Adelic Convex Geometry of Numbers

Dissertation
zur Erlangung des akademischen Grades

doctor rerum naturalium
(Dr. rer. nat.)

von Dipl.-Math. Carsten Thiel
geb. am 19. 08. 1982 in Marburg

genehmigt durch die Fakultät für Mathematik
der Otto-von-Guericke-Universität Magdeburg

Gutachter: Prof. Dr. Martin Henk
Prof. Lenny Fukshansky, Ph.D.

eingereicht am: 27. 11. 2013
Verteidigung am: 28. 02. 2014

Zusammenfassung

Wir untersuchen Fragestellungen zu Gitterpunkten und konvexen Körpern in *Adelischer Geometrie der Zahlen* (bezüglich Zahlkörpern). Diese Theorie wurde von Bombieri und Vaaler sowie unabhängig von McFeat für Verallgemeinerungen von Siegels Lemma etabliert. Letzterer Begriff wird gewöhnlich für Ergebnisse verwendet, die die Norm von nicht-trivialen ganzzahligen Lösungen eines Gleichungssystems mit ganzen Koeffizienten beschränken. Geometrisch gesprochen fragt das Lemma nach kleinen Dilatationen konvexer Körper, die nicht-triviale Gitterpunkte enthalten.

Im ersten Teil präsentieren wir Schranken für die sukzessiven Minima eines 0-symmetrischen Konvexkörpers unter der Einschränkung, dass die Gitterpunkte, die die sukzessiven Minima realisieren, nicht in einer Sammlung von verbotenen Untergittern enthalten sind. Dies kann als eine inverse Variante von Siegels Lemma aufgefasst werden. Unsere Untersuchungen ergänzen frühere Resultate von Fukshansky und Gaudron und erweitern Arbeiten zu verbotenen niederdimensionalen Gittern auf den volldimensionalen Fall sowie auf alle sukzessiven Minima.

Die weiteren Teile dieser Arbeit behandeln vorwiegend Probleme in Adelischer Geometrie der Zahlen. Wir beginnen mit einem Kapitel, das den Begriff der *konvexen Hülle* im adelischen Raum einführt und sich dann adelischen Gitterpunktproblemen zuwendet. Hier verallgemeinern wir klassische Zählschranken von van der Corput und Blichfeldt, aber auch neuere Resultate von Betke et al. sowie von Henze.

Im Anschluss wenden wir uns dem Problem der adelischen Polarität zu und verallgemeinern klassische Übertragungsprinzipien aus der Geometrie der Zahlen auf die adelische Situation. Das zentrale Resultat ist hier eine adelische Version des klassischen Mahlerprodukts der sukzessiven Minima. Dazu führen wir einen mehr geometrisch motivierten Begriff eines adelisches polaren Körpers ein, im Vergleich zu einem eher algebraischen Ansatz von Burger und Rothlisberger in diesen Kontext.

Das letzte Kapitel behandelt adelische Verallgemeinerungen der Überdeckungsminima, erstmals von Kannan und Lovász definiert. Weiterhin verallgemeinern wir unsere Resultate zu eingeschränkten sukzessiven Minima aus dem ersten Teil auf den adelischen Fall. Die bewiesenen Schranken zu eingeschränkten sukzessiven Minima ergänzen auch auf diesem Gebiet frühere Arbeiten von Fukshansky und Gaudron.

Abstract

We study problems concerning lattice points and convex bodies in *Adelic Geometry of Numbers* (with respect to number fields). This theory was established by Bombieri and Vaaler and independently by McFeat to generalise Siegel's Lemma. The latter term is commonly applied to results on bounds on the norm of non-trivial integral solutions to a system of linear equations with integral coefficients. In geometric terms, it asks for small dilations of convex bodies containing non-trivial lattice points.

In the first part, we present bounds on the successive minima of a 0-symmetric convex body under the restriction that the lattice points realising the successive minima are not contained in a collection of forbidden sublattices. This can be regarded as an inverse variant of Siegel's Lemma. Our investigations complement former results by Fukshansky and Gaudron. Furthermore, we extend previous work on forbidden lower-dimensional lattices to the full-dimensional case and to all successive minima.

The remaining parts of this thesis deal with problems in Adelic Geometry of Numbers. We start with a chapter introducing the notion of *convex hull* into adele space and focusing on adelic lattice point problems, generalising classical counting estimates by van der Corput and Blichfeldt and also more recent results by Betke et al. and by Henze.

Subsequently we turn to the problem of adelic polarity and generalise classical transference theorems from the Geometry of Numbers to the adelic setting. The central result here is an adelic version of the classical Mahler product of successive minima. To this end, we introduce a more geometrically motivated notion of polarity for adelic convex bodies, compared to the rather algebraic approach considered by Burger and Rothlisberger in this context.

The last chapter deals with adelic generalisations of the covering minima, first introduced by Kannan and Lovász. Moreover, we generalise our results on restricted successive minima from the first part to the adelic case, proving bounds on restricted adelic successive minima which complement also in this setting previous work by Fukshansky and Gaudron.

Contents

Introduction

The study of *Adelic Convex Geometry of Numbers* was originally introduced by Bombieri and Vaaler [BV83] and independently by McFeat [McF71]. The theory was established to study generalisations of Siegel's Lemma.

The term *Siegel's Lemma* is generally applied to a collection of results dealing with the question of bounding the norm of a non-trivial solution to a system of linear equations. Since Siegel [Sie29] was the first to formally state it in the following way, it is usually associated with his name. Given a system of linear equations $A\mathbf{x} = 0$ with $A \in \mathbb{Z}^{r\times m}$, $\operatorname{rank}(A) = r < m$, there exists a non-trivial integral solution $x \in \ker(A) \cap \mathbb{Z}^m \setminus \{0\}$ with

$$\|x\|_\infty \leq \left\lfloor (m\|A\|_\infty)^{r/(m-r)} \right\rfloor ,$$

where $\|\cdot\|_\infty$ denotes the maximum norm, i.e. the maximal absolute value of the entries of the argument. Notice that $\ker(A)$ is an $(m-r)$-dimensional subspace of $\mathbb{R}^m$ and thus Siegel's Lemma can be seen as asking for the smallest edge-length of a cube in $\mathbb{R}^m$, that contains a lattice point from a certain subspace.

The statement can be proven using the Dirichlet box principle or using elementary tools from the Geometry of Numbers. The basic idea was first studied by Thue [Thu09].

Siegel's Lemma has been generalised and applied in various ways. One is to ask for a full basis of solutions, see e.g. Aliev [Ali08] and Faltings [Fal91]. Another possibility is to allow the coefficients and solutions of the equations to come from more general domains, e.g. the ring of integers of an algebraic number field K.

Since the elementary methods for the proof of Siegel's Lemma fail in a domain of the latter kind, the methods from Geometry of Numbers need to be applied. To this end Bombieri and Vaaler and McFeat introduced the notion of *Adelic Geometry* over the ring $K_{\mathbb{A}}$ of adeles of K. In this framework, the field K embeds into $K_{\mathbb{A}}$ as a discrete submodule and thus plays the role of the lattice. Given a suitable notion of an adelic convex body C, introduced in Definition 1.2.7, we can ask for small dilations of C that contain field elements, similar to the cube of smallest edge-length mentioned above.

In recent years a new kind of reverse version of Siegel's Lemma has been studied. This line of research was initiated by Lenny Fukshansky [Fuk06a], and it asks for lattice points of small norm that are not included in a union of proper sublattices, the *restrictions*.

The central results of our work focus on both Euclidean and Adelic Geometry. We will introduce both settings in Chapter 1. Chapter 2 is then set in the Euclidean space, while Chapters 3–5 contain work in Adelic Geometry, ultimately also generalising our own results of Chapter 2. We now provide an overview of the contents and structure of this thesis.

Throughout the thesis, we will generalise both classical and also more recent results from the Geometry of Numbers in Euclidean space. We will also extend them to the Geometry of Numbers over the ring of adeles of an algebraic number field. While in both situations we are dealing with compact convex sets and discrete subspaces or lattices, the two cases are fundamentally different. In the classical setting we work in a finite-dimensional vector space over the field of real numbers $\mathbb{R}$, the second is merely a module over a ring.

One of the core notions in classical Euclidean Geometry of Numbers is that of the i-th successive minimum $\lambda_i(C,\Lambda)$ for $1 \le i \le m$ of a convex body C with respect to a lattice Λ. It is the smallest dilation factor $\lambda > 0$, such that $\lambda \cdot C$ contains i linearly independent lattice points, i.e.

$$\lambda_i(C,\Lambda) := \inf\left\{ \lambda > 0 \,\middle|\, \dim_{\mathbb{R}}(\lambda C \cap \Lambda) \ge i \right\}.$$

Siegel's Lemma, in its formulation above, can thus be read as bounding $\lambda_1(\mathrm{C}_m,\Lambda)$ for the cube $\mathrm{C}_m = [-1,1]^m$ and the lattice $\Lambda = \ker(A) \cap \mathbb{Z}^m$. Fukshansky's problem of restrictions can then be expressed similarly. Given a convex body C and a lattice Λ as well as some restrictions $\Lambda_1,\ldots,\Lambda_s$, we ask for a small dilation of C, that contains a point of Λ, which is not contained in any of the forbidden lattices $\Lambda_1,\ldots,\Lambda_s$.

This is the first topic we consider. Its discussion is the content of Chapter 2, which constitutes joint work with Martin Henk and has been published previously as [HT13]. The main result of Section 2.1, complementing previous work by Fukshansky and Gaudron, then is the following.

Theorem 2.1.1. *Let $C \subset \mathbb{R}^m$ be a convex body and $\Lambda \subset \mathbb{R}^m$ a lattice of rank m and $m \geq 2$. Let further $\Lambda_1, \dots, \Lambda_s \subset \Lambda$ be a non-trivial collection of sublattices with* $\operatorname{rank}\Lambda_j \leq m-1$, $1 \leq j \leq s$. *Then*

$$\lambda_1\Big(C, \Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\Big) < 6^{m-1} \frac{\det \Lambda}{\lambda_1(C,\Lambda)^{m-2}\operatorname{vol}(C)} \left(\sum_{j=1}^{s} \frac{1}{\lambda_1(C,\Lambda_j)}\right) + \sqrt[m]{2^m \frac{\det \Lambda}{\operatorname{vol}(C)}}.$$

Supplementing Fukshansky's original problem, we also consider the case of full-dimensional restrictions. Our main result for this case is the following.

Theorem 2.2.5. *Let $C \subset \mathbb{R}^m$ be a convex body and $\Lambda \subset \mathbb{R}^m$ a lattice of rank m and $m \geq 2$. Let further $\Lambda_1, \dots, \Lambda_s \subset \Lambda$ be a collection of sublattices with* $\operatorname{rank}\Lambda_j = m$, $1 \leq j \leq s$, *such that* $\bigcup_{j=1}^{s} \Lambda_j \neq \Lambda$. *Then*

$$\lambda_1\Big(C, \Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\Big) < \frac{2^m \det \Lambda}{\lambda_1(C,\overline{\Lambda})^{m-1}\operatorname{vol}(C)} \left(\sum_{j=1}^{s} \frac{\det \overline{\Lambda}}{\det \Lambda_j} - s + 1\right) + \lambda_1(C,\overline{\Lambda}),$$

where $\overline{\Lambda} = \bigcap_{j=1}^{s} \Lambda_j$.

In both cases, we also provide extensions of the results for several linearly independent points of Λ outside $\Lambda_1, \dots, \Lambda_s$.

When dealing with a convex body C and a lattice Λ, it is a natural question to consider the number

$$|C \cap \Lambda|$$

of lattice points contained in C and to give upper and lower bounds on it, depending on the body and lattice. Classical results are in terms of the volume of C, the determinant of Λ or their successive minima. In fact, these bounds play a central role in the arguments employed in Chapter 2 and Theorem 2.1.1 in particular.

In Chapter 3, we consider this problem in the adelic setting, i.e. the standard module $K_{\mathbb{A}}^n$ of rank n over the adeles $K_{\mathbb{A}}$ for a number field K of degree d and discriminant Δ_K. Here, the discrete subset K^n will play the role of the lattice and an adelic notion of a convex body C, following Bombieri and Vaaler, is introduced. After some preliminary results on the adelic convex hull and adelic polytopes in Section 3.1, we provide bounds on $|C \cap K^n|$ in analogy to classical results by Blichfeldt and van der Corput. The first result is a lower bound on this number.

Theorem 3.2.1. *Let $n \geq 2$ and C be a symmetric adelic convex body, then*

$$\left|C \cap K^n\right| > \frac{\mathrm{vol}_{\mathbb{A}}(C)}{2^{nd-1}\left(\sqrt{|\Delta_K|}\right)^n} - 1 .$$

We also provide two upper bounds. The first one is valid for arbitrary bodies, dropping the requirement of symmetry, but with a restriction on the field K.

Theorem 3.3.1. *Let K be a totally real number field of degree $d = [K : \mathbb{Q}]$. Let C be an adelic convex body with $\dim_K(C \cap K^n) = n$. Then*

$$\left|C \cap K^n\right| \leq (n!)^d \,\mathrm{vol}_{\mathbb{A}}(C) + n .$$

We also give a symmetric variant for arbitrary K, using a recent result by Henze [Hen13].

Theorem 3.3.6. *Let K be an algebraic number field of degree d and let C be a symmetric adelic convex body with $\dim_{\mathbb{Q}}(C \cap K^n) = nd$. Then*

$$\left|C \cap K^n\right| \leq \frac{(nd)!}{2^{nd}} L_{nd}(2) \frac{\mathrm{vol}_{\mathbb{A}}(C)}{\left(\sqrt{|\Delta_K|}\right)^n} ,$$

where L_{nd} is the (nd)-th Laguerre polynomial, $L_{nd}(x) = \sum_{k=0}^{nd} \binom{nd}{k} \frac{x^k}{k!}$.

Classically, to a convex body $C \subset \mathbb{R}^m$ and a lattice $\Lambda \subset \mathbb{R}^m$, one can associate the polar body and the polar lattice

$$C^\star := \left\{ x \in \mathbb{R}^m \,\middle|\, \langle x, y \rangle \leq 1 \;\forall\, y \in C \right\} \quad \text{and} \quad \Lambda^\star := \left\{ x \in \mathbb{R}^m \,\middle|\, \langle x, y \rangle \in \mathbb{Z} \;\forall\, y \in \Lambda \right\},$$

where $\langle \cdot , \cdot \rangle$ denotes the standard scalar product on $\mathbb{R}^m$.

In Chapter 4, we introduce a notion of adelic polarity, that allows us to prove an adelic version of the Mahler inequality

$$1 \leq \lambda_i(C, \Lambda)\lambda_{m-i+1}(C^\star, \Lambda^\star) \leq m^{3/2} , \quad 1 \leq i \leq m .$$

We provide the following adelic version for both the upper and lower bound.

Theorem 4.2.1. *Let C be an adelic convex body and $C^\star$ its polar. Let further $\lambda_i(C)$ and $\lambda_j(C^\star)$ $(1 \leq i, j \leq n)$ be the successive minima of C and $C^\star$, respectively. Then for $1 \leq \ell \leq n$*

$$\lambda_\ell(C)\lambda_{n-\ell+1}(C^\star) \leq (nd)^{3/2} .$$

Theorem 4.2.3. *Let K be totally real or a CM-field and let C be an adelic convex body which is c-symmetric. Let $C^\star$ be its polar and let $\lambda_i(C)$, $\lambda_j(C^\star)$ $(1 \le i, j \le n)$ be the successive minima of C and $C^\star$, respectively. Then for $1 \le \ell \le n$*

$$\frac{1}{\sqrt[d]{|\Delta_K|}} \le \lambda_\ell(C)\lambda_{n-\ell+1}(C^\star)\,.$$

While the above theorems have been previously published in [Thi12], we conclude Chapter 4 with a new section containing some further transference results.

The final chapter consists of two parts. In Section 5.1 we deal with an adelic version of the covering minima $\mu_i(C,\Lambda)$, $1 \le i \le m$, which were first introduced by Kannan and Lovász [KL88] and extend the classical covering radius

$$\mu(C,\Lambda) = \inf\{\mu > 0 \,|\, \mathbb{R}^m \subseteq (\mu C + \Lambda)\}\,.$$

Here we prove the following adelic analogue to their classical result.

Theorem 5.1.2. *Let C be an adelic convex body. Then for each j, $1 \le j < n$,*

$$\mu_{j+1}(C) \le \mu_j(C) + \nu(K)\lambda_{n-j}(C)\,,$$

where $\nu(K)$ is a constant depending on the field K.

Finally, in Section 5.2, we generalise the results of Chapter 2 to the adelic setting, employing the counting estimates of Chapter 3, in particular Theorem 3.2.1. The central result is the following generalisation of Theorem 2.1.1, complementing previous results by Fukshansky [Fuk06b] and Gaudron [Gau09].

Theorem 5.2.2. *Let C be an adelic convex body and $L_1,\ldots,L_s \subset K^n$ linear subspaces with $n_j = \dim_K L_j < n$. Then*

$$\lambda_1\Big(C, K^n \setminus \bigcup_{j=1}^{s} L_j\Big) \le \frac{6^{nd-1}\big(\sqrt{|\Delta_K|}\big)^n}{3^{d-1}\,\lambda_1(C)^{nd-2}\,\mathrm{vol}_{\mathbb{A}}(C)}\left(\sum_{j=1}^{s}\frac{1}{\lambda_1(C,L_j)}\right) + 2\,\frac{\big(\sqrt{|\Delta_K|}\big)^{1/d}}{\sqrt[nd]{\mathrm{vol}_{\mathbb{A}}(C)}}\,.$$

Similar to the classical case of Chapter 2, we extend this result to several linearly independent lattice points and discuss the question of full-dimensional restrictions.

Acknowledgements

First of all, I wish to express my gratitude to my advisor Martin Henk. Only his encouragement, support and inspiration made this thesis possible. Moreover, I thank Lenny Fukshansky for co-refereeing this thesis.

I am also very thankful to my colleagues Matthias Henze, Eva Linke and Eugenia Saorín Gómez for all I learned from them, as well as to Gohar Kyureghyan, Anton Malevich, Frank Pfeuffer, Kai-Uwe Schmidt and Yue Zhou for a pleasant time and atmosphere throughout the past four years. Particular thanks go to Matthias and Frank for their extensive proof-reading of this work.

Furthermore, I thank Lenny Fukshansky, Benjamin Klopsch, Jörg Jahnel and Ulrich Stuhler for several helpful discussions on topics related to this thesis.

I am also deeply indebted to my friends Wilfried Keller, Ulrich Pennig, Daniel Rettstadt, Henrik Schumacher, Kirstin Strokorb and Kristin Stroth for their continuous support.

Last but not least I thank my parents Karola and Stefan and my sister Laura, as well as Kristin for simply everything.

1

Fundamentals from Geometry

1.1 Geometry of Numbers in Euclidean Space

The classical geometry of numbers is set in the m-dimensional Euclidean space $\mathbb{R}^m$. In order to keep the thesis self-contained, we start with an introductory overview of this theory. For a more in-depth discussion and proofs we refer to [Gru07], especially Sections 21–23. We will provide further references as needed.

Throughout this section, we state classical results which we later generalise to the adelic setting as theorems, for easier reference and comparison. We do however usually omit their proofs.

For a set $A \subseteq \mathbb{R}^m$ we define the *linear hull* and *affine hull* of A as

$$\begin{aligned}\operatorname{lin}(A) = \operatorname{lin}_{\mathbb{R}}(A) &= \left\{ \sum_{i=1}^{r} \lambda_i a_i \;\middle|\; \lambda_i \in \mathbb{R}, a_i \in A, r \in \mathbb{N} \right\} \\ \operatorname{aff}_{\mathbb{R}}(A) &= \left\{ \sum_{i=1}^{r} \lambda_i a_i \;\middle|\; \lambda_i \in \mathbb{R}, \sum_{i=1}^{r} \lambda_i = 1, a_i \in A, r \in \mathbb{N} \right\}\end{aligned}$$

and the *convex hull* of A as

$$\operatorname{conv}(A) = \left\{ \sum_{i=1}^{r} \lambda_i a_i \;\middle|\; \lambda_i \in [0,1], \sum_{i=1}^{r} \lambda_i = 1, a_i \in A, r \in \mathbb{N} \right\}.$$

We further call $\dim_{\mathbb{R}}(A) = \dim_{\mathbb{R}}(\operatorname{aff}_{\mathbb{R}}(A))$ the dimension of A. In the same way $\operatorname{lin}_{\mathbb{Z}}(A)$ consists of all linear combinations of elements of A with integral coefficients.

By a convex body $C \subset \mathbb{R}^m$ we mean a compact and convex set with non-empty interior. Standard examples are the unit ball B_m and the unit cube C_m,

$$\mathrm{B}_m = \left\{ x \in \mathbb{R}^m \;\middle|\; x_1^2 + \ldots + x_m^2 \le 1 \right\} \quad \text{and} \quad \mathrm{C}_m = \left\{ x \in \mathbb{R}^m \;\middle|\; |x_1|, \ldots, |x_m| \le 1 \right\}.$$

For a body C, we denote by $-C = \{-x \mid x \in C\}$ its reflection in the origin, and C is called 0-symmetric if $C = -C$. The family of all convex bodies will be denoted by $\mathscr{K}^m$ and $\mathscr{K}_0^m$ are the 0-symmetric bodies. An important subclass is that of polytopes, the convex hull of a finite number of points. The cube C_m is one of three prominent examples. The second one is the simplex, which is the convex hull of $m+1$ points in general position, i.e. $m+1$ points whose affine hull is the whole space. Finally, a body

$$\operatorname{conv}\{\pm a_1,\ldots,\pm a_m\} \quad \text{for linearly independent } a_i \in \mathbb{R}^m$$

is known as a generalised cross-polytope. For a convex body $C \in \mathscr{K}^r$, we denote by $\operatorname{vol}_r(C)$ its r-dimensional volume, i.e. its Lebesgue measure. If the dimension of C is clear from the context or equals m, we usually just write $\operatorname{vol}(C)$.

A lattice $\Lambda \subset \mathbb{R}^m$ is a free $\mathbb{Z}$-submodule of $\mathbb{R}^m$, i.e. given linearly independent $b_1,\ldots,b_r \in \mathbb{R}^m$, the set $\operatorname{lin}_{\mathbb{Z}}\{b_1,\ldots,b_r\}$ forms a lattice of rank $r = \operatorname{rank}\Lambda$ and every lattice can be written in this way. Therefore $\Lambda = B\mathbb{Z}^m$ for the matrix $B \in \mathbb{R}^{r\times m}$ with columns $b_1,\ldots,b_r$. For a matrix M, we denote by $M^\intercal$ its transpose and by $M^{-\intercal}$ its inverse transpose, if it exists. Denote by $\det\Lambda = \sqrt{\det(B^\intercal B)}$ the determinant of Λ. Notice that for full-dimensional Λ, i.e. $\operatorname{rank}\Lambda = m$, $B \in \mathrm{GL}_m(\mathbb{R})$ and $\det\Lambda = |\det B|$. The family of all lattices of rank at most m is denoted by $\mathscr{L}^m$. We call a subspace $W \subset \mathbb{R}^m$ a Λ-*lattice plane* for $\Lambda \in \mathscr{L}^m$ if $W = \operatorname{lin}(W \cap \Lambda)$.

If $(b_1,\ldots,b_r)$ with $b_j \in \mathbb{R}^m$ is a basis of Λ, then

$$P_\Lambda = \left\{ \sum_{i=1}^{r} \alpha_i b_i \,\middle|\, 0 \le \alpha_i \le 1 \right\}$$

is a polytope, the *fundamental cell* of Λ, with $\operatorname{vol}(P_\Lambda) = \det\Lambda$. Let further

$$\mathrm{Gr}(\Lambda_B) = \left(\det B_j \,\middle|\, B_j \text{ a } r\times r \text{ submatrix of } B = (b_1,\ldots,b_r)\right) \in \mathbb{R}^{\binom{m}{r}} \tag{1.1}$$

be the vector with entries $\det B_j$. Then up to the order and signs of the coordinates the vector is independent of the given basis, as for two bases B_1 and B_2 of $\Lambda = B_1\mathbb{Z}^r = B_2\mathbb{Z}^r$, there exists a change-of-basis matrix $A \in \mathrm{GL}_m(\mathbb{Z})$ with $B_2 = B_1 A$ and thus B_1 and $B_1 A$ have the same subdeterminants. By the Cauchy-Binet formula,

$$\det(\Lambda) = \|\mathrm{Gr}(\Lambda_B)\|_2 = \left(\left(\det B_1\right)^2 + \ldots + \left(\det B_{\binom{m}{r}}\right)^2\right)^{1/2}.$$

Finally, if $\Lambda \subset \mathbb{Z}^m$ is a lattice of rank m, then

$$\det(\Lambda) = \left[\,\mathbb{Z}^m : \Lambda\,\right], \tag{1.2}$$

the index of Λ in $\mathbb{Z}^m$ as an abelian subgroup.

Given a convex body C and a lattice Λ in $\mathbb{R}^m$ of rank m, the *i-th successive minimum* $\lambda_i(C,\Lambda)$ for $1 \le i \le m$ of C with respect to Λ is defined as the smallest dilation factor, such that $\lambda_i(C,\Lambda)\cdot C$ contains i linearly independent lattice points, i.e.

$$\lambda_i(C,\Lambda) := \inf\big\{\lambda > 0 \,\big|\, \dim_{\mathbb{R}}(\lambda C \cap \Lambda) \ge i\big\}.$$

Since C is compact and Λ discrete, the infimum is in fact a minimum. Further $\lambda_i(C,\Lambda) \le \lambda_j(C,\Lambda)$ for $i < j$, and for $C' \subseteq C$ and $\Lambda' \subseteq \Lambda$ we obviously have $\lambda_i(C,\Lambda) \le \lambda_i(C',\Lambda)$ and $\lambda_i(C,\Lambda) \le \lambda_i(C,\Lambda')$ for all i. Finally, they behave nicely with respect to dilations, as for $\gamma > 0$ we have

$$\lambda_i(\gamma C,\Lambda) = \lambda_i\big(C, \tfrac{1}{\gamma}\Lambda\big) = \tfrac{1}{\gamma}\lambda_i(C,\Lambda).$$

Minkowski's First Fundamental Theorem on successive minima establishes an upper bound on the first successive minimum in terms of the volume of a convex body. More precisely, for $C \in \mathscr{K}_0^m$ and $\Lambda \in \mathscr{L}^m$ with $\operatorname{rank}\Lambda = r$ it may be formulated as

$$\lambda_1(C,\Lambda)^r \operatorname{vol}_r(C \cap \operatorname{lin}\Lambda) \le 2^r \det\Lambda, \tag{1.3}$$

establishing that a convex body of sufficiently large volume, i.e. greater than $2^r \det\Lambda$, always contains a non-trivial lattice point.

Apart from its importance for the Geometry of Numbers, the theorem also has applications in classical and algebraic number theory. It can be used to prove, amongst others, Lagrange's four squares theorem and Minkowski's bound on the discriminant of an algebraic number field.

The theorem can be refined to state Minkowski's Second Fundamental Theorem on successive minima. For $C \in \mathscr{K}_0^m$ and $\Lambda \in \mathscr{L}^m$ with $\operatorname{rank}\Lambda = r$

$$\frac{2^r \det(\Lambda)}{r!} \le \lambda_1(C,\Lambda)\cdots\lambda_r(C,\Lambda)\cdot\operatorname{vol}_r(C \cap \operatorname{lin}\Lambda) \le 2^r \det(\Lambda), \tag{1.4}$$

directly implying the first theorem on account of the monotonicity of the successive minima. This result and in particular the upper bound is of high importance in the Geometry of Numbers, being used for many standard results and also several of our proofs later on.

Denoting by $\langle\cdot,\cdot\rangle$ the standard scalar product on $\mathbb{R}^m$, we can associate a convex body $C \in \mathscr{K}^m$ with its polar body

$$C^\star := \big\{x \in \mathbb{R}^m \,\big|\, \langle x,y\rangle \le 1 \ \forall\, y \in C\big\} \tag{1.5}$$

and a lattice $\Lambda \in \mathscr{L}^m$ of rank m with its polar lattice

$$\Lambda^\star := \left\{ x \in \mathbb{R}^m \,\middle|\, \langle x, y\rangle \in \mathbb{Z} \ \forall\, y \in \Lambda \right\}. \tag{1.6}$$

In this context, we call C and Λ the primal body and lattice, respectively. We have $\left(\mathbb{Z}^m\right)^\star = \mathbb{Z}^m$ and $\mathrm{B}_m^\star = \mathrm{B}_m$ for the Euclidean unit ball. For general bodies $C \in \mathscr{K}^m$, we always have $C \subseteq \left(C^\star\right)^\star$ with equality, if $0 \in C$, as we assumed $\dim_{\mathbb{R}}(C) = m$. It follows directly from the definition that $(AC)^\star = A^{-\top} C^\star$ for $A \in \mathrm{GL}_m(\mathbb{R})$.

A classical inequality, first investigated by Mahler [Mah39] and with an improved upper bound by Banaszczyk [Ban93, Thm. 2.1], is the following transference result.

Theorem 1.1.1 (Mahler, Banaszczyk)**.** *Let $C \in \mathscr{K}_0^m$ and $\Lambda \in \mathscr{L}^m$ of rank m, then for all* $1 \leq i \leq m$

$$1 \leq \lambda_i(C, \Lambda)\lambda_{m-i+1}(C^\star, \Lambda^\star) \leq m^{3/2}.$$

The generalisation of this result to the adelic setting will be the topic of Section 4.2.

We further define the *covering radius* or *inhomogeneous minimum*, $\mu(C, \Lambda)$, of $C \in \mathscr{K}^m$ and $\Lambda \in \mathscr{L}^m$ as the smallest dilation of C, such that the union of all lattice translates of $\mu(C, \Lambda) \cdot C$ covers $\mathbb{R}^m$, i.e.

$$\mu(C, \Lambda) = \inf\left\{ \mu > 0 \,\middle|\, \mathbb{R}^m \subseteq \left(\mu C + \Lambda\right) \right\}. \tag{1.7}$$

Again, as C is compact and Λ discrete, the infimum is in fact a minimum.

One important relation between the successive minima and the inhomogeneous minimum of a convex body $C \in \mathscr{K}_0^m$ and a lattice $\Lambda \in \mathscr{L}^m$ of rank m is given by Jarník's inequality,

$$\tfrac{1}{2}\lambda_m(C, \Lambda) \leq \mu(C, \Lambda) \leq \tfrac{1}{2}\left(\lambda_1(C, \Lambda) + \ldots + \lambda_m(C, \Lambda)\right). \tag{1.8}$$

As the covering radius is the smallest $\gamma > 0$ such that $\gamma C + \Lambda$ covers $\mathbb{R}^m$, we can conversely ask for the biggest $\gamma > 0$ such that $\gamma C + \Lambda$ forms a packing, i.e. no two lattice translates of γC overlap in the interior:

$$\left(\gamma(\operatorname{int} C) + a\right) \cap \left(\gamma(\operatorname{int} C) + b\right) = \emptyset \text{ for all } a \neq b \in \Lambda. \tag{1.9}$$

This number is called the *packing radius* of C with respect to Λ. Thus by definition, for symmetric $C \in \mathscr{K}_0^n$ and $\gamma \geq 0$ the set $\gamma C + \Lambda$ is a packing, if and only if

$$\gamma \leq \lambda_1(C, \Lambda)/2. \tag{1.10}$$

A sequence of numbers extending the inhomogeneous minimum of a body $C \in \mathscr{K}^m$ and a lattice $\Lambda \in \mathscr{L}^m$ are the *covering minima* $\mu_i(C,\Lambda)$ for $1 \le i \le m$, introduced by Kannan and Lovász [KL88],

$$\mu_i(C,\Lambda) = \inf\{\mu > 0 \mid \mu C + \Lambda \text{ intersects every } (m-i)\text{-dimensional affine subspace}\}.$$

Observe that $\mu_m(C,\Lambda) = \mu(C,\Lambda)$ and, as for the covering radius before, the infimum is in fact a minimum.

