varphi_{\alpha,1}^j(t)|\right)\\ &\le \nu_0^* + e^{-|t|^\alpha\cos(\alpha\pi/2)}\left(1 + \sum_{j=2}^{s}|A_j|\,|t|^{\alpha j}\right) < 1 \quad \text{for large } |t|.\end{aligned}$$

Finally, from Lemma A.16 it follows that $\sup_{|t|>\varepsilon}|f(t)| < 1$ for each $\varepsilon > 0$.

Remark 3.35. Note that there exists such $n_0 \in \mathbb{N}$ that $n^{r/\alpha} Q_n \le 1$ for all $n \ge n_0$, since $\nu_0^* < 1$ and, as a result, $\sup_{|t|>\widetilde{\varepsilon}}|f(t)| < 1$ (see Remark 3.34).

Example 3.11. Let us consider a Pareto-distributed random variable X with $\alpha = 1/2$ and $\kappa = c_1^2$, where $c_1 = C_1(1/2) = 1/\sqrt{\pi}$ is defined in (58). The distribution function F of X has the form:

$$1 - F(x) = \frac{1}{\sqrt{\pi}\sqrt{x}}, \quad x \ge 1/\pi.$$

Comparing this representation with representation (85) we can make the following conclusions: s can be chosen equal to 3 (since $3\alpha \ge 1 + \alpha$), $u(x) = 0$ and $d_3 = 0$. Coefficient d_2 can be chosen arbitrarily, since $c_2 = C_2(1/2) = 0$ (see formula (58)). Let us put $d_2 = 3/4$.

In order to apply Theorem 3.26 we have to check the condition $\nu_0^* < 1$ and to decide for which r we have $\gamma_r^* < \infty$ and $\nu_r^* < \infty$. According to the definition (80) we have

$$\nu_r^* = \int_0^\infty x^r\left|d\left(F - \widetilde{G}_{1/2}\right)(x)\right|, \quad r \ge 0,$$

where

$$\widetilde{G}_{1/2}(x) = G_{1/2,1}(x) + \sum_{j=2}^{3} A_j\, G^{(0,j)}(x,1)$$

with coefficients $A_2 = -1/8$ and $A_3 = -1/24$ (see (69) and (70)), chosen in such a way, that

$$F(x) - \widetilde{G}_{1/2}(x) = O\left(x^{-5/2}\right) \quad \text{as} \quad x \to \infty.$$

Note that we obtain $O(x^{-5/2})$ and not just $O(x^{-(s+1)\alpha}) = O(x^{-2})$, since $C_4(1/2) = 0$ in (57). Therefore, $\gamma^*_{5/2} < \infty$ and $\nu^*_r < \infty$ for $r < 5/2$. So, we apply Theorem 3.26 with $r = 5/2$.

Let us check whether the condition $\nu^*_0 < 1$ is satisfied. Note that the stable distribution corresponding to $G_{1/2,1}(x, \lambda)$ is a Lévy distribution which has an explicit density function

$$p_{1/2,1}(x, \lambda) = \frac{\lambda}{2\sqrt{\pi}}\, e^{-\frac{\lambda^2}{4\,x}}\, x^{-\frac{3}{2}}, \quad x > 0,\ \lambda > 0.$$

Using this we get

$$\nu^*_0 = \int_0^\infty \left| d\left(F - \widetilde{G}_{1/2}\right)(x)\right| = \int_0^\infty \left| p(x) - \widetilde{p}_{1/2}(x)\right| dx, \quad \text{where}$$

$$\widetilde{p}_{1/2}(x) = \frac{1}{384\sqrt{\pi}}\, e^{-\frac{1}{4\,x}}\, x^{-\frac{9}{2}}\, \left(192\, x^3 + 48\, x^2 - 18\, x + 1\right).$$

Using software *Mathematica* we obtain $\nu^*_0 \approx 0.32 < 1$. Thus, Theorem 3.26 is applicable. Now, let us see now how the correction term $\widetilde{W}_{5/2,n}(x)$ looks like in this particular case.

$$\widetilde{W}_{5/2,n}(x) = \sum_{k=2}^{12} \frac{c_{k,n}}{n^k} G^{(0,k)}(x, 1) + \sum_{k=0}^{8} \sum_{\ell=1}^{1+[4-k/2]} \binom{n}{\ell} \frac{c_{k,n-\ell}}{n^k} \sum_{u=\ell}^{m_{\ell,k}} \sum_{v=0}^{p_{u,\ell,k}} G^{(u,k+v)}(x, 1) \frac{(-\ell/n)^v\, (-1)^u}{v!}\, n^{-u/\alpha}\, \widetilde{C}_{u,\ell},$$

where $m_{\ell,k} = \left[3 + \frac{1}{2}\left(\ell - 1 - \frac{k}{2}\right)\right]$, $p_{u,\ell,k} = \max\left\{0, \left[2\,(3-u) + \ell - 1 - \frac{k}{2}\right]\right\}$ and

$$c_{r,\rho} = \sum_{\substack{k_0+k_2+k_3=\rho \\ 2\,k_2+3\,k_3=r}} \frac{\rho!}{k_0!\, k_2!\, k_3!} \left(\frac{-1}{8}\right)^{k_2} \left(\frac{-1}{24}\right)^{k_3}, \qquad r, \rho \in \mathbb{N}_0,$$

$$\widetilde{C}_{u,\ell} = \sum_{\substack{k_1+2\,k_2=u \\ k_1+k_2=\ell}} \frac{\ell!}{k_1!\, k_2!} \left(\frac{\mu^*_1}{1!}\right)^{k_1} \left(\frac{\mu^*_2}{2!}\right)^{k_2}, \qquad u = \ell, ..., m_{\ell,k}.$$

The first and the second pseudomoments μ^*_1, μ^*_2 can be found precisely using the same method as in Example 3.10. The only difference is that we do not put d_2 equal to 0. We get

$$\mu^*_1 = \int_0^\infty x\, d\left(F - \widetilde{G}_{1/2}\right)(x) = \frac{c_1^{1/\alpha}\,\alpha}{\alpha - 1} + \left(\frac{1}{2} + A_2\right) = \frac{3}{8} - \frac{1}{\pi},$$

$$\mu^*_2 = \int_0^\infty x^2\, d\left(F - \widetilde{G}_{1/2}\right)(x) = 2!\left(\frac{c_1^{2/\alpha}\,\alpha}{2(\alpha - 2)} - \left(\frac{1}{24} + \frac{1}{2}\,A_2 + A_3\right)\right) = \frac{1}{8} - \frac{1}{3\,\pi^2}.$$

From Theorem 3.26 it follows that for all $x \in \mathbb{R}$ and $n \geq 2$ we have

$$\left| F_n(x) - G_{1/2,1}(x) - \widetilde{W}_{5/2,n}(x) \right| \leq C \, (1 + |x|)^{-5/2} \, n^{-4} \, \Big(1 + n^5 \, Q_n \Big) \, ,$$

where $Q_n = \nu_0^{* n-1} + \Big(\sup_{|t| > \widetilde{\varepsilon}} |f(t)| + 2 \, \gamma_r^{*} n^{-5/(2\,\alpha)} \Big)^{n-1}$ with $\widetilde{\varepsilon}$ defined by (91). $\square$

4 Proof of the main result

4.1 Some auxiliary functions and plan of the proof

In order to prove Theorem 3.26 we need to introduce some auxiliary functions. The first one is a truncated distribution function. We define it for any fixed $n \in \mathbb{N}$ and $\xi \in [0, \infty)$ as follows:

$$\overline{\overline{F}}_{n,\xi}(y) = \begin{cases} F(y) & \text{for} \quad y \leq n^{1/\alpha}(1+\xi), \\ \widetilde{G}_\alpha(y) & \text{for} \quad y > n^{1/\alpha}(1+\xi), \end{cases} \qquad y \in \mathbb{R}. \tag{92}$$

Thus, we have a family of truncated functions: $\left(\overline{\overline{F}}_{n,\xi}\right)_{n\in\mathbb{N},\xi\in[0,\infty)}$.

We denote $\overline{\overline{H}}_{n,\xi}(x) := \overline{\overline{F}}_{n,\xi}(x) - \widetilde{G}_\alpha(x)$, i.e.

$$\overline{\overline{H}}_{n,\xi}(x) = \begin{cases} \widetilde{H}(x) = F(x) - \widetilde{G}_\alpha(x) & \text{for} \quad x \leq n^{1/\alpha}(1+\xi), \\ 0 & \text{for} \quad x > n^{1/\alpha}(1+\xi), \end{cases} \qquad x \in \mathbb{R}, \tag{93}$$

and consider pseudomoments $\overline{\overline{\mu}}_{i,n,\xi} = \mu_i\left(\overline{\overline{H}}_{n,\xi}\right)$ and $\overline{\overline{\nu}}_{r,n,\xi} = \nu_r\left(\overline{\overline{H}}_{n,\xi}\right)$. They are well-defined for the same reasons as those for which μ_i^* and ν_r^* are well-defined (see Remark 3.33). Moreover, for any $i \in \mathbb{N}_0$ and any $r \geq 0$ we have $|\overline{\overline{\mu}}_{i,n,\xi}| < \infty$ and $\overline{\overline{\nu}}_{r,n,\xi} < \infty$. Indeed, putting $N = n^{1/\alpha}(1+\xi)$, taking into account the possible jump of $\overline{\overline{H}}(x)$ at point $x = N$, and using the boundedness of ν_0^* (see Lemma 3.24 *(ii)*) and of $\widetilde{G}_\alpha$ (see (72)), we obtain

$$\begin{aligned} \overline{\overline{\nu}}_{r,n,\xi} &= \nu_r\left(\overline{\overline{H}}_{n,\xi}\right) = \int_{-\infty}^{+\infty} |x|^r \left|d\overline{\overline{H}}_{n,\xi}(x)\right| = \int_0^N x^r \left|d\widetilde{H}(x)\right| + N^r \left|\widetilde{H}(N)\right| \\ &\leq N^r \left(\int_0^N \left|d\widetilde{H}(x)\right| + \left|\widetilde{H}(N)\right|\right) \leq N^r \left(\nu_0^* + \left|F(N) - \widetilde{G}_\alpha(N)\right|\right) \\ &\leq N^r \left(\nu_0^* + 1 + \left|\widetilde{G}_\alpha(N)\right|\right) < \infty \quad \forall\, r \geq 0. \end{aligned} \tag{94}$$

From (94) and from Lemma 3.17 it follows that $|\overline{\overline{\mu}}_{i,n,\xi}| \leq \overline{\overline{\nu}}_{i,n,\xi} < \infty$ for any $i \in \mathbb{N}_0$. Note also that for all $n \in \mathbb{N}$ and $\xi \geq 0$ we have

$$\overline{\overline{F}}_{n,\xi}(x) = \overline{\overline{H}}_{n,\xi}(x) = 0 \quad \text{for} \quad x < 0. \tag{95}$$

This follows from Lemma 3.22 (i) and from the fact that we consider only non-negative random variables, i.e. we have $F(x) = 0$ for $x < 0$.

Let us consider one more function. We denote $\overline{\overline{M}}_{n,\xi}(x) := \overline{\overline{H}}_{n,\xi}(x) - \overline{\overline{H}}_{n,0}(x)$. Using the definition of $\overline{\overline{H}}_{n,\xi}$ we obtain

$$\overline{\overline{M}}_{n,\xi}(x) = \begin{cases} F(x) - \widetilde{G}_\alpha(x), & \text{if} \quad x \in \left(n^{1/\alpha},\, n^{1/\alpha}(1+\xi)\right], \\ 0, & \text{otherwise}, \end{cases} \qquad x \in \mathbb{R}. \tag{96}$$

Note that $\overline{\overline{M}}_{n,\xi}$ and $\overline{\overline{H}}_{n,\xi}$ are functions of bounded variation. This follows from the fact that $\widetilde{H}(x) = F(x) - \widetilde{G}_\alpha(x)$ is a function of bounded variation (see Remark 3.33). Thus, pseudomoments $\mu_k\left(\overline{\overline{M}}_{n,\xi}\right)$, $k \in \mathbb{N}_0$, and absolute pseudomoments $\nu_r\left(\overline{\overline{M}}_{n,\xi}\right)$,

$r \geq 0$, are well-defined. From the definition of $\overline{\overline{H}}_{n,\xi}$, $\overline{\overline{M}}_{n,\xi}$, Lemma 3.17 and inequality (94) it follows that

$$\left|\mu_i(\overline{\overline{M}}_{n,\xi})\right| \leq \nu_i(\overline{\overline{M}}_{n,\xi}) \leq \nu_i(\overline{\overline{H}}_{n,\xi}) = \overline{\overline{\nu}}_{i,n,\xi} < \infty, \quad i \in \mathbb{N}_0. \tag{97}$$

From Lemma 3.22 (*iii*) and from the definition of $\overline{\overline{M}}_{n,\xi}$ it follows that

$$\lim_{x\to+\infty} \overline{\overline{M}}_{n,\xi}(x) = 0 \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad \xi \geq 0. \tag{98}$$

The following two lemmata give some properties of pseudomoments $\overline{\overline{\mu}}_{i,n,\xi}$, $\mu_i(\overline{\overline{M}}_{n,\xi})$ and $\overline{\overline{\nu}}_{r,n,\xi}$ as well as the connection between them and pseudomoments μ_i^* and ν_r^*. Recall that for any $r \in \mathbb{R}_+$ we denote

$$R = \begin{cases} [r], & \text{if} \quad [r] \neq r, \\ r-1, & \text{if} \quad [r] = r. \end{cases} \tag{99}$$

Lemma 4.1. *For pseudomoments $\overline{\overline{\mu}}_{i,n,\xi}$, $\mu_i\left(\overline{\overline{M}}_{n,\xi}\right)$ and μ_i^* the following statements hold true.*

(i) $\overline{\overline{\mu}}_{0,n,\xi} = \mu_0\left(\overline{\overline{M}}_{n,\xi}\right) = 0$ *for all $n \in \mathbb{N}$ and $\xi \geq 0$;*

(ii) If for $r > 1$ we have $0 < \gamma_r^* < \infty$, then

$$\left|\overline{\overline{\mu}}_{u,n,\xi} - \mu_u^*\right| \leq 2\left(n^{1/\alpha}(1+\xi)\right)^{u-r}\gamma_r^*, \qquad u = 1,\dots,R,$$

where R is given by (99).

Proof. *(i)* We put $N = n^{1/\alpha}(1+\xi)$. Then,

$$\begin{aligned}
\overline{\overline{\mu}}_{0,n,\xi} &= \int_{-\infty}^{+\infty} d\overline{\overline{H}}_{n,\xi}(x) = \int_0^{+\infty} d\overline{\overline{H}}_{n,\xi}(x) = \int_0^{+\infty} d\overline{\overline{F}}_{n,\xi}(x) - \int_0^{+\infty} d\widetilde{G}_\alpha(x) \\
&= \int_0^N dF(x) + \int_N^{+\infty} d\widetilde{G}_\alpha(x) + \widetilde{G}_\alpha(N) - F(N) - \int_0^{+\infty} d\widetilde{G}_\alpha(x) \\
&= \int_0^N dF(x) - F(N) = 0.
\end{aligned}$$

Using the last equality and the definition of $\overline{\overline{M}}_{n,\xi}(x) = \overline{\overline{H}}_{n,\xi}(x) - \overline{\overline{H}}_{n,0}(x)$ we obtain

$$\mu_0\left(\overline{\overline{M}}_{n,\xi}\right) = \int_{-\infty}^{+\infty} d\overline{\overline{M}}_{n,\xi}(x) = \int_{-\infty}^{+\infty} d\overline{\overline{H}}_{n,\xi}(x) - \int_{-\infty}^{+\infty} d\overline{\overline{H}}_{n,0}(x) = \overline{\overline{\mu}}_{0,n,\xi} - \overline{\overline{\mu}}_{0,n,0} = 0.$$

(ii) Using the fact that $\lim_{x\to+\infty} \widetilde{H}(x) = 0$, which follows from equality (73), we have for $u = 1,\dots,R$:

$$\begin{aligned}
\left|\overline{\overline{\mu}}_{u,n,\xi} - \mu_u^*\right| &= \left|\int_0^{+\infty} y^u d\left(\left(\overline{\overline{H}}_{n,\xi} - \widetilde{H}\right)(y)\right)\right| = \left|\int_N^{+\infty} y^u d\widetilde{H}(y) + N^u\,\widetilde{H}(N)\right| \\
&\leq \int_N^{+\infty} y^u \left|d\widetilde{H}(y)\right| + \left|N^u \int_N^{+\infty} d\widetilde{H}(y)\right|
\end{aligned}$$

$$\begin{aligned} &\le \int_N^{+\infty} y^u \left| d\widetilde{H}(y) \right| + N^u \int_N^{+\infty} \left| d\widetilde{H}(y) \right| \\ &\le 2 \int_N^{+\infty} y^u \left| d\widetilde{H}(y) \right|. \end{aligned}$$

We distinguish two cases. For $[r] \neq r$ we obtain

$$\left| \overline{\overline{\mu}}_{u,n,\xi} - \mu_u^* \right| \le 2 \int_N^{+\infty} y^u \left| d\widetilde{H}(y) \right| \le 2\, N^{u-r}\, N^{r-[r]} \int_N^{+\infty} y^{[r]} \left| d\widetilde{H}(y) \right| \le 2\, N^{u-r} \gamma_r^*.$$

For $[r] = r$ we have

$$\begin{aligned} \left| \overline{\overline{\mu}}_{u,n,\xi} - \mu_u^* \right| &\le 2 \int_N^{+\infty} y^u \left| d\widetilde{H}(y) \right| \le 2\, N^{u-r}\, N \int_N^{+\infty} y^{r-1} \left| d\widetilde{H}(y) \right| \\ &\le 2\, N^{u-r} \left(\left| \int_0^N x^r d\widetilde{H}(x) \right| + N \int_N^{+\infty} y^{r-1} \left| d\widetilde{H}(y) \right| \right) \\ &\le 2\, N^{u-r} \gamma_r^*. \end{aligned}$$

This completes the proof of the lemma. □

Lemma 4.2. *If for $r > 1$ we have $0 < \gamma_r^* < \infty$, then*

(i)

$$\overline{\overline{\nu}}_{k,n,\xi} \le \nu_k^* \le (\nu_0^* + 1) \gamma_r^{*\, k/r}, \quad k = 1, \ldots, R,$$

where R is defined by (99).

(ii)

$$\overline{\overline{\nu}}_{q,n,\xi} \le C \left(n^{1/\alpha} (1+\xi) \right)^{q-r} \gamma_r^*, \quad q > r, \quad q \in \mathbb{R}, \tag{100}$$

where C is some constant, which depends only on q and r.

Proof. *(i)* As always we put $N = n^{1/\alpha}(1+\xi)$. Using the fact that $\lim_{x \to +\infty} \widetilde{H}(x) = 0$ we have for $k = 1, \ldots, R$:

$$\begin{aligned} \overline{\overline{\nu}}_{k,n,\xi} &= \int_{-\infty}^{+\infty} x^k \left| d\overline{\overline{H}}_{n,\xi}(x) \right| = \int_0^{+\infty} x^k \left| d\overline{\overline{H}}_{n,\xi}(x) \right| = \int_0^N x^k \left| d\widetilde{H}(x) \right| + N^k \left| \widetilde{H}(N) \right| \\ &= \int_0^N x^k \left| d\widetilde{H}(x) \right| + N^k \left| \int_N^{+\infty} d\widetilde{H}(x) \right| \\ &\le \int_0^N x^k \left| d\widetilde{H}(x) \right| + \int_N^{+\infty} x^k \left| d\widetilde{H}(x) \right| = \nu_k^*. \end{aligned}$$

The other inequality from *(i)* we obtain as follows with $z = \gamma_r^{*1/r}$:

$$\begin{aligned} \nu_k^* &= \int_0^{+\infty} x^k \left| d\widetilde{H}(x) \right| = \int_0^z x^k \left| d\widetilde{H}(x) \right| + \int_z^{+\infty} x^k \left| d\widetilde{H}(x) \right| \\ &\le z^k \int_0^z \left| d\widetilde{H}(x) \right| + \begin{cases} z^{k-r}\, z^{r-[r]} \int_z^{+\infty} x^{[r]} \left| d\widetilde{H}(x) \right|, & \text{if } [r] \neq r, \\ z^{k-r}\, z \int_z^{+\infty} x^{r-1} \left| d\widetilde{H}(x) \right|, & \text{if } [r] = r \end{cases} \\ &\le z^k \nu_0^* + z^{k-r} \gamma_r^* \le (\nu_0^* + 1) \gamma_r^{*\, k/r}. \end{aligned}$$

(ii) We denote $T(x) = -\int_x^{+\infty} z^R \left|d\widetilde{H}(z)\right|$. Using integration by parts, we obtain

$$\begin{aligned}
\overline{\overline{\nu}}_{q,n,\xi} &= \int_{-\infty}^{+\infty} x^q \left|d\overline{\overline{H}}_{n,\xi}(x)\right| = \int_0^{+\infty} x^q \left|d\overline{\overline{H}}_{n,\xi}(x)\right| = \int_0^N x^{q-R} dT(x) + N^q \left|\widetilde{H}(N)\right| \\
&= \left(x^{q-R}T(x)\right)\Big|_{x=0}^{x=N} - \int_0^N (q-R)\, x^{q-R-1}\, T(x)\, dx + N^{q-R} \int_N^{+\infty} x^R \left|d\widetilde{H}(x)\right| \\
&\le N^{q-R}T(N) + \int_0^N (q-R)\, x^{q-R-1} \int_x^{+\infty} z^R \left|d\widetilde{H}(z)\right| \, dx - N^{q-R}T(N) \\
&= (q-R)\cdot \begin{cases} \displaystyle\int_0^N x^{q-r-1}\, x^{r-[r]} \int_x^{+\infty} z^{[r]} \left|d\widetilde{H}(z)\right| \, dx, & \text{if} \quad r \neq [r], \\ \displaystyle\int_0^N x^{q-r-1}\, x \int_x^{+\infty} z^{r-1} \left|d\widetilde{H}(z)\right| \, dx, & \text{if} \quad r = [r] \end{cases} \\
&\le (q-R)\, \gamma_r^* \int_0^N x^{q-r-1} dx \le \frac{q-R}{q-r}\, N^{q-r} \gamma_r^*.
\end{aligned}$$

The lemma is proved. □

The following lemma gives us the estimation of absolute pseudomoments of $\overline{\overline{H}}{}_{n,\xi}^{\ell*}$ in terms of pseudomoments of $\overline{\overline{H}}_{n,\xi}$.

Lemma 4.3. *For all $n, \ell \in \mathbb{N}$, $\xi \ge 0$ and all $q \in \mathbb{N}_0$ we have*

$$\nu_q\left(\overline{\overline{H}}{}_{n,\xi}^{\ell*}\right) \le \ell^q\, \overline{\overline{\nu}}_{q,n,\xi}\, \overline{\overline{\nu}}{}_{0,n,\xi}^{\ell-1} < \infty. \tag{101}$$

In particular, the absolute pseudomoment $\nu_q\left(\overline{\overline{H}}{}_{n,\xi}^{\ell*}\right)$ is finite.

Proof. Using (95), the definition of the ℓ-fold convolution of $\overline{\overline{H}}_{n,\xi}$, inequality (148) from Lemma A.6 and the fact that $\overline{\overline{\nu}}_{r,n,\xi}$ is finite for any $r \ge 0$ (see (94)), we obtain for $q \in \mathbb{N}$:

$$\begin{aligned}
\nu_q\left(\overline{\overline{H}}{}_{n,\xi}^{\ell*}\right) &= \int_0^{+\infty} x^q \left|d\overline{\overline{H}}{}_{n,\xi}^{\ell*}(x)\right| = \int_0^{+\infty} x^q \left|d\int_0^{+\infty} \overline{\overline{H}}{}_{n,\xi}^{(\ell-1)*}(x-y_1)\, d\overline{\overline{H}}_{n,\xi}(y_1)\right| \\
&\le \underbrace{\int_0^{+\infty} \cdots \int_0^{+\infty}}_{\ell} (y_1 + \cdots + y_\ell)^q \left|d\overline{\overline{H}}_{n,\xi}(y_1)\right| \ldots \left|d\overline{\overline{H}}_{n,\xi}(y_\ell)\right| \\
&\le \ell^q \left(\int_0^{+\infty} \left|d\overline{\overline{H}}_{n,\xi}(x)\right|\right)^{\ell-1} \int_0^{+\infty} x^q \left|d\overline{\overline{H}}_{n,\xi}(x)\right| \le \ell^q\, \overline{\overline{\nu}}_{q,n,\xi}\, \overline{\overline{\nu}}{}_{0,n,\xi}^{\ell-1} < \infty.
\end{aligned}$$

Similarly we can show that $\nu_0\left(\overline{\overline{H}}{}_{n,\xi}^{\ell*}\right) \le \overline{\overline{\nu}}{}_{0,n,\xi}^{\ell}$. The lemma is proved. □

Below we give one more useful lemma about the pseudomoments of $\overline{\overline{H}}{}_{n,\xi}^{\ell*}$.

Lemma 4.4. *The following properties hold true.*

(i) For all $u, \ell \in \mathbb{N}$ we have

$$\mu_u\left(\overline{\overline{H}}{}_{n,\xi}^{\ell*}\right) = \sum_{k_1+k_2+\cdots+k_\ell=u} \frac{u!}{k_1! \cdots k_\ell!} \overline{\overline{\mu}}_{k_1,n,\xi} \cdots \overline{\overline{\mu}}_{k_\ell,n,\xi}.$$

(ii) For all $u, \ell \in \mathbb{N}$ with $u < \ell$ we have

$$\mu_u\left(\overline{\overline{H}}^{\ell *}_{n,\xi}\right) = 0.$$

(iii) For fixed $R \geq 1$, $\ell \in \mathbb{N}$ and $u = 1, ..., R$ we have

$$\mu_u\left(\overline{\overline{H}}^{\ell *}_{n,\xi}\right) = u! \sum_{\substack{k_1+k_2+\cdots+k_R=\ell \\ k_1+2k_2+\ldots+Rk_R=u}} \frac{\ell!}{k_1!\cdots k_R!}\left(\frac{\overline{\overline{\mu}}_{1,n,\xi}}{1!}\right)^{k_1}\cdots\left(\frac{\overline{\overline{\mu}}_{R,n,\xi}}{R!}\right)^{k_R}.$$

Proof. (i) Using (95) and rewriting $\mu_u\left(\overline{\overline{H}}^{\ell *}_{n,\xi}\right)$ and using formula (147) from Lemma A.5 we obtain

$$\begin{aligned}
\mu_u\left(\overline{\overline{H}}^{\ell *}_{n,\xi}\right) &= \int_0^{+\infty} y^u d\overline{\overline{H}}^{\ell *}_{n,\xi}(y) \\
&= \underbrace{\int_0^{+\infty}\ldots\int_0^{+\infty}}_{\ell}(z_1+\cdots+z_\ell)^u d\overline{\overline{H}}_{n,\xi}(z_1)\cdots d\overline{\overline{H}}_{n,\xi}(z_\ell) \\
&= \underbrace{\int_0^{+\infty}\ldots\int_0^{+\infty}}_{\ell}\sum_{k_1+k_2+\cdots+k_\ell=u}\frac{u!}{k_1!\cdots k_\ell!}z_1^{k_1}\cdots z_\ell^{k_\ell}d\overline{\overline{H}}_{n,\xi}(z_1)\ldots d\overline{\overline{H}}_{n,\xi}(z_\ell) \\
&= \sum_{k_1+k_2+\cdots+k_\ell=u}\frac{u!}{k_1!\cdots k_\ell!}\overline{\overline{\mu}}_{k_1,n,\xi}\cdots\overline{\overline{\mu}}_{k_\ell,n,\xi}.
\end{aligned}$$

(ii) If $u < \ell$, then there exists such $i \in \{1, \ldots, \ell\}$ that $k_i = 0$. Therefore each product $\overline{\overline{\mu}}_{k_1,n,\xi}\cdots\overline{\overline{\mu}}_{k_\ell,n,\xi}$ from above contains $\overline{\overline{\mu}}_{0,n,\xi}$. According to Lemma 4.1 (i) we have $\overline{\overline{\mu}}_{0,n,\xi} = 0$. This implies $\mu_u\left(\overline{\overline{H}}^{\ell *}_{n,\xi}\right) = 0$ for $u < \ell$.
(iii) Note that if $u < \ell$, then the right side of the equality in (iii) is equal to 0 since we have to sum up over the empty set. This fact together with statement (ii) gives us the statement of (iii) for $u < \ell$. Let us consider now $u \geq \ell$. Using the properties of the inverse Fourier transform $\overline{\overline{h}}_{n,\xi}(t)$ of $\overline{\overline{H}}_{n,\xi}(x)$ (similarly to Lemma 3.25) we find

$$\overline{\overline{h}}_{n,\xi}(t) := \int_{-\infty}^{+\infty} e^{itx} d\overline{\overline{H}}_{n,\xi}(x) = \frac{it}{1!}\overline{\overline{\mu}}_{1,n,\xi} + \cdots + \frac{(it)^R}{R!}\overline{\overline{\mu}}_{R,n,\xi} + O(|t|^{R+1}), \quad t \to 0.$$

Using this expansion and formula (147) from Lemma A.5 we obtain for $t \to 0$:

$$\begin{aligned}
\overline{\overline{h}}^{\ell}_{n,\xi}(t) &= \left(\frac{it}{1!}\overline{\overline{\mu}}_{1,n,\xi} + \ldots + \frac{(it)^R}{R!}\overline{\overline{\mu}}_{R,n,\xi}\right)^{\ell} + O(|t|^{R+1}) \\
&= \sum_{k_1+\ldots+k_R=\ell}\frac{\ell!}{k_1!\ldots k_R!}\left(\frac{\mu_{1,n,\xi}}{1!}\right)^{k_1}\ldots\left(\frac{\mu_{R,n,\xi}}{R!}\right)^{k_R}(it)^{k_1+2k_2+\ldots+Rk_R} + O(|t|^{R+1}) \\
&= \sum_{u=\ell}^{R}\frac{(it)^u}{u!}\,u!\sum_{\substack{k_1+k_2+\cdots+k_R=\ell \\ k_1+2k_2+\ldots+Rk_R=u}}\frac{\ell!}{k_1!\cdots k_R!}\left(\frac{\overline{\overline{\mu}}_{1,n,\xi}}{1!}\right)^{k_1}\cdots\left(\frac{\overline{\overline{\mu}}_{R,n,\xi}}{R!}\right)^{k_R} + O(|t|^{R+1}).
\end{aligned}$$

On the other hand we can consider $\overline{\overline{h}}^{\ell}_{n,\xi}(t)$ as the inverse Fourier transform of $\overline{\overline{H}}^{\ell *}_{n,\xi}(x)$, i.e. we have

$$\overline{\overline{h}}^{\ell}_{n,\xi}(t) = \frac{it}{1!}\,\mu_1\left(\overline{\overline{H}}^{\ell *}_{n,\xi}\right) + \cdots + \frac{(it)^R}{R!}\,\mu_R\left(\overline{\overline{H}}^{\ell *}_{n,\xi}\right) + O(|t|^{R+1}), \quad t \to 0.$$

Taking into account property (ii) and comparing coefficients in both representations of $\overline{\overline{h}}^{\ell}_{n,\xi}$ we get the statement of (iii). The lemma is proved. □

We need one more auxiliary function, which is similar to the function $\widetilde{W}_{r,n}$ from (87). For $r \in \mathbb{R}_+,\ r > 1,\ n \in \mathbb{N}$ and $\xi \in [0,\infty)$ we define

$$\begin{aligned}\overline{\overline{W}}_{r,n,\xi}(x) &= \sum_{k=2}^{\rho} \frac{c_{k,n}}{n^k} G^{(0,k)}(x,1) + \overline{\overline{W}}^{*}_{n,\xi}(x) \\ &+ \sum_{k=0}^{p}\sum_{\ell=1}^{m_k} \binom{n}{\ell} \frac{c_{k,n-\ell}}{n^k} \sum_{u=\ell}^{m_{\ell,k}} \sum_{v=0}^{p_{u,\ell,k}} G^{(u,k+v)}(x,1) \frac{(-\ell/n)^v\,(-1)^u}{v!}\, n^{-u/\alpha}\, \overline{\overline{C}}_{u,\ell},\end{aligned} \tag{102}$$

where $\rho = [2(R+1)/\alpha],\ p = [2\,R/\alpha],\ m_{\ell,k} = [R+1+\alpha(\ell-1-k/2)],$ $m_k = 1 + [(R-\alpha k/2)/(1-\alpha)],\ p_{u,\ell,k} = \max\{0, [(R+1-u)/\alpha + \ell - 1 - k/2]\}$ with

$$R = \begin{cases} [r], & \text{if} \quad [r] \neq r \\ r-1, & \text{if} \quad [r] = r \end{cases},$$

$$c_{r,\rho} = \sum_{\substack{k_0+k_2+\cdots+k_s=\rho \\ 2k_2+\cdots+sk_s=r}} \frac{\rho!}{k_0!k_2!\cdots k_s!} A_2^{k_2}\cdots A_s^{k_s}, \qquad r,\rho \in \mathbb{N}_0, \tag{103}$$

s and $A_j,\ j = 2, ..., s$, are from (86) and (70), respectively,

$$\overline{\overline{C}}_{u,\ell} = \sum_{\substack{k_1+2k_2+...+Rk_R=u \\ k_1+k_2+...+k_R=\ell}} \frac{\ell!}{k_1!...k_R!} \left(\frac{\overline{\overline{\mu}}_{1,n,\xi}}{1!}\right)^{k_1} ... \left(\frac{\overline{\overline{\mu}}_{R,n,\xi}}{R!}\right)^{k_R}, \tag{104}$$

and

$$\begin{aligned}\overline{\overline{W}}^{*}_{n,\xi}(x) &= n\,G^{(R+1,0)}(x,1)\,\frac{(-1)^{R+1}}{(R+1)!}\,n^{-\frac{R+1}{\alpha}}\,\overline{\overline{\mu}}_{R+1,n,\xi} \\ &+ n\left(G_{\alpha,1}(\,\cdot\,,n) * \overline{\overline{M}}_{n,\xi}\right)(xn^{1/\alpha}) - \sum_{w=0}^{R+1} n\,G^{(w,0)}(x,1)\,\frac{(-1)^w}{w!} n^{-w/\alpha}\mu_w\left(\overline{\overline{M}}_{n,\xi}\right)\end{aligned} \tag{105}$$

with $\overline{\overline{M}}_{n,\xi}(x) = \overline{\overline{H}}_{n,\xi}(x) - \overline{\overline{H}}_{n,0}(x)$.

Lemma 4.5. *For the function* $\overline{\overline{W}}_{r,n,\xi}$ *defined by* (102) *the following holds true.*

(i) $\overline{\overline{W}}_{r,n,\xi}$ *is absolutely continuous and differentiable on* $\mathbb{R}$ *with* $\overline{\overline{W}}_{r,n,\xi}(x) = 0$ *for* $x < 0$.

(ii) There exists such constant $\overline{\overline{W}} > 0$ *that* $\left|\overline{\overline{W}}_{r,n,\xi}(x)\right| \leq \overline{\overline{W}}$ *for all* $x \in \mathbb{R}$.

(iii) $\lim_{x\to+\infty} \overline{\overline{W}}_{r,n,\xi}(x) = 0$.

Proof. (i) According to the definition, $\overline{\overline{W}}_{r,n,\xi}$ is a linear combination of the term $\left(G_{\alpha,1}(\,\cdot\,,n) * \overline{\overline{M}}_{n,\xi}\right)(xn^{1/\alpha})$ and some derivatives of a stable distribution function $G_{\alpha,1}(x,\lambda)$. From Lemma 3.14 it follows that $G_{\alpha,1}(x,\lambda)$ is infinitely differentiable with respect to x and λ, and its derivatives are absolutely continuous. The function $G_{\alpha,1}(\,\cdot\,,n) * \overline{\overline{M}}_{n,\xi}$ is absolutely continuous and differentiable as a convolution of the absolutely continuous and infinitely differentiable function $G_{\alpha,1}$ and the function of bounded variation $\overline{\overline{M}}_{n,\xi}$. These facts together with the property $G_{\alpha,1}(x,\lambda) = 0$ for $x < 0$ and $\alpha < 1$ (see Section 3.3) and the fact that $\overline{\overline{M}}_{n,\xi}(x) = 0$ for $x < 0$ give the statement (i) of the lemma.

(ii) Let us consider $G_{\alpha,1}(\,\cdot\,,n) * \overline{\overline{M}}_{n,\xi}$. Using (97) and the fact that $G_{\alpha,1}$ is the distribution function of a stable random variable we have

$$\left|\left(G_{\alpha,1}(\,\cdot\,,n) * \overline{\overline{M}}_{n,\xi}\right)(xn^{1/\alpha})\right| = \left|\int_{-\infty}^{+\infty} G_{\alpha,1}\left(xn^{1/\alpha} - y, n\right) d\overline{\overline{M}}_{n,\xi}(y)\right|$$
$$\leq \int_{-\infty}^{+\infty} 1 \left|d\overline{\overline{M}}_{n,\xi}(y)\right| = \nu_0(\overline{\overline{M}}_{n,\xi}) \leq \nu_0(\overline{\overline{H}}_{n,\xi}) = \nu_{0,n,\xi} < \infty.$$

Using the last estimate and estimate (59) from Lemma 3.14 we obtain the inequality from (ii) for all $x \in \mathbb{R}$:

$$\left|\overline{\overline{W}}_{r,n,\xi}(x)\right| \leq \sum_{k=2}^{p} \frac{|c_{k,n}|}{n^k} D_{0,k} + \sum_{k=0}^{p}\sum_{\ell=1}^{m_k} \binom{n}{\ell} \frac{|c_{k,n-\ell}|}{n^k} \sum_{u=\ell}^{m_{\ell,k}} \sum_{v=0}^{p_{u,\ell,k}} D_{u,k+v} \frac{(\ell/n)^v}{v!}\, n^{-u/\alpha} \left|\overline{\overline{C}}_{u,\ell}\right|$$
$$+ n\,\overline{\overline{\nu}}_{0,n,\xi} + n\,D_{R+1,0}\,\frac{1}{(R+1)!}\, n^{-\frac{R+1}{\alpha}} \left|\overline{\overline{\mu}}_{R+1,n,\xi}\right| + \sum_{w=0}^{R+1} n\,D_{w,0}\,\frac{n^{-w/\alpha}}{w!} \left|\mu_w(\overline{\overline{M}}_{n,\xi})\right| =: \overline{\overline{W}}.$$

Note that all pseudomoments occurring in $\overline{\overline{W}}$ are finite (see (94) and (97)).

(iii) Let us show that $\lim\limits_{x\to+\infty} \left(G_{\alpha,1}(\,\cdot\,,n) * \overline{\overline{M}}_{n,\xi}\right)(xn^{1/\alpha}) = 0$. Using the definition of $G_{\alpha,1}$ and Lemma 4.1 (i) we obtain

$$\lim_{x\to+\infty} \left(G_{\alpha,1}(\,\cdot\,,n) * \overline{\overline{M}}_{n,\xi}\right)(xn^{1/\alpha}) = \lim_{x\to+\infty} \int_{-\infty}^{+\infty} G_{\alpha,1}\left(xn^{1/\alpha} - y, n\right) d\overline{\overline{M}}_{n,\xi}(y)$$
$$= \int_{-\infty}^{+\infty} \lim_{x\to+\infty} G_{\alpha,1}\left(xn^{1/\alpha} - y, n\right) d\overline{\overline{M}}_{n,\xi}(y) = \int_{-\infty}^{+\infty} 1\, d\overline{\overline{M}}_{n,\xi}(y) = \mu_0(\overline{\overline{M}}_{n,\xi}) = 0.$$

The convergence of all other terms to 0 as $x \to \infty$ can be proved in the same way as in Lemma 3.22 (iii). This completes the proof of the lemma.

□

Now we are ready to give a plan of the proof of our main result (Theorem (3.26)), which we repeat here for the sake of readability. Recall that

$$F_n(x) = P\left(X_1 + \cdots + X_n \leq xn^{1/\alpha}\right) = F^{n*}(xn^{1/\alpha}).$$

Theorem. *If for $r > 1$ we have $0 < \gamma_r^* < \infty$ and $\nu_0^* < 1$, then for all $x \in \mathbb{R}$ and all integers $n \geq 2$ the following inequality holds:*

$$\left|F_n(x) - G_{\alpha,1}(x) - \widetilde{W}_{r,n}(x)\right| \leq C\,(1+|x|)^{-r}\, n^{-\frac{r-\alpha}{\alpha}} \left(1 + n^{\frac{r}{\alpha}}\, Q_n\right),$$

where $\widetilde{W}_{r,n}(x)$ is defined by (87), $Q_n = \nu_0^{*\,n-1} + \left(\sup_{|t|>\widetilde{\varepsilon}} |f(t)| + 2\,\gamma_r^* n^{-r/\alpha}\right)^{n-1}$ *with $\widetilde{\varepsilon}$ defined by* (91) *and constant C does not depend on x and n.*

Plan of the proof.

In order to estimate $\left|F_n(x) - G_{\alpha,1}(x) - \widetilde{W}_{r,n}(x)\right|$ we add and subtract the auxiliary functions $\overline{\overline{F}}^{n*}_{n,\xi}$ and $\overline{\overline{W}}_{r,n,\xi}$ with some fixed $\xi \in [0,\infty)$, and come to the following inequality.

$$\left|F_n(x) - G_{\alpha,1}(x) - \widetilde{W}_{r,n}(x)\right| \leq \left|F_n(x) - \overline{\overline{F}}^{n*}_{n,\xi}\left(n^{1/\alpha}x\right)\right| + \left|\overline{\overline{F}}^{n*}_{n,\xi}\left(n^{1/\alpha}x\right) - G_{\alpha,1}(x) - \overline{\overline{W}}_{r,n,\xi}(x)\right| + \left|\overline{\overline{W}}_{r,n,\xi}(x) - \widetilde{W}_{r,n}(x)\right|.$$

We discuss and estimate each of the three summands on the right-hand side separately in the subsequent subsections. Combining the results we will then obtain the estimation from the theorem.

4.2 Estimation of $\left|F_n(x) - \overline{\overline{F}}^{n*}_{n,\xi}\left(n^{1/\alpha}x\right)\right|$

Recall that $\overline{\overline{F}}_{n,\xi}$ is defined by (92) and depends on $n \in \mathbb{N}$ and $\xi \in [0,\infty)$. First, we prove the following auxiliary lemma.

Lemma 4.6. *If for $r > 1$ we have $0 < \gamma_r^* < \infty$, then*

(i) $\displaystyle\int_0^{+\infty} \left|d\overline{\overline{F}}_{n,\xi}(y)\right| \leq 1 + 2\gamma_r^* n^{-r/\alpha}, \quad n \in \mathbb{N};$

(ii) $\displaystyle\sup_y \left|F^{n*}(y) - \overline{\overline{F}}^{n*}_{n,\xi}(y)\right| \leq C\gamma_r^* n^{-(r-\alpha)/\alpha}(1+\xi)^{-r}, \quad n \geq 2,$
where C is some constant that does not depend on n and ξ.

Proof. *(i)* Putting $N = n^{1/\alpha}(1+\xi)$ and using the inequality $|x| - |y| \leq |x - y|$, we obtain

$$\begin{aligned}
\int_0^{+\infty} \left|d\overline{\overline{F}}_{n,\xi}(y)\right| &\leq \int_0^N dF(y) + \int_N^{+\infty} \left|d\widetilde{G}_\alpha(y)\right| + \left|F(N) - \widetilde{G}_\alpha(N)\right| \\
&= 1 - \int_N^{+\infty} dF(y) + \int_N^{+\infty} \left|d\left(\widetilde{G}_\alpha(y) \pm F(y)\right)\right| + \left|\widetilde{H}(N)\right| \\
&\leq 1 + \int_N^{+\infty} \left|d\left(F(y) - \widetilde{G}_\alpha(y)\right)\right| + \left|\int_N^{+\infty} d\widetilde{H}(x)\right| \\
&\leq 1 + 2\int_N^{+\infty} \left|d\widetilde{H}(y)\right| \leq 1 + 2N^{-r}\gamma_r^* \\
&= 1 + 2n^{-r/\alpha}(1+\xi)^{-r}\gamma_r^* \leq 1 + 2n^{-r/\alpha}\gamma_r^*.
\end{aligned}$$

(ii) First, we represent the difference $F^{n*} - \overline{\overline{F}}^{n*}_{n,\xi}$ in another form. Using the fact that $F(x) = \overline{\overline{F}}_{n,\xi}(x) = 0$ for $x < 0$ we get

$$\begin{aligned}
&F^{n*}(y) - \overline{\overline{F}}^{n*}_{n,\xi}(y) = \\
&= F^{n*}(y) \pm F^{(n-1)*} * \overline{\overline{F}}_{n,\xi}(y) \pm \cdots \pm F * \overline{\overline{F}}^{(n-1)*}_{n,\xi}(y) - \overline{\overline{F}}^{n*}_{n,\xi}(y) \\
&= \sum_{j=0}^{n-1} \left(F^{(n-j)*} * \overline{\overline{F}}^{j*}_{n,\xi}(y) - F^{(n-j-1)*} * \overline{\overline{F}}^{(j+1)*}_{n,\xi}(y)\right) \qquad (106) \\
&= \sum_{j=0}^{n-1} \int_0^{+\infty} \left(F^{(n-j-1)*} * \overline{\overline{F}}^{j*}_{n,\xi}(y-u)\right) d\left(F - \overline{\overline{F}}_{n,\xi}\right)(u).
\end{aligned}$$

Using Lemma 4.6 (i) we estimate the m-fold convolutions of $\overline{\overline{F}}_{n,\xi}$ for $m = 1, 2, ..., n$, $n \in \mathbb{N}$:

$$\begin{aligned}
\left|\overline{\overline{F}}^{m*}_{n,\xi}(y)\right| &= \left|\underbrace{\int_0^{+\infty} \cdots \int_0^{+\infty}}_{m-1} \overline{\overline{F}}_{n,\xi}(y - u_1 - \cdots - u_{m-1})\, d\overline{\overline{F}}_{n,\xi}(u_1) \cdots d\overline{\overline{F}}_{n,\xi}(u_{m-1})\right| \\
&\leq \max\left\{1, \widetilde{G}\right\} \left(\int_0^{+\infty} \left|d\overline{\overline{F}}_{n,\xi}(y)\right|\right)^{m-1} \leq A\left(1 + 2\gamma_r^* n^{-r/\alpha}\right)^{m-1}, \qquad (107)
\end{aligned}$$

where $\widetilde{G}$ is defined in (72) and $A := \max\left\{1, \widetilde{G}\right\}$. Moreover, since F is a distribution function,

$$|F^{m*}(y)| \leq 1 \quad \text{for all} \quad m \in \mathbb{N}_0.$$

The above estimations lead us for $j = 1, \ldots, n-2$ to

$$\left|\left(F^{(n-1-j)*} * \overline{\overline{F}}^{j*}_{n,\xi}\right)(y)\right| \leq \sup_{y\in\mathbb{R}} \left|\overline{\overline{F}}^{j*}_{n,\xi}(y)\right| \leq A\left(1 + 2\gamma_r^* n^{-r/\alpha}\right)^{j-1}.$$

Using

$$\int_0^{+\infty} \left|d\left(F - \overline{\overline{F}}_{n,\xi}\right)(y)\right| = \int_N^{+\infty} \left|d\widetilde{H}(y)\right| + \left|\widetilde{H}(N)\right| \leq 2N^{-r}\gamma_r^* \qquad (108)$$

and the three above estimates, we obtain with $C = 2A$

$$\begin{aligned}
\left|F^{n*}(y) - \overline{\overline{F}}^{n*}_{n,\xi}(y)\right| &\leq 2\gamma_r^* N^{-r}\left(\sum_{j=1}^{n-1} A\left(1 + 2\gamma_r^* n^{-r/\alpha}\right)^{j-1} + 1\right) \\
&\leq 2\gamma_r^* N^{-r}\left(A\left(1 + 2\gamma_r^* n^{-r/\alpha}\right)^{n-2}(n-1) + 1\right) \\
&\leq 2A\gamma_r^* n N^{-r}\left(1 + 2\gamma_r n^{-r/\alpha}\right)^{n-2} \\
&= C\gamma_r^* n^{-(r-\alpha)/\alpha}\left(1 + 2\gamma_r^* n^{-r/\alpha}\right)^{n-2}(1+\xi)^{-r}.
\end{aligned}$$

Since the function $f(x) = (x-2)/x^{r/\alpha}$, $x \in (0, \infty)$, with $r > 1 > \alpha$ takes its maximum at $x_{\max} = 2r/(r-\alpha)$, and since $\gamma_r^* > 0$ and $n \geq 2$, we have the following estimation:

$$\left(1 + 2\gamma_r^* n^{-r/\alpha}\right)^{n-2} \leq \exp\left(\frac{2\gamma_r^*(n-2)}{n^{r/\alpha}}\right) \leq \exp\left(2\gamma_r^*\ f\left(\frac{2r}{r-\alpha}\right)\right) =: C < \infty.$$

This completes the proof of the lemma. □

Theorem 4.7. *If for $r > 1$ we have $0 < \gamma_r^* < \infty$, then for all $x \in \mathbb{R}$, $\xi \in [0, \infty)$ and all integers $n \geq 2$ the following inequality holds:*

$$\left|F_n(x) - \overline{\overline{F}}^{n*}_{n,\xi}\left(n^{1/\alpha}x\right)\right| \leq C\gamma_r^* n^{-(r-\alpha)/\alpha}(1+\xi)^{-r},$$

where C is some constant that does not depend on n and ξ.

Proof. The statement follows from the fact that

$$F_n(x) - \overline{\overline{F}}^{n*}_{n,\xi}\left(n^{1/\alpha}x\right) = F^{n*}\left(n^{1/\alpha}x\right) - \overline{\overline{F}}^{n*}_{n,\xi}\left(n^{1/\alpha}x\right)$$

and from Lemma 4.6 (ii). □

4.3 Estimation of $\left|\overline{\overline{F}}^{n*}_{n,\xi}\left(n^{1/\alpha}x\right) - G_{\alpha,1}(x) - \overline{\overline{W}}_{r,n,\xi}(x)\right|$

Estimation of this term is the most difficult part of the proof. That is why we start this subsection by giving the general plan of our actions.

Instead of the required difference we consider the function $A_{n,\xi} : \mathbb{R} \to \mathbb{R}$ with

$$A_{n,\xi}(x) = \overline{\overline{F}}^{n*}_{n,\xi}\Big(n^{1/\alpha}x\Big) - G_{\alpha,1}(x) - \overline{\overline{W}}_{r,n,\xi}(x) - \overline{\overline{H}}^{n*}_{n,\xi}\Big(n^{1/\alpha}x\Big). \tag{109}$$

Remark 4.1. Note that $A_{n,\xi}(x) = 0$ for $x < 0$, since all the components of the function are equal to 0 for $x < 0$.

The first goal of this subsection is to apply the following result of Christoph and Wolf [12] to $A_{n,\xi}(x)$.

Lemma 4.8 ([12]). *Let the function $A : \mathbb{R} \to \mathbb{R}$ be given and let $s \in \mathbb{N}_0$. If*

(i) $A(x)$ is absolutely continuous with $A(x) \to 0$ as $|x| \to \infty$,

(ii) $\int_{-\infty}^{+\infty} |x|^s\, |dA(x)| < \infty$,

(iii) and there exists a constant $K_s \geq 0$ such that

$$|A^{'}(x)| \leq K_s\,(1 + |x|)^{-s} \quad \text{for all } x \in \mathbb{R},$$

then

$$|A(x)| \leq C_s\,(1 + |x|)^{-s}\,\Big(I_0(T) + I_s(T) + K_s T^{-1}\Big) \qquad \text{for all } x \in \mathbb{R} \text{ and } T \geq 1,$$

where $C_s \geq 0$ depends only on s, and

$$I_m(T) = \int_{|t|\leq T} |d_m(t)|\,|t|^{-1}dt \quad \text{with} \quad d_m(t) = \int_{-\infty}^{+\infty} e^{itx} d\left(x^m A(x)\right),\ m \in \{0, s\}.$$

Proof. See [12, Theorem 1.16]. □

We will check each of the three conditions of Lemma 4.8 for $A = A_{n,\xi}$ separately. Conditions (i) and (ii) are easy to show. Verifying (iii) is fairly technical. We will divide the proof of it into several parts. After that we estimate $I_0(T)$ and $I_s(T)$ defined in the lemma and obtain the estimation of $A_{n,\xi}(x)$.
According to

$$\left|\overline{\overline{F}}^{n*}_{n,\xi}\left(n^{1/\alpha}x\right) - G_{\alpha,1}(x) - \overline{\overline{W}}_{r,n,\xi}(x)\right| \leq \left|A_{n,\xi}(x)\right| + \left|\overline{\overline{H}}^{n*}_{n,\xi}\Big(n^{1/\alpha}x\Big)\right|$$

we will need the estimation of $\overline{\overline{H}}^{n*}_{n,\xi}\Big(n^{1/\alpha}x\Big)$ as well. And as the last step we will combine everything in order to estimate $\left|\overline{\overline{F}}^{n*}_{n,\xi}\left(n^{1/\alpha}x\right) - G_{\alpha,1}(x) - \overline{\overline{W}}_{r,n,\xi}(x)\right|$.

4.3.1 Condition (i)

Let us show that condition (i) of Lemma 4.8 is satisfied.

Lemma 4.9. *The function $A_{n,\xi}(x)$ defined by* (109) *is absolutely continuous and differentiable on $\mathbb{R}$ with $A_{n,\xi}(x) \to 0$ as $|x| \to \infty$.*

Proof. First, we show that $A_{n,\xi}$ is an absolutely continuous and differentiable function. According to the definition of $\overline{\overline{H}}_{n,\xi}$ we have $\overline{\overline{F}}_{n,\xi} = \widetilde{G}_\alpha + \overline{\overline{H}}_{n,\xi}$. Using this and expanding the n-fold convolution of the sum we obtain

$$\begin{aligned} A_{n,\xi}(x) &= \left(\widetilde{G}_\alpha + \overline{\overline{H}}_{n,\xi}\right)^{n*}\left(n^{1/\alpha}x\right) - G_{\alpha,1}(x) - \overline{\overline{W}}_{r,n,\xi}(x) - \overline{\overline{H}}^{n*}_{n,\xi}\left(n^{1/\alpha}x\right) \\ &= \sum_{\ell=0}^{n-1}\binom{n}{\ell}\left(\widetilde{G}_\alpha^{(n-\ell)*} * \overline{\overline{H}}^{\ell*}_{n,\xi}\right)\left(n^{1/\alpha}x\right) - G_{\alpha,1}(x) - \overline{\overline{W}}_{r,n,\xi}(x). \end{aligned} \tag{110}$$

All terms after the last equality sign are absolutely continuous and differentiable on $\mathbb{R}$. This follows from the properties of convolution and Lemmata 3.22 (i) and 4.5 (i) for $\widetilde{G}_\alpha$ and $\overline{\overline{W}}_{r,n,\xi}$.

Now we show that $A_{n,\xi}(x) \to 0$ as $x \to \pm\infty$. Since $A(x) = 0$ for $x < 0$ (see Remark 4.1), it remains to show that $A_{n,\xi}(x) \to 0$ as $x \to +\infty$. From Lemma 3.22 (iii) it follows that $\widetilde{G}_\alpha(x) \to 1$ as $x \to +\infty$. Using this fact and the definitions of $\overline{\overline{F}}_{n,\xi}$ and $\overline{\overline{H}}_{n,\xi}$ we obtain

$$\overline{\overline{F}}^{n*}_{n,\xi}\left(n^{1/\alpha}x\right) \to 1, \quad x \to +\infty,$$
$$\overline{\overline{H}}^{n*}_{n,\xi}\left(n^{1/\alpha}x\right) \to 0, \quad x \to +\infty.$$

Function $G_{\alpha,1}$ is a distribution function of a stable random variable, so $G_{\alpha,1}(x) \to 1$. From Lemma 4.5 (iii) it follows that $\overline{\overline{W}}_{r,n,\xi}(x) \to 0$ as $x \to +\infty$. Combining all these limit equalities we obtain that $A_{n,\xi}(x) \to 0$ as $x \to \pm\infty$. The lemma is proved. □

4.3.2 Condition (ii)

In this subsection we will show that condition (ii) of Lemma 4.8 is satisfied.

Lemma 4.10. *If for $r > 1$ we have $0 < \gamma_r^* < \infty$, then for any $n \in \mathbb{N}$ and $\xi \in [0, \infty)$*

$$\int_{-\infty}^{+\infty} |x|^{R+1}\,|dA_{n,\xi}(x)| < \infty, \qquad \text{where } R = \begin{cases} [r], & r \neq [r], \\ r-1, & r = [r], \end{cases}$$

and $A_{n,\xi}$ is given by (109).

Proof. Since $A_{n,\xi}(x) = 0$ for $x < 0$ (see Remark 4.1), we have to consider only non-negative x and show that $\int_0^{+\infty} x^{R+1}\,|dA_{n,\xi}(x)| < \infty$. We split the last integral into

$$I_1 = \int_0^a x^{R+1}\,|dA_{n,\xi}(x)| \qquad \text{and} \qquad I_2 = \int_a^{+\infty} x^{R+1}\,|dA_{n,\xi}(x)|\,,$$

where $a \in (0, \infty)$ is a constant chosen as follows:

$$a = 2\,\max\left\{1, \max_{i,j}\{A_{i,j}\}\right\}, \quad i = 1, ..., R+3+[\alpha(n-2)],\ j = 0, 1, ..., sn + \left[\frac{R+1}{\alpha}\right],$$

where $s \in \mathbb{N}$ is such that $s\alpha \geq 1+\alpha$ (for details see Section 3.5.1, formula (85)), and $A_{i,j}$ are defined in Lemma 3.14.

Integral I_1. Since $A_{n,\xi}$ is absolutely continuous (see Lemma 4.9) it follows that $A_{n,\xi}$ is of bounded variation on $[0,a]$ (see [47, Lemma 5.11, p.108]). Using this fact and [27, Kapitel XV/574] we obtain that $I_1 = \int_0^a x^{R+1}\,|dA_{n,\xi}(x)| < \infty$.

Integral I_2. We start this case with rewriting $A_{n,\xi}(x)$ in another form. Using representation (110) of $A_{n,\xi}$ and applying Lemma 3.23 to $\widetilde{G}_\alpha^{n*}$ we obtain

$$\begin{aligned}
A_{n,\xi}(x) &= \widetilde{G}_\alpha^{n*}(n^{1/\alpha}x) + \sum_{\ell=1}^{n-1}\binom{n}{\ell}\left(\widetilde{G}_\alpha^{(n-\ell)*} * \overline{\overline{H}}{}_{n,\xi}^{\ell*}\right)(n^{1/\alpha}x) - G_{\alpha,1}(x) - \overline{\overline{W}}_{r,n,\xi}(x)\\
&= G_{\alpha,1}(x) + \sum_{k=2}^{sn}\frac{c_{k,n}}{n^k}G^{(0,k)}(x,1) - G_{\alpha,1}(x) - \overline{\overline{W}}_{r,n,\xi}(x)\\
&+ \sum_{\ell=1}^{n-1}\binom{n}{\ell}\int_{-\infty}^{+\infty}\widetilde{G}_\alpha^{(n-\ell)*}\left(n^{1/\alpha}(x-y)\right)d\overline{\overline{H}}{}_{n,\xi}^{\ell*}(n^{1/\alpha}y),
\end{aligned}$$

where coefficients $c_{k,n}$ are defined by (76). Taking into account that $c_{0,\rho}=1$ and $c_{1,\rho}=0$ for all $\rho = 1,\dots,n$, we apply Lemma 3.23 to $\widetilde{G}_\alpha^{(n-\ell)*}$ and come to the following representation:

$$\begin{aligned}
A_{n,\xi}(x) &= \sum_{k=2}^{sn}\frac{c_{k,n}}{n^k}G^{(0,k)}(x,1) - \overline{\overline{W}}_{r,n,\xi}(x)\\
&+ \sum_{\ell=1}^{n-1}\binom{n}{\ell}\sum_{k=0}^{s(n-\ell)}\frac{c_{k,n-\ell}}{n^k}\int_{-\infty}^{\infty}G^{(0,k)}\left(x-y,\frac{n-\ell}{n}\right)d\overline{\overline{H}}{}_{n,\xi}^{\ell*}\left(n^{1/\alpha}y\right).
\end{aligned}\tag{111}$$

Let us consider the integral $\int_{-\infty}^{\infty}G^{(0,k)}\left(x-y,\frac{n-\ell}{n}\right)d\overline{\overline{H}}{}_{n,\xi}^{\ell*}\left(n^{1/\alpha}y\right)$. Using estimate (59) from Lemma 3.14 and the fact that $x > a \geq 2$ we have for $|y| > x/2$:

$$\begin{aligned}
&\int_a^\infty x^{R+1}\int_{|y|>x/2}\left|G^{(1,k)}\left(x-y,\frac{n-\ell}{n}\right)\right|\left|d\overline{\overline{H}}{}_{n,\xi}^{\ell*}\left(n^{1/\alpha}y\right)\right|dx\\
&\leq \int_a^\infty x^{R+1}\int_{|y|>x/2}D_{1,k}\left(\frac{n-\ell}{n}\right)^{-k-1/\alpha}\left|d\overline{\overline{H}}{}_{n,\xi}^{\ell*}\left(n^{1/\alpha}y\right)\right|dx\\
&\leq C\int_a^\infty x^{R+1}\int_{|y|>x/2}y^{-R-3}\,y^{R+3}\left|d\overline{\overline{H}}{}_{n,\xi}^{\ell*}\left(n^{1/\alpha}y\right)\right|dx\\
&\leq C\int_a^\infty x^{R+1}\,(x/2)^{-R-3}\int_{|y|>x/2}y^{R+3}\left|d\overline{\overline{H}}{}_{n,\xi}^{\ell*}\left(n^{1/\alpha}y\right)\right|dx\\
&\leq C\,\nu_{R+3}\left(\overline{\overline{H}}{}_{n,\xi}^{\ell*}\right)\int_a^\infty x^{-2}dx < \infty,
\end{aligned}\tag{112}$$

where the last inequality holds, since the absolute $(R+3)$-pseudomoment of $\overline{\overline{H}}{}_{n,\xi}^{\ell*}$ is finite (see Lemma 4.3). Now consider the case $|y| \leq x/2$. Recall the notation

$$G^{(u,k)}\left(x-y,\frac{n-\ell}{n}\right) = \frac{d^{u+k}}{dz^u\,d\lambda^k}G_{\alpha,1}(z,\lambda)\bigg|_{\lambda=\frac{n-\ell}{n},\,z=x-y}.$$

Using Lemma A.8 with $z = x - y$ and $a = x$ we obtain the Taylor expansion for $G^{(1,k)}\left(x - y, \frac{n-\ell}{n}\right)$ with respect to the *first* variable at the point $a = x$:

$$G^{(1,k)}\left(x - y, \frac{n-\ell}{n}\right) = \sum_{u=0}^{m_{\ell,k}} G^{(u+1,k)}\left(x, \frac{n-\ell}{n}\right)\frac{(-y)^u}{u!} + G^{(m_{\ell,k}+2,k)}\left(x - \theta y, \frac{n-\ell}{n}\right)\frac{(-y)^{m_{\ell,k}+1}}{(m_{\ell,k}+1)!},$$

where $\theta \in (0,1)$ and $m_{\ell,k} = [R + 1 + \alpha(\ell - 1 - k/2)]$. It is easy to see that for all ℓ and k we have $m_{\ell,k} + 1 + \alpha k > R + 1$. Such choice of $m_{\ell,k}$ provides the convergence of the integral considered below. Indeed, using estimate (60) from Lemma 3.14 and the fact that $x - \theta y > x/2 > a/2 \geq \max\{1, \max_k\{A_{m_{\ell,k}+2,k}\}\}$ for $|y| \leq x/2$ we have:

$$\begin{aligned}
&\int_a^\infty x^{R+1}\int_{|y|\leq x/2}\left|G^{(m_{\ell,k}+2,k)}\left(x - \theta y, \frac{n-\ell}{n}\right)\frac{(-y)^{m_{\ell,k}+1}}{(m_{\ell,k}+1)!}\right|\left|d\overline{\overline{H}}^{\ell*}_{n,\xi}\left(n^{1/\alpha}y\right)\right|dx \\
&\leq C\int_a^\infty x^{R+1}\int_{|y|\leq x/2}(x-\theta y)^{-m_{\ell,k}-2-\alpha k}\,|y|^{m_{\ell,k}+1}\left|d\overline{\overline{H}}^{\ell*}_{n,\xi}\left(n^{1/\alpha}y\right)\right|dx \\
&\leq C\int_a^\infty x^{R-m_{\ell,k}-\alpha k-1}\int_{|y|\leq x/2}|y|^{m_{\ell,k}+1}\left|d\overline{\overline{H}}^{\ell*}_{n,\xi}\left(n^{1/\alpha}y\right)\right|dx \\
&\leq C\,\nu_{m_{\ell,k}+1}\left(\overline{\overline{H}}^{\ell*}_{n,\xi}\right)\int_a^\infty x^{-1-\delta}dx < \infty \qquad \text{with some} \quad \delta > 0.
\end{aligned}$$

Using the expansion of $G^{(1,k)}\left(x - y, \frac{n-\ell}{n}\right)$ considered above we can rewrite

$$\begin{aligned}
&\int_{|y|\leq x/2} G^{(1,k)}\left(x - y, \frac{n-\ell}{n}\right)d\overline{\overline{H}}^{\ell*}_{n,\xi}\left(n^{1/\alpha}y\right) \\
&\quad= \sum_{u=0}^{m_{\ell,k}}\int_{-\infty}^{+\infty} G^{(u+1,k)}\left(x, \frac{n-\ell}{n}\right)\frac{(-y)^u}{u!}\,d\overline{\overline{H}}^{\ell*}_{n,\xi}\left(n^{1/\alpha}y\right) \\
&\quad- \sum_{u=0}^{m_{\ell,k}}\int_{|y|>x/2} G^{(u+1,k)}\left(x, \frac{n-\ell}{n}\right)\frac{(-y)^u}{u!}\,d\overline{\overline{H}}^{\ell*}_{n,\xi}\left(n^{1/\alpha}y\right) \\
&\quad+ \int_{|y|\leq x/2} G^{(m_{\ell,k}+2,k)}\left(x - \theta y, \frac{n-\ell}{n}\right)\frac{(-y)^{m_{\ell,k}+1}}{(m_{\ell,k}+1)!}\,d\overline{\overline{H}}^{\ell*}_{n,\xi}\left(n^{1/\alpha}y\right), \qquad (113)
\end{aligned}$$

where for the second sum we have the following estimation (by using the same procedure as in (112)):

$$\int_a^\infty x^{R+1}\sum_{u=0}^{m_{\ell,k}}\int_{|y|>x/2}\left|G^{(u+1,k)}\left(x, \frac{n-\ell}{n}\right)\frac{(-y)^u}{u!}\right|\left|d\overline{\overline{H}}^{\ell*}_{n,\xi}\left(n^{1/\alpha}y\right)\right|dx < \infty. \qquad (114)$$

Now for each $n \in \mathbb{N}$ and each $\ell = 1, \ldots, n-1$ we apply Lemma A.8 with $z = 1 - \frac{\ell}{n}$, $a = 1$, and obtain the Taylor expansion for $G^{(u+1,k)}\left(x, \frac{n-\ell}{n}\right)$ with respect to the *second* variable at the point 1:

$$G^{(u+1,k)}\left(x, \frac{n-\ell}{n}\right) = \sum_{v=0}^{p_{u,\ell,k}} G^{(u+1,k+v)}(x,1)\frac{(-\ell/n)^v}{v!} + G^{(u+1,k+p_{u,\ell,k}+1)}\left(x, 1 - \theta\frac{\ell}{n}\right)\frac{(-\ell/n)^{p_{u,\ell,k}+1}}{(p_{u,\ell,k}+1)!}, \qquad (115)$$

where $\theta \in (0,1)$ and $p_{u,\ell,k} = \max\{0, [(R+1-u)/\alpha + \ell - 1 - k/2]\}$. It is easy to see that $u + \alpha(k + p_{u,\ell,k} + 1) > R + 1$. Such choice of $p_{u,\ell,k}$ together with estimate (60) for $x > a$ provides the finiteness of the following integral:

$$\begin{aligned} &\int_a^\infty x^{R+1} \int_{-\infty}^{+\infty} \left| G^{(u+1,k+p_{u,\ell,k}+1)}\left(x, 1-\theta\frac{\ell}{n}\right) \frac{(-\ell/n)^{p_{u,\ell,k}+1}(-y)^u}{(p_{u,\ell,k}+1)!\; u!} \right| \left| d\overline{\overline{H}}^{\ell *}_{n,\xi}\left(n^{1/\alpha}y\right)\right| dx \\ &\quad \le C \int_a^\infty x^{R+1}\, x^{-u-1-\alpha(k+p_{u,\ell,k}+1)}\, dx \int_{-\infty}^{+\infty} |y|^u \left| d\overline{\overline{H}}^{\ell *}_{n,\xi}\left(n^{1/\alpha}y\right)\right| \\ &\quad \le C\, \nu_u \left(\overline{\overline{H}}^{\ell *}_{n,\xi}\right) \int_a^\infty x^{-1-\delta} dx < \infty \qquad \text{with some} \quad \delta > 0. \end{aligned} \tag{116}$$

Define $P_{u,\ell,n} := \int_{-\infty}^{\infty} y^u d\overline{\overline{H}}^{\ell *}_{n,\xi}\left(n^{1/\alpha}y\right) = n^{-u/\alpha}\mu_u\left(\overline{\overline{H}}^{\ell *}_{n,\xi}\right)$. Note that $P_{u,\ell,n} = 0$ for all $u < \ell$ (it follows from Lemma 4.4 (ii)) and that $P_{u,\ell,n}/u! = n^{-u/\alpha}\,\overline{\overline{C}}_{u,\ell}$ for $u = 1, ..., R$ (see Lemma 4.4 (iii) and formula (104)). Substituting representation (113) and expansion (115) in formula (111), and using the definition of $\overline{\overline{W}}_{r,n,\xi}(x)$ in (102), $\overline{\overline{W}}^{*}_{n,\xi}(x)$ in (105), and two facts from above about $P_{u,\ell,n}$ we get

$$\begin{aligned} dA_{n,\xi}(x) = &\sum_{k=\rho+1}^{sn} \frac{c_{k,n}}{n^k} G^{(1,k)}(x,1)\, dx - \overline{\overline{W}}^{*\prime}_{n,\xi}(x)\, dx + \sum_{\ell=1}^{n-1} \binom{n}{\ell} \sum_{k=0}^{s(n-\ell)} \frac{c_{k,n-\ell}}{n^k} \cdot \\ &\cdot \left[\int_{|y|>x/2} \left(G^{(1,k)}\left(x-y, \frac{n-\ell}{n}\right) - \sum_{u=0}^{m_{\ell,k}} G^{(u+1,k)}\left(x, \frac{n-\ell}{n}\right) \frac{(-y)^u}{u!} \right) d\overline{\overline{H}}^{\ell *}_{n,\xi}\left(n^{1/\alpha}y\right) \right. \\ &+ \int_{|y|\le x/2} G^{(m_{\ell,k}+2,k)}\left(x-\theta y, \frac{n-\ell}{n}\right) \frac{(-y)^{m_{\ell,k}+1}}{(m_{\ell,k}+1)!} d\overline{\overline{H}}^{\ell *}_{n,\xi}\left(n^{1/\alpha}y\right) \\ &\left. + \sum_{u=0}^{m_{\ell,k}} G^{(u+1,k+p_{u,\ell,k}+1)}\left(x, 1-\theta\frac{\ell}{n}\right) \frac{(-\ell/n)^{p_{u,\ell,k}+1}}{(p_{u,\ell,k}+1)!} \frac{(-1)^u}{u!} P_{u,\ell,n} \right] dx \\ &+ \left(\sum_{\ell=1}^{n-1} \sum_{k=0}^{s(n-\ell)} \sum_{u=R+1}^{m_{\ell,k}} + \sum_{\ell=1}^{n-1} \sum_{k=p+1}^{s(n-\ell)} \sum_{u=\ell}^{R} + \sum_{k=0}^{p} \sum_{\ell=m_k+1}^{n-1} \sum_{u=\ell}^{R} \right) \binom{n}{\ell} \frac{c_{k,n-\ell}}{n^k} \\ &\qquad \cdot \sum_{v=0}^{p_{u,\ell,k}} \frac{\left(-\frac{\ell}{n}\right)^v (-1)^u}{v!u!} P_{u,\ell,n}\, G^{(u+1,k+v)}(x,1)\, dx \\ &- \sum_{k=0}^{p} \sum_{\ell=1}^{m_k} \binom{n}{\ell} \frac{c_{k,n-\ell}}{n^k} \sum_{u=R+1}^{m_{\ell,k}} \sum_{v=0}^{p_{u,\ell,k}} \frac{\left(-\frac{\ell}{n}\right)^v (-1)^u}{v!}\, n^{-u/\alpha}\, \overline{\overline{C}}_{u,\ell}\, G^{(u+1,k+v)}(x,1)\, dx. \end{aligned} \tag{117}$$

We need to show that $\int_a^{+\infty} x^{R+1} |dA_{n,\xi}(x)| < \infty$. We have already shown above (see (112), (114), (116)) that there is no problem with the summands in square brackets from representation (117). Let us check the convergence of the integral for the first term of (117). Taking into account that $\rho \in \mathbb{N}$ is chosen in such a way that $(\rho+1)\alpha > R+1$, and using estimate (60) from Lemma 3.14 for $x > a$ we obtain:

$$\int_a^{+\infty} x^{R+1} \left| \sum_{k=\rho+1}^{sn} \frac{c_{k,n}}{n^k} G^{(1,k)}(x,1)\, dx \right| \le \sum_{k=\rho+1}^{sn} \frac{|c_{k,n}|}{n^k} \int_a^{+\infty} x^{R+1} \left| G^{(1,k)}(x,1) \right| dx$$

$$\leq C \sum_{k=\rho+1}^{sn} \frac{|c_{k,n}|}{n^k} \int_a^{+\infty} x^{R+1-(1+\alpha k)} dx \leq C \sum_{k=\rho+1}^{sn} \frac{|c_{k,n}|}{n^k} \int_a^{+\infty} x^{-1-\delta} dx < \infty,$$

where $\delta > 0$. In a similar way we can show the convergence of the integral for all terms after the square brackets in (117). Namely, for the first and the last sums with $u \geq R+1$ we apply Lemma 3.14 and obtain for $x > a$:

$$\int_a^{+\infty} x^{R+1} \left|G^{(u+1,k+v)}(x,1)\right| dx \leq C \int_a^{+\infty} x^{R+1-(u+1+\alpha)} dx \leq C \int_a^{+\infty} x^{-1-\alpha} dx < \infty.$$

For $k \geq p+1$ with $p = [2R/\alpha]$ we have $R-u-\alpha(k+v) \leq R-1-\alpha(p+1) \leq -R-1$, and the integral of the second sum converges too. Finally, for $\ell \geq m_k+1$ and $k \leq p$ we get $R-u-\alpha(k+v) \leq R-\ell-\alpha k \leq R-(m_k+1)-\alpha k < -1$ (see estimation of (163) from Appendix B). Using this fact and Lemma 3.14 we obtain the following for the third sum:

$$\int_a^{+\infty} x^{R+1} \left|G^{(u+1,k+v)}(x,1)\right| dx \leq C \int_a^{+\infty} x^{R+1-(u+1+\alpha(k+v))} dx < \infty.$$

It remains to show that $\int_a^{+\infty} x^{R+1} \left|\overline{\overline{W}}^{*\prime}_{n,\xi}(x)\right| dx < \infty$. From (105) it follows that

$$\overline{\overline{W}}^{*\prime}_{n,\xi}(x) = n\, G^{(R+2,0)}(x,1) \frac{(-1)^{R+1}}{(R+1)!} n^{-\frac{R+1}{\alpha}} \overline{\overline{\mu}}_{R+1,n,\xi}$$
$$+ n \frac{d}{dx}\left(G_{\alpha,1}(\,\cdot\,,n) * \overline{\overline{M}}_{n,\xi}\right)(xn^{1/\alpha}) - \sum_{w=0}^{R+1} n\, G^{(w+1,0)}(x,1) \frac{(-1)^w}{w!} n^{-w/\alpha} \mu_w\left(\overline{\overline{M}}_{n,\xi}\right).$$

From Lemma 3.14 it follows that

$$\int_a^{+\infty} x^{R+1} \left|G^{(R+2,0)}(x,1)\right| dx \leq C \int_a^{+\infty} x^{R+1-(R+2+\alpha)} dx < \infty.$$

Let us consider the second term of $\overline{\overline{W}}^{*}_{n,\xi}(x)$. Using the definition of the convolution and formula (55) we get

$$\left(G_{\alpha,1}(\,\cdot\,,n) * \overline{\overline{M}}_{n,\xi}\right)(xn^{1/\alpha}) = \int_{-\infty}^{+\infty} G_{\alpha,1}(x-y,1)\, d\overline{\overline{M}}_{n,\xi}(yn^{1/\alpha}).$$

Now we distinguish two cases: $|y| > x/2$ and $|y| \leq x/2$. Using estimate (59) from Lemma 3.14, the fact that $x > a \geq 2$ and acting in the same way as in (112) we obtain for $|y| > x/2$:

$$\int_a^{\infty} x^{R+1} \int_{|y|>x/2} \left|G^{(1,0)}(x-y,1)\right| \left|d\overline{\overline{M}}_{n,\xi}\left(n^{1/\alpha}y\right)\right| dx \leq C\, \nu_{R+3}\left(\overline{\overline{M}}_{n,\xi}\right) \int_a^{\infty} x^{-2} dx < \infty,$$

where the last inequality holds, since the absolute $(R+3)$-pseudomoment of $\overline{\overline{M}}_{n,\xi}$ is finite (see (97)). Now consider the case $|y| \leq x/2$. Using Lemma A.8 with $z = x-y$ and $a = x$ we obtain the Taylor expansion for $G^{(1,0)}(x-y,1)$:

$$G^{(1,0)}(x-y,1) = \sum_{w=0}^{R+1} G^{(w+1,0)}(x,1) \frac{(-y)^w}{w!} + G^{(R+3,0)}(x-\theta y,1) \frac{(-y)^{R+2}}{(R+2)!},$$

where $\theta \in (0,1)$. Taking into account that $x-\theta y > x/2$ for $|y| \le x/2$ we get

$$\begin{aligned}
&\int_a^\infty x^{R+1} \int_{|y|\le x/2} \left| G^{(R+3,0)}(x-\theta y, 1) \frac{(-y)^{R+2}}{(R+2)!} \right| \left| d\overline{\overline{M}}_{n,\xi}\left(n^{1/\alpha} y\right)\right| dx \\
&\le C \int_a^\infty x^{R+1} \int_{|y|\le x/2} (x-\theta y)^{-R-3} |y|^{R+2} \left| d\overline{\overline{M}}_{n,\xi}\left(n^{1/\alpha} y\right)\right| dx \\
&\le C \int_a^\infty x^{R+1-R-3} \int_{|y|\le x/2} |y|^{R+2} \left| d\overline{\overline{M}}_{n,\xi}\left(n^{1/\alpha} y\right)\right| dx \\
&\le C\ \nu_{R+2}\left(\overline{\overline{M}}_{n,\xi}\right) \int_a^\infty x^{-2} dx < \infty.
\end{aligned}$$

Plugging the Taylor expansion into the formula for $\overline{\overline{W}}^{*\prime}_{n,\xi}$ and taking into account inequalities from above we get $I_2 = \int_a^{+\infty} x^{R+1} \left|\overline{\overline{W}}^{*\prime}_{n,\xi}(x)\right| dx < \infty$. This completes the proof of the lemma.

□

4.3.3 Condition *(iii)*

This subsection is devoted to the estimation of $A'_{n,\xi}(x)$. Our goal is to prove the following theorem, which states that condition *(iii)* of Lemma 4.8 is satisfied.

Theorem 4.11. *If for $r>1$ we have $0<\gamma_r^*<\infty$, then for all $x\in\mathbb{R}$, $n\in\mathbb{N}$ and $\xi\in[0,\infty)$ the following inequality holds:*

$$\left|A'_{n,\xi}(x)\right| \le K(n,\xi)(1+|x|)^{-R-1}, \qquad \textit{where } R = \begin{cases} [r], & r\ne [r], \\ r-1, & r=[r], \end{cases}$$

$$K(n,\xi) = C\,(1+\xi)^{R+1-r}\, n^{-\frac{r-\alpha}{\alpha}} \left(1 + n^{\frac{r+1}{\alpha}}\, Q_n\right)$$

with $Q_n = \nu_0^{*n-1} + \left(\sup_{|t|>\widetilde{\varepsilon}} |f(t)| + 2\,\gamma_r^* n^{-r/\alpha}\right)^{n-1}$ and constant C not depending on n and ξ, but depending on the pseudomoments. The constant $\widetilde{\varepsilon}$ will be defined in the proof (see formula (123)).

Remark 4.2. Since $A_{n,\xi}(x)=0$ for $x\le 0$ (see Remark 4.1), we need to consider only the case $x>0$ and show that $\left|A'_{n,\xi}(x)\right| \le K(n,\xi)(1+x)^{-R-1}$ for $x>0$.

Plan of the proof. The proof of this theorem is very long and technical. In order to make it more comprehensible for the reader we will at first give a rough sketch. Denote $a_{n,\xi}(t) := \int_{-\infty}^{+\infty} e^{itx} dA_{n,\xi}(x)$. Using the inversion formula for the Fourier transform and the Riemann-Lebesgue lemma we will come to the inequality

$$\left|A'_{n,\xi}(x)\right| \le \frac{x^{-R-1}}{2\pi} \left(\int_{|t|\le\varepsilon_n} + \int_{|t|>\varepsilon_n} \right) \left| a_{n,\xi}^{(R+1)}(t) \right| dt \qquad (118)$$

with some $\varepsilon_n>0$. Next step is the estimation of $\left|a_{n,\xi}^{(R+1)}(t)\right|$. Note that different methods will be used for that in the cases $|t|\le\varepsilon_n$ and $|t|>\varepsilon_n$. After that we will estimate each of the two integrals from (118) and summarize, which will lead us to the statement of Theorem 4.11.

Proof. The proof is carried out in four steps.

Step 1: Getting to the integrals

For $a_{n,\xi}(t) = \int_{-\infty}^{+\infty} e^{itx} dA_{n,\xi}(x)$ the following lemma holds true.

Lemma 4.12. *Function $a_{n,\xi}(t)$ is an $(R+1)$-times differentiable function and*

$$a^{(k)}_{n,\xi}(t) \to 0 \quad as \quad t \to \pm\infty \tag{119}$$

for all $k = 0, 1, \ldots, R+1$.

Proof. The first statement of the lemma follows from Lemma 4.10. Namely,

$$\left|a^{(k)}_{n,\xi}(t)\right| \leq \int_{-\infty}^{+\infty} |x|^k \, |dA_{n,\xi}(x)| < \infty \qquad \text{for } k = 0, 1, \ldots, R+1.$$

Denote $v_k(x) := (ix)^k A'_{n,\xi}(x)$ for $k = 0, 1, \ldots, R+1$. It follows from Lemma 4.10 that $v_k \in L^1(\mathbb{R})$, since

$$\int_{-\infty}^{+\infty} |v_k(x)| \, dx = \int_{-\infty}^{+\infty} |x|^k \left|A'_{n,\xi}(x)\right| dx = \int_{-\infty}^{+\infty} |x|^k \, |dA_{n,\xi}(x)| < \infty$$

for all $k = 0, 1, \ldots, R+1$. This makes the Riemann-Lebesgue lemma (see Lemma A.12) applicable and we obtain

$$a^{(k)}_{n,\xi}(t) = \int_{-\infty}^{+\infty} e^{itx} (ix)^k A'_{n,\xi}(x) dx \to 0 \qquad \text{as} \qquad |t| \to \infty.$$

□

Since the function $A_{n,\xi}$ is absolutely continuous and differentiable on $\mathbb{R}$ (see Lemma 4.9), its derivative can be represented as follows

$$A'_{n,\xi}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itx} a_{n,\xi}(t) dt \qquad \text{for all} \quad x \in \mathbb{R}.$$

Using integration by parts $(R+1)$ times and property (119) we obtain for $x > 0$:

$$\begin{aligned} A'_{n,\xi}(x) &= -\frac{i}{2\pi x} \int_{-\infty}^{+\infty} e^{-itx} a'_{n,\xi}(t) dt = \frac{i^2}{2\pi x^2} \int_{-\infty}^{+\infty} e^{-itx} a''_{n,\xi}(t) dt = \cdots \\ &= \frac{(-i)^{R+1}}{2\pi x^{R+1}} \int_{-\infty}^{+\infty} e^{-itx} a^{(R+1)}_{n,\xi}(t) dt. \end{aligned} \tag{120}$$

In order to prove Theorem 4.11 we need to estimate the integrals from the following inequality

$$\left|A'_{n,\xi}(x)\right| \leq \frac{x^{-R-1}}{2\pi} \int_{-\infty}^{+\infty} \left|a^{(R+1)}_{n,\xi}(t)\right| dt = \frac{x^{-R-1}}{2\pi} \left(\int_{|t| \leq \varepsilon_n} + \int_{|t| > \varepsilon_n} \right) \left|a^{(R+1)}_{n,\xi}(t)\right| dt$$

with some constant $\varepsilon_n > 0$. This will be done in the following two steps.

Step 2: Estimation of the first integral

Our task is to estimate the $(R+1)$-st derivative of $a_{n,\xi}(t)$ for $|t| \le \varepsilon_n$, and then the integral $\int_{|t|\le\varepsilon_n} \left|a_{n,\xi}^{(R+1)}(t)\right| dt$ itself.

First, let us represent $a_{n,\xi}(t)$ in another form. Denote $\overline{\overline{f}}_{n,\xi}(t) = \int_{-\infty}^{+\infty} e^{itx} d\overline{\overline{F}}_{n,\xi}(x)$, $\overline{\overline{w}}_{r,n,\xi}(t) = \int_{-\infty}^{+\infty} e^{itx} d\overline{\overline{W}}_{r,n,\xi}(x)$ and $\overline{\overline{h}}_{n,\xi}(t) = \int_{-\infty}^{+\infty} e^{itx} d\overline{\overline{H}}_{n,\xi}(x)$. Then by (109) we have

$$a_{n,\xi}(t) = \overline{\overline{f}}^{n}_{n,\xi}\Big(tn^{-1/\alpha}\Big) - g_{\alpha,1}(t) - \overline{\overline{w}}_{r,n,\xi}(t) - \overline{\overline{h}}^{n}_{n,\xi}\Big(tn^{-1/\alpha}\Big),$$

where $g_{\alpha,1}(t) = \exp(\varphi_{\alpha,1}(t))$ is the characteristic function of stable distribution $G_{\alpha,1}(x)$. Recall (see Lemma 3.25) that the inverse Fourier transform of $\widetilde{G}_\alpha(x)$ is

$$\widetilde{g}_\alpha(t) = \int_{-\infty}^{+\infty} e^{itx} d\widetilde{G}_\alpha(x) = g_{\alpha,1}(t) + \sum_{k=2}^{s} A_k g_{\alpha,1}(t)\varphi^k_{\alpha,1}(t).$$

Using the definition of $\overline{\overline{W}}_{r,n,\xi}(x)$ and the fact that the inverse Fourier transform of $G^{(u,k)}(x,1)$ is equal to $g_{\alpha,1}(t)\varphi^k_{\alpha,1}(t)(-it)^u$ we obtain

$$\begin{aligned}\overline{\overline{w}}_{r,n,\xi}(t) &= \int_{-\infty}^{+\infty} e^{itx} d\overline{\overline{W}}_{r,n,\xi}(x) = \sum_{k=2}^{\rho} \frac{c_{k,n}}{n^k} g_{\alpha,1}(t)\varphi^k_{\alpha,1}(t) + \overline{\overline{w}}^{*}_{n,\xi}(t)\\ &+ \sum_{k=0}^{p}\sum_{\ell=1}^{m_k} \binom{n}{\ell} \frac{c_{k,n-\ell}}{n^k} \sum_{u=\ell}^{m_{\ell,k}} \sum_{v=0}^{p_{u,\ell,k}} g_{\alpha,1}(t)\, \varphi^{k+v}_{\alpha,1}(t)\, (it)^u \frac{(-\ell/n)^v}{v!} n^{-u/\alpha}\, \overline{\overline{C}}_{u,\ell},\end{aligned} \tag{121}$$

where $c_{k,n}$, $\overline{\overline{C}}_{u,\ell}$, ρ, p, m_k, $m_{\ell,k}$ and $p_{u,\ell,k}$ are defined after formula (102), and

$$\begin{aligned}\overline{\overline{w}}^{*}_{n,\xi}(t) &= \int_{-\infty}^{+\infty} e^{itx} d\overline{\overline{W}}^{*}_{n,\xi}(x) = n\, g_{\alpha,1}(t) \frac{(it)^{R+1}}{(R+1)!} n^{-\frac{R+1}{\alpha}} \overline{\overline{\mu}}_{R+1,n,\xi}\\ &+ n\, g_{\alpha,1}(t) \int_{-\infty}^{+\infty} \left(e^{ixtn^{-\frac{1}{\alpha}}} - \sum_{w=0}^{R+1} \frac{(ixtn^{-\frac{1}{\alpha}})^w}{w!} \right) d\left(\overline{\overline{H}}_{n,\xi} - \overline{\overline{H}}_{n,0}\right)(x).\end{aligned}$$

Using the fact that $\overline{\overline{f}}_{n,\xi}(t) = \widetilde{g}_\alpha(t) + \overline{\overline{h}}_{n,\xi}(t)$ we have

$$\begin{aligned}a_{n,\xi}(t) &= \Big(\widetilde{g}_\alpha\Big(tn^{-1/\alpha}\Big) + \overline{\overline{h}}_{n,\xi}\Big(tn^{-1/\alpha}\Big)\Big)^n - g_{\alpha,1}(t) - \overline{\overline{w}}_{r,n,\xi}(t) - \overline{\overline{h}}^{n}_{n,\xi}\Big(tn^{-1/\alpha}\Big)\\ &= \widetilde{g}^n_\alpha\Big(tn^{-1/\alpha}\Big) + \sum_{\ell=1}^{n-1}\binom{n}{\ell} \widetilde{g}^{n-\ell}_\alpha\Big(tn^{-1/\alpha}\Big)\overline{\overline{h}}^{\ell}_{n,\xi}\Big(tn^{-1/\alpha}\Big) - g_{\alpha,1}(t) - \overline{\overline{w}}_{r,n,\xi}(t).\end{aligned}$$

Applying formula (78) for $\widetilde{g}^\rho_\alpha$ and using the representation of $\overline{\overline{w}}_{r,n,\xi}$ from (121) we continue

$$\begin{aligned}a_{n,\xi}(t) &= g_{\alpha,1}(t) + \sum_{k=2}^{sn} \frac{c_{k,n}}{n^k} g_{\alpha,1}(t)\varphi^k_{\alpha,1}(t)\\ &+ \sum_{\ell=1}^{n-1}\binom{n}{\ell} \sum_{k=0}^{s(n-\ell)} \frac{c_{k,n-\ell}}{(n-\ell)^k} g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right) \varphi^k_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right) \overline{\overline{h}}^{\ell}_{n,\xi}\Big(tn^{-\frac{1}{\alpha}}\Big)\\ &- g_{\alpha,1}(t) - \sum_{k=2}^{\rho} \frac{c_{k,n}}{n^k} g_{\alpha,1}(t)\varphi^k_{\alpha,1}(t) - \overline{\overline{w}}^{*}_{n,\xi}(t) -\end{aligned}$$

$$
\begin{aligned}
& - \sum_{k=0}^{p} \sum_{\ell=1}^{m_k} \binom{n}{\ell} \frac{c_{k,n-\ell}}{n^k} \sum_{u=\ell}^{m_{\ell,k}} \sum_{v=0}^{p_{u,\ell,k}} g_{\alpha,1}(t)\, \varphi_{\alpha,1}^{k+v}(t)\, (it)^u \frac{(-\ell/n)^v}{v!} n^{-u/\alpha} \overline{\overline{C}}_{u,\ell} \\
= & \sum_{k=\rho+1}^{sn} \frac{c_{k,n}}{n^k} g_{\alpha,1}(t) \varphi_{\alpha,1}^k(t) \\
& + \sum_{\ell=1}^{n-1} \binom{n}{\ell} \sum_{k=0}^{s(n-\ell)} \frac{c_{k,n-\ell}}{n^k} g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right) \varphi_{\alpha,1}^k(t)\, \overline{\overline{h}}_{n,\xi}^{\ell}\left(tn^{-\frac{1}{\alpha}}\right) - \overline{\overline{w}}_{n,\xi}^{*}(t) \\
& - \sum_{k=0}^{p} \sum_{\ell=1}^{m_k} \binom{n}{\ell} \frac{c_{k,n-\ell}}{n^k} \sum_{u=\ell}^{m_{\ell,k}} \sum_{v=0}^{p_{u,\ell,k}} g_{\alpha,1}(t)\, \varphi_{\alpha,1}^{k+v}(t)\, (it)^u \frac{(-\ell/n)^v}{v!} n^{-u/\alpha} \overline{\overline{C}}_{u,\ell} \\
= & \sum_{k=\rho+1}^{sn} \frac{c_{k,n}}{n^k} g_{\alpha,1}(t) \varphi_{\alpha,1}^k(t) \\
& + \sum_{k=0}^{p} \sum_{\ell=m_k+1}^{n-1} \binom{n}{\ell} \frac{c_{k,n-\ell}}{n^k} \varphi_{\alpha,1}^k(t)\, g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right) \overline{\overline{h}}_{n,\xi}^{\ell}\left(tn^{-\frac{1}{\alpha}}\right) \\
& + \sum_{\ell=1}^{n-1} \binom{n}{\ell} \sum_{k=p+1}^{s(n-\ell)} \frac{c_{k,n-\ell}}{n^k} \varphi_{\alpha,1}^k(t)\, g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right) \overline{\overline{h}}_{n,\xi}^{\ell}\left(tn^{-\frac{1}{\alpha}}\right) \\
& + \sum_{k=0}^{p} \sum_{\ell=1}^{m_k} \binom{n}{\ell} \frac{c_{k,n-\ell}}{n^k} \Bigg[g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right) \varphi_{\alpha,1}^k(t)\, \overline{\overline{h}}_{n,\xi}^{\ell}\left(tn^{-\frac{1}{\alpha}}\right) \\
& \qquad - \sum_{u=\ell}^{m_{\ell,k}} \sum_{v=0}^{p_{u,\ell,k}} g_{\alpha,1}(t)\, \varphi_{\alpha,1}^{k+v}(t)\, (it)^u \frac{(-\ell/n)^v}{v!} n^{-u/\alpha} \overline{\overline{C}}_{u,\ell} \Bigg] - \overline{\overline{w}}_{n,\xi}^{*}(t). \qquad (122)
\end{aligned}
$$

As we will show in Appendix B each of the four terms of $a_{n,\xi}(t)$ in (122) is differentiable. Now we differentiate and estimate the derivatives of these terms. The results are presented in the following four lemmata (their proofs are given in Appendix B). Recall that we consider $|t| \le \varepsilon_n$ with some $\varepsilon_n > 0$. Also note that in the following we use constants C that do not depend on n and ξ, but may depend on the pseudomoments.