Kannan and Lovász showed that

$$\mu_i(C,\Lambda) = \sup\{\mu(C|W,\Lambda|W) \mid W \text{ is an } (m-i)\text{-dimensional lattice plane}\}, \tag{1.11}$$

where $C|W$ and $\Lambda|W$ are the images of C and Λ under the orthogonal projection onto $W^\perp$, the orthogonal complement of W in $\mathbb{R}^m$ with respect to the standard scalar product. They further showed that the supremum is in fact a maximum. It is well-known, that the orthogonal projection onto a subspace $U \subseteq \mathbb{R}^m$ can be written as $P_U = A(A^\intercal A)^{-1}A^\intercal$, where A is any matrix whose columns form a basis of U and if A is an orthonormal basis of U, this reduces to $P_U = AA^\intercal$. Conversely, $\overline{P}_U = (1_m - P_U)$ is the orthogonal projection onto $U^\perp$, where 1_m is the $m \times m$-identity matrix. Thus, in our situation, the map P_W is the projection onto W and $C|W = \overline{P}_W(C)$.

The following Theorem is one of the results of Kannan and Lovász' paper. We provide a proof of it here, using the description (1.11) for $\mu_i(C,\Lambda)$ instead of the definition, since this will later help us prove its adelic generalisation in Section 5.1.

Theorem 1.1.2 (Kannan, Lovász [KL88, Lemma 2.5])**.** *For each j, $1 \le j < m$,*

$$\mu_{j+1}(C,\Lambda) \le \mu_j(C,\Lambda) + \tfrac{1}{2}\lambda_{m-j}(C,\Lambda).$$

Proof. We start with the case $j = m-1$. We write $\lambda_1 = \lambda_1(C,\Lambda)$ and $\mu_{m-1} = \mu_{m-1}(C,\Lambda)$ for short.

Let $w \in \Lambda \cap \lambda_1 \cdot C$ and denote by P the orthogonal projection onto $\operatorname{lin}_{\mathbb{R}}(w)$ and by $\overline{P}$ that onto its orthogonal complement, as defined above. Let further V be the $(m-1)$-dimensional orthogonal complement of w, i.e. $V = \overline{P}(\mathbb{R}^m)$. Let now $x \in \mathbb{R}^m$. Then $x = \overline{P}(x) + P(x)$ with $\overline{P}(x) = x|w \in V$ and $P(x) = \alpha \cdot w$ for some $\alpha \in \mathbb{R}$. Further $\overline{P}(x) \in \mu_{m-1} \cdot C + \Lambda$ by definition of μ_{m-1}.

Now let $\lceil\alpha\rfloor \in \mathbb{Z}$ be a closest integer of α, i.e. $|\alpha - \lceil\alpha\rfloor| \le \frac{1}{2}$, then

$$x = \overline{P}(x) + \alpha w = \underbrace{\overline{P}(x) + \lceil\alpha\rfloor\, w}_{\in \mu_{m-1}C+\Lambda} + \underbrace{(\alpha - \lceil\alpha\rfloor)\, w}_{\in \frac{1}{2}\lambda_1 C} \in \left(\mu_{m-1} + \tfrac{1}{2}\lambda_1\right)C + \Lambda\,,$$

which shows the first case.

For general j let W be a $(m-j-1)$-dimensional lattice plane. Then by the first part

$$\mu_{j+1}(C|W, \Lambda|W) \le \mu_j(C|W, \Lambda|W) + \tfrac{1}{2}\lambda_1(C|W, \Lambda|W)\,.$$

Now $\mu_j(C|W, \Lambda|W) \le \mu_j(C, \Lambda)$ by definition and $\lambda_1(C|W, \Lambda|W) \le \lambda_{m-j}(C, \Lambda)$, since the kernel of the projection has dimension $m-j-1$. Thus we also have

$$\mu_{j+1}(C|W, \Lambda|W) \le \mu_j(C, \Lambda) + \tfrac{1}{2}\lambda_{m-j}(C, \Lambda)\,. \tag{1.12}$$

But since (1.12) holds for all subspaces W, it also holds for their supremum $\mu_{j+1}(C, \Lambda)$. □

Kannan and Lovász also proved [KL88, Lemma 2.3]

$$\mu_1(C, \Lambda) = \frac{1}{2\lambda_1(C^\star, \Lambda^\star)}\,. \tag{1.13}$$

We note that due to the symmetry of C we have

$$\min_{b\in\Lambda^\star\setminus\{0\}} \left(\max_{x\in C} \langle b, x\rangle - \min_{x\in C} \langle b, x\rangle \right) = \min_{b\in\Lambda^\star\setminus\{0\}} 2 \max_{x\in C} |\langle b, x\rangle| = 2\lambda_1(C^\star, \Lambda^\star)\,,$$

establishing that $\mu_1(C, \Lambda)$ is the inverse of the so-called *lattice width*, which measures the minimal distance between two parallel lattice hyperplanes touching the body at antipodal points.

Lattice Point Problems

An important quantity is the number of points from a lattice $\Lambda \in \mathscr{L}^m$ of rank m contained in a convex body $C \in \mathscr{K}^m$,

$$|C \cap \Lambda|\,. \tag{1.14}$$

This number has been studied in various ways, see e.g. [GW93] for an introduction. In the following, we will list some important facts and results that will be used and generalised in this thesis.

Observe that the number is invariant under linear transformations, since if $\Lambda = M\mathbb{Z}^m$ for some $M \in \mathrm{GL}_m(\mathbb{R})$, then

$$C \cap \Lambda = M\big(M^{-1}C \cap \mathbb{Z}^m\big) \quad \text{and} \quad \big|C \cap \Lambda\big| = \big|M^{-1}C \cap \mathbb{Z}^m\big|. \tag{1.15}$$

We can thus restrict to the case of $\Lambda = \mathbb{Z}^m$ and replace C by $M^{-1}C$ when investigating this number, although we usually state all results for arbitrary $\Lambda \in \mathscr{L}^m$.

Classical upper and lower bounds on this number in terms of the volume of the body C were proved by Blichfeldt and van der Corput.

Theorem 1.1.3 (van der Corput)**.** *For $C \in \mathscr{K}_0^m$ and $\Lambda \in \mathscr{L}^m$ of rank m,*

$$|C \cap \Lambda| \geq \left\lfloor \frac{\mathrm{vol}(C)}{2^{m-1}\det\Lambda} \right\rfloor > \frac{\mathrm{vol}(C)}{2^{m-1}\det\Lambda} - 1,$$

where $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$.

As the bound expressed in this form is not readily available in the literature, we provide a short proof.

Proof. Van der Corput's original result, [GL87, Theorem 1, Sec 6.1], states that if $X \subset \mathbb{R}^m$ is bounded and Jordan-measurable with $\mathrm{vol}(X) > k\det\Lambda$, then

$$\exists\ x_1, \ldots, x_{k+1} \in X \text{ pairwise different such that } x_i - x_j \in \Lambda. \tag{1.16}$$

Choose $k \in \mathbb{N}_0$ and $0 \leq \varepsilon < 1$ such that $\mathrm{vol}(C) = 2^m(k+\varepsilon)\det\Lambda$. By the compactness of C and the discreteness of Λ without loss of generality we may assume $\varepsilon > 0$. For $k = 0$ the statement $|C \cap \Lambda| \geq 1$ is trivially true. Otherwise we apply (1.16) to $X = \frac{1}{2}C$ and get $x_1, \ldots, x_{k+1} \in X$. Let $u_i = x_{k+1} - x_i$ for $i = 1, \ldots, k$, then $\pm u_i \in \Lambda \setminus \{0\}$ are pairwise different and also $u_i \in \frac{1}{2}C + \frac{1}{2}C = C$ by the 0-symmetry.

Therefore

$$|C \cap \Lambda| \geq 2k + 1 \geq \lfloor 2(k+\varepsilon) \rfloor = \left\lfloor \frac{\mathrm{vol}(C)}{2^{m-1}\det\Lambda} \right\rfloor.$$

The second inequality is obvious. □

Theorem 1.1.4 (Blichfeldt [GL87, p. 62])**.** *Let $C \in \mathscr{K}^m$ and $\Lambda \in \mathscr{L}^m$ with* $\dim_{\mathbb{R}}(C \cap \Lambda) = m$*. Then*

$$|C \cap \Lambda| \leq m!\,\frac{\mathrm{vol}(C)}{\det\Lambda} + m.$$

The bound is sharp for $\Lambda = \mathbb{Z}^m$ and simplices of the form $\mathrm{conv}\{0, \ell e_1, e_2, \ldots, e_m\}$, *where $\ell \in \mathbb{N}$ and $e_1, \ldots, e_m$ are the standard unit vectors.*

The additional requirement on the dimension is necessary, as an axis-parallel box with very small edge length in one direction can contain a large number of lattice points, while still having arbitrarily small volume.

Blichfeldt's inequality has recently been improved for symmetric $C \in \mathcal{K}_0^m$ by Henze.

Theorem 1.1.5 (Henze [Hen13, (2.4)]). *Let $C \in \mathcal{K}_0^m$ and $\Lambda \in \mathcal{L}^m$ with $\dim_{\mathbb{R}}(C \cap \Lambda) \geq m$. Then*

$$|C \cap \Lambda| \leq \frac{m!}{2^m} L_m(2) \frac{\mathrm{vol}_m(C)}{\det \Lambda},$$

where L_m is the m-th Laguerre polynomial, $L_m(x) = \sum_{k=0}^{m} \binom{m}{k} \frac{x^k}{k!}$.
The bound is asymptotically sharp for certain cross-polytopes.

Instead of correlating the number of lattice points in $C \cap \Lambda$ with the volume of the body C, Betke, Henk and Wills [BHW93] suggested correlating this number with the successive minima of C with respect to Λ.

They showed, assuming $\lambda_1(C,\Lambda) \leq 1$,

$$|C \cap \Lambda| \geq \frac{2^m}{m!} \left(1 - \frac{\lambda_1(C,\Lambda)}{2}\right)^m \prod_{i=1}^{m} \frac{1}{\lambda_i(C,\Lambda)}. \tag{1.17}$$

In general the constants cannot be improved. They also gave the following upper bound, which can be interpreted as replacing the volume of C in Minkowski's First Theorem (1.3) with the number of lattice points.

Theorem 1.1.6 (Betke, Henk, Wills [BHW93, (2.1)]). *Let $C \in \mathcal{K}_0^m$ of full dimension and $\Lambda \in \mathcal{L}^m$ of full rank, then*

$$|C \cap \Lambda| \leq \left\lfloor \frac{2}{\lambda_1(C,\Lambda)} + 1 \right\rfloor^m.$$

The inequality can not be improved.

Betke, Henk and Wills further conjectured and proved for $m = 2$

$$|C \cap \Lambda| \leq \prod_{i=1}^{m} \left\lfloor \frac{2}{\lambda_i(C,\Lambda)} + 1 \right\rfloor.$$

It was proved by Malikiosis [Mal12a] for $m = 3$ and for ellipsoids of all dimensions [Mal12b]. Weaker variants for arbitrary bodies include a dimensional factor c_m on the right and were proved by Henk [Hen02] with $c_m = 2^{m-1}$ and also improved by Malikiosis [Mal10] with $c_m \approx 1.644^m$.

1.2 Introduction to Adelic Geometry

The main topic of this thesis is the Geometry of Numbers over the ring of adeles of a number field. We first give an overview of classical results from algebraic number theory. Following that, we present the generalisation of the Geometry of Numbers and its embedding into real space.

Background from Algebraic Number Theory

We shall first fix some notation that will be used throughout the thesis. By K we always denote an algebraic number field, i.e. a finite extension of $\mathbb{Q}$ of degree $[K:\mathbb{Q}] = d$. We further denote by $\mathcal{O}$ the ring of algebraic integers of K and by Δ_K its field discriminant.

The following is a short summary of the construction and properties of the adele ring associated to K. For more details and proofs we recommend the exhaustive references [Wei95, Ch. IV–V] and [Kna07, Ch. VI].

Every number field of degree d can be embedded into $\mathbb{C}$ in exactly d ways. Some of the images might be real, while complex embeddings always occur in pairs, where one is the complex conjugate of the other. Denoting by r_1 the number of real embeddings and by r_2 the number of pairs of complex embeddings we have $d = r_1 + 2r_2$.

Example 1.2.1. Our standard example of an algebraic number field will be $\mathbb{Q}[\sqrt{2}]$. Despite the notation we do not view it as a subfield of the real numbers. In fact $\mathbb{Q}[\sqrt{2}] \cong \mathbb{Q}[X]/(X^2-2)$, where (X^2-2) is the maximal ideal generated by X^2-2 in the polynomial ring $\mathbb{Q}[X]$. This is a galois extension of degree 2 of $\mathbb{Q}$. It has two real embeddings, defined by

$$a + b\sqrt{2} \mapsto a + b\sqrt{2} \in \mathbb{R} \quad \text{and} \quad a + b\sqrt{2} \mapsto a - b\sqrt{2} \in \mathbb{R} \quad \text{for } a, b \in \mathbb{Q}.$$

The ring of integers is $\mathcal{O} = \mathbb{Z}[\sqrt{2}]$ and the discriminant is $\Delta_{\mathbb{Q}[\sqrt{2}]} = \sqrt{8}$.

Example 1.2.2. An example of a different kind is the field $\mathbb{Q}[\sqrt[3]{2}]$ of degree 3, which has the following three embeddings, one with real image and two with complex images,

$$a + b\sqrt[3]{2} \mapsto \begin{cases} a + b\ \sqrt[3]{2} \\ a + b\xi\sqrt[3]{2} \\ a + b\xi^2\sqrt[3]{2} \end{cases},$$

where $\xi \in \mathbb{C}$ is a primitive 3rd root of unity. Since $\xi^2 = \overline{\xi}$, the complex images are conjugates of one another, but do not coincide. This is possible, as the extension is not galois.

Recall that an absolute value is a map $|\cdot| : K \to \mathbb{R}_{\geq 0}$ with

$$|x| = 0 \Leftrightarrow x = 0, \quad |xy| = |x||y| \quad \text{and} \quad |x+y| \leq |x| + |y|$$

for all $x, y \in K$. The last condition is known as the *triangle inequality*, while the condition

$$|x+y| \leq \max\{|x|, |y|\} \quad \text{for all } x, y \in K$$

is called the *strong triangle inequality*. An absolute value satisfying the strong triangle inequality shall be called a *non-archimedean* absolute value. Otherwise it is an *archimedean* absolute value. We remark that $\{|n| \mid n \in \mathbb{N}\}$ is bounded if and only if $|\cdot|$ is non-archimedean.

We will always exclude the trivial absolute value $|x| = 1 \Leftrightarrow x \neq 0$.

Any absolute value induces a metric and thus a topology on K. If two absolute values $|\cdot|_1$ and $|\cdot|_2$ induce the same topology, they are called *equivalent*. This is the case if and only if there exists an $a \in \mathbb{R}$ with $|x|_1 = |x|_2^a$ for all $x \in K$. An equivalence class of absolute values is called a *place* and we denote by $M(K)$ the set of all places of K.

Example 1.2.3. For $K = \mathbb{Q}$ and p a prime, write $x = p^e \frac{a}{b}$ with $p \nmid ab$ and $a, b, e \in \mathbb{Z}$. Then

$$|x|_p = \frac{1}{p^e}$$

defines the non-archimedean p-adic absolute value on $\mathbb{Q}$. On the other hand

$$|x|_\infty = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

is the usual archimedean absolute value on $\mathbb{Q}$. None of the above are equivalent and there is no other non-trivial absolute value on $\mathbb{Q}$, up to equivalence. This result is commonly known as Ostrowski's Theorem.

Recall that given an absolute value $|\cdot|$, a sequence $(x_i)_{i\in\mathbb{N}}$ of elements $x_i \in K$ satisfying

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \text{ s.t. } \forall i, j > N : |x_i - x_j| < \varepsilon$$

is called a Cauchy sequence with respect to the given absolute value. Then

$$\mathscr{C} = \{(x_i)_i \mid x_i \in K, \text{ Cauchy sequence }\}$$

is a ring with component-wise addition and multiplication and

$$\mathscr{N} = \{(x_i)_i \in \mathscr{C} \mid \text{ converges to } 0 \}$$

is a maximal ideal in $\mathscr{C}$ and thus $\mathscr{R} = \mathscr{C}/\mathscr{N}$ is a field. Mapping $x \in K$ to the constant sequence $(x)_i \in \mathscr{R}$, the field K embeds into $\mathscr{R}$ diagonally. We call $\mathscr{R}$ the *completion* of K with respect to $|\cdot|$.

Example 1.2.4. The possible completions of the rationals $\mathbb{Q}$ are classified by Ostrowski's Theorem, see Example 1.2.3.

- The completion of $\mathbb{Q}$ with respect to the p-adic absolute value $|\cdot|_p$ for a prime p is the field of *p-adic numbers* $\mathbb{Q}_p$.
- The completion of $\mathbb{Q}$ with respect to the usual absolute value $|\cdot|_\infty$ is of course the field of real numbers $\mathbb{R} = \mathbb{Q}_\infty$.

We will therefore refer to a place of $\mathbb{Q}$ by a prime p or the symbol ∞. All of those fields are uncountable, but topologically distinct. For an elementary introduction to the field $\mathbb{Q}_p$ of p-adic numbers we refer to [Gou97].

Given a number field K, any absolute value on K restricted to the subfield $\mathbb{Q}$ induces one of the known absolute values (up to equivalence). On the other hand, any absolute value on $\mathbb{Q}$ can be extended to an absolute value on the field extension $K/\mathbb{Q}$.

To explicitly construct the absolute values, fix a place p of $\mathbb{Q}$ (i.e. a prime or ∞) and let f be the irreducible monic polynomial generating the extension, that is $K \cong \mathbb{Q}[x]/(f)\mathbb{Q}[x]$ where (f) is the maximal ideal generated by f in the polynomial ring $\mathbb{Q}[x]$, cf. Example 1.2.1 above. Then f will factor over $\mathbb{Q}_p$ as $f = f_1 \dots f_r$ with some polynomials f_i that are irreducible over $\mathbb{Q}_p$. Every f_i defines a finite extension $K_{(p,i)}$ of the complete field $\mathbb{Q}_p$ with a unique absolute value extending the p-adic absolute value of $\mathbb{Q}_p$. Since K embeds into $K_{(p,i)}$, this also defines an absolute value on K. See Example 1.2.5 below for an explicit construction.

We shall write $v \nmid \infty$ for non-archimedean places and $v \mid \infty$ for the archimedean ones. Alternatively we will also speak of finite and infinite places. We write $|\cdot|_v$ for the corresponding absolute value on K.

We shall denote by K_v the completion of K with respect to v as described above. It is a local field, i.e. a locally compact topological field with respect to the topology induced by the absolute value $|\cdot|_v$. Further, K_v is a finite extension of $\mathbb{Q}_v$, the completion of $\mathbb{Q}$ with respect to the absolute value $|\cdot|_v$ restricted to $\mathbb{Q}$.

For $v \nmid \infty$, the field K_v is the field of fractions of an integral domain $\mathscr{O}_v$ with a unique maximal ideal $\mathfrak{m}_v$. Furthermore, this ideal is generated by some

$\omega_v \in \mathcal{O}_v$, i.e. $\mathfrak{m}_v = \omega_v \mathcal{O}_v$. The ring $\mathcal{O}_v$ is the ring of integers of K_v and the field K_v has a local discriminant $\mathfrak{D}_v$. We can also think of $\mathcal{O}_v$ and $\mathfrak{m}_v$ in the topological sense, i.e.

$$\mathcal{O}_v = \left\{ x \in K_v \,\middle|\, |x|_v \leq 1 \right\} \quad \text{and} \quad \mathfrak{m}_v = \left\{ x \in K_v \,\middle|\, |x|_v < 1 \right\}.$$

Then every fractional ideal $\mathfrak{J} \subset K_v$ can be realised as $\mathfrak{J} = \mathfrak{m}_v^\alpha = \omega_v^\alpha \mathcal{O}_v$ for some $\alpha \in \mathbb{Z}$. This implies that, despite their descriptions, $\mathcal{O}_v$ and $\mathfrak{m}_v$ are both open and closed. Finally, the group of units of $\mathcal{O}_v$ is then

$$\mathcal{O}_v^* = \left\{ x \in K_v \,\middle|\, |x|_v = 1 \right\} = \mathcal{O}_v \setminus \mathfrak{m}_v.$$

The connection between prime ideals and absolute values is given by discrete valuations. A *discrete valuation* of K is a map $v : K \to \mathbb{Z} \cup \{\infty\}$ with

$$v(x) = \infty \Leftrightarrow x = 0, \quad v(xy) = v(x) + v(y) \quad \text{and} \quad v(x+y) \geq \min\left\{ v(x), v(y) \right\}.$$

We now fix some prime ideal $\mathfrak{P}_0 \subset \mathcal{O}$. For any $a \in K \setminus \{0\}$ there exist prime ideals $\mathfrak{P}_1, \ldots, \mathfrak{P}_r \subset \mathcal{O}$ such that the (fractional) principal ideal $a\mathcal{O}$ has a unique factorisation (up to order)

$$a\mathcal{O} = \mathfrak{P}_0^{e_0} \cdot \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}, \quad \text{where } e_i \in \mathbb{Z} \text{ and possibly } e_0 = 0.$$

Using this decomposition, we set $v_{\mathfrak{P}_0}(a) = e_0$ for all $a \in K$, and thus define a discrete valuation $v_{\mathfrak{P}_0} : K \to \mathbb{Z} \cup \{\infty\}$.

Now for a prime ideal $\mathfrak{P} \subset \mathcal{O}$ and a prime $p \in \mathbb{Z}$ such that $e = v_{\mathfrak{P}}(p) \neq 0$,

$$|a|_{\mathfrak{P}} = p^{-v_{\mathfrak{P}}(a)/e}$$

defines an absolute value on K. Moreover, this absolute value coincides with the p-adic absolute value on $\mathbb{Q}$, as $|p|_{\mathfrak{P}} = p^{-v_{\mathfrak{P}}(p)/e} = p^{-e/e} = p^{-1}$. All non-archimedean absolute values on K are equivalent to one of this form and for a fixed prime p, the possibilities of extending $|\cdot|_p$ to K correspond to the factors of $p\mathcal{O} = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_r^{e_r}$ into prime ideals.

Thus every prime ideal $\mathfrak{P} \subset \mathcal{O}$ corresponds to a place v of K. We write $v \mid p$, if $e = v_{\mathfrak{P}}(p) \neq 0$, that is $\mathfrak{P}$ is a factor of $p\mathcal{O}$, and we have $|\omega_v|_v = p^{-1/e}$. Locally the extension $|\cdot|_v$ of $|\cdot|_p$ is given by

$$|a|_v = \sqrt[d]{|\mathrm{N}_{K_v/\mathbb{Q}_p}(a)|_p},$$

where $\mathrm{N}_{K_v/\mathbb{Q}_p}$ is the field norm of K_v over $\mathbb{Q}_p$, i.e. $\mathrm{N}_{K_v/\mathbb{Q}_p}(a)$ is the determinant of the $\mathbb{Q}_p$-linear map $m_a : K_v \to K_v$, given by multiplication with $a \in K_v$.

We will always assume that our absolute values are normalised in this way, that is for $v \mid p$ we have $|x|_v = |x|_p$ for $x \in \mathbb{Q}$. This normalisation is also possible for $p = \infty$.

Example 1.2.5. We continue our Example 1.2.1, $K = \mathbb{Q}[\sqrt{2}]$. For the archimedean absolute value $|\cdot|_\infty$ on $\mathbb{Q}$ we get two possible extensions to K given by

$$|a+b\sqrt{2}|_1 = |a+b\sqrt{2}| \quad \text{and} \quad |a+b\sqrt{2}|_2 = |a-b\sqrt{2}|.$$

For an odd prime p, 2 is a square in $\mathbb{Q}_p$ if and only if $p^2 \equiv 1 \bmod 8$.

Thus if $p^2 \equiv 1 \bmod 8$, there are two possible extensions of $|\cdot|_p$ to K, given by

$$|a+b\sqrt{2}|_1 = |a+b\gamma| \quad \text{and} \quad |a+b\sqrt{2}|_2 = |a-b\gamma|,$$

where γ is root of 2 in $\mathbb{Q}_p$, i.e. a solution of $X^2-2=0$.

For p odd and $p^2 \not\equiv 1 \bmod 8$, the unique extension of $|\cdot|_p$ to K is given by

$$|a+b\sqrt{2}|_{\mathfrak{p}} = |a^2-2b^2|_p^{1/2}.$$

For the case $p=2$, note that the ideal $2\mathcal{O} = (\sqrt{2}\mathcal{O})^2$ splits and therefore, there is exactly one extension of $|\cdot|_2$ to K, as in the previous case.

We can now define the *adele ring* $K_{\mathbb{A}}$ of K as the restricted direct product of $\{K_v \mid v \in M(K)\}$ with respect to $\{\mathcal{O}_v \mid v \in M(K)\}$, i.e.

$$K_{\mathbb{A}} = \left\{ (x_v)_{v\in M(K)} \,\middle|\, x_v \in K_v, \text{ all but finitely many } x_v \in \mathcal{O}_v \right\}.$$

We then take $K_{\mathbb{A}}^n$ to be the standard module of rank n over $K_{\mathbb{A}}$, i.e. the n-fold product of adeles.

For any $v \in M(K)$ let $d_v = [K_v : \mathbb{Q}_v]$ be the local degree. Then for all primes $p \in \mathbb{Z}$

$$d = \sum_{v\mid p} d_v \quad \text{and} \quad d = \sum_{v\mid\infty} d_v. \tag{1.18}$$

We also have the *product formula*

$$\prod_{v\in M(K)} |a|_v^{d_v} = 1 \tag{1.19}$$

for all non-zero $a \in K$. For $K = \mathbb{Q}$ this can be easily seen, as $|a|_p$ is the inverse p-power of the prime p contained in $a \in \mathbb{Q}$ and thus (1.19) reduces to

$$\prod_{p \text{ prime}} |a|_p = |a|_\infty^{-1}.$$

This extends to any field K by our normalisations above. Finally

$$\prod_{v\nmid\infty} |\mathfrak{D}_v|_v = |\Delta_K|^{-1}. \tag{1.20}$$

For any fractional ideal $\mathfrak{J} \subset K$ there is a map $\mathfrak{a}\colon M(K) \to \mathbb{Z}$ such that $\mathfrak{J}$ can be written as

$$\mathfrak{J} = \bigcap_{v\nmid\infty} (K \cap \mathfrak{m}_v^{\mathfrak{a}(v)}), \tag{1.21}$$

where almost all $\mathfrak{a}(v) = 0$, and $\mathfrak{m}_v$ is the unique maximal ideal in $\mathcal{O}_v$.

The general linear group $\mathrm{GL}_n(K_{\mathbb{A}})$ consists of elements of the form

$$A = \prod_{v\nmid\infty} A_v \times \prod_{v\mid\infty} A_v$$

where $A_v \in \mathrm{GL}_n(K_v)$ for all v and $A_v \in \mathrm{GL}_n(\mathcal{O}_v)$ for almost all $v \nmid \infty$. We write

$$A^\intercal = \prod_{v\in M(K)} A_v^\intercal \qquad \text{and} \qquad |\det A|_{\mathbb{A}} = \left(\prod_{v\in M(K)} |\det A_v|_v^{d_v}\right)^{1/d}.$$

If $A \in \mathrm{GL}_n(K)$, then by the product formula (1.19), $|\det A|_{\mathbb{A}} = 1$.

Adelic Geometry

The Theory of Adelic Geometry of Numbers was first introduced independently by Bombieri and Vaaler [BV83] and MacFeat [McF71], see also [BV84]. For more details and proofs of the following see e.g. [BG06, Appendix C].

Of course, K can be identified with its image under inclusion $K \hookrightarrow K_v$ for any place v, as described above. Furthermore, using this embedding for every place, K can also be identified with its image under the diagonal embedding $K \hookrightarrow K_{\mathbb{A}}$. We will usually not distinguish between these copies of K.

Now the inclusion $K \hookrightarrow K_{\mathbb{A}}$ is discrete, i.e. any two $a, b \in K$ can be separated by disjoint open neighbourhoods $U_a, U_b \subset K_{\mathbb{A}}$ of a and b, respectively. On the other hand, the set $K_{\mathbb{A}}/K$ is compact. The same holds of course for $K^n \subset K_{\mathbb{A}}^n$ and $K_{\mathbb{A}}^n/K^n$. We can therefore think of a subspace $W \subseteq K^n$ of dimension ℓ as a rank-ℓ-lattice in $K_{\mathbb{A}}^n$.

Remark 1.2.6. *We mention the trivial but rather advantageous fact, that a set $\{x_1, \dots, x_\ell\} \subset W$ of linearly independent vectors of an ℓ-dimensional subspace W, i.e. a rank-ℓ-lattice in $K_{\mathbb{A}}^n$, forms a basis of $W \subseteq K^n$. This is of course not true for linearly independent points of a lattice in Euclidean space.*

On the other hand, up to isomorphism, there is only one vector space of dimension n over K, that is K^n. Because of this, we will almost always fix K^n as the lattice. This also means that there are no full-dimensional strict adelic sublattices of K^n.

We shall now define our notion of convexity. As mentioned above, any $\mathscr{O}_v$-module C_v is in fact free and thus of the form $C_v = A_v^{-1}\mathscr{O}_v^n$ for some matrix $A_v \in \mathrm{GL}_n(K_v)$, which is unique up to elements of $\mathrm{GL}_n(\mathscr{O}_v)$. On the other hand $\mathscr{O}_v$ is the unit disc of K_v. This motivates the following definition, where we think of $\mathscr{O}_v$ as convex, even though topologically it is not connected.

Definition 1.2.7. For each $v \nmid \infty$ let $A_v \in \mathrm{GL}_n(K_v)$ such that $C_v = A_v^{-1}\mathscr{O}_v^n$, where A_v is the identity for all but finitely many v. For $v \mid \infty$ let $C_v \subset K_v^n$ be a 0-symmetric compact convex body with non-empty interior in $K_v^n \cong \mathbb{R}^n$ or $K_v^n \cong \mathbb{C}^n \cong \mathbb{R}^{2n}$, respectively. Then the set

$$C = \prod_{v\nmid\infty} C_v \times \prod_{v\mid\infty} C_v = \prod_{v\nmid\infty} A_v^{-1}\mathscr{O}_v^n \times \prod_{v\mid\infty} C_v$$

is called a *symmetric adelic convex body*. Unless stated otherwise, we assume all adelic convex bodies to be symmetric and thus refer to them simply as adelic convex bodies. If necessary, we denote $C_\infty = \prod_{v\mid\infty} C_v$. As C_v is compact for all $v \in M(K)$ and $C_v = \mathscr{O}_v^n$ for almost all $v \nmid \infty$, C is itself compact and thus $C \cap K^n$ is a finite set.

Example 1.2.8. Let $\Lambda = M\mathbb{Z}^n \subset \mathbb{R}^n$ be a rational lattice, i.e. $M \in \mathrm{GL}_n(\mathbb{Q})$. And let $C_\infty \subset \mathbb{R}^n$ be a 0-symmetric convex body. By the inclusion $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$, we can think of $M \in \mathrm{GL}_n(\mathbb{Q}_p)$ for all primes p.