Lemma 4.13. *Define* $d_{1n}(t) = \sum_{k=\rho+1}^{sn} \frac{c_{k,n}}{n^k} \varphi_{\alpha,1}^k(t)\, g_{\alpha,1}(t)$ *with* $\rho = [2(R+1)/\alpha]$. *Then for* $|t| \le \varepsilon n^{1/\alpha}$ *and* $q = 0, 1, ..., R+1$ *we have*

$$\left|d_{1n}^{(q)}(t)\right| \le C\, e^{-\frac{1}{4}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} n^{-\frac{R+1}{\alpha}}\ |t|^{-q} \left(|t|^{\alpha\left(\frac{\rho}{2}+\frac{1}{2}\right)} + |t|^{\alpha(\rho+2)}\right)$$

with constant C *not depending on* n, *and*

$$\varepsilon = \min\left\{1, \frac{1}{(2D)^{1/\alpha}}, \frac{1}{D^{1/\alpha}} \left(\frac{\cos\left(\frac{\alpha\pi}{2}\right)}{8D}\right)^{\frac{2+\rho/2}{\alpha}}\right\}, \qquad D = \max_{2\le j\le s}\{2\,|A_j|^{1/j}\},$$

where s *and* A_j, $j = 2, ..., s$, *are defined in* (85) *and* (70).

Lemma 4.14. *Define*

$$d_{2n}(t) = \sum_{k=0}^{p} \sum_{\ell=m_k+1}^{n-1} \binom{n}{\ell} \frac{c_{k,n-\ell}}{n^k} \varphi_{\alpha,1}^k(t)\, g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right) \overline{\overline{h}}_{n,\xi}^{\ell}\left(tn^{-\frac{1}{\alpha}}\right),$$

where $p = [2R/\alpha]$ *and* $m_k = 1 + \left[\left(R - \frac{\alpha k}{2}\right)/(1-\alpha)\right]$. *Then for* $|t| \le \varepsilon n^{1/\alpha}$ *we have*

$$\left|d_{2n}^{(q)}(t)\right| \le C\, e^{-\frac{|t|^\alpha}{4}\cos\left(\frac{\alpha\pi}{2}\right)} n^{-\frac{R+1-\alpha}{\alpha}} |t|^{R+1-q} \left(|t|^{\theta_1} + |t|^{\theta_2}\right), \qquad q = 0, 1, ..., R,$$
$$\left|d_{2n}^{(R+1)}(t)\right| \le C\, e^{-\frac{|t|^\alpha}{4}\cos\left(\frac{\alpha\pi}{2}\right)} n^{-\frac{r-\alpha}{\alpha}} \left(|t|^{\theta_1} + |t|^{\theta_2} + |t|^{\alpha p+1}\right) (1+\xi)^{R+1-r},$$

where $\theta_{1(2)} = \min_k(\max_k)\{m_k + \alpha k - R\} > 0$, r *comes from Theorem 3.26,* $\varepsilon = \min\left\{1, \frac{1}{c_0}, \left(\frac{\cos\left(\frac{\alpha\pi}{2}\right)}{8\,e\,c_0}\right)^{1/(1-\alpha)}\right\}$ *with* $c_0 = (\nu_0^* + 1)\gamma_r^{*\,1/r}$, *and constants* C *do not depend on* n *and* ξ.

Lemma 4.15. *Define*

$$d_{3n}(t) = \sum_{\ell=1}^{n-1} \binom{n}{\ell} \sum_{k=p+1}^{s(n-\ell)} \frac{c_{k,n-\ell}}{n^k}\, \varphi_{\alpha,1}^k(t)\, g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right) \overline{\overline{h}}_{n,\xi}^{\ell}\left(tn^{-\frac{1}{\alpha}}\right),$$

where $p = [2\,R/\alpha]$. *Then for* $|t| \le \varepsilon n^{1/\alpha}$ *we have*

$$\left|d_{3n}^{(q)}(t)\right| \le C\, e^{-\frac{|t|^\alpha}{8}\cos\left(\frac{\alpha\pi}{2}\right)} n^{-\frac{R+1-\alpha}{\alpha}} |t|^{1-q} \left(|t|^{\alpha\left(\frac{p}{2}+\frac{1}{2}\right)} + |t|^{\alpha(p+2)}\right), \quad q = 0, 1, ..., R,$$
$$\left|d_{3n}^{(R+1)}(t)\right| \le C\, e^{-\frac{|t|^\alpha}{8}\cos\left(\frac{\alpha\pi}{2}\right)} n^{-\frac{r-\alpha}{\alpha}} \left(|t|^{\frac{\alpha(p+1)}{2}-R} + |t|^{\alpha(p+2)+1}\right) (1+\xi)^{R+1-r},$$

where r *comes from Theorem 3.26, constants* C *do not depend on* n *and* ξ, *and*

$$\varepsilon = \min\left\{1, \frac{1}{c_0}, \frac{1}{(2D)^{1/\alpha}}, \frac{1}{D^{1/\alpha}} \left(\frac{\cos\left(\frac{\alpha\pi}{2}\right)}{8D}\right)^{\frac{2+p/2}{\alpha}}, \left(\frac{\cos\left(\frac{\alpha\pi}{2}\right)}{16\,e\,c_0}\right)^{1/(1-\alpha)}\right\}$$

with $D = \max_{2\le j\le s}\{2\,|A_j|^{1/j}\}$, $c_0 = (\nu_0^* + 1)\gamma_r^{*\,1/r}$, s *and* A_j *defined in* (85) *and* (70).

Lemma 4.16. *Define*

$$d_{4n}(t) = \sum_{k=0}^{p}\sum_{\ell=1}^{m_k} \binom{n}{\ell} \frac{c_{k,n-\ell}}{n^k} \Bigg[g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right) \varphi_{\alpha,1}^k(t)\, \overline{\overline{h}}_{n,\xi}^{\ell}\left(tn^{-\frac{1}{\alpha}}\right) - \sum_{u=\ell}^{m_{\ell,k}} \sum_{v=0}^{p_{u,\ell,k}} \overline{\overline{C}}_{u,\ell}\, \frac{(-\ell/n)^v}{v!}\, n^{-u/\alpha}\, g_{\alpha,1}(t)\, \varphi_{\alpha,1}^{k+v}(t)\, (it)^u \Bigg] - \overline{\overline{w}}_{n,\xi}^{*}(t),$$

where $p = [2R/\alpha]$, $m_k = 1 + [(R - \alpha k/2)/(1-\alpha)]$, $m_{\ell,k} = [R + 1 + \alpha(\ell - 1 - k/2)]$ *and* $p_{u,\ell,k} = \max\{0, [(R+1-u)/\alpha + \ell - 1 - k/2]\}$ *and*

$$\overline{\overline{C}}_{u,\ell} = \sum_{\substack{k_1+2k_2+...+Rk_R=u \\ k_1+k_2+...+k_R=\ell}} \frac{\ell!}{k_1!...k_R!} \left(\frac{\overline{\overline{\mu}}_{1,n,\xi}}{1!}\right)^{k_1} ... \left(\frac{\overline{\overline{\mu}}_{R,n,\xi}}{R!}\right)^{k_R},$$

$$\overline{\overline{w}}_{n,\xi}^{*}(t) = n\, g_{\alpha,1}(t) \int_{-\infty}^{+\infty} \left(e^{ixtn^{-\frac{1}{\alpha}}} - \sum_{w=0}^{R+1} \frac{(ixtn^{-\frac{1}{\alpha}})^w}{w!} \right) d\left(\overline{\overline{H}}_{n,\xi} - \overline{\overline{H}}_{n,0}\right)(x)$$
$$+ n\, g_{\alpha,1}(t) \frac{(it)^{R+1}}{(R+1)!}\, n^{-\frac{R+1}{\alpha}}\, \overline{\overline{\mu}}_{R+1,n,\xi}.$$

Then for $|t| \le \varepsilon\, n^{1/\alpha}$ and $q = 0, 1, ..., R+1$ we have

$$\left|d_{4n}^{(q)}(t)\right| \le C\, e^{-\frac{1}{2}|t|^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)}\, n^{-\frac{r-\alpha}{\alpha}}\, (1+\xi)^{R+1-r}\, |t|^{R+1-q} \left(|t|^{\theta} + |t|^{R\max\left\{1, \frac{\alpha}{1-\alpha}\right\}}\right),$$

where $\theta = \min_u\{u + \alpha(p_{u,1,0}+1) - (R+1)\} \in (0, \alpha]$, $\varepsilon = \min\left\{1,\, c_0^{-1}\right\}$ with $c_0 = (\nu_0^* + 1)\gamma_r^{*\,1/r}$, pseudomoments ν_0^*, γ_r^* are defined in (80)*, r comes from Theorem 3.26, R is defined by* (99) *and constant C does not depend on n and ξ.*

The proofs of Lemmata 4.13 - 4.16 are given in Appendix B.

Now we come back to the main task of this step, namely, to the estimation of the integral $\int_{|t|\le\varepsilon_n}\left|a_{n,\xi}^{(R+1)}(t)\right| dt$. Using representation (122) and Lemmata 4.13 - 4.16 we obtain

$$\int_{|t|\le\varepsilon_n} \left|a_{n,\xi}^{(R+1)}(t)\right| dt \le \sum_{k=1}^{4} \int_{|t|\le\varepsilon_n} \left|d_{kn}^{(R+1)}(t)\right| dt.$$

Let us specify ε_n. We have to find such a region $|t| \le \varepsilon_n$ that Lemmata 4.13 - 4.16 are satisfied. Let us take

$$\varepsilon_n = \widetilde{\varepsilon}\, n^{1/\alpha} \quad \text{with}$$

$$\widetilde{\varepsilon} = \min\left\{1,\ \frac{1}{c_0},\ \frac{1}{(2D)^{1/\alpha}},\ \frac{1}{D^{1/\alpha}} \left(\frac{\cos\left(\frac{\alpha\pi}{2}\right)}{8D}\right)^{\frac{2+\rho/2}{\alpha}},\ \left(\frac{\cos\left(\frac{\alpha\pi}{2}\right)}{16\, e\, c_0}\right)^{1/(1-\alpha)}\right\}, \tag{123}$$

where $c_0 = (\nu_0^* + 1)\gamma_r^{*\,1/r},\ D = \max_{2\le j\le s}\{2\,|A_j|^{1/j}\},\ \rho = [2(R+1)/\alpha]$.

When estimating the integrals of $d_{kn}^{(R+1)}(t)$, $k = 1, ..., 4$, we have to deal with only one pattern: $|t|^s \exp\{-b\,|t|^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)\}$ with different powers $s > 0$ and constants $b > 0$. Using Lemma A.14 we obtain

$$\int_{|t|\le\varepsilon_n} e^{-b\,|t|^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)} |t|^s dt \le \int_{-\infty}^{+\infty} e^{-b\,|t|^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)}\, |t|^s dt \le C,$$

where $C > 0$ depends on α, b and s. Now, using this inequality we get

$$\int_{|t|\le\varepsilon_n} \left|a_{n,\xi}^{(R+1)}(t)\right| dt \le C\, n^{-\frac{R+1}{\alpha}} + C\,(1+\xi)^{R+1-r}\, n^{-\frac{r-\alpha}{\alpha}} \le C\,(1+\xi)^{R+1-r}\, n^{-\frac{r-\alpha}{\alpha}}.$$

This completes the second step.

Step 3: Estimation of the second integral.

Our task is to estimate the $(R+1)$-st derivative of $a_{n,\xi}(t)$ for $|t| > \varepsilon_n$, and then the integral $\int_{|t|>\varepsilon_n} \left| a_{n,\xi}^{(R+1)}(t) \right| dt$ itself.

Using the definition of $a_{n,\xi}$ and the fact that $\overline{\overline{f}}_{n,\xi}(t) - \overline{\overline{h}}_{n,\xi}(t) = \widetilde{g}_\alpha(t)$ we obtain

$$\begin{aligned} a_{n,\xi}(t) &= \overline{\overline{f}}^{n}_{n,\xi}\left(tn^{-1/\alpha}\right) - g_{\alpha,1}(t) - \overline{\overline{w}}_{r,n,\xi}(t) - \overline{\overline{h}}^{n}_{n,\xi}\left(tn^{-1/\alpha}\right) \\ &= \underbrace{\widetilde{g}_\alpha\left(tn^{-1/\alpha}\right) \sum_{\ell=0}^{n-1} \overline{\overline{f}}^{n-1-\ell}_{n,\xi}\left(tn^{-1/\alpha}\right) \overline{\overline{h}}^{\ell}_{n,\xi}\left(tn^{-1/\alpha}\right)}_{=:b_{1n}(t)} - \underbrace{\left(g_{\alpha,1}(t) + \overline{\overline{w}}_{r,n,\xi}(t)\right)}_{=:b_{2n}(t)}. \end{aligned} \tag{124}$$

Next we differentiate $b_{1n}(t)$ and $b_{2n}(t)$ separately and get the following lemmata. Their proofs are given in Appendix B.

Lemma 4.17. *Define* $b_{1n}(t) = \sum_{\ell=0}^{n-1} \widetilde{g}_\alpha\left(tn^{-1/\alpha}\right) \overline{\overline{f}}^{n-1-\ell}_{n,\xi}\left(tn^{-1/\alpha}\right) \overline{\overline{h}}^{\ell}_{n,\xi}\left(tn^{-1/\alpha}\right)$. *Then for* $|t| > \varepsilon_n$ *we have*

$$\left| b_{1n}^{(q)}(t) \right| \le \begin{cases} C\, e^{-\frac{|t|^\alpha}{4n}\cos\left(\alpha\frac{\pi}{2}\right)} \left(\nu_0^{*n-1} + \overline{\overline{Q}}^{n-1}_{n,\xi} \right) n^{q\,(1-\frac{1}{\alpha})}, & q = 0, ..., R; \\ C\, e^{-\frac{|t|^\alpha}{4n}\cos\left(\alpha\frac{\pi}{2}\right)} \left(\nu_0^{*n-1} + \overline{\overline{Q}}^{n-1}_{n,\xi} \right) n^{R+1-\frac{r}{\alpha}} \, (1+\xi)^{R+1-r}, & q = R+1, \end{cases}$$

where $\overline{\overline{Q}}_{n,\xi} = \sup_{|t|>\widetilde{\varepsilon}} \left| \overline{\overline{f}}_{n,\xi}(t) \right|$, r *comes from Theorem 3.26,* $\varepsilon_n = \widetilde{\varepsilon} n^{1/\alpha}$ *with* $\widetilde{\varepsilon}$ *defined in* (123), *and constants* C *do not depend on* n *and* ξ.

Lemma 4.18. *Define* $b_{2n}(t) = g_{\alpha,1}(t) + \overline{\overline{w}}_{r,n,\xi}(t)$. *Then for* $|t| > \varepsilon_n$ *we have*

$$\left| b_{2n}^{(q)}(t) \right| \le C\, e^{-\frac{1}{4}|t|^\alpha \cos\left(\alpha\frac{\pi}{2}\right)} n^{-\frac{r-\alpha}{\alpha}} \, (1+\xi)^{R+1-r}, \qquad q = 0, 1, ..., R+1,$$

where r *comes from Theorem 3.26,* $\varepsilon_n = \widetilde{\varepsilon} n^{1/\alpha}$ *with* $\widetilde{\varepsilon}$ *defined in* (123) *and constant* C *does not depend on* n *and* ξ.

Using Lemmata 4.17, 4.18 and formula (157) for $\overline{\overline{Q}}_{n,\xi}$ we obtain

$$\begin{aligned} \left| a_{n,\xi}^{(R+1)}(t) \right| &\le \left| b_{1n}^{(R+1)} \right| + \left| b_{2n}^{(R+1)} \right| \le C\,(1+\xi)^{R+1-r}\, n^{-\frac{r-\alpha}{\alpha}} \\ &\cdot \left(e^{-\frac{|t|^\alpha}{4}\cos\left(\alpha\frac{\pi}{2}\right)} + e^{-\frac{|t|^\alpha}{4n}\cos\left(\alpha\frac{\pi}{2}\right)} n^R \left(\nu_0^{*n-1} + \left[\sup_{|t|>\widetilde{\varepsilon}} |f(t)| + 2\,\gamma_r^* n^{-r/\alpha} \right]^{n-1} \right) \right). \end{aligned}$$

From Lemma A.14 it follows that

$$\begin{aligned} \int_{|t|>\varepsilon_n} e^{-\frac{|t|^\alpha}{4n}\cos\left(\alpha\frac{\pi}{2}\right)} dt &\le \int_{-\infty}^{+\infty} e^{-\frac{\left|t\,n^{-1/\alpha}\right|^\alpha}{4}\cos\left(\alpha\frac{\pi}{2}\right)}\, n^{1/\alpha}\, d(t\,n^{-1/\alpha}) \\ &\le n^{1/\alpha} \int_{-\infty}^{+\infty} e^{-\frac{|y|^\alpha}{4}\cos\left(\alpha\frac{\pi}{2}\right)}\, dy \le C\, n^{1/\alpha}. \end{aligned}$$

Combining this and the above estimation of $\left|a_{n,\xi}^{(R+1)}(t)\right|$ we get

$$\int_{|t|>\varepsilon_n} \left|a_{n,\xi}^{(R+1)}(t)\right| dt \le C\,(1+\xi)^{R+1-r}\, n^{-\frac{r-\alpha}{\alpha}} \left(1 + n^{R+\frac{1}{\alpha}}\, Q_n\right),$$

where

$$Q_n = \nu_0^{*n-1} + \left(\sup_{|t|>\widetilde{\varepsilon}} |f(t)| + 2\,\gamma_r^* n^{-r/\alpha}\right)^{n-1}. \tag{125}$$

This completes the third step.

Step 4: Summarizing the previous steps.

Finally we come to the last step of the proof of Theorem 4.11. According to our plan we use the results of Step 2 and 3 in order to get the following:

$$\begin{aligned}\left|A'_{n,\xi}(x)\right| &\le \frac{x^{-R-1}}{2\pi}\left(\int_{|t|\le\varepsilon_n} + \int_{|t|>\varepsilon_n}\right)\left|a_{n,\xi}^{(R+1)}(t)\right| dt \\ &\le C\,x^{-R-1}\,(1+\xi)^{R+1-r}\, n^{-\frac{r-\alpha}{\alpha}} \left(1 + n^{R+\frac{1}{\alpha}}\, Q_n\right),\end{aligned} \tag{126}$$

where Q_n is defined by (125). Moreover, from formula (120) it follows that for all $q = 0, ..., R$ we have

$$\left|A'_{n,\xi}(x)\right| \le \frac{x^{-q}}{2\pi}\int_{-\infty}^{+\infty} \left|a_{n,\xi}^{(q)}(t)\right| dt = \frac{x^{-q}}{2\pi}\left(\int_{|t|\le\varepsilon_n} + \int_{|t|>\varepsilon_n}\right)\left|a_{n,\xi}^{(q)}(t)\right| dt,$$

where ε_n is defined by (123). Using Lemma 4.13-4.18 and acting in the same way as in Steps 2 and 3 we get

$$\left|A'_{n,\xi}(x)\right| \le C\,x^{-q}\,(1+\xi)^{R+1-r}\, n^{-\frac{r-\alpha}{\alpha}} \left(1 + n^{\frac{r+1-\alpha}{\alpha}+q\left(1-\frac{1}{\alpha}\right)}\, Q_n\right), \tag{127}$$

where Q_n is from (125). Combining inequality (127) with $q = 0$ and inequality (126) we get for $x > 0$:

$$(1 + x^{R+1})\left|A'_{n,\xi}(x)\right| \le C\,(1+\xi)^{R+1-r}\, n^{-\frac{r-\alpha}{\alpha}} \left(1 + n^{\frac{r+1}{\alpha}}\, Q_n\right).$$

Taking into account Remark 4.2 and the fact that $1 + x^{R+1} \ge 2^{-R}\,(1+x)^{R+1}$ for $x > 0$ we obtain

$$\left|A'_{n,\xi}(x)\right| \le K(n,\xi)\,(1+|x|)^{-R-1}$$

with $K(n,\xi) = C\,(1+\xi)^{R+1-r}\, n^{-\frac{r-\alpha}{\alpha}} \left(1 + n^{\frac{r+1}{\alpha}}\, Q_n\right)$. This completes the proof of Theorem 4.11.

□

4.3.4 Final results

In this subsection we estimate the function $A_{n,\xi}(x)$ defined by (109) and finally get the estimate of $\left|\overline{\overline{F}}^{n*}_{n,\xi}\left(n^{1/\alpha}x\right) - G_{\alpha,1}(x) - \overline{\overline{W}}_{r,n,\xi}(x)\right|$.

Theorem 4.19. *If for $r > 1$ we have $0 < \gamma_r^* < \infty$, then for all $x \in \mathbb{R}$, $n \in \mathbb{N}$ and $\xi \in [0, \infty)$ the following inequality holds:*

$$|A_{n,\xi}(x)| \le C\,(1+|x|)^{-R-1}\,(1+\xi)^{R+1-r}\,n^{-\frac{r-\alpha}{\alpha}}\left(1 + n^{\frac{r}{\alpha}}\,Q_n\right),$$

where $Q_n = \nu_0^{*n-1} + \left(\sup_{|t|>\widetilde{\varepsilon}}|f(t)| + 2\,\gamma_r^* n^{-r/\alpha}\right)^{n-1}$ with $\widetilde{\varepsilon}$ defined in (123), constant C does not depend on n and ξ, $R = [r]$ if $r \ne [r]$, and $R = r - 1$ otherwise.

Proof. From Lemmata 4.9, 4.10 and Theorem 4.11 it follows that all requirements of Lemma 4.8 with $s = R+1$ and $A(x) = A_{n,\xi}(x)$ are satisfied. This means that we can estimate $A_{n,\xi}(x)$ as follows:

$$|A_{n,\xi}(x)| \le C\,(1+|x|)^{-R-1}\left(I_0(T) + I_{R+1}(T) + K(n,\xi)\,T^{-1}\right)$$

for all $x \in \mathbb{R}$ and $T \ge 1$. Here $C \ge 0$ depends only on R, $K(n,\xi)$ is defined in Theorem 4.11, and

$$I_m(T) = \int_{|t|\le T} |d_m(t)|\,|t|^{-1}dt \quad \text{with} \quad d_m(t) = \int_{-\infty}^{+\infty} e^{itx} d\left(x^m A_{n,\xi}(x)\right)$$

for $m \in \{0, R+1\}$. Let us take $T = n^{1/\alpha}$ and estimate $I_0(T)$ and $I_{R+1}(T)$. Using the definition of $a_{n,\xi}(t)$, representation (122) for $|t| \le \varepsilon_n$ and representation (124) for $|t| > \varepsilon_n$, we obtain for $I_0(n^{1/\alpha})$:

$$\begin{aligned} I_0(n^{1/\alpha}) &= \int_{|t|\le n^{1/\alpha}} |t|^{-1}\left|\int_{-\infty}^{+\infty} e^{itx} dA_{n,\xi}(x)\right| dt = \int_{|t|\le n^{1/\alpha}} |t|^{-1}\,|a_{n,\xi}(t)|\,dt \\ &\le \left(\int_{|t|\le\varepsilon_n} + \int_{|t|>\varepsilon_n}\right)\frac{|a_{n,\xi}(t)|}{|t|}dt \le \sum_{k=1}^{4}\int_{|t|\le\varepsilon_n}\frac{|d_{kn}(t)|}{|t|}dt + \sum_{k=1}^{2}\varepsilon_n^{-1}\int_{|t|>\varepsilon_n}|b_{kn}(t)|\,dt, \end{aligned}$$

where ε_n is given in (123). Now applying Lemmata 4.13 - 4.18 with $q = 0$ and dealing with the integrals in the same way as in Step 2 and Step 3 of Theorem 4.11 we get

$$I_0(n^{1/\alpha}) \le C\,n^{-\frac{r-\alpha}{\alpha}}\,(1+\xi)^{R+1-r}\left(1 + n^{\frac{r-\alpha}{\alpha}}Q_n\right),$$

where Q_n is defined by (125).

Now let us consider $I_{R+1}(T)$. From Lemmata 4.9 and 4.10 it follows that Lemma A.15 with $G(x) = A_{n,\xi}(x)$ and $m = R+1$ holds true and we get

$$(-it)^{R+1}\int_{-\infty}^{+\infty} e^{itx} d\left(x^{R+1}A_{n,\xi}(x)\right) = (R+1)!\sum_{q=0}^{R+1}\frac{(-t)^q}{q!}\,a_{n,\xi}^{(q)}(t).$$

Using this equality we repeat what we have done above for $I_0(T)$:

$$I_{R+1}(n^{1/\alpha}) = \int_{|t|\le n^{1/\alpha}} |t|^{-1}\left|\int_{-\infty}^{+\infty} e^{itx} d\left(x^{R+1}A_{n,\xi}(x)\right)\right| dt$$

$$\leq C \sum_{q=0}^{R+1} \int_{|t|\leq n^{1/\alpha}} \frac{\left|a^{(q)}_{n,\xi}(t)\right|}{|t|^{R+2-q}}\, dt \leq C \sum_{q=0}^{R+1} \left(\int_{|t|\leq \varepsilon_n} + \int_{|t|>\varepsilon_n}\right) \frac{\left|a^{(q)}_{n,\xi}(t)\right|}{|t|^{R+2-q}}\, dt$$
$$\leq C \sum_{q=0}^{R+1} \left(\sum_{k=1}^{4} \int_{|t|\leq \varepsilon_n} \frac{\left|d^{(q)}_{kn}(t)\right|}{|t|^{R+2-q}}\, dt + \sum_{k=1}^{2} \varepsilon_n^{q-R-2} \int_{|t|>\varepsilon_n} \left|b^{(q)}_{kn}(t)\right| dt\right).$$

Recall that $\varepsilon_n = \widetilde{\varepsilon}\, n^{1/\alpha}$. The integrals of $b^{(q)}_{kn}(t)$ can be estimated as in Step 3 of the proof of Theorem 4.11. For $d^{(q)}_{kn}$ we apply Lemmata 4.13 - 4.16 with $q = 0, 1, ..., R+1$ and use Lemma A.14 for the estimation of integrals. This yields

$$I_{R+1}(n^{1/\alpha}) \leq C\, n^{-\frac{r-\alpha}{\alpha}} (1+\xi)^{R+1-r} \left(1 + n^R\, Q_n\right),$$

where Q_n is the same as before, i.e. defined by (125). Summarizing everything we obtain the following estimate of $A_{n,\xi}(x)$:

$$|A_{n,\xi}(x)| \leq C\, (1+|x|)^{-R-1} \left(I_0(n^{1/\alpha}) + I_{R+1}(n^{1/\alpha}) + K(n,\xi)\, n^{-1/\alpha}\right)$$
$$\leq C\, (1+\xi)^{R+1-r} (1+|x|)^{-R-1}\, n^{-\frac{r-\alpha}{\alpha}} \left(1 + n^{\frac{r}{\alpha}}\, Q_n\right).$$

This completes the proof of Theorem 4.19.

□

In order to prove our next theorem we will need the following auxiliary lemma.

Lemma 4.20. *If for $r > 1$ we have $0 < \gamma_r^* < \infty$, then for all $x \in \mathbb{R}$, $n \in \mathbb{N}$ and $\xi \in [0,\infty)$ the following inequality holds:*

$$\left|\overline{\overline{H}}^{n*}_{n,\xi}(n^{1/\alpha}x)\right| \leq C\, (1+|x|)^{-R-1} (1+\xi)^{R+1-r}\, n\, \nu_0^{*n-1},$$

where constant C does not depend on n and ξ, $R = [r]$, if $r \neq [r]$, and $R = r-1$ otherwise.

Proof. From the definition of $\overline{\overline{H}}_{n,\xi}$ it follows that $\overline{\overline{H}}^{n*}_{n,\xi}(n^{1/\alpha}x) = 0$ for $x \leq 0$, which means that the lemma holds true for $x \leq 0$. Using $\overline{\overline{H}}^{n*}_{n,\xi}(z) = -\int_z^\infty d\overline{\overline{H}}^{n*}_{n,\xi}(y)$, $z \in \mathbb{R}$, and Lemmata 4.2, 4.3 we get for $x > 0$:

$$\left|\overline{\overline{H}}^{n*}_{n,\xi}(n^{1/\alpha}x)\right| \leq \int_{n^{1/\alpha}x}^\infty \left|d\overline{\overline{H}}^{n*}_{n,\xi}(y)\right| \leq \nu_0\left(\overline{\overline{H}}^{n*}_{n,\xi}\right) \leq \overline{\overline{\nu}}^{\,n}_{0,n,\xi} \leq \nu_0^{*n},$$

and for $x\, n^{1/\alpha} > 1$ we can estimate as follows:

$$\left|\overline{\overline{H}}^{n*}_{n,\xi}(n^{1/\alpha}x)\right| \leq \int_{n^{1/\alpha}x}^\infty y^{-R-1}\, y^{R+1} \left|d\overline{\overline{H}}^{n*}_{n,\xi}(y)\right| \leq \left(n^{1/\alpha}x\right)^{-R-1} \nu_{R+1}\left(\overline{\overline{H}}^{n*}_{n,\xi}\right)$$
$$\leq \left(n^{1/\alpha}x\right)^{-R-1} n^{R+1}\, \overline{\overline{\nu}}_{R+1,n,\xi}\, \overline{\overline{\nu}}^{\,n-1}_{0,n,\xi} \leq C\, x^{-R-1} (1+\xi)^{R+1-r}\, n^{R+1-\frac{r}{\alpha}}\, \nu_0^{*n-1}.$$

Combining the last two inequalities we obtain for $x > n^{-1/\alpha}$:

$$\left(1 + x^{R+1}\right) \left|\overline{\overline{H}}^{n*}_{n,\xi}(n^{1/\alpha}x)\right| \leq C\, (1+\xi)^{R+1-r}\, n\, \nu_0^{*n-1},$$

which, together with $1 + x^{R+1} \geq 2^{-R}(1+x)^{R+1}$ for $x > 0$, gives the following estimate for $x > n^{-1/\alpha}$:

$$\left|\overline{\overline{H}}_{n,\xi}^{n*}(n^{1/\alpha}x)\right| \leq C\,(1+x)^{-R-1}\,(1+\xi)^{R+1-r}\,n\,\nu_0^{*n-1}.$$

For $0 < x \leq n^{-1/\alpha}$ we estimate as follows;

$$\left|\overline{\overline{H}}_{n,\xi}^{n*}(n^{1/\alpha}x)\right| \leq \nu_0^{*n}\,(1+x)^{-R-1}\,(1+n^{-1/\alpha})^{R+1} \leq C\,(1+x)^{-R-1}\,\nu_0^{*n-1}.$$

The last two inequalities give us the estimate of $\left|\overline{\overline{H}}_{n,\xi}^{n*}(n^{1/\alpha}x)\right|$ for all $x > 0$:

$$\left|\overline{\overline{H}}_{n,\xi}^{n*}(n^{1/\alpha}x)\right| \leq C\,(1+x)^{-R-1}\,(1+\xi)^{R+1-r}\,n\,\nu_0^{*n-1}.$$

This completes the proof of the lemma. □

Theorem 4.21. *If for $r > 1$ we have $0 < \gamma_r^* < \infty$, then for all $x \in \mathbb{R}$, $n \in \mathbb{N}$ and $\xi \in [0,\infty)$ the following inequality holds:*

$$\left|\overline{\overline{F}}_{n,\xi}^{n*}(n^{1/\alpha}x) - G_{\alpha,1}(x) - \overline{\overline{W}}_{r,n,\xi}(x)\right| \leq C\,(1+|x|)^{-R-1}\,(1+\xi)^{R+1-r}\,n^{-\frac{r-\alpha}{\alpha}}\,\left(1 + n^{\frac{r}{\alpha}}\,Q_n\right),$$

where $Q_n = \nu_0^{*n-1} + \left(\sup_{|t|>\widetilde{\varepsilon}}|f(t)| + 2\,\gamma_r^* n^{-r/\alpha}\right)^{n-1}$ with $\widetilde{\varepsilon}$ defined in (123), constant C does not depend on n and ξ, $R = [r]$ if $r \neq [r]$, and $R = r-1$ otherwise.

Proof. From definition (109) of $A_{n,\xi}(x)$ it follows that

$$\left|\overline{\overline{F}}_{n,\xi}^{n*}\left(n^{1/\alpha}x\right) - G_{\alpha,1}(x) - \overline{\overline{W}}_{r,n,\xi}(x)\right| \leq \left|A_{n,\xi}(x)\right| + \left|\overline{\overline{H}}_{n,\xi}^{n*}\left(n^{1/\alpha}x\right)\right|.$$

Using the estimates from Theorem 4.19 for $A_{n,\xi}$ and from Lemma 4.20 for $\overline{\overline{H}}_{n,\xi}^{n*}$ we obtain the statement of the theorem. □

4.4 Estimation of $\left|\overline{\overline{W}}_{r,n,\xi}(x) - \widetilde{W}_{r,n}(x)\right|$

Theorem 4.22. *If for $r > 1$ we have $0 < \gamma_r^* < \infty$, then for all $x \in \mathbb{R}$, $n \in \mathbb{N}$ and $\xi \in [0,\infty)$ the following inequality holds:*

$$\left|\overline{\overline{W}}_{r,n,\xi}(x) - \widetilde{W}_{r,n}(x)\right| \leq C\,n^{-\frac{r-\alpha}{\alpha}}\sum_{k=0}^{p}\sum_{\ell=1}^{m_k}\sum_{u=\ell}^{m_{\ell,k}}(1+|x|)^{-u}\,(1+\xi)^{u-r},$$

where $p = [2\,R/\alpha]$, $m_k = 1 + [(R-\alpha k/2)/(1-\alpha)]$, $m_{\ell,k} = [R+1+\alpha(\ell-1-k/2)]$ with $R = [r]$ if $r \neq [r]$ and $R = r-1$ otherwise, and constant C does not depend on n and ξ.

Proof. Using definition (102) of $\overline{\overline{W}}_{r,n,\xi}$ and definition (87) of $\widetilde{W}_{r,n}$ we obtain

$$\left|\overline{\overline{W}}_{r,n,\xi}(x) - \widetilde{W}_{r,n}(x)\right| \leq \left|\overline{\overline{W}}_{n,\xi}^{*}(x)\right| \tag{128}$$

$$+\sum_{k=0}^{p}\sum_{\ell=1}^{m_k}\binom{n}{\ell}\frac{|c_{k,n-\ell}|}{n^k}\sum_{u=\ell}^{m_{\ell,k}}\sum_{v=0}^{p_{u,\ell,k}}\left|G^{(u,k+v)}(x,1)\right|\frac{(\ell/n)^v}{v!}\,n^{-u/\alpha}\,\left|\overline{\overline{C}}_{u,\ell} - \widetilde{C}_{u,\ell}\right|,$$

where $p = [2\,R/\alpha]$, $m_k = 1 + [(R - \alpha k/2)\,/(1-\alpha)]$, $m_{\ell,k} = [R + 1 + \alpha(\ell - 1 - k/2)]$, $p_{u,\ell,k} = \max\{0, [(R+1-u)/\alpha + \ell - 1 - k/2]\}$ with

$$R = \begin{cases} [r], & \text{if } \; [r] \neq r \\ r-1, & \text{if } \; [r] = r \end{cases},$$

$$\widetilde{C}_{u,\ell} = \sum_{\substack{k_1+2k_2+...+Rk_R=u \\ k_1+k_2+...+k_R=\ell}} \frac{\ell!}{k_1!...k_R!} \left(\frac{\mu_1^*}{1!}\right)^{k_1} ... \left(\frac{\mu_R^*}{R!}\right)^{k_R},$$

$$\overline{\overline{C}}_{u,\ell} = \sum_{\substack{k_1+2k_2+...+Rk_R=u \\ k_1+k_2+...+k_R=\ell}} \frac{\ell!}{k_1!...k_R!} \left(\frac{\overline{\overline{\mu}}_{1,n,\xi}}{1!}\right)^{k_1} ... \left(\frac{\overline{\overline{\mu}}_{R,n,\xi}}{R!}\right)^{k_R},$$

$\overline{\overline{W}}^*_{n,\xi}(x)$ and $c_{k,n-\ell}$ are given by formulas (105) and (103), respectively.

First, let us consider the second term from (128). From the definition of $c_{k,n-\ell}$ it is easy to see that for $k = 0, ..., p$ and $\ell = 1, ..., m_k$ there always exists a constant $C > 0$ such that

$$\frac{|c_{k,n-\ell}|}{n^k} \leq \sum_{\substack{k_0+k_2+\cdots+k_s=n-\ell \\ 2k_2+\cdots+sk_s=k}} \frac{(n-\ell)!}{k_0!k_2!\cdots k_s!\,n^k} \, |A_2|^{k_2} \cdots |A_s|^{k_s} \leq \frac{C}{n^{k/2}},$$

where $k_0, k_2, ..., k_s \in \mathbb{N}_0$ and C does not depend on n. Using Lemma 3.14 and Lemma A.4 for $\left|G^{(u,k+v)}(x,1)\right|$ we obtain

$$\left|G^{(u,k+v)}(x,1)\right| \leq C_u\,(1+|x|)^{-u}, \qquad x \in \mathbb{R}, \tag{129}$$

where $C_u = 4\,D^{\max}(1+A^{\max})^u$, $D^{\max} = \max\limits_{u,k,v}\{D_{u,k+v}\}$ and $A^{\max} = \max\limits_{u,k,v}\{A_{u,k+v}\} > 1$ with $D_{u,k+v}$, $A_{u,k+v}$ defined in Lemma 3.14 and all possible combination of u, k, v that can appear in the sum from (128).

Let us estimate the difference $\left|\overline{\overline{C}}_{u,\ell} - \widetilde{C}_{u,\ell}\right|$. Using the formulas given above we get

$$\left|\overline{\overline{C}}_{u,\ell} - \widetilde{C}_{u,\ell}\right| \leq \sum_{\substack{k_1+k_2+...+k_R=\ell \\ k_1+2k_2+...+Rk_R=u}} \frac{\ell!}{k_1!...k_R!} \left| \left(\frac{\overline{\overline{\mu}}_{1,n,\xi}}{1!}\right)^{k_1} ... \left(\frac{\overline{\overline{\mu}}_{R,n,\xi}}{R!}\right)^{k_R} - \left(\frac{\mu_1^*}{1!}\right)^{k_1} ... \left(\frac{\mu_R^*}{R!}\right)^{k_R} \right|.$$

First of all note that $u = \ell, \ell+1, ..., m_{\ell,k}$ and $m_{\ell,k} \leq \ell + R$ (see the definition of $m_{\ell,k}$). Let us fix an arbitrary $\ell \in \{1, ..., m_k\}$ and observe what happens with the sum considered above for different $u \in \{\ell, \ell+1, ..., \ell+R\}$. For $u = \ell$ only one solution is possible for the system of equation $k_1 + ... + k_R = \ell$, $k_1 + 2k_2 + ... + Rk_R = u$. Namely, $k_1 = \ell$, $k_2 = ... = k_R = 0$. If $u = \ell+1$ we obtain $k_1 = \ell - 1$, $k_2 = 1$, $k_3 = ... = k_R = 0$ and so on. Let us take $u = \ell + j$ with arbitrary $j \in \{0, ..., R\}$. It follows from

$$\begin{cases} k_1 + \cdots + k_R = \ell \\ k_1 + 2k_2 + \cdots + Rk_R = \ell + j \end{cases} \iff \begin{cases} k_1 + \cdots + k_R = \ell \\ k_2 + 2k_3 + \cdots + (R-1)k_R = j \end{cases}$$

that the largest $i \in \{1, ..., R\}$ such that $k_i \neq 0$ cannot be greater than $j+1 = u-\ell+1$. In other words,

$$k_i = 0 \quad \text{for} \quad i > u - \ell + 1. \tag{130}$$

Note that for $u \in \{\ell, ..., m_{\ell,k}\}$ and $\ell \in \{1, ..., m_k\}$ we have $u - \ell + 1 \in \{1, ..., R+1\}$. The case $u - \ell + 1 = R + 1$ is only possible if $k = 0$, $\ell = 1$ and $u = m_{1,0} = R + 1$. But for $\ell = 1$ and $u = R + 1$ we have $\overline{\overline{C}}_{u,\ell} = \widetilde{C}_{u,\ell} = 0$. Therefore we exclude this case and consider $u - \ell + 1 \in \{1, ..., R\}$ in what follows.
Denote $q := u - \ell + 1$. Using (130) we obtain for $\left|\overline{\overline{C}}_{u,\ell} - \widetilde{C}_{u,\ell}\right|$:

$$\left|\overline{\overline{C}}_{u,\ell} - \widetilde{C}_{u,\ell}\right| \leq \sum_{\substack{k_1+k_2+...+k_q=\ell \\ k_1+2k_2+...+qk_q=u \\ u-\ell+1\in\{1,...,R\}}} \frac{\ell!}{k_1!...k_q!} \left| \left(\frac{\overline{\overline{\mu}}_{1,n,\xi}}{1!}\right)^{k_1} ... \left(\frac{\overline{\overline{\mu}}_{q,n,\xi}}{q!}\right)^{k_q} - \left(\frac{\mu_1^*}{1!}\right)^{k_1} ... \left(\frac{\mu_q^*}{q!}\right)^{k_q} \right|.$$

From Lemma 4.2 (i) it follows that $\left|\overline{\overline{\mu}}_{j,n,\xi}\right| \leq \overline{\overline{\nu}}_{j,n,\xi} \leq \nu_j^* \leq C$ and $\left|\mu_j^*\right| \leq \nu_j^* \leq C$ for all $j \in \{1, ..., R\}$. Using this fact and the fact that $k_j \in \{0, 1, ..., m_k\}$ with finite m_k we apply Lemma A.7 and get

$$\left|\overline{\overline{C}}_{u,\ell} - \widetilde{C}_{u,\ell}\right| \leq C \sum_{j=1}^{q} \left|\overline{\overline{\mu}}_{j,n,\xi} - \mu_j^*\right| = C \sum_{j=1}^{u-\ell+1} \left|\overline{\overline{\mu}}_{j,n,\xi} - \mu_j^*\right|.$$

Finally, applying Lemma 4.1 (ii) and taking into account that $u - \ell + 1 \in \{1, ..., R\}$ and that $R < r$ we obtain

$$\left|\overline{\overline{C}}_{u,\ell} - \widetilde{C}_{u,\ell}\right| \leq C \sum_{j=1}^{u-\ell+1} 2 \left(n^{1/\alpha}(1+\xi)\right)^{j-r} \gamma_r^* \leq C \left(n^{1/\alpha}(1+\xi)\right)^{u-\ell+1-r}.$$

Using this estimate, the estimates for $\frac{|c_{k,n-\ell}|}{n^k}$ and $\left|G^{(u,k+v)}(x,1)\right|$ and the facts that $\alpha \in (0,1)$, $\ell \geq 1$, we get

$$\begin{aligned}
&\sum_{k=0}^{p}\sum_{\ell=1}^{m_k} \binom{n}{\ell} \frac{|c_{k,n-\ell}|}{n^k} \sum_{u=\ell}^{m_{\ell,k}} \sum_{v=0}^{p_{u,\ell,k}} \left|G^{(u,k+v)}(x,1)\right| \frac{(\ell/n)^v}{v!} n^{-u/\alpha} \left|\overline{\overline{C}}_{u,\ell} - \widetilde{C}_{u,\ell}\right| \\
&\quad \leq C \sum_{k=0}^{p}\sum_{\ell=1}^{m_k}\sum_{u=\ell}^{m_{\ell,k}} n^{\ell-k/2-u/\alpha} (1+|x|)^{-u} \left(n^{1/\alpha}(1+\xi)\right)^{u-\ell+1-r} \\
&\quad \leq C\, n^{-\frac{r-\alpha}{\alpha}} \sum_{k=0}^{p}\sum_{\ell=1}^{m_k}\sum_{u=\ell}^{m_{\ell,k}} (1+|x|)^{-u} (1+\xi)^{u-r}.
\end{aligned} \tag{131}$$

Now let us estimate the term $\left|\overline{\overline{W}}^*_{n,\xi}(x)\right|$ from (128). Recall from (105) that

$$\begin{aligned}
\overline{\overline{W}}^*_{n,\xi}(x) = &\underbrace{n\, G^{(R+1,0)}(x,1) \frac{(-1)^{R+1}}{(R+1)!} n^{-\frac{R+1}{\alpha}} \overline{\overline{\mu}}_{R+1,n,\xi}}_{=:\overline{\overline{W}}^*_1(x)} \\
&+ \underbrace{n \left(G_{\alpha,1}(\,\cdot\,,n) * \overline{\overline{M}}_{n,\xi}\right)(xn^{1/\alpha}) - \sum_{w=0}^{R+1} n\, G^{(w,0)}(x,1) \frac{(-1)^w}{w!} n^{-w/\alpha} \mu_w\left(\overline{\overline{M}}_{n,\xi}\right)}_{:=\overline{\overline{W}}^*_2(x)},
\end{aligned}$$

where $\overline{\overline{M}}_{n,\xi}(x) = \overline{\overline{H}}_{n,\xi}(x) - \overline{\overline{H}}_{n,0}(x)$. From estimate (129) and from Lemma 4.2 (ii) for $\overline{\nu}_{R+1,n,\xi} \geq |\overline{\mu}_{R+1,n,\xi}|$ it follows that

$$\left|\overline{\overline{W}}^*_1(x)\right| \leq C\, n^{1-\frac{R+1}{\alpha}}\,(1+|x|)^{-R-1}\,\left(n^{1/\alpha}\,(1+\xi)\right)^{R+1-r}$$
$$\leq C\, n^{-\frac{r-\alpha}{\alpha}}\,(1+|x|)^{-R-1}\,(1+\xi)^{R+1-r}, \qquad x \in \mathbb{R}.$$

Let us estimate $\left|\overline{\overline{W}}^*_2(x)\right|$. We start with a slight transformation of $\overline{\overline{W}}^*_2(x)$. Using the definition of pseudomoments $\mu_w(\overline{\overline{M}}_{n,\xi})$ and property (55) we obtain

$$\overline{\overline{W}}^*_2(x) = n\int_{-\infty}^{+\infty}\left(G_{\alpha,1}(x-y,1) - \sum_{w=0}^{R+1} G^{(w,0)}(x,1)\,\frac{(-y)^w}{w!}\right) d\overline{\overline{M}}_{n,\xi}(yn^{1/\alpha}).$$

First of all note that for $x \leq 0$ we have $G_{\alpha,1}(x,\lambda) = \overline{\overline{M}}_{n,\xi}(x) = 0$ and, consequently, $\overline{\overline{W}}^*_2(x) = 0$. Let $x > 0$ in what follows. We consider separately two cases: $x > 0$ and $x > A$, where $A := 2 \max\limits_{j\in\{1,\ldots,R+2\}}\{A_{j,0}\}$ with $A_{j,0}$ defined in Lemma 3.14.

Case $x > A$. For $|y| \leq x/2$ we use Lemma A.8 with $z = x-y$, $a = x$ and $\theta \in (0,1)$:

$$n\left|\int_{|y|\leq x/2}\left(G_{\alpha,1}(x-y,1) - \sum_{w=0}^{R+1} G^{(w,0)}(x,1)\,\frac{(-y)^w}{w!}\right) d\overline{\overline{M}}_{n,\xi}(yn^{1/\alpha})\right|$$
$$\leq n\int_{|y|\leq x/2}\left|G^{(R+2,0)}(x-\theta y,1)\right|\,\frac{|y|^{R+2}}{(R+2)!}\left|d\overline{\overline{M}}_{n,\xi}(yn^{1/\alpha})\right| = \text{Ⓢ}.$$

Note that $|x-\theta y| \geq x/2$ for $|y| \leq x/2$. Using the fact that $x > A$ and applying formulas (60), (97) and (100) we continue

$$\text{Ⓢ} \leq C\,n\int_{|y|\leq x/2}|x-\theta y|^{-R-2}\,|y|^{R+2}\left|d\overline{\overline{M}}_{n,\xi}(yn^{1/\alpha})\right|$$
$$\leq C\,n\,(x/2)^{-R-1}\int_{|y|\leq x/2}|y|^{R+1}\left|d\overline{\overline{M}}_{n,\xi}(yn^{1/\alpha})\right|$$
$$\leq C\,n\,(x/2)^{-R-1}\,n^{-\frac{R+1}{\alpha}}\,\nu_{R+1}\left(\overline{\overline{M}}_{n,\xi}\right) \leq C\,n^{1-\frac{R+1}{\alpha}}\,x^{-R-1}\,\overline{\nu}_{R+1,n,\xi}$$
$$\leq C\,n^{1-\frac{R+1}{\alpha}}\,x^{-R-1}\,(n^{1/\alpha}\,(1+\xi))^{R+1-r} \leq C\,n^{-\frac{r-\alpha}{\alpha}}\,x^{-R-1}\,(1+\xi)^{R+1-r}.$$

For $|y| > x/2$ we again use formulas (60), (97) and (100) and obtain

$$n\left|\int_{|y|>x/2}\left(G_{\alpha,1}(x-y,1) - \sum_{w=0}^{R+1} G^{(w,0)}(x,1)\,\frac{(-y)^w}{w!}\right) d\overline{\overline{M}}_{n,\xi}(yn^{1/\alpha})\right|$$
$$\leq C\,n\int_{|y|>x/2}\left|d\overline{\overline{M}}_{n,\xi}(yn^{1/\alpha})\right| + C\,n\sum_{w=1}^{R+1}x^{-w}\int_{|y|>x/2}|y|^w\left|d\overline{\overline{M}}_{n,\xi}(yn^{1/\alpha})\right|$$
$$\leq C\,n\left(\frac{x}{2}\right)^{-R-1}\int_{|y|>x/2}|y|^{R+1}\left|d\overline{\overline{M}}_{n,\xi}(yn^{1/\alpha})\right|$$
$$+\,C\,n\sum_{w=1}^{R+1}x^{-w}\left(\frac{x}{2}\right)^{w-R-1}\int_{|y|>x/2}|y|^{R+1}\left|d\overline{\overline{M}}_{n,\xi}(yn^{1/\alpha})\right|$$
$$\leq C\,n\,x^{-R-1}\,n^{-\frac{R+1}{\alpha}}\,\nu_{R+1}\left(\overline{\overline{M}}_{n,\xi}\right) \leq C\,n^{-\frac{r-\alpha}{\alpha}}\,x^{-R-1}\,(1+\xi)^{R+1-r}.$$

Summarizing the cases $|y| > x/2$ and $|y| \le x/2$ we have

$$\left|\overline{\overline{W}}{}^*_2(x)\right| \le C\, n^{-\frac{r-\alpha}{\alpha}}\, x^{-R-1}\,(1+\xi)^{R+1-r}, \qquad x > A. \tag{132}$$

Case $x > 0$. From the definition of $\overline{\overline{M}}_{n,\xi}$ (see (96)) it follows that $\overline{\overline{M}}_{n,\xi}(x) = 0$ if $x \notin (n^{1/\alpha}, n^{1/\alpha}(1+\xi)]$. Using this fact and formulas (59), (97) and (100) we estimate

$$\begin{aligned}\left|\overline{\overline{W}}{}^*_2(x)\right| &\le n \int_1^{1+\xi} \left| G_{\alpha,1}(x-y,1) - \sum_{w=0}^{R+1} G^{(w,0)}(x,1)\,\frac{(-y)^w}{w!}\right| \left| d\overline{\overline{M}}_{n,\xi}(yn^{1/\alpha})\right| \\ &\le C\,n \int_1^{1+\xi} \frac{y^{R+1}}{y^{R+1}} \left| d\overline{\overline{M}}_{n,\xi}(yn^{1/\alpha})\right| + C\,n \sum_{w=1}^{R+1} \int_1^{1+\xi} |y|^w\, \frac{y^{R+1}}{y^{R+1}} \left| d\overline{\overline{M}}_{n,\xi}(yn^{1/\alpha})\right| \\ &\le C\,n^{1-\frac{R+1}{\alpha}}\, \nu_{R+1}\left(\overline{\overline{M}}_{n,\xi}\right) \le C\,n^{-\frac{r-\alpha}{\alpha}}\,(1+\xi)^{R+1-r}, \qquad x > 0.\end{aligned}$$

Note that the last inequality holds for all $x \in \mathbb{R}$, since $\overline{\overline{W}}{}^*_2(x) = 0$ for $x \le 0$. This means that condition (i) from Lemma A.4 is satisfied for $\overline{\overline{W}}{}^*_2(x)$. Estimate (132) (which is also true for $x < -A$) provides us condition (ii) from Lemma A.4 and we obtain

$$\left|\overline{\overline{W}}{}^*_2(x)\right| \le C\, n^{-\frac{r-\alpha}{\alpha}}\,(1+|x|)^{-R-1}\,(1+\xi)^{R+1-r}, \quad x \in \mathbb{R}.$$

Using this estimate and the estimate for $\overline{\overline{W}}{}^*_1(x)$ we get

$$\left|\overline{\overline{W}}{}^*_{n,\xi}(x)\right| \le \left|\overline{\overline{W}}{}^*_1(x)\right| + \left|\overline{\overline{W}}{}^*_2(x)\right| \le C\,n^{-\frac{r-\alpha}{\alpha}}\,(1+|x|)^{-R-1}\,(1+\xi)^{R+1-r}, \qquad x \in \mathbb{R}.$$

Plugging the last inequality together with estimate (131) into (128) we get the statement of the theorem:

$$\left|\overline{\overline{W}}_{r,n,\xi}(x) - \widetilde{W}_{r,n}(x)\right| \le C\,n^{-\frac{r-\alpha}{\alpha}} \sum_{k=0}^{p}\sum_{\ell=1}^{m_k}\sum_{u=\ell}^{m_{\ell,k}} (1+|x|)^{-u}\,(1+\xi)^{u-r}.$$

□

4.5 Proof of Theorem 3.26

According to our plan we have estimated each of the three summands on the right-hand side of the following inequality:

$$\begin{aligned}\left|F_n(x) - G_{\alpha,1}(x) - \widetilde{W}_{r,n}(x)\right| &\le \left|F_n(x) - \overline{\overline{F}}{}^{n*}_{n,\xi}\left(n^{1/\alpha}x\right)\right| \\ &+ \left|\overline{\overline{F}}{}^{n*}_{n,\xi}\left(n^{1/\alpha}x\right) - G_{\alpha,1}(x) - \overline{\overline{W}}_{r,n,\xi}(x)\right| + \left|\overline{\overline{W}}_{r,n,\xi}(x) - \widetilde{W}_{r,n}(x)\right|.\end{aligned}$$

If $0 < \gamma_r^* < \infty$ for some $r > 1$, then we can apply Theorems 4.7, 4.21 and 4.22. Thus, for all $x \in \mathbb{R}$, $\xi \in [0,\infty)$ and all integers $n \ge 2$ we obtain

$$\begin{aligned}\left|F_n(x) - G_{\alpha,1}(x) - \widetilde{W}_{r,n}(x)\right| &\le C\,n^{-\frac{r-\alpha}{\alpha}}\,(1+\xi)^{-r}\,\gamma_r^* \\ &+ C\,(1+|x|)^{-R-1}\,(1+\xi)^{R+1-r}\,n^{-\frac{r-\alpha}{\alpha}}\,\left(1 + n^{\frac{r}{\alpha}}\,Q_n\right)\end{aligned}$$

$$+ C\, n^{-\frac{r-\alpha}{\alpha}} \sum_{k=0}^{p} \sum_{\ell=1}^{m_k} \sum_{u=\ell}^{m_{\ell,k}} (1+|x|)^{-u}\, (1+\xi)^{u-r},$$

where $Q_n = \nu_0^{*n-1} + \left(\sup_{|t|>\widetilde{\varepsilon}} |f(t)| + 2\, \gamma_r^* n^{-r/\alpha}\right)^{n-1}$ with $\widetilde{\varepsilon}$ defined in (123), p, m_k, $m_{\ell,k}$ defined after formula (87), $R = [r]$ if $r \neq [r]$, and $R = r - 1$ otherwise.
Note that constants C do not depend on n and ξ. Let now $\xi := |x|$ for each $x \in \mathbb{R}$. This gives us the following estimate

$$\left| F_n(x) - G_{\alpha,1}(x) - \widetilde{W}_{r,n}(x) \right| \leq C\, (1+|x|)^{-r}\, n^{-\frac{r-\alpha}{\alpha}} \left(1 + n^{\frac{r}{\alpha}}\, Q_n\right).$$

This completes the proof of Theorem 3.26.

5 Further asymptotic results for Pareto-like distributions

In Section 2.2 we introduced a class of Pareto-like distributions and gave some known asymptotic results concerning this class. In this section we want to continue investigating the asymptotic behavior of sums of Pareto-like distributed random variables. Here we consider some special cases and obtain some new results.

5.1 Two special cases

First of all, let us recall that a Pareto-like distributed random variable $X \geq 0$ with parameter $\alpha > 0$ has the distribution function F of the form (see Definition 2.9)

$$1 - F(x) = C(\alpha)\, x^{-\alpha} + O(x^{-r}), \quad \text{as} \quad x \to +\infty$$

for some $r > \alpha$ and some $C(\alpha) > 0$. We consider a random sum $S_\nu = X_1 + \cdots + X_\nu$, $S_0 = 0$, where $X_1, X_2, \ldots$ is a sequence of independent and identically Pareto-like distributed random variables and ν is an integer-valued counting random variable with $p_n = P(\nu = n)$, $n \in \mathbb{N}_0$. Theorem 2.11 states that under some technical conditions for $\alpha \in (0,1)$ we have

$$\Delta(x) = \frac{P(S_\nu > x)}{P(X > x)} - E\nu = O\left(x^{-\alpha}\right) \text{ as } x \to \infty. \tag{133}$$

In the special case of $\alpha = 1/2$ this asymptotic result can be improved as follows.

Theorem 5.1 ([10])**.** *Let $\alpha = 1/2$. Suppose $u_{1/2}(x) := 1 - F(x) - (\pi x)^{-1/2} = O(x^{-r})$ as $x \to \infty$ for $3/2 < r \leq 5/2$, and, additionally,* (39) *holds if $r = 2$. If $E\nu^4 < \infty$, then*

$$\Delta(x) = \frac{1}{2x}\left(\mu_1\, E\nu^2 - \frac{1}{6}\, E\nu^3 - \left(\mu_1 - \frac{1}{6}\right) E\nu\right) + O\left(x^{-(r-1/2)}\right), \quad x \to \infty,$$

where $\mu_1 = \int_{-\infty}^{+\infty} x d\left(F - G_{1/2,1}\right)(x)$ is the first pseudomoment (see Definition 3.15).

Remark 5.1. From Section 3.1 we know that the Pareto-like distribution function F with parameter $\alpha = 1/2$ belongs to the domain of normal attraction of stable distribution function $G_{1/2,1}$, i.e. $F \in \mathrm{DNA}(G_{1/2,1})$. That is how the first pseudomoment appears in Theorem 5.1. But why is the case $\alpha = 1/2$ special? For other $\alpha \in (0,1)$ we use the same method with stable distributions but have only (133) for $\Delta(x)$. The answer lies in the representation of stable $G_{\alpha,1}$ with $\alpha \in (0,1)$ (see formula (57)):

$$1 - G_{\alpha,1}(x) = C_1(\alpha)\, x^{-\alpha} + C_2(\alpha)\, x^{-2\alpha} + C_3(\alpha)\, x^{-3\alpha} + C_4(\alpha)\, x^{-4\alpha} + O(x^{-5\alpha}) \tag{134}$$

as $x \to \infty$ with coefficients $C_j(\alpha)$ defined in (58). For $\alpha = 1/2$ we have $C_2(\alpha) = C_4(\alpha) = 0$. Therefore,

$$F(x) - G_{1/2,1}(x) = C_3\,(1/2)\, x^{-3/2} + O(x^{-r}) \quad \text{as} \quad x \to \infty$$

and the first pseudomoment is finite if $r > 3/2$. This helps to estimate $\Delta(x)$ more precisely.

Another special case concerns the class of stable distributions (see Definition 3.1). First, we consider stable distributions with parameter $\alpha \in (0,1)$. From representation (134) it follows that each stable distribution is Pareto-like. Therefore, Theorem 2.11 from Section 2.2 may be applied. But we can deal with stable distributions in a different way. Namely, we can use stability property (56): for $x > 0$ we have

$$G_{\alpha,1}^{n*}(x) = G_{\alpha,1}(n^{-1/\alpha}x) \quad \text{if} \quad \alpha \neq 1. \tag{135}$$

Using this Christoph [9] obtained not only the first-order asymptotic result for $\Delta(x)$ but also results of higher orders.

Theorem 5.2 ([9])**.** *Suppose $0 < \alpha < 1$. Let $X, X_1, X_2, \dots$ be independent and identically distributed random variables with stable distribution function $G_{\alpha,1}$. If $E\nu^4 < \infty$, then as $x \to \infty$*

$$\begin{aligned}\Delta(x) \quad &= \quad \frac{P(S_\nu > x)}{P(X > x)} - E\nu = \Big(E\nu^2 - E\nu\Big)\,\frac{C_2(\alpha)}{C_1(\alpha)}\,x^{-\alpha} \\ &\quad + \left\{\Big(E\nu^3 - E\nu\Big)\,\frac{C_3(\alpha)}{C_1(\alpha)} - \Big(E\nu^2 - E\nu\Big)\,\frac{C_2^2(\alpha)}{C_1^2(\alpha)}\right\} x^{-2\alpha} + O(x^{-3\alpha}),\end{aligned}$$

where $C_j(\alpha)$, $j = 1,2,3$, are defined by (58)*.*

Proof. For a detailed proof see [9, Proposition 1]. □

Remark 5.2. In order to obtain more terms in the asymptotic expansion of $\Delta(x)$ we have to consider more terms in representation (134).

For stable distributions with parameter $\alpha \in (1,2)$ stability property (135) is also satisfied and allows to improve asymptotic results from Section 2.2 as follows.

Theorem 5.3 ([9])**.** *Suppose $1 < \alpha < 2$. Let $X, X_1, X_2, \dots$ be independent and identically distributed random variables with stable distribution function $G_{\alpha,1}(\cdot;1,\gamma)$ for some $\gamma > 0$. If $E\nu^3 < \infty$, then as $x \to \infty$*

$$\Delta(x) = \alpha\,\gamma\,\Big(E\nu^2 - E\nu\Big)\,x^{-1} + \Big(E\nu^2 - E\nu\Big)\,\frac{C_2(\alpha)}{C_1(\alpha)}\,x^{-\alpha} + O(x^{-2}),$$

where $C_j(\alpha)$, $j = 1,2$, are defined by (58)*.*

Proof. The proof is given in [9, Proposition 2]. □

Remark 5.3. Similarly to the previous theorem it is possible to obtain more terms in the expansion of $\Delta(x)$ by considering more terms in the expansion of $G_{\alpha,1}(x;1,\gamma)$. The latter can be obtained from formula (55) and representation (134).

We do not give here an asymptotic result in the case of stable distributions with parameter $\alpha = 1$. It was obtained by Christoph and can be found in [9, Proposition 3].

Independently from Christoph, Omey and Willekens have obtained the following asymptotic results for $\Delta(x)$ in the case of stable $X_1, X_2, \dots$.

Theorem 5.4 ([41]). *Let $X_1, X_2, \ldots$ be independent and identically distributed positive random variables with a stable distribution function $G := G_{\alpha,\beta}(\cdot\,; \lambda, \gamma)$ of index $\alpha \in (0,1)$ and let $p = G'$. If $\sum_{n=0}^{\infty} p_n(1+\varepsilon)^n < \infty$ for some $\varepsilon > 0$ then*

$$\text{(i)}\ \lim_{x\to\infty} \frac{\Delta(x)\,(1-G(x))}{p(x)\,\int_0^x (1-G(y))\,dy} = c(\alpha+1)\,E\,(\nu(\nu-1)) \quad \text{if} \quad \alpha \neq \frac{1}{2},$$

$$\text{(ii)}\ \lim_{x\to\infty} \frac{\Delta(x)\,(1-G(x))}{p(x)} = \frac{1}{6}\,E\left(\nu(\nu^2-1)\right) \quad \text{if} \quad \alpha = \frac{1}{2},$$

$$\text{(iii)}\ \lim_{x\to\infty} \frac{\Delta(x)\,(1-G(x)) - c(\alpha+1)\,E\,(\nu(\nu-1))\,p(x)\,\int_0^x (1-G(y))\,dy}{(1-G(x))^3} = \frac{1}{6}\,d(\alpha+1)\,E\left(\nu(\nu^2-1)\right),$$

where

$$d(\alpha+1) = -\frac{1}{4}\left(\frac{\alpha}{1-\alpha}\right)^2 c^2(\alpha+1)\,\frac{\Gamma(3\alpha)\,\Gamma(\alpha)\,(1+2\,\cos(2\,\alpha\,\pi))}{\Gamma^2(2\alpha)\,\cos^2(\alpha\,\pi)}$$

and $c(\alpha+1)$ is defined by (32).

Remark 5.4. In the paper [41] Omey and Willekens require the condition of analyticity of the function $P_\nu(z) := \sum_{n=0}^{\infty} p_n z^n$ at $z = 1$ instead of $\sum_{n=0}^{\infty} p_n(1+\varepsilon)^n < \infty$ for some $\varepsilon > 0$. These two conditions are equivalent (see [18, Remark on p. 45]).

5.2 Pareto-like with $\alpha \in (0,1)$. Improvement

In this section we give a new asymptotic result for $\Delta(x)$ in the case of Pareto-like distributed random variables $X, X_1, X_2, \ldots$ with parameter $\alpha \in (0,1)$ and with distribution function F in a special form. Namely,

$$1 - F(x) = \frac{c_1}{x^\alpha} + \frac{c_2\,d_2}{x^{2\alpha}} + \cdots + \frac{c_s\,d_s}{x^{s\alpha}} + u(x), \quad x \to \infty, \tag{136}$$

where $c_k := C_k(\alpha)$ are defined by (58), $s \in \mathbb{N}$ and $u(x)$ are such that

$$1 + \alpha \leq s\alpha < 1 + 2\,\alpha, \qquad \int_0^{+\infty} x^q |du(x)| < \infty \quad \text{for some } q > s\alpha,$$

and $d_i \in \mathbb{R}$ are arbitrary constants for $i = 2, \ldots, s$. Such a representation of F implies that there exists an $r \in (s\alpha, (s+1)\alpha]$, such that $\gamma_r^* < \infty$, and allows to apply Theorem 3.26. For more details about pseudomoments see Section 3.5.

Recall that we consider a random sum $S_\nu = X_1 + \cdots + X_\nu$, $S_0 = 0$, where $X_1, X_2, \ldots$ is a sequence of nonnegative independent and identically distributed random variables with distribution function F, and ν is an integer-valued counting random variable with $p_n = P(\nu = n)$, $n \in \mathbb{N}_0$. Our task is to get more terms in asymptotic expansion (133) of

$$\Delta(x) = \frac{P(S_\nu > x)}{P(X > x)} - E\nu \qquad \text{as} \quad x \to \infty.$$

Theorem 5.5. *Let $\alpha \in (0,1)$ and let F satisfy condition* (136) *from above. If $\nu_0^* < 1$ and $E\nu^j < \infty$ for all $j \in \mathbb{N}$, then*

$$\Delta(x) = \frac{P(S_\nu > x)}{P(X > x)} - E\nu = \frac{c_2\,(E\nu^2 - E\nu)}{c_1\,x^\alpha} + \frac{m_2}{x^{2\alpha}} + \cdots + \frac{m_{s-1}}{x^{(s-1)\alpha}} + \frac{\alpha\,\mu_1^*\,(E\nu^2 - E\nu)}{x} + O(x^{-1-\delta}), \quad x \to +\infty,$$

where $\delta > 0$ is some constant and m_i, $i = 2, ..., s-1$, depend only on $E\nu, ..., E\nu^s$, $d_2, ..., d_s$, $c_1, ..., c_s$ and α. Namely, m_i, $i = 2, ..., s-1$, can be found as follows:

$$m_i = \sum_{n=2}^{\infty} p_n\, b_i \qquad \text{with} \qquad b_0 = n, \quad b_1 = c_2(n^2 - n)/c_1, \tag{137}$$

$$b_{k-1} = \frac{1}{c_1}\left(c_k n^k \left(1 + \sum_{j=2}^{k} \frac{c_{j,n}}{n^j}\,\frac{k!}{(k-j)!}\right) - \sum_{j=2}^{k} c_j\, d_j\, b_{k-j}\right), \quad k = 3, ..., s,$$

where c_j, d_j are from (136)*, $c_{j,n}$ is defined by* (89)*.*

Proof. Recall that we consider nonnegative random variables $X_1, X_2, ...$ Therefore, for $x < 0$ we have $\Delta(x) = 1 - E\nu$. Let $x \geq 0$ in what follows. Using equality (1) and the fact that $\sum_{n=0}^{\infty} p_n = 1$ we obtain for $x \geq 0$:

$$\frac{P(S_\nu > x)}{P(X > x)} = \frac{1 - \sum\limits_{n=0}^{\infty} p_n F^{n*}(x)}{1 - F(x)} = \frac{\sum\limits_{n=0}^{\infty} p_n\,(1 - F^{n*}(x))}{1 - F(x)}$$

$$= \underbrace{\frac{p_0\left(1 - \mathbb{1}_{[0,+\infty)}(x)\right)}{1 - F(x)}}_{=0} + p_1 + \sum_{n=2}^{\infty} p_n \frac{1 - F^{n*}(x)}{1 - F(x)}.$$

We want to apply Theorem 3.26 with $r \in (s\alpha, (s+1)\alpha)$. Therefore, using the notation $F_n(x) = F^{n*}(n^{1/\alpha}x)$ we add and subtract $G_{\alpha,1}(xn^{-1/\alpha}) + \widetilde{W}_{r,n}(xn^{-1/\alpha})$ with $\widetilde{W}_{r,n}(x)$ defined by (87). We get

$$\frac{P(S_\nu > x)}{P(X > x)} = p_1 + \sum_{n=2}^{\infty} p_n \frac{1 - F_n(xn^{-1/\alpha}) \pm \left(G_{\alpha,1}(xn^{-1/\alpha}) + \widetilde{W}_{r,n}(xn^{-1/\alpha})\right)}{1 - F(x)}$$

$$= p_1 + \sum_{n=2}^{\infty} p_n \frac{1 - G_{\alpha,1}(xn^{-1/\alpha}) - \widetilde{W}_{r,n}(xn^{-1/\alpha})}{1 - F(x)} + \sum_{n=2}^{\infty} p_n \frac{G_{\alpha,1}(xn^{-1/\alpha}) + \widetilde{W}_{r,n}(xn^{-1/\alpha}) - F_n(xn^{-1/\alpha})}{1 - F(x)}.$$

Let us show that

$$\sum_{n=2}^{\infty} p_n \frac{G_{\alpha,1}(xn^{-1/\alpha}) + \widetilde{W}_{r,n}(xn^{-1/\alpha}) - F_n(xn^{-1/\alpha})}{1 - F(x)} = O(x^{\alpha - r}), \quad x \to \infty,$$

where $r > s\alpha \geq 1+\alpha$. From (136) it follows that there always exists such a constant $C > 0$ that for some large $A > 0$ we have

$$1 - F(x) \geq \frac{C}{x^{\alpha}}, \qquad x \geq A. \tag{138}$$

Using the estimate from Theorem 3.26 we obtain

$$\begin{aligned}\left|G_{\alpha,1}(xn^{-1/\alpha}) + \widetilde{W}_{r,n}(xn^{-1/\alpha}) - F_n(xn^{-1/\alpha})\right| &\leq C(1+xn^{-1/\alpha})^{-r} n^{-\frac{r-\alpha}{\alpha}} \left(1 + n^{\frac{r}{\alpha}}\, Q_n\right) \\ \leq C\,(n^{-1/\alpha} + xn^{-1/\alpha})^{-r}\, n^{-\frac{r-\alpha}{\alpha}} \left(1 + n^{\frac{r}{\alpha}}\, Q_n\right) &\leq C\,(1+x)^{-r}\, n \left(1 + n^{\frac{r}{\alpha}}\, Q_n\right),\end{aligned}$$

where Q_n is defined in Theorem 3.26. Note that since $\nu_0^* < 1$ there always exists such $n_0 \in \mathbb{N}$ that $n^{\frac{r}{\alpha}}\, Q_n < 1$ for all $n > n_0$. Using this fact and the last two estimates we get

$$\begin{aligned}\sum_{n=2}^{\infty} p_n \frac{\left|G_{\alpha,1}(xn^{-\frac{1}{\alpha}}) + \widetilde{W}_{r,n}(xn^{-\frac{1}{\alpha}}) - F_n(xn^{-\frac{1}{\alpha}})\right|}{(1-F(x))\, x^{\alpha-r}} &\leq \sum_{n=2}^{\infty} p_n \frac{C(1+x)^{-r}\, n \left(1 + n^{\frac{r}{\alpha}}\, Q_n\right)}{C\, x^{-\alpha}\, x^{\alpha-r}} \\ \leq \frac{C}{(1+1/x)^r} \left(\underbrace{\sum_{n=2}^{n_0} p_n\, n \left(1 + n^{\frac{r}{\alpha}}\, Q_n\right)}_{\leq C} + \sum_{n=n_0+1}^{\infty} p_n\, n\, \underbrace{\left(1 + n^{\frac{r}{\alpha}}\, Q_n\right)}_{\leq 2} \right) \\ \leq \frac{C}{(1+1/x)^r} \left(1 + \sum_{n=n_0+1}^{\infty} p_n\, n\right) \leq \frac{C\,(1+E\nu)}{(1+1/x)^r} \leq C.\end{aligned}$$

Thus, we proved that

$$\frac{P(S_\nu > x)}{P(X > x)} = p_1 + \sum_{n=2}^{\infty} p_n \frac{1 - G_{\alpha,1}(xn^{-1/\alpha}) - \widetilde{W}_{r,n}(xn^{-1/\alpha})}{1-F(x)} + O(x^{\alpha-r}), \quad x \to \infty. \tag{139}$$

The next step is to show that for $x \to \infty$

$$1 - G_{\alpha,1}(xn^{-1/\alpha}) - \widetilde{W}_{r,n}(xn^{-1/\alpha}) = \frac{a_1}{x^{\alpha}} + \cdots + \frac{a_s}{x^{s\alpha}} + \frac{a_{s+1}}{x^{\alpha+1}} + O(x^{-2\alpha-1}), \tag{140}$$

where $a_1 = c_1 n$, $a_{s+1} = \alpha\, c_1 \mu_1^*(n^2 - n)$, $a_k = c_k n^k \left(1 + \sum_{j=2}^{k} \frac{c_{j,n}}{n^j} \frac{k!}{(k-j)!}\right)$, $k = 2, ..., s$, with $c_j = C_j(\alpha)$ defined by (58) and $c_{j,n}$ defined by (89). Recall from (87) that

$$\begin{aligned}\widetilde{W}_{r,n}(x) \;=\; & \sum_{k=2}^{\rho} \frac{c_{k,n}}{n^k} G^{(0,k)}(x,1) \\ & + \sum_{k=0}^{p} \sum_{\ell=1}^{m_k} \binom{n}{\ell} \frac{c_{k,n-\ell}}{n^k} \sum_{u=\ell}^{m_{\ell,k}} \sum_{v=0}^{p_{u,\ell,k}} G^{(u,k+v)}(x,1) \frac{(-\ell/n)^v\,(-1)^u}{v!}\, n^{-u/\alpha}\, \widetilde{C}_{u,\ell}.\end{aligned}$$

In order to show (140) we use asymptotic expansion (57) of $G_{\alpha,1}(x,\lambda)$:

$$G_{\alpha,1}(x,\lambda) = 1 - c_1\,\lambda\, x^{-\alpha} - c_2\,\lambda^2\, x^{-2\alpha} - \cdots - c_s\,\lambda^s\, x^{-s\alpha} + O(x^{-(s+1)\alpha})$$

as $x \to \infty$ with $c_j = C_j(\alpha)$ defined by (58). Since $G_{\alpha,1}(x,\lambda)$ has bounded derivatives of all orders with respect to x and λ (see Lemma 3.14) we also have:

$$\begin{aligned} G^{(0,k)}(x,\lambda) &= -k!\, c_k\, x^{-k\alpha} - \frac{(k+1)!}{1!}\, c_{k+1}\, \lambda\, x^{-(k+1)\alpha} - ... \\ &\qquad - \frac{s!}{(s-k)!}\, c_s\, \lambda^{s-k}\, x^{-s\alpha} + O(x^{-(s+1)\alpha}) \\ &= -\sum_{j=k}^{s} \frac{j!}{(j-k)!}\, c_j\, \lambda^{j-k}\, x^{-j\alpha} + O(x^{-(s+1)\alpha}), \quad x \to \infty, \quad k = 1, ..., s, \end{aligned}$$

and

$$\begin{aligned} G^{(1,0)}(x,\lambda) &= \alpha\, c_1\, \lambda\, x^{-\alpha-1} + O(x^{-2\alpha-1}), \quad x \to \infty, \\ G^{(1,0)}(x,\lambda) &= \alpha\, c_1\, x^{-\alpha-1} + O(x^{-2\alpha-1}), \quad x \to \infty. \end{aligned}$$

Recall the notation $G^{(u,k)}(x,\lambda) = \frac{\partial^{u+k}}{\partial x^u \partial \lambda^k} G_{\alpha,1}(x,\lambda)$. Using the same considerations as above we obtain that

$$\begin{aligned} G^{(0,k)}(x,\lambda) &= O(x^{-(s+1)\alpha}), \quad x \to \infty, \qquad \text{for} \quad k \geq s+1, \\ G^{(u,v)}(x,\lambda) &= O(x^{-2\alpha-1}), \quad x \to \infty, \qquad \text{for} \quad u = 1, v \geq 2 \quad \text{or} \quad u \geq 2, v \geq 0. \end{aligned}$$

Thus, some terms from $\widetilde{W}_{r,n}(xn^{-1/\alpha})$ do not have to be considered at all, since they go directly into $O(x^{-(s+1)\alpha})$ or $O(x^{-2\alpha-1})$ for large x. Taking into account that $(s+1)\alpha \geq 2\alpha + 1$ we have

$$\begin{aligned} &1 - G_{\alpha,1}(xn^{-1/\alpha}) - \widetilde{W}_{r,n}(xn^{-1/\alpha}) = 1 - G_{\alpha,1}(xn^{-1/\alpha}) - \sum_{k=2}^{s} \frac{c_{k,n}}{n^k} G^{(0,k)}(xn^{-1/\alpha}, 1) \\ &- \binom{n}{1} \frac{c_{0,n-1}}{n^0}\, G^{(1,0)}(xn^{-1/\alpha}, 1) \frac{(-1)}{0!}\, n^{-1/\alpha}\, \widetilde{C}_{1,1} \\ &- \binom{n}{1} G^{(1,1)}(xn^{-1/\alpha}, 1) n^{-1/\alpha}\, \widetilde{C}_{1,1} \left(\frac{c_{0,n-1}}{n^0} \frac{1/n}{1!} + \frac{c_{1,n-1}}{n^1} \frac{(-1)}{0!} \right) + O(x^{-2\alpha-1}),\ x \to \infty, \end{aligned}$$

where $c_{0,n-1} = 1$, $c_{1,n-1} = 0$ and $\widetilde{C}_{1,1} = \mu_1^*$ (see formulas (89), (90)). Now, using the asymptotic expansions for $G_{\alpha,1}(x,1)$, $G^{(0,k)}(x,1)$, $k = 2, ..., s$, and $G^{(1,0)}(x,1)$, $G^{(1,1)}(x,1)$ considered above and simplifying, we obtain

$$\begin{aligned} 1 - G_{\alpha,1}(xn^{-1/\alpha}) - \widetilde{W}_{r,n}(xn^{-1/\alpha}) &= c_1 n x^{-\alpha} + \sum_{k=2}^{s} c_k n^k \left(1 + \sum_{j=2}^{k} \frac{c_{j,n}}{n^j} \frac{k!}{(k-j)!} \right) x^{-k\alpha} \\ &\quad + \alpha\, c_1\, \mu_1^*\, (n^2 - n)\, x^{-\alpha-1} + O(x^{-2\alpha-1}), \quad x \to \infty. \end{aligned}$$

Finally, we want to show that for $x \to \infty$

$$\frac{1 - G_{\alpha,1}(xn^{-1/\alpha}) - \widetilde{W}_{r,n}(xn^{-1/\alpha})}{1 - F(x)} = b_0 + \frac{b_1}{x^\alpha} + \cdots + \frac{b_{s-1}}{x^{(s-1)\alpha}} + \frac{b_s}{x} + O(x^{-1-\delta}), \tag{141}$$

where $\delta > 0$ is some constant and

$$b_0 = n, \qquad b_s = \alpha\,\mu_1^*\,(n^2 - n), \tag{142}$$

$$b_{k-1} = \frac{1}{c_1}\left(c_k n^k\left(1 + \sum_{j=2}^{k}\frac{c_{j,n}}{n^j}\,\frac{k!}{(k-j)!}\right) - \sum_{j=2}^{k} c_j\,d_j\,b_{k-j}\right), \quad k = 2, ..., s.$$

Note that d_j are coming from the representation (136) of $F(x)$. According to the definition we have to prove that for large x we have

$$\frac{\left|1 - G_{\alpha,1}(xn^{-1/\alpha}) - \widetilde{W}_{r,n}(xn^{-1/\alpha}) - (1-F(x))\left(b_0 + \frac{b_1}{x^\alpha} + \cdots + \frac{b_{s-1}}{x^{(s-1)\alpha}} + \frac{b_s}{x}\right)\right|}{(1-F(x))\,x^{-1-\delta}} \le C. \tag{143}$$

From (136) it follows that

$$(1-F(x))\left(b_0 + \frac{b_1}{x^\alpha} + \cdots + \frac{b_{s-1}}{x^{(s-1)\alpha}} + \frac{b_s}{x}\right) = c_1\,b_0\,x^{-\alpha} + c_1\,b_s\,x^{-1-\alpha}$$
$$+ \sum_{k=2}^{s}\left(c_1\,b_{k-1} + \sum_{j=2}^{k} c_j\,d_j\,b_{k-j}\right)x^{-k\alpha} + O(x^{-1-\alpha-\delta}), \quad x \to \infty,$$

where $\delta > 0$ is some constant. Since b_j are chosen in such a way that

$$a_1 = c_1\,b_0, \qquad a_{s+1} = c_1\,b_s,$$
$$a_k = c_1\,b_{k-1} + \sum_{j=2}^{k} c_j\,d_j\,b_{k-j}, \quad k = 2, ..., s,$$

with a_j from (140), we obtain for large x:

$$1 - G_{\alpha,1}(xn^{-1/\alpha}) - \widetilde{W}_{r,n}(xn^{-1/\alpha}) - (1-F(x))\left(b_0 + \frac{b_1}{x^\alpha} + \cdots + \frac{b_{s-1}}{x^{(s-1)\alpha}} + \frac{b_s}{x}\right)$$
$$= O(x^{-1-\alpha-\delta}), \quad x \to \infty.$$

From this fact and from estimate (138) for $1 - F(x)$ for large x it follows that inequality (143) and, consequently, (141) hold true.

From asymptotic equality (141) with b_j defined by (142) it follows that

$$\sum_{n=2}^{\infty} p_n\,\frac{1 - G_{\alpha,1}(xn^{-1/\alpha}) - \widetilde{W}_{r,n}(xn^{-1/\alpha})}{1-F(x)} = \sum_{n=2}^{\infty} p_n\,n + \frac{\sum_{n=2}^{\infty} p_n\,b_1}{x^\alpha} + \cdots$$
$$+ \frac{\sum_{n=2}^{\infty} p_n\,b_{s-1}}{x^{(s-1)\alpha}} + \frac{\alpha\,\mu_1^*\,\sum_{n=2}^{\infty} p_n\,(n^2-n)}{x} + O(x^{-1-\delta}), \quad x \to \infty.$$

Now, plugging the last formula into formula (139) and using the definition of $E\nu^j$, $j = 1, 2...$, we obtain

$$\frac{P(S_\nu > x)}{P(X > x)} = E\nu + \frac{\sum_{n=2}^{\infty} p_n\,b_1}{x^\alpha} + \cdots + \frac{\sum_{n=2}^{\infty} p_n\,b_{s-1}}{x^{(s-1)\alpha}} + \frac{\alpha\,\mu_1^*\,(E\nu^2 - E\nu)}{x} + O(x^{-1-\delta})$$

as $x \to \infty$. Using formula (142) we can calculate all b_j, $j = 1, ..., s$. For example, $b_1 = c_2\,(n^2 - n)/c_1$. Unfortunately, there is no explicit form for b_j. Using the notation $m_j = \sum_{n=2}^{\infty} p_n\, b_j$, $j = 2, ..., s-1$, and the fact that all moments of ν are finite we get

$$\Delta(x) = \frac{c_2\,(E\nu^2 - E\nu)}{c_1\,x^\alpha} + \frac{m_2}{x^{2\alpha}} + \cdots + \frac{m_{s-1}}{x^{(s-1)\alpha}} + \frac{\alpha\,\mu_1^*\,(E\nu^2 - E\nu)}{x} + O(x^{-1-\delta}), \quad x \to \infty.$$

This completes the proof of the theorem.

□

5.3 Application: Cramér-Lundberg model

Consider the Cramér-Lundberg model introduced in Section 1.3. Under some conditions Theorem 5.5 allows to obtain a result similar to Theorem 1.7. Suppose that the integrated tail distribution $F_I(x)$ (see formula (15)) can be represented in the form

$$1 - F_I(x) = \frac{c_1}{x^\alpha} + \frac{c_2\,d_2}{x^{2\alpha}} + \cdots + \frac{c_s\,d_s}{x^{s\alpha}} + u(x), \quad x \to \infty, \tag{144}$$

where $c_k := C_k(\alpha)$ are defined by (58), $s \in \mathbb{N}$ and $u(x)$ are such that

$$1 + \alpha \le s\alpha < 1 + 2\,\alpha, \qquad \int_0^{+\infty} x^q |du(x)| < \infty \quad \text{for some } q > s\alpha,$$

and $d_i \in \mathbb{R}$ are suitable constants for $i = 2, ..., s$. Such a representation of F_I implies that there exists an $r \in (s\alpha, (s+1)\alpha]$, such that $\gamma_r^* < \infty$, and allows to apply Theorem 3.26.

We are interested in the asymptotic behavior of the ruin probability ψ defined by (12). From (14) in Section 1.3 we know that the non-ruin probability $1-\psi$ can be interpreted as the distribution function of the sum $S_{\nu^*}^* = X_1^* + X_2^* + \cdots + X_{\nu^*}^*$, $S_0^* = 0$, where $X_1^*, X_2^*, \ldots$ are i.i.d. random variables with common distribution function $F_I(u)$ and ν^* is a counting random variable defined by $P(\nu^* = n) = \rho\,(1+\rho)^{-(n+1)}$ for $n \in \mathbb{N}_0$. In other words, we have $\psi(u) = P(S_{\nu^*}^* > u)$. Then $\psi(u)/(1 - F_I(u))$ considered in Theorem 1.7 can be written as

$$\frac{\psi(u)}{1 - F_I(u)} = \frac{P(S_{\nu^*}^* > u)}{P(X^* > u)} = \Delta^*(u) + E\nu^*.$$

Applying Theorem 5.5 we obtain the following result.

Corollary 5.6. *Consider the Cramér-Lundberg model with the net profit condition $\rho > 0$. Let F_I satisfy condition* (144) *with $\alpha \in (0,1)$. If $\nu_0^* < 1$, then*

$$\frac{\psi(u)}{1 - F_I(u)} = \rho^{-1} + \frac{2\,c_2}{c_1\,\rho^2\,x^\alpha} + \frac{m_2}{x^{2\alpha}} + \cdots + \frac{m_{s-1}}{x^{(s-1)\alpha}} + \frac{2\,\alpha\,\mu_1^*}{\rho^2\,x} + O(x^{-1-\delta}), \quad x \to \infty,$$

where $\delta > 0$ is some constant and m_i, $i = 2, ..., s-1$, depend only on $d_2, \ldots, d_s$, $c_1, ..., c_s$, ρ and α. Namely, m_i, $i = 2, ..., s-1$, can be found as follows:

$$m_i = \sum_{n=2}^{\infty} \rho\,(1+\rho)^{-(n+1)}\,b_i \qquad \textit{with} \qquad b_0 = n, \quad b_1 = c_2\,(n^2 - n)/c_1,$$

$$b_{k-1} = \frac{1}{c_1}\left(c_k n^k \left(1 + \sum_{j=2}^{k} \frac{c_{j,n}}{n^j}\,\frac{k!}{(k-j)!}\right) - \sum_{j=2}^{k} c_j\, d_j\, b_{k-j}\right), \quad k = 3, ..., s,$$

where c_j, d_j *are from* (144), $c_{j,n}$ *is defined by* (89).

Proof. Let us show that the moments of ν^* of all orders are finite. For any $j \in \mathbb{N}$ we have

$$E\nu^j = \sum_{n=0}^{\infty} n^j\, P(\nu^* = n) = \rho \sum_{n=0}^{\infty} n^j\,(1+\rho)^{-(n+1)}.$$

According to the ratio convergence test, the latter infinite series converges, since $\limsup_{n\to\infty} a_{n+1}/a_n = \limsup_{n\to\infty}(1+1/n)^j/(1+\rho) = 1/(1+\rho) < 1$.
Moreover, for $j = 1, 2$ we have $E\nu = 1/\rho$ and $E\nu^2 = (2+\rho)/\rho^2$. Using these facts and Theorem 5.5 we get the statement of the corollary.

□

5.4 Examples

In this section we consider some examples for which we can apply Theorem 5.5.

Example 5.1. Let us consider a Pareto-distributed random variable X with $\alpha = 1/3$ and $\kappa = c_1^3$, where $c_1 = C_1(1/3) = \sqrt{3}\,\Gamma(1/3)/(2\pi)$ is defined in (58). The distribution function F of X has the form:

$$1 - F(x) = \frac{c_1}{x^{1/3}}, \quad x \geq c_1^3.$$

Comparing this representation with representation (136) we can make the following conclusions: s can be chosen equal to 4 (since $4\alpha \geq 1+\alpha$), $u(x) = 0$ and $d_2 = d_4 = 0$. Coefficient d_3 can be chosen arbitrarily, since $c_3 = C_3(1/3) = 0$ (see formula (58)). Let us put $d_3 = -0.98$.

In order to apply Theorem 5.5 we have to check the condition $\nu_0^* < 1$. First, let us see how pseudomoment ν_0^* can be calculated. According to definition (80) we have

$$\nu_0^* = \int_0^{\infty} \left| d\left(F - \widetilde{G}_{1/3}\right)(x)\right|,$$

where

$$\widetilde{G}_{1/3}(x) = G_{1/3,1}(x) + \sum_{j=2}^{4} A_j\, G^{(0,j)}(x,1)$$

with coefficients $A_2 = -1/2$, $A_3 = 17/100$ and $A_4 = 23/600$, chosen in such a way, that

$$F(x) - \widetilde{G}_{1/3}(x) = O\left(x^{-5/3}\right) \quad \text{as} \quad x \to \infty.$$

This implies the finiteness of the first pseudomoment μ_1^*. Moreover, it can be found precisely:

$$\mu_1^* = \int_0^{\infty} x\, d\left(F - \widetilde{G}_{1/3}\right)(x) = \frac{c_1^{1/\alpha}\,\alpha}{\alpha - 1} - \left(\frac{1}{6} + A_2 + A_3\right) = \frac{49}{300} - \frac{3\sqrt{3}}{16\pi^3}\,\Gamma^3(1/3).$$

When calculating μ_1^* we used the same method as in Example 3.10. The only difference is that we do not put all d_2, d_3, d_4 equal to 0. The absolute pseudomoment ν_0^* was calculated using software *Mathematica* and has approximate value $\nu_0^* \approx 0.995$, which is less than 1. Thus, Theorem 5.5 is applicable.