Then

$$C = \prod_{p \text{ prime}} M\mathbb{Z}_p^n \times C_\infty$$

is a convex body in $\mathbb{Q}_{\mathbb{A}}^n$ and conversely every adelic convex body in $\mathbb{Q}_{\mathbb{A}}^n$ corresponds to a lattice and convex body in $\mathbb{R}^n$. We will see the case for arbitrary K later in the section on embedding, page 26.

Remark 1.2.9. *Please note that our definition of symmetric adelic convex bodies agrees with the one by Bombieri and Vaaler [BV83, p. III.1] and by Gaudron [Gau09], but is slightly weaker than the one by McFeat [McF71, p. 4.2]. In fact, McFeat defines convexity not for each finite place $v \nmid \infty$ individually, but for the space $K_p^n = \prod_{v\mid p} K_v^n$, where p is a prime. In fact, our bodies are convex in McFeat's sense, but his notion is more general for ramified primes. Since the space K_p^n does not play a role in our investigations, we restrict to the more widely used definition above.*

For some results we need a stronger requirement on the symmetry of the adelic convex body than the one of Definition 1.2.7, which just mirrors the Euclidean case, i.e. $C = -C$.

Definition 1.2.10. Let $C = \prod_{v \in M(K)} C_v$ be an adelic convex body and let $v \nmid \infty$ be a complex place of K. Then C_v is *c-symmetric* if

$$\alpha C_v = C_v \quad \text{for all } \alpha \in \mathbb{C} \text{ with } |\alpha| = 1,$$

extending the usual symmetry condition for $\alpha \in \{-1, 1\} \subset \mathbb{R}$ to complex space. An adelic convex body is then c-symmetric, if and only if, for all complex places $v \nmid \infty$, the local convex body C_v is c-symmetric.

Example 1.2.11. We now define two adelic bodies, that will play an important role later on. For a fixed field K, let

$$\mathrm{B}_{\mathbb{A}} = \prod_{v \nmid \infty} \mathcal{O}_v^n \times \prod_{v \mid \infty} \left\{ x \in K_v^n \,\middle|\, \left(|x_1|_v^2 + \ldots + |x_n|_v^2 \right)^{1/2} \le 1 \right\}$$

and

$$\mathrm{C}_{\mathbb{A}} = \prod_{v \nmid \infty} \mathcal{O}_v^n \times \prod_{v \mid \infty} \left\{ x \in K_v^n \,\middle|\, \max\{ |x_1|_v, \ldots, |x_n|_v \} \le 1 \right\}$$

be the *adelic unit ball* and *adelic unit cube*.

The ring $K_{\mathbb{A}}$ also has a Haar measure, which can be described as follows.

For $v \nmid \infty$ the field K_v has itself a unique local Haar measure vol_v, which we normalise to $\mathrm{vol}_v(\mathcal{O}_v) = 1$, then $\mathrm{vol}_v(\alpha\mathcal{O}_v) = |\alpha|_v$ or conversely $|x|_v \le \mathrm{vol}_v(\mathfrak{J})$ for $x \in \mathfrak{J}$, a fractional ideal in K_v.

For $v \mid \infty$ real let vol_v be the usual Lebesgue measure.

For $v \mid \infty$ complex let vol_v be the usual Lebesgue measure multiplied by 2, $\mathrm{vol}_v\left(\left\{ x \in \mathbb{C} \,\middle|\, |x| \le 1 \right\}\right) = 2\pi$. We benefit from this normalisation later, e.g. (1.28).

We now get a Haar measure on K_v^n by taking the n-fold product measure. Then $\mathrm{vol}_v(A_v C_v) = |\det(A_v)|_v \cdot \mathrm{vol}_v(C_v)$ for some measurable $C_v \subset K_v^n$ and a matrix $A_v \in \mathrm{GL}_n(K_v)$.

Now let

$$\mathrm{vol}_{\mathbb{A}} = \prod_{v \in M(K)} \mathrm{vol}_v$$

be the Haar measure or *adelic volume* on $K_{\mathbb{A}}$ or $K_{\mathbb{A}}^n$, respectively. Notice that by construction, the measure of an adelic convex body will always be a finite product, as for almost all v we have $\mathrm{vol}_v(C_v) = 1$. Finally, cf. [BV93, p. 205],

$$\mathrm{vol}_{\mathbb{A}}(AC) = |\det A|_{\mathbb{A}}^d \, \mathrm{vol}_{\mathbb{A}}(C)$$

for any measurable set C and $A \in \mathrm{GL}_n(K_{\mathbb{A}})$.

We remark, that the above normalisation is chosen in precisely that way in which the compact quotient of $K_{\mathbb{A}}$ with respect to our discrete subset $K \subset K_{\mathbb{A}}$ can be shown to have measure $\sqrt{|\Delta_K|}$, i.e. $\mathrm{vol}_{\mathbb{A}}(K_{\mathbb{A}}/K) = \sqrt{|\Delta_K|}$. Some authors, e.g. Bombieri and Vaaler [BV83], normalise it to $\mathrm{vol}_{\mathbb{A}}(K_{\mathbb{A}}/K) = 1$ instead.

Example 1.2.12. For the bodies $\mathrm{B}_{\mathbb{A}}$ and $\mathrm{C}_{\mathbb{A}}$ from Example 1.2.11 we get

$$\mathrm{vol}_{\mathbb{A}}(\mathrm{B}_{\mathbb{A}}) = \kappa_n^{r_1} 2^{nr_2} \kappa_{2n}^{r_2} \quad \text{and} \quad \mathrm{vol}_{\mathbb{A}}(\mathrm{C}_{\mathbb{A}}) = 2^{nr_1}(2\pi)^{nr_2},$$

where κ_m is the volume of the m-dimensional Euclidean unit ball and r_1 the number of real and r_2 the number of complex places of K, since the infinite parts of the bodies are real balls or cubes of the respective dimensions and edge-lengths and thus the product follows immediately from the definition of $\mathrm{vol}_{\mathbb{A}}$. We note that $\kappa_m = \frac{\pi^{m/2}}{\Gamma(\frac{m}{2}+1)}$ for the gamma function $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\,\mathrm{d}t$.

For $(x_v)_v \in K_{\mathbb{A}}^n$ we define the scalar multiple $(y_v)_v = \lambda(x_v)_v$ for $\lambda \in \mathbb{R}^+$ by

$$y_v := \begin{cases} x_v, & \text{if } v \nmid \infty, \\ \lambda x_v, & \text{if } v \mid \infty. \end{cases}$$

Definition 1.2.13. The *i-th successive minimum* of the adelic convex body C with respect to the linear subspace $L \subseteq K^n$ is

$$\lambda_i(C, L) = \inf\left\{ \lambda > 0 \,\middle|\, \dim_K\big(C \cap L\big) \geq i \right\}$$

for $1 \leq i \leq n$. We write $\lambda_i(C) = \inf\{\lambda > 0 \mid \dim_K\big(C \cap K^n\big) \geq i\}$ for short. Observe that for any L we have by construction $\lambda_i(C, L) \leq \lambda_j(C, L)$ for $i \leq j$ and $\lambda_i(C, L) \leq \lambda_i(C', L)$ for $C' \subseteq C$ and all i.

On account of Example 1.2.8, for $K = \mathbb{Q}$ this reduces to the classical case.

Bombieri and Vaaler proved the following adelic variant of Minkowski's second theorem (1.4).

Theorem 1.2.14 (Bombieri-Vaaler [BV83], (McFeat, Thunder [McF71; Thu02]))**.** *Let C be an adelic convex body which is c-symmetric, i.e. satisfies the additional requirement at the complex places, Definition 1.2.10. Then*

$$\frac{2^{nd}\pi^{nr_2}}{(n!)^{r_1}((2n)!)^{r_2}} \leq \lambda_1(C)^d \cdots \lambda_n(C)^d \cdot \mathrm{vol}_{\mathbb{A}}(C) \leq 2^{nd}\left(\sqrt{|\Delta_K|}\right)^n. \tag{1.22}$$

The condition of c-symmetry is only required for the lower bound.

Just as in the classical Euclidean setting, we can define the inhomogeneous minimum of an adelic convex body.

Definition 1.2.15. The *inhomogeneous minimum* of an adelic convex body C is

$$\mu(C) := \inf\left\{ \mu > 0 \,\middle|\, K_{\mathbb{A}}^n = \bigcup_{\zeta \in K^n} (\mu C + \zeta) \right\}.$$

We further define the *adelic field constant*

$$\nu(K) = \mu(\mathrm{B}_{\mathbb{A}}^1)\,, \tag{1.23}$$

where $\mathrm{B}_{\mathbb{A}}^1$ is the 1-dimensional adelic unit ball of Example 1.2.11. Note that $\nu(K)$ depends only on K. The constant can be interpreted as expressing how good an algebraically integral point can be approximated by an arbitrary field element for all archimedean places simultaneously. In particular, $\nu(\mathbb{Q}) = \frac{1}{2}$, as the distance from any real number to the nearest integer is at most $\frac{1}{2}$.

We will calculate $\nu(\mathbb{Q}[\sqrt{2}]) = \frac{1+\sqrt{2}}{2}$ in Example 1.2.17 below.

For a general field K, O'Leary and Vaaler [OV93, Thm. 6] showed, if $K \not\cong \mathbb{Q}[\mathrm{i}]$ and $K \not\cong \mathbb{Q}[\sqrt{-3}]$, then

$$\nu(K) \le \frac{1}{2}\left(\frac{2}{\pi}\right)^{r_2} \left|\Delta_K\right|^{1/2}$$

and

$$\nu(\mathbb{Q}[\mathrm{i}]) = \tfrac{1}{\sqrt{8}} \left|\Delta_{\mathbb{Q}[\mathrm{i}]}\right|^{1/2} \quad \text{and} \quad \nu(\mathbb{Q}[\sqrt{-3}]) = \tfrac{1}{3} \left|\Delta_{\mathbb{Q}[\sqrt{-3}]}\right|^{1/2}.$$

They also proved the adelic variant of Jarník's inequality, cf. [OV93, Thm. 5]. Let C be an adelic convex body, then

$$\tfrac{1}{2}\lambda_n(C) \le \mu(C) \le \nu(K)\big(\lambda_1(C) + \ldots + \lambda_n(C)\big). \tag{1.24}$$

As before, for $K = \mathbb{Q}$ and with $\nu(\mathbb{Q}) = \frac{1}{2}$, (1.24) reduces to the classical result (1.8).

Embedding into Euclidean space

Denote by σ_i, $1 \le i \le r_1$, the embeddings of K into $\mathbb{R}$ and by $\sigma_{r_1+i} = \overline{\sigma}_{r_1+i+r_2}$, $1 \le i \le r_2$, the pairs of embeddings of K into $\mathbb{C}$, so $d = r_1 + 2r_2$. We call K *totally real*, if $r_2 = 0$, and we call K a *CM-field*, if it is a quadratic extension of a totally real field with $r_1 = 0$.

While $\mathbb{Q}[\sqrt{2}]$ is of course totally real, examples of CM-fields are $\mathbb{Q}[\sqrt{-3}]$ and $\mathbb{Q}[\mathrm{i}]$ and in fact $\mathbb{Q}[\zeta_k]$ for any primitive k-th root of unity ζ_k. The field $\mathbb{Q}[\zeta_k]$ is an imaginary quadratic extension of the totally real field $\mathbb{Q}[\zeta_k+\zeta_k^{-1}]$.

If K is a CM-field, there exists a unique non-trivial automorphism τ_K of K, such that $\sigma(\tau_K(x)) = \overline{\sigma(x)}$ for any embedding $\sigma\colon K\to\mathbb{C}$, where $\bar{\cdot}$ denotes complex conjugation in $\mathbb{C}$, cf. [BL78]. Then

$$\iota\colon x\mapsto \big(\sigma_1(x),\dots,\sigma_{r_1}(x),\sigma_{r_1+1}(x),\dots,\sigma_{r_1+r_2}(x)\big)$$

and

$$\bar{\iota}\colon x\mapsto \big(\sigma_1(x),\dots,\sigma_{r_1}(x),\overline{\sigma}_{r_1+1}(x),\dots,\overline{\sigma}_{r_1+r_2}(x)\big)$$

are embeddings of K into $K_\infty=\prod_{v\mid\infty}K_v$.

There is a canonical isomorphism $\rho\colon K_\infty\to\mathbb{R}^d$ with

$$\begin{aligned}\rho\big(x_1,\dots,x_{r_1},x_{r_1+1},\dots,x_{r_1+r_2}\big) &=\\ \big(x_1,\dots,x_{r_1},\Re(x_{r_1+1}),\Im(x_{r_1+1}),\dots,\Re(x_{r_1+r_2}),\Im(x_{r_1+r_2})\big)&.\end{aligned}\tag{1.25}$$

Here $\Re$ and $\Im$ denote real and imaginary parts, respectively.

Together we get a map $(\rho\circ\iota)\colon K\hookrightarrow\mathbb{R}^d$, that sends a field element to the vector whose entries are the images under the real and complex embeddings, splitting the latter points into real and imaginary part,

$$x\mapsto\big(\sigma_1(x),\dots,\sigma_{r_1}(x),\Re(\sigma_{r_1+1}(x)),\Im(\sigma_{r_1+1}(x)),\dots,\Re(\sigma_{r_1+r_2}(x)),\Im(\sigma_{r_1+r_2}(x))\big).$$

In the rank-n-case let $K_\infty^n=\prod_{v\mid\infty}K_v^n$,

$$\iota^n:=(\sigma_1^n,\dots,\sigma_{r_1}^n,\sigma_{r_1+1}^n,\dots,\sigma_{r_1+r_2}^n)\colon K^n\to K_\infty^n,\quad \bar{\iota}^n \text{ respectively,}$$

where the σ_i act component-wise. Similarly $\rho^n\colon K_\infty^n\to\mathbb{R}^{nd}$. To simplify notation, we usually write ρ and ι instead of ρ^n and ι^n.

Let C be an adelic convex body and define

$$\mathfrak{M}:=\bigcap_{v\nmid\infty}\big(C_v\cap K^n\big)=\bigcap_{v\nmid\infty}\big((A_v\mathcal{O}_v^n)\cap K^n\big),\tag{1.26}$$

which is the $\mathcal{O}$-lattice in K^n containing all points that lie in all finite factors of C and thus in some dilate of C, see also Example 1.2.16 below.

In fact, by [Thu02, Lemma], $\rho(\iota(\mathfrak{M}))\subset\mathbb{R}^{nd}$ is a lattice of full rank and determinant

$$\det(\rho(\iota(\mathfrak{M})))=\frac{\big(2^{-r_2}\sqrt{|\Delta_K|}\big)^n}{\prod_{v\nmid\infty}\mathrm{vol}_v(C_v)}.\tag{1.27}$$

Due to our normalisations above, we then get

$$\begin{aligned}\operatorname{vol}_{nd}\big(\rho(C_\infty)\big) &= 2^{-r_2 n}\prod_{v\mid\infty}\operatorname{vol}_v(C_v)\\ &= \operatorname{vol}_{\mathbb{A}}\Big(\prod_{v\nmid\infty}C_v\times\prod_{v\mid\infty}C_v\Big)\cdot\frac{\det\big(\rho(\iota(\mathfrak{M}))\big)}{\big(\sqrt{|\Delta_K|}\big)^n},\end{aligned}\tag{1.28}$$

where vol_{nd} is the Lebesgue measure on $\mathbb{R}^{nd}$.

Denote by $\widehat{\lambda}_i\big(\rho(C_\infty),\rho(\iota(\mathfrak{M}))\big)$, $1\le i\le nd$, the classical successive minima of the body $\rho(C_\infty)$ with respect to the lattice $\rho(\iota(\mathfrak{M}))$. Then

$$\widehat{\lambda}_i\big(\rho(C_\infty),\rho(\iota(\mathfrak{M}))\big)=\inf\Big\{\lambda>0\;\Big|\;\dim_{\mathbb{Q}}(C\cap K^n)\ge i\Big\}\tag{1.29}$$

for $i=1,\ldots,nd$. This follows directly from the definitions, as ρ is a $\mathbb{Q}$-linear map and injectively maps K^n into $\mathbb{R}^{nd}$.

We further have for $\ell=0,\ldots,n-1$,

$$\lambda_{\ell+1}(C)\le\widehat{\lambda}_{\ell d+1}\big(\rho(C_\infty),\rho(\iota(\mathfrak{M}))\big).\tag{1.30}$$

This is a direct consequence of (1.29). Let $x_1,\ldots,x_{\ell d+1}$ be linearly independent over $\mathbb{Q}$ and assume that $x_{i_1},\ldots,x_{i_\ell}$ is a selection of ℓ points that are linearly independent over K. Then

$$\dim_{\mathbb{Q}}\big(\operatorname{lin}_K(x_{i_1},\ldots,x_{i_\ell})\big)\le\ell d,$$

showing that there must be another point $x_{i_\ell+1}$ that is K-linearly independent from the first ℓ points.

Example 1.2.16. Let $K=\mathbb{Q}[\sqrt{2}]$ and $n=2$. We define a adelic convex body C locally, i.e. for each place. For $v\nmid\infty$ let $C_v=\mathcal{O}_v^2$ and for $v\mid\infty$ let $C_v=[-1,1]^2$. Thus $\mathfrak{M}=\bigcap_{v\nmid\infty}\mathcal{O}_v^2=\mathcal{O}^2$ and $C_\infty=[-1,1]^4$. Now if $(a+b\sqrt{2},c+d\sqrt{2})\in C\setminus\{0\}$, then $a+b\sqrt{2},c+d\sqrt{2}\in\mathcal{O}$ and thus

$$\mathrm{N}_{K/\mathbb{Q}}\big(a+b\sqrt{2}\big)=\big|a^2-2b^2\big|,\ \mathrm{N}_{K/\mathbb{Q}}\big(c+d\sqrt{2}\big)=\big|c^2-2d^2\big|\ge1.$$

On the other hand, by definition of the infinite parts of C,

$$\big|a+b\sqrt{2}\big|,\big|a-b\sqrt{2}\big|,\big|c+d\sqrt{2}\big|,\big|c-d\sqrt{2}\big|\le1$$

and this combined

$$\big|a+b\sqrt{2}\big|=\big|a-b\sqrt{2}\big|=\big|c+d\sqrt{2}\big|=\big|c-d\sqrt{2}\big|=1$$

or $a+b\sqrt{2}, c+d\sqrt{2} \in \mathscr{O}^*$, implying $a=\pm 1$, $c=\pm 1$ and $b=d=0$ and therefore

$$C \cap K^2 = \{(0,0),(\pm 1,0),(0,\pm 1),(\pm 1,\pm 1)\}.$$

So obviously $\lambda_1(C) = \lambda_2(C) = 1$. Due to the description (1.29), as $(\pm 1,0)$ and $(0,\pm 1)$ are also $\mathbb{Q}$-linearly independent, we have $\widehat{\lambda}_1\big(\rho(C_\infty),\rho(\iota(\mathfrak{M}))\big) = \widehat{\lambda}_2\big(\rho(C_\infty),\rho(\iota(\mathfrak{M}))\big) = 1$, but obviously $\widehat{\lambda}_3\big(\rho(C_\infty),\rho(\iota(\mathfrak{M}))\big) > 1$.

As another example, define the adelic body D by $D_v = \mathscr{O}_v^2$ for $v \nmid \infty$ and $D_v = \left[-\frac{1}{2},\frac{1}{2}\right] \times \left[-\sqrt{2},\sqrt{2}\right]$ for $v \mid \infty$. Then

$$D \cap K^2 = \left\{(0,0),(0,\pm 1),(0,\pm\sqrt{2})\right\},$$

as $\mathrm{N}_{K/\mathbb{Q}}\big(c+d\sqrt{2}\big) = \left|c^2-2d^2\right| \le 2$ has no other integral solution than $c,d \in \{0,\pm 1\}$ and $\left|1+\sqrt{2}\right| > \sqrt{2}$. Therefore we get

$$\lambda_1(D) = \frac{1}{\sqrt{2}}, \quad \lambda_2(D) = 2$$

as we need to dilate with 2 to reach $(\pm 1,0)$. But since $(0,\pm 1)$ and $(0,\pm\sqrt{2})$ are linearly independent over $\mathbb{Q}$, we have

$$\widehat{\lambda}_1\big(\rho(C_\infty),\rho(\iota(\mathfrak{M}))\big) = \frac{1}{\sqrt{2}}, \quad \widehat{\lambda}_2\big(\rho(C_\infty),\rho(\iota(\mathfrak{M}))\big) = 1, \quad \widehat{\lambda}_3\big(\rho(C_\infty),\rho(\iota(\mathfrak{M}))\big) = 2.$$

These connections between the adelic setting and the situation we get after the embedding into K_∞ are used by Thunder [Thu02, p. 256] and Gaudron [Gau09, (12)]. In fact it is also known as the Minkowski-embedding and has applications in algebraic number theory, e.g. Minkowski's discriminant theorem. The construction above reduces to Example 1.2.8 for $K = \mathbb{Q}$.

As ι injects K^n into K_∞^n and ρ identifies the latter with $\mathbb{R}^{nd}$, we also have the following direct connection between the adelic inhomogeneous minimum $\mu(C)$ and that of the embedded body and lattice,

$$\mu(C) = \widehat{\mu}\big(\rho(C_\infty),\rho(\iota(\mathfrak{M}))\big). \tag{1.31}$$

Here $\widehat{\mu}\big(\rho(C_\infty),\rho(\iota(\mathfrak{M}))\big)$ is the classical inhomogeneous minimum, (1.7), of the body $\rho(C_\infty)$ and the lattice $\rho(\iota(\mathfrak{M}))$. To see the equality, observe that to compute the number γ on the left, we have to cover K_∞^n by copies of γC_∞ with the sections of the K^n-translates of γC in $K_\mathbb{A}^n$ corresponding to the finite places. These are exactly the translates of γC_∞ by $\iota(\mathfrak{M})$. Applying the isomorphism ρ, we see that this is precisely the number on the right.

Example 1.2.17. Let again $K = \mathbb{Q}[\sqrt{2}]$ and consider the embedding into $\mathbb{R}^2$. Figure 1.1 shows the lattice $\rho(\iota(\mathbb{Z}[\sqrt{2}]))$ in $\mathbb{R}^2$ and four copies of the dilate of $C_\infty = [-1,1]^2$ by $\frac{1+\sqrt{2}}{2}$, translated to $\rho(0) = (0,0)$, $\rho(1) = (1,1)$, $\rho(\sqrt{2}) = (\sqrt{2}, -\sqrt{2})$ and $\rho(1+\sqrt{2}) = (\sqrt{2}+1, 1-\sqrt{2})$.

Evidently, this dilation of the square corresponds to a covering of $\mathbb{R}^2$ and any smaller dilation does not. Thus $\nu(K) = \frac{1+\sqrt{2}}{2}$.

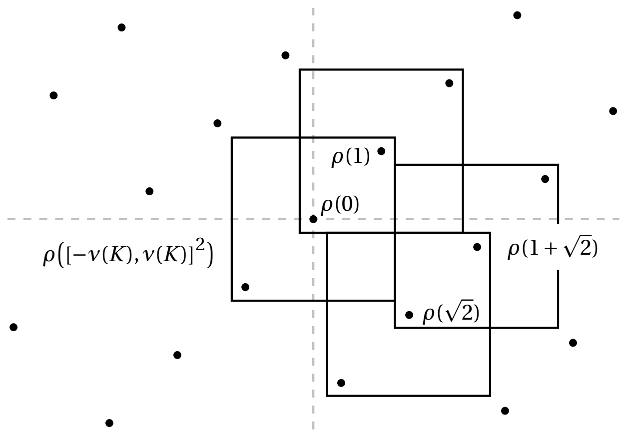


Figure 1.1: $\nu\big(\mathbb{Q}[\sqrt{2}]\big)$

The Notion of Heights

We conclude our preliminary overview with a very brief introduction to the theory of heights. Heights are an important tool in Diophantine Geometry, and many results and generalisations in connection with Siegel's Lemma use this language, since, by a famous result of Northcott [Nor49], bounding a height on K^n defines a finite set similar to our adelic convex bodies. Heights do however allow a unified approach, using the same language also for function fields and varieties or quaternion algebras. The theory has been used to study questions similar to those we consider in this thesis only for number fields in more generality and for all global fields in the works of Fukshansky [Fuk06b; Fuk10], Fukshansky and Henshaw [FH13], Gaudron [Gau09], Gaudron and Rémond [GR12; GR13]. For an in-depth introduction to the theory of heights, we recommend the book by Bombieri and Gubler [BG06].

Definition 1.2.18. We define the following norms on K_v^n: For all $v \in M(K)$ let

$$\big|(x_1,\ldots,x_n)\big|_v = \max\{|x_1|_v,\ldots,|x_n|_v\}$$

and for $v \mid \infty$ we also define the standard ℓ^2-norm

$$\left\|(x_1,\dots,x_n)\right\|_v = \left(|x_1|_v^2 + \ldots + |x_n|_v^2\right)^{1/2}.$$

This allows us to define the *height*

$$H(x) = \left(\prod_{v\nmid\infty} |x|_v^{d_v} \times \prod_{v\mid\infty} \|x\|_v^{d_v}\right)^{1/d} \quad \text{for } x \in K^n.$$

We can also define an alternative height, using the maximum norm at the infinite places,

$$\mathcal{H}(x) = \left(\prod_{v\nmid\infty} |x|_v^{d_v} \times \prod_{v\mid\infty} |x|_v^{d_v}\right)^{1/d} \quad \text{for } x \in K^n.$$

Heights are projective in the sense that on account of the product formula (1.19), we have $H(ax) = H(x)$ and $\mathcal{H}(ax) = \mathcal{H}(x)$ for all $x \in K^n$ and $a \in K \setminus \{0\}$. Furthermore, using the equivalence of the ℓ^2- and max-norms, the two heights are also equivalent, i.e.

$$1 \le \mathcal{H}(x) \le H(x) \le \sqrt{n}\,\mathcal{H}(x).$$

Let $x \in \mathcal{O}^n$. Then $H(x) \le 1$ if and only if x is contained in the adelic unit ball $\mathrm{B}_{\mathbb{A}}$, and $\mathcal{H}(x) \le 1$ if and only if x is contained in the adelic unit cube $\mathrm{C}_{\mathbb{A}}$. Conversely for $y \in \mathrm{B}_{\mathbb{A}}$ we always have $H(y) \le 1$ and also $\mathcal{H}(z) \le 1$ for $z \in \mathrm{C}_{\mathbb{A}}$. For the purpose of this thesis, it suffices to consider only the height H. Notice that for general $x \in K^n$ with $H(x) = 1$, there might be a $v \in M(K)$ with $\|x\|_v$ or $|x|_v$ arbitrarily large, e.g. let p be a prime and $z \in \mathbb{Z}$ and consider $\left(p^z, p^z\right) \in \mathbb{Q}^2$, then $H\left(p^z, p^z\right) = 1$ on account of the product formula (1.19), but $\left|\left(p^z, p^z\right)\right|_p = p^{-z}$ and $\left|\left(p^z, p^z\right)\right|_\infty = p^z$, one of which is large and the other small. Using the projectivity of H, one might try finding a $a \in K$ with $H(ax) = 1$ and $\|x\|_v, |x|_v \le \mathfrak{z}$ for all $v \in M(K)$ and some constant $\mathfrak{z} \ge 1$, e.g. $a = 1, \mathfrak{z} = p^{|z|}$ in the example above. This is however non-trivial in general, if $\mathcal{O}$ is not a principal ideal domain.

We can also define the height of a subspace $L \subseteq K^n$. Let $b_1, \dots, b_r$ be a basis of L and similar to (1.1) let

$$\mathrm{Gr}(L) = \left(\det B_j \,\middle|\, B_j \text{ a } r \times r \text{ submatrix of } B = (b_1,\dots,b_r)\right) \in K^{\binom{n}{r}} \tag{1.32}$$

be the vector with entries $\det B_j$. Then the height of $\mathrm{Gr}(L)$ is again independent of the chosen basis and thus we call $H(L) = H(\mathrm{Gr}(L))$ the *height* of L.

A more general notion is that of twisted heights introduced by Roy and Thunder [RT96]. They also play an important role in the work of Rothlisberger [Rot10] and have a much closer connection to our approach to Adelic Geometry than the general definition.

Definition 1.2.19 (Roy-Thunder)**.** Let $A \in \mathrm{GL}_n(K_{\mathbb{A}})$, then the *twisted height* of $x \in K^n$ is defined as

$$H_A(x) = \left(\prod_{v \nmid \infty} |A_v(x)|_v^{d_v} \times \prod_{v \mid \infty} \|A_v(x)\|_v^{d_v} \right)^{1/d}.$$

Observe that $H_I(x) = H(x)$ for the identity I. For general $A \in \mathrm{GL}_n(K\mathbb{A})$, by [RT96, Proposition 4.1], there exist constants $C_1, C_2 \in \mathbb{R}$, depending only on A, such that $C_1 H(x) \leq H_A(x) \leq C_2 H(x)$ for all $x \in K^n$.

We can define successive minima also in terms of twisted heights.

Definition 1.2.20. For any $A \in \mathrm{GL}_n(K_{\mathbb{A}})$ and any $i \in \mathbb{Z}$ with $1 \leq i \leq n$ we define the *i-th successive minimum in terms of heights* as

$$\tilde{\lambda}_i(A) = \inf\{\lambda > 0 \,|\, \exists x_1, \ldots, x_i \in K^n \text{ lin. indep. over } K \text{ s.t. } H_A(x_j) \leq \lambda \text{ for all } j\}.$$

To avoid confusion, when necessary we refer to the successive minima of Definition 1.2.13 as *successive minima in terms of dilations*. The two notions are related, as we will see below.

Let $C = \prod_{v \nmid \infty} A_v^{-1} \mathcal{O}_v^n \times \prod_{v \mid \infty} C_v$ be an adelic convex body, and consider $v \mid \infty$. We denote by B_v the unit ball with respect to the ℓ^2-norm on $\mathbb{R}^n$ or $\mathbb{C}^n$, respectively. By John's Theorem, cf. [Gru07, § 11.1], there is an $A_v \in \mathrm{GL}_n(K_v)$ such that

$$A_v^{-1}\mathrm{B}_v \subset C_v \subset \begin{cases} \sqrt{n}\, A_v^{-1}\mathrm{B}_v, & v \text{ real}, \\ \sqrt{2n}\, A_v^{-1}\mathrm{B}_v, & v \text{ complex, identifying } \mathbb{C}^n \cong \mathbb{R}^{2n}. \end{cases}$$

The convex body $A_v^{-1}\mathrm{B}_v$ is an ellipsoid and in fact A_v can be chosen, such that $A_v^{-1}\mathrm{B}_v$ is the unique ellipsoid of maximal volume among all ellipsoids contained in C_v. This ellipsoid is known as the Löwner-John-ellipsoid, see [Hen12] for a detailed account. Define

$$A = \prod_{v \nmid \infty} A_v \times \prod_{v \mid \infty} A_v \in \mathrm{GL}_n(K_{\mathbb{A}}), \quad \text{and recall} \quad \mathrm{B}_{\mathbb{A}} = \prod_{v \nmid \infty} \mathcal{O}_v^n \times \prod_{v \mid \infty} \mathrm{B}_v$$

from Example 1.2.11. Then

$$A^{-1}\mathrm{B}_{\mathbb{A}} = \prod_{v\nmid\infty} A_v^{-1}\mathscr{O}_v^n \times \prod_{v\mid\infty} A_v^{-1}\mathrm{B}_v \subset \prod_{v\nmid\infty} C_v \times \prod_{v\mid\infty} C_v = C \subset \prod_{v\nmid\infty} A_v^{-1}\mathscr{O}_v^n \times \prod_{v\mid\infty} \sqrt{2n}\, A_v^{-1}\mathrm{B}_v =: \sqrt{2n}\, A^{-1}\mathrm{B}_{\mathbb{A}}\,. \tag{1.33}$$

Thus $A^{-1}\mathrm{B}_{\mathbb{A}}$ can be seen the unique maximal adelic ellipsoid contained in C and we therefore also have an adelic version of John's Theorem.

The following lemma shows the connection between our notion of successive minima of convex bodies in terms of dilations and the successive minima in terms of twisted heights.

Lemma 1.2.21. *We have $\tilde{\lambda}_i(A) \le \lambda_i(A^{-1}\mathrm{B}_{\mathbb{A}})$ for $1 \le i \le n$ and every $A \in \mathrm{GL}_n(K_{\mathbb{A}})$.*

Proof. Let $\overline{\lambda} := \lambda_i(A^{-1}\mathrm{B}_{\mathbb{A}})$ and let $x_1,\dots,x_i \in \overline{\lambda}A^{-1}\mathrm{B}_{\mathbb{A}}$ be linearly independent points and let $x \in \{x_1,\dots,x_i\}$. Then

$$H_A(x)^d = \prod_{v\nmid\infty} |A_v x_v|_v^{d_v} \cdot \prod_{v\mid\infty} \|A_v x_v\|_v^{d_v}\,.$$

Consider $v \nmid \infty$. From $x \in \overline{\lambda}A^{-1}\mathrm{B}_{\mathbb{A}}$ we get

$$x_v \in A_v^{-1}\mathscr{O}_v^n \Rightarrow A_v x_v \in \mathscr{O}_v^n \Rightarrow |A_v x_v|_v \le 1\,.$$

Now consider $v \mid \infty$. Again from $x \in \overline{\lambda}A^{-1}\mathrm{B}_{\mathbb{A}}$ we get

$$x_v \in \overline{\lambda}A_v^{-1}\mathrm{B}_v \Rightarrow A_v x_v \in \overline{\lambda}\mathrm{B}_v \Rightarrow \|A_v x_v\|_v \le \overline{\lambda}\,.$$

This gives

$$H_A(x) \le 1 \cdot \left(\overline{\lambda}^d\right)^{1/d} = \lambda_i(A^{-1}\mathrm{B}_{\mathbb{A}})$$

and since the $\tilde{\lambda}_i$ are the infima of the heights, the assertion follows. □

Finally, we state the following inequality by Roy and Thunder [RT96, Theorem 6.3] for later reference.

Assume $n \ge 2$, and let $A \in \mathrm{GL}_n(K_{\mathbb{A}})$, then

$$|\det A|_{\mathbb{A}} \le \prod_{i=1}^{n} \tilde{\lambda}_i(A) \le 2^{n(n-1)/2} |\det A|_{\mathbb{A}}\,. \tag{1.34}$$

2

Restricted Successive Minima

In this Chapter we study a generalisation of Siegel's Lemma that introduces additional restrictions, as already described in the introduction. Recall that in its original form Siegel's lemma guarantees the existence of a non-trivial integral solution $x \in \ker(A) \cap \mathbb{Z}^m \setminus \{0\}$ to a given system of linear equations $A\mathbf{x} = 0$ with $A \in \mathbb{Z}^{r\times m}$ and $\operatorname{rank}(A) = r < m$, such that

$$\|x\|_\infty \leq \left\lfloor (m\|A\|_\infty)^{r/(m-r)} \right\rfloor ,$$

or equivalently it states in terms of successive minima

$$\lambda_1\Big([-1,1]^m, \ker(A) \cap \mathbb{Z}^m\Big) \leq \left\lfloor (m\|A\|_\infty)^{r/(m-r)} \right\rfloor .$$

Motivated by questions in Diophantine approximation, Fukshansky studied in [Fuk06a] an inverse problem to that addressed in Siegel's Lemma, namely to bound the norm of lattice points which are not contained in the union of proper sublattices.

In other words, given a lattice $\Lambda \in \mathscr{L}^m$ and a collection $\Lambda_1, \dots, \Lambda_s \subset \Lambda$ of sublattices, called *restrictions*, Fukshansky proved the existence of a point $x \in \Lambda \setminus (\Lambda_1 \cup \ldots \cup \Lambda_s)$, whose norm is bounded by a constant depending on the lattices $\Lambda, \Lambda_1, \dots, \Lambda_s$ and the dimensions involved.

We now introduce the following functional, that generalises the notion of successive minima introduced in Section 1.1 to include these restrictions.

Definition 2.0.1. Let $\Lambda \in \mathscr{L}^m$ be a lattice and $C \in \mathscr{K}^m$ a convex body.

For a collection of sublattices $\Lambda_j \subset \Lambda$, $1 \leq j \leq s$, with $\bigcup_{j=1}^s \Lambda_j \neq \Lambda$ we call

$$\lambda_i\big(C, \Lambda \setminus \bigcup_{j=1}^s \Lambda_j\big) = \min\big\{\lambda > 0 \,\big|\, \dim_{\mathbb{R}}\big(\lambda C \cap \Lambda \setminus \bigcup_{j=1}^s \Lambda_j\big) \geq i\big\}, \quad 1 \leq i \leq \operatorname{rank}\Lambda,$$

the *i-th restricted successive minimum* of C with respect to $\Lambda \setminus \bigcup_{j=1}^s \Lambda_j$.

Observe that by the compactness of C and the discreteness of $\Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j$ these minima are well-defined.

This naturally extends the classical definition of successive minima to more general discrete sets, as $\lambda_i(C,\Lambda) = \lambda_i(C,\Lambda \setminus \{0\})$ for the trivial restriction. Furthermore, the properties exhibited by the classical minima are still true, $\lambda_i\big(C,\Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\big) \leq \lambda_k\big(C,\Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\big)$ for $i<k$ and for $\gamma > 0$ we have

$$\lambda_i\big(\gamma C,\Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\big) = \lambda_i\big(C,\tfrac{1}{\gamma}\big(\Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\big)\big) = \tfrac{1}{\gamma}\lambda_i\big(C,\Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\big). \tag{2.1}$$

With the notation as in Section 1.1, Fukshansky proved [Fuk06a, Theorem 1.1]

$$\lambda_1\Big([-1,1]^n,\Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\Big) \leq \left(\frac{3}{2}\right)^{r-1} r^r \left(\sum_{j=1}^{s} \frac{1}{\|\mathrm{Gr}(\Lambda_j)\|_\infty} + \sqrt{s}\right) \|\mathrm{Gr}(\Lambda)\|_\infty + 1, \tag{2.2}$$

for proper sublattices Λ_j, i.e. $\operatorname{rank}\Lambda_j < \operatorname{rank}\Lambda = r$, $1 \leq j \leq s$. Here, $\|\mathrm{Gr}(\Lambda)\|_\infty$ and $\|\mathrm{Gr}(\Lambda_j)\|_\infty$ are in the light of (1.32) also called the heights of the lattices Λ and Λ_j, respectively.

This result was generalised and improved in various ways by Gaudron [Gau09] and Gaudron & Rémond [GR12]. In particular, (2.2) has been extended to all 0-symmetric bodies. For instance, the following is a simplified version of [Gau09, Theorem 6.1] when we assume that $\operatorname{rank}\Lambda_j = \operatorname{rank}\Lambda - 1 = r-1$ (see also [GR12, Theorem 2.2, Corollary 3.3])

$$\begin{aligned}\lambda_1\Big(C,\Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\Big) \leq \nu \max_{1\leq j\leq s}\Bigg\{1\,,\ \frac{\nu^{r-1}\operatorname{vol}(C\cap \operatorname{lin}\Lambda_j)}{\kappa_r \det\Lambda_j}\,,\\ \left(\frac{\nu}{\lambda_1(C,\Lambda\cap\operatorname{lin}\Lambda_j)}\right)^{\frac{r-2}{2}}\Bigg\},\end{aligned} \tag{2.3}$$

where $\nu = 7\,r(s\kappa_r \det\Lambda/\operatorname{vol}(C))^{1/r}$ and κ_r is again the volume of the r-dimensional Euclidean unit ball.

In the first section of this chapter we complement these results on forbidden lower-dimensional lattices by bounds which take care of the size or the structure of the individual forbidden sublattices and also ask for more than one linearly independent lattice point outside of the restrictions.

In Section 2.2 we extend the investigation to the case of restrictions of full rank.

All results of this chapter were obtained in joint work with Martin Henk [HT13].

Both Fukshansky and Gaudron also gave adelic generalisations of their results. We will come back to this in Section 5.2.

2.1 Avoiding Lower-dimensional Sublattices

The central argument, used throughout this section, is that the number of lattice points of any lattice $\Lambda' \in \{\Lambda, \Lambda_1, \ldots, \Lambda_s\}$ in the dilated body λC grows roughly with the power $\operatorname{rank}\Lambda'$ of λ for large λ. That is

$$\left|\lambda C \cap \Lambda'\right| \approx \lambda^{\operatorname{rank}\Lambda'} \cdot \operatorname{vol}\left(C \cap \operatorname{lin}\Lambda'\right) \qquad \text{as } \lambda \to \infty.$$

But in this section we will always assume $\operatorname{rank}\Lambda_j < \operatorname{rank}\Lambda = m$ for the forbidden lattices $\Lambda_1, \ldots, \Lambda_s$. Therefore the numbers $\left|\lambda C \cap \Lambda_j\right|$ grow at most like λ^{n-1}, while the number of points $\left|\lambda C \cap \Lambda\right|$ grows like λ^n.

Thus for very large $\overline{\lambda} \gg 1$ we surely get

$$\left|\overline{\lambda} C \cap \Lambda\right| > \sum_{j=1}^{s} \left|\overline{\lambda} C \cap \Lambda_j\right|.$$

The main idea used in the proofs of the results given below is to find a small $\overline{\lambda}$ that satisfies this inequality.

Theorem 2.1.1. *Let* $C \in \mathcal{K}_0^m$ *and* $\Lambda \in \mathcal{L}^m$ *with* $\operatorname{rank}\Lambda = m \geq 2$. *Let further* $\Lambda_1, \ldots, \Lambda_s \subset \Lambda$ *be a non-trivial collection of sublattices with* $\operatorname{rank}\Lambda_j \leq m-1$, $1 \leq j \leq s$. *Then*

$$\lambda_1\left(C, \Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\right) < 6^{m-1} \frac{\det\Lambda}{\lambda_1(C,\Lambda)^{m-2} \operatorname{vol}(C)} \left(\sum_{j=1}^{s} \frac{1}{\lambda_1(C,\Lambda_j)}\right) + \sqrt[m]{2^m \frac{\det\Lambda}{\operatorname{vol}(C)}}.$$

Observe that if $\lambda_1(C,\Lambda_j)$ is very large, which means that the restriction imposed by Λ_j is of little importance, the first summand of our bound will become smaller. And in fact it converges to the case of the trivial collection $s = 0$ or all $\Lambda_j = \{0\}$, for which the strict inequality has to be relaxed and reduces to Minkowski's First Theorem (1.3).

Proof. Notice that our bound exhibits the same behaviour as $\lambda_1\left(C, \Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\right)$ under scaling of C, cf. (2.1). Therefore, without loss of generality we may assume that $\lambda_1(C,\Lambda) = 1$, i.e. C contains no non-trivial lattice point in its interior and by assumption also $\lambda_1(C,\Lambda_j) \geq \lambda_1(C,\Lambda) = 1$ for $1 \leq j \leq m$.

For all $1 \le j \le m$ let $m_j = \operatorname{rank}\Lambda_j < m$. By Theorem 1.1.6 for every $\gamma \ge 1$ we now get

$$\left|\gamma C \setminus \{0\} \cap \Lambda_j\right| \le \left(\gamma \frac{2}{\lambda_1(C,\Lambda_j)} + 1\right)^{m_j} - 1 \tag{2.4}$$
$$\le \gamma^{m_j} \sum_{k=0}^{m_j} \binom{m_j}{k} \left(\frac{2}{\lambda_1(C,\Lambda_j)}\right)^k 1^{m_j-k} - 1$$
$$\le \gamma^{m_j} \sum_{k=1}^{m_j} \binom{m_j}{k} \frac{2^k}{\lambda_1(C,\Lambda_j)}$$
$$< \gamma^{m_j} 3^{m_j} \frac{1}{\lambda_1(C,\Lambda_j)}$$
$$\le \gamma^{m-1} 3^{m-1} \frac{1}{\lambda_1(C,\Lambda_j)}.$$

Hence, for $\gamma \ge 1$ we have

$$\left|\gamma C \setminus \{0\} \cap \left(\bigcup_{j=1}^{s} \Lambda_j\right)\right| < \gamma^{m-1} 3^{m-1} \sum_{j=1}^{s} \frac{1}{\lambda_1(C,\Lambda_j)}. \tag{2.5}$$

We now combine this bound with van der Corput's upper bound of Theorem 1.1.3. This leads, again for $\gamma \ge 1$, to the estimate

$$\left|\gamma C \setminus \{0\} \cap \Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\right| \ge \left|\gamma C \setminus \{0\} \cap \Lambda\right| - \left|\gamma C \setminus \{0\} \cap \left(\bigcup_{j=1}^{s} \Lambda_j\right)\right|$$
$$> \gamma^m \frac{\operatorname{vol}(C)}{2^{m-1} \det\Lambda} - 2 - \gamma^{m-1} 3^{m-1} \left(\sum_{j=1}^{s} \frac{1}{\lambda_1(C,\Lambda_j)}\right)$$
$$= \frac{\operatorname{vol}(C)}{2^{m-1} \det\Lambda} \left(\gamma^m - \gamma^{m-1}\beta - \rho\right), \tag{2.6}$$

where

$$\beta = 6^{m-1} \frac{\det\Lambda}{\operatorname{vol}(C)} \left(\sum_{j=1}^{s} \frac{1}{\lambda_1(C,\Lambda_j)}\right) > 0, \quad \rho = 2^m \frac{\det\Lambda}{\operatorname{vol}(C)}.$$

Observe that γC contains a non-trivial lattice point of $\Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j$ if the lower bound (2.6) is strictly positive. Hence, we have to find $\gamma \ge 1$, depending on β and ρ, such that $\gamma^m - \gamma^{m-1}\beta - \rho > 0$. To this end let $\overline{\gamma} = \beta + \rho^{1/m}$. Then

$$\overline{\gamma}^m - \overline{\gamma}^{m-1}\beta = \left(\beta + \rho^{1/m}\right)^m - \left(\beta + \rho^{1/m}\right)^{m-1}\beta \tag{2.7}$$
$$= \left(\beta + \rho^{1/m} - \beta\right) \cdot \left(\beta + \rho^{1/m}\right)^{m-1}$$
$$> \rho^{1/m} \cdot \rho^{(m-1)/m}$$
$$= \rho.$$

Finally, we observe that

$$\overline{\gamma} > \rho^{1/m} = \left(2^m \det\Lambda / \operatorname{vol}(C)\right)^{1/m} \geq \lambda_1(C,\Lambda) = 1$$

by Minkowski's First Theorem (1.3) and our assumption. Hence, $\overline{\gamma} > 1$ and in view of (2.7) we have $\lambda_1(C, \Lambda \setminus \bigcup_{j=1}^s \Lambda_j) < \overline{\gamma}$, which by the definition of $\overline{\gamma}$ yields the desired bound of the theorem with respect to our normalisation $\lambda_1(C,\Lambda) = 1$. □

We remark that inequality (2.5) is not optimal if the restricted lattices do intersect non-trivially. However, as we do not know how the sublattices intersect, taking these intersections into account is beyond the topic of this thesis. In any case, solving a more general inequality of degree n would still involve considering the worst case of lattices of degree $n-1$.

Theorem 2.1.1 can easily be extended inductively to higher restricted successive minima $\lambda_{i+1}\left(C, \Lambda \setminus \bigcup_{j=1}^s \Lambda_j\right)$, $1 \leq i \leq m-1$, by avoiding, in addition to $\Lambda_1, \ldots, \Lambda_s$, an i-dimensional lattice containing i linearly independent lattice points corresponding to the successive minima $\lambda_k\left(C, \Lambda \setminus \bigcup_{j=1}^s \Lambda_j\right)$, $1 \leq k \leq i$.

Corollary 2.1.2. *Let* $C \in \mathcal{K}_0^m$ *and* $\Lambda \in \mathcal{L}^m$ *with* $\operatorname{rank}\Lambda = m \geq 2$. *Let further* $\Lambda_1, \ldots, \Lambda_s \subset \Lambda$ *be a non-trivial collection of sublattices with* $\operatorname{rank}\Lambda_j \leq m-1$, $1 \leq j \leq s$. *Then for all* $i = 0, \ldots, m-1$

$$\lambda_{i+1}\Big(C, \Lambda \setminus \bigcup_{j=1}^s \Lambda_j\Big) < 6^{m-1} \frac{\det\Lambda}{\lambda_1(C,\Lambda)^{m-2}\operatorname{vol}(C)} \left(\sum_{j=1}^s \frac{1}{\lambda_1(C,\Lambda_j)}\right) + \left(\frac{3^i}{\lambda_1(C,\Lambda)^i} 2^{m-1} \frac{\det\Lambda}{\operatorname{vol}(C)} + \left(2^m \frac{\det\Lambda}{\operatorname{vol}(C)}\right)^{\frac{m-i}{m}}\right)^{\frac{1}{m-i}}.$$

Proof. We proceed by induction. The case $i = 0$ is a weaker version of Theorem 2.1.1. For $i \geq 1$ we proceed as follows. Let $z_k \in \lambda_k\left(C, \Lambda \setminus \bigcup_{j=1}^s \Lambda_j\right) \cdot C \cap \Lambda$ for $1 \leq k \leq i$ be linearly independent, and let $\overline{\Lambda} \subset \Lambda$ be the i-dimensional lattice $\overline{\Lambda} = \Lambda \cap \operatorname{lin}\{z_1, \ldots, z_i\}$. Then

$$\lambda_{i+1}\Big(C, \Lambda \setminus \bigcup_{j=1}^s \Lambda_j\Big) = \lambda_1\Big(C, \Lambda \setminus \Big(\bigcup_{j=1}^s \Lambda_j \cup \overline{\Lambda}\Big)\Big) \tag{2.8}$$

and we now follow the proof of Theorem 2.1.1. In particular, we assume $\lambda_1(C,\Lambda) = 1$.

In addition to the upper bounds on $|\gamma C\setminus\{0\}\cap\Lambda_j|$, $1\le j\le s$, in (2.4), we also use for $\gamma\ge\lambda_1(C,\overline{\Lambda})\ge\lambda_1(C,\Lambda)=1$ the bound

$$|\gamma C\setminus\{0\}\cap\overline{\Lambda}|<\left(\frac{2\gamma}{\lambda_1(C,\overline{\Lambda})}+1\right)^i\le 3^i\left(\frac{\gamma}{\lambda_1(C,\overline{\Lambda})}\right)^i. \tag{2.9}$$

Combining this bound with van der Corput's Theorem 1.1.3 leads for all $\gamma\ge\lambda_1(C,\overline{\Lambda})$ to the estimate

$$\begin{aligned}
\Big|\gamma C\setminus\{0\}\cap\Lambda\setminus\big(\bigcup_{j=1}^{s}\Lambda_j\cup\overline{\Lambda}\big)\Big| &> |\gamma C\setminus\{0\}\cap\Lambda|-\Big|\gamma C\setminus\{0\}\cap\big(\bigcup_{j=1}^{s}\Lambda_j\big)\Big|-|\gamma C\setminus\{0\}\cap\overline{\Lambda}|\\
&>\gamma^m\frac{\mathrm{vol}(C)}{2^{m-1}\det\Lambda}-2 \qquad (2.10)\\
&\quad -\gamma^{m-1}3^{m-1}\left(\sum_{j=1}^{s}\frac{1}{\lambda_1(C,\Lambda_j)}\right)-3^i\left(\frac{\gamma}{\lambda_1(C,\overline{\Lambda})}\right)^i\\
&=\frac{\mathrm{vol}(C)}{2^{m-1}\det\Lambda}\left(\gamma^m-\gamma^{m-1}\beta-\gamma^i\alpha-\rho\right),
\end{aligned}$$

with

$$\beta=6^{m-1}\frac{\det\Lambda}{\mathrm{vol}(C)}\left(\sum_{j=1}^{s}\frac{1}{\lambda_1(C,\Lambda_j)}\right),\qquad \alpha=\frac{3^i}{\lambda_1(C,\overline{\Lambda})^i}2^{m-1}\frac{\det\Lambda}{\mathrm{vol}(C)}$$

and

$$\rho=2^m\frac{\det\Lambda}{\mathrm{vol}(C)}\ge 1 \text{ by } (1.3).$$

Setting now $\overline{\gamma}=\beta+\big(\alpha+\rho^{\frac{m-i}{m}}\big)^{\frac{1}{m-i}}\ge\rho^{\frac{1}{m}}$ we see, as in the proof of Theorem 2.1.1, that

$$\begin{aligned}
\overline{\gamma}^m-\overline{\gamma}^{m-1}\beta-\overline{\gamma}^i\alpha-\rho &=\overline{\gamma}^i\big(\overline{\gamma}^{m-i}-\beta\overline{\gamma}^{m-i-1}-\alpha\big)-\rho\\
&=\overline{\gamma}^i\rho^{\frac{m-i}{m}}-\rho,
\end{aligned} \tag{2.11}$$

since

$$\begin{aligned}
&\overline{\gamma}^{m-i}-\beta\overline{\gamma}^{m-i-1}-\alpha\\
&=\left(\beta+\left(\alpha+\rho^{\frac{m-i}{m}}\right)^{\frac{1}{m-i}}\right)^{m-i}-\beta\left(\beta+\left(\alpha+\rho^{\frac{m-i}{m}}\right)^{\frac{1}{m-i}}\right)^{m-i-1}-\alpha\\
&=\sum_{k=0}^{m-i}\left(\alpha+\rho^{\frac{m-i}{m}}\right)^{\frac{m-i-k}{m-i}}\beta^k-\beta\sum_{k=0}^{m-i-1}\left(\alpha+\rho^{\frac{m-i}{m}}\right)^{\frac{m-i-1-k}{m-i}}\beta^k-\alpha\\
&=\left(\alpha+\rho^{\frac{m-i}{m}}\right)+\sum_{k=1}^{m-i}\left(\alpha+\rho^{\frac{m-i}{m}}\right)^{\frac{m-i-k}{m-i}}\beta^k-\sum_{k=0}^{m-i-1}\left(\alpha+\rho^{\frac{m-i}{m}}\right)^{\frac{m-i-(k+1)}{m-i}}\beta^{k+1}-\alpha\\
&=\rho^{\frac{m-i}{m}}.
\end{aligned}$$

And since $\rho \geq 1$ by (1.3) and $\beta > 0$ and $\alpha > 0$, we have $\overline{\gamma} > \rho^{\frac{1}{m}}$. Thus $\overline{\gamma}^i \rho^{\frac{m-i}{m}} > \rho^{\frac{i}{m}} \rho^{\frac{m-i}{m}} = \rho$ and so with (2.11) we get

$$\overline{\gamma}^m - \overline{\gamma}^{m-1}\beta - \overline{\gamma}^i \alpha - \rho > 0\,. \tag{2.12}$$

Since $\overline{\gamma} > \beta + \rho^{\frac{1}{m}}$ and, as we saw in the proof of Theorem 2.1.1, $\beta + \rho^{\frac{1}{m}}$ is itself an upper bound on $\lambda_1(C, \overline{\Lambda}) = \lambda_1\left(C, \Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\right)$, we also have $\overline{\gamma} > \lambda_1(C, \overline{\Lambda})$. Therefore $\overline{\gamma}$ satisfies the requirement for (2.10). So with (2.12) we conclude $\lambda_{i+1}\left(C, \Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\right) < \overline{\gamma}$ and by definition $\overline{\gamma}$ is the required upper bound with respect to the normalisation $\lambda_1(C, \Lambda) = 1$. The general statement follows together with $\lambda_1(C, \overline{\Lambda}) \geq \lambda_1(C, \Lambda)$. □

The bounds given in Theorem 2.1.1 and Corollary 2.1.2 are in terms of the successive minima. The following variant gives bounds in terms of the determinants of the lattices involved. They also depend on the intersections of C with hyperplanes spanned by sublattices of Λ and the forbidden lattices $\Lambda_1, \ldots, \Lambda_s \subset \Lambda$. The dimensions and volumes of these intersections can in general not be controlled.

Theorem 2.1.3. *Let $C \in \mathcal{K}_0^m$ and $\Lambda \in \mathcal{L}^m$ with* $\operatorname{rank}\Lambda = m \geq 2$. *Let further $\Lambda_1, \ldots, \Lambda_s \subset \Lambda$ be a collection of sublattices with $m_j = \operatorname{rank}\Lambda_j \leq m-1$, $1 \leq j \leq s$. Denote $\overline{\lambda}_1 = \lambda_1(C, \Lambda)$. Then*

$$\lambda_1\Big(C, \Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\Big) \leq \frac{2^{m-1}\det(\Lambda)}{\overline{\lambda}_1^{m-1}\operatorname{vol}_m(C)} \left(\sum_{j=1}^{s} \max_{k=1,\ldots,m_j} \frac{k!\, L_k(2)\, \overline{\lambda}_1^{k} \operatorname{vol}_k\big(C \cap H_k^{(j)}\big)}{2^k \det\big(\Lambda_j \cap H_k^{(j)}\big)}\right) + \sqrt[m]{2^m \frac{\det(\Lambda)}{\operatorname{vol}_m(C)}}\,,$$

where

$$H_k^{(j)} = \operatorname{lin}\{x_1, \ldots, x_k\}$$

with

$$x_i \in \big(\lambda_i(C, \Lambda_j) \cdot C\big) \cap \Lambda_j \text{ for } i = 1, \ldots, m_j\,, \quad \dim\Big(\operatorname{lin}\{x_1, \ldots, x_{m_j}\}\Big) = m_j$$

for $k = 1, \ldots, m_j$ and $j = 1, \ldots, s$ and $L_k(x) = \sum_{\ell=0}^{k} \binom{k}{\ell} \frac{x^\ell}{\ell!}$ is the k-th Laguerre polynomial.

By construction, for a fixed Λ_j, $1 \leq j \leq s$, the linear spaces $H_k^{(j)}$ for $k = 1, \ldots, m_j$ do in fact form a flag $H_1^{(j)} \subsetneq H_2^{(j)} \subsetneq \ldots \subsetneq H_{m_j}^{(j)}$, corresponding to a list of vectors $x_1, \ldots, x_{m_j} \in \Lambda_j$ representing the successive minima of Λ_j.

Proof. The main idea of the argument mirros the proof of Theorem 2.1.1, but uses different bounds on the number of lattice points. Again, we can assume $\overline{\lambda}_1 = \lambda_1(C,\Lambda) = 1$.

So by Henze's variant of Blichfeldt, Theorem 1.1.5, if $\gamma \geq \lambda_k(C,\Lambda_j)$,

$$\left|\gamma C \cap \Lambda_j \cap H_k^{(j)}\right| \leq \frac{k!\, L_k(2)\, \mathrm{vol}_k\big(\gamma C \cap H_k^{(j)}\big)}{2^k \det\big(\Lambda_j \cap H_k^{(j)}\big)} = \gamma^k \frac{k!\, L_k(2)\, \mathrm{vol}_k\big(C \cap H_k^{(j)}\big)}{2^k \det\big(\Lambda_j \cap H_k^{(j)}\big)},$$

but also for $\gamma < \lambda_1(C,\Lambda_j)$ the inequality

$$0 = \left|\gamma C \setminus \{0\} \cap \Lambda_j \cap H_1^{(j)}\right| \leq \frac{1!\, L_1(2)\, \mathrm{vol}_1\big(\gamma C \cap H_1^{(j)}\big)}{2^1 \det\big(\Lambda_j \cap H_1^{(j)}\big)}$$

is still satisfied.

Combining the cases we get for all $\gamma \geq 1$ the estimate

$$\begin{aligned} |\gamma C \setminus \{0\} \cap \Lambda_j| &\leq \max_{k=1,\dots,m_j} \gamma^k \frac{k!\, L_k(2)\, \mathrm{vol}_k\big(C \cap H_k^{(j)}\big)}{2^k \det\big(\Lambda_j \cap H_k^{(j)}\big)} \\ &\leq \gamma^{m-1} \max_{k=1,\dots,m_j} \frac{k!\, L_k(2)\, \mathrm{vol}_k\big(C \cap H_k^{(j)}\big)}{2^k \det\big(\Lambda_j \cap H_k^{(j)}\big)}. \end{aligned}$$

The above combined with van der Corput's Theorem 1.1.3 gives

$$\begin{aligned} \Big|\gamma C \setminus \{0\} \cap \Lambda \setminus \big(\bigcup_{j=1}^{s} \Lambda_j\big)\Big| &\geq \big|\gamma C \setminus \{0\} \cap \Lambda\big| - \sum_{j=1}^{s} \big|\gamma C \setminus \{0\} \cap \Lambda_j\big| \\ &\geq \gamma^m \frac{\mathrm{vol}_m(C)}{2^{m-1}\det(\Lambda)} - 2 \\ &\quad - \sum_{j=1}^{s} \gamma^{m-1} \max_{k=1,\dots,m_j} \frac{k!\, L_k(2)\, \mathrm{vol}_k\big(C \cap H_k^{(j)}\big)}{2^k \det\big(\Lambda_j \cap H_k^{(j)}\big)} \\ &= \frac{\mathrm{vol}_m(C)}{2^{m-1}\det\Lambda}\big(\gamma^m - \gamma^{m-1}\beta - \rho\big), \end{aligned} \tag{2.13}$$

where

$$\beta = \frac{2^{m-1}\det\Lambda}{\mathrm{vol}(C)} \left(\sum_{j=1}^{s} \max_{k=1,\dots,m_j} \frac{k!\, L_k(2)\, \mathrm{vol}_k\big(C \cap H_k^{(j)}\big)}{2^k \det\big(\Lambda_j \cap H_k^{(j)}\big)} \right), \quad \rho = 2^m \frac{\det\Lambda}{\mathrm{vol}(C)}.$$

Notice that up to the definition of β, (2.13) is exactly the same as (2.6).

Therefore, the rest of the argument from the proof of Theorem 2.1.1 can be applied verbatim. Thus $\overline{\gamma} = \beta + \rho^{1/m}$ again yields the desired bound with respect to the normalisation $\lambda_1(C, \Lambda) = 1$. □

To conclude this section, we give an upper bound on $\lambda_1\big(C, \Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\big)$ of a different kind, involving the inhomogeneous minimum $\mu(C, \Lambda)$, see (1.7), of C with respect to Λ.

Proposition 2.1.4. *Let* $C \in \mathscr{K}_0^m$ *and* $\Lambda \in \mathscr{L}^m$ *with* $\operatorname{rank}\Lambda = m \geq 2$. *Let further* $\Lambda_1, \ldots, \Lambda_s \subset \Lambda$ *be a collection of sublattices with* $\operatorname{rank}\Lambda_j \leq m-1$, $1 \leq j \leq s$. *Then*

$$\lambda_1\Big(C, \Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\Big) \leq (s+1)\, \mu(C, \Lambda)$$

and hence, $\lambda_i\big(C, \Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\big) \leq (s+2)\, \mu(C, \Lambda)$ *for* $2 \leq i \leq m$.

Proof. Observe that on account of (2.8) the bound for $i \geq 2$ follows from the one for $\lambda_1\big(C, \Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\big)$. For the proof in the case $i = 1$ let $H_j = \operatorname{lin}\Lambda_j$ for $1 \leq j \leq s$, and for short we write $\overline{\mu}$ instead of $\mu(C, \Lambda)$. By Ball's [Bal91] solution of the affine plank problem for 0-symmetric convex bodies, applied to $\overline{\mu}C$, we know that there exists a $z \in \mathbb{R}^m$ such that

$$\big(z + \tfrac{1}{s+1}\overline{\mu}C\big) \subset \overline{\mu}C \qquad \text{and} \qquad \operatorname{int}\big(z + \tfrac{1}{s+1}\overline{\mu}C\big) \cap H_j = \emptyset \text{ for } 1 \leq j \leq s,$$

where $\operatorname{int}(\cdot)$ denotes the interior. Thus, for any $\varepsilon > 0$ the body $(s+1+\varepsilon)\overline{\mu}C$ contains a translate $z_\varepsilon + \overline{\mu}C$ having no points in common with H_j for all $1 \leq j \leq s$.