If $E\nu^j < \infty$ for all $j \in \mathbb{N}$, then according to Theorem 5.5 we have

$$\Delta(x) = \frac{c_2\,(E\nu^2 - E\nu)}{c_1\,x^{1/3}} + \frac{m_2}{x^{2/3}} + \frac{m_3}{x} + \frac{\mu_1^*\,(E\nu^2 - E\nu)}{3\,x} + O\left(\frac{1}{x^{1+\delta}}\right), \quad x \to \infty,$$

with some $\delta > 0$. Let us calculate m_2 and m_3. Using formulas (137) we have $m_2 = \sum_{n=2}^{\infty} p_n\, b_2$ and $m_3 = \sum_{n=2}^{\infty} p_n\, b_3$ with

$$b_2 = \frac{1}{c_1}\left(c_3 n^3\left(1 + \sum_{j=2}^{3} \frac{c_{j,n}}{n^j}\,\frac{3!}{(3-j)!}\right) - \sum_{j=2}^{3} c_j\, d_j\, b_{3-j}\right),$$

$$b_3 = \frac{1}{c_1}\left(c_4 n^4\left(1 + \sum_{j=2}^{4} \frac{c_{j,n}}{n^j}\,\frac{4!}{(4-j)!}\right) - \sum_{j=2}^{4} c_j\, d_j\, b_{4-j}\right).$$

Using formula (89) for $c_{j,n}$, $j = 2, 3, 4$, we obtain

$$c_{2,n} = -\frac{n}{2}, \qquad c_{3,n} = \frac{17\,n}{100}, \qquad c_{4,n} = \frac{75\,n^2 - 52\,n}{600}.$$

Using this and the fact that $d_2 = c_3 = d_4 = 0$ we get

$$b_2 = \frac{c_3}{50\,c_1}\left(50\,n^3 - 150\,n^2 + 51\,n\right) = 0, \;\; b_3 = \frac{c_4}{25\,c_1}\left(25\,n^4 - 150\,n^3 + 177\,n^2 - 52\,n\right).$$

Thus, $m_2 = \sum_{n=2}^{\infty} p_n b_2 = 0$. From the definition of $E\nu^j$, $j = 1, ..., 4$, and the fact that $25\,n^4 - 150\,n^3 + 177\,n^2 - 52\,n = 0$ for $n = 0, 1$, we obtain

$$\begin{aligned} m_3 &= \frac{c_4}{25\,c_1} \sum_{n=2}^{\infty} p_n \left(25\,n^4 - 150\,n^3 + 177\,n^2 - 52\,n\right) \\ &= \frac{c_4}{25\,c_1}\left(25\,E\nu^4 - 150\,E\nu^3 + 177\,E\nu^2 - 52\,E\nu\right). \end{aligned}$$

Using formula (58) for $c_k = C_k(1/3)$, $k = 1, ..., 4$, we calculate

$$c_1 = \frac{\sqrt{3}}{2\pi}\,\Gamma(1/3), \qquad c_2 = -\frac{\sqrt{3}}{4\pi}\,\Gamma(2/3), \qquad c_3 = 0, \qquad c_4 = \frac{\sqrt{3}}{144\pi}\,\Gamma(1/3).$$

Plugging c_1, c_2, m_2, m_3 and μ_1^* into the formula for $\Delta(x)$ we obtain

$$\begin{aligned} \Delta(x) &= \frac{\Gamma(2/3)\,(E\nu - E\nu^2)}{2\,\Gamma(1/3)\,x^{1/3}} \\ &+ \frac{2\pi^3(E\nu^4 - 6E\nu^3) + \left(22\pi^3 - 9\sqrt{3}\,\Gamma^3(\tfrac{1}{3})\right) E\nu^2 - \left(12\pi^3 - 9\sqrt{3}\,\Gamma^3(\tfrac{1}{3})\right) E\nu}{144\,\pi^3\,x} \\ &+ O\left(\frac{1}{x^{1+\delta}}\right), \quad x \to \infty, \end{aligned}$$

with some $\delta > 0$. □

Example 5.2. Consider a nonnegative random variable X with distribution function F and density function p:

$$p(x) = \begin{cases} 0, & \text{for} \quad x < 0, \\ 1 - A, & \text{for} \quad 0 \le x < 1, \\ \dfrac{1}{2\sqrt{\pi}} x^{-3/2} \left(1 - \sum\limits_{k=2}^{\infty} \mathbb{1}_{[k - \frac{1}{k^2}; k + \frac{1}{k^2})}(x) \right), & \text{for} \quad x \ge 1. \end{cases}$$

where $A\,(\approx 0.492)$ is chosen such that $\int_{-\infty}^{+\infty} p(x)dx = 1$. In Example 2.8 we have shown that

$$1 - F(x) = \frac{1}{\sqrt{\pi}\,\sqrt{x}} + O(x^{-5/2}), \quad x \to \infty.$$

Comparing this representation with representation (136) we can make the following conclusions: $\alpha = 1/2$, $c_1 = 1/\sqrt{\pi}$, s can be chosen equal to 3 (since $3\alpha \ge 1 + \alpha$) and $d_3 = 0$. Coefficient d_2 can be chosen arbitrarily, since $c_2 = C_2(1/2) = 0$ (see formula (58)). Let us put $d_2 = 3/4$. Such a choice of d_2 makes the pseudomoment ν_0^* less than 1. Recall how ν_0^* can be calculated. According to definition (80) we have

$$\nu_0^* = \int_0^{\infty} \left| d \left(F - \widetilde{G}_{1/2} \right)(x) \right| \quad \text{with} \quad \widetilde{G}_{1/2}(x) = G_{1/2,1}(x) + \sum_{j=2}^{3} A_j \, G^{(0,j)}(x, 1)$$

and coefficients $A_2 = -1/8$ and $A_3 = -1/24$. Note that the stable distribution corresponding to $G_{1/2,1}(x, \lambda)$ (Lévy distribution) has an explicit density function

$$p_{1/2,1}(x, \lambda) = \frac{\lambda}{2\sqrt{\pi}} \, e^{-\frac{\lambda^2}{4\,x}} \, x^{-\frac{3}{2}}, \quad x > 0, \, \lambda > 0,$$

which makes all calculations much easier. Using software *Mathematica* we obtain $\nu_0^* \le 0.32 < 1$. Thus, Theorem 5.5 is applicable.

If $E\nu^j < \infty$ for all $j \in \mathbb{N}$, then according to Theorem 5.5 we have

$$\Delta(x) = \frac{c_2 \, (E\nu^2 - E\nu)}{c_1 \, x^{1/2}} + \frac{m_2}{x} + \frac{\mu_1^* \, (E\nu^2 - E\nu)}{2\,x} + O\left(\frac{1}{x^{1+\delta}} \right), \quad x \to \infty,$$

with some $\delta > 0$. Let us calculate m_2. Using formulas (137) we have $m_2 = \sum\limits_{n=2}^{\infty} p_n \, b_2$ with

$$b_2 = \frac{1}{c_1} \left(c_3 n^3 \left(1 + \sum_{j=2}^{3} \frac{c_{j,n}}{n^j} \, \frac{3!}{(3-j)!} \right) - \sum_{j=2}^{3} c_j \, d_j \, b_{3-j} \right).$$

Using formula (89) for $c_{j,n}$, $j = 2, 3$, we obtain

$$c_{2,n} = -\frac{n}{8}, \qquad c_{3,n} = -\frac{n}{24}.$$

Using this and the fact that $c_2 = d_3 = 0$ we get

$$b_2 = \frac{c_3}{4\,c_1} \left(4\,n^3 - 3\,n^2 - \, n \right).$$

From the definition of $E\nu^j$, $j = 1, ..., 3$, and the fact that $4\,n^3 - 3\,n^2 - n = 0$ for $n = 0, 1$, we obtain

$$m_2 = \frac{c_3}{4\,c_1} \sum_{n=2}^{\infty} p_n \left(4\,n^3 - 3\,n^2 - n\right) = \frac{c_3}{4\,c_1} \left(4\,E\nu^3 - 3\,E\nu^2 - E\nu\right).$$

Using formula (58) for $c_k = C_k(1/3)$, $k = 1, ..., 3$, we calculate

$$c_1 = \frac{1}{\sqrt{\pi}}, \qquad c_2 = 0, \qquad c_3 = -\frac{1}{12\,\sqrt{\pi}}.$$

Plugging c_1, c_2, c_3 and m_2 into the formula for $\Delta(x)$ we obtain

$$\Delta(x) = \frac{-\,(4\,E\nu^3 - 3\,E\nu^2 - E\nu) + 24\mu_1^*\,(E\nu^2 - E\nu)}{48\,x} + O\left(\frac{1}{x^{1+\delta}}\right), \quad x \to \infty,$$

with some $\delta > 0$. Pseudomoment μ_1^* was also approximately calculated using *Mathematica* and has approximate value $\mu_1^* \approx -0.128$. □

For more examples see [11].

Appendix A Some useful results

Lemma A.1 ([33, p. 87])**.** *For $\alpha \in (0,1)$ we have*

$$\int_0^{+\infty} \left\{ \begin{array}{c} \sin x \\ \cos x \end{array} \right\} x^{-\alpha} dx = \Gamma(1-\alpha) \left\{ \begin{array}{c} \cos(\pi\alpha/2) \\ \sin(\pi\alpha/2) \end{array} \right\}. \tag{145}$$

Lemma A.2 (Euler's reflection formula; see [32, pp. 58–59])**.**
For all $\alpha \in \mathbb{R}_+$ we have

$$\Gamma(1-\alpha)\Gamma(\alpha) = \pi/\sin(\pi\alpha). \tag{146}$$

Lemma A.3. *Let $m \in \mathbb{N}_0$.*

(i) For any $y \in \mathbb{R}$ we have

$$\left| e^{iy} - \sum_{j=0}^{m} \frac{(iy)^j}{j!} \right| \leq \min\left\{ \frac{|y|^{m+1}}{(m+1)!}, \frac{2\,|y|^m}{m!} \right\}.$$

(ii) For $z \in \mathbb{C}$ with $|z| \leq 1$ we have

$$\left| e^{z} - \sum_{j=0}^{m} \frac{z^j}{j!} \right| \leq \frac{C(m)\,|z|^{m+1}}{(m+1)!},$$

where $C(m) = m(e+1) + e + 2$.

Proof. Part (i) of the lemma is [31, Appendix A, Lemma 1.2].

In order to show part (ii) we consider the general form of the remainder in the Taylor's expansion (see [1, Chapter IV, §3, 3.2]). Using $|z| \leq 1$ we get

$$\left| e^{z} - \sum_{j=0}^{m+1} \frac{z^j}{j!} \right| \leq \frac{|z|^{m+1}}{m!} \sup_{0\leq\theta\leq 1} \left| e^{\theta z} - 1 \right| \leq \frac{(e+1)\,|z|^{m+1}}{m!}.$$

Now, using the triangle inequality $|a+b| - |b| \leq |a|$ with $a = e^z - \sum_{j=0}^{m+1} \frac{z^j}{j!}$ and $b = \frac{z^{m+1}}{(m+1)!}$ we obtain

$$\left| e^{z} - \sum_{j=0}^{m} \frac{z^j}{j!} \right| - \frac{|z|^{m+1}}{(m+1)!} \leq \left| e^{z} - \sum_{j=0}^{m+1} \frac{z^j}{j!} \right| \leq \frac{(e+1)\,|z|^{m+1}}{m!}$$

and, finally,

$$\left| e^{z} - \sum_{j=0}^{m} \frac{z^j}{j!} \right| \leq \frac{(e+1)\,|z|^{m+1}}{m!} + \frac{|z|^{m+1}}{(m+1)!} = \frac{C(m)\,|z|^{m+1}}{(m+1)!},$$

where $C(m) = m(e+1) + e + 2$. □

Lemma A.4. *If for some function $f : \mathbb{R} \to \mathbb{R}$ and $a, b, d \in \mathbb{R}$, $r \in \mathbb{R}_+$ we have*

(i) $|f(x)| \le d \quad$ *for all* $x \in \mathbb{R}$,

(ii) $|f(x)| \le b\,|x|^{-r} \quad$ *for* $|x| > a$,

then

$$|f(x)| \le C(1+|x|)^{-r} \quad \text{for all} \quad x \in \mathbb{R},$$

where $C = \max\{d\,(1+a)^r, b\,(1+1/a)^r\}$.

Proof. Using condition (i) and the definition of C we have for $|x| \le a$

$$C(1+|x|)^{-r} \ge C(1+a)^{-r} \ge d \ge |f(x)|.$$

From condition (ii) for $|x| > a$ it follows that

$$C(1+|x|)^{-r} = C\,|x|^{-r}(1+1/|x|)^{-r} \ge C\,|x|^{-r}(1+1/a)^{-r} \ge b\,|x|^{-r} \ge |f(x)|.$$

This completes the proof. □

Lemma A.5 (see [1, Chapter I, §8, 8.5])**.**
For $n, v \in \mathbb{N}$ and all $y_1, \ldots, y_n \in \mathbb{C}$ the following equality holds

$$(y_1 + \cdots + y_n)^v = \sum_{k_1+\cdots+k_n=v} \frac{v!}{k_1! \ldots k_n!} y_1^{k_1} \ldots y_n^{k_n}, \tag{147}$$

where the summation is carried out over all non-negative integer solutions $k_1, k_2, \ldots, k_n$ of the equation $k_1 + k_2 + \cdots + k_n = v$.

Lemma A.6. *For $n, v \in \mathbb{N}$, $y_i \in \mathbb{R}$, $y_i \ge 0$, $i = 1, ..., n$, we have*

$$(y_1 + \cdots + y_n)^v \le n^{v-1}(y_1^v + \cdots + y_n^v). \tag{148}$$

Proof. This is application of Jensen's inequality (see [25, Chapter IV]) to the convex function $x \mapsto x^v$. □

Lemma A.7. *Let $s, K \in \mathbb{N}$. If $|a_j| \le a$ and $|b_j| \le b$ for $j \in \{1, ..., s\}$ and some constants $a, b > 0$, then*

$$\left| a_1^{k_1}\, a_2^{k_2} \cdots a_s^{k_s} - b_1^{k_1}\, b_2^{k_2} \cdots b_s^{k_s} \right| \le C(a, b, s, K) \sum_{j=1}^{s} |a_j - b_j|\,,$$

where $k_j \in \{0, 1, ..., K\}$ for $j \in \{1, ..., s\}$ and $C(a, b, s, K)$ is some constant depending only on a, b, s and K.

Proof. We prove by induction on s.

For $s = 1$ we have

$$\begin{aligned}\left|a_1^{k_1} - b_1^{k_1}\right| &= \left|(a_1 - b_1)\sum_{j=0}^{k_1-1} a_1^j b_1^{k_1-1-j}\right| \\ &\le |a_1 - b_1| \sum_{j=0}^{k_1-1} a^j b^{k_1-1-j} \le |a_1 - b_1|\, C(a,b,1,K),\end{aligned}$$

where $C(a,b,1,K) = K a^K b^K$.

Now, suppose that the inequality from the assertion of the lemma holds for some $s \le n$ and consider the case $s = n+1$:

$$\begin{aligned}&\left|a_1^{k_1}\cdots a_n^{k_n} a_{n+1}^{k_{n+1}} - b_1^{k_1}\cdots b_n^{k_n} b_{n+1}^{k_{n+1}}\right| \\ &\quad\le \left|a_1^{k_1}\cdots a_n^{k_n} a_{n+1}^{k_{n+1}} \pm a_1^{k_1}\cdots a_n^{k_n} b_{n+1}^{k_{n+1}} - b_1^{k_1}\cdots b_n^{k_n} b_{n+1}^{k_{n+1}}\right| \\ &\quad\le \left|a_1^{k_1}\cdots a_n^{k_n} \underbrace{\left(a_{n+1}^{k_{n+1}} - b_{n+1}^{k_{n+1}}\right)}_{\text{case } s=1}\right| + \left|\underbrace{\left(a_1^{k_1}\cdots a_n^{k_n} - b_1^{k_1}\cdots b_n^{k_n}\right)}_{\text{case } s=n} b_{n+1}^{k_{n+1}}\right| \\ &\quad\le |a|^{nK}\, C(a,b,1,K)\, |a_{n+1} - b_{n+1}| + |b|^K\, C(a,b,n,K) \sum_{j=1}^{n} |a_j - b_j| \\ &\quad\le C(a,b,n+1,K) \sum_{j=1}^{n+1} |a_j - b_j|\,,\end{aligned}$$

where $C(a,b,n+1,K) = \max\left\{|a|^{nK}\, C(a,b,1,K), |b|^K\, C(a,b,n,K)\right\}$. □

Lemma A.8 (see [1, Chapter IV, §3, 3.8])**.**
Let $f : D \to \mathbb{R}$ be $(n+1)$ times differentiable on D, where $D \subseteq \mathbb{R}$ is convex with $a \in D$. Then for any $x \in D$ we have

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(a + \theta(x-a)), \text{ with } \theta \in (0,1).$$

Lemma A.9 ([44, Chapter VI, §1])**.**
Let the functions $y : \mathbb{R} \to \mathbb{C}$ and $z : \mathbb{C} \to \mathbb{C}$ have derivatives of order $\nu \in \mathbb{N}$ and let $0^0 = 1$. Then

$$\frac{d^\nu}{dt^\nu} z(y(t)) = \nu! \sum \left.\frac{d^s z(y)}{dy^s}\right|_{y=y(t)} \prod_{m=1}^{\nu} \frac{1}{k_m!} \left(\frac{1}{m!}\frac{d^m y(t)}{dt^m}\right)^{k_m}, \tag{149}$$

where the summation is carried out over all non-negative integer solutions $(k_1, k_2, \ldots, k_\nu)$ of the equation $k_1 + 2k_2 + \cdots + \nu k_\nu = \nu$, and $s = k_1 + k_2 + \cdots + k_\nu$.

Remark A.1. If $y : \mathbb{R} \to \mathbb{C}$ has a derivative of order $\nu \ge 1$, then

$$\frac{d^\nu}{dt^\nu} e^{y(t)} = \nu!\, e^{y(t)} \sum_{k_1+2k_2+\ldots+\nu k_\nu=\nu} \prod_{m=1}^{\nu} \frac{1}{k_m!} \left(\frac{1}{m!}\frac{d^m y(t)}{dt^m}\right)^{k_m}. \tag{150}$$

This follows from Lemma A.9 with $z(y) = e^y$.

Lemma A.10. *If the functions $f_1, \dots, f_s : \mathbb{R} \to \mathbb{C}$ have derivatives of order $n \in \mathbb{N}$, then*

$$(f_1 \cdot f_2 \cdot \ldots \cdot f_s)^{(n)}(t) = \sum_{k_1+\ldots+k_s=n} \frac{n!}{k_1! \ldots k_s!} f_1^{(k_1)}(t) \cdot \ldots \cdot f_s^{(k_s)}(t). \tag{151}$$

Proof. We get the assertion by inductively applying the Leibniz rule of differentiation (product rule for higher-order derivatives, see [1, Chapter IV, §1, 1.12]). Compare also to Lemma A.5. □

Lemma A.11. *For each $a \in (0,1)$ and $s \in \mathbb{N}$ there exists a constant $C = C(s,a)$ such that*

$$C\ (|y| + |y|^s)\ e^{-a|y|} \leq 1 \qquad \textit{for all} \quad y \in \mathbb{R}.$$

Proof. We need to show that the function $f(x) = (x + x^s)e^{-ax}$ is bounded on $[0, \infty)$. This follows from $\lim_{x\to\infty} f(x) = 0$ and from the continuity of f on $[0, \infty)$. □

Lemma A.12 (Riemann-Lebesgue lemma; see [3, Prop. 5.7.1])**.**
Let $L^1(\mathbb{R}) = \{v \mid v : \mathbb{R} \to \mathbb{C}$ Borel measurable, $\int_{\mathbb{R}} |v|\, d\lambda < \infty\}$, where λ is Lebesgue measure. If $v \in L^1(\mathbb{R})$, then

$$\int_{\mathbb{R}} v(x) e^{itx} \lambda(dx) \to 0 \qquad \textit{as} \qquad |t| \to \infty.$$

Lemma A.13. *Suppose b, $s > 0$ and $\alpha \in (0,1)$ are fixed. Then for all $t \in \mathbb{R}$ we have*

$$z(t) := \exp\left(-b\,|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)\right) |t|^s \leq \left(\frac{s}{e\,\alpha\, b\, \cos\left(\frac{\alpha\pi}{2}\right)}\right)^{s/\alpha}.$$

Proof. If $t = 0$, then $z(t) = 0$ and the statement of the lemma is satisfied. Now we consider the case $t \in \mathbb{R} \setminus \{0\}$. Denote $y := b\,|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)$. Then we obtain $z(y) = e^{-y}\, y^{s/\alpha} \left(b \cos\left(\frac{\alpha\pi}{2}\right)\right)^{-s/\alpha}$, $y > 0$. It is easy to see that the function $e^{-y}\, y^{s/\alpha}$ achieves its maximum (on $\mathbb{R}_+$) at the point $y_0 = s/\alpha$. Thus,

$$z(y) \leq e^{-s/\alpha}\,(s/\alpha)^{s/\alpha} \left(b \cos\left(\frac{\alpha\pi}{2}\right)\right)^{-s/\alpha} \leq \left(s^{-1}\, e\, \alpha\, b\, \cos\left(\frac{\alpha\pi}{2}\right)\right)^{-s/\alpha}.$$

This completes the proof of the lemma. □

Lemma A.14. *For $\alpha \in (0,1)$, $b > 0$ and $d > -1$ we have*

$$\int_{-\infty}^{+\infty} e^{-b\,|t|^\alpha}\, |t|^d\, dt = 2\,\alpha^{-1}\, b^{-\frac{1+d}{\alpha}}\ \Gamma\left(\frac{1+d}{\alpha}\right) < \infty.$$

Proof. Consider $t \geq 0$ and denote $y := b\, t^{\alpha}$. Then we have $t = \left(b^{-1}\, y\right)^{1/\alpha}$. Integrating by substitution and using the definition of the gamma function we obtain

$$\int_{-\infty}^{+\infty} e^{-b\,|t|^{\alpha}}\, |t|^{d}\, dt = 2 \int_{0}^{+\infty} e^{-b\, t^{\alpha}}\, t^{d}\, dt = 2 \int_{0}^{+\infty} e^{-y} \left(b^{-1}\, y\right)^{d/\alpha} \alpha^{-1}\, b^{-1/\alpha}\, y^{\frac{1}{\alpha}-1}\, dy$$

$$= 2\, \alpha^{-1}\, b^{-\frac{1+d}{\alpha}} \int_{0}^{+\infty} e^{-y}\, y^{\frac{1+d}{\alpha}-1}\, dy = 2\, \alpha^{-1}\, b^{-\frac{1+d}{\alpha}}\ \Gamma\left(\frac{1+d}{\alpha}\right).$$

Note that $\Gamma\left(\frac{1+d}{\alpha}\right) < \infty$ for $d > -1$. This completes the proof of the lemma. □

Lemma A.15 ([44, Chapter VI.2, Lemma 7])**.**
Let $G(x)$ be a function of bounded variation on the real line and let $g(t)$ be its Fourier-Stieltjes transform. Suppose that $\lim_{|x|\to\infty} G(x) = 0$ and

$$\int_{-\infty}^{+\infty} |x|^{m}\, |dG(x)| < \infty$$

for some integer $m \geq 1$. Then $x^{m}G(x)$ is a function of bounded variation on the real line and we have

$$(-it)^{m} \int_{-\infty}^{+\infty} e^{itx} d\left(x^{m}G(x)\right) = m! \sum_{\nu=0}^{m} \frac{(-t)^{\nu}}{\nu!}\, \frac{d^{\nu}}{dt^{\nu}}\, g(t).$$

Lemma A.16 ([44, Chapter I.2, Theorem 1])**.**
Let $f(t)$ be a characteristic function and let $b > 0$ and $c \in (0,1)$. If $|f(t)| \leq c$ for $|t| \geq b$, then

$$|f(t)| \leq 1 - \frac{1-c^{2}}{8\, b^{2}}\, t^{2} \qquad \textit{for all} \quad |t| < b.$$

Appendix B Proofs of Lemmata 4.13 – 4.18

In this appendix we often use estimate (i) from Lemma 4.2 for pseudomoments $\overline{\overline{\nu}}_{k,n,\xi}$:

$$\overline{\overline{\nu}}_{k,n,\xi} \leq (\nu_0^* + 1)\gamma_r^{*\,k/r}, \quad k = 1, \ldots, R,$$

where r and R come from Theorem 3.26. Since $\gamma_r^* < \infty$ and ν_0^* are fixed and can be calculated, we can estimate all $\overline{\overline{\nu}}_{k,n,\xi}$ by some constant $C > 0$:

$$\overline{\overline{\nu}}_{k,n,\xi} \leq C \qquad \text{for} \quad k = 1, ..., R.$$

Therefore, in the formulations and proofs of some lemmata from this section constants C depend on the pseudomoments.

B.1 Some auxiliary results

Lemma B.1. *For $k \in \mathbb{N}$, $s \in \mathbb{N}_0$ and $\varphi_{\alpha,1}(t) = -|t|^\alpha e^{-i\,\alpha\,\frac{\pi}{2}\,\text{sign}\,t}$, $t \in \mathbb{R}$, we have*

$$\left|\frac{d^s}{dt^s}\varphi_{\alpha,1}^k(t)\right| = \prod_{j=0}^{s-1} |\alpha k - j|\ |t|^{\alpha k - s} \quad \textit{for} \quad t \in \mathbb{R}\setminus\{0\}.$$

Moreover, if $s < \alpha k$ then

$$\left|\frac{d^s}{dt^s}\varphi_{\alpha,1}^k(t)\right| = \prod_{j=0}^{s-1}(\alpha k - j)\,|t|^{\alpha k - s} \quad \textit{for} \quad t \in \mathbb{R}.$$

Proof. For $k \in \mathbb{N}$ we obtain

$$\varphi_{\alpha,1}^k(t) = \Big(-|t|^\alpha\Big)^k e^{-i\,\alpha\,k\,\frac{\pi}{2}\,\text{sign}\,t} = \begin{cases} (-1)^k t^{\alpha k} e^{-i\alpha k\frac{\pi}{2}}, & \text{if } t > 0, \\ 0, & \text{if } t = 0, \\ (-1)^k (-t)^{\alpha k} e^{i\alpha k\frac{\pi}{2}}, & \text{if } t < 0. \end{cases}$$

If $s = 0$, then the assertion of the lemma is immediately true. We consider $s \in \mathbb{N}$ in what follows. For $t > 0$ we have

$$\frac{d^s}{dt^s}\varphi_{\alpha,1}^k(t) = (-1)^k \alpha k(\alpha k - 1)\cdots\Big(\alpha k - (s-1)\Big)t^{\alpha k - s}e^{-i\alpha k\frac{\pi}{2}}.$$

For $t < 0$ we get

$$\frac{d^s}{dt^s}\varphi_{\alpha,1}^k(t) = (-1)^{k+s}\alpha k(\alpha k - 1)\cdots\Big(\alpha k - (s-1)\Big)(-t)^{\alpha k - s}e^{i\alpha k\frac{\pi}{2}}.$$

Finally, we compute the derivatives at $t = 0$. With $\alpha k - s > 0$ we obtain for $s = 1$:

$$\Big(\varphi_{\alpha,1}^k\Big)'_+(0) = \lim_{h\to 0+}\frac{\varphi_{\alpha,1}^k(0+h) - \varphi_{\alpha,1}^k(0)}{h} = \lim_{h\to 0+}(-1)^k h^{\alpha k - 1}e^{-i\alpha k\frac{\pi}{2}} = 0,$$

$$\Big(\varphi_{\alpha,1}^k\Big)'_-(0) = \lim_{h\to 0-}\frac{\varphi_{\alpha,1}^k(0+h) - \varphi_{\alpha,1}^k(0)}{h} = \lim_{h\to 0-}(-1)^{k-1}(-h)^{\alpha k - 1}e^{i\alpha k\frac{\pi}{2}} = 0.$$

Thus, $\Big(\varphi_{\alpha,1}^k\Big)'(0) = 0$. Similarly it can be shown that $\Big(\varphi_{\alpha,1}^k\Big)^{(s)}(0) = 0$ for all $s < \alpha k$. Combining all three cases we get the assertion of the lemma. □

Lemma B.2. *Let $g_{\alpha,1}(t) = e^{\varphi_{\alpha,1}(t)}$ with $\varphi_{\alpha,1}(t) = -|t|^\alpha e^{-i\alpha\frac{\pi}{2}\operatorname{sign} t}$, $t \in \mathbb{R}$. For $s \in \mathbb{N}_0$, some constant $N \in \mathbb{R}_+$ and $t \in \mathbb{R}\setminus\{0\}$ we have*

$$\left|\frac{d^s}{dt^s} g_{\alpha,1}(Nt)\right| \le C(s)\, e^{-|Nt|^\alpha \cos\left(\alpha\frac{\pi}{2}\right)/2}\, |t|^{-s},$$

where $C(s)$ is some constant not depending on N.

Proof. For $s = 0$ the assertion of the lemma is obviously true. We consider the case $s \in \mathbb{N}$ in what follows. Note that $g_{\alpha,1}(t)$ is infinitely differentiable for $t \in \mathbb{R}\setminus\{0\}$. Using formula (150) with $y(t) = \varphi_{\alpha,1}(Nt)$ and Lemma B.1 we obtain

$$\begin{aligned}\left|\frac{d^s}{dt^s} g_{\alpha,1}(Nt)\right| &= \left|s!\, g_{\alpha,1}(Nt) \sum_{k_1+2k_2+\ldots+sk_s=s} \prod_{m=1}^{s} \frac{1}{k_m!}\left(\frac{1}{m!}\frac{d^m}{dt^m}\varphi_{\alpha,1}(Nt)\right)^{k_m}\right| \\ &\le s!\, |g_{\alpha,1}(Nt)| \sum_{k_1+2k_2+\ldots+sk_s=s} \prod_{m=1}^{s} \frac{1}{k_m!}\left(\frac{1}{m!}\right)^{k_m} N^{\alpha k_m}\left|\frac{d^m}{dt^m}\varphi_{\alpha,1}(t)\right|^{k_m}.\end{aligned}$$

Removing the product sign and taking into account that $1 \le k_1 + k_2 + \cdots + k_s \le k_1 + 2k_2 + \cdots + sk_s = s$ for $s \ge 1$, we find

$$\begin{aligned}\left|\frac{d^s}{dt^s} g_{\alpha,1}(Nt)\right| &\le s!\, |g_{\alpha,1}(Nt)| \sum_{k_1+2k_2+\ldots+sk_s=s} \frac{1}{k_1!\cdots k_s!}\left(\frac{1}{1!}\right)^{k_1}\cdots\left(\frac{1}{s!}\right)^{k_s} \times \\ &\quad\times N^{\alpha(k_1+\ldots+k_s)} \underbrace{\left|\frac{d}{dt}\varphi_{\alpha,1}(t)\right|^{k_1}}_{=\left(\alpha|t|^{\alpha-1}\right)^{k_1}} \cdots \underbrace{\left|\frac{d^s}{dt^s}\varphi_{\alpha,1}(t)\right|^{k_s}}_{=\left(\prod_{j=0}^{s-1}|\alpha-j|\,|t|^{\alpha-s}\right)^{k_s}} \\ &\le s!\, |g_{\alpha,1}(Nt)| \sum_{k_1+2k_2+\ldots+sk_s=s} C(k_1,..,k_s) N^{\alpha(k_1+\ldots+k_s)}\, |t|^{\alpha(k_1+\ldots+k_s)-(k_1+2k_2+\ldots+sk_s)} \\ &= s!\, |g_{\alpha,1}(Nt)| \sum_{k_1+2k_2+\ldots+sk_s=s} C(k_1,..,k_s) N^{\alpha(k_1+\ldots+k_s)}\, |t|^{\alpha(k_1+\ldots+k_s)-s} \\ &\le C(s)\, e^{-|Nt|^\alpha\cos\left(\alpha\frac{\pi}{2}\right)}\left(N^\alpha |t|^{\alpha-s} + N^{s\alpha}|t|^{\alpha s-s}\right).\end{aligned}$$

Applying Lemma A.11 with $|y| = |Nt|^\alpha$ and $a = \cos(\frac{\alpha\pi}{2})/2$ to the last expression we get the assertion of the lemma. □

Lemma B.3. *For $\rho \in \mathbb{N}_0$ and $t \in \mathbb{R}\setminus\{0\}$ we have*

$$\left|\frac{d^\rho}{dt^\rho}\widetilde{g}_\alpha\left(tn^{-1/\alpha}\right)\right| \le C(\rho)\, e^{-\frac{|t|^\alpha}{4n}\cos\left(\alpha\frac{\pi}{2}\right)}\, |t|^{-\rho},$$

where $\widetilde{g}_\alpha$ is given by (82) *and $C(\rho) > 0$ is some constant not depending on n.*

Proof. Recall that $\widetilde{g}_\alpha(t) = \int_{-\infty}^{+\infty} e^{itx} d\widetilde{G}_\alpha(x) = g_{\alpha,1}(t) + \sum_{k=2}^{s} A_k\, g_{\alpha,1}(t)\, \varphi_{\alpha,1}^k(t)$, $t \in \mathbb{R}$, with A_k, $k = 2, ..., s$, from (70). Define $A_0 := 1$ and $A_1 := 0$. Then we have

$$\widetilde{g}_\alpha(tn^{-1/\alpha}) = \sum_{k=0}^{s} A_k\, g_{\alpha,1}(tn^{-1/\alpha})\, \varphi_{\alpha,1}^k(tn^{-1/\alpha}) = \sum_{k=0}^{s}\frac{A_k}{n^k}\, g_{\alpha,1}(tn^{-1/\alpha})\, \varphi_{\alpha,1}^k(t).$$

For $t \neq 0$ functions $g_{\alpha,1}$ and $\varphi^k_{\alpha,1}$ are infinitely differentiable and Lemma A.10 can be applied:

$$\frac{d^\rho}{dt^\rho}\,\widetilde{g}_\alpha\Big(tn^{-1/\alpha}\Big) = \sum_{k=0}^{s}\frac{A_k}{n^k}\sum_{\substack{n_1+n_2=\rho\\ n_1,n_2\in\mathbb{N}_0}}\frac{d^{n_1}}{dt^{n_1}}\,g_{\alpha,1}\Big(tn^{-1/\alpha}\Big)\frac{d^{n_2}}{dt^{n_2}}\varphi^k_{\alpha,1}(t).$$

Using the above equality and Lemmata B.1 and B.2 we get

$$\begin{aligned}\left|\frac{d^\rho}{dt^\rho}\,\widetilde{g}_\alpha\Big(tn^{-1/\alpha}\Big)\right| &\le \sum_{k=0}^{s}\frac{|A_k|}{n^k}\sum_{n_1+n_2=\rho}\left(C(n_1)\,e^{-\frac{\left|tn^{-1/\alpha}\right|^\alpha}{2}\cos\left(\alpha\frac{\pi}{2}\right)}\,|t|^{-n_1}\right)\\ &\cdot\left(\prod_{i=0}^{n_2-1}|\alpha k - i|\;|t|^{\alpha k-n_2}\right) \le C(\rho)\,e^{-\frac{\left|tn^{-1/\alpha}\right|^\alpha}{2}\cos\left(\alpha\frac{\pi}{2}\right)}\,|t|^{-\rho}\sum_{k=0}^{s}\frac{|t|^{\alpha k}}{n^k}\\ &\le C(\rho)\,e^{-\frac{\left|tn^{-1/\alpha}\right|^\alpha}{4}\cos\left(\alpha\frac{\pi}{2}\right)}\,|t|^{-\rho}\left[e^{-\frac{\left|tn^{-1/\alpha}\right|^\alpha}{4}\cos\left(\alpha\frac{\pi}{2}\right)}\sum_{k=0}^{s}\left|tn^{-1/\alpha}\right|^{\alpha k}\right].\end{aligned}$$

It follows from Lemma A.13 that the expression in square brackets in the last inequality can be estimated by a constant not depending on n. Thus, we obtain

$$\left|\frac{d^\rho}{dt^\rho}\,\widetilde{g}_\alpha\Big(tn^{-1/\alpha}\Big)\right| \le C(\rho)\,e^{-\frac{|t|^\alpha}{4n}\cos\left(\alpha\frac{\pi}{2}\right)}\,|t|^{-\rho}.$$

This completes the proof of the lemma. □

Lemma B.4. *Let $s, p, u \in \mathbb{N}_0$, $N \in \mathbb{R}$ and $t \in \mathbb{R}\setminus\{0\}$. For $|N||t|^\alpha \le 1$ we have*

$$\left|\frac{d^s}{dt^s}\left(t^u\left(e^{N\varphi_{\alpha,1}(t)} - \sum_{v=0}^{p}\frac{N^v}{v!}\varphi^v_{\alpha,1}(t)\right)\right)\right| \le C(s,u,p)\,|N|^{p+1}\,|t|^{\alpha(p+1)+u-s},$$

where $\varphi_{\alpha,1}(t) = -\,|t|^\alpha\,e^{-i\,\alpha\,\frac{\pi}{2}\,\operatorname{sign} t}$, $t\in\mathbb{R}$, and $C(s,u,p)$ is some constant not depending on N.

Proof. For $s = 0$ Lemma A.3 provides the assertion of the lemma. We consider the case $s \in \mathbb{N}$ in what follows. Note that $t^u\left(e^{N\varphi_{\alpha,1}(t)} - \sum_{v=0}^{p}\frac{N^v}{v!}\varphi^v_{\alpha,1}(t)\right)$ is infinitely differentiable for $t \in \mathbb{R}\setminus\{0\}$. Using formula (151) from Lemma A.10 we obtain

$$\begin{aligned}&\left|\frac{d^s}{dt^s}\left(t^u\left(e^{N\varphi_{\alpha,1}(t)} - \sum_{v=0}^{p}\frac{N^v}{v!}\varphi^v_{\alpha,1}(t)\right)\right)\right|\\ &\qquad\le \sum_{j=(s-u)^+}^{s}\binom{s}{j}\left|\frac{d^j}{dt^j}\left(e^{N\varphi_{\alpha,1}(t)} - \sum_{v=0}^{p}\frac{N^v}{v!}\varphi^v_{\alpha,1}(t)\right)\right|\cdot\left|\frac{d^{s-j}\,t^u}{dt^{s-j}}\right| = ⓢ.\end{aligned}$$

Now we apply formula (149) with $y(t) = N\varphi_{\alpha,1}(t)$ and $z(y) = e^y - \sum_{v=0}^{p}\frac{y^v}{v!}$:

$$\begin{aligned}\left|\frac{d^j}{dt^j}\,z(y(t))\right| &\le j!\sum_{\substack{k_1,\dots,k_j\in\mathbb{N}_0\\ k_1+2k_2+\dots+jk_j=j\\ q:=k_1+k_2+\dots+k_j}}\left|\left.\frac{d^q z(y)}{dy^q}\right|_{y=y(t)}\right|\prod_{m=1}^{j}\frac{1}{k_m!}\left(\frac{1}{m!}\left|\frac{d^m y(t)}{dt^m}\right|\right)^{k_m}\\ &\le C(j)\sum_{\substack{k_1+2k_2+\dots+jk_j=j\\ q:=k_1+k_2+\dots+k_j}}\left|e^{y(t)} - \sum_{v=0}^{p-q}\frac{y(t)^v}{v!}\right|\;|N|^{k_1+\dots+k_j}\prod_{m=1}^{j}\left|\frac{d^m\varphi_{\alpha,1}(t))}{dt^m}\right|^{k_m}.\end{aligned}$$

Taking into account that $|N||t|^\alpha \le 1$ and using Lemmata B.1 and A.3 we get

$$\left|\frac{d^j}{dt^j}\, z(y(t))\right| \le C(j) \sum_{\substack{k_1+2k_2+\ldots+jk_j=j\\ q:=k_1+k_2+\ldots+k_j}} |N\varphi_{\alpha,1}(t)|^{(p-q+1)^+}\ |N|^q\,|t|^{\alpha(k_1+\ldots+k_j)-k_1-2k_2-\ldots-jk_j}$$

$$\le C(j)\,|t|^{-j} \sum_{\substack{k_1+2k_2+\ldots+jk_j=j\\ q:=k_1+k_2+\ldots+k_j}} (|N||t|^\alpha)^{(p-q+1)^+}\ (|N||t|^\alpha)^q \le C(j)\,|N|^{p+1}\,|t|^{\alpha(p+1)-j}.$$

Using the last estimation we continue

$$\begin{aligned} Ⓢ &\le \sum_{j=(s-u)^+}^{s} \binom{s}{j} C(j)\,|N|^{p+1}\,|t|^{\alpha(p+1)-j}\ C(u,s,j)\,|t|^{u-(s-j)} \\ &\le C(s,u)\,|N|^{p+1}\,|t|^{\alpha(p+1)+u-s}. \end{aligned}$$

This completes the proof of the lemma.

□

Lemma B.5. *For $s\in\mathbb{N}_0$, $\ell\in\mathbb{N}$ and some constant $N\in\mathbb{R}_+$ we have*

$$\left|\frac{d^s}{dt^s}\overline{\overline{h}}^{\ell}_{n,\xi}(Nt)\right| \le (N\ell)^s\ \overline{\overline{\nu}}_{s,n,\xi}\ \overline{\overline{\nu}}^{\,\ell-1}_{0,n,\xi},$$

where $\overline{\overline{h}}_{n,\xi}(t) = \int_{-\infty}^{+\infty} e^{itx}\, d\overline{\overline{H}}_{n,\xi}(x)$ for $t\in\mathbb{R}$ with $\overline{\overline{H}}_{n,\xi}$ defined by (93) *and pseudo-moments $\overline{\overline{\nu}}_{s,n,\xi}$ are given immediately after formula* (93).

Proof. Using the properties of Fourier transform we have

$$\overline{\overline{h}}^{\ell}_{n,\xi}(Nt) = \int_{-\infty}^{+\infty} e^{itNx}\, d\overline{\overline{H}}^{\ell*}_{n,\xi}(x).$$

From Lemma 4.3 it follows that $\overline{\overline{h}}^{\ell}_{n,\xi}(Nt)$ is s times differentiable, and for $s\in\mathbb{N}_0$ we obtain using (101):

$$\begin{aligned} \left|\frac{d^s}{dt^s}\overline{\overline{h}}^{\ell}_{n,\xi}(Nt)\right| &= \left|\int_{-\infty}^{+\infty} (ixN)^s e^{itxN}\, d\overline{\overline{H}}^{\ell*}_{n,\xi}(x)\right| \le N^s \int_{-\infty}^{+\infty} |x|^s \left|d\overline{\overline{H}}^{\ell*}_{n,\xi}(x)\right| \\ &= N^s \nu_s\left(\overline{\overline{H}}^{\ell*}_{n,\xi}(x)\right) \le (N\ell)^s\ \overline{\overline{\nu}}_{s,n,\xi}\ \overline{\overline{\nu}}^{\,\ell-1}_{0,n,\xi}. \end{aligned}$$

This completes the proof of the lemma. □

Lemma B.6. *Let $\overline{\overline{h}}_{n,\xi}$ be as in Lemma B.5. For $s\in\mathbb{N}_0$, $\ell\in\mathbb{N}$ and some constant $N\in\mathbb{R}_+$ we have*

$$\left|\frac{d^s}{dt^s}\overline{\overline{h}}^{\ell}_{n,\xi}(Nt)\right| \le \begin{cases} C(s)N^s \displaystyle\sum_{j=1}^{\min\{s,\ell\}} \ell^j\,\left(N\,|t|\,\overline{\overline{\nu}}_{1,n,\xi}\right)^{\ell-j}\,\overline{\overline{\nu}}_{s-j+1,n,\xi}, & \text{if } s\in\mathbb{N},\\ \left(N\,|t|\,\overline{\overline{\nu}}_{1,n,\xi}\right)^{\ell}, & \text{if } s=0. \end{cases} \tag{152}$$

In particular, for $s=0,1,\ldots,R+1$ with R defined by (99) *and $N\,|t|\le 1$ we have*

$$\left|\frac{d^s}{dt^s}\overline{\overline{h}}^{\ell}_{n,\xi}(Nt)\right| \le C\,(N\,\ell)^s\ \left(N\,|t|\,\overline{\overline{\nu}}_{1,n,\xi}\right)^{(\ell-s)+}\left(1+(N\,|t|)^{(\min\{\ell,s\}-1)^+}\ \overline{\overline{\nu}}_{s,n,\xi}\right), \tag{153}$$

where constants C, $C(s)$ do not depend on n,ℓ,ξ or N, and pseudomoments $\overline{\overline{\nu}}_{s,n,\xi}$ are defined immediately after formula (93).

Proof. Consider $s = 0$. Using the fact that $\overline{\overline{\mu}}_{0,n,\xi} = \int_{-\infty}^{+\infty} 1\, d\overline{\overline{H}}_{n,\xi}(x) = 0$ (see Lemma 4.1 (i)) we obtain for $\ell \in \mathbb{N}$:

$$\begin{aligned}\left|\overline{\overline{h}}^{\ell}_{n,\xi}(Nt)\right| &= \left|\int_{-\infty}^{+\infty} e^{itNx}\, d\overline{\overline{H}}_{n,\xi}(x)\right|^{\ell} = \left|\int_{-\infty}^{+\infty} \left(e^{itNx} - 1\right)\, d\overline{\overline{H}}_{n,\xi}(x)\right|^{\ell} \\ &\leq \left(\int_{-\infty}^{+\infty} |itNx| \left|d\overline{\overline{H}}_{n,\xi}(x)\right|\right)^{\ell} = \left(N\,|t|\,\overline{\overline{\nu}}_{1,n,\xi}\right)^{\ell}.\end{aligned} \tag{154}$$

Now let $s \in \mathbb{N}$. Using formula (149) with $z(y) = y^{\ell}$ and $y(t) = \overline{\overline{h}}_{n,\xi}(Nt)$ we obtain

$$\begin{aligned}\frac{d^s}{dt^s}\overline{\overline{h}}^{\ell}_{n,\xi}(Nt) &= s! \sum_{\substack{k_1+2k_2+\cdots+sk_s=s \\ \rho:=k_1+\cdots+k_s}} \frac{d^{\rho}}{dy^{\rho}} y^{\ell}\Big|_{y=\overline{\overline{h}}_{n,\xi}(Nt)} \prod_{m=1}^{s} \frac{1}{k_m!}\left(\frac{1}{m!}\frac{d^m}{dt^m}\overline{\overline{h}}_{n,\xi}(Nt)\right)^{k_m} \\ &= s! \sum_{j=1}^{\min\{s,\ell\}} \sum_{\substack{k_1+2k_2+\cdots+sk_s=s \\ k_1+\cdots+k_s=j}} \frac{\ell!}{(\ell-j)!}\overline{\overline{h}}^{\ell-j}_{n,\xi}(Nt) \prod_{m=1}^{s} \frac{1}{k_m!}\left(\frac{1}{m!}\frac{d^m}{dt^m}\overline{\overline{h}}_{n,\xi}(Nt)\right)^{k_m}.\end{aligned} \tag{155}$$

From Lemma B.5 we obtain

$$\left|\frac{d^m}{dt^m}\overline{\overline{h}}_{n,\xi}(Nt)\right| \leq N^m\, \overline{\overline{\nu}}_{m,n,\xi}, \qquad \text{for} \quad m \in \mathbb{N}.$$

Plugging the last estimation and estimation (154) in (155) we come to the following:

$$\begin{aligned}\left|\frac{d^s}{dt^s}\overline{\overline{h}}^{\ell}_{n,\xi}(Nt)\right| &\leq s! \sum_{j=1}^{\min\{s,\ell\}} \sum_{\substack{k_1+2k_2+\cdots+sk_s=s \\ k_1+\cdots+k_s=j}} \frac{\ell!}{(\ell-j)!}\left(N\,|t|\,\overline{\overline{\nu}}_{1,n,\xi}\right)^{\ell-j} \times \\ &\quad\times \prod_{m=1}^{s}\frac{1}{k_m!}\left(\frac{1}{m!}\right)^{k_m}\left(N^1\overline{\overline{\nu}}_{1,n,\xi}\right)^{k_1}\cdots\left(N^s\overline{\overline{\nu}}_{s,n,\xi}\right)^{k_s} \\ &\leq C(s)N^s \sum_{j=1}^{\min\{s,\ell\}} \ell^j \left(N\,|t|\,\overline{\overline{\nu}}_{1,n,\xi}\right)^{\ell-j} \sum_{\substack{k_1+2k_2+\cdots+sk_s=s \\ k_1+\cdots+k_s=j}} \overline{\overline{\nu}}^{k_1}_{1,n,\xi}\cdots\overline{\overline{\nu}}^{k_s}_{s,n,\xi}.\end{aligned}$$

Note that if $j = 1$, then $k_s = 1$. From $j = 2$ it follows that $k_s = 0$. For each $j \geq 3$ we always have $k_s = k_{s-1} = ... = k_{s-j+2} = 0$. This means that the "largest-order"-pseudomoment that can appear above is $\overline{\overline{\nu}}_{s-j+1,n,\xi}$. Using this fact and Lemma 4.2 we can estimate the sum of pseudomoments in the last expression by some constant times the largest-order pseudomoment $\overline{\overline{\nu}}_{s-j+1,n,\xi}$. Finally, combining the cases $s = 0$ and $s \in \mathbb{N}$ we get (152).