Hence, together with the definition of the covering radius, we have

$$(z_\varepsilon + \overline{\mu}C) \cap \Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j \neq \emptyset$$

and so

$$\lambda_1\Big(C, \Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\Big) \leq (s+1+\varepsilon)\overline{\mu}.$$

By the arbitrariness of ε and the compactness of C the assertion follows. □

Notice that in contrast to the previous results, this bound depends only on C and Λ and the number s of restrictions, but not the forbidden lattices $\Lambda_1, \ldots, \Lambda_s$ themselves. We also get the following consequence.

Corollary 2.1.5. *Let* $C \in \mathcal{K}_0^m$ *and* $\Lambda \in \mathcal{L}^m$ *with* $\operatorname{rank}\Lambda = m \geq 2$. *Let further* $\Lambda_1, \dots, \Lambda_s \subset \Lambda$ *be a collection of sublattices with* $\operatorname{rank}\Lambda_j \leq m-1$, $1 \leq j \leq s$. *Then*

$$\lambda_1\Big(C, \Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\Big) \leq \tfrac{1}{2}(s+1) \sum_{i=1}^{m} \lambda_i(C,\Lambda) \leq \tfrac{1}{2}\, m\,(s+1)\,\lambda_m(C,\Lambda)$$

and hence, $\lambda_i\big(C, \Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\big) \leq \frac{1}{2}\, m\,(s+2)\,\lambda_m(C,\Lambda)$ *for* $2 \leq i \leq m$.

Proof. The result is a direct consequence of Proposition 2.1.4 and Jarník's inequality (1.8), i.e.

$$\mu(C,\Lambda) \leq \tfrac{1}{2} \sum_{i=1}^{m} \lambda_i(C,\Lambda) \leq \tfrac{1}{2}\, m\,\lambda_m(C,\Lambda)\,. \qquad \square$$

Comparison to Previous Results

As already mentioned, our bounds all share the property of reducing to Minkowski's First Theorem (1.3) for vanishing restrictions, i.e. $\lambda_1(C,\Lambda_j) \to \infty$ or $\det(\Lambda_j)/\operatorname{vol}_k(C \cap H_k^{(j)}) \to \infty$ for all j and k. In this case the bounds of Fukshansky in (2.2) and Gaudron in (2.3) still have a dependency on s of order $\sqrt{s}$ and $s^{1/r}$, respectively.

In the case of Theorem 2.1.1, the dependence on $\lambda_1(C,\Lambda_j)$ for all j instead of the respective determinants also better reflects the restriction imposed by Λ_j. In general, the determinant of Λ_j can be arbitrarily large, while it contains a small lattice point. But this is the case exactly if $\lambda_1(C,\Lambda_j)$ is small.

We remark that restricted successive minima have also been investigated from an algorithmic point of view. Blömer and Naewe [BN09] studied the complexity of computing $\lambda_1\left(C, \Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\right)$ for $s = 1$ and when C is the unit ball of an l_p-norm. Among other things, they show that some of the well-known lattice problems, like the shortest or closest lattice vector problem, are polynomial reducible to computing/approximating $\lambda_1(C, \Lambda \setminus \Lambda_1)$. Moreover, as in the case of these lattice problems an LLL-reduced basis (cf. [GLS93, Chap. 5]) can be used to find in polynomial time a lattice vector b which approximates $\lambda_1(B_m, \Lambda \setminus \Lambda_1)$ up to a factor of 2^{m-1} [BN09, Theorem 3.9]. Here B_m is again the unit ball of the Euclidean norm. Hence, Theorem 2.1.1 implies (cf. [GLS93, Thm. 5.3.13 a)] for a similar result in the standard setting, $s = 0$) the following algorithmic result.

Corollary 2.1.6. *Let* $\Lambda \in \mathcal{L}^m$ *with* $\operatorname{rank}\Lambda = m \geq 2$ *and let* $\Lambda_1 \subset \Lambda$ *be a sublattice with* $\operatorname{rank}\Lambda_1 \leq m-1$. *Then there exists a polynomial time algorithm for* *computing a lattice point $b \in \Lambda \setminus \Lambda_1$ of Euclidean length*

$$\|b\| < 2^{m-1}\left(6^{m-1}\frac{\det\Lambda}{\lambda_1(\mathrm{B}_m,\Lambda)^{m-2}\operatorname{vol}(\mathrm{B}_m)}\frac{1}{\lambda_1(\mathrm{B}_m,\Lambda_1)} + \sqrt[m]{2^m\frac{\det\Lambda}{\operatorname{vol}(\mathrm{B}_m)}}\right).$$

It seems to be a challenging problem to extend this result to more than one forbidden sublattice.

We also mention a closely related problem, namely to cover the lattice points $C \cap \Lambda$ of a body $C \in \mathcal{K}_0^n$ and a lattice $\Lambda \in \mathcal{L}^m$ by a minimal number $\gamma(C,\Lambda)$ of lattice hyperplanes. Obviously, given $\nu > 0$ with $\gamma(\nu C,\Lambda) \geq s+1$ we get

$$\lambda_1\Big(C, \Lambda \setminus \bigcup_{i=1}^{s} \Lambda_i\Big) \leq \nu$$

for any collection $\Lambda_1,\ldots,\Lambda_s$ of lower dimensional sublattices. Bárány et al. [BHPT01] showed

$$\gamma(C,\mathbb{Z}^m) \geq \frac{1-\lambda_m(C,\mathbb{Z}^m)}{16\,m^2}\min_{0<r<m}\left(\lambda_r(C,\mathbb{Z}^m)\cdots\lambda_m(C,\mathbb{Z}^m)\right)^{\frac{-1}{m-r}}$$

and thus, if

$$\min_{0<r<m}\left(\nu^{-(m-r+1)}\lambda_r(C,\mathbb{Z}^m)\cdots\lambda_m(C,\mathbb{Z}^m)\right)^{\frac{-1}{m-r}} \geq \frac{16m^2(s+1)\nu}{\nu-\lambda_m(C,\mathbb{Z}^m)},$$

we can ensure $\gamma(\nu C,\Lambda) \geq s+1$, which for the cube $C = \mathrm{C}_m$ reduces to

$$\nu^{\frac{m}{m-1}} - \nu^{\frac{1}{m-1}} \geq 16m^2(s+1),$$

not improving the bound of Proposition 2.1.4. Further bounds, also in terms of other functionals from the Geometry of Numbers, have been studied by Bezdek and Hausel [BH94] and Bezdek and Litvak [BL09].

2.2 Avoiding Full-dimensional Sublattices

In contrast to the last section, we will now assume that $\operatorname{rank}\Lambda_j = \operatorname{rank}\Lambda$ for all forbidden Λ_j. Therefore, we can no longer use the techniques applied before.

The tool we are using in this full-dimensional case is the torus group $\mathbb{R}^m/\overline{\Lambda}$ for a certain lattice $\overline{\Lambda}$. For a more detailed discussion we refer to [Gru07, Section 26].

As $\overline{\Lambda} \subset \mathbb{R}^m$ is an additive subgroup, $\mathbb{R}^m/\overline{\Lambda}$, equipped with the quotient topology is a compact abelian topological group. We may identify $\mathbb{R}^m/\overline{\Lambda}$ with a fundamental parallelepiped P of $\overline{\Lambda}$, i.e.

$$\mathbb{R}^m/\overline{\Lambda} \cong P = \left\{ \alpha_1 b_1 + \cdots + \alpha_m b_m \,\middle|\, 0 \le \alpha_i < 1 \right\},$$

where $b_1, \ldots, b_m$ form a basis of $\overline{\Lambda}$. Via this identification the set $X \subset \mathbb{R}^m$ maps to X modulo $\overline{\Lambda}$, written $X/\overline{\Lambda}$, and can be described (thought of) as

$$X/\overline{\Lambda} = \left\{ y \in P \,\middle|\, \exists b \in \overline{\Lambda} \text{ s.t. } y + b \in X \right\} = (\overline{\Lambda} + X) \cap P$$

and we can think of $\overline{X} \subseteq \mathbb{R}^m/\overline{\Lambda}$ as its image under inclusion into $\mathbb{R}^m$. In the same spirit we may identify the addition $\oplus$ in $\mathbb{R}^m/\overline{\Lambda}$ with the corresponding operation in $\mathbb{R}^m$, i.e. for $\overline{X}_1, \overline{X}_2 \subset \mathbb{R}^m/\overline{\Lambda}$ we have

$$\overline{X}_1 \oplus \overline{X}_2 = \left(\left(\overline{X}_1 + \overline{X}_2 \right) + \overline{\Lambda} \right) \cap P .$$

As $T = \mathbb{R}^m/\overline{\Lambda}$ is a compact abelian group, it has a unique Haar measure $\mathrm{vol}_T(\cdot)$, normalised to $\mathrm{vol}_T(\mathbb{R}^n/\overline{\Lambda}) = \det \overline{\Lambda}$. For a measurable subset $X \subset \mathbb{R}^m$ or $\overline{X} \subset \mathbb{R}^m/\overline{\Lambda}$ we have

$$\mathrm{vol}_T(X/\overline{\Lambda}) = \mathrm{vol}((\overline{\Lambda} + X) \cap P) \quad \text{and} \quad \mathrm{vol}_T(\overline{X}) = \mathrm{vol}((\overline{\Lambda} + \overline{X}) \cap P) .$$

Regarding the volume of the sum of two sets $\overline{X}_1, \overline{X}_2 \subset \mathbb{R}^m/\overline{\Lambda}$ we have the following classical so-called sum theorem of Kneser and Macbeath.

Theorem 2.2.1 (Kneser-Macbeath [Gru07, Theorem 26.1])**.** *Let $\overline{X}_1, \overline{X}_2 \subset \mathbb{R}^m/\overline{\Lambda}$, such that $\overline{X}_1, \overline{X}_2$ and $\overline{X}_1 + \overline{X}_2$ are measurable, then*

$$\mathrm{vol}_T\left(\overline{X}_1 \oplus \overline{X}_2\right) \ge \min\left\{ \mathrm{vol}_T\left(\overline{X}_1\right) + \mathrm{vol}_T\left(\overline{X}_2\right), \det \overline{\Lambda} \right\} .$$

As we have established in (1.10), for $0 \le \lambda \le \lambda_1(C, \overline{\Lambda})/2$, λC is a packing with respect to $\overline{\Lambda}$ and therefore

$$\mathrm{vol}_T((\lambda C)/\overline{\Lambda})) = \mathrm{vol}((\lambda C + \overline{\Lambda}) \cap P) = \lambda^m \mathrm{vol}(C) . \tag{2.14}$$

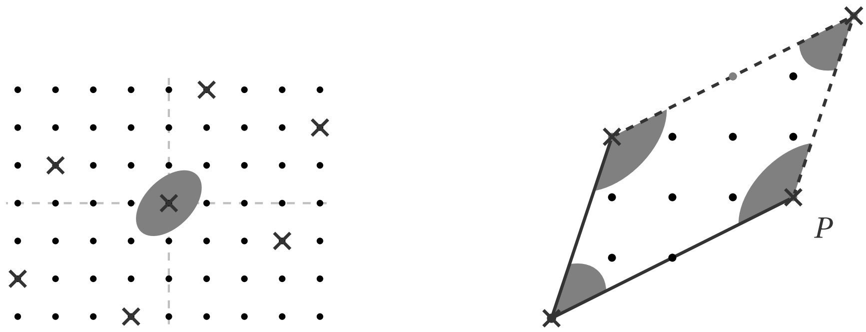
Furthermore, we also need a version of van der Corput's result, Theorem 1.1.3, for a convex body in the torus group.

Lemma 2.2.2. *Let $C \in \mathcal{K}_0^m$, $\Lambda \in \mathcal{L}^m$, $\mathrm{rank}\,\Lambda = m$ and let $\overline{\Lambda} \subsetneq \Lambda$ be a sublattice with $\mathrm{rank}\,\overline{\Lambda} = m$, and let $k \in \mathbb{N}$ with $k \det \Lambda < \det \overline{\Lambda}$.*

If $\mathrm{vol}_T\left(\frac{1}{2} C/\overline{\Lambda}\right) \ge k \det \Lambda$ then

$$\left| C/\overline{\Lambda} \cap \Lambda \right| \ge k + 1,$$

i.e. C contains at least $k+1$ lattice points of Λ belonging to different cosets modulo $\overline{\Lambda}$.



C and $\mathbb{Z}^m$ with $\overline{\Lambda}$ (crosses) in $\mathbb{R}^m$ $\qquad$ C and $\mathbb{Z}^m$ in $\mathbb{R}^m/\overline{\Lambda} \cong P$

Figure 2.1: the torus group

Proof. First, as C is compact and the lattice $\overline{\Lambda}$ discrete, without loss of generality we may assume $\mathrm{vol}_T\big(\frac{1}{2}C/\overline{\Lambda}\big) > k \det \Lambda$. Let P be a fundamental parallelepiped of the lattice $\overline{\Lambda}$. Then, equivalently, $\mathrm{vol}\, X > k \det \Lambda$ for the measurable set $X = \big(\frac{1}{2}C + \overline{\Lambda}\big) \cap P$. According to van der Corput's original result (1.16) there exist pairwise different $x_i \in X$, $1 \le i \le k+1$, such that $x_i - x_j \in \Lambda$. By the 0-symmetry and convexity of C, we have $(X - X) = (C + \overline{\Lambda}) \cap (P - P)$ and since $(P - P) \cap \overline{\Lambda} = \{0\}$ we conclude

$$x_i - x_j \in \Big(\big(C + \overline{\Lambda}\big) \cap \Lambda\Big) \setminus \overline{\Lambda}, \quad \text{for } i \ne j.$$

Hence the k points $x_i - x_1 \in C + \overline{\Lambda}$, $i = 2, \ldots, k+1$, belong to different non-trivial cosets of Λ modulo $\overline{\Lambda}$. Together with the trivial coset represented by the origin, $\big|C/\overline{\Lambda} \cap \Lambda\big| \ge k+1$. □

The next lemma states some simple facts on the intersection of full-dimensional sublattices.

Lemma 2.2.3. *Let* $\Lambda \in \mathscr{L}^m$, $\Lambda_j \subseteq \Lambda$, $1 \le j \le s$, $\mathrm{rank}\,\Lambda_j = \mathrm{rank}\,\Lambda = m$, *and let* $\overline{\Lambda} = \bigcap_{j=1}^{s} \Lambda_j$. *Then* $\overline{\Lambda} \in \mathscr{L}^m$ *with* $\mathrm{rank}\,\overline{\Lambda} = m$, *and*

$$\max_{1 \le j \le s} \det \Lambda_j \le \det \overline{\Lambda} \le (\det \Lambda)^{1-s} \det \Lambda_1 \cdot \ldots \cdot \det \Lambda_s.$$

Moreover, with $k = \displaystyle\sum_{j=1}^{s} \frac{\det \overline{\Lambda}}{\det \Lambda_j} - s + 1$ *we have*

i) The union $\bigcup_{j=1}^{s} \Lambda_j$ *is covered by at most* k *cosets of* Λ *modulo* $\overline{\Lambda}$.

ii) If $\dfrac{\det\overline{\Lambda}}{\det\Lambda} \geq k+1$*, then* $\Lambda \neq \bigcup_{j=1}^{s}\Lambda_j$.

Proof. In order to show that $\overline{\Lambda}$ is a full-dimensional lattice, it suffices to consider the case $s=2$, the general statement then follows inductively.

Obviously, $\Lambda_1 \cap \Lambda_2$ is a discrete subgroup of Λ and it also contains m linearly independent points, e.g. $(\det\Lambda_2)a_1, \ldots, (\det\Lambda_2)a_m$, where $a_1, \ldots, a_m$ is a basis of Λ_1. Hence $\overline{\Lambda}$ is a full-dimensional lattice, cf. [GL87, Theorem 2, Sec 3.2].

The lower bound on $\det\overline{\Lambda}$ is clear from the inclusion $\overline{\Lambda} \subseteq \Lambda_j$ for all $1 \leq j \leq s$.

For the upper bound we interpret the determinants as group indices, cf. (1.2), and count the cosets of $\Lambda' \in \{\Lambda, \Lambda_1, \ldots, \Lambda_s\}$ modulo $\overline{\Lambda}$. These are

$$\left[\Lambda' : \overline{\Lambda}\right] = \frac{\left[\mathbb{Z}^m : \Lambda'\right]}{\left[\mathbb{Z}^m : \Lambda'\right]}\left[\Lambda' : \overline{\Lambda}\right] = \frac{\left[\mathbb{Z}^m : \overline{\Lambda}\right]}{\det\Lambda'} = \frac{\det\overline{\Lambda}}{\det\Lambda'}.$$

Now observe that two points $g, h \in \Lambda$ belong to different cosets modulo $\overline{\Lambda}$ if and only if g and h belong to different cosets of Λ modulo at least one Λ_j. Therefore

$$\frac{\det\overline{\Lambda}}{\det\Lambda} \leq \frac{\det\Lambda_1}{\det\Lambda}\cdots\frac{\det\Lambda_s}{\det\Lambda}.$$

For i) we note that each Λ_j is the union of $\left[\Lambda_j : \overline{\Lambda}\right] = \det\overline{\Lambda}/\det\Lambda_j$ many of its cosets modulo $\overline{\Lambda}$. The union of all Λ_j is certainly covered by the union of all of these cosets and so there are no more than

$$\sum_{j=1}^{s}\left[\Lambda_j : \overline{\Lambda}\right] = \sum_{j=1}^{s}\frac{\det\overline{\Lambda}}{\det\Lambda_j}$$

many cosets of Λ modulo $\overline{\Lambda}$ covering $\bigcup_{j=1}^{s}\Lambda_j$. But here we have counted the trivial coset at least s times.

Finally, ii) is a direct consequence of the above, as $\det\overline{\Lambda}/\det\Lambda = \left[\Lambda : \overline{\Lambda}\right] \geq k+1$ implies that at least one coset of Λ and thus one of its elements must be outside of $\bigcup_{j=1}^{s}\Lambda_j$. □

Lemma 2.2.3 ii) implies, in particular, that the union of two strict sublattices can never be the whole lattice. This is no longer true for three sublattices, as we see in the next example, which also shows that Lemma 2.2.3 ii) is not an equivalence.

Example 2.2.4. Let $\Lambda = \mathbb{Z}^2$, and let $\Lambda_1, \dots, \Lambda_4 \subset \mathbb{Z}^2$ be the sublattices

$$\Lambda_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbb{Z}^2, \qquad \Lambda_2 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \mathbb{Z}^2,$$
$$\Lambda_3 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \mathbb{Z}^2, \qquad \Lambda_4 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \mathbb{Z}^2,$$

so $\Lambda_4 = \left\{ (z_1, z_2)^\intercal \in \mathbb{Z}^2 \,\middle|\, z_1 \equiv z_2 \bmod 2 \right\}$. Then by construction $\Lambda_1 \cup \Lambda_2 \cup \Lambda_4 = \Lambda$, but $\Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \neq \Lambda$. Furthermore $\det \Lambda = 1$, while $\det \Lambda_1 = \det \Lambda_2 = \det \Lambda_4 = 2$, $\det \Lambda_3 = 3$ and

$$\overline{\Lambda} = \Lambda_1 \cap \Lambda_2 \cap \Lambda_3 = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix} \mathbb{Z}^2,$$

with $\det \overline{\Lambda} = 12$, while $\displaystyle\sum_{j=1}^{3} \frac{\det \overline{\Lambda}}{\det \Lambda_j} - 1 = 15.$

We now proceed to the main theory of this section.

Theorem 2.2.5. *Let $C \in \mathcal{K}_0^m$ and $\Lambda \in \mathcal{L}^m$ with* $\operatorname{rank} \Lambda = m \geq 2$. *Let further $\Lambda_1, \dots, \Lambda_s \subset \Lambda$ be a collection of sublattices with* $\operatorname{rank} \Lambda_j = m$, *$1 \leq j \leq s$, such that $\bigcup_{j=1}^{s} \Lambda_j \neq \Lambda$. Then*

$$\lambda_1\Big(C, \Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\Big) < \frac{2^m \det \Lambda}{\lambda_1(C, \overline{\Lambda})^{m-1} \operatorname{vol}(C)} \Big(\sum_{j=1}^{s} \frac{\det \overline{\Lambda}}{\det \Lambda_j} - s + 1 \Big) + \lambda_1(C, \overline{\Lambda}),$$

where $\overline{\Lambda} = \bigcap_{j=1}^{s} \Lambda_j$.

Proof. Let

$$k = \min \left\{ \sum_{j=1}^{s} \frac{\det \overline{\Lambda}}{\det \Lambda_j} - s + 1 \,,\, \frac{\det \overline{\Lambda}}{\det \Lambda} \right\}.$$

We first assume that $\lambda > 0$ satisfies

$$(*) \qquad \operatorname{vol}_T\big((\lambda \tfrac{1}{2} K) / \overline{\Lambda} \big) \geq k \det \Lambda$$

and show that this implies

$$(2.15) \qquad \lambda_1\big(K, \Lambda \setminus \textstyle\bigcup_{j=1}^{s} \Lambda_j\big) \leq \lambda \,.$$

In order to verify this claim, we start with the case

$$k = \sum_{j=1}^{s} \frac{\det \overline{\Lambda}}{\det \Lambda_j} - s + 1 < \frac{\det \overline{\Lambda}}{\det \Lambda}.$$

Then, by Lemma 2.2.2, λC contains $k+1$ lattice points of Λ belonging to different cosets with respect to $\overline{\Lambda}$. But by Lemma 2.2.3 i), the union $\bigcup_{j=1}^{s} \Lambda_j$ is covered by at most k cosets of Λ modulo $\overline{\Lambda}$. Thus λC contains a lattice point of $\Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j$.

Next suppose that $k = \det\overline{\Lambda}/\det\Lambda$. Then

$$\mathrm{vol}_T\left(\left(\lambda \tfrac{1}{2}C\right)/\overline{\Lambda}\right) = \det\overline{\Lambda} = \mathrm{vol}_T\left(\mathbb{R}^m/\overline{\Lambda}\right)$$

and, in particular, λC contains a representative of each coset of Λ modulo $\overline{\Lambda}$. By assumption there exists a coset containing a point $a \in \Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j$ and hence, all points of this coset, i.e. $a+\overline{\Lambda}$, lie in $\Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j$.

This verifies (2.15) and it remains to compute a λ that satisfies $(*)$.

To this end we set $\lambda_1 = \lambda_1(C, \overline{\Lambda})$ and we write an arbitrary $\lambda > 0$ modulo λ_1 in the form

$$\lambda = \lambda_1\left(\left\lfloor \frac{\lambda}{\lambda_1} \right\rfloor + \frac{\lambda}{\lambda_1} - \left\lfloor \frac{\lambda}{\lambda_1} \right\rfloor\right) = \lambda_1\left(\left\lfloor \frac{\lambda}{\lambda_1} \right\rfloor + \rho\right) \quad \text{with } \rho \in [0,1)\,.$$

Here $\lfloor x \rfloor$ denotes the greatest integer smaller than or equal to x.

By the packing property (2.14) of λ_1 with respect to $\frac{1}{2}C$,

$$\mathrm{vol}_T\left(\tfrac{\lambda_1}{2}C/\overline{\Lambda}\right) = \mathrm{vol}\left(\tfrac{\lambda_1}{2}C\right) = \left(\tfrac{\lambda_1}{2}\right)^m \mathrm{vol}(C)$$

and

$$\mathrm{vol}_T\left(\tfrac{\rho\lambda_1}{2}C/\overline{\Lambda}\right) = \mathrm{vol}\left(\tfrac{\rho\lambda_1}{2}C\right) = \left(\tfrac{\rho\lambda_1}{2}\right)^m \mathrm{vol}(C)\,.$$

Hence, in view of the sum theorem of Kneser and Macbeath, Theorem 2.2.1, we may write

$$\begin{aligned}
(2.16)\quad \mathrm{vol}_T\left(\left(\lambda \tfrac{1}{2}C\right)/\overline{\Lambda}\right) &= \mathrm{vol}_T\left(\left(\left(\left\lfloor \frac{\lambda}{\lambda_1} \right\rfloor \frac{\lambda_1}{2} + \rho\,\frac{\lambda_1}{2}\right)C\right)/\overline{\Lambda}\right) \\
&= \mathrm{vol}_T\left(\underbrace{\left(\frac{\lambda_1}{2}C\right)/\overline{\Lambda} \oplus \cdots \oplus \left(\frac{\lambda_1}{2}C\right)/\overline{\Lambda}}_{\lfloor \lambda/\lambda_1 \rfloor} \oplus \left(\frac{\rho\lambda_1}{2}C\right)/\overline{\Lambda}\right) \\
&\geq \min\left\{\left\lfloor \frac{\lambda}{\lambda_1} \right\rfloor \mathrm{vol}\left(\frac{\lambda_1}{2}C\right) + \mathrm{vol}\left(\frac{\rho\lambda_1}{2}C\right), \det\overline{\Lambda}\right\} \\
&= \min\left\{\left\lfloor \frac{\lambda}{\lambda_1} \right\rfloor \left(\frac{\lambda_1}{2}\right)^m \mathrm{vol}(C) + \left(\frac{\rho\lambda_1}{2}\right)^m \mathrm{vol}(C), \det\overline{\Lambda}\right\} \\
&= \min\left\{\left(\left\lfloor \frac{\lambda}{\lambda_1} \right\rfloor + \rho^m\right)\left(\frac{\lambda_1}{2}\right)^m \mathrm{vol}(C), \det\overline{\Lambda}\right\}.
\end{aligned}$$

Therefore, $(*)$ is certainly satisfied for a $\overline{\lambda}$ with

$$\left(\left\lfloor\frac{\overline{\lambda}}{\lambda_1}\right\rfloor+\rho^m\right)\left(\frac{\lambda_1}{2}\right)^m \operatorname{vol}(C)=\left(\sum_{j=1}^{s}\frac{\det\overline{\Lambda}}{\det\Lambda_j}-s+1\right)\det\Lambda .$$

Using $0<\rho^m<1$ and thus

$$\left\lfloor\frac{\overline{\lambda}}{\lambda_1}\right\rfloor+\rho^m>\frac{\overline{\lambda}}{\lambda_1}-1=\frac{\overline{\lambda}-\lambda_1}{\lambda_1}, \tag{2.17}$$

we find

$$\overline{\lambda}-\lambda_1<\frac{2^m\det\Lambda}{\lambda_1^{m-1}\operatorname{vol}(C)}\left(\sum_{j=1}^{s}\frac{\det\overline{\Lambda}}{\det\Lambda_j}-s+1\right),$$

proving $\lambda_1(C,\Lambda\setminus\bigcup_{j=1}^{s}\Lambda_j)\le\overline{\lambda}$. □

In the special case $s=1$ the theorem above can be formulated as

Corollary 2.2.6. *Let* $C\in\mathscr{K}_0^n$, $\Lambda\in\mathscr{L}^m$ *with* $\operatorname{rank}\Lambda=m\ge 2$ *and let* $\Lambda_1\subsetneq\Lambda$ *be a strict sublattice with* $\operatorname{rank}\Lambda_1=m$. *Then*

$$\lambda_1\left(C,\Lambda\setminus\Lambda_1\right)\le\frac{2^m\det\Lambda}{\lambda_1(C,\Lambda_1)^{m-1}\operatorname{vol}(C)}+\lambda_1(C,\Lambda).$$

Indeed, the corollary is just an immediate consequence of Theorem 2.2.5, since in this case $\overline{\Lambda}=\Lambda_1$ and we may assume $\lambda_1(C,\Lambda_1)=\lambda_1(C,\Lambda)$, as otherwise $\lambda_1(C,\Lambda\setminus\Lambda_1)=\lambda_1(C,\Lambda)$ is a trivial bound.

The following example shows that the bound in Theorem 2.2.5 as well as the one of the corollary above cannot be improved in general by a multiplicative factor.

Example 2.2.7. Let $C\in\mathscr{K}_0^2$ be the rectangle $C=[-1,1]\times[-\alpha,\alpha]$ of edge-lengths 2 and 2α for some $\alpha\le 1$. Then $\operatorname{vol}(C)=4\alpha$. Let $\Lambda=\mathbb{Z}^2$, and let $\Lambda_1,\Lambda_2\subset\mathbb{Z}^2$ be the sublattices

$$\Lambda_1=\begin{pmatrix}1&0\\0&2\end{pmatrix}\mathbb{Z}^2\quad\text{and}\quad\Lambda_2=\begin{pmatrix}p&0\\0&1\end{pmatrix}\mathbb{Z}^2$$

where $p>2$ is a prime. Then $\det\Lambda=1$, $\det\Lambda_1=2$, $\det\Lambda_2=p$, and

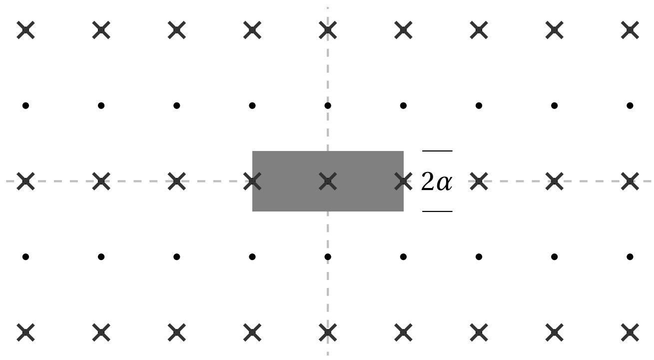
$$\overline{\Lambda}=\Lambda_1\cap\Lambda_2=\begin{pmatrix}p&0\\0&2\end{pmatrix}\mathbb{Z}^2$$

with $\det \overline{\Lambda} = 2p$.

For $\alpha \le 2/p$ we therefore have $\lambda_1(K, \overline{\Lambda}) = p$. Regarding the set $\Lambda \setminus (\Lambda_1 \cup \Lambda_2)$ we observe that the lattice points on the axes are forbidden, but not $(1,1)^\intercal$ and so $\lambda_1(K, \Lambda \setminus (\Lambda_1 \cup \Lambda_2)) = 1/\alpha$. Putting everything together, the bound in Theorem 2.2.5 evaluates for $\alpha \le 2/p$ to

$$\frac{1}{\alpha} = \lambda_1(\Lambda \setminus (\Lambda_1 \cup \Lambda_2)) < \frac{4}{p4\alpha}(p+1) + p = \frac{1}{\alpha} + \frac{1}{p\alpha} + p.$$

Hence for $\alpha = 2/p^2$ and $p \to \infty$ the bound cannot be improved by a multiplicative factor. The following picture shows the body C and the lattice $\Lambda = \mathbb{Z}^2$, as well as the restriction $\overline{\Lambda}$ as crosses.



In the situation of Corollary 2.2.6, i.e. we consider only the forbidden lattice Λ_1, the upper bound in the corollary evaluates to $\frac{1}{\alpha} + 1$, whereas, as before, $\lambda_1(C, \Lambda \setminus \Lambda_1) = 1/\alpha$.

The example holds in any dimension m, by taking the body $C \times [-1,1]^{m-2}$ and the lattices $\Lambda_i \times \mathbb{Z}^{m-2}$ for $i = 1,2$.

Remark 2.2.8. *The bound in Theorem 2.2.5 can slightly be improved in small dimensions by noticing that in* (2.17) *we may replace*

$$\frac{\overline{\lambda} - \lambda_1}{\lambda_1} \qquad \textit{by} \qquad \frac{\overline{\lambda}}{\lambda_1} - \rho + \rho^m.$$

Since $\rho - \rho^m$ takes its maximum at $\rho = (1/m)^{1/(m-1)}$ with value

$$m^{\frac{-1}{m-1}} - m^{\frac{-m}{m-1}} = m^{\frac{-1}{m-1}}\left(1 - m^{\frac{-m+1}{m-1}}\right) = m^{\frac{-1}{m-1}}\left(\tfrac{m}{m} - \tfrac{1}{m}\right),$$

we get

$$\lambda_1\left(C, \Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\right) \le \frac{2^m \det \Lambda}{\lambda_1(C, \overline{\Lambda})^{m-1} \operatorname{vol}(C)} \left(\sum_{j=1}^{s} \frac{\det \overline{\Lambda}}{\det \Lambda_j} - s + 1\right) + m^{\frac{-1}{m-1}} \left(\frac{m-1}{m}\right) \lambda_1(C, \overline{\Lambda}).$$

Now $m^{\frac{-1}{m-1}}\left(\frac{m-1}{m}\right) \ge \frac{1}{4}$ *for* $m \ge 2$ *and it converges to* 1 *from below as* $m \to \infty$*, improving the previous bound for small* m*.*

There is a straightforward way to extend Theorem 2.2.5 to higher successive minima which we will first present in the special case $s = 1$.

Corollary 2.2.9. *Under the assumptions of Corollary 2.2.6, for all* $1 \le i \le m$ *we have*

$$\lambda_i\left(C, \Lambda \setminus \Lambda_1\right) \le \frac{2^m \det \Lambda}{\lambda_1(C, \Lambda_1)^{m-1} \operatorname{vol}(C)} + \lambda_1(C, \Lambda) + \lambda_i(C, \Lambda).$$

Proof. In view of Corollary 2.2.6 it suffices to show

$$\lambda_i\left(C, \Lambda \setminus \Lambda_1\right) \le \lambda_1(C, \Lambda \setminus \Lambda_1) + \lambda_i(C, \Lambda) \quad \text{for } i = 2, \ldots, m.$$

To this end let $a \in \lambda_1(C, \Lambda \setminus \Lambda_1) \cdot C \cap (\Lambda \setminus \Lambda_1)$ and let $b_1, \ldots, b_m$ be linearly independent with $b_k \in \lambda_k(C, \Lambda)\, C \cap \Lambda$ for $k = 1, \ldots, m$. Since $a \notin \Lambda_1$, not both of b_k and $a + b_k$ can belong to the forbidden sublattice Λ_1. Therefore we can select from each such pair b_k and $a + b_k$ $(1 \le k \le m)$ one contained in $\Lambda \setminus \Lambda_1$. Let these points be denoted by $\overline{b}_k$ for $k = 1, \ldots, m$. Then

$$a, \overline{b}_k \in \left(\lambda_1(C, \Lambda \setminus \Lambda_1) + \lambda_k(C, \Lambda)\right) C \quad \text{for } 1 \le k \le m.$$

Now choose ℓ such that $a \notin \operatorname{lin}(\{b_1, \ldots, b_m\} \setminus \{b_\ell\})$. Then the lattice points $\overline{b}_1, \ldots \overline{b}_{\ell-1}, a, \overline{b}_{\ell+1}, \ldots, \overline{b}_m$ are linearly independent and we are done. □

For the general case, i.e. for a collection of $s \ge 2$ forbidden sublattices, we have the following result.

Corollary 2.2.10. *Under the assumptions of Theorem 2.2.5, we have for* $1 \le i \le m$*,*

$$\lambda_i\left(C, \Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\right) \le \frac{2^m \det \Lambda}{\lambda_1(C, \overline{\Lambda})^{m-1} \operatorname{vol}(C)} \left(\sum_{j=1}^{s} \frac{\det \overline{\Lambda}}{\det \Lambda_j} - s + 1\right) + \lambda_1(C, \overline{\Lambda}) + \lambda_i(C, \overline{\Lambda}),$$

where again $\overline{\Lambda} = \bigcap_{j=1}^{s} \Lambda_j$*.*

Proof. The argument is similar to that used for Corollary 2.2.9. But as for $s \geq 2$ the set $\bigcup_{j=1}^{s} \Lambda_j$ is in general not a lattice, we choose the vectors b_k, $1 \leq k \leq m$ from the lattice $\overline{\Lambda} = \bigcap_{j=1}^{s} \Lambda_j$ instead. Then for $a \in \Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j$ we have

$$a, a+b_1, a+b_2, \ldots, a+b_m \in \Lambda \setminus \bigcup_{j=1}^{s} \Lambda_j\,,$$

and we can choose m linearly independent points among them and argue as before. □

Remark 2.2.11. *It is also possible to extend lower-dimensional lattices to lattices of full rank by adjoining "sufficiently large" vectors, i.e. for each Λ_j of rank $m_j < m$ choose linearly independent $z_{j,m_j+1}, \ldots, z_{j,m} \in \Lambda \setminus \Lambda_j$ and consider the lattice $\overline{\Lambda}_j$ spanned by Λ_j and $z_{j,m_j+1}, \ldots, z_{j,m}$.*

If $z_{j,i}$ are such that $\lambda_i(C, \overline{\Lambda}_j)$ is very large for $i > m_j$, one can apply the results from this section to the collection $\overline{\Lambda}_j$, $1 \leq j \leq s$.

However, the bounds obtained in this way are in general weaker, with one exception in the case $s = 1$ for the bound on $\lambda_1(C, \Lambda \setminus \Lambda_1)$. Here we get

$$\lambda_1\left(C, \Lambda \setminus \Lambda_1\right) \leq \frac{2^m \det \Lambda}{\lambda_1(C, \Lambda_1)^{m-1} \operatorname{vol}(C)} + \lambda_1(C, \Lambda)$$

for $\Lambda_1 \subsetneq \Lambda$ with $\operatorname{rank} \Lambda_1 < m$, which improves on Theorem 2.1.1.

3

Adelic Lattice Point Problems

In Section 1.1 we introduced the problem of bounding the number of lattice points contained within a convex body C, cf. (1.14), in terms of the volume or successive minima of C.

In this chapter, we will generalise this problem to the rank-n module over the adeles, $K_{\mathbb{A}}^n$. Using an adelic convex body $C = \prod_v C_v$, cf. Definition 1.2.7, we can, similarly to the classical Geometry of Numbers, ask for elements of the discrete subset $K^n \subset K_{\mathbb{A}}^n$ contained in C, i.e. investigate bounds of the number

$$\left|C \cap K^n\right|. \tag{3.1}$$

We remark that in spite of the nice connection between adelic successive minima in terms of dilations and in terms of heights, Lemma 1.2.21, there is no direct connection between the number (3.1) and the number of algebraic points of bounded heights, mentioned in the introduction to heights on page 30. While the motivation is very similar, the latter counts 1-dimensional subspaces and not single elements of K^n.

Throughout the chapter, we use the following notation as introduced in Section 1.2

$$C_\infty = \prod_{v \mid \infty} C_v \quad \text{and} \quad \mathfrak{M} = \bigcap_{v \nmid \infty} \left(C_v \cap K^n\right).$$

Observe that our standard embedding $\rho \circ \iota : K^n \hookrightarrow \mathbb{R}^{nd}$, cf. (1.25) and thereafter, is injective, and therefore

$$\left|C \cap K^n\right| = \left|\rho\Big(\prod_{v \mid \infty} C_v\Big) \cap \rho(\iota(\mathfrak{M}))\right|. \tag{3.2}$$

We will use this important connection for some of the proofs below.

The chapter is organised as follows. In Section 3.1 we introduce different notions of convex hull. We will in particular discuss the differences between the real and complex spaces $\mathbb{R}^n$ and $\mathbb{C}^n$. In Sections 3.2 and 3.3 we give bounds on the number (3.1) analogous to the classical bounds by van der Corput and Blichfeldt and its improvements and generalisations by Henze and Betke, Henk, Wills, i.e. Theorems 1.1.3, 1.1.4, 1.1.5 and 1.1.6.

3.1 The Adelic Convex Hull

Remark 3.1.1. *The condition of* 0*-symmetry at the infinite places in Definition 1.2.7 can be dropped in favour of a more general definition as follows.*

Let C_v *be a* K_v*-lattice in* K_v^n *for* $v \nmid \infty$ *and for* $v \mid \infty$ *let* C_v *be any compact convex body with non-empty interior in* K_v^n*. Then*

$$C = \prod_{v \nmid \infty} C_v \times \prod_{v \mid \infty} C_v$$

is a general adelic convex body.

Dropping the assumption of symmetry now allows us to introduce several notions of convex bodies that are in some sense minimal.

We start by giving local definitions of *convex hull* of points $\overline{a}_0, \ldots, \overline{a}_m \in K_{\mathbb{A}}^n$. To exclude degenerate cases, we require that for all $v \in M(K)$ we have

$$\operatorname{lin}_{K_v}\{\overline{a}_{k,v} \mid 0 \le k \le m\} = K_v^n,$$

and for all but finitely many $v \nmid \infty$, the entries of $\overline{a}_{k,v}$ are in $\mathcal{O}_v^*$. Here $\overline{a}_{k,v}$ is the v-entry of $\overline{a}_k$.

For $v \nmid \infty$ define the module

$$C_v = \operatorname{conv}_v\{\overline{a}_{0,v}, \ldots, \overline{a}_{m,v}\} = \mathcal{O}_v \overline{a}_{0,v} + \ldots + \mathcal{O}_v \overline{a}_{m,v}. \tag{3.3}$$

Note that C_v is an $\mathcal{O}_v$-module in K_v^n of full rank. In fact, we have $C_v = \operatorname{lin}_{\mathcal{O}_v}\{\overline{a}_{0,v}, \ldots, \overline{a}_{m,v}\}$, the minimal $\mathcal{O}_v$-module in K_v^n containing all the points.

Since the K_v are local fields, there exist $A_v \in \mathrm{GL}_n(K_v)$, such that $C_v = A_v \mathcal{O}_v^n$.

For $v \mid \infty$ real, let

$$\begin{aligned} C_v &= \operatorname{conv}_{\mathbb{R}}\{\overline{a}_{0,v}, \ldots, \overline{a}_{m,v}\} \\ &= \left\{ \sum_{i=0}^m \lambda_i \overline{a}_{i,v} \,\middle|\, \lambda_i \in \mathbb{R},\ 0 \le \lambda_i \le 1,\ \sum_{i=0}^m \lambda_i = 1 \right\} \subset \mathbb{R}^n \end{aligned} \tag{3.4}$$

and

$$C_v^\diamond = \mathrm{conv}_{\mathbb{R}}\{\pm\overline{a}_{0,v},\dots,\pm\overline{a}_{m,v}\} = \left\{ \sum_{i=0}^m \lambda_i \overline{a}_{i,v} \,\middle|\, \lambda_i \in \mathbb{R},\ 0 \le |\lambda_i| \le 1,\ \sum_{i=0}^m |\lambda_i| \le 1 \right\} \subset \mathbb{R}^n. \tag{3.5}$$

These are the standard convex hull of points in real space and its symmetric variant. They are equivalent to defining the bodies as the intersection of all (symmetric) convex bodies containing the points $\overline{a}_{0,v},\dots,\overline{a}_{m,v}$.

For $v \mid \infty$ complex, we only define the symmetric body

$$C_v^\diamond = \mathrm{conv}_{\mathbb{C}}\{\overline{a}_{0,v},\dots,\overline{a}_{m,v}\} = \left\{ \sum_{i=0}^m \lambda_i \overline{a}_{i,v} \,\middle|\, \lambda_i \in \mathbb{C},\ 0 \le |\lambda_i| \le 1,\ \sum_{i=0}^m |\lambda_i| \le 1 \right\} \subset \mathbb{C}^n. \tag{3.6}$$

By construction, this is the intersection of all c-symmetric convex bodies containing the points $\overline{a}_{0,v},\dots,\overline{a}_{m,v}$. We are not aware of any more general notion of convex hull in complex spaces. When identifying $\mathbb{C}^n \cong \mathbb{R}^{2n}$, we can use the definitions used in the real case, but the bodies obtained in this way lie in a real (affine) subspace of $\mathbb{R}$-dimension n in $\mathbb{R}^{2n}$ and do thus not define an adelic convex body.

Using our constructions of convex hull, we can now define the following special classes of adelic convex bodies.

Definition 3.1.2. Given $\overline{a}_0,\dots,\overline{a}_m \in K_{\mathbb{A}}^n$ as before, we define

$$C^\diamond = \mathrm{conv}_{\mathbb{A}}^\diamond\{\overline{a}_0,\dots,\overline{a}_m\} = \prod_{v\nmid\infty} A_v\mathcal{O}_v^n \times \prod_{v\mid\infty} C_v^\diamond$$

with A_v implicitly defined by (3.3) above and $C_v^\diamond$ as in (3.5) and (3.6), as the *symmetric adelic convex hull* of $\overline{a}_0,\dots,\overline{a}_m$. If K is totally real, we define the *adelic convex hull* of $\overline{a}_0,\dots,\overline{a}_m$ as

$$C = \mathrm{conv}_{\mathbb{A}}\{\overline{a}_0,\dots,\overline{a}_m\} = \prod_{v\nmid\infty} A_v\mathcal{O}_v^n \times \prod_{v\mid\infty} C_v,$$

where C_v is defined as in (3.4). In case $m = n$, we speak of the *adelic cross-polytope* and *adelic simplex* respectively. All of these bodies will also be called *adelic polytopes* and if C is an adelic polytope, it can be written as $C = \prod_{v\in M(K)} C_v$ with local bodies C_v defined as above.

Not surprisingly, only the symmetric convex hull defines an adelic convex body in the sense of Definition 1.2.7. The more general convex hull defines a general adelic convex body in the sense of Remark 3.1.1.

Denote by $\sigma_v\colon K \to K_v$ the inclusion of K into K_v and by abuse of notation also $\sigma_v\colon K^n \to K_v^n$ for all places v of K.

Definition 3.1.3. Given points $a_0, \ldots, a_m \in K^n$ that span K^n, identify

$$a_k = \overline{a}_k = \left(\sigma_v(a_k) \,\middle|\, v \in M(K)\right).$$

Then

$$C = \operatorname{conv}_{\mathbb{A}}\{a_0, \ldots, a_m\} \quad \text{and} \quad C^{\diamond} = \operatorname{conv}^{\diamond}_{\mathbb{A}}\{a_0, \ldots, a_m\}$$

are the *adelic lattice polytope* and *symmetric adelic lattice polytope* containing $a_0, \ldots, a_m$, respectively. The body C is of course again only defined for K totally real. Again, for $m = n$ we call C an *adelic lattice simplex* and if additionally $a_0 = 0$, call $C^{\diamond}$ an *adelic lattice cross-polytope.*

Remark 3.1.4. *The intersection of two adelic polytopes is again an adelic polytope, since this property holds for all v and two adelic polytopes differ only for finitely many v and for arbitrary sets X, Y and Z we have*

$$(X \times Y) \cap (X \times Z) = \left\{(x, y) \,\middle|\, x \in X,\ y \in Y,\ y \in Z\right\} = X \times (Y \cap Z).$$

The intersection of two adelic lattice polytopes however is of course not an adelic lattice polytope in general, see Example 3.3.3 below.

Remark 3.1.5. *Adelic polytopes are not as nice as their classical counterparts. In Euclidean space, a polytope $P \in \mathcal{K}^m$ can be written as both the convex hull of a finite number of points or as the intersection of a finite number of closed half-spaces. Given a linear functional $\ell : \mathbb{R}^m \to \mathbb{R}$, the kernel of ℓ is a hyperplane H and using the ordering of $\mathbb{R}$, we decide whether two points $x_1, x_2 \in \mathbb{R}^m$ lie on the same side of H by comparing the signs of $\ell(x_1)$ and $\ell(x_2)$ and thereby also defining two half-spaces.*

Such a construction is not possible in the adelic setting, as we do not have an ordering on $K_{\mathbb{A}}$.

Given a polytope $P = \operatorname{conv}\{v_1, \ldots, v_s\} \in \mathcal{K}^m$ with vertices $v_1, \ldots, v_s \in \Lambda$, where $\Lambda \in \mathcal{L}^m$ is a lattice of rank m, by a famous result of Ehrhart [Ehr62], the number of lattice points in the dilate kP for a positive integer k is given by a polynomial of degree m, the Ehrhart polynomial

$$\left|kP \cap \Lambda\right| = \sum_{i=1}^{m} G_i(P,\Lambda)k^i\,, \quad k \in \mathbb{N}.$$

The polynomial is unique and the coefficients depend only on P and Λ. The behaviour and properties of this polynomial for given polytope and lattice have been studied intensively, see Beck and Robins [BR07] for an overview as well as e.g. McMullen [McM79] and Linke [Lin11] for more specific results.

Now consider an adelic lattice polytope $C = \prod_v C_v$, and let C_∞ and $\mathfrak{M}$ be as before, cf. (1.26). Then $\rho(C_\infty)$ is a polytope in $\mathbb{R}^{nd}$, as the factors of C_∞ are polytopes. On account of (3.2), the number $\left|kC \cap K^n\right|$ has to grow like k^{nd}. On the other hand, the body $\rho(C_\infty)$ is in general not a lattice polytope with respect to the lattice $\rho(\iota(\mathfrak{M}))$.

Consider for example $K = \mathbb{Q}[\sqrt{2}]$ for $n = 1$ and the body $C = \prod_{v\nmid\infty} \mathcal{O}_v \times [-1,1]^2$, which is an adelic lattice polytope in the sense above, as $C = \mathrm{conv}_{\mathbb{A}}\{\pm 1\}$. Observe, that $\mathfrak{M} = \mathcal{O}$. Figure 1.1 on page 30 shows the embedding of a dilate of C. It is evident from the figure, that of the four vertices of $\rho\big([-1,1]^2\big)$ only $(1,1)$ and $(-1,-1)$ are lattice points but not $(1,-1)$ and $(-1,1)$ and thus $\rho(C_\infty)$ is not a lattice polytope with respect to $\rho(\iota(\mathcal{O}))$. However, the infinite part of any adelic convex body has to be of the form $C_\infty = [-a,a] \times [-b,b]$ for $a,b \in \mathbb{R}$. But since $\rho(\iota(\mathcal{O}))$ is generated by $(1,1), (-\sqrt{2},\sqrt{2}) \in \mathbb{R}^2$, no box can have only lattice points as vertices. The image of C under embedding into $\mathbb{R}^2$ can thus not be a lattice polytope. We can therefore not find an adelic analogue of the classical Ehrhart polynomial.

Lemma 3.1.6. *Let $e_1,\ldots,e_n$ be any basis of K^n, then the volume of the adelic lattice cross-polytope $C^\diamond = \mathrm{conv}_{\mathbb{A}}^\diamond\{e_1,\ldots,e_n\}$ is*

$$\mathrm{vol}_{\mathbb{A}}(C^\diamond) = \frac{2^{dn}\pi^{r_2 n}}{(n!)^{r_1}((2n)!)^{r_2}}\,.$$

Observe, that this body gives the equality case for the lower bound in the Bombieri-Vaaler version of Minkowski's second theorem, Theorem 1.2.14. The additional requirement of c-symmetry there guarantees the inclusion $C^\diamond \subseteq C$.

Proof. Without loss of generality we can assume $e_1,\ldots,e_n$ is the standard basis of unit vectors, as $|\det M|_{\mathbb{A}} = 1$ for the corresponding change-of-basis matrix $M \in \mathrm{GL}_n(K)$, and $\mathrm{vol}_{\mathbb{A}}(MC^\diamond) = |\det M|_{\mathbb{A}}\,\mathrm{vol}_{\mathbb{A}}(C^\diamond)$.

As $\operatorname{vol}_v(\mathscr{O}_v^n) = 1$ for all $v \nmid \infty$, we only have to consider the infinite places. Let $v \mid \infty$ be a real place, then

$$\begin{aligned}
\operatorname{vol}_v(C_v) &= \operatorname{vol}\left(\left\{ x \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^n |x_i| \le 1 \right\}\right) \\
&= 2^n \int_0^1 \int_0^{1-x_1} \cdots \int_0^{1-x_1-\ldots-x_{n-1}} 1 \, \mathrm{d}x_n \cdots \mathrm{d}x_1 \\
&= 2^n \int_0^1 \int_0^{1-x_1} \cdots \int_0^{1-x_1-\ldots-x_{n-2}} (1 - x_1 - \ldots - x_{n-1}) \, \mathrm{d}x_{n-1} \cdots \mathrm{d}x_1 \\
&= 2^n \int_0^1 \int_0^{1-x_1} \cdots \int_0^{1-x_1-\ldots-x_{n-3}} \tfrac{1}{2}(1 - x_1 - \ldots - x_{n-2})^2 \, \mathrm{d}x_{n-2} \cdots \mathrm{d}x_1 \\
&= 2^n \int_0^1 \int_0^{1-x_1} \cdots \int_0^{1-x_1-\ldots-x_{n-4}} \tfrac{1}{2}\tfrac{1}{3}(1 - x_1 - \ldots - x_{n-3})^3 \, \mathrm{d}x_{n-3} \cdots \mathrm{d}x_1 \\
&\;\vdots \\
&= 2^n \cdot \frac{1}{n!}.
\end{aligned}$$

Let $v \mid \infty$ be a complex place, then, using the standard transformation $(\vartheta, \varphi) \mapsto (\vartheta \cos\varphi, \vartheta \sin\varphi)$ to polar coordinates, we get

$$\begin{aligned}
\operatorname{vol}_v(C_v) &= 2^n \operatorname{vol}\left(\left\{ z \in \mathbb{C}^n \,\middle|\, \sum_{i=1}^n |z_i|_{\mathbb{C}} \le 1 \right\}\right) \\
&= 2^n \int_0^1 \int_{-\pi}^{\pi} \int_0^{1-\vartheta_1} \int_{-\pi}^{\pi} \cdots \int_0^{1-\ldots-\vartheta_{n-1}} \int_{-\pi}^{\pi} \vartheta_1 \cdots \vartheta_n \, \mathrm{d}\vartheta_n \, \mathrm{d}\varphi_n \cdots \mathrm{d}\vartheta_1 \, \mathrm{d}\varphi_1 \\
&= (2\pi)^n 2^n \int_0^1 \int_0^{1-\vartheta_1} \cdots \int_0^{1-\ldots-\vartheta_{n-1}} \vartheta_1 \cdots \vartheta_n \, \mathrm{d}\vartheta_n \cdots \mathrm{d}\vartheta_1 \\
&= (4\pi)^n \int_0^1 \int_0^{1-\vartheta_1} \cdots \int_0^{1-\ldots-\vartheta_{n-2}} \tfrac{1}{2} \vartheta_1 \cdots \vartheta_{n-1} (1 - \ldots - \vartheta_{n-1})^2 \, \mathrm{d}\vartheta_{n-1} \cdots \mathrm{d}\vartheta_1 \\
&= (4\pi)^n \int_0^1 \int_0^{1-\vartheta_1} \cdots \int_0^{1-\ldots-\vartheta_{n-3}} \tfrac{1}{2}\tfrac{1}{3}\tfrac{1}{4} \vartheta_1 \cdots \vartheta_{n-2} (1 - \ldots - \vartheta_{n-2})^4 \, \mathrm{d}\vartheta_{n-2} \cdots \mathrm{d}\vartheta_1 \\
&\;\vdots \\
&= (4\pi)^n \frac{1}{(2n)!}.
\end{aligned}$$

The conclusion follows, as $2^{r_1} 4^{r_2} = 2^{r_1 + 2r_2} = 2^d$. □

The proof includes the following special case.

Corollary 3.1.7. *Let K be totally real and let $S = \prod_{v\in M(K)} S_v$ be an adelic lattice simplex. Then*

$$\mathrm{vol}_v(S_v) = \frac{1}{n!} \qquad \text{for } v \mid \infty \text{ real.}$$

Proposition 3.1.8. *Let K be totally real and $S = \mathrm{conv}_{\mathbb{A}}\{a_0,\ldots,a_n\}$ an adelic lattice simplex. Then*

$$\mathrm{vol}_{\mathbb{A}}(S) \geq \frac{1}{(n!)^d}.$$

Proof. Notice that for $v \nmid \infty$, in general

$$\mathrm{conv}_v\{a_0+t,\ldots,a_n+t\} \neq \mathrm{conv}_v\{a_0,\ldots,a_n\}+t$$

for $t \in K^n$, but if $t \in \{a_0,\ldots,a_n\}$, certainly

$$\mathrm{conv}_v\{a_0+t,\ldots,a_n+t\} \subseteq \mathrm{conv}_v\{a_0,\ldots,a_n\},$$

as the points on the left are contained in the $\mathbb{Z}$-span of the points on the right.
For $v \mid \infty$, i.e. v real, however

$$\mathrm{conv}_v\{a_0+t,\ldots,a_n+t\} = \mathrm{conv}_v\{a_0,\ldots,a_n\}+t$$

for any $t \in K^n$. So we may assume w.l.o.g. $a_0 = 0$, possibly switching to a subset at some finite places.

Denote by $A = (a_1 \ldots a_n)$ the matrix whose columns are $a_1,\ldots,a_n$. In fact, just as in Lemma 3.1.6, we may assume that $a_1,\ldots,a_n$ is the standard basis, $|\det A|_{\mathbb{A}} = 1$.

For $v \mid \infty$ we get from Corollary 3.1.7 that

$$\begin{aligned}\mathrm{vol}_v\big(\mathrm{conv}_{\mathbb{R}}\{\sigma_v(a_1,),\ldots,\sigma_v(a_n)\}\big) &= \frac{\big|\det\big(\sigma_v(a_1)\ldots\sigma_v(a_n)\big)\big|_\infty}{n!}\\ &= \frac{\big|\sigma_v\big(\det(a_1\ldots a_n)\big)\big|_\infty}{n!}.\end{aligned}$$

On the other hand, for $v \nmid \infty$, we have

$$C_v = \mathcal{O}_v\sigma_v(a_1)+\ldots+\mathcal{O}_v\sigma_v(a_n) = \big(\sigma_v(a_1)\ldots\sigma_v(a_n)\big)\mathcal{O}_v^n$$

and thus

$$\mathrm{vol}_v(C_v) = \big|\det\big(\sigma_v(a_1)\ldots\sigma_v(a_n)\big)\big|_v.$$

Therefore

$$\begin{aligned}
\mathrm{vol}_{\mathbb{A}}(C) &= \prod_{v\nmid\infty} \mathrm{vol}_v(C_v) \cdot \prod_{v\mid\infty} \mathrm{vol}_v(C_v) \\
&= \prod_{v\nmid\infty} |\det(a_1 \ldots a_n)|_v \cdot \frac{1}{(n!)^d} \prod_{v\mid\infty} |\det(a_1 \ldots a_n)|_v \\
&= \frac{1}{(n!)^d} \cdot 1\,.
\end{aligned}$$

For $a_0 \neq 0$, this is in fact an inequality and not necessarily tight. □

3.2 Adelic van-der-Corput-type Inequalities

We start by giving our adelic generalisation of the classical van der Corput inequality of Theorem 1.1.3.

Theorem 3.2.1. *Let $n \geq 2$ and C be a symmetric adelic convex body, then*

$$\left|C \cap K^n\right| \geq \left\lfloor \frac{\mathrm{vol}_{\mathbb{A}}(C)}{2^{nd-1}\left(\sqrt{|\Delta_K|}\right)^n} \right\rfloor > \frac{\mathrm{vol}_{\mathbb{A}}(C)}{2^{nd-1}\left(\sqrt{|\Delta_K|}\right)^n} - 1\,.$$

Proof. Using $\mathfrak{M}$ and C_∞ as before, cf. (1.26), we consider the lattice $\rho(\iota(\mathfrak{M})) \subset \mathbb{R}^{nd}$ and the body $\rho(C_\infty) \subset \mathbb{R}^{nd}$. On account of (3.2), we apply the classical Theorem 1.1.3 together with (1.27) and (1.28). Thus

$$\begin{aligned}
\left|C \cap K^n\right| &= \left|\rho\Big(\prod_{v\mid\infty} C_v\Big) \cap \rho(\iota(\mathfrak{M}))\right| \\
&= |\rho\big(C_\infty\big) \cap \rho(\iota(\mathfrak{M}))| \\
&\geq \left\lfloor \frac{\mathrm{vol}_{nd}\big(\rho\big(C_\infty\big)\big)}{2^{nd-1}|\det(\rho(\iota(\mathfrak{M})))|} \right\rfloor \\
&= \left\lfloor \frac{\mathrm{vol}_{\mathbb{A}}(C)}{2^{nd-1}\left(\sqrt{|\Delta_K|}\right)^n} \right\rfloor .
\end{aligned}$$

The second inequality is obvious. □

It is apparent from the construction of the proof that on account of Example 1.2.8, this result reduces to the classical statement for $K = \mathbb{Q}$.

We now give an alternative lower bound on $\left|C \cap K^n\right|$, replacing the volume of the adelic convex body C by its successive minima. The approach used for the proof was inspired by a visit of Lenny Fukshansky to Magdeburg.