Let now $s = 1, \ldots, R+1$ and $N\,|t| \leq 1$. From Lemma 4.2 (i) and from inequality (152) it follows for $\ell \leq s$ that

$$\begin{aligned}\left|\frac{d^s}{dt^s}\overline{\overline{h}}^{\ell}_{n,\xi}(Nt)\right| &\leq C\,N^s \sum_{j=1}^{\ell} \ell^j \left(N\,|t|\,\overline{\overline{\nu}}_{1,n,\xi}\right)^{\ell-j}\, \overline{\overline{\nu}}_{s-j+1,n,\xi} \\ &\leq C\,(N\ell)^s \left((N\,|t|)^{\ell-1}\,\overline{\overline{\nu}}_{s,n,\xi} + 1\right).\end{aligned}$$

For $\ell > s$ we have

$$\left|\frac{d^s}{dt^s}\overline{\overline{h}}^{\ell}_{n,\xi}(Nt)\right| \le C\,N^s \sum_{j=1}^{s} \ell^j \left(N\,|t|\,\overline{\nu}_{1,n,\xi}\right)^{\ell-j}\,\overline{\nu}_{s-j+1,n,\xi}$$
$$\le C\,(N\,\ell)^s \left(\left(N\,|t|\,\overline{\nu}_{1,n,\xi}\right)^{\ell-1}\,\overline{\nu}_{s,n,\xi} + \sum_{j=2}^{s}\left(N\,|t|\,\overline{\nu}_{1,n,\xi}\right)^{\ell-j}\,\overline{\nu}_{s-j+1,n,\xi}\right)$$
$$\le C\,(N\,\ell)^s \left(N\,|t|\,\overline{\nu}_{1,n,\xi}\right)^{\ell-s}\left(\left(N\,|t|\,\overline{\nu}_{1,n,\xi}\right)^{s-1}\,\overline{\nu}_{s,n,\xi} + \sum_{j=2}^{s}\Big(\underbrace{N\,|t|}_{\le 1}\,\overline{\nu}_{1,n,\xi}\Big)^{s-j}\right)$$
$$\le C\,(N\,\ell)^s \left(N\,|t|\,\overline{\nu}_{1,n,\xi}\right)^{\ell-s}\left(\left(N\,|t|\right)^{s-1}\,\overline{\nu}_{s,n,\xi} + 1\right).$$

Combining the cases $\ell \le s$ and $\ell > s$ we obtain formula (153), which also holds true for $s = 0$. This completes the proof of the lemma. □

Lemma B.7. *Let $g_{\alpha,1}$, $\varphi_{\alpha,1}$ and $\overline{\overline{h}}_{n,\xi}$ be as in Lemmata B.2 and B.5. Further, let $k \in \mathbb{N}_0$, $\ell = 0, 1, ..., n$, $n \in \mathbb{N}$ and $|t| \le \varepsilon\, n^{1/\alpha}$, $t \neq 0$, where $\varepsilon = \min\{1, c_0^{-1}\}$ with $c_0 = (\nu_0^* + 1)\gamma_r^{*\,1/r}$.*

If $q = 0, 1, ..., R$ or if $q = R+1$ and $\ell = 0$, then

$$\left|\frac{d^q}{dt^q}\left(\varphi^k_{\alpha,1}(t)\cdot g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right)\cdot \overline{\overline{h}}^{\ell}_{n,\xi}\left(tn^{-\frac{1}{\alpha}}\right)\right)\right|$$
$$\le C\,e^{-\frac{n-\ell}{2n}|t|^\alpha\cos\left(\frac{\alpha\pi}{2}\right)}\,(\max\{1,k\}\,\max\{1,\ell\})^q\,|t|^{\alpha k - q}\left(c_0\;|t|\;n^{-\frac{1}{\alpha}}\right)^{\ell}.$$

If $q = R+1$ and $\ell \ge 1$, then

$$\left|\frac{d^{R+1}}{dt^{R+1}}\left(\varphi^k_{\alpha,1}(t)\,g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right)\,\overline{\overline{h}}^{\ell}_{n,\xi}\left(tn^{-\frac{1}{\alpha}}\right)\right)\right| \le Ce^{-\frac{n-\ell}{2n}|t|^\alpha\cos\left(\frac{\alpha\pi}{2}\right)}(\ell\max\{1,k\})^{R+1}$$
$$\cdot\,|t|^{\alpha k-(R+1)}\left(\left(c_0\,|t|\,n^{-\frac{1}{\alpha}}\right)^{\ell} + \left(c_0\,|t|\,n^{-\frac{1}{\alpha}}\right)^{R+1+(\ell-(R+1))^+}\left(n^{-\frac{1}{\alpha}}|t|\right)^{\min\{1,\ell-1\}}\overline{\overline{\nu}}_{R+1,n,\xi}\right),$$

where pseudomoments ν_0^*, γ_r^* are given by (80), pseudomoment $\overline{\overline{\nu}}_{R+1,n,\xi}$ is defined immediately after formula (93), r comes from Theorem 3.26, R is defined by (99) and constants C (in both cases) do not depend on n, ℓ, ξ.

Proof. Let $\ell, k \in \mathbb{N}$. For $t \neq 0$ the functions $\varphi^k_{\alpha,1}$, $g_{\alpha,1}$ and $\overline{\overline{h}}^{\ell}_{n,\xi}$ are infinitely differentiable and we can apply formula (151) from Lemma A.10:

$$\frac{d^q}{dt^q}\left(\varphi^k_{\alpha,1}(t)\cdot g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right)\cdot \overline{\overline{h}}^{\ell}_{n,\xi}\left(tn^{-\frac{1}{\alpha}}\right)\right)$$
$$= \sum_{\substack{n_1,n_2,n_3\in\mathbb{N}_0\\ n_1+n_2+n_3=q}} \frac{q!}{n_1!n_2!n_3!}\,\frac{d^{n_1}}{dt^{n_1}}\left(\varphi^k_{\alpha,1}(t)\right)\frac{d^{n_2}}{dt^{n_2}}\left(g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right)\right)\frac{d^{n_3}}{dt^{n_3}}\left(\overline{\overline{h}}^{\ell}_{n,\xi}\left(tn^{-\frac{1}{\alpha}}\right)\right).$$

Applying Lemmata B.1, B.2 and (153) in B.6 we get

$$\left|\frac{d^q}{dt^q}\left(\varphi^k_{\alpha,1}(t)\cdot g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right)\cdot \overline{\overline{h}}^{\ell}_{n,\xi}\left(tn^{-\frac{1}{\alpha}}\right)\right)\right|$$

$$\leq \sum_{n_1+n_2+n_3=q} \frac{q!}{n_1!n_2!n_3!} \left(\prod_{j=0}^{n_1-1} |\alpha k - j| \, |t|^{\alpha k - n_1} \right) \cdot \left(C\, e^{-\frac{n-\ell}{2n}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} \, |t|^{-n_2} \right)$$
$$\cdot \left(C\, (n^{-\frac{1}{\alpha}}\,\ell)^{n_3} \, \left(n^{-\frac{1}{\alpha}}\, |t| \, \overline{\nu}_{1,n,\xi} \right)^{(\ell-n_3)^+} \left(1 + \left(n^{-\frac{1}{\alpha}}\, |t| \right)^{(\min\{\ell,n_3\}-1)^+} \overline{\overline{\nu}}_{n_3,n,\xi} \right) \right)$$
$$\leq C\, e^{-\frac{n-\ell}{2n}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} \, (\ell\, k)^q \sum_{n_1+n_2+n_3=q} |t|^{\alpha k - n_1 - n_2 - n_3}$$
$$\cdot \left(|t| \; n^{-\frac{1}{\alpha}} \right)^{n_3} \left(n^{-\frac{1}{\alpha}} \overline{\overline{\nu}}_{1,n,\xi} \, |t| \right)^{(\ell-n_3)^+} \left(1 + \left(n^{-\frac{1}{\alpha}}\, |t| \right)^{(\min\{\ell,n_3\}-1)^+} \overline{\overline{\nu}}_{n_3,n,\xi} \right). \quad (156)$$

Denote

$$\chi(t) := \left(|t| \; n^{-\frac{1}{\alpha}} \right)^{n_3} \left(n^{-\frac{1}{\alpha}} \overline{\overline{\nu}}_{1,n,\xi} \, |t| \right)^{(\ell-n_3)^+} \left(1 + \left(n^{-\frac{1}{\alpha}}\, |t| \right)^{(\min\{\ell,n_3\}-1)^+} \overline{\overline{\nu}}_{n_3,n,\xi} \right).$$

At this point we have to distinguish two cases: $q = R+1$ and $q = 0, 1, ..., R$.
Case 1: $q = R+1$. If $q = R+1$, then n_3 can take the values $0, 1, ..., R+1$.
If $n_3 = 0, 1, .., R$, then according to Lemma 4.2 pseudomoments $\overline{\overline{\nu}}_{n_3,n,\xi}$ can be estimated by $(\nu_0^*+1)\gamma_r^{*\,n_3/r}$. Denote $c_0 := (\nu_0^*+1)\gamma_r^{*\,1/r}$. Taking into account that $c_0\, |t| n^{-1/\alpha} \leq 1$ and $|t| n^{-1/\alpha} \leq 1$ we get

$$\chi(t) \leq \begin{cases} 2 \left(|t| \; n^{-\frac{1}{\alpha}} \right)^{\ell} (\nu_0^*+1)^{\ell+1-n_3} \, \gamma_r^{*\,\ell/r}, & \text{if} \quad \ell > n_3, \\ 2 \left(|t| \; n^{-\frac{1}{\alpha}} \right)^{n_3} \, (\nu_0^*+1)\, \gamma_r^{*\,n_3/r} \leq 2 \left((\nu_0^*+1)\gamma_r^{*\,1/r}\, |t| \; n^{-\frac{1}{\alpha}} \right)^{n_3}, & \text{if} \quad \ell \leq n_3 \end{cases}$$
$$\leq C \left(c_0 \; |t| \; n^{-\frac{1}{\alpha}} \right)^{\ell} \qquad \text{for all} \quad \ell = 1, ..., n.$$

Consider $n_3 = R+1$. Again, applying Lemma 4.2 for estimation of $\overline{\overline{\nu}}_{1,n,\xi}$ we obtain

$$\chi(t) \leq \begin{cases} C \left((\nu_0^*+1)\, \gamma_r^{*\,1/r} \; |t| \; n^{-\frac{1}{\alpha}} \right)^{\ell} \left(1 + \left(n^{-\frac{1}{\alpha}}\, |t| \right)^{R} \overline{\overline{\nu}}_{R+1,n,\xi} \right), & \text{if} \quad \ell > R+1, \\ \left(|t| \; n^{-\frac{1}{\alpha}} \right)^{R+1} \left(1 + \left(n^{-\frac{1}{\alpha}}\, |t| \right)^{\ell-1} \overline{\overline{\nu}}_{R+1,n,\xi} \right), & \text{if} \quad \ell \leq R+1 \end{cases}$$
$$\leq C \left(c_0 \; |t| \; n^{-\frac{1}{\alpha}} \right)^{R+1+(\ell-(R+1))^+} \left(1 + \left(n^{-\frac{1}{\alpha}}\, |t| \right)^{\min\{1,\ell-1\}} \overline{\overline{\nu}}_{R+1,n,\xi} \right), \quad \ell = 1, ..., n.$$

Combining these two cases we obtain

$$\left| \frac{d^{R+1}}{dt^{R+1}} \left(\varphi_{\alpha,1}^k(t) \cdot g_{\alpha,1} \left(t \left(\tfrac{n-\ell}{n} \right)^{\frac{1}{\alpha}} \right) \cdot \overline{\overline{h}}_{n,\xi}^{\ell} \left(t n^{-\frac{1}{\alpha}} \right) \right) \right| \leq C\, e^{-\frac{n-\ell}{2n}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} \; |t|^{\alpha k - (R+1)}$$
$$\cdot\, (\ell\, k)^{R+1} \left(\left(c_0 \; |t| \; n^{-\frac{1}{\alpha}} \right)^{\ell} + \left(c_0 \; |t| \; n^{-\frac{1}{\alpha}} \right)^{R+1+(\ell-(R+1))^+} \left(n^{-\frac{1}{\alpha}}\, |t| \right)^{\min\{1,\ell-1\}} \overline{\overline{\nu}}_{R+1,n,\xi} \right).$$

Now we consider the second case.
Case 2: $q = 0, 1, ..., R$. If $q = 0, 1, ..., R$, then $n_3 \leq R$ in (156) and we can use the estimate $\chi(t) \leq C \left(c_0 \; |t| \; n^{-\frac{1}{\alpha}} \right)^{\ell}$ as above. From (156) we obtain

$$\left| \frac{d^{q}}{dt^{q}} \left(\varphi_{\alpha,1}^k(t) \cdot g_{\alpha,1} \left(t \left(\tfrac{n-\ell}{n} \right)^{\frac{1}{\alpha}} \right) \cdot \overline{\overline{h}}_{n,\xi}^{\ell} \left(t n^{-\frac{1}{\alpha}} \right) \right) \right|$$
$$\leq C\, e^{-\frac{n-\ell}{2n}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} \, (k\, \ell)^q \, |t|^{\alpha k - q} \left(c_0 \; |t| \; n^{-\frac{1}{\alpha}} \right)^{\ell},$$

which proves the statement of the lemma for this case.

Using the same method as above it is easy to see that the statement of the lemma holds true if $k = 0$ and/or $\ell = 0$. □

Lemma B.8. *Let $\ell = 1, 2, ..., n$, $n \in \mathbb{N}$. For $s = 0, 1, ..., R+1$ and $|t| > \varepsilon_n$ we have*

$$\left| \frac{d^s}{dt^s} \overline{\overline{f}}^{\ell}_{n,\xi}\left(tn^{-1/\alpha}\right) \right| \le C\, \ell^s\, n^{-s/\alpha} \max\{1, \overline{\overline{\nu}}_{s,n,\xi}\}\, \overline{\overline{Q}}_{n,\xi}^{(\ell-s)^+}$$

where $\overline{\overline{f}}_{n,\xi}(t) = \int_{-\infty}^{+\infty} e^{itx} d\overline{\overline{F}}_{n,\xi}(x)$ with $\overline{\overline{F}}_{n,\xi}$ defined in (92), $\varepsilon_n = \widetilde{\varepsilon}\, n^{1/\alpha}$ with $\widetilde{\varepsilon}$ given by (123), $\overline{\overline{Q}}_{n,\xi} = \sup_{|t| > \widetilde{\varepsilon}} \left| \overline{\overline{f}}_{n,\xi}(t) \right|$, pseudomoments $\overline{\overline{\nu}}_{s,n,\xi}$ are defined immediately after formula (93), and constant C does not depend on n, ℓ, ξ.

Remark B.1. Note that $\overline{\overline{Q}}_{n,\xi}$ can be estimated by a constant not depending on n and ξ. It follows from the definition of $\overline{\overline{f}}_{n,\xi}$, f and estimate (108) with $N = n^{1/\alpha}(1+\xi)$ that

$$\begin{aligned} \left|\overline{\overline{f}}_{n,\xi}(t)\right| &= \left| \int_{-\infty}^{+\infty} e^{itx} d\overline{\overline{F}}_{n,\xi}(x) \right| \le \left| \int_{-\infty}^{+\infty} e^{itx} dF(x) \right| + \left| \int_{-\infty}^{+\infty} e^{itx} d\left(\overline{\overline{F}}_{n,\xi} - F\right)(x) \right| \\ &\le |f(t)| + \int_0^{+\infty} \left| d\left(\overline{\overline{F}}_{n,\xi} - F\right)(x) \right| \le |f(t)| + 2\,\gamma_r^* n^{-r/\alpha} \le C, \quad t \in \mathbb{R}. \end{aligned}$$

Hence,

$$\overline{\overline{Q}}_{n,\xi} \le \sup_{|t| > \widetilde{\varepsilon}} |f(t)| + 2\,\gamma_r^* n^{-r/\alpha} \le C. \tag{157}$$

Proof. Consider $s = 0$. Using the definition of $\overline{\overline{Q}}_{n,\xi}$ and the fact that $\overline{\overline{\nu}}_{0,n,\xi}$ can be estimated by a constant (see (94) with $r = 0$) we obtain the statement of the lemma.

Now let $s \ge 1$. According to the definition we have $\overline{\overline{f}}_{n,\xi}(t) = \widetilde{g}_\alpha(t) + \overline{\overline{h}}_{n,\xi}(t)$. From Lemma B.3 and Lemma B.5 it follows that for $m = 1, ..., R+1$ and $|t| > \varepsilon_n$ we have

$$\begin{aligned} \left| \frac{d^m}{dt^m} \overline{\overline{f}}_{n,\xi}\left(tn^{-\frac{1}{\alpha}}\right) \right| &\le \left| \frac{d^m}{dt^m} \widetilde{g}_\alpha\left(tn^{-\frac{1}{\alpha}}\right) \right| + \left| \frac{d^m}{dt^m} \overline{\overline{h}}_{n,\xi}\left(tn^{-\frac{1}{\alpha}}\right) \right| \le C\, e^{-\frac{|t|^\alpha}{4n}\cos\left(\alpha\frac{\pi}{2}\right)}\, |t|^{-m} \\ &+ n^{-m/\alpha}\, \overline{\overline{\nu}}_{m,n,\xi} \le C\, n^{-m/\alpha} + n^{-m/\alpha}\, \overline{\overline{\nu}}_{m,n,\xi} \le C\, n^{-m/\alpha} \max\{1, \overline{\overline{\nu}}_{m,n,\xi}\}. \end{aligned} \tag{158}$$

Using formula (149) from Lemma A.9 with $z(y) = y^\ell$ and $y(t) = \overline{\overline{f}}_{n,\xi}(Nt)$ we obtain

$$\begin{aligned} \frac{d^s}{dt^s} \overline{\overline{f}}^{\ell}_{n,\xi}(Nt) &= s! \sum_{\substack{k_1+2k_2+\dots+sk_s=s \\ \rho := k_1+\dots+k_s}} \frac{d^\rho}{dy^\rho} y^\ell \Big|_{y=\overline{\overline{f}}_{n,\xi}(Nt)} \prod_{m=1}^{s} \frac{1}{k_m!} \left(\frac{1}{m!} \frac{d^m}{dt^m} \overline{\overline{f}}_{n,\xi}(Nt) \right)^{k_m} \\ &= s! \sum_{j=1}^{\min\{s,\ell\}} \sum_{\substack{k_1+2k_2+\dots+sk_s=s \\ k_1+\dots+k_s=j}} \frac{\ell!}{(\ell-j)!} \overline{\overline{f}}^{\ell-j}_{n,\xi}(Nt) \prod_{m=1}^{s} \frac{1}{k_m!} \left(\frac{1}{m!} \frac{d^m}{dt^m} \overline{\overline{f}}_{n,\xi}(Nt) \right)^{k_m}. \end{aligned} \tag{159}$$

Plugging estimation (158) in (159) with $N = n^{-1/\alpha}$ we come to the following:

$$\left|\frac{d^s}{dt^s}\overline{\overline{f}}^{\,\ell}_{n,\xi}\left(tn^{-\frac{1}{\alpha}}\right)\right| \le s! \sum_{j=1}^{\min\{s,\ell\}} \sum_{\substack{k_1+2k_2+\cdots+sk_s=s\\ k_1+\cdots+k_s=j}} \frac{\ell!}{(\ell-j)!}\,\overline{\overline{Q}}^{\,\ell-j}_{n,\xi} \prod_{m=1}^{s}\frac{1}{k_m!}\left(\frac{1}{m!}\right)^{k_m} \times$$

$$\times\left(C\,n^{-1/\alpha}\max\{1,\overline{\overline{\nu}}_{1,n,\xi}\}\right)^{k_1}\cdots\left(C\,n^{-s/\alpha}\max\{1,\overline{\overline{\nu}}_{s,n,\xi}\}\right)^{k_s} = \text{Ⓢ}.$$

Note that $k_s \le 1$. Then taking into account Remark B.1, the fact that $s = 1, ..., R+1$ and that all pseudomoments $\overline{\overline{\nu}}_{s,n,\xi}$ can be estimated by some constant for $s = 1, ..., R$ (see Lemma 4.2 (i)), we continue

$$\text{Ⓢ} \le C \sum_{j=1}^{\min\{s,\ell\}} \sum_{\substack{k_1+2k_2+\cdots+sk_s=s\\ k_1+\cdots+k_s=j}} \frac{\ell!}{(\ell-j)!}\,\overline{\overline{Q}}^{\,\ell-j}_{n,\xi}\, n^{-s/\alpha}\max\{1,\overline{\overline{\nu}}_{s,n,\xi}\}$$

$$\le C\,\ell^s\, n^{-s/\alpha}\max\{1,\overline{\overline{\nu}}_{s,n,\xi}\}\,\overline{\overline{Q}}^{\,(\ell-s)^+}_{n,\xi}.$$

This completes the proof of the lemma. □

Lemma B.9. *Let $r > 1$ such that $0 < \gamma_r^* < \infty$. Define R as in* (99) *and, further, for $n \in \mathbb{N}$, $\xi \in [0,\infty)$ and $t \in \mathbb{R}$ define*

$$\chi_{R,n,\xi}(t) = \frac{\overline{\overline{\mu}}_{1,n,\xi}(itn^{-1/\alpha})}{1!} + \frac{\overline{\overline{\mu}}_{2,n,\xi}(itn^{-1/\alpha})^2}{2!} + \cdots + \frac{\overline{\overline{\mu}}_{R,n,\xi}(itn^{-1/\alpha})^R}{R!}$$

where $\overline{\overline{\mu}}_{k,n,\xi}$, $k = 1, ..., R$, are given immediately after formula (93).
Then for $s = 0, 1, ..., R+1$ and $|t| \le n^{1/\alpha}$ we have

$$\left|\frac{d^s}{dt^s}\,\chi^{\ell}_{R,n,\xi}(t)\right| \le C(\ell)\,n^{-s/\alpha}\left(|t|\,n^{-1/\alpha}\right)^{\ell-s},$$

where $C(\ell)$ is some constant not depending on n and ξ.

Proof. Consider $s = 0$. Taking into account that $\left|\overline{\overline{\mu}}_{k,n,\xi}\right| \le \overline{\overline{\nu}}_{k,n,\xi} \le (\nu_0^* + 1){\gamma_r^*}^{k/r}$ for $k = 1, ..., R$ (see Lemma 4.2) and that $|t|\,n^{-1/\alpha} \le 1$ we obtain

$$\left|\chi^{\ell}_{R,n,\xi}(t)\right| \le \left(|t|\,n^{-1/\alpha}\right)^{\ell}\left(\frac{\overline{\overline{\nu}}_{1,n,\xi}}{1!} + \cdots + \frac{\overline{\overline{\nu}}_{R,n,\xi}(|t|\,n^{-1/\alpha})^{R-1}}{R!}\right)^{\ell}$$

$$\le (\nu_0^*+1)^{\ell}\left(\frac{{\gamma_r^*}^{1/r}}{1!} + \cdots + \frac{{\gamma_r^*}^{R/r}}{R!}\right)^{\ell}\left(|t|\,n^{-1/\alpha}\right)^{\ell} \le C(\ell)\,\left(|t|\,n^{-1/\alpha}\right)^{\ell},$$

which means that for $s = 0$ the assertion of the lemma is true. We consider the case $s = 1, 2, ..., R+1$ in what follows. Using formula (149), the above estimate of $\chi^{\ell}_{R,n,\xi}(t)$ and keeping in mind that $|t|\,n^{-1/\alpha} \le 1$ we obtain

$$\left|\frac{d^s}{dt^s}\,\chi^{\ell}_{R,n,\xi}(t)\right| = \left|s!\sum_{j=1}^{\min\{s,\ell\}} \sum_{\substack{k_1+2k_2+\cdots+sk_s=s\\ k_1+\cdots+k_s=j}} \frac{\ell!\,\chi^{\ell-j}_{R,n,\xi}(t)}{(\ell-j)!}\prod_{m=1}^{s}\frac{1}{k_m!}\left(\frac{1}{m!}\frac{d^m}{dt^m}\chi_{R,n,\xi}(t)\right)^{k_m}\right|$$

$$\le C \sum_{j=1}^{\min\{s,\ell\}} \sum_{\substack{k_1+2k_2+\cdots+sk_s=s \\ k_1+\cdots+k_s=j}} \ell^j\, C(\ell,j)\, \left(|t|\, n^{-1/\alpha}\right)^{\ell-j} \frac{1}{k_1!...k_s!} \left(\frac{1}{1!}\right)^{k_1} \cdot ... \cdot \left(\frac{1}{s!}\right)^{k_s}$$

$$\cdot \left|(in^{-1/\alpha})^1 \sum_{u=1}^{R} \frac{\overline{\overline{\mu}}_{u,n,\xi}(itn^{-1/\alpha})^{u-1}}{(u-1)!}\right|^{k_1} \cdot ... \cdot \left|(in^{-1/\alpha})^s \sum_{u=s}^{R} \frac{\overline{\overline{\mu}}_{u,n,\xi}(itn^{-1/\alpha})^{u-s}}{(u-s)!}\right|^{k_s}$$

$$\le C(\ell) \sum_{j=1}^{\min\{s,\ell\}} \left(|t|\, n^{-1/\alpha}\right)^{\ell-j} \sum_{\substack{k_1+2k_2+\cdots+sk_s=s \\ k_1+\cdots+k_s=j}} \left[\sum_{u=1}^{R} \frac{\gamma_r^{*u/r}}{(u-1)!}\right]^{k_1} \cdots \left[\sum_{u=s}^{R} \frac{\gamma_r^{*u/r}}{(u-s)!}\right]^{k_s}$$

$$\cdot\, n^{-s/\alpha}\,(\nu_0^*+1)^j \le C(\ell)\, n^{-s/\alpha} \sum_{j=1}^{\min\{s,\ell\}} \left(|t|\, n^{-1/\alpha}\right)^{\ell-j} \le C(\ell)\, n^{-s/\alpha} \left(|t|\, n^{-1/\alpha}\right)^{\ell-\min\{s,\ell\}}.$$

Since $|t|\, n^{-1/\alpha} \le 1$ we have $\left(|t|\, n^{-1/\alpha}\right)^{\ell-\min\{s,\ell\}} \le \left(|t|\, n^{-1/\alpha}\right)^{\ell-s}$. This completes the proof of the lemma. □

Below we give the technical proofs of Lemmata 4.13 – 4.18. For reader's convenience we also repeat the statements of the lemmata.

B.2 Proof of Lemma 4.13

Lemma 4.13. *Define* $d_{1n}(t) = \sum_{k=\rho+1}^{sn} \frac{c_{k,n}}{n^k} \varphi_{\alpha,1}^k(t)\, g_{\alpha,1}(t)$ *with* $\rho = [2(R+1)/\alpha]$. *Then for* $|t| \le \varepsilon n^{1/\alpha}$ *and* $q = 0, 1, ..., R+1$ *we have*

$$\left|d_{1n}^{(q)}(t)\right| \le C\, e^{-\frac{1}{4}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)}\, n^{-\frac{R+1}{\alpha}}\, |t|^{-q} \left(|t|^{\alpha\left(\frac{\rho}{2}+\frac{1}{2}\right)} + |t|^{\alpha(\rho+2)}\right)$$

with constant C *not depending on* n, *and*

$$\varepsilon = \min\left\{1, \frac{1}{(2D)^{1/\alpha}}, \frac{1}{D^{1/\alpha}} \left(\frac{\cos\left(\frac{\alpha\pi}{2}\right)}{8D}\right)^{\frac{2+\rho/2}{\alpha}}\right\}, \qquad D = \max_{2\le j\le s}\{2\,|A_j|^{1/j}\},$$

where s *and* A_j, $j = 2, ..., s$, *are defined in* (85) *and* (70).

Proof. For $t \ne 0$ the functions $\varphi_{\alpha,1}^k(t)$ and $g_{\alpha,1}(t)$ are infinitely differentiable. Applying formula (151) from Lemma A.10 we obtain for $t \ne 0$:

$$\left(\varphi_{\alpha,1}^k \cdot g_{\alpha,1}\right)^{(q)}(t) = \sum_{\substack{n_1+n_2=q \\ n_1,n_2\in\mathbb{N}_0}} \frac{q!}{n_1!n_2!} \left(\varphi_{\alpha,1}^k\right)^{(n_1)}(t) \cdot g_{\alpha,1}^{(n_2)}(t).$$

Taking into account that $k \ge \rho+1 > \frac{2(R+1)}{\alpha} > \frac{q}{\alpha}$ and using Lemmata B.1 and B.2 we can estimate

$$\left|\left(\varphi_{\alpha,1}^k \cdot g_{\alpha,1}\right)^{(q)}(t)\right| \le \sum_{n_1+n_2=q} \frac{q!}{n_1!n_2!} \left|\left(\varphi_{\alpha,1}^k\right)^{(n_1)}(t)\right| \cdot \left|g_{\alpha,1}^{(n_2)}(t)\right|$$

$$\le \sum_{n_1+n_2=q} \frac{q!}{n_1!n_2!} \prod_{j=0}^{n_1-1} (\alpha k - j)\, |t|^{\alpha k - n_1} \cdot C(n_2) e^{-\frac{1}{2}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)}\, |t|^{-n_2}$$

$$\leq e^{-\frac{1}{2}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} |t|^{\alpha k - q} \sum_{n_1+n_2=q} C(n_2) \frac{q!}{n_1! n_2!} \prod_{j=0}^{n_1-1} (\alpha k - j)$$
$$\leq C\, k^q\, e^{-\frac{1}{2}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} |t|^{\alpha k - q}. \tag{160}$$

Let us consider the case $t = 0$. We start with the first derivative of $\varphi_{\alpha,1}^k\, g_{\alpha,1}$. Taking into account that $\alpha k > R + 1$ we get

$$\left(\varphi_{\alpha,1}^k\, g_{\alpha,1}\right)'_+ (0) = \lim_{h\to 0+} \frac{\left(\varphi_{\alpha,1}^k\, g_{\alpha,1}\right)(h) - \left(\varphi_{\alpha,1}^k\, g_{\alpha,1}\right)(0)}{h}$$
$$= \lim_{h\to 0+} \left(e^{-h^\alpha e^{-i\alpha\frac{\pi}{2}}} (-1)^k h^{\alpha k - 1} e^{-i\alpha\frac{\pi}{2}k} \right) = 0$$

as well as

$$\left(\varphi_{\alpha,1}^k\, g_{\alpha,1}\right)'_- (0) = \lim_{h\to 0-} \frac{\left(\varphi_{\alpha,1}^k\, g_{\alpha,1}\right)(h) - \left(\varphi_{\alpha,1}^k\, g_{\alpha,1}\right)(0)}{h}$$
$$= \lim_{h\to 0-} \left(e^{-|h|^\alpha e^{i\alpha\frac{\pi}{2}}} (-1)^{k-1} |h|^{\alpha k - 1} e^{i\alpha\frac{\pi}{2}k} \right) = 0.$$

Thus, the first derivative of $\varphi_{\alpha,1}^k\, g_{\alpha,1}$ exists at point 0 and $\left(\varphi_{\alpha,1}^k\, g_{\alpha,1}\right)'(0) = 0$. Similarly, it can be shown that $\left(\varphi_{\alpha,1}^k\, g_{\alpha,1}\right)^{(i)}(0) = 0$ for $i = 1, \ldots, R+1$.

Thus, inequality (160) holds for all $t \in \mathbb{R}$. Using it and the fact that there always exists a constant $C > 0$ such that $k^q \leq k^{R+1} \leq C\, 2^k$ for all $k \in \mathbb{N}$, we estimate

$$\left| d_{1n}^{(q)}(t) \right| \leq \sum_{k=\rho+1}^{sn} \frac{|c_{k,n}|}{n^k} \left| \left(\varphi_{\alpha,1}^k \cdot g_{\alpha,1}\right)^{(q)}(t) \right|$$
$$\leq C \sum_{k=\rho+1}^{sn} \frac{|c_{k,n}|}{n^k} k^q\, |t|^{\alpha k - q}\, e^{-\frac{1}{2}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)}$$
$$\leq C\, |t|^{-q}\, e^{-\frac{1}{2}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} \sum_{k=\rho+1}^{sn} \frac{c_{k,n}^*}{\left(\frac{n}{2}\right)^k} |t|^{\alpha k},$$

where the notation $c_{k,n}^*$ comes from the following:

$$|c_{k,n}| \leq c_{k,n}^* := \sum_{\substack{k_0+k_2+\cdots+k_s=n \\ 2k_2+\cdots+sk_s=k}} \frac{n!}{k_0! k_2! \cdots k_s!} |A_2|^{k_2} \cdots |A_s|^{k_s}.$$

Note that $c_{0,n} = c_{0,n}^* = 1$ and $c_{1,n} = c_{1,n}^* = 0$ for all $n \in \mathbb{N}$. We have to find an upper bound for the sum $\sum_{k=\rho+1}^{sn} \frac{c_{k,n}^*}{\left(n/2\right)^k} |t|^{\alpha k}$. Using formula (147) from Lemma A.5 we obtain

$$\left(1 + \frac{|A_2|\,|t|^{2\alpha}}{\left(\frac{n}{2}\right)^2} + \cdots + \frac{|A_s|\,|t|^{s\alpha}}{\left(\frac{n}{2}\right)^s}\right)^n$$
$$= \sum_{k_0+k_2+\cdots+k_s=n} \frac{n!}{k_0! k_2! \cdots k_s!} \left(\frac{|A_2|\,|t|^{2\alpha}}{\left(\frac{n}{2}\right)^2}\right)^{k_2} \cdots \left(\frac{|A_s|\,|t|^{s\alpha}}{\left(\frac{n}{2}\right)^s}\right)^{k_s}$$

$$= \sum_{k_0+k_2+\cdots+k_s=n} \frac{n!}{k_0!k_2!\cdots k_s!} |A_2|^{k_2} \cdots |A_s|^{k_s} \frac{|t|^{\left(2k_2+\ldots+sk_s\right)\alpha}}{\left(\frac{n}{2}\right)^{2k_2+\ldots+sk_s}}$$

$$= \sum_{k=0}^{sn} \frac{|t|^{\alpha k}}{\left(\frac{n}{2}\right)^k} \underbrace{\sum_{\substack{k_0+k_2+\cdots+k_s=n \\ 2k_2+\cdots+sk_s=k}} \frac{n!}{k_0!k_2!\cdots k_s!} |A_2|^{k_2} \cdots |A_s|^{k_s}}_{c^*_{k,n}} = \sum_{k=0}^{sn} \frac{c^*_{k,n}}{\left(\frac{n}{2}\right)^k} |t|^{\alpha k}.$$

Thus, taking into account that $c^*_{0,n} = 1$ and $c^*_{1,n} = 0$ we get

$$\sum_{k=\rho+1}^{sn} \frac{c^*_{k,n}}{\left(\frac{n}{2}\right)^k} |t|^{\alpha k} = \left(1 + \frac{|A_2|\,|t|^{2\alpha}}{\left(\frac{n}{2}\right)^2} + \cdots + \frac{|A_s|\,|t|^{s\alpha}}{\left(\frac{n}{2}\right)^s}\right)^n - 1 - \sum_{k=2}^{\rho} \frac{c^*_{k,n}}{\left(\frac{n}{2}\right)^k} |t|^{\alpha k}.$$

In other words, from $\left(1 + \frac{|A_2||t|^{2\alpha}}{(\frac{n}{2})^2} + \cdots + \frac{|A_s||t|^{s\alpha}}{(\frac{n}{2})^s}\right)^n$ we subtract 1 and all summands containing $|t|^{2\alpha}$, $|t|^{3\alpha}, \ldots, |t|^{\rho\alpha}$. In order to see the structure of the remainder we do the following. Let $\widetilde{\rho} = \begin{cases} \rho/2, & \text{if } \rho \text{ even} \\ (\rho-1)/2, & \text{if } \rho \text{ odd} \end{cases}$. Note that $\widetilde{\rho} \in \mathbb{N}$ is the smallest number such that $2\alpha(\widetilde{\rho}+1) > \alpha\rho$. Then we have

$$\left(1 + \frac{|A_2|\,|t|^{2\alpha}}{\left(\frac{n}{2}\right)^2} + \cdots + \frac{|A_s|\,|t|^{s\alpha}}{\left(\frac{n}{2}\right)^s}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{|A_2|\,|t|^{2\alpha}}{\left(\frac{n}{2}\right)^2} + \cdots + \frac{|A_s|\,|t|^{s\alpha}}{\left(\frac{n}{2}\right)^s}\right)^k$$

$$= 1 + n\left(\frac{|A_2|\,|t|^{2\alpha}}{\left(\frac{n}{2}\right)^2} + \cdots + \frac{|A_s|\,|t|^{s\alpha}}{\left(\frac{n}{2}\right)^s}\right)$$

$$+ \frac{n(n-1)}{2!}\left(\frac{|A_2|\,|t|^{2\alpha}}{\left(\frac{n}{2}\right)^2} + \cdots + \frac{|A_s|\,|t|^{s\alpha}}{\left(\frac{n}{2}\right)^s}\right)^2 + \cdots$$

$$+ \frac{n(n-1)\cdots\left(n-(\widetilde{\rho}-1)\right)}{\widetilde{\rho}!}\left(\frac{|A_2|\,|t|^{2\alpha}}{\left(\frac{n}{2}\right)^2} + \cdots + \frac{|A_s|\,|t|^{s\alpha}}{\left(\frac{n}{2}\right)^s}\right)^{\widetilde{\rho}}$$

$$+ \sum_{k=\widetilde{\rho}+1}^{n} \binom{n}{k}\left(\frac{|A_2|\,|t|^{2\alpha}}{\left(\frac{n}{2}\right)^2} + \cdots + \frac{|A_s|\,|t|^{s\alpha}}{\left(\frac{n}{2}\right)^s}\right)^k.$$

With some constants $c_1, \ldots, c_{\widetilde{\rho}}$ not depending on n we obtain

$$\left(1 + \frac{|A_2|\,|t|^{2\alpha}}{\left(\frac{n}{2}\right)^2} + \cdots + \frac{|A_s|\,|t|^{s\alpha}}{\left(\frac{n}{2}\right)^s}\right)^n - \sum_{k=0}^{\rho} \frac{c^*_{k,n}}{\left(\frac{n}{2}\right)^k} |t|^{\alpha k}$$

$$\leq \sum_{k=\rho+1}^{s\widetilde{\rho}} \left(\frac{c_1}{n^{k-1}} + \cdots + \frac{c_{\widetilde{\rho}}}{n^{k-\widetilde{\rho}}}\right) |t|^{\alpha k} + \sum_{k=\widetilde{\rho}+1}^{n} \binom{n}{k}\left(\frac{|A_2|\,|t|^{2\alpha}}{\left(\frac{n}{2}\right)^2} + \cdots + \frac{|A_s|\,|t|^{s\alpha}}{\left(\frac{n}{2}\right)^s}\right)^k.$$

Let us try to estimate the second sum from the above expression. Consider $k = \widetilde{\rho}+1$. Taking into account that $D\,|t|^\alpha/n \le 1/2$ with $D = \max\limits_{2\le j\le s}\{2\,|A_j|^{1/j}\}$ and using the definition of ρ and $\widetilde{\rho}$ we obtain

$$\binom{n}{\widetilde{\rho}+1}\left(\frac{|A_2|\,|t|^{2\alpha}}{\left(\frac{n}{2}\right)^2}+\ldots+\frac{|A_s|\,|t|^{s\alpha}}{\left(\frac{n}{2}\right)^s}\right)^{\widetilde{\rho}+1} \le n^{\widetilde{\rho}+1}\left(\left[\frac{D\,|t|^\alpha}{n}\right]^2+\ldots+\left[\frac{D\,|t|^\alpha}{n}\right]^s\right)^{\widetilde{\rho}+1}$$
$$\le n^{\widetilde{\rho}+1}\left((s-1)\left(\frac{D\,|t|^\alpha}{n}\right)^2\right)^{\widetilde{\rho}+1} \le C\,\frac{|t|^{2\alpha(\widetilde{\rho}+1)}}{n^{\widetilde{\rho}+1}} \le C\,n^{-\frac{R+1}{\alpha}}\left(|t|^{\alpha(\rho+1)}+|t|^{\alpha(\rho+2)}\right).$$

Now let $k \ge \widetilde{\rho}+2$. Denote $U := U_n(t) := D\,|t|^\alpha/n$ with D defined above. Using the formula for the infinite geometric series together with the fact that $U \le 1/2$ we have

$$\sum_{k=\widetilde{\rho}+2}^{n}\binom{n}{k}\left(\frac{|A_2|\,|t|^{2\alpha}}{\left(\frac{n}{2}\right)^2}+\cdots+\frac{|A_s|\,|t|^{s\alpha}}{\left(\frac{n}{2}\right)^s}\right)^k \le \sum_{k=\widetilde{\rho}+2}^{n}\binom{n}{k}\left(U^2+\cdots+U^s\right)^k$$
$$\le \sum_{k=\widetilde{\rho}+2}^{n}\binom{n}{k}\left(U^{\frac{\widetilde{\rho}+1}{\widetilde{\rho}+2}}\,U^{\frac{1}{\widetilde{\rho}+2}}\right)^k\left(U^1+\cdots+U^{s-1}\right)^k$$
$$\le U^{\widetilde{\rho}+1}\sum_{k=\widetilde{p}+2}^{n}\binom{n}{k}U^{\frac{k}{\widetilde{\rho}+2}}\left(\frac{U}{1-U}\right)^k \le U^{\widetilde{\rho}+1}\left(1+\frac{U^{\frac{1}{\widetilde{\rho}+2}}\,U}{1-U}\right)^n = \text{Ⓢ}.$$

Now using fact that $U \le \min\{1/2,\varepsilon_0\}$ with $\varepsilon_0 = \left(\frac{1}{8D}\cos\left(\frac{\alpha\pi}{2}\right)\right)^{\widetilde{\rho}+2}$ and plugging the formula for U we continue

$$\text{Ⓢ} \le U^{\widetilde{\rho}+1}\left(1+\frac{\varepsilon_0^{\frac{1}{\widetilde{\rho}+2}}\,\frac{D|t|^\alpha}{n}}{1/2}\right)^n \le \left(\frac{D|t|^\alpha}{n}\right)^{\widetilde{\rho}+1}\left(1+\frac{|t|^\alpha\cos\left(\frac{\alpha\pi}{2}\right)}{4n}\right)^n$$
$$\le C\,\frac{|t|^{\alpha(\widetilde{\rho}+1)}}{n^{\widetilde{\rho}+1}}e^{\frac{1}{4}\,|t|^\alpha\cos\left(\frac{\alpha\pi}{2}\right)} \le C\,e^{\frac{1}{4}\,|t|^\alpha\cos\left(\frac{\alpha\pi}{2}\right)}n^{-\frac{R+1}{\alpha}}\left(|t|^{\alpha\left(\frac{\varrho}{2}+1\right)}+|t|^{\alpha\left(\frac{\varrho}{2}+\frac{1}{2}\right)}\right).$$

Now we have all we need for the estimation of the sum $\sum_{k=\rho+1}^{sn}\frac{c^*_{k,n}}{\left(n/2\right)^k}|t|^{\alpha k}$. Thus, taking into account the definition of ρ, $\widetilde{\rho}$ and the fact that $|t| \le \varepsilon\,n^{\frac{1}{\alpha}}$ we have

$$\sum_{k=\rho+1}^{sn}\frac{c^*_{k,n}}{\left(\frac{n}{2}\right)^k}|t|^{\alpha k} \le \sum_{k=\rho+1}^{s\widetilde{\rho}}\left(\frac{c_1}{n^{k-1}}+\ldots+\frac{c_{\widetilde{\rho}}}{n^{k-\widetilde{\rho}}}\right)|t|^{\alpha k}$$
$$+\sum_{k=\widetilde{\rho}+1}^{n}\binom{n}{k}\left(\frac{|A_2|\,|t|^{2\alpha}}{\left(\frac{n}{2}\right)^2}+\ldots+\frac{|A_s|\,|t|^{s\alpha}}{\left(\frac{n}{2}\right)^s}\right)^k \le \frac{C}{n^{-\widetilde{\rho}}}\sum_{k=\rho+1}^{s\widetilde{\rho}}\left(\frac{|t|^\alpha}{n}\right)^k$$
$$+C\,n^{-\frac{R+1}{\alpha}}\left(|t|^{\alpha(\rho+1)}+|t|^{\alpha(\rho+2)}\right)+C\,e^{\frac{1}{4}\,|t|^\alpha\cos\left(\frac{\alpha\pi}{2}\right)}n^{-\frac{R+1}{\alpha}}\left(|t|^{\alpha\left(\frac{\varrho}{2}+1\right)}+|t|^{\alpha\left(\frac{\varrho}{2}+\frac{1}{2}\right)}\right)$$

$$\le C\, n^{\widetilde{\rho}-\rho-1}\, |t|^{\alpha(\rho+1)} + C\, e^{\frac{1}{4}\,|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} n^{-\frac{R+1}{\alpha}} \left(|t|^{\alpha\left(\frac{\rho}{2}+\frac{1}{2}\right)} + |t|^{\alpha(\rho+2)}\right)$$

$$\le C\, e^{\frac{1}{4}\,|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} n^{-\frac{R+1}{\alpha}} \left(|t|^{\alpha\left(\frac{\rho}{2}+\frac{1}{2}\right)} + |t|^{\alpha(\rho+2)}\right). \tag{161}$$

Finally we come back to the estimation of $d_{1n}^{(q)}(t)$. Combining everything we obtain

$$\begin{aligned}\left|d_{1n}^{(q)}(t)\right| &\le C\, |t|^{-q}\, e^{-\frac{1}{2}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} \sum_{k=\rho+1}^{sn} \frac{c_{k,n}^*}{\left(\frac{n}{2}\right)^k} |t|^{\alpha k} \\ &\le C\, e^{-\frac{1}{4}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)}\, n^{-\frac{R+1}{\alpha}}\, |t|^{-q} \left(|t|^{\alpha\left(\frac{\rho}{2}+\frac{1}{2}\right)} + |t|^{\alpha(\rho+2)}\right),\end{aligned}$$

where $\alpha(\rho+1)/2 - q \ge \alpha(\rho+1)/2 - (R+1) > 0$ according to the definition of ρ. This completes the proof of the lemma. □

B.3 Proof of Lemma 4.14

Lemma 4.14. *Define*

$$d_{2n}(t) = \sum_{k=0}^{p} \sum_{\ell=m_k+1}^{n-1} \binom{n}{\ell} \frac{c_{k,n-\ell}}{n^k} \varphi_{\alpha,1}^k(t)\, g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right) \overline{\overline{h}}_{n,\xi}^{\ell}\left(tn^{-\frac{1}{\alpha}}\right),$$

where $p = [2R/\alpha]$ *and* $m_k = 1 + \left[\left(R - \frac{\alpha k}{2}\right)/(1-\alpha)\right]$. *Then for* $|t| \le \varepsilon n^{1/\alpha}$ *we have*

$$\begin{aligned}\left|d_{2n}^{(q)}(t)\right| &\le C\ e^{-\frac{|t|^\alpha}{4}\cos\left(\frac{\alpha\pi}{2}\right)}\, n^{-\frac{R+1-\alpha}{\alpha}}\, |t|^{R+1-q} \left(|t|^{\theta_1} + |t|^{\theta_2}\right), \qquad q = 0, 1, ..., R, \\ \left|d_{2n}^{(R+1)}(t)\right| &\le C\ e^{-\frac{|t|^\alpha}{4}\cos\left(\frac{\alpha\pi}{2}\right)}\, n^{-\frac{r-\alpha}{\alpha}} \left(|t|^{\theta_1} + |t|^{\theta_2} + |t|^{\alpha p+1}\right) (1+\xi)^{R+1-r},\end{aligned}$$

where $\theta_{1(2)} = \min_k(\max_k)\{m_k + \alpha k - R\} > 0$, *r comes from Theorem 3.26,* $\varepsilon = \min\left\{1, \frac{1}{c_0}, \left(\frac{\cos\left(\frac{\alpha\pi}{2}\right)}{8\,e\,c_0}\right)^{1/(1-\alpha)}\right\}$ *with* $c_0 = (\nu_0^* + 1)\gamma_r^{*\,1/r}$, *and constants C do not depend on n and* ξ.