Theorem 3.2.2. *Let $n \geq 2$ and C be a symmetric adelic convex body, then*

$$\left|C \cap K^n\right| \geq \left\lfloor \frac{2 \cdot \pi^{nd/2}}{(2n)^{nd} \cdot \prod_{i=1}^n \lambda_i(C)^d \left(\sqrt{|\Delta_K|}\right)^n} \right\rfloor .$$

Proof. Let $\mathrm{B}_{\mathbb{A}}$ be the adelic unit ball and $A \in \mathrm{GL}_n(K_{\mathbb{A}})$ such that $A^{-1}\mathrm{B}_{\mathbb{A}}$ is the adelic John ellipsoid for C, i.e. $A^{-1}\mathrm{B}_{\mathbb{A}} \subseteq C$ is volume maximal, cf. (1.33). Then

$$\left|C \cap K^n\right| \geq \left|A^{-1}\mathrm{B}_{\mathbb{A}} \cap K^n\right| \quad \text{and} \quad \mathrm{vol}_{\mathbb{A}}\left(A^{-1}\mathrm{B}_{\mathbb{A}}\right) = \frac{1}{|\det A|_{\mathbb{A}}^d} \mathrm{vol}_{\mathbb{A}}(\mathrm{B}_{\mathbb{A}}) .$$

We now apply (1.34) and Lemma 1.2.21 and get

$$\frac{1}{|\det A|_{\mathbb{A}}} \geq \frac{1}{\prod_i \tilde{\lambda}_i(A)} \geq \frac{1}{\prod_i \lambda_i(A^{-1}\mathrm{B}_{\mathbb{A}})} = \frac{1}{\prod_i (2n)^{1/2} \lambda_i\left(\sqrt{2n}\, A^{-1}\mathrm{B}_{\mathbb{A}}\right)} .$$

Combining this with the properties of λ_i, cf. Definition 1.2.13, and the adelic John's Theorem (1.33) we have

$$\frac{1}{|\det A|_{\mathbb{A}}} \geq \frac{1}{(2n)^{n/2} \prod_i \lambda_i(C)} .$$

With $\mathfrak{M}$ as before, (1.26), using $A^{-1}\mathrm{B}_{\mathbb{A}}$ in place of C, we get

$$\begin{aligned}
\left|A^{-1}\mathrm{B}_{\mathbb{A}} \cap K^n\right| &\geq \left\lfloor \frac{\mathrm{vol}_{\mathbb{A}}(A^{-1}\mathrm{B}_{\mathbb{A}})}{2^{nd-1}\left(\sqrt{|\Delta_K|}\right)^n} \right\rfloor = \left\lfloor \frac{\mathrm{vol}_{\mathbb{A}}(\mathrm{B}_{\mathbb{A}})}{2^{nd-1}|\det(A)|_{\mathbb{A}}^d\left(\sqrt{|\Delta_K|}\right)^n} \right\rfloor \\
&\geq \left\lfloor \frac{\mathrm{vol}_{\mathbb{A}}(\mathrm{B}_{\mathbb{A}})}{2^{nd-1}(2n)^{nd/2} \prod_i \lambda_i(C)^d \left(\sqrt{|\Delta_K|}\right)^n} \right\rfloor \qquad (3.7) \\
&= \left\lfloor \frac{\left(\frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}\right)^{r_1} \left(\frac{\pi^{2n/2}}{\Gamma\left(\frac{2n}{2}+1\right)}\right)^{r_2} 2^{nr_2}}{2^{nd-1}(2n)^{nd/2} \prod_i \lambda_i(C)^d \left(\sqrt{|\Delta_K|}\right)^n} \right\rfloor .
\end{aligned}$$

Using $\Gamma(x+1) \leq x^x$ for $x \geq 1$, cf. [Gau09, Lemma 2.20], we get

$$\begin{aligned}
\left|A^{-1}\mathrm{B}_{\mathbb{A}} \cap K^n\right| &\geq \left\lfloor \frac{\pi^{n(r_1+2r_2)/2}\, 2^{nr_2}}{2^{nd-1}(2n)^{nd/2}\left(\frac{n}{2}\right)^{r_1 n/2}\left(\frac{2n}{2}\right)^{2r_2 n/2} \prod_i \lambda_i(C)^d \left(\sqrt{|\Delta_K|}\right)^n} \right\rfloor \\
&= \left\lfloor \frac{\pi^{nd/2}\, 2^{2nr_2/2}}{2^{nd-1} 2^{nd/2} n^{nd/2} 2^{-r_1 n/2} n^{r_1 n/2} n^{2r_2 n/2} \prod_i \lambda_i(C)^d \left(\sqrt{|\Delta_K|}\right)^n} \right\rfloor
\end{aligned}$$

$$= \left\lfloor \frac{2\cdot\pi^{nd/2}2^{dn/2}}{2^{nd}2^{nd/2}n^{nd/2}n^{dn/2}\prod_i\lambda_i(C)^d\left(\sqrt{|\Delta_K|}\right)^n} \right\rfloor$$
$$= \left\lfloor \frac{2\cdot\pi^{nd/2}}{(2n)^{nd}\prod_i\lambda_i(C)^d\left(\sqrt{|\Delta_K|}\right)^n} \right\rfloor .$$

This establishes the Theorem. □

If K is totally real, i.e. $K_v \cong \mathbb{R}$ for all $v \mid \infty$, we can take $\sqrt{n}\,A^{-1}B$ instead of $\sqrt{2n}\,A^{-1}B$ in (1.33) and therefore replace $(2n)^{n/2}$ in (3.7) with $n^{n/2}$ and of course also in the bound of Theorem 3.2.2.

Corollary 3.2.3. *In the situation of Theorem 3.2.2 for the special case of $K = \mathbb{Q}$ we get*

$$\left|C\cap\mathbb{Q}^n\right| \geq \left\lfloor \frac{2}{n^n}\left(\frac{\sqrt{\pi}}{2}\right)^n\prod_{i=1}^n\frac{1}{\lambda_i(C)} \right\rfloor .$$

Remark 3.2.4. *Let C be a c-symmetric adelic convex body. Then, instead of applying the adelic John's Theorem* (1.33)*, we can use the Bombieri-Vaaler version of Minkowski's second theorem, Theorem 1.2.14 and get*

$$\left|C\cap K^n\right| \geq \left\lfloor \frac{\mathrm{vol}_{\mathbb{A}}(C)}{2^{nd-1}\left(\sqrt{|\Delta_K|}\right)^n} \right\rfloor \geq \left\lfloor \frac{2\cdot\pi^{nr_2}}{\left(\sqrt{|\Delta_K|}\right)^n(n!)^{r_1}((2n)!)^{r_2}\prod_i\lambda_i(C)^d} \right\rfloor .$$

The right-hand side of the inequality is greater than our bound in Theorem 3.2.2.

3.3 Adelic Blichfeldt-type Inequalities

For this section, we assume that K is totally real, i.e. $K_v = \mathbb{R}$ for all $v \mid \infty$.

Given $n+m$ lattice points $a_1,\dots,a_{n+m} \in K^n$, fix an embedding $\overline{v}\colon K^n \to \mathbb{R}^n$. Then

$$P_{\overline{v}} = \mathrm{conv}\left\{\overline{v}(a_1),\dots,\overline{v}(a_{n+m})\right\} \subset \mathbb{R}^n$$

is a polytope and there exists a triangulation $T_1,\dots,T_k$ of $P_{\overline{v}}$ with $k \geq m$ full-dimensional simplices, whose vertices are among the $\overline{v}(a_i)$, see e.g. [DLRS10, Sec. 2.2]. An element T_j of this triangulation, i.e. an $(n+1)$-element set from the $n+m$ points, gives rise to an adelic simplex

$$S_j = \mathrm{conv}_{\mathbb{A}}\left\{a_i \,\middle|\, i\in T_j\right\} = \mathrm{conv}_{\mathbb{A}}\left\{a_i \,\middle|\, \overline{v}(a_i) \text{ is a vertex of } T_j\right\}$$

with $S_{j,\overline{v}} = T_j$. Since the triangulation fulfills $\dim(T_j \cap T_i) < n$ for $j \neq i$, we get $\mathrm{vol}(S_{j,\overline{v}} \cap S_{i,\overline{v}}) = 0$ and thus $\mathrm{vol}(S_j \cap S_i) = 0$ for $i \neq j$. On the other hand,

$$P = \mathrm{conv}_{\mathbb{A}}\{a_1, \ldots, a_{n+m}\} \supset S_1 \cup \ldots \cup S_k$$

and therefore, by Proposition 3.1.8,

$$\mathrm{vol}_{\mathbb{A}}(P) \geq k \cdot \frac{1}{(n!)^d} \geq \frac{m}{(n!)^d}. \tag{3.8}$$

From this, we immediately get the following adelic Blichfeldt-type inequality, generalising the classical Theorem 1.1.4.

Theorem 3.3.1. *Let K be a totally real number field of degree $d = [K : \mathbb{Q}]$. Let C be an adelic convex body with* $\dim_K(C \cap K^n) = n$. *Then*

$$\left|C \cap K^n\right| \leq (n!)^d \mathrm{vol}_{\mathbb{A}}(C) + n.$$

However, this construction does not give a triangulation of P, since in general

$$S_1 \cup \ldots \cup S_k \subsetneq P,$$

see Examples 3.3.4 and 3.3.5 below. Example 3.3.5 does also show that the dependence on m can not be improved in general, whereas a minimal triangulation of $P_{\overline{v}}$ does not necessarily give rise to a minimal set of adelic simplices, as Example 3.3.4 shows.

Remark 3.3.2. *Recall that an adelic convex body C in $\mathbb{Q}_{\mathbb{A}}^n$ can be realised as a rational lattice $\Lambda \subset \mathbb{Q}^n \subset \mathbb{R}^n$ and a convex body $C_\infty \subset \mathbb{R}^n$, cf. Example 1.2.8.*

Consider elements $a_0, \ldots, a_m \in \mathbb{Q}^n$, m arbitrary, that span $\mathbb{R}^n$. Now the adelic convex hull C of $a_0, \ldots, a_m \in \mathbb{Q}^n$ can be interpreted as the convex hull

$$C_\infty = \mathrm{conv}_{\mathbb{R}}\{a_0, \ldots, a_m\} \subset \mathbb{R}^n$$

together with the lattice

$$\Lambda = \mathbb{Z}a_0 + \ldots + \mathbb{Z}a_m.$$

Then $\mathrm{vol}(C) = \det(\Lambda)^{-1} \mathrm{vol}(C_\infty)$.

Example 3.3.3. Let $K = \mathbb{Q}$ and $n = 1$. Consider the points $2, 3, 4 \in \mathbb{Q}$ and

$$S_1 = \mathrm{conv}_{\mathbb{A}}\{2,3\} = \prod_{p \text{ prime}} \mathbb{Z}_p \times [2,3], \quad S_2 = \mathrm{conv}_{\mathbb{A}}\{3,4\} = \prod_{p \text{ prime}} \mathbb{Z}_p \times [3,4]$$

and

$$S_3 = \mathrm{conv}_{\mathbb{A}}\{2,4\} = 2\mathbb{Z}_2 \times \prod_{p>2} \mathbb{Z}_p \times [2,4] \ , \quad P = \mathrm{conv}_{\mathbb{A}}\{2,3,4\} = \prod_{p \text{ prime}} \mathbb{Z}_p \times [2,4] \ .$$

Now $\mathrm{vol}(S_1) = \mathrm{vol}(S_2) = \mathrm{vol}(S_3) = 1$, $\mathrm{vol}(P) = 2$ and $3 \notin S_3$. We get

$$S_1 \cap S_2 = \prod_{p \text{ prime}} \mathbb{Z}_p \times [3,3]$$

and

$$S_1 \cap S_3 = 2\mathbb{Z}_2 \times \prod_{p>2} \mathbb{Z}_p \times [2,3] \ , \quad S_2 \cap S_3 = 2\mathbb{Z}_2 \times \prod_{p>2} \mathbb{Z}_p \times [3,4] \ .$$

Thus $\mathrm{vol}(S_1 \cap S_2) = 0 = \mathrm{vol}(S_1 \cap S_2 \cap S_3)$. Further

$$\mathrm{vol}(P) = \mathrm{vol}(S_1 \cup S_2) = \mathrm{vol}(S_1) + \mathrm{vol}(S_2) - \mathrm{vol}(S_1 \cap S_2) = 1 + 1 - 0$$

and

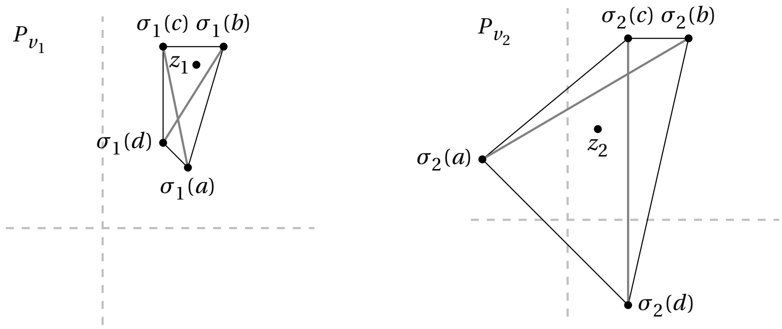
$$\begin{aligned} \mathrm{vol}(P) &= \mathrm{vol}(S_1 \cup S_2 \cup S_3) \\ &= \mathrm{vol}(S_1) + \mathrm{vol}(S_2) + \mathrm{vol}(S_3) - \mathrm{vol}(S_1 \cap S_2) - \mathrm{vol}(S_1 \cap S_3) - \mathrm{vol}(S_2 \cap S_3) \\ &\qquad + \mathrm{vol}(S_1 \cap S_2 \cap S_3) \\ &= 1 + 1 + 1 - 0 - \tfrac{1}{2} - \tfrac{1}{2} + 0 = 2 \, . \end{aligned}$$

However, even though these sets satisfy inclusion-exclusion, $S_1 \cap S_3$ and $S_2 \cap S_3$ are no adelic lattice polytopes.

Example 3.3.4. Let $K = \mathbb{Q}[\sqrt{2}]$ and $n = 2$. Let P be the adelic convex hull of

$$a = (\sqrt{2}, 1) \, , \quad b = (1,3) \, , \quad c = (2,3) \quad \text{and} \quad d = (1, \sqrt{2}) \in K^2 \, .$$

Then for $v \nmid \infty$ we get $P_v = \mathcal{O}_v^2$ and the two convex bodies at the infinite places v_1 and v_2 with corresponding real embeddings σ_1 and σ_2 are



The adelic simplices

$$S_1 = \mathrm{conv}_{\mathbb{A}}\{a,b,c\} \qquad S_2 = \mathrm{conv}_{\mathbb{A}}\{a,b,d\}$$
$$S_3 = \mathrm{conv}_{\mathbb{A}}\{a,c,d\} \qquad S_4 = \mathrm{conv}_{\mathbb{A}}\{b,c,d\}$$

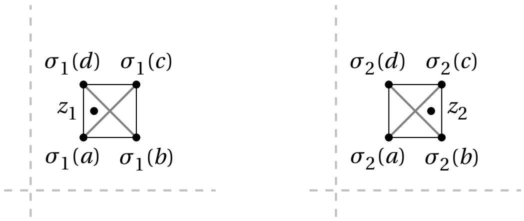
all satisfy $S_{j,v} = \mathcal{O}_v^2$ for $v \nmid \infty$ and $\mathrm{vol}(S_j \cap S_i) = 0$ for all $j \neq i$ since the intersection at one infinite place is always lower-dimensional. Therefore (3.8) is not optimal in this case as P contains four disjoint simplices, even though the indicated triangulations of P_{v_1} and P_{v_2} are minimal.

The point $z = (z_1, z_2)$ indicated in the picture is not contained in any S_j.

Example 3.3.5. Consider again $K = \mathbb{Q}[\sqrt{2}]$ and let P be the adelic convex hull of

$$a = (1,1), \quad b = (2,1), \quad c = (2,2) \quad \text{and} \quad d = (1,2) \in K^2 .$$

Then for $v \nmid \infty$ we get $P_v = \mathcal{O}_v^2$ and the two convex bodies at the infinite places v_1 and v_2 with corresponding real embeddings σ_1 and σ_2 are



As before, for an adelic simplex S with vertices from a,b,c,d it still holds $S_v = \mathcal{O}_v^2$ for $v \nmid \infty$. Any selection of more than two simplices will contain a pair whose infinite parts have non-trivial intersection, thus (3.8) is best possible. Again, the point $z = (z_1, z_2)$ indicated in the picture is not contained in any of the four adelic simplices.

The Symmetric Case

In this section, K is again an arbitrary algebraic number field.

We now prove our adelic version of Henze's symmetric Blichfeldt-inequality, Theorem 1.1.5.

Theorem 3.3.6. *Let K be an algebraic number field of degree d and let C be a symmetric adelic convex body with $\dim_{\mathbb{Q}}(C \cap K^n) = nd$. Then*

$$\left|C \cap K^n\right| \leq \frac{(nd)!}{2^{nd}} L_{nd}(2) \frac{\mathrm{vol}_{\mathbb{A}}(C)}{\left(\sqrt{|\Delta_K|}\right)^n},$$

where L_{nd} is the (nd)-th Laguerre polynomial, $L_{nd}(x) = \sum_{k=0}^{nd} \binom{nd}{k} \frac{x^k}{k!}$.

Proof. Again, we use $\rho \circ \iota : K^n \to \mathbb{R}^{nd}$ and (3.2). By Henze's Blichfeldt-type inequality of Theorem 1.1.5, with

$$\dim_{\mathbb{Q}}\big(C \cap K^n\big) = \dim_{\mathbb{R}}\big(\rho(C_\infty) \cap \rho\big(\iota(\mathfrak{M})\big)\big) \geq nd\,,$$

we get

$$\begin{aligned}\big|\rho\big(\iota(\mathfrak{M})\big) \cap \rho(C_\infty)\big| &\leq \frac{(nd)!}{2^{nd}} L_{nd}(2) \frac{\mathrm{vol}_{nd}\big(\rho(C_\infty)\big)}{\det\big(\rho\big(\iota(\mathfrak{M})\big)\big)} \\ &= \frac{(nd)!}{2^{nd}} L_{nd}(2) \frac{\mathrm{vol}_{\mathbb{A}}(C)}{\big(\sqrt{|\Delta_K|}\big)^n}\,. \end{aligned}$$ □

As $L_{nd}(2)/2^{nd} < 1$ for $nd \geq 8$, and in all cases $L_{nd}(2)/2^{nd} < \sqrt[nd]{9}$, we directly get the following consequence.

Corollary 3.3.7. *Let C be a symmetric adelic convex body and assume further, that* $\dim_{\mathbb{Q}}(C \cap K^n) = nd$*, then*

$$\big|C \cap K^n\big| < 9^{(1/nd)}\,(nd)!\,\frac{\mathrm{vol}_{\mathbb{A}}(C)}{\big(\sqrt{|\Delta_K|}\big)^n}\,.$$

Another adelic Blichfelt-type inequality was proved by Gaudron, [Gau09, p. 173], using the language of heights and vector bundles. Let C be a symmetric adelic convex body and let $\dim_K(C \cap K^n) = n$, then

$$\big|C \cap K^n\big| < (5n)^{nd}\,\mathrm{vol}_{\mathbb{A}}(C)\,. \tag{3.9}$$

While for $d \leq 5$ or very large n, the dimensional constant is bigger in Gaudron's result, the assumption $\dim_{\mathbb{Q}}(C \cap K^n) = nd$ in our Corollary 3.3.7 is a very technical one and not as nice as we would like. In the same spirit we can, of course, directly apply Blichfeldt's original result and get the following variant.

Corollary 3.3.8. *Let C be a symmetric adelic convex body and assume further that* $\dim_{\mathbb{Q}}(C \cap K^n) = nd$*, then*

$$\big|C \cap K^n\big| \leq (nd)!\,\frac{\mathrm{vol}_{\mathbb{A}}(C)}{\big(\sqrt{|\Delta_K|}\big)^n} + nd\,.$$

The corollary has again the assumption $\dim_{\mathbb{Q}}(C \cap K^n) = nd$, but is stated for all fields K, while our previous Theorem 3.3.1 only works for K totally real.

Proof. Again using the notation of the proof of Theorem 3.2.1, by assumption there are $x_1,\dots,x_{nd} \in C\cap K^n$ such that

$$\rho\big(\iota(x_1)\big),\dots,\rho\big(\iota(x_{nd})\big) \in \rho\big(\iota(\mathfrak{M})\big)\cap\rho(C_\infty)$$

are linearly independent over $\mathbb{Q}$. Since $0\in C$ we can apply Theorem 1.1.4 and get

$$\begin{aligned}\left|C\cap K^n\right| &= \left|\rho\big(\iota(\mathfrak{M})\big)\cap\rho\big(C_\infty\big)\right| \\ &\le (nd)!\left(\frac{\mathrm{vol}_{nd}\big(\rho\big(C_\infty\big)\big)}{\left|\det\big(\rho\big(\iota(\mathfrak{M})\big)\big)\right|}\right)+nd \\ &= (nd)!\,\frac{\mathrm{vol}_{\mathbb{A}}(C)}{\big(\sqrt{|\Delta_K|}\big)^n}+nd\,. \end{aligned}$$ □

This can alternatively be formulated in an affine way as follows.

Corollary 3.3.9. *Let C be a general adelic body that contains $nd+1$ elements $x_1,\dots,x_{nd+1}$ of K^n, affinely independent over $\mathbb{Q}$, then*

$$\left|C\cap K^n\right| \le (nd)!\,\frac{\mathrm{vol}_{\mathbb{A}}(C)}{\big(\sqrt{|\Delta_K|}\big)^n}+nd\,.$$

Proof. Without loss of generality we may assume the elements

$$x_1-x_{nd+1}\,,\,\dots\,,\,x_{nd}-x_{nd+1}$$

to be linearly independent. Using the same notation and argument as before, their images under ρ are distinct and linearly independent. Therefore

$$\rho(x_1),\dots,\rho(x_{nd+1})\in\rho(\mathfrak{M})\cap\rho(C_\infty)$$

are affinely independent over $\mathbb{Q}$ and we apply Theorem 1.1.4. □

Finally we give the following straight-forward generalisation of the Betke-Henk-Wills-bound in Theorem 1.1.6.

Proposition 3.3.10. *Let C be a symmetric adelic convex body, then*

$$\left|C\cap K^n\right| \le \left\lfloor \frac{2}{\lambda_1(C)}+1\right\rfloor^{nd}.$$

Proof. As

$$\left|C\cap K^n\right| = \left|\rho(C_\infty)\cap\rho(\iota(\mathfrak{M}))\right| \quad\text{and}\quad \lambda_1(C)=\lambda_1\big(\rho(C_\infty),\rho(\iota(\mathfrak{M}))\big)\,,$$

the result follows directly from Theorem 1.1.6 for the body $\rho(C_\infty)$ and the lattice $\rho(\iota(\mathfrak{M}))$. □

Symmetrisation

A famous result by Rogers and Shephard relates the volume of an arbitrary convex body to that of its symmetrisation.

Theorem 3.3.11 (Rogers, Shephard [RS57]). *Let $v \mid \infty$ and $C_v \subset K_v^n$ be any compact convex body with non-empty interior, then*

$$\operatorname{vol}_v(C_v) \le \operatorname{vol}_v\Big(\tfrac{1}{2}(C_v - C_v)\Big) \le \begin{cases} \dfrac{1}{2^n}\dbinom{2n}{n}\operatorname{vol}_v(C_v), & v \text{ real},\\[2ex] \dfrac{1}{2^{2n}}\dbinom{4n}{2n}\operatorname{vol}_v(C_v), & v \text{ complex}, \end{cases}$$

where $\mathcal{D}(C_v) = \frac{1}{2}(C_v - C_v)$ is called the central symmetrisation *or* difference body *with $\mathcal{D}(C_v) = C_v$ for 0-symmetric C_v.*

This notion of symmetrisation can also be used in the adelic case.

Definition 3.3.12. Let C be a general adelic convex body. Then

$$\mathcal{D}(C) = \tfrac{1}{2}(C - C) = \prod_{v \nmid \infty}\left(C_v - C_v\right) \times \prod_{v \mid \infty}\tfrac{1}{2}\left(C_v - C_v\right)$$

is called the *adelic central symmetrisation* or *adelic difference body* of C. This is an adelic convex body in the original sense, Defintion 1.2.7, as $C_v - C_v = C_v$ for the $\mathcal{O}_v$-modules at $v \nmid \infty$. And $\mathcal{D}(C) = C$ for symmetric C.

Using this definition, we immediately get the following adelic generalisation of the Rogers-Shepard bound of Theorem 3.3.11.

Corollary 3.3.13. *Let C be a general adelic convex body. Then*

$$\operatorname{vol}_{\mathbb{A}}(C) \le \operatorname{vol}_{\mathbb{A}}\left(\tfrac{1}{2}(C - C)\right) \le \frac{1}{2^{nd}}\binom{2n}{n}^{r_1}\binom{4n}{2n}^{r_2}\operatorname{vol}_{\mathbb{A}}(C),$$

since $r_1 + 2r_2 = d$.

4

Adelic Polarity

In this Chapter, we generalise the notion of polarity, introduced for the Euclidean space in Section 1.1, to adelic geometry. This allows a generalisation of the Mahler inequality of Theorem 1.1.1 to the adelic setting.

Sections 4.1 and 4.2 have been previously published in [Thi12]. Only after publication, I have been made aware of the work by Burger [Bur92], which also contains a notion of adelic polarity and its application to the Mahler inequality. His notion is however different from the one considered here, as we will see below.

Our common approach is to directly generalise the classical notion of polarity for each place and thus construct a polar body that is contained in the same space as the primal body. Recall that in the Euclidean setting of Section 1.1, the polar body and lattice were defined as subsets of the same space as the primal set, which is often very convenient for geometrical interpretations. But they can also be seen as sets in the dual space $\left(\mathbb{R}^m\right)^\star$ of $\mathbb{R}^m$, which is isomorphic to the space itself.

In the adelic setting, however, we are not working in a finite-dimensional vector space and we thus do not have an isomorphism of this kind. We therefore employ a more algebraic approach to introduce an adelic polar body and prove some transference results.

Further adelic formulations in terms of twisted heights have been formulated by Roy and Thunder [RT96; RT99] and improved by Rothlisberger [Rot10]. Comparisons of these results will be provided.

For completeness, we also briefly mention the work by Gillet and Soulé [GS91], who use a very different approach, involving the language of Hermitian modules. Our adelic convex bodies can be translated into Hermitian modules as used by Gillet and Soulé. However, their polar set lives in the dual space of $K_{\mathbb{A}}^n$, i.e. the module of $K_{\mathbb{A}}$-linear maps from $K_{\mathbb{A}}^n$ to $K_{\mathbb{A}}$. We will therefore not include it in the discussion below. We also exclude a recent generalisation of the problem to general abelian varieties by Gaudron and Rémond [GR13].

4.1 The Notion of Adelic Polarity

In order to define our notion of adelic polarity we first recall some further background from Algebraic Number Theory. We refer to the standard references [Kna06], [Kna07], [Neu92] or [Wei95] for details.

It is well known that

$$T(x,y) := \mathrm{Tr}_{K/\mathbb{Q}}(xy)$$

is a non-degenerate symmetric $\mathbb{Q}$-bilinear form on K. Here $\mathrm{Tr}_{K/\mathbb{Q}}$ denotes the field trace, i.e. $\mathrm{Tr}_{K/\mathbb{Q}}(a)$ is the trace of the $\mathbb{Q}$-linear map $m_a : K \to K$, given by multiplication with $a \in K$. This allows to define

$$(4.1) \qquad {}^\star\mathcal{O} = \left\{ x \in K \,\middle|\, \mathrm{Tr}_{K/\mathbb{Q}}(xy) \in \mathbb{Z} \ \forall\, y \in \mathcal{O} \right\},$$

the *codifferent*, sometimes also referred to as complementary module. This is a fractional ideal in K, its inverse is the so-called *different* $\mathfrak{d}$. Note that we write the codifferent ${}^\star\mathcal{O}$ with the star on the left, to distinguish it from the group $\mathcal{O}^*$ of units of $\mathcal{O}$. The same construction can of course be applied locally, i.e., for all $v \nmid \infty$ we get

$${}^\star\mathcal{O}_v = \left\{ x \in K_v \,\middle|\, \mathrm{Tr}_{K_v/\mathbb{Q}_v}(xy) \in \mathbb{Z} \ \forall\, y \in \mathcal{O}_v \right\}$$

and its inverse ideal, the local different $\mathfrak{d}_v$. Then by construction

$$(4.2) \qquad \mathrm{vol}_v\left({}^\star\mathcal{O}_v\right) = \mathrm{vol}_v\left(\mathfrak{d}_v\right)^{-1} = \left|\mathcal{D}_v\right|_v^{-1},$$

where $\mathcal{D}_v$ is the local discriminant of K_v. The bilinear form above can be extended to yield a bilinear form on K^n given by

$$T_n(x,y) = \sum_{j=1}^{n} \mathrm{Tr}_{K/\mathbb{Q}}(x_j y_j).$$

The following Lemma is a special case of 1.21.

Lemma 4.1.1. *We have*

$${}^\star\mathcal{O} = \bigcap_{v \nmid \infty} \left({}^\star\mathcal{O}_v \cap K\right).$$

For all but finitely many $v \nmid \infty$, we have ${}^\star\mathcal{O}_v = \mathcal{O}_v$.

Proof. By their definitions (cf. [Kna07, p. 377 (⋆)]) we have

$$^{\star}\mathcal{O}_v \cap K = {}^{\star}\mathcal{O}_{(v)} := \left\{ \tfrac{a}{b} \,\middle|\, a \in {}^{\star}\mathcal{O}, b \in \mathcal{O} \setminus (v) \right\} \supseteq {}^{\star}\mathcal{O},$$

where ${}^{\star}\mathcal{O}_{(v)}$ is the localisation of ${}^{\star}\mathcal{O}$ at the ideal (v) corresponding to v.

For the converse inclusion we follow an idea suggested to us by Jörg Jahnel. Let $M = \bigcap_{v \nmid \infty} {}^{\star}\mathcal{O}_{(v)}$ and $x \in M$, and consider the "ideal of denominators"

$$I = \left\{ b \in \mathcal{O} \,\middle|\, bx \in M \right\}.$$

Since $x \in K \cap {}^{\star}\mathcal{O}_v = {}^{\star}\mathcal{O}_{(v)}$, we have $I \not\subset (v)$ for the ideal in K corresponding to the finite place v. Since this holds for all v, we have $I = \mathcal{O}$. Therefore $x \in {}^{\star}\mathcal{O}$.

The second statement follows from [Kna07, Lemma 6.48], since only finitely many primes are ramified in K. Furthermore, if p is not ramified in K, we get $\mathfrak{D}_v = 1$ for all $v \mid p$. □

The following lemma shows that the construction of the codifferent, (4.1), is compatible with the rank-n-case and the form T_n in the expected sense.

Lemma 4.1.2. *Let $A \in \mathrm{GL}_n(K)$ and $A_v \in \mathrm{GL}_n(K_v)$ for any finite v. Then*

$$^{\star}\left(A\mathcal{O}^n\right) = A^{-\top}\left({}^{\star}\mathcal{O}\right)^n \quad \textit{and} \quad {}^{\star}\left(A_v\mathcal{O}_v^n\right) = A_v^{-\top}\left({}^{\star}\mathcal{O}_v\right)^n,$$

where $A^{-\top}$ and $A_v^{-\top}$ are the transpose of A^{-1} and A_v^{-1}, respectively.

Proof. Notice that

$$^{\star}\left(\mathcal{O}^n\right) := \left\{ x \in K^n \,\middle|\, T_n(x,y) \in \mathbb{Z} \;\forall\, y \in \mathcal{O}^n \right\} \supseteq \left({}^{\star}\mathcal{O}\right)^n.$$

Suppose they are not the same, i.e. $\exists\, a \in {}^{\star}\left(\mathcal{O}^n\right) \setminus \left({}^{\star}\mathcal{O}\right)^n$. Then for some i we must have $a_i \not\in {}^{\star}\mathcal{O}$. So there is some $b_i \in \mathcal{O}$, such that $\mathrm{Tr}_{K/\mathbb{Q}}(a_i b_i) \not\in \mathbb{Z}$ by definition of ${}^{\star}\mathcal{O}$. But then $T_n\big(a, (0,\dots,0,b_i,0,\dots,0)\big) \not\in \mathbb{Z}$ giving a contradiction.

Now let $(a_{ij})_{ij} = A \in \mathrm{GL}_n(K)$ and $x, y \in K^n$. Then

$$\begin{aligned}
T_n(x, Ay) &= \sum_i \mathrm{Tr}_{K/\mathbb{Q}}\big(x_i (Ay)_i\big) = \sum_i \mathrm{Tr}_{K/\mathbb{Q}}\big(x_i \big(\textstyle\sum_j a_{ij} y_j\big)\big) \\
&= \sum_i \sum_j \mathrm{Tr}_{K/\mathbb{Q}}\big(x_i (a_{ij} y_j)\big) = \sum_j \sum_i \mathrm{Tr}_{K/\mathbb{Q}}\big((a_{ij} x_i) y_j\big) \\
&= \sum_j \mathrm{Tr}_{K/\mathbb{Q}}\big(\big(A^\top x\big)_j y_j\big) = T_n\big(A^\top x, y\big).
\end{aligned}$$

For the second statement we can apply the above argument verbatim for $x, y \in K_v^n$ and $A_v \in \mathrm{GL}_n(K_v)$ using $\mathrm{Tr}_{K_v/\mathbb{Q}_v}$ instead. □

We now define a scalar product on $\mathbb{R}^d = \mathbb{R}^{r_1+2r_2}$ by

$$(x,y) = \sum_{j=1}^{r_1} x_j y_j + 2\sum_{j=r_1+1}^{2r_2} x_j y_j\,. \tag{4.3}$$

Since $\rho : K_\infty \cong \mathbb{R}^d$, (1.25), we can also express this in terms of the following scalar product, see also [Neu92, p. III.3],

$$(x,y) = (\rho(x),\rho(y)) = \sum_{v\ \text{real}} x_v y_v + \sum_{v\ \text{complex}} (x_v \overline{y}_v + \overline{x}_v y_v) \tag{4.4}$$

on K_∞. Here $\overline{x}_v$ is the complex conjugate of $x_v \in \mathbb{C}$. To see the equivalence, observe that

$$\begin{aligned}(a+b\mathrm{i})(\overline{c+d\mathrm{i}}) + (\overline{a+b\mathrm{i}})(c+d\mathrm{i}) &= (a+b\mathrm{i})(c-d\mathrm{i}) + (a-b\mathrm{i})(c+d\mathrm{i})\\ &= ac - ad\mathrm{i} + bc\mathrm{i} - bd\mathrm{i}^2 + ac + ab\mathrm{i} - bc\mathrm{i} - bd\mathrm{i}^2 = ac+bd\,.\end{aligned}$$

We now proof the vital Lemma, establishing the link between the bilinear forms on K and on $\mathbb{R}^d$.