Proof. We will distinguish two cases.
Case 1: $q = R + 1$. Using Lemma B.7 and taking into account that $k = 0, 1, ..., p$, $\ell \ge 1$ and that $|t|n^{-1/\alpha} \le 1$ we obtain for $|t| \le \varepsilon n^{1/\alpha}$, $t \ne 0$:

$$\begin{aligned}&\left|\frac{d^{R+1}}{dt^{R+1}}\left(\varphi_{\alpha,1}^k(t)\cdot g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right)\cdot \overline{\overline{h}}_{n,\xi}^{\ell}\left(tn^{-\frac{1}{\alpha}}\right)\right)\right| \le C\, e^{-\frac{n-\ell}{2n}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} \\ &\quad\cdot \ell^{R+1}\, |t|^{\alpha k-(R+1)} \left(\left(c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^\ell + \left(c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^{R+1+(\ell-(R+1))^+} \overline{\overline{\nu}}_{R+1,n,\xi}\right).\end{aligned} \tag{162}$$

Let us consider the case $t = 0$. Using the definition of m_k and p we can show that $\alpha k + \ell > R + 1$ for $k = 0, ..., p$ and $\ell = m_k + 1, ..., n-1$. Indeed,

$$\alpha k + \ell \ge \alpha k + m_k + 1 > \alpha k + \frac{R - \alpha k/2}{1-\alpha} + 1 = \frac{R+1-\alpha}{1-\alpha} + \frac{\alpha k\,(1-2\,\alpha)}{2\,(1-\alpha)} = \circledS \tag{163}$$

For $\alpha < 1/2$ the expression Ⓢ considered above is increasing with respect to k and takes the smallest value at point $k = 0$. Therefore, Ⓢ $\geq \frac{R+1-\alpha}{1-\alpha} > R+1$. If $\alpha \geq 1/2$, then expression Ⓢ is decreasing with respect to k and takes the smallest value at point $k = p$. From the definition of p it follows that Ⓢ $\geq 2R+1 > R+1$. Thus, $\alpha k + \ell > R+1$ for all $\alpha \in (0,1)$ and k, ℓ considered in the lemma. Keeping this fact in mind let us calculate the first derivative of $\varphi^k_{\alpha,1}(t)\, g_{\alpha,1}\left(t\left(\frac{n-\ell}{n}\right)^{1/\alpha}\right) \overline{\overline{h}}^{\ell}_{n,\xi}\left(tn^{-1/\alpha}\right)$. Using the definition of $\overline{\overline{h}}_{n,\xi}$ and the fact that $\overline{\overline{\mu}}_{0,n,\xi} = \int_{-\infty}^{+\infty} 1\, d\overline{\overline{H}}_{n,\xi}(x) = 0$ (see Lemma 4.1) we obtain

$$\left(\varphi^k_{\alpha,1}(\cdot)\; g_{\alpha,1}\left(\cdot\left(\tfrac{n-\ell}{n}\right)^{1/\alpha}\right) \overline{\overline{h}}^{\ell}_{n,\xi}\left(\cdot\, n^{-1/\alpha}\right)\right)'_{+}(0)$$

$$= \lim_{h\to 0+} \frac{e^{-\frac{n-\ell}{n}h^{\alpha}e^{-i\alpha\frac{\pi}{2}}}(-1)^k h^{\alpha k} e^{-i\alpha\frac{\pi}{2}k}\left(\int\limits_{-\infty}^{\infty}\left(e^{ihn^{-1/\alpha}x}-1\right) d\overline{\overline{H}}_{n,\xi}(x)\right)^{\ell} - 0}{h}$$

$$= \lim_{h\to 0+} e^{-\frac{n-\ell}{n}h^{\alpha}e^{-i\alpha\frac{\pi}{2}}}(-1)^k h^{\alpha k} e^{-i\alpha\frac{\pi}{2}k}\left(\int_{-\infty}^{\infty}\frac{e^{ihn^{-1/\alpha}x}-1}{h^{1/\ell}}\, d\overline{\overline{H}}_{n,\xi}(x)\right)^{\ell} = 0$$

as well as

$$\left(\varphi^k_{\alpha,1}(\cdot)\; g_{\alpha,1}\left(\cdot\left(\tfrac{n-\ell}{n}\right)^{1/\alpha}\right) \overline{\overline{h}}^{\ell}_{n,\xi}\left(\cdot\, n^{-1/\alpha}\right)\right)'_{-}(0)$$

$$= \lim_{h\to 0-} e^{-\frac{n-\ell}{n}|h|^{\alpha}e^{i\alpha\frac{\pi}{2}}}(-1)^{k-1}\,|h|^{\alpha k}\, e^{i\alpha\frac{\pi}{2}k}\left(\int_{-\infty}^{\infty}\frac{e^{ihn^{-1/\alpha}x}-1}{|h|^{1/\ell}}\, d\overline{\overline{H}}_{n,\xi}(x)\right)^{\ell} = 0.$$

Thus, the first derivative at point 0 exists and is equal to 0. Similarly, using the fact that $\alpha k + \ell > R+1$ we can show that

$$\frac{d^j}{dt^j}\left(\varphi^k_{\alpha,1}(t)\; g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{1/\alpha}\right) \overline{\overline{h}}^{\ell}_{n,\xi}\left(tn^{-1/\alpha}\right)\right)\Bigg|_{t=0} = 0 \qquad \text{for} \quad j = 1, ..., R+1.$$

Thus, inequality (162) holds for all $|t| \leq \varepsilon n^{1/\alpha}$. Using it we estimate

$$\begin{aligned}\left|d^{(R+1)}_{2n}(t)\right| &\leq \sum_{k=0}^{p}\sum_{\ell=m_k+1}^{n-1}\binom{n}{\ell}\frac{c^*_{k,n-\ell}}{n^k}\; C\, e^{-\frac{n-\ell}{2n}|t|^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)}\, \ell^{R+1}\, |t|^{\alpha k-(R+1)} \\ &\quad\cdot\left(\left(c_0\,|t|\,n^{-\frac{1}{\alpha}}\right)^{\ell} + \left(c_0\,|t|\,n^{-\frac{1}{\alpha}}\right)^{R+1+(\ell-(R+1))^+}\overline{\overline{\nu}}_{R+1,n,\xi}\right) \\ &= C\, e^{-\frac{|t|^{\alpha}}{2}\cos\left(\frac{\alpha\pi}{2}\right)}\sum_{k=0}^{p}|t|^{\alpha k-(R+1)}\sum_{\ell=m_k+1}^{n-1}\binom{n}{\ell}\frac{c^*_{k,n-\ell}}{n^k}\left(e^{\frac{|t|^{\alpha}}{2n}\cos\left(\frac{\alpha\pi}{2}\right)}\right)^{\ell} \\ &\quad\cdot\,\ell^{R+1}\left(\left(c_0\,|t|\,n^{-\frac{1}{\alpha}}\right)^{\ell} + \left(c_0\,|t|\,n^{-\frac{1}{\alpha}}\right)^{R+1+(\ell-(R+1))^+}\overline{\overline{\nu}}_{R+1,n,\xi}\right).\end{aligned} \tag{164}$$

It is easy to see that for $k = 0, \ldots, p$ and $\ell = 1, \ldots, n-1$ there always exists a constant $C > 0$ such that

$$\frac{c^*_{k,n-\ell}}{n^k} = \sum_{\substack{k_0+k_2+\cdots+k_s=n-\ell\\ 2k_2+\cdots+sk_s=k}}\frac{(n-\ell)!}{k_0!k_2!\cdots k_s!\,n^k}\,|A_2|^{k_2}\cdots|A_s|^{k_s} \leq \frac{C}{n^{k/2}}, \tag{165}$$

where $k_0, k_2, ..., k_s \in \mathbb{N}_0$ and C does not depend on n.

Note also that $e^{\frac{|t|^\alpha}{2n}\cos\left(\frac{\alpha\pi}{2}\right)} \leq e$, since $|t| \leq n^{1/\alpha}$. Using the last two inequalities and the fact that there exists a constant $C > 0$, such that $\ell^{R+1} \leq C\,2^\ell$ for all $\ell \in \mathbb{N}$, we obtain

$$\begin{aligned}
&\sum_{\ell=m_k+1}^{n-1} \binom{n}{\ell} \frac{c^*_{k,n-\ell}}{n^k} \left(e^{\frac{|t|^\alpha}{2n}\cos\left(\frac{\alpha\pi}{2}\right)}\right)^\ell \ell^{R+1} \left(\left(c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^\ell + \cdots\right)\\
&\leq C \sum_{\ell=m_k+1}^{n-1} \binom{n}{\ell} \frac{(2\,e)^\ell}{n^{k/2}} \left(\left(c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^\ell + \left(c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^{R+1+(\ell-(R+1))^+} \overline{\overline{\nu}}_{R+1,n,\xi}\right)\\
&= C \sum_{\ell=m_k+1}^{R+1} \binom{n}{\ell} \frac{(2\,e)^\ell}{n^{k/2}} \left(\left(c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^\ell + \left(c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^{R+1} \overline{\overline{\nu}}_{R+1,n,\xi}\right)\\
&\quad + C \sum_{\ell=\max\{m_k+1,R+2\}}^{n-1} \binom{n}{\ell} \frac{(2\,e)^\ell}{n^{k/2}} \left(\left(c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^\ell (1+\overline{\overline{\nu}}_{R+1,n,\xi})\right) = \circledS.
\end{aligned}$$

Note that if $k \leq 2R$ then the first sum in $\circledS$ is equal to 0, since $m_k+1 > R+1$ (this follows from the definition of m_k). This means, one considers the first sum only for $k > 2R$. We denote $\ell_0 := \max\{m_k+1, R+2\}$ and continue

$$\begin{aligned}
\circledS &\leq C \sum_{\ell=m_k+1}^{R+1} n^{\ell-k/2-\ell/\alpha}\,|t|^\ell + C\,\left(n^{-\frac{1}{\alpha}}\,|t|\right)^{R+1} \overline{\overline{\nu}}_{R+1,n,\xi} \sum_{\ell=m_k+1}^{R+1} n^{\ell-k/2}\\
&\quad + C\,\max\{1,\overline{\overline{\nu}}_{R+1,n,\xi}\}\ n^{-k/2}\,\left(|t|\ n^{-\frac{1}{\alpha}}\right)^{\ell_0} \sum_{\ell=\ell_0}^{n-1} \binom{n}{\ell} \left(2\,e\,c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^{\ell-\ell_0} = \circledS.
\end{aligned}$$

Using the definition of m_k we obtain for $\ell \geq m_k+1$:

$$\ell(1-\tfrac{1}{\alpha}) - k/2 \leq (m_k+1)(1-\tfrac{1}{\alpha}) - k/2 < \left(1+\tfrac{R-k\alpha/2}{1-\alpha}\right)\tfrac{\alpha-1}{\alpha} - k/2 = -\tfrac{R+1-\alpha}{\alpha}.$$

Also, for $\ell = m_k+1, ..., R+1$ with $m_k+1 \leq R+1$ we have

$$\ell - k/2 \leq R+1-k/2 < R+1-2R/2 = 1.$$

Taking into account that $|t| \leq \left(\cos\left(\frac{\alpha\pi}{2}\right)/(8\,e\,c_0)\right)^{1/(1-\alpha)} n^{1/\alpha}$ and using the properties $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ and $(1+a/n)^n \leq e^a$, $a \in \mathbb{R}_+$, we have

$$\begin{aligned}
&\sum_{\ell=\ell_0}^{n-1} \binom{n}{\ell} \left(2\,e\,c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^{\ell-\ell_0} = \sum_{j=0}^{n-1-\ell_0} \binom{n}{j+\ell_0} \left(2\,e\,c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^j\\
&= \sum_{j=0}^{n-1-\ell_0} \frac{n(n-1)\cdots(n-\ell_0+1)}{(j+\ell_0)\cdots(j+1)} \binom{n-\ell_0}{j} \left(2\,e\,c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^j\\
&\leq n^{\ell_0}\left(1+2\,e\,c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^{n-\ell_0} \leq n^{\ell_0}\left(1+\frac{2\,e\,c_0\,|t|^\alpha}{n}\left(\frac{|t|^\alpha}{n}\right)^{\frac{1-\alpha}{\alpha}}\right)^n\\
&\leq n^{\ell_0}\left(1+\frac{\cos\left(\frac{\alpha\,\pi}{2}\right)\ |t|^\alpha}{4\,n}\right)^n \leq n^{\ell_0}\,e^{\frac{|t|^\alpha}{4}\cos\left(\frac{\alpha\,\pi}{2}\right)}.
\end{aligned} \tag{166}$$

Denote $D = \{m \in \mathbb{N} : m > 2\,R\}$. Using the last three inequalities considered above, the fact that $\ell_0 \left(1 - \frac{1}{\alpha}\right) - k/2 \le (m_k + 1)\left(1 - \frac{1}{\alpha}\right) - k/2 < -(R + 1 - \alpha)/\alpha$ and Lemma 4.2 for $\overline{\overline{\nu}}_{R+1,n,\xi}$, we finally obtain

$$\begin{aligned}
Ⓢ \le C \left(n^{-\frac{R+1-\alpha}{\alpha}} \left(|t|^{m_k+1} + |t|^{R+1} \right) + n^{-\frac{r-\alpha}{\alpha}} \, |t|^{R+1} \, (1+\xi)^{R+1-r} \right) \mathbb{1}_D(k) \\
+ C\, e^{\frac{|t|^\alpha}{4} \cos\left(\frac{\alpha\,\pi}{2}\right)} \, n^{\ell_0 \left(1-\frac{1}{\alpha}\right) - k/2} \, |t|^{\ell_0} \, n^{\frac{R+1-r}{\alpha}} (1+\xi)^{R+1-r} \\
\le C\, e^{\frac{|t|^\alpha}{4} \cos\left(\frac{\alpha\,\pi}{2}\right)} \, n^{-\frac{r-\alpha}{\alpha}} \, \left(\left(|t|^{m_k+1} + |t|^{R+1} \right) \mathbb{1}_D(k) + |t|^{\ell_0} \right) (1+\xi)^{R+1-r}.
\end{aligned} \tag{167}$$

Now we come back to the estimation of $d_{2n}^{(R+1)}(t)$. Plugging the upper bound (167) of the sum $\sum_{\ell=m_k+1}^{n-1} \binom{n}{\ell} \frac{c^*_{k,n-\ell}}{n^k} \ldots$ into formula (164) we obtain

$$\begin{aligned}
&\left| d_{2n}^{(R+1)}(t) \right| \le C\; e^{-\frac{|t|^\alpha}{2} \cos\left(\frac{\alpha\pi}{2}\right)} \sum_{k=0}^{p} |t|^{\alpha k - (R+1)} \sum_{\ell=m_k+1}^{n-1} \binom{n}{\ell} \frac{c^*_{k,n-\ell}}{n^k} \left(e^{\frac{|t|^\alpha}{2n} \cos\left(\frac{\alpha\pi}{2}\right)} \right)^{\ell} \\
&\cdot \ell^{R+1} \left(\left(c_0\, |t|\, n^{-\frac{1}{\alpha}} \right)^{\ell} + \left(c_0\, |t|\, n^{-\frac{1}{\alpha}} \right)^{R+1+(\ell-(R+1))^+} \overline{\overline{\nu}}_{R+1,n,\xi} \right) \\
&\le C\; e^{-\frac{|t|^\alpha}{4} \cos\left(\frac{\alpha\pi}{2}\right)} \, n^{-\frac{r-\alpha}{\alpha}} \, (1+\xi)^{R+1-r} \sum_{k=0}^{p} |t|^{\alpha k - (R+1)} \left(\left(|t|^{m_k+1} + |t|^{R+1} \right) \mathbb{1}_D(k) + |t|^{\ell_0} \right) \\
&\le C\; e^{-\frac{|t|^\alpha}{4} \cos\left(\frac{\alpha\pi}{2}\right)} \, n^{-\frac{r-\alpha}{\alpha}} \, \left(|t|^{\theta_1} + |t|^{\theta_2} + |t|^{\alpha p+1} \right) (1+\xi)^{R+1-r},
\end{aligned}$$

where $\theta_1 := \min_{0 \le k \le p} \{m_k + \alpha k - R\}$ and $\theta_2 := \max_{0 \le k \le p} \{m_k + \alpha k - R\}$. From inequality (163) it follows that $\theta_1, \theta_2 > 0$.

Now we come back to the second case, which is much simpler.

Case 2: $q = 0, 1, ..., R$. Again, using Lemma B.7 we obtain:

$$\begin{aligned}
&\left| \frac{d^q}{dt^q} \left(\varphi^k_{\alpha,1}(t) \cdot g_{\alpha,1} \left(t \left(\tfrac{n-\ell}{n} \right)^{\frac{1}{\alpha}} \right) \cdot \overline{\overline{h}}{}^{\ell}_{n,\xi} \left(t n^{-\frac{1}{\alpha}} \right) \right) \right| \\
&\qquad \le C\, e^{-\frac{n-\ell}{2n} |t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} \, \ell^q \, |t|^{\alpha k - q} \, \left(c_0\, |t|\, n^{-\frac{1}{\alpha}} \right)^{\ell}.
\end{aligned}$$

Now we repeat the same procedure as in the case $q = R + 1$. Using the above estimate we obtain

$$\begin{aligned}
\left| d_{2n}^{(q)}(t) \right| &\le \sum_{k=0}^{p} \sum_{\ell=m_k+1}^{n-1} \binom{n}{\ell} \frac{c^*_{k,n-\ell}}{n^k} \; C\, e^{-\frac{n-\ell}{2n} |t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} \, \ell^q \, |t|^{\alpha k - q} \, \left(c_0\, |t|\, n^{-\frac{1}{\alpha}} \right)^{\ell} \\
&= C\; e^{-\frac{|t|^\alpha}{2} \cos\left(\frac{\alpha\pi}{2}\right)} \sum_{k=0}^{p} |t|^{\alpha k - q} \sum_{\ell=m_k+1}^{n-1} \binom{n}{\ell} \frac{c^*_{k,n-\ell}}{n^k} \left(e^{\frac{|t|^\alpha}{2n} \cos\left(\frac{\alpha\pi}{2}\right)} \right)^{\ell} \\
&\quad \cdot \ell^q \left(c_0\, |t|\, n^{-\frac{1}{\alpha}} \right)^{\ell}.
\end{aligned} \tag{168}$$

Using (166) with $m_k + 1$ instead of ℓ_0 we estimate the sum $\sum_{\ell=m_k+1}^{n-1} \binom{n}{\ell} \frac{c^*_{k,n-\ell}}{n^k} \ldots$ as follows:

$$\sum_{\ell=m_k+1}^{n-1}\binom{n}{\ell}\frac{c^*_{k,n-\ell}}{n^k}\left(e^{\frac{|t|^\alpha}{2n}\cos\left(\frac{\alpha\pi}{2}\right)}\right)^\ell \ell^q\left(c_0\;|t|\;n^{-\frac{1}{\alpha}}\right)^\ell$$
$$\le C\sum_{\ell=m_k+1}^{n-1}\binom{n}{\ell}\frac{(2\,e)^\ell}{n^{k/2}}\left(c_0\;|t|\;n^{-\frac{1}{\alpha}}\right)^\ell$$
$$\le C\left(|t|\;n^{-\frac{1}{\alpha}}\right)^{m_k+1}n^{-k/2}\sum_{\ell=m_k+1}^{n-1}\binom{n}{\ell}\left(2\,e\,c_0\;|t|\;n^{-\frac{1}{\alpha}}\right)^{\ell-(m_k+1)}$$
$$\le C\,e^{\frac{|t|^\alpha}{4}\cos\left(\frac{\alpha\,\pi}{2}\right)}\;|t|^{m_k+1}\;n^{(m_k+1)\left(1-\frac{1}{\alpha}\right)-k/2}\le C\,e^{\frac{|t|^\alpha}{4}\cos\left(\frac{\alpha\,\pi}{2}\right)}\;|t|^{m_k+1}\;n^{-\frac{R+1-\alpha}{\alpha}}. \tag{169}$$

Plugging (169) into formula (168) we obtain

$$\left|d^{(q)}_{2n}(t)\right|\le C\;e^{-\frac{|t|^\alpha}{2}\cos\left(\frac{\alpha\pi}{2}\right)}\sum_{k=0}^{p}|t|^{\alpha k-q}\sum_{\ell=m_k+1}^{n-1}\binom{n}{\ell}\frac{c^*_{k,n-\ell}}{n^k}\left(e^{\frac{|t|^\alpha}{2n}\cos\left(\frac{\alpha\pi}{2}\right)}\right)^\ell\;\ell^q$$
$$\cdot\left(c_0\;|t|\;n^{-\frac{1}{\alpha}}\right)^\ell\le C\;e^{-\frac{|t|^\alpha}{4}\cos\left(\frac{\alpha\pi}{2}\right)}n^{-\frac{R+1-\alpha}{\alpha}}\sum_{k=0}^{p}|t|^{\alpha k+(m_k+1)-q}$$
$$\le C\;e^{-\frac{|t|^\alpha}{4}\cos\left(\frac{\alpha\pi}{2}\right)}n^{-\frac{R+1-\alpha}{\alpha}}\;|t|^{R+1-q}\;\left(|t|^{\theta_1}+|t|^{\theta_2}\right).$$

where $\theta_{1(2)}=\min_k(\max_k)\{\alpha k+m_k-R\}>0$. This completes the proof of the lemma.

□

B.4 Proof of Lemma 4.15

Lemma 4.15. *Define*

$$d_{3n}(t)=\sum_{\ell=1}^{n-1}\binom{n}{\ell}\sum_{k=p+1}^{s(n-\ell)}\frac{c_{k,n-\ell}}{n^k}\,\varphi^k_{\alpha,1}(t)\,g_{\alpha,1}\left(t\left(\frac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right)\overline{\overline{h}}^{\ell}_{n,\xi}\left(tn^{-\frac{1}{\alpha}}\right),$$

where $p=[2\,R/\alpha]$. *Then for* $|t|\le\varepsilon n^{1/\alpha}$ *we have*

$$\left|d^{(q)}_{3n}(t)\right|\le C\;e^{-\frac{|t|^\alpha}{8}\cos\left(\frac{\alpha\pi}{2}\right)}n^{-\frac{R+1-\alpha}{\alpha}}\;|t|^{1-q}\;\left(|t|^{\alpha\left(\frac{p}{2}+\frac{1}{2}\right)}+|t|^{\alpha(p+2)}\right),\quad q=0,1,...,R,$$
$$\left|d^{(R+1)}_{3n}(t)\right|\le C\;e^{-\frac{|t|^\alpha}{8}\cos\left(\frac{\alpha\pi}{2}\right)}n^{-\frac{r-\alpha}{\alpha}}\;\left(|t|^{\frac{\alpha(p+1)}{2}-R}+|t|^{\alpha(p+2)+1}\right)(1+\xi)^{R+1-r},$$

where r *comes from Theorem 3.26, constants* C *do not depend on* n *and* ξ*, and*

$$\varepsilon=\min\left\{1,\;\frac{1}{c_0},\;\frac{1}{(2D)^{1/\alpha}},\;\frac{1}{D^{1/\alpha}}\left(\frac{\cos\left(\frac{\alpha\pi}{2}\right)}{8D}\right)^{\frac{2+p/2}{\alpha}},\;\left(\frac{\cos\left(\frac{\alpha\pi}{2}\right)}{16\,e\,c_0}\right)^{1/(1-\alpha)}\right\}$$

with $D=\max_{2\le j\le s}\{2\,|A_j|^{1/j}\}$, $c_0=(\nu^*_0+1)\gamma^{*\;1/r}_r$, *s and* A_j *defined in* (85) *and* (70).

Proof. We distinguish two cases: $q = R+1$ and $q = 0, 1, ..., R$.
Case 1: $q = R+1$. Using Lemma B.7 and taking into account that $\ell, k \geq 1$ and that $|t|n^{-1/\alpha} \leq 1$ we obtain for $|t| \leq \varepsilon n^{1/\alpha}$, $t \neq 0$:

$$\left| \frac{d^{R+1}}{dt^{R+1}} \left(\varphi^k_{\alpha,1}(t)\, g_{\alpha,1} \left(t \left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}} \right) \overline{\overline{h}}^{\ell}_{n,\xi} \left(tn^{-\frac{1}{\alpha}} \right) \right) \right| \leq C\, e^{-\frac{n-\ell}{2n}|t|^{\alpha} \cos\left(\frac{\alpha\pi}{2}\right)} (k\,\ell)^{R+1} \cdot |t|^{\alpha k-(R+1)} \left(\left(c_0\, |t|\, n^{-\frac{1}{\alpha}} \right)^{\ell} + \left(c_0\, |t|\, n^{-\frac{1}{\alpha}} \right)^{R+1+(\ell-(R+1))^+} \overline{\overline{\nu}}_{R+1,n,\xi} \right). \tag{170}$$

Now let us consider the case $t = 0$. In the same way as in Lemma 4.13 or Lemma 4.14 (i.e. using the definition of the derivative) we can show that

$$\frac{d^j}{dt^j} \left(\varphi^k_{\alpha,1}(t)\; g_{\alpha,1} \left(t \left(\tfrac{n-\ell}{n}\right)^{1/\alpha} \right) \overline{\overline{h}}^{\ell}_{n,\xi} \left(tn^{-1/\alpha} \right) \right) \Bigg|_{t=0} = 0$$

for $j = 1, \dots, R+1$. When proving this one takes into account that $\alpha k > R+1$ for $k \geq p+1$. This follows from the definition of p.
Thus, inequality (170) holds for all $|t| \leq \varepsilon n^{1/\alpha}$. Using it, the estimate $e^{\frac{|t|^{\alpha}}{2n} \cos\left(\frac{\alpha\pi}{2}\right)} \leq e$ (since $|t| \leq n^{1/\alpha}$), and the fact that there always exists a constant $C > 0$ such that $k^{R+1} \leq C\; 2^k$ for all $k \in \mathbb{N}$, we obtain

$$\begin{aligned}
\left| d^{(R+1)}_{3n}(t) \right| &\leq \sum_{\ell=1}^{n-1} \binom{n}{\ell} \sum_{k=p+1}^{s(n-\ell)} \frac{|c_{k,n-\ell}|}{n^k} C\, e^{-\frac{n-\ell}{2n}|t|^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)} (k\,\ell)^{R+1} |t|^{\alpha k-(R+1)} \\
&\cdot \left(\left(c_0\, |t|\, n^{-\frac{1}{\alpha}} \right)^{\ell} + \left(c_0\, |t|\, n^{-\frac{1}{\alpha}} \right)^{R+1+(\ell-(R+1))^+} \overline{\overline{\nu}}_{R+1,n,\xi} \right) \\
&\leq C\; e^{-\frac{|t|^{\alpha}}{2}\cos\left(\frac{\alpha\pi}{2}\right)}\; |t|^{-(R+1)} \sum_{\ell=1}^{n-1} \binom{n}{\ell} \left(e^{\frac{|t|^{\alpha}}{2n}\cos\left(\frac{\alpha\pi}{2}\right)} \right)^{\ell} \ell^{R+1} \\
&\cdot \left(\left(c_0\, |t|\, n^{-\frac{1}{\alpha}} \right)^{\ell} + \left(c_0\, |t|\, n^{-\frac{1}{\alpha}} \right)^{R+1+(\ell-(R+1))^+} \overline{\overline{\nu}}_{R+1,n,\xi} \right) \sum_{k=p+1}^{s(n-\ell)} \frac{c^*_{k,n-\ell}}{n^k} k^{R+1}\, |t|^{\alpha k} \\
&\leq C\; e^{-\frac{|t|^{\alpha}}{2}\cos\left(\frac{\alpha\pi}{2}\right)}\; |t|^{-(R+1)} \sum_{\ell=1}^{n-1} \binom{n}{\ell} (2e)^{\ell} \\
&\cdot \left(\left(c_0\, |t|\, n^{-\frac{1}{\alpha}} \right)^{\ell} + \left(c_0\, |t|\, n^{-\frac{1}{\alpha}} \right)^{R+1+(\ell-(R+1))^+} \overline{\overline{\nu}}_{R+1,n,\xi} \right) \sum_{k=p+1}^{s(n-\ell)} \frac{c^*_{k,n-\ell}}{\left(\frac{n}{2}\right)^k} |t|^{\alpha k}.
\end{aligned}$$

Now we find an upper bound for the sum $\sum_{k=p+1}^{s(n-\ell)} \frac{c^*_{k,n-\ell}}{(n/2)^k} |t|^{\alpha k}$. Note that we consider only $|t| \leq \varepsilon n^{1/\alpha}$, where ε is defined in the statement of the lemma. Recall that

$$c^*_{k,n-\ell} = \sum_{\substack{k_0+k_2+\dots+k_s=n-\ell \\ 2k_2+\dots+sk_s=k}} \frac{(n-\ell)!}{k_0!k_2!\cdots k_s!} |A_2|^{k_2} \cdots |A_s|^{k_s},$$

where $k_0, k_2, ..., k_s \in \mathbb{N}_0$. It is easy to see that $c^*_{k,n-\ell} \leq c^*_{k,n}$ for all $\ell = 1, ..., n-1$. Using this fact first and then doing the same procedure as we did in order to get inequality (161) from Lemma 4.13 we obtain

$$\sum_{k=p+1}^{s(n-\ell)} \frac{c^*_{k,n-\ell}}{\left(\frac{n}{2}\right)^k} |t|^{\alpha k} \leq \sum_{k=p+1}^{sn} \frac{c^*_{k,n}}{\left(\frac{n}{2}\right)^k} |t|^{\alpha k} \leq C\, e^{\frac{1}{4}|t|^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)} n^{-\frac{R}{\alpha}} \left(|t|^{\alpha\left(\frac{p}{2}+\frac{1}{2}\right)} + |t|^{\alpha(p+2)} \right).$$

Using the last inequality we continue estimating $d_{3n}^{(R+1)}(t)$:

$$\left|d_{3n}^{(R+1)}(t)\right| \le C\ e^{-\frac{|t|^\alpha}{4}\cos\left(\frac{\alpha\pi}{2}\right)}\, n^{-\frac{R}{\alpha}}\ |t|^{-(R+1)}\ \left(|t|^{\alpha\left(\frac{p}{2}+\frac{1}{2}\right)} + |t|^{\alpha(p+2)}\right)$$
$$\cdot \sum_{\ell=1}^{n-1}\binom{n}{\ell}(2e)^\ell \left(\left(c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^\ell + \left(c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^{R+1+(\ell-(R+1))^+}\ \overline{\overline{\nu}}_{R+1,n,\xi}\right). \tag{171}$$

For the sum from (171) we distinguish two cases $\ell \le R+1$ and $\ell \ge R+2$ and get

$$\sum_{\ell=1}^{n-1}\binom{n}{\ell}(2e)^\ell \left(\left(c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^\ell + \left(c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^{R+1+(\ell-(R+1))^+}\ \overline{\overline{\nu}}_{R+1,n,\xi}\right)$$
$$\le \sum_{\ell=1}^{R+1}\binom{n}{\ell}(2e)^\ell \left(\left(c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^\ell + \left(c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^{R+1}\ \overline{\overline{\nu}}_{R+1,n,\xi}\right)$$
$$+ \sum_{\ell=R+2}^{n-1}\binom{n}{\ell}(2e)^\ell \left(\left(c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^\ell\ (1+\overline{\overline{\nu}}_{R+1,n,\xi})\right) = \text{Ⓢ}.$$

Now using the estimation for $\overline{\overline{\nu}}_{R+1,n,\xi}$ from Lemma 4.2 and repeating the same procedure as in (166) for $|t| \le \left(\cos\left(\frac{\alpha\pi}{2}\right)/(16\,e\,c_0)\right)^{1/(1-\alpha)} n^{1/\alpha}$, we continue

$$\text{Ⓢ} \le C\sum_{\ell=1}^{R+1} n^{\ell\left(1-\frac{1}{\alpha}\right)}\ |t|^\ell + C\,n^{R+1}\ \left(|t|\ n^{-\frac{1}{\alpha}}\right)^{R+1}\ \left(n^{\frac{1}{\alpha}}(1+\xi)\right)^{R+1-r}$$
$$+ C\ \left(n^{\frac{1}{\alpha}}(1+\xi)\right)^{R+1-r}\ \left(|t|\ n^{-\frac{1}{\alpha}}\right)^{R+2}\sum_{\ell=R+2}^{n-1}\binom{n}{\ell}\left(2\,e\,c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^{\ell-(R+2)}$$
$$\le C\ \left(|t| + |t|^{R+1}\right) n^{1-\frac{1}{\alpha}} + C\,|t|^{R+1}\,(1+\xi)^{R+1-r}\,n^{R+1-\frac{r}{\alpha}}$$
$$+ C\ e^{\frac{|t|^\alpha}{8}\cos\left(\frac{\alpha\pi}{2}\right)}\ |t|^{R+2}\,(1+\xi)^{R+1-r}\,n^{R+2-\frac{r+1}{\alpha}}$$
$$\le C\ e^{\frac{|t|^\alpha}{8}\cos\left(\frac{\alpha\pi}{2}\right)}\ \left(|t| + |t|^{R+2}\right)\,(1+\xi)^{R+1-r}\,n^{\max\left\{R+1-\frac{r}{\alpha},1-\frac{1}{\alpha}\right\}} \tag{172}$$

Finally, combining inequalities (171) and (172) we obtain

$$\left|d_{3n}^{(R+1)}(t)\right| \le C\ e^{-\frac{|t|^\alpha}{8}\cos\left(\frac{\alpha\pi}{2}\right)}\,(1+\xi)^{R+1-r}\,n^{-\frac{r-\alpha}{\alpha}}\ \left(|t|^{\frac{\alpha(p+1)}{2}-R} + |t|^{\alpha(p+2)+1}\right),$$

where $\alpha(p+1)/2 - R > 0$ according to the definition of p.

Now we consider the second case.

Case 2: $q = 0,1,...,R$. It follows from Lemma B.7 that in this case we have

$$\left|\frac{d^q}{dt^q}\left(\varphi_{\alpha,1}^k(t)\cdot g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right)\cdot \overline{\overline{h}}{}_{n,\xi}^{\ell}\left(tn^{-\frac{1}{\alpha}}\right)\right)\right|$$
$$\le C\,e^{-\frac{n-\ell}{2n}|t|^\alpha\cos\left(\frac{\alpha\pi}{2}\right)}\,(k\,\ell)^q\,|t|^{\alpha k-q}\ \left(c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^\ell. \tag{173}$$

Now we repeat the same procedure as in the case $q = R+1$. Using the estimate (173) we obtain

$$\left|d_{3n}^{(q)}(t)\right| \le \sum_{\ell=1}^{n-1}\binom{n}{\ell}\sum_{k=p+1}^{s(n-\ell)}\frac{|c_{k,n-\ell}|}{n^k}\,C\,e^{-\frac{n-\ell}{2n}|t|^\alpha\cos\left(\frac{\alpha\pi}{2}\right)}\,(k\,\ell)^q\,|t|^{\alpha k-q}\left(c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^\ell$$
$$\le C\ e^{-\frac{|t|^\alpha}{2}\cos\left(\frac{\alpha\pi}{2}\right)}\ |t|^{-q}\sum_{\ell=1}^{n-1}\binom{n}{\ell}\ \left(2\,e\,c_0\ |t|\ n^{-\frac{1}{\alpha}}\right)^\ell\ \sum_{k=p+1}^{s(n-\ell)}\frac{c^*_{k,n-\ell}}{\left(\frac{n}{2}\right)^k}\,|t|^{\alpha k}.$$

We already showed above that

$$\sum_{k=p+1}^{s(n-\ell)} \frac{C^*_{k,n-\ell}}{\left(\frac{n}{2}\right)^k} |t|^{\alpha k} \le C\, e^{\frac{1}{4}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} n^{-\frac{R}{\alpha}} \left(|t|^{\alpha\left(\frac{p}{2}+\frac{1}{2}\right)} + |t|^{\alpha(p+2)}\right).$$

Using the same method as in (166) for $|t| \le \left(\cos\left(\frac{\alpha\pi}{2}\right)/(16\, e\, c_0)\right)^{1/(1-\alpha)} n^{1/\alpha}$ we obtain

$$\sum_{\ell=1}^{n-1} \binom{n}{\ell} \left(2\, e\, c_0\, |t|\, n^{-\frac{1}{\alpha}}\right)^\ell \le C\, n^{-\frac{1}{\alpha}} |t| \sum_{\ell=1}^{n-1} \binom{n}{\ell} \left(2\, e\, c_0\, |t|\, n^{-\frac{1}{\alpha}}\right)^{\ell-1}$$
$$\le C\, e^{\frac{|t|^\alpha}{8}\cos\left(\frac{\alpha\pi}{2}\right)} n^{1-\frac{1}{\alpha}} |t|.$$

Using these two estimates we have

$$\left|d^{(q)}_{3n}(t)\right| \le C\ e^{-\frac{|t|^\alpha}{8}\cos\left(\frac{\alpha\pi}{2}\right)} n^{-\frac{R+1-\alpha}{\alpha}}\ |t|^{1-q} \left(|t|^{\alpha\left(\frac{p}{2}+\frac{1}{2}\right)} + |t|^{\alpha(p+2)}\right),$$

where $1-q+\alpha(p+1)/2 \ge \alpha(p+1)/2 - R > 0$ according to the definition of p. This completes the proof of the lemma. □

B.5 Proof of Lemma 4.16

Lemma 4.16. *Define*

$$d_{4n}(t) = \sum_{k=0}^{p}\sum_{\ell=1}^{m_k} \binom{n}{\ell} \frac{c_{k,n-\ell}}{n^k} \left[g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right) \varphi^k_{\alpha,1}(t)\, \overline{\overline{h}}^{\ell}_{n,\xi}\left(tn^{-\frac{1}{\alpha}}\right) \right.$$
$$\left. - \sum_{u=\ell}^{m_{\ell,k}} \sum_{v=0}^{p_{u,\ell,k}} \overline{\overline{C}}_{u,\ell}\, \frac{(-\ell/n)^v}{v!}\, n^{-u/\alpha}\, g_{\alpha,1}(t)\, \varphi^{k+v}_{\alpha,1}(t)\, (it)^u \right] - \overline{\overline{w}}^*_{n,\xi}(t),$$

where $p = [2R/\alpha]$, $m_k = 1 + [(R-\alpha k/2)/(1-\alpha)]$, $m_{\ell,k} = [R+1+\alpha(\ell-1-k/2)]$ *and* $p_{u,\ell,k} = \max\{0, [(R+1-u)/\alpha + \ell - 1 - k/2]\}$ *and*

$$\overline{\overline{C}}_{u,\ell} = \sum_{\substack{k_1+2k_2+\ldots+Rk_R=u \\ k_1+k_2+\ldots+k_R=\ell}} \frac{\ell!}{k_1!\ldots k_R!} \left(\frac{\overline{\overline{\mu}}_{1,n,\xi}}{1!}\right)^{k_1} \ldots \left(\frac{\overline{\overline{\mu}}_{R,n,\xi}}{R!}\right)^{k_R},$$

$$\overline{\overline{w}}^*_{n,\xi}(t) = n\, g_{\alpha,1}(t) \int_{-\infty}^{+\infty} \left(e^{ixtn^{-\frac{1}{\alpha}}} - \sum_{w=0}^{R+1} \frac{(ixtn^{-\frac{1}{\alpha}})^w}{w!}\right) d\left(\overline{\overline{H}}_{n,\xi} - \overline{\overline{H}}_{n,0}\right)(x)$$
$$+ n\, g_{\alpha,1}(t) \frac{(it)^{R+1}}{(R+1)!} n^{-\frac{R+1}{\alpha}}\, \overline{\overline{\mu}}_{R+1,n,\xi}.$$

Then for $|t| \le \varepsilon\, n^{1/\alpha}$ *and* $q = 0, 1, ..., R+1$ *we have*

$$\left|d^{(q)}_{4n}(t)\right| \le C\, e^{-\frac{1}{2}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} n^{-\frac{r-\alpha}{\alpha}} (1+\xi)^{R+1-r} |t|^{R+1-q} \left(|t|^\theta + |t|^{R\max\left\{1, \frac{\alpha}{1-\alpha}\right\}}\right),$$

where $\theta = \min_u\{u + \alpha(p_{u,1,0}+1) - (R+1)\} \in (0,\alpha]$, $\varepsilon = \min\left\{1,\, c_0^{-1}\right\}$ *with* $c_0 = (\nu^*_0+1)\gamma^{*\,1/r}_r$, *pseudomoments* ν^*_0, γ^*_r *are defined in* (80), r *comes from Theorem 3.26,* R *is defined by* (99) *and constant* C *does not depend on* n *and* ξ.

Proof. We start with some transformations of $d_{4n}(t)$. We denote the expression in square brackets from the definition of $d_{4n}(t)$ by $I_{k,\ell}(t)$. We will distinguish three cases: (1) $k=0, \ell=1$; (2) $k=0, \ell=2,...,m_k$ and (3) $k \geq 1, \ell=1,...,m_k$. Taking into account that $c_{0,n}=1$ and $c_{1,n}=0$ for all $n \in \mathbb{N}$ we obtain

$$d_{4n}(t) = \sum_{k=0}^{p}\sum_{\ell=1}^{m_k}\binom{n}{\ell}\frac{c_{k,n-\ell}}{n^k}\, I_{k,\ell}(t) - \overline{\overline{w}}^*_{n,\xi}(t) = \Big(n\, I_{0,1}(t) - \overline{\overline{w}}^*_{n,\xi}(t)\Big)$$
$$+\sum_{\ell=2}^{m_0}\binom{n}{\ell} I_{0,\ell}(t) + \sum_{k=2}^{p}\sum_{\ell=1}^{m_k}\binom{n}{\ell}\frac{c_{k,n-\ell}}{n^k}\, I_{k,\ell}(t). \tag{174}$$

We need to estimate the q-order derivative of $I_{k,\ell}(t)$ for $q=0,1,...,R+1$. Before differentiating we start with some transformations of $I_{k,\ell}(t)$. We consider

$$\chi_{R,n,\xi}(t) = \frac{\overline{\overline{\mu}}_{1,n,\xi}(itn^{-1/\alpha})}{1!} + \frac{\overline{\overline{\mu}}_{2,n,\xi}(itn^{-1/\alpha})^2}{2!} + \cdots + \frac{\overline{\overline{\mu}}_{R,n,\xi}(itn^{-1/\alpha})^R}{R!}.$$

Taking into account that $g_{\alpha,1}\left(t\left(\frac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right) = e^{\frac{n-\ell}{n}\varphi_{\alpha,1}(t)}$ we obtain:

$$I_{k,\ell}(t) = \underbrace{g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right)\varphi^k_{\alpha,1}(t)\left(\overline{\overline{h}}^{\ell}_{n,\xi}\Big(tn^{-\frac{1}{\alpha}}\Big) - \chi^{\ell}_{R,n,\xi}(t)\right)}_{=:S_1(t)} +$$
$$+\underbrace{g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right)\varphi^k_{\alpha,1}(t)\left(\chi^{\ell}_{R,n,\xi}(t) - \sum_{u=\ell}^{m_{\ell,k}}\overline{\overline{C}}_{u,\ell}\, n^{-u/\alpha}\,(it)^u\right)}_{=:S_2(t)}$$
$$+\underbrace{g_{\alpha,1}(t)\,\varphi^k_{\alpha,1}(t)\sum_{u=\ell}^{m_{\ell,k}}\overline{\overline{C}}_{u,\ell}\, n^{-u/\alpha}\,(it)^u\left(e^{-\frac{\ell}{n}\varphi_{\alpha,1}(t)} - \sum_{v=0}^{p_{u,\ell,k}}\frac{(-\ell/n)^v}{v!}\varphi^v_{\alpha,1}(t)\right)}_{=:S_3(t)}$$
$$+\Bigg[g_{\alpha,1}(t)\,\varphi^k_{\alpha,1}(t)\sum_{u=\ell}^{m_{\ell,k}}\overline{\overline{C}}_{u,\ell}\, n^{-u/\alpha}\,(it)^u\sum_{v=0}^{p_{u,\ell,k}}\frac{(-\ell/n)^v}{v!}\varphi^v_{\alpha,1}(t)$$
$$-\sum_{u=\ell}^{m_{\ell,k}}\sum_{v=0}^{p_{u,\ell,k}}\overline{\overline{C}}_{u,\ell}\,\frac{(-\ell/n)^v}{v!}\, n^{-u/\alpha}\, g_{\alpha,1}(t)\,\varphi^{k+v}_{\alpha,1}(t)\,(it)^u\Bigg] = S_1(t)+S_2(t)+S_3(t).$$

Note that the expression in the square brackets in the last formula is equal to 0. Applying formula $a^n - b^n = (a-b)\sum_{j=0}^{n-1} a^j\, b^{n-1-j}$ and taking into account that $\overline{\overline{h}}_{n,\xi}\Big(tn^{-\frac{1}{\alpha}}\Big) = \int_{-\infty}^{+\infty} e^{ixtn^{-\frac{1}{\alpha}}}\, d\overline{\overline{H}}_{n,\xi}(x)$, $\overline{\overline{\mu}}_{k,n,\xi} = \int_{-\infty}^{+\infty} x^k d\overline{\overline{H}}_{n,\xi}(x)$ and $\overline{\overline{\mu}}_{0,n,\xi} = 0$ for all $n \in \mathbb{N}$, $\xi \geq 0$ (see Lemma 4.1 (*i*)), we rewrite $S_1(t)$ as follows:

$$S_1(t) = g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right)\varphi^k_{\alpha,1}(t)\left(\overline{\overline{h}}_{n,\xi}\Big(tn^{-\frac{1}{\alpha}}\Big) - \chi_{R,n,\xi}(t)\right)\sum_{j=0}^{\ell-1}\overline{\overline{h}}^{j}_{n,\xi}\Big(tn^{-\frac{1}{\alpha}}\Big)\,\chi^{\ell-1-j}_{R,n,\xi}(t)$$
$$= \int_{-\infty}^{+\infty}\left(e^{ixtn^{-\frac{1}{\alpha}}} - \sum_{w=0}^{R}\frac{(ixtn^{-\frac{1}{\alpha}})^w}{w!}\right)d\overline{\overline{H}}_{n,\xi}(x)\cdot$$
$$\cdot\sum_{j=0}^{\ell-1} g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right)\varphi^k_{\alpha,1}(t)\,\overline{\overline{h}}^{j}_{n,\xi}\Big(tn^{-\frac{1}{\alpha}}\Big)\,\chi^{\ell-1-j}_{R,n,\xi}(t).$$

Now let us differentiate $S_1(t)$ and estimate the q-order derivative. We distinguish two cases: $q = R+1$ and $q = 0, 1, ..., R$.

Case 1: $q = R+1$. For $t \neq 0$ we can apply formula (151) and get

$$\left|\frac{d^{R+1}}{dt^{R+1}} S_1(t)\right| \leq \sum_{\substack{n_1,n_2,n_3\in\mathbb{N}_0\\ n_1+n_2+n_3=R+1}} \frac{(R+1)!}{n_1!\,n_2!\,n_3!} \left|\frac{d^{n_1}}{dt^{n_1}} \int_{-\infty}^{+\infty} \left(e^{ixtn^{-\frac{1}{\alpha}}} - \sum_{w=0}^{R} \frac{(ixt)^w}{n^{w/\alpha}\,w!}\right) d\overline{\overline{H}}_{n,\xi}(x)\right|$$
$$\cdot \sum_{j=0}^{\ell-1} \left|\frac{d^{n_2}}{dt^{n_2}} \left(g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right) \varphi^k_{\alpha,1}(t)\, \overline{\overline{h}}^j_{n,\xi}\left(tn^{-\frac{1}{\alpha}}\right)\right)\right| \left|\frac{d^{n_3}}{dt^{n_3}} \chi^{\ell-1-j}_{R,n,\xi}(t)\right| = \text{Ⓢ}.$$

We consider the cases $n_1 = R+1$, $n_2 = n_3 = 0$ and $n_2 = R+1$, $n_1 = n_3 = 0$ separately. Taking into account that k, ℓ are finite and using Lemmata A.3, B.7, B.9 we continue

$$\text{Ⓢ} \leq C \sum_{\substack{n_1+n_2+n_3=R+1\\ n_1,n_2\neq R+1}} \int_{-\infty}^{+\infty} \left(|x|n^{-1/\alpha}\right)^{n_1} \underbrace{\left|e^{ixtn^{-\frac{1}{\alpha}}} - \sum_{w=0}^{R-n_1} \frac{(ixt)^w}{n^{w/\alpha}\,w!}\right|}_{\leq C\,|txn^{-1/\alpha}|^{R+1-n_1}} \left|d\overline{\overline{H}}_{n,\xi}(x)\right|$$
$$\cdot \sum_{j=0}^{\ell-1} e^{-\frac{n-\ell}{2n}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} |t|^{\alpha k-n_2} \left(c_0\,|t|\,n^{-\frac{1}{\alpha}}\right)^j n^{-n_3/\alpha} \left(|t|\,n^{-\frac{1}{\alpha}}\right)^{\ell-1-j-n_3}$$
$$+ C \int_{-\infty}^{+\infty} \left(|x|n^{-\frac{1}{\alpha}}\right)^{R+1} \left|d\overline{\overline{H}}_{n,\xi}(x)\right| \sum_{j=0}^{\ell-1} e^{-\frac{n-\ell}{2n}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} |t|^{\alpha k} \left(c_0\,|t|\,n^{-\frac{1}{\alpha}}\right)^j$$
$$\cdot \left(|t|\,n^{-\frac{1}{\alpha}}\right)^{\ell-1-j} + C \sum_{j=0}^{\ell-1} \left(|t|\,n^{-\frac{1}{\alpha}}\right)^{\ell-1-j} e^{-\frac{n-\ell}{2n}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} |t|^{\alpha k-(R+1)}$$
$$\cdot \left(\left(c_0\,|t|\,n^{-\frac{1}{\alpha}}\right)^j \int_{-\infty}^{+\infty} \underbrace{\left|e^{ixtn^{-\frac{1}{\alpha}}} - \sum_{w=0}^{R} \frac{(ixt)^w}{n^{w/\alpha}\,w!}\right|}_{\leq C\,|txn^{-1/\alpha}|^{R+1}} \left|d\overline{\overline{H}}_{n,\xi}(x)\right| + \left(n^{-\frac{1}{\alpha}}|t|\right)^{\min\{1,j-1\}}\right.$$
$$\left.\cdot \left(c_0\,|t|\,n^{-\frac{1}{\alpha}}\right)^{R+1+(j-(R+1))^+} \overline{\overline{\nu}}_{R+1,n,\xi} \int_{-\infty}^{+\infty} \underbrace{\left|e^{ixtn^{-\frac{1}{\alpha}}} - \sum_{w=0}^{R} \frac{(ixt)^w}{n^{w/\alpha}\,w!}\right|}_{\leq C\,|txn^{-1/\alpha}|^{R}} \left|d\overline{\overline{H}}_{n,\xi}(x)\right|\right),$$

where $c_0 = (\nu_0^* + 1)\gamma_r^{*\,1/r}$. Taking into account that $c_0\,|t|\,n^{-\frac{1}{\alpha}} \leq 1$ and $|t|\,n^{-\frac{1}{\alpha}} \leq 1$, and using Lemma 4.2 for the estimation of $\overline{\overline{\nu}}_{k,n,\xi}$ with $k = R, R+1$ we obtain:

$$\left|\frac{d^{R+1}}{dt^{R+1}} S_1(t)\right| \leq C\, e^{-\frac{n-\ell}{2n}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} \left(|t|\,n^{-\frac{1}{\alpha}}\right)^{\ell-1} \left[|t|^{\alpha k} n^{-\frac{R+1}{\alpha}} \overline{\overline{\nu}}_{R+1,n,\xi}\right.$$
$$\left.+ |t|^{\alpha k-1} n^{-\frac{R}{\alpha}} \overline{\overline{\nu}}_{R,n,\xi}\, \overline{\overline{\nu}}_{R+1,n,\xi} \sum_{j=0}^{\ell-1} \left(c_0\,|t|\,n^{-\frac{1}{\alpha}}\right)^{R+1+(j-(R+1))^+-j} \left(n^{-\frac{1}{\alpha}}|t|\right)^{\min\{1,j-1\}}\right]$$
$$\leq C\, e^{-\frac{n-\ell}{2n}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} n^{-\frac{R+1}{\alpha}} \overline{\overline{\nu}}_{R+1,n,\xi}\, |t|^{\alpha k} \left(|t|\,n^{-\frac{1}{\alpha}}\right)^{\ell-1}$$
$$\leq C\, e^{-\frac{n-\ell}{2n}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} n^{-\frac{r}{\alpha}} (1+\xi)^{R+1-r} |t|^{\alpha k} \left(|t|\,n^{-\frac{1}{\alpha}}\right)^{\ell-1}.$$

Case 2: $q = 0, 1, ..., R$. Acting in the same way as in Case 1 we obtain for $|t| \le \varepsilon n^{1/\alpha}$ (ε is defined in the statement of the lemma) and $t \neq 0$:

$$\left| \frac{d^q}{dt^q} S_1(t) \right| \le C\, e^{-\frac{n-\ell}{2n}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)}\, n^{-\frac{r}{\alpha}}\, (1+\xi)^{R+1-r}\, |t|^{\alpha k+R+1-q}\, \left(|t|\, n^{-\frac{1}{\alpha}}\right)^{\ell-1}.$$

Combining Case 1 and Case 2 we get for $t \neq 0$ and $q = 0, 1, ..., R+1$:

$$\left| \frac{d^q}{dt^q} S_1(t) \right| \le C\, e^{-\frac{n-\ell}{2n}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)}\, n^{-\frac{r}{\alpha}}\, (1+\xi)^{R+1-r}\, |t|^{\alpha k+R+1-q}\, \left(|t|\, n^{-\frac{1}{\alpha}}\right)^{\ell-1}. \quad (175)$$

Similarly to $d_{1n}(t)$ and $d_{2n}(t)$ (i.e. using the definition) we can show that $S_1(t)$ is $(R+1)$-times differentiable at point $t = 0$ and its derivatives are equal to 0. Thus, inequality (175) holds for all $|t| \le \varepsilon\, n^{1/\alpha}$.

Now we consider $S_2(t)$. From (147) in Lemma A.5 it follows that

$$S_2(t) = g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right) \varphi^k_{\alpha,1}(t) \sum_{u=m_{\ell,k}+1}^{\ell R} \overline{\overline{C}}_{u,\ell}\, n^{-u/\alpha}\, (it)^u$$

Note that from the fact that $|\overline{\overline{\mu}}_{k,n,\xi}| \le \overline{\overline{\nu}}_{k,n,\xi}$ and from Lemma 4.2 it follows that $|\overline{\overline{C}}_{u,\ell}| \le C$, where C is some constant not depending on n and ξ. Let us differentiate $S_2(t)$ and estimate the q-order derivative. For $|t| \le \varepsilon n^{1/\alpha}$, $t \neq 0$ we apply formulas (151), (160) and after some transformations we get

$$\begin{aligned} \left| \frac{d^q}{dt^q} S_2(t) \right| &\le \sum_{u=m_{\ell,k}+1}^{\ell R} \left|\overline{\overline{C}}_{u,\ell}\right|\, n^{-u/\alpha} \sum_{j=0}^{q} \binom{q}{j} \left| \frac{d^j}{dt^j}\left(g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right) \varphi^k_{\alpha,1}(t)\right)\right| \left|\frac{d^{q-j}}{dt^{q-j}} t^u\right| \\ &\le C\, e^{-\frac{n-\ell}{2n}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)}\, |t|^{\alpha k+m_{\ell,k}+1-q}\, n^{-\frac{m_{\ell,k}+1}{\alpha}}. \quad (176) \end{aligned}$$

Now let us differentiate $S_3(t)$. Note that $p_{u,\ell,k}+1 > \frac{R+1-u}{\alpha} + \ell - 1 - \frac{k}{2}$. Again, for $|t| \le \varepsilon n^{1/\alpha}$, $t \neq 0$ we apply formulas (151), (160) and Lemma B.4, and get

$$\begin{aligned} \left| \frac{d^q}{dt^q} S_3(t) \right| &\le \sum_{u=\ell}^{m_{\ell,k}} \left|\overline{\overline{C}}_{u,\ell}\right|\, n^{-u/\alpha} \sum_{j=0}^{q} \binom{q}{j} \left| \frac{d^j}{dt^j}\left(g_{\alpha,1}(t)\, \varphi^k_{\alpha,1}(t)\right)\right| \\ &\quad \cdot \left| \frac{d^{q-j}}{dt^{q-j}}\left((it)^u \left(e^{-\frac{\ell}{n}\varphi_{\alpha,1}(t)} - \sum_{v=0}^{p_{u,\ell,k}} \frac{(-\ell/n)^v}{v!} \varphi^v_{\alpha,1}(t)\right)\right)\right| \\ &\le C\, e^{-\frac{1}{2}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)}\, |t|^{\alpha k-q} \sum_{u=\ell}^{m_{\ell,k}} \left(|t| n^{-1/\alpha}\right)^{u+\alpha(p_{u,\ell,k}+1)} \quad (177) \\ &\le C\, e^{-\frac{1}{2}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)}\, |t|^{\alpha k-q} \left(|t| n^{-1/\alpha}\right)^{R+1+\alpha\left(\ell-1-\frac{k}{2}\right)}. \quad (178) \end{aligned}$$

Using the definition of the derivative we can show that $S_2^{(i)}(0) = 0$ and $S_3^{(i)}(0) = 0$ for $i = 1, ..., R+1$. Thus, estimations (176), (178) hold true for all $|t| \le \varepsilon\, n^{1/\alpha}$.

Since $\ell \le m_p$ and $|t| \le n^{1/\alpha}$, we have $e^{-\frac{n-\ell}{2n}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} \le C\, e^{-\frac{1}{2}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)}$. Note also that $m_{\ell,k}+1 > R+1+\alpha(\ell-1-k/2)$. Using these facts and combining (175), (176) and (178) we obtain for $q = 0, 1, ..., R+1$ and $|t| \le \varepsilon n^{1/\alpha}$:

$$\left| \frac{d^q}{dt^q} I_{k,\ell}(t) \right| \le \left| \frac{d^q}{dt^q} S_1(t) \right| + \left| \frac{d^q}{dt^q} S_2(t) \right| + \left| \frac{d^q}{dt^q} S_3(t) \right| \le C\, e^{-\frac{1}{2}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)}\, |t|^{\alpha k-q}.$$

$$\cdot\left(n^{-\frac{r}{\alpha}}(1+\xi)^{R+1-r}\,|t|^{R+1}\left(|t|\,n^{-\frac{1}{\alpha}}\right)^{\ell-1}+\left(|t|n^{-\frac{1}{\alpha}}\right)^{m_{\ell,k}+1}\right.$$
$$\left.+\left(|t|n^{-\frac{1}{\alpha}}\right)^{R+1+\alpha\left(\ell-1-\frac{k}{2}\right)}\right)\le C\,e^{-\frac{1}{2}|t|^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)}\;|t|^{\alpha k-q}$$
$$\cdot\left(n^{-\frac{r}{\alpha}}(1+\xi)^{R+1-r}\,|t|^{R+1}\left(|t|\,n^{-\frac{1}{\alpha}}\right)^{\ell-1}+\left(|t|n^{-\frac{1}{\alpha}}\right)^{R+1+\alpha\left(\ell-1-\frac{k}{2}\right)}\right). \quad (179)$$

Now we come back to the estimation of $d_{4n}^{(q)}(t)$. From (174) it follows that

$$\left|\frac{d^q}{dt^q}\,d_{4n}(t)\right|\le\left|\frac{d^q}{dt^q}\,\left(n\,I_{0,1}(t)-\overline{\overline{w}}^{*}_{n,\xi}(t)\right)\right|+\sum_{\ell=2}^{m_0}\binom{n}{\ell}\left|\frac{d^q}{dt^q}\,I_{0,\ell}(t)\right|$$
$$+\sum_{k=2}^{p}\sum_{\ell=1}^{m_k}\binom{n}{\ell}\frac{|c_{k,n-\ell}|}{n^k}\left|\frac{d^q}{dt^q}\,I_{k,\ell}(t)\right|. \quad (180)$$

Using estimation (179), the facts that $|t|n^{-1/\alpha}\le 1$, $\binom{n}{\ell}\le n^{\ell}$ and the definitions of r, R, m_0 we obtain

$$\sum_{\ell=2}^{m_0}\binom{n}{\ell}\left|\frac{d^q}{dt^q}\,I_{0,\ell}(t)\right|\le C\,e^{-\frac{1}{2}|t|^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)}\,|t|^{R+1-q}\,n^{-\frac{r}{\alpha}}\,(1+\xi)^{R+1-r}$$
$$\cdot\sum_{\ell=2}^{m_0}n^{\ell}\left(\left(|t|\,n^{-\frac{1}{\alpha}}\right)^{\ell-1}+\left(|t|n^{-\frac{1}{\alpha}}\right)^{\alpha(\ell-1)}\right)$$
$$\le C\,e^{-\frac{1}{2}|t|^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)}\,|t|^{R+1-q}\,n^{-\frac{r}{\alpha}}(1+\xi)^{R+1-r}\sum_{\ell=2}^{m_0}n^{\ell}\left(|t|n^{-\frac{1}{\alpha}}\right)^{\alpha(\ell-1)}$$
$$\le C\,e^{-\frac{1}{2}|t|^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)}\,|t|^{R+1-q}\,n^{-\frac{r-\alpha}{\alpha}}\,(1+\xi)^{R+1-r}\left(|t|^{\alpha}+|t|^{\frac{\alpha R}{1-\alpha}}\right).$$

Using estimations (165), (179), the definition of m_k and the fact that $|t|n^{-1/\alpha}\le 1$ we show in the same way as above that

$$\sum_{k=2}^{p}\sum_{\ell=1}^{m_k}\binom{n}{\ell}\frac{|c_{k,n-\ell}|}{n^k}\left|\frac{d^q}{dt^q}\,I_{k,\ell}(t)\right|\le C\,e^{-\frac{1}{2}|t|^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)}\,|t|^{R+1-q}\,n^{-\frac{r}{\alpha}}\,(1+\xi)^{R+1-r}$$
$$\cdot\sum_{k=2}^{p}\sum_{\ell=1}^{m_k}n^{\ell-\frac{k}{2}}\,|t|^{\alpha k}\left(\left(|t|\,n^{-\frac{1}{\alpha}}\right)^{\ell-1}+\left(|t|n^{-\frac{1}{\alpha}}\right)^{\alpha\left(\ell-1-\frac{k}{2}\right)}\right)$$
$$\le C\,e^{-\frac{1}{2}|t|^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)}\,|t|^{R+1-q}\,n^{-\frac{r}{\alpha}}(1+\xi)^{R+1-r}\sum_{k=2}^{p}\sum_{\ell=1}^{m_k}n^{\ell-\frac{k}{2}}\,|t|^{\alpha k}\left(|t|n^{-\frac{1}{\alpha}}\right)^{\alpha\left(\ell-1-\frac{k}{2}\right)}$$
$$\le C\,e^{-\frac{1}{2}|t|^{\alpha}\cos\left(\frac{\alpha\pi}{2}\right)}\,|t|^{R+1-q}\,n^{-\frac{r-\alpha}{\alpha}}\,(1+\xi)^{R+1-r}\left(|t|^{\alpha}+|t|^{R}+|t|^{\frac{\alpha R}{1-\alpha}}\right).$$

Now, it only remains to estimate the first term from (180). In this case we transform $I_{0,1}(t)$ a little bit differently. Note that if $k=0$, $\ell=1$, then $m_{\ell,k}=m_{1,0}=R+1$, $\overline{\overline{C}}_{R+1,1}=0$ and for $u=1,...,R$ we have

$$\overline{\overline{C}}_{u,\ell}=\overline{\overline{C}}_{u,1}=\sum_{\substack{k_1+2k_2+...+Rk_R=u\\k_1+k_2+...+k_R=1}}\frac{1}{k_1!...k_R!}\left(\frac{\overline{\overline{\mu}}_{1,n,\xi}}{1!}\right)^{k_1}\cdots\left(\frac{\overline{\overline{\mu}}_{R,n,\xi}}{R!}\right)^{k_R}=\frac{\overline{\overline{\mu}}_{u,n,\xi}}{u!},$$

which allows the following representation

$$\chi_{R,n,\xi}(t) = \sum_{u=1}^{m_{1,0}} \overline{\overline{C}}_{u,1}\, n^{-u/\alpha}\, (it)^u.$$

Using this and the fact that $g_{\alpha,1}\left(t\left(\frac{n-1}{n}\right)^{\frac{1}{\alpha}}\right) = e^{\frac{n-1}{n}\varphi_{\alpha,1}(t)} = g_{\alpha,1}(t)\, e^{-\frac{1}{n}\varphi_{\alpha,1}(t)}$ we obtain

$$\begin{aligned} I_{0,1}(t) &= g_{\alpha,1}\left(t\left(\tfrac{n-\ell}{n}\right)^{\frac{1}{\alpha}}\right)\overline{\overline{h}}_{n,\xi}\Big(tn^{-\frac{1}{\alpha}}\Big) - \sum_{u=1}^{m_{1,0}}\sum_{v=0}^{p_{u,1,0}} \overline{\overline{C}}_{u,1}\,\frac{(-1/n)^v}{v!}\, n^{-u/\alpha} g_{\alpha,1}(t)\varphi^v_{\alpha,1}(t)(it)^u \\ &= \underbrace{g_{\alpha,1}(t)\left(e^{-\frac{1}{n}\varphi_{\alpha,1}(t)} - 1\right)\left(\overline{\overline{h}}_{n,\xi}\Big(tn^{-\frac{1}{\alpha}}\Big) - \chi_{R,n,\xi}(t)\right)}_{=:V_1(t)} \\ &\quad + \underbrace{g_{\alpha,1}(t)\left(\overline{\overline{h}}_{n,\xi}\Big(tn^{-\frac{1}{\alpha}}\Big) - \chi_{R,n,\xi}(t)\right)}_{=:V_2(t)} \\ &\quad + \underbrace{g_{\alpha,1}(t)\sum_{u=1}^{m_{1,0}} \overline{\overline{C}}_{u,1}\, n^{-u/\alpha}\,(it)^u\left(e^{-\frac{1}{n}\varphi_{\alpha,1}(t)} - \sum_{v=0}^{p_{u,1,0}}\frac{(-1/n)^v}{v!}\varphi^v_{\alpha,1}(t)\right)}_{=:V_3(t)} \\ &= V_1(t) + V_2(t) + V_3(t). \end{aligned}$$

Note that $V_3(t)$ is a particular case ($k = 0$, $\ell = 1$) of $S_3(t)$ considered above. Therefore, using (177) and taking into account that $p_{u,1,0} + 1 > \frac{R+1-u}{\alpha}$ and $|t|\, n^{-1/\alpha} \le 1$ we obtain for $q = 0, 1, ..., R + 1$:

$$\begin{aligned} \left|\frac{d^q}{dt^q} V_3(t)\right| &\le C\, e^{-\frac{1}{2}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)}\, |t|^{-q} \sum_{u=1}^{m_{1,0}} \left(|t|n^{-1/\alpha}\right)^{u+\alpha(p_{u,1,0}+1)} \\ &\le C\, e^{-\frac{1}{2}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)}\, |t|^{-q} \left(|t|n^{-1/\alpha}\right)^{R+1+\theta}, \end{aligned}$$

where $\theta = \min_u\{u + \alpha(p_{u,1,0} + 1) - (R + 1)\} \in (0, \alpha]$. Acting in the same way as with $S_1(t)$ above we get for $q = 0, 1, ..., R + 1$:

$$\left|\frac{d^q}{dt^q} V_1(t)\right| \le C\, e^{-\frac{1}{2}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)}\, n^{-\frac{r+\alpha}{\alpha}}\, (1+\xi)^{R+1-r}\, |t|^{\alpha+R+1-q}.$$

Let us consider $V_2(t)$. Taking into account that $\overline{\overline{h}}_{n,\xi}\Big(tn^{-\frac{1}{\alpha}}\Big) = \int_{-\infty}^{+\infty} e^{ixtn^{-\frac{1}{\alpha}}}\, d\overline{\overline{H}}_{n,\xi}(x)$, $\overline{\overline{\mu}}_{k,n,\xi} = \int_{-\infty}^{+\infty} x^k d\overline{\overline{H}}_{n,\xi}(x)$ and $\overline{\overline{\mu}}_{0,n,\xi} = 0$ (see Lemma 4.1 (i)), we obtain

$$\begin{aligned} V_2(t) &= g_{\alpha,1}(t)\left(\overline{\overline{h}}_{n,\xi}\Big(tn^{-\frac{1}{\alpha}}\Big) - \chi_{R,n,\xi}(t)\right) \\ &= g_{\alpha,1}(t)\int_{-\infty}^{+\infty}\left(e^{ixtn^{-\frac{1}{\alpha}}} - \sum_{w=0}^{R}\frac{(ixtn^{-\frac{1}{\alpha}})^w}{w!}\right) d\overline{\overline{H}}_{n,\xi}(x) \\ &= g_{\alpha,1}(t)\int_{-\infty}^{+\infty}\left(e^{ixtn^{-\frac{1}{\alpha}}} - \sum_{w=0}^{R+1}\frac{(ixtn^{-\frac{1}{\alpha}})^w}{w!}\right) d\overline{\overline{H}}_{n,\xi}(x) + g_{\alpha,1}(t)\frac{\left(itn^{-\frac{1}{\alpha}}\right)^{R+1}}{(R+1)!}\overline{\overline{\mu}}_{R+1,n,\xi}. \end{aligned}$$

Using the last equality and the definition of $\overline{\overline{w}}^*_{n,\xi}(t)$ we get

$$nV_2(t) - \overline{\overline{w}}^*_{n,\xi}(t) = n\, g_{\alpha,1}(t) \int_{-\infty}^{+\infty} \left(e^{ixtn^{-\frac{1}{\alpha}}} - \sum_{w=0}^{R+1} \frac{(ixtn^{-\frac{1}{\alpha}})^w}{w!} \right) d\overline{\overline{H}}_{n,0}(x).$$

Using Lemmata B.2, A.3 and Lemma 4.2 (ii) for estimation $\overline{\overline{\nu}}_{R+2,n,0}$ we obtain

$$\left| \frac{d^q}{dt^q} \left(nV_2(t) - \overline{\overline{w}}^*_{n,\xi}(t) \right) \right| \leq C\, n\, e^{-\frac{1}{2}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} \left|t\right|^{R+2-q} n^{-\frac{R+2}{\alpha}}\; \overline{\overline{\nu}}_{R+2,n,0}$$
$$\leq C\, e^{-\frac{1}{2}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} n^{-\frac{r-\alpha}{\alpha}} \left|t\right|^{R+2-q}.$$

Now, using the the last estimate and the estimates for $\left|\frac{d^q}{dt^q} V_1(t)\right|$ and $\left|\frac{d^q}{dt^q} V_3(t)\right|$ found above we obtain for $q = 0, 1, ..., R+1$:

$$\left| \frac{d^q}{dt^q} \left(n\, I_{0,1}(t) - \overline{\overline{w}}^*_{n,\xi}(t) \right) \right| \leq n \left(\left| \frac{d^q}{dt^q} V_1(t) \right| + \left| \frac{d^q}{dt^q} V_3(t) \right| \right) + \left| \frac{d^q}{dt^q} \left(nV_2(t) - \overline{\overline{w}}^*_{n,\xi}(t) \right) \right|$$
$$\leq C\, e^{-\frac{1}{2}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} n^{-\frac{r-\alpha}{\alpha}} (1+\xi)^{R+1-r} \left|t\right|^{R+1-q} \left(|t|^\theta + |t| \right),$$

where $\theta = \min_u \{u + \alpha(p_{u,1,0} + 1) - (R+1)\} \in (0, \alpha]$.
Now we summarize everything for $d_{4n}(t)$ and get

$$\left| \frac{d^q}{dt^q} d_{4n}(t) \right| \leq C\, e^{-\frac{1}{2}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} n^{-\frac{r-\alpha}{\alpha}} (1+\xi)^{R+1-r} \left|t\right|^{R+1-q} \left(|t|^\theta + |t|^{R \max\left\{1, \frac{\alpha}{1-\alpha}\right\}} \right).$$

This completes the proof of the lemma. □

B.6 Proof of Lemma 4.17

Lemma 4.17. *Define* $b_{1n}(t) = \sum_{\ell=0}^{n-1} \widetilde{g}_\alpha\left(tn^{-1/\alpha}\right) \overline{\overline{f}}^{\,n-1-\ell}_{n,\xi}\left(tn^{-1/\alpha}\right) \overline{\overline{h}}^{\,\ell}_{n,\xi}\left(tn^{-1/\alpha}\right)$. *Then for* $|t| > \varepsilon_n$ *we have*

$$\left| b^{(q)}_{1n}(t) \right| \leq \begin{cases} C\, e^{-\frac{|t|^\alpha}{4n} \cos\left(\alpha\frac{\pi}{2}\right)} \left(\nu_0^{*n-1} + \overline{\overline{Q}}^{\,n-1}_{n,\xi} \right) n^{q\,(1-\frac{1}{\alpha})}, & q = 0, ..., R; \\ C\, e^{-\frac{|t|^\alpha}{4n} \cos\left(\alpha\frac{\pi}{2}\right)} \left(\nu_0^{*n-1} + \overline{\overline{Q}}^{\,n-1}_{n,\xi} \right) n^{R+1-\frac{r}{\alpha}} (1+\xi)^{R+1-r}, & q = R+1, \end{cases}$$

where $\overline{\overline{Q}}_{n,\xi} = \sup_{|t|>\widetilde{\varepsilon}} \left| \overline{\overline{f}}_{n,\xi}(t) \right|$, r *comes from Theorem 3.26,* $\varepsilon_n = \widetilde{\varepsilon} n^{1/\alpha}$ *with* $\widetilde{\varepsilon}$ *defined in* (123)*, and constants* C *do not depend on* n *and* ξ*.*

Proof. For $|t| > \varepsilon_n$ the functions $\widetilde{g}_\alpha$, $\overline{\overline{f}}^{\,n-1-\ell}_{n,\xi}$ and $\overline{\overline{h}}^{\,\ell}_{n,\xi}$ are infinitely differentiable and we can apply formula (151) from Lemma A.10:

$$\frac{d^q}{dt^q} \left(\widetilde{g}_\alpha \left(tn^{-\frac{1}{\alpha}} \right) \cdot \overline{\overline{f}}^{\,n-1-\ell}_{n,\xi} \left(tn^{-\frac{1}{\alpha}} \right) \cdot \overline{\overline{h}}^{\,\ell}_{n,\xi} \left(tn^{-\frac{1}{\alpha}} \right) \right) = \sum_{\substack{n_1,n_2,n_3 \in \mathbb{N}_0 \\ n_1+n_2+n_3=q}} \frac{q!}{n_1! n_2! n_3!}$$
$$\cdot \frac{d^{n_1}}{dt^{n_1}} \left(\widetilde{g}_\alpha \left(tn^{-\frac{1}{\alpha}} \right) \right) \frac{d^{n_2}}{dt^{n_2}} \left(\overline{\overline{f}}^{\,n-1-\ell}_{n,\xi} \left(tn^{-\frac{1}{\alpha}} \right) \right) \frac{d^{n_3}}{dt^{n_3}} \left(\overline{\overline{h}}^{\,\ell}_{n,\xi} \left(tn^{-\frac{1}{\alpha}} \right) \right).$$

Using estimations from Lemmata B.3, B.5 and B.8 we get for $\ell = 1, ..., n-2$:

$$\begin{aligned}&\left|\frac{d^q}{dt^q}\left(\widetilde{g}_\alpha\left(tn^{-\frac{1}{\alpha}}\right)\overline{\overline{f}}_{n,\xi}^{\,n-1-\ell}\left(tn^{-\frac{1}{\alpha}}\right)\overline{\overline{h}}_{n,\xi}^{\,\ell}\left(tn^{-\frac{1}{\alpha}}\right)\right)\right|\\ &\le \sum_{\substack{n_1,n_2,n_3\in\mathbb{N}_0\\ n_1+n_2+n_3=q}} \frac{q!}{n_1!n_2!n_3!}\left(C\,e^{-\frac{|t|^\alpha}{4n}\cos\left(\alpha\frac{\pi}{2}\right)}|t|^{-n_1}\right)\left(\left(\ell\,n^{-\frac{1}{\alpha}}\right)^{n_3}\ \overline{\overline{\nu}}_{n_3,n,\xi}\,\overline{\overline{\nu}}_{0,n,\xi}^{\,\ell-1}\right)\\ &\quad\cdot\left(C\,(n-1-\ell)^{n_2}\,n^{-n_2/\alpha}\max\{1,\overline{\overline{\nu}}_{n_2,n,\xi}\}\ \overline{\overline{Q}}_{n,\xi}^{\,(n-1-\ell-n_2)^+}\right) = \text{Ⓢ}.\end{aligned}$$

Distinguishing the cases $n_2 = q$ and $n_3 = q$, using Lemma 4.2 for estimation of $\overline{\overline{\nu}}_{n_i,n,\xi}$ and keeping Remark B.1 in mind we obtain

$$\begin{aligned}\text{Ⓢ} &\le C\,e^{-\frac{|t|^\alpha}{4n}\cos\left(\alpha\frac{\pi}{2}\right)}\ \overline{\overline{\nu}}_{0,n,\xi}^{\,\ell-1}\ \overline{\overline{Q}}_{n,\xi}^{\,n-1-\ell}\sum_{\substack{n_1,n_2,n_3\in\mathbb{N}_0\\ n_1+n_2+n_3=R+1}}|t|^{-n_1}\left(\left(\ell\,n^{-\frac{1}{\alpha}}\right)^{n_3}\ \overline{\overline{\nu}}_{n_3,n,\xi}\right)\\ &\quad\cdot\left(C\,(n-1-\ell)^{n_2}\,n^{-n_2/\alpha}\ \max\{1,\overline{\overline{\nu}}_{n_2,n,\xi}\}\right)\\ &\le C\,e^{-\frac{|t|^\alpha}{4n}\cos\left(\alpha\frac{\pi}{2}\right)}\,\nu_0^{*\ell-1}\ \overline{\overline{Q}}_{n,\xi}^{\,n-1-\ell}\ n^{q\,(1-\frac{1}{\alpha})}\ \max\{1,\overline{\overline{\nu}}_{q,n,\xi}\}.\end{aligned}$$

In the same way we obtain the estimations for $\ell = 0$ and $\ell = n-1$:

$$\left|\frac{d^q}{dt^q}\left(\widetilde{g}_\alpha\left(tn^{-\frac{1}{\alpha}}\right)\overline{\overline{f}}_{n,\xi}^{\,n-1}\left(tn^{-\frac{1}{\alpha}}\right)\right)\right| \le C\,e^{-\frac{|t|^\alpha}{4n}\cos\left(\alpha\frac{\pi}{2}\right)}\ \overline{\overline{Q}}_{n,\xi}^{\,n-1}\ n^{q\,(1-\frac{1}{\alpha})}\ \max\{1,\overline{\overline{\nu}}_{q,n,\xi}\},$$

and

$$\left|\frac{d^q}{dt^q}\left(\widetilde{g}_\alpha\left(tn^{-\frac{1}{\alpha}}\right)\overline{\overline{h}}_{n,\xi}^{\,n-1}\left(tn^{-\frac{1}{\alpha}}\right)\right)\right| \le C\,e^{-\frac{|t|^\alpha}{4n}\cos\left(\alpha\frac{\pi}{2}\right)}\ \nu_0^{*n-2}\,n^{q\,(1-\frac{1}{\alpha})}\ \overline{\overline{\nu}}_{q,n,\xi}.$$

Combining everything and using inequality $|a|^u\,|b|^v \le |a|^{u+v} + |b|^{u+v}$ for $a, b \in \mathbb{R}$ we get

$$\begin{aligned}\left|b_{1n}^{(q)}(t)\right| &\le C\,e^{-\frac{|t|^\alpha}{4n}\cos\left(\alpha\frac{\pi}{2}\right)}\ n^{q\,(1-\frac{1}{\alpha})}\ \max\{1,\overline{\overline{\nu}}_{q,n,\xi}\}\ \nu_0^{*-1}\sum_{\ell=0}^{n-1}\nu_0^{*\ell}\ \overline{\overline{Q}}_{n,\xi}^{\,n-1-\ell}\\ &\le C\,e^{-\frac{|t|^\alpha}{4n}\cos\left(\alpha\frac{\pi}{2}\right)}\ n^{q\,(1-\frac{1}{\alpha})}\ \max\{1,\overline{\overline{\nu}}_{q,n,\xi}\}\ \left(\nu_0^{*n-1}+\overline{\overline{Q}}_{n,\xi}^{\,n-1}\right).\end{aligned}$$

Now, distinguishing the cases $q = 0, 1, ..., R$ and $q = R+1$ and applying Lemma 4.2 for the estimation of $\overline{\overline{\nu}}_{q,n,\xi}$ we obtain

$$\left|b_{1n}^{(q)}(t)\right| \le \begin{cases} C\,e^{-\frac{|t|^\alpha}{4n}\cos\left(\alpha\frac{\pi}{2}\right)}\ \left(\nu_0^{*n-1}+\overline{\overline{Q}}_{n,\xi}^{\,n-1}\right)\ n^{q\,(1-\frac{1}{\alpha})}, & q = 0, ..., R;\\ C\,e^{-\frac{|t|^\alpha}{4n}\cos\left(\alpha\frac{\pi}{2}\right)}\ \left(\nu_0^{*n-1}+\overline{\overline{Q}}_{n,\xi}^{\,n-1}\right)\ n^{R+1-\frac{r}{\alpha}}\ (1+\xi)^{R+1-r}, & q = R+1.\end{cases}$$

This completes the proof of the lemma. □

B.7 Proof of Lemma 4.18

Lemma 4.18. *Define* $b_{2n}(t) = g_{\alpha,1}(t) + \overline{\overline{w}}_{r,n,\xi}(t)$*. Then for* $|t| > \varepsilon_n$ *we have*

$$\left|b_{2n}^{(q)}(t)\right| \le C\, e^{-\frac{1}{4}|t|^\alpha \cos\left(\alpha\frac{\pi}{2}\right)} n^{-\frac{r-\alpha}{\alpha}} (1+\xi)^{R+1-r}, \qquad q = 0,1,...,R+1,$$

where r *comes from Theorem 3.26,* $\varepsilon_n = \widetilde{\varepsilon} n^{1/\alpha}$ *with* $\widetilde{\varepsilon}$ *defined in* (123) *and constant* C *does not depend on* n *and* ξ*.*

Proof. From (121) we know that

$$\overline{\overline{w}}_{r,n,\xi}(t) = \sum_{k=2}^{\rho} \frac{c_{k,n}}{n^k} g_{\alpha,1}(t)\varphi_{\alpha,1}^k(t) + \overline{\overline{w}}_{n,\xi}^*(t)$$
$$+ \sum_{k=0}^{p}\sum_{\ell=1}^{m_k} \binom{n}{\ell} \frac{c_{k,n-\ell}}{n^k} \sum_{u=\ell}^{m_{\ell,k}} \sum_{v=0}^{p_{u,\ell,k}} g_{\alpha,1}(t)\, \varphi_{\alpha,1}^{k+v}(t)\, (it)^u \frac{(-\ell/n)^v}{v!}\, n^{-u/\alpha}\, \overline{\overline{C}}_{u,\ell},$$

where $c_{k,n}$, $\overline{\overline{C}}_{u,\ell}$, ρ, p, m_k, $m_{\ell,k}$ and $p_{u,\ell,k}$ are defined after formula (102), and

$$\overline{\overline{w}}_{n,\xi}^*(t) = \int_{-\infty}^{+\infty} e^{itx} d\overline{\overline{W}}_{n,\xi}^*(x) = n\, g_{\alpha,1}(t) \frac{(it)^{R+1}}{(R+1)!} n^{-\frac{R+1}{\alpha}} \overline{\overline{\mu}}_{R+1,n,\xi}$$
$$+ n\, g_{\alpha,1}(t) \int_{-\infty}^{+\infty} \left(e^{ixtn^{-\frac{1}{\alpha}}} - \sum_{w=0}^{R+1} \frac{(ixtn^{-\frac{1}{\alpha}})^w}{w!} \right) d\left(\overline{\overline{H}}_{n,\xi} - \overline{\overline{H}}_{n,0}\right)(x).$$

Using this representation of $\overline{\overline{w}}_{r,n,\xi}$ we obtain

$$\left|b_{2n}^{(q)}(t)\right| \le \left|g_{\alpha,1}^{(q)}(t)\right| + \sum_{k=2}^{\rho} \frac{|c_{k,n}|}{n^k} \left|\frac{d^q}{dt^q}\left(g_{\alpha,1}(t)\varphi_{\alpha,1}^k(t)\right)\right| + \left|\frac{d^q}{dt^q}\, \overline{\overline{w}}_{n,\xi}^*(t)\right|$$
$$+ \sum_{k=0}^{p}\sum_{\ell=1}^{m_k} \binom{n}{\ell} \frac{|c_{k,n-\ell}|}{n^k} \sum_{u=\ell}^{m_{\ell,k}} \sum_{v=0}^{p_{u,\ell,k}} \left|\frac{d^q}{dt^q}\left(g_{\alpha,1}(t)\varphi_{\alpha,1}^{k+v}(t)\, t^u\right)\right| \frac{(\ell/n)^v}{v!}\, n^{-u/\alpha}\, |\overline{\overline{C}}_{u,\ell}|.$$

For $\frac{d^q}{dt^q}\left(g_{\alpha,1}(t)\varphi_{\alpha,1}^{k+v}(t)\, t^u\right)$ we can apply formula (151) from Lemma A.10, since the functions $g_{\alpha,1}$, $\varphi_{\alpha,1}^{k+v}$ and t^u are infinitely differentiable for $|t| > \varepsilon_n$:

$$\frac{d^q}{dt^q}\left(g_{\alpha,1}(t)\varphi_{\alpha,1}^{k+v}(t)\, t^u\right) = \sum_{j=(q-u)^+}^{q} \binom{q}{j} \frac{d^j}{dt^j}\left(g_{\alpha,1}(t)\varphi_{\alpha,1}^{k+v}(t)\right) \frac{d^{q-j}}{dt^{q-j}} t^u.$$

Plugging the last equality into the estimate for $\left|b_{2n}^{(q)}(t)\right|$, and applying Lemma B.2 and estimate (160) from Lemma 4.13 we continue

$$\left|b_{2n}^{(q)}(t)\right| - \left|\overline{\overline{w}}^{*(q)}_{n,\xi}(t)\right| \le C\, e^{-\frac{1}{2}|t|^\alpha \cos\left(\alpha\frac{\pi}{2}\right)} |t|^{-q} + \sum_{k=2}^{\rho} \frac{|c_{k,n}|}{n^k} \left(C\, k^q\, e^{-\frac{1}{2}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} |t|^{\alpha k - q} \right)$$
$$+ \sum_{k=0}^{p}\sum_{\ell=1}^{m_k} \binom{n}{\ell} \frac{|c_{k,n-\ell}|}{n^k} \sum_{u=\ell}^{m_{\ell,k}} \sum_{v=0}^{p_{u,\ell,k}} \frac{(\ell/n)^v}{v!}\, n^{-u/\alpha}\, |\overline{\overline{C}}_{u,\ell}| \sum_{j=(q-u)^+}^{q} \binom{q}{j} C\, |t|^{u-(q-j)}$$
$$\cdot \left(C\,(k+v)^j\, e^{-\frac{1}{2}|t|^\alpha \cos\left(\frac{\alpha\pi}{2}\right)} |t|^{\alpha(k+v)-j} \right) \le C\, e^{-\frac{1}{2}|t|^\alpha \cos\left(\alpha\frac{\pi}{2}\right)} \left(|t|^{-q} + \sum_{k=2}^{\rho} \frac{|c_{k,n}|}{n^k} |t|^{\alpha k - q} \right.$$
$$\left. + \sum_{k=0}^{p}\sum_{\ell=1}^{m_k} \binom{n}{\ell} \frac{|c_{k,n-\ell}|}{n^k} \sum_{u=\ell}^{m_{\ell,k}} \sum_{v=0}^{p_{u,\ell,k}} \frac{(\ell/n)^v}{v!}\, n^{-u/\alpha}\, |\overline{\overline{C}}_{u,\ell}|\, |t|^{\alpha(k+v)+u-q} \right) = \text{Ⓢ}.$$

Now we split $e^{-\frac{1}{2}|t|^\alpha \cos\left(\alpha\frac{\pi}{2}\right)}$ into two equal parts as a product. Multiplying and dividing each term in the brackets by $|t|^{R+1}$ and using Lemma A.13 we continue

$$\text{Ⓢ} \le C\, e^{-\frac{1}{4}|t|^\alpha \cos\left(\alpha\frac{\pi}{2}\right)} |t|^{-(R+1)} \Bigg(C(q) + \sum_{k=2}^{\rho} \frac{|c_{k,n}|}{n^k} C(k,q)$$
$$+ \sum_{k=0}^{p}\sum_{\ell=1}^{m_k} \binom{n}{\ell} \frac{|c_{k,n-\ell}|}{n^k} \sum_{u=\ell}^{m_{\ell,k}} \sum_{v=0}^{p_{u,\ell,k}} \frac{(\ell/n)^v}{v!}\, n^{-u/\alpha}\, |\overline{\overline{C}}_{u,\ell}|\, C(k,v,u,q) \Bigg).$$

Recall that for $c_{k,n-\ell}$ with $k = 0, ..., p$ and $\ell = 1, ..., n-1$, $n \in \mathbb{N}$ we have the following estimate (see (165)):

$$\frac{|c_{k,n-\ell}|}{n^k} \le \frac{c^*_{k,n-\ell}}{n^k} = \sum_{\substack{k_0+k_2+\cdots+k_s=n-\ell \\ 2k_2+\cdots+sk_s=k}} \frac{(n-\ell)!}{k_0!k_2!\cdots k_s!\, n^k}\, |A_2|^{k_2} \cdots |A_s|^{k_s} \le \frac{C}{n^{k/2}}.$$

In the same way we obtain that $|c_{k,n}|/n^k \le c^*_{k,n}/n^k \le C\, n^{-k/2}$ for $k = 2, ..., \rho$ and $n \in \mathbb{N}$. Using the fact that $|t| > \varepsilon\, n^{1/\alpha}$, definition (104) of $\overline{\overline{C}}_{u,\ell}$ together with Lemmata 3.17 and 4.2 (i) we obtain

$$\left|b^{(q)}_{2n}(t)\right| - \left|\frac{d^q}{dt^q}\overline{\overline{w}}^*_{n,\xi}(t)\right| \le C\, e^{-\frac{1}{4}|t|^\alpha \cos\left(\alpha\frac{\pi}{2}\right)} n^{-\frac{R+1}{\alpha}} \left(1 + \sum_{k=0}^{p}\sum_{\ell=1}^{m_k} n^\ell\, n^{-k/2} \sum_{u=\ell}^{m_{\ell,k}} n^{-u/\alpha}\right).$$

Let us consider the sum in the brackets from the expression above. Taking into account that $\alpha \in (0,1)$ we obtain

$$\sum_{k=0}^{p}\sum_{\ell=1}^{m_k} n^\ell\, n^{-k/2} \sum_{u=\ell}^{m_{\ell,k}} n^{-u/\alpha} \le C \sum_{k=0}^{p}\sum_{\ell=1}^{m_k} n^\ell\, n^{-k/2}\, n^{-\ell/\alpha} = C \sum_{k=0}^{p}\sum_{\ell=1}^{m_k} n^{\ell\left(1-\frac{1}{\alpha}\right)}\, n^{-k/2} < C.$$

Using the the last estimate we get

$$\left|b^{(q)}_{2n}(t)\right| - \left|\frac{d^q}{dt^q}\overline{\overline{w}}^*_{n,\xi}(t)\right| \le C\, e^{-\frac{1}{4}|t|^\alpha \cos\left(\alpha\frac{\pi}{2}\right)} n^{-\frac{R+1}{\alpha}}.$$

Let us estimate $\left|\frac{d^q}{dt^q}\overline{\overline{w}}^*_{n,\xi}(t)\right|$. Recall that $\overline{\overline{M}}_{n,\xi}(x) = \overline{\overline{H}}_{n,\xi}(x) - \overline{\overline{H}}_{n,0}(x)$. Using this fact, Lemma B.2 and inequalities (97), (100) we obtain for $q = 0, 1, ..., R+1$:

$$\left|\frac{d^q}{dt^q}\overline{\overline{w}}^*_{n,\xi}(t)\right| \le n^{-\frac{R+1-\alpha}{\alpha}} \frac{|\overline{\overline{\mu}}_{R+1,n,\xi}|}{(R+1)!} \sum_{j=0}^{q} \binom{q}{j} \left|\frac{d^j}{dt^j} g_{\alpha,1}(t)\right| \left|\frac{d^{q-j}}{dt^{q-j}} t^{R+1}\right|$$
$$+ n \sum_{j=0}^{q} \binom{q}{j} \left|\frac{d^{q-j}}{dt^{q-j}} g_{\alpha,1}(t)\right| \left|\int_{-\infty}^{+\infty} \left|\frac{d^j}{dt^j}\left(e^{ixtn^{-\frac{1}{\alpha}}} - \sum_{w=0}^{R+1}\frac{(ixtn^{-\frac{1}{\alpha}})^w}{w!}\right)\right| \left|d\overline{\overline{M}}_{n,\xi}(x)\right|\right.$$
$$\le C\, e^{-\frac{1}{2}|t|^\alpha \cos\left(\alpha\frac{\pi}{2}\right)} |t|^{R+1-q}\, n^{-\frac{R+1-\alpha}{\alpha}} \left(\overline{\overline{\nu}}_{R+1,n,\xi} + \nu_{R+1}\left(\overline{\overline{M}}_{n,\xi}\right)\right)$$
$$\le C\, e^{-\frac{1}{2}|t|^\alpha \cos\left(\alpha\frac{\pi}{2}\right)} |t|^{R+1-q}\, n^{-\frac{r-\alpha}{\alpha}} (1+\xi)^{R+1-r}$$
$$\le C\, e^{-\frac{1}{4}|t|^\alpha \cos\left(\alpha\frac{\pi}{2}\right)} n^{-\frac{r-\alpha}{\alpha}} (1+\xi)^{R+1-r}.$$

Combining the last two estimates above we get

$$\left|b^{(q)}_{2n}(t)\right| \le C\, e^{-\frac{1}{4}|t|^\alpha \cos\left(\alpha\frac{\pi}{2}\right)} n^{-\frac{r-\alpha}{\alpha}} (1+\xi)^{R+1-r},$$

which completes the proof of the lemma. □

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