Lemma 4.1.3. *For all $x,y \in K$*

$$\mathrm{Tr}_{K/\mathbb{Q}}(xy) = \big(\rho(\iota(x)), \rho(\overline{\iota}(y))\big)\,.$$

Proof. Let $x,y\in K$, then

$$\begin{aligned}&\big(\rho(\iota(x)),\rho(\overline{\iota}(y))\big)\\ &= \sum_{j=1}^{r_1}\sigma_j(x)\sigma_j(y) + \sum_{j=1}^{r_2} 2\Big(\Re\big(\sigma_{r_1+j}(x)\big)\Re\big(\overline{\sigma}_{r_1+j}(y)\big) + \Im\big(\sigma_{r_1+j}(x)\big)\Im\big(\overline{\sigma}_{r_1+j}(y)\big)\Big)\\ &= \sum_{j=1}^{r_1}\sigma_j(x)\sigma_j(y) + 2\sum_{j=1}^{r_2}\Big(\Re\big(\sigma_{r_1+j}(x)\big)\Re\big(\sigma_{r_1+j}(y)\big) - \Im\big(\sigma_{r_1+j}(x)\big)\Im\big(\sigma_{r_1+j}(y)\big)\Big).\end{aligned}$$

Recall that $\mathrm{Tr}_{K/\mathbb{Q}}(x) = \sum_\sigma \sigma(x)$, where the sum ranges over all embeddings $\sigma\colon K \hookrightarrow \overline{\mathbb{Q}}$. And since all complex embeddings appear in conjugate pairs,

$$\begin{aligned}\mathrm{Tr}_{K/\mathbb{Q}}(xy) &= \sum_{j=1}^{r_1}\sigma_j(xy) + \sum_{j=1}^{r_2}\sigma_{r_1+j}(xy) + \sum_{j=1}^{r_2}\overline{\sigma}_{r_1+j}(xy)\\ &= \sum_{j=1}^{r_1}\sigma_j(x)\sigma_j(y) + 2\sum_{j=1}^{r_2}\Re\big(\sigma_{r_1+j}(x)\sigma_{r_1+j}(y)\big).\end{aligned}$$

Finally, we apply the relation $\Re(ab) = \Re(a)\Re(b) - \Im(a)\Im(b)$ for $a,b\in\mathbb{C}$, to see that both expressions do in fact coincide. □

Alternatively, we express this link in terms of polarity.

Corollary 4.1.4. *For any algebraic number field K with ring of integers $\mathcal{O}$ and embeddings ρ and ι as above, we have*

$$\rho\big(\iota(\mathcal{O})\big)^\star = \rho\big(\bar\iota({}^\star\mathcal{O})\big),$$

where $(\cdot)^\star$ on the left is the polar with respect to the form in (4.3).

Example 4.1.5. We consider again the field $K=\mathbb{Q}[\sqrt{2}]$ with $\mathcal{O}=\mathbb{Z}[\sqrt{2}]$. Consider $x=a+b\sqrt{2}\in\mathbb{Q}[\sqrt{2}]$ and $y=c+d\sqrt{2}\in\mathbb{Z}[\sqrt{2}]$. Then

$$xy=(a+b\sqrt{2})(c+d\sqrt{2})=ac+2bd+(ad+bc)\sqrt{2}.$$

Therefore

$$\mathrm{Tr}(xy)=\mathrm{Tr}\begin{pmatrix} ac+2bd & 2ad+2bc \\ ad+bc & ac+2bd \end{pmatrix}=2ac+4bd$$

and this is an integer if $a\in\frac{1}{2}\mathbb{Z}$ and $b\in\frac{1}{4}\mathbb{Z}$. Therefore ${}^\star\mathcal{O}=\frac{1}{2}\mathbb{Z}+\frac{\sqrt{2}}{4}\mathbb{Z}$.

Now $\rho(\iota(\mathcal{O})),\rho(\iota({}^\star\mathcal{O}))\subset\mathbb{R}^2$ are lattices of rank 2, more precisely

$$\rho(\iota(\mathcal{O}))=\begin{pmatrix} 1 & \sqrt{2} \\ 1 & -\sqrt{2} \end{pmatrix}\mathbb{Z}^2, \qquad \rho(\iota({}^\star\mathcal{O}))=\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{2}}{4} \\ \frac{1}{2} & -\frac{\sqrt{2}}{4} \end{pmatrix}\mathbb{Z}^2,$$

and we see that $\rho(\iota(\mathcal{O}))^\star=\rho(\iota({}^\star\mathcal{O}))$. This follows easily from the fact, that the matrices are the inverse transpose of one another.

As we want to work in the rank-n-case over the adeles, we extend the scalar product from (4.3) to $\mathbb{R}^{nd}$ in the obvious way: The scalar product $(\cdot,\cdot)$ on $\mathbb{R}^{nd}$ is also defined as the sum of the components of each copy of $\mathbb{R}^d$. Notice that we get the standard scalar product at the real places and the real scalar product multiplied by 2 at the complex places. By direct consequence of Lemma 4.1.2, Corollary 4.1.4 and (1.21) this leads to the following generalisation.

Corollary 4.1.6. *For any algebraic number field K with ring of integers $\mathcal{O}$ and embeddings ρ^n and ι^n as above, we have for all $n\in\mathbb{N}$*

$$\rho^n\big(\iota^n\big(A^{-1}\mathcal{O}^n\big)\big)^\star=\rho^n\big(\bar\iota^n\big(A^\intercal\big({}^\star\mathcal{O}\big)^n\big)\big)$$

and

$$\rho^n\Big(\iota^n\Big(\bigcap_{v\nmid\infty}\big(A_v^{-1}\mathcal{O}_v^n\cap K^n\big)\Big)\Big)^\star=\rho^n\Big(\bar\iota^n\Big(\bigcap_{v\nmid\infty}\big(A_v^\intercal\big({}^\star\mathcal{O}_v\big)^n\cap K^n\big)\Big)\Big),$$

where $A\in\mathrm{GL}_n(K)$ and $A_v\in\mathrm{GL}_n(K_v)$ for all $v\nmid\infty$.

Proposition 4.1.7. *Consider a finite number d of 0-symmetric convex bodies $C_i \subset \mathbb{R}^{m_i}$. Then, using the classical notion of polarity, see* (1.5)*,*

$$\Big(\prod_{i=1}^{d} C_i\Big)^\star \subseteq \prod_{i=1}^{d} C_i^\star \subseteq d\Big(\prod_{i=1}^{d} C_i\Big)^\star .$$

Proof. Let $x \in \big(\prod_i C_i\big)^\star$, then $\langle x, y\rangle \le 1$ for all $y \in \prod_i C_i$. So especially for any $1 \le i \le d$ we have $\langle x, (0,\dots,0,y_i,0,\dots,0)\rangle \le 1$ for all $y_i \in C_i$. But that implies $\langle x_i, y_i\rangle \le 1$ for all i, which defines the middle set in the chain of includions. Summing over all d components we get the second inclusion.

Using polarity as defined by the scalar product $(\cdot,\cdot)$ instead of $\langle\cdot,\cdot\rangle$, we get $\langle x, (0,\dots,0,y_i,0,\dots,0)\rangle \le \frac{1}{2}$ and $\langle x_i, y_i\rangle \le \frac{1}{2}$ for the complex places $(x_i, y_i \in \mathbb{C})$, so inclusions hold as well. □

Due to Corollary 4.1.6, we are now in the situation to define our notion of adelic polarity.

Definition 4.1.8. Let $C = \prod_{v\nmid\infty} A_v^{-1}\mathscr{O}_v^n \times \prod_{v\mid\infty} C_v$ be a symmetric adelic convex body. The *polar adelic body* of C is

$$C^\star := \prod_{v\nmid\infty} A_v^\intercal\big({}^\star\mathscr{O}_v\big)^n \times \prod_{v\mid\infty} C_v^\star ,$$

where for $v \nmid \infty$ we let $C_v^\star$ be the polar body of C_v with respect to the restriction of (4.3). In other words, for real $v \nmid \infty$ we take the classical polar body in $\mathbb{R}^n$, but for complex $v \nmid \infty$ we take the usual polar body in $\mathbb{R}^{2d}$ and scale it by 2. Since $\mathscr{O}_v = {}^\star\mathscr{O}_v$ for almost all $v \nmid \infty$ by Lemma 4.1.1, $C^\star$ is again an adelic convex body.

From the description of $C^\star$ we immediately we get the following consequence, mirroring the classical statement.

Corollary 4.1.9. *For an adelic convex body C and $A \in \mathrm{GL}_n(K_\mathbb{A})$ we get*

$$(A^{-1}C)^\star = A^\intercal C^\star .$$

Lemma 4.1.10. *Let C be an adelic convex body, then $\big(C^\star\big)^\star = C$.*

Proof. For the convex bodies at the infinite places, the statement follows from the classical result, page 11.

For a finite place $v \nmid \infty$, notice that ${}^\star\big({}^\star\mathscr{O}_v\big) = \mathscr{O}_v$ by definition. □

4.2 The Adelic Mahler inequality

We now apply the results of the previous section, especially Corollary 4.1.6, to prove the main results on adelic polarity. We start with the following upper bound.

Theorem 4.2.1. *Let C be an adelic convex body and $C^\star$ its polar. Then for $1 \le \ell \le n$*

$$\lambda_\ell(C)\lambda_{n-\ell+1}(C^\star) \le (nd)^{3/2}.$$

Proof. Let

$$\mathfrak{M} = \bigcap_{v\nmid\infty} \left(A_v^{-1}\mathcal{O}_v^n \cap K^n\right) \quad \text{and} \quad \mathfrak{M}^\star = \bigcap_{v\nmid\infty} \left(A_v^{\intercal}({}^\star\mathcal{O}_v)^n \cap K^n\right).$$

Then, cf. (1.27), $\rho(\iota(\mathfrak{M}))$ and $\rho(\bar\iota(\mathfrak{M}^\star))$ are lattices of full rank in $\mathbb{R}^{nd}$. By Corollary 4.1.6, they are polar to each other.

Denote by C_∞ and $C_\infty^\star$ the infinite parts of C and $C^\star$, respectively. By Proposition 4.1.7 we have

$$(\rho(C_\infty))^\star \subset \rho(C_\infty^\star). \tag{4.5}$$

Denote by $\lambda_\ell(C)$ and $\lambda_\ell(C^\star)$ the adelic successive minima of C and $C^\star$, respectively and by $\widehat\lambda_i(D,\Lambda)$ the classical successive minima of the convex body D and the lattice Λ in $\mathbb{R}^{nd}$. Then, by (1.30), for $\ell = 1,\dots,n$

$$\lambda_\ell(C) \le \widehat\lambda_{(\ell-1)d+1}\big(\rho(C_\infty),\rho(\iota(\mathfrak{M}))\big)$$

and

$$\lambda_\ell(C^\star) \le \widehat\lambda_{(\ell-1)d+1}\big(\rho(C_\infty^\star),\rho(\bar\iota(\mathfrak{M}^\star))\big) \le \widehat\lambda_{(\ell-1)d+1}\big(\rho(C_\infty)^\star,\rho(\bar\iota(\mathfrak{M}^\star))\big),$$

where the last inequality follows from (4.5).

Finally

$$\begin{aligned}\lambda_\ell(C)\lambda_{n-\ell+1}(C^\star) &\le \widehat\lambda_{(\ell-1)d+1}\big(\rho(C_\infty),\rho(\iota(\mathfrak{M}))\big)\widehat\lambda_{((n-\ell+1)-1)d+1}\big(\rho(C_\infty)^\star,\rho(\bar\iota(\mathfrak{M}^\star))\big)\\ &\le \widehat\lambda_{(\ell-1)d+1}\big(\rho(C_\infty),\rho(\iota(\mathfrak{M}))\big)\widehat\lambda_{(n-\ell)d+d}\big(\rho(C_\infty)^\star,\rho(\bar\iota(\mathfrak{M}^\star))\big)\\ &\le (nd)^{3/2},\end{aligned}$$

applying the classical Mahler inequality, Theorem 1.1.1. □

Corollary 4.2.2. *Let C and $C^\star$ be as in Theorem 4.2.1 and let $\mu(C^\star)$ be the inhomogeneous minimum of $C^\star$. Then*

$$\lambda_1(C)\cdot\mu(C^\star) \le \mathfrak{z}\, nd(1+\log nd),$$

where $\mathfrak{z}$ is a universal constant.

Proof. As in the proof of Theorem 4.2.1, we have $\lambda_1(C) = \widehat{\lambda}_1(\rho(\iota(\mathfrak{M})), \rho(C_\infty))$ and by (4.5) we get

$$\widehat{\mu}(\rho(C_\infty^\star), \Lambda) \le \widehat{\mu}(\rho(C_\infty)^\star, \Lambda)$$

for any lattice $\Lambda \subset \mathbb{R}^{nd}$.

Therefore

$$\lambda_1(C) \cdot \mu(C^\star) \le \widehat{\lambda}_1(\rho(C_\infty), \rho(\iota(\mathfrak{M}))) \cdot \widehat{\mu}(\rho(C_\infty)^\star, \rho(\bar{\iota}(\mathfrak{M}^\star))) \le \mathfrak{z}\, nd(1 + \log nd),$$

by [Ban96, Corollary 1] with some universal constant $\mathfrak{z}$. □

In view of the classical result of Theorem 1.1.1 we are also interested in a lower bound. While the classical bound is comparatively easy to prove, this is not the case in the adelic setting, and we also cannot prove our bound in full generality. For a special class of adelic convex bodies and for K totally real or a CM-field (i.e. a field of complex multiplication) we get the following estimate.

Theorem 4.2.3. *Let K be totally real or a CM-field, and let C be an adelic convex body which is c-symmetric, Definition 1.2.10, and let $C^\star$ be its polar. Then for $1 \le \ell \le n$*

$$\frac{1}{\sqrt[d]{|\Delta_K|}} \le \lambda_\ell(C)\lambda_{n-\ell+1}(C^\star).$$

Proof. We use the standard bilinear form on K^n:

$$b(x, y) = \sum_{i=1}^{n} x_i \overline{y}_i,$$

where $\bar{\cdot}$ is the identity for K totally real and if K is a CM-field, it is the unique non-trivial automorphism of K, that corresponds to complex conjugation in $\mathbb{C}$.

Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be K-bases of K^n such that $u_i \in \lambda_i(C)C$ and $v_j \in \lambda_j(C^\star)C^\star$ for all i, j. Notice that for $u_i \in \mathcal{O}^n$ and $v_j \in ({}^\star\mathcal{O})^n$, we have $b(A^{-1}u_j, \overline{A^\intercal v_j}) = b(u_j, \overline{v}_j) \in {}^\star\mathcal{O}$, using that ${}^\star\mathcal{O}$ is a fractional ideal in K. On account of Lemma 4.1.1, we can also view $b(u_j, \overline{v}_j) \in {}^\star\mathcal{O}_v$. Now since ${}^\star\mathcal{O}_v$ is the inverse fractional ideal of $\mathfrak{d}_v$, on account of (4.2) and the definition of the local volume, we have $|x|_v \le |\mathfrak{D}_v|_v^{-1}$ for $x \in {}^\star\mathcal{O}_v$.

Since b is non-degenerate, there are $i \in \{1,\dots,\ell\}$ and $j \in \{1,\dots,n-\ell+1\}$ such that $b(u_i,\overline{v}_j) \neq 0$. Then by the product formula (1.19)

$$\begin{aligned}1 &= \prod_v \left|b(u_i,\overline{v}_j)\right|_v^{d_v} \cdot \left(\frac{\lambda_i(C)\lambda_j(C^\star)}{\lambda_i(C)\lambda_j(C^\star)}\right)^d \\ &= \prod_{v\nmid\infty} \left|b(u_i,\overline{v}_j)\right|_v^{d_v} \cdot \left(\lambda_i(C)\lambda_j(C^\star)\right)^d \cdot \prod_{v\mid\infty} \left|b\left(\tfrac{1}{\lambda_i(C)}u_i, \tfrac{1}{\lambda_j(C^\star)}\overline{v}_j\right)\right|_v^{d_v}.\end{aligned}$$

Now for any finite v we have $b(u_i,\overline{v}_j) \in {}^\star\mathcal{O}_v$, therefore $\left|b(u_i,\overline{v}_j)\right|_v^{d_v} \le |\mathfrak{D}_v|^{-d_v}$. Finally $\prod_{v\nmid\infty}|\mathfrak{D}_v|^{-d_v} = |\Delta_K|$ by (1.20).

To conclude the proof, we consider the factors at the infinite places. By assumption they are either all real or all complex. Fix some $v \mid \infty$. Let $x := \frac{1}{\lambda_i(C)}u_i$ and $y := \frac{1}{\lambda_j(C^\star)}v_j$. If K is totally real, i.e. v is real, we have

$$\left|b(x,\overline{y})\right|_v^{d_v} = \left|b(x,y)\right|_v^{1} = \left|\sigma_v\left(\textstyle\sum_i x_i y_i\right)\right| = \left|\textstyle\sum_i \sigma_v(x_i)\sigma_v(y_i)\right| \le 1,$$

by definition of $C_v^\star$.

If K is a CM-field, i.e. v is complex, we get

$$\begin{aligned}\left|b(x,\overline{y})\right|_v^{d_v} &= \left|\sigma_v\left(\textstyle\sum_i x_i \overline{y}_i\right)\right|^2 = \left|\textstyle\sum_i \sigma_v(x_i)\overline{\sigma_v(y_i)}\right|^2 \\ &\le \left(\left|\Re(\textstyle\sum_i \sigma_v(x_i)\overline{\sigma_v(y_i)})\right| + \left|\mathrm{i}\Im(\textstyle\sum_i \sigma_v(x_i)\overline{\sigma_v(y_i)})\right|\right)^2 \le \left(\left|\tfrac{1}{2}\right| + 1\left|\tfrac{1}{2}\right|\right)^2 = 1,\end{aligned}$$

by definition of $C_v^\star$, since $\mathrm{i}\Im(x) = \mathrm{i}\Re(\mathrm{i}x)$ for all $x \in \mathbb{C}$ and from $(\sigma_v(x_i))_i \in C_v$ we get $\mathrm{i}(\sigma_v(x_i))_i \in C_v$ by c-symmetry.

The conclusion follows from the monotonicity of the minima. □

Example 4.2.4. We continue our Example 4.1.5. Thus again $K = \mathbb{Q}[\sqrt{2}]$, with $\mathcal{O} = \mathbb{Z}[\sqrt{2}] = \mathbb{Z} + \sqrt{2}\mathbb{Z}$ and ${}^\star\mathcal{O} = \frac{1}{2}\mathbb{Z} + \frac{\sqrt{2}}{4}\mathbb{Z}$. The field discriminant is $|\Delta_K| = \sqrt{8}$.

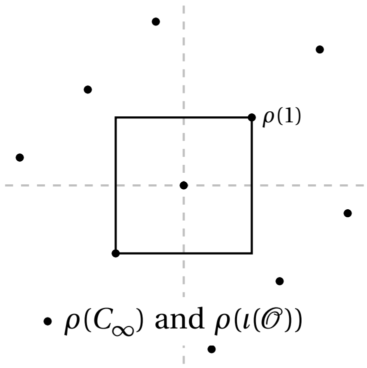
We consider the case $n = 1$, and define our adelic body by taking $C_v = \mathcal{O}_v$ for $v \nmid \infty$ and $C_v = [-1,1]$, the 1-dimensional unit ball, at both infinite places. We see that

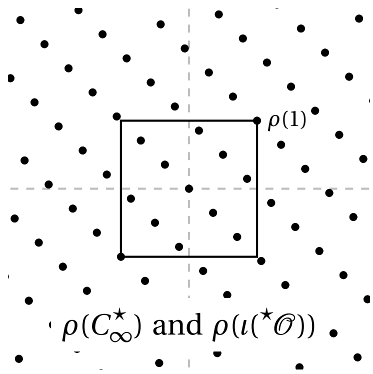
$$C = \prod_{v\nmid\infty} \mathcal{O}_v \times \prod_{v\mid\infty} [-1,1] \quad \text{and} \quad C^\star = \prod_{v\nmid\infty} {}^\star\mathcal{O}_v \times \prod_{v\mid\infty} [-1,1]$$

are polar. Obviously $\lambda_1(C) \le 1$ and we determine the first minimum of $[-1,1]^2$ with respect to the lattice

$$\rho(\iota({}^\star\mathcal{O})) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{2}}{4} \\ \frac{1}{2} & -\frac{\sqrt{2}}{4} \end{pmatrix} \mathbb{Z}^2.$$

Since $\frac{\sqrt{2}}{4} < \frac{1}{2}$, we have $\lambda_1(C^\star) \le \frac{\sqrt{2}}{4}$. This gives equality for the lower bound in Theorem 4.2.3. The following picture shows the images of the embedded bodies C and $C^\star$, similar to Figure 1.1.





Comparison to Previous Results

The adelic variant of Mahler's inequality was already studied by Roy and Thunder [RT96] using the successive minima in terms of heights, Definition 1.2.20. For $A \in \mathrm{GL}_n(K_\mathbb{A})$ and $1 \le \ell \le n$ their result reads

$$\tilde{\lambda}_\ell(A)\tilde{\lambda}_{n-\ell+1}(A^{-\top}) \le 2^{n(n-1)},$$

which was improved by Rothlisberger [Rot10, Theorem 4.6], using an adelic generalisation of so-called Korkin-Zolotarev-reduced bases. He showed for $A \in \mathrm{GL}_n(K_\mathbb{A})$ and $1 \le \ell \le n$

$$\tilde{\lambda}_\ell(A)\tilde{\lambda}_{n-\ell+1}(A^{-\top}) \le d_3(K)\left(1 + \tfrac{n-1}{2}\tau_K^2\right)\gamma_K^*(n),$$

for some implicit constants $d_3(K)$ and τ_K depending only on the field K, and $\gamma_K^*(n)$ is a generalised Hermite constant, which can be bounded linearly in n. Therefore the bound is essentially of order n^2, up to constants. However, combining Lemma 1.2.21 and Theorem 4.2.1, we get

$$\tilde{\lambda}_\ell(A)\tilde{\lambda}_{n-\ell+1}(A^{-\top}) \le \lambda_\ell(A^{-1}\mathrm{B}_\mathbb{A})\lambda_{n-\ell+1}(A^\top\mathrm{B}_\mathbb{A}) \le (nd)^{3/2}.$$

Thus for a fixed field K, our bound is a slight improvement. A direct generalisation of Banaszczyk's appraoch to the adelic setting, as proposed by Rothlisberger [Rot10, Section 1.1], is still to be considered.

An approach to adelic polarity more closely related to our version was first studied by Burger [Bur92]. His definition of the polar body $C^\vee$ of the adelic convex body C is however not compatible with our Definition 4.1.8, as for the finite places $v \nmid \infty$ he defines the local polar body of C_v as

$$C_v^\vee = \left\{ x \in K_v \,\middle|\, \left| \sum_{i=1}^{n} x_i y_i \right|_v \leq 1 \text{ for all } y \in C_v \right\}.$$

Then, [Bur92, Theorem 3.7], for $1 \leq \ell \leq n$

$$1 \leq \lambda_\ell(C)\lambda_{n-\ell+1}(C^\vee) \leq \frac{\left(n!\right)^{2r_1}\left((2n)!\right)^{2r_2}|\Delta_K|^n}{4^{nr_2}}.$$

4.3 More Adelic Transference Results

We now briefly turn to some results in connection with the Mahler volume, i.e. the product

$$\mathrm{vol}_m(C)\,\mathrm{vol}_m(C^\star)$$

for $C \in \mathcal{K}_0^m$.

The Blaschke-Santaló inequality [San49] gives the upper bound

$$\mathrm{vol}_m(C)\,\mathrm{vol}_m(C^\star) \leq \kappa_m^2, \tag{4.6}$$

where κ_m is the volume of the m-dimensional Euclidean ball. On the other hand, Bourgain and Milman established, [BM87],

$$\mathrm{vol}_m(C)\,\mathrm{vol}_m(C^\star) \geq \frac{\mathfrak{z}^m}{m!} \tag{4.7}$$

for some absolute constant $0 < \mathfrak{z} \leq 4$. Observe, that for the cube $C = \mathrm{C}_m$ we have $\mathrm{vol}_m(\mathrm{C}_m)\,\mathrm{vol}_m(\mathrm{C}_m^\star) = 4^m/m!$ and this is the conjectured optimal value. More recently Kuperberg [Kup08] established (4.7) with $\mathfrak{z} = \pi$ and also gave an asymptotic improvement, replacing $\mathfrak{z}$ by a constant depending on m.

Proposition 4.3.1. *Let C be an adelic convex body. Then*

$$\frac{\mathfrak{z}^{nd}4^{nr_2}|\Delta_K|^n}{(n!)^{r_1}((2n)!)^{r_2}} \leq \mathrm{vol}_{\mathbb{A}}(C)\,\mathrm{vol}_{\mathbb{A}}(C^\star) \leq \kappa_n^{2r_1}4^{nr_2}\kappa_{2n}^{2r_2},$$

where r_1 is the number of real, r_2 the number of complex places of K and $\pi \leq \mathfrak{z} \leq 4$ is an absolute constant.

Proof. We compare the factors locally, i.e. for all $v \in M(K)$.

Let $v \mid \infty$, then $C_v, C_v^\star \subset \mathbb{R}^{nd_v}$ and (4.7) yields

$$\mathrm{vol}_v(C_v)\,\mathrm{vol}_v(C_v^\star) = \mathrm{vol}_n(C_v)\,\mathrm{vol}_n(C_v^\star) \geq \frac{\mathfrak{z}^n}{(n)!}$$

for v real, and

$$\mathrm{vol}_v(C_v)\,\mathrm{vol}_v(C_v^\star) = 2^n\,\mathrm{vol}_{2n}(C_v)\,2^n\,\mathrm{vol}_{2n}(C_v^\star) \geq \frac{\mathfrak{z}^{2n}4^n}{(2n)!}$$

for v complex. On the other hand, (4.6) gives

$$\mathrm{vol}_v(C_v)\,\mathrm{vol}_v(C_v^\star) \leq \kappa_n^2 \qquad \text{and} \qquad \mathrm{vol}_v(C_v)\,\mathrm{vol}_v(C_v^\star) \leq 4^n\kappa_{2n}^2$$

for real and complex v respectively. Now for $v \nmid \infty$ we get

$$\mathrm{vol}_v(C_v)\,\mathrm{vol}_v(C_v^\star) = \mathrm{vol}_v(\mathscr{O}_v^n)\,\mathrm{vol}_v({}^\star\mathscr{O}_v^n) = 1 \cdot \left(|\mathscr{D}_v|_v^{-1}\right)^n.$$

The assertion follows with (1.20). □

A similar result has also been proven by Burger [Bur92, Theorem 3.6] for his polar body $C^\vee$ with slightly different constants.

An immediate consequence is the following adelic variant of a pair of inequalities by Kannan and Lovász [KL88, Lemma 1.2].

Their inequalities for any $C \in \mathscr{K}_0^m$ and $\Lambda \in \mathscr{L}^m$ are

$$\lambda_1(C,\Lambda)\,\lambda_1(C^\star,\Lambda^\star) \leq \mathfrak{z}_0 m$$

and

$$\sqrt[m]{\lambda_1(C,\Lambda)\cdots\lambda_m(C,\Lambda)}\;\sqrt[m]{\lambda_1(C^\star,\Lambda^\star)\cdots\lambda_m(C^\star,\Lambda^\star)} \leq \mathfrak{z}_0 m$$

for some absolute constant $\mathfrak{z}_0$.

Our adelic variant is the following.

Proposition 4.3.2. *Let C be adelic convex body. Then*

$$\lambda_1(C)\lambda_1(C^\star) \leq \left(\frac{8}{\mathfrak{z}}\right)\frac{n^2}{4^{r_2/d}}$$

and

$$\sqrt[n]{\lambda_1(C)\cdots\lambda_n(C)}\cdot\sqrt[n]{\lambda_1(C^\star)\cdots\lambda_n(C^\star)} \leq \left(\frac{8}{\mathfrak{z}}\right)\frac{n^2}{4^{r_2/d}},$$

where $\pi \leq \mathfrak{z} \leq 4$ is an absolute constant and r_2 the number of complex places of K. The factor n^2 can be replaced by n if K is totally real or a CM-field.

Proof. First, by the Bombieri-Vaaler-Theorem 1.2.14, we get

$$\lambda_1(C) \le 2\big(\mathrm{vol}_{\mathbb{A}}(C)\big)^{-1/nd}\big(\sqrt{|\Delta_K|}\big)^{1/d}$$
$$\lambda_1(C^\star) \le 2\big(\mathrm{vol}_{\mathbb{A}}(C^\star)\big)^{-1/nd}\big(\sqrt{|\Delta_K|}\big)^{1/d}$$

and

$$\sqrt[n]{\lambda_1(C)}\cdots\sqrt[n]{\lambda_n(C)} \le 2\big(\mathrm{vol}_{\mathbb{A}}(C)\big)^{-1/nd}\big(\sqrt{|\Delta_K|}\big)^{1/d}$$
$$\sqrt[n]{\lambda_1(C^\star)}\cdots\sqrt[n]{\lambda_n(C^\star)} \le 2\big(\mathrm{vol}_{\mathbb{A}}(C^\star)\big)^{-1/nd}\big(\sqrt{|\Delta_K|}\big)^{1/d}.$$

Now we apply Theorem 4.3.1 to the respective products,

$$\lambda_1(C)\lambda_1(C^\star) \le \left(\frac{4}{\mathfrak{z}}\right)\sqrt[nd]{\frac{(n!)^{r_1}((2n)!)^{r_2}}{4^{nr_2}|\Delta_K|^n}}\big(\sqrt{|\Delta_K|}\big)^{1/d} \le \left(\frac{4}{\mathfrak{z}}\right)\sqrt[nd]{\frac{(n!)^{r_1}((2n)!)^{r_2}}{4^{nr_2}}}$$

and similarly

$$\sqrt[n]{\lambda_1(C)\cdots\lambda_n(C)}\cdot\sqrt[n]{\lambda_1(C^\star)\cdots\lambda_n(C^\star)} \le \left(\frac{4}{\mathfrak{z}}\right)\sqrt[nd]{\frac{(n!)^{r_1}((2n)!)^{r_2}}{4^{nr_2}}}.$$

Finally

$$(n!)^{r_1/nd} \le (n!)^{1/n} \le n$$

and

$$((2n)!)^{r_2/nd} \le ((2n)!)^{1/2n} \le 2n$$

as $r_1 \le d$ and $2r_2 \le d$. In the special cases either $r_1 = 0$ or $r_2 = 0$. □

We now generalise a result by Gillet and Soulé [GS91], see also [GS09]. For any $C \in \mathcal{K}^m$ they showed

$$\frac{1}{6^m} \le \frac{\big|C \cap \mathbb{Z}^m\big|}{\big|C^\star \cap \mathbb{Z}^m\big|\cdot \mathrm{vol}_m(C)} \le \frac{6^m}{\mathfrak{z}_0^m \kappa_m^2}, \tag{4.8}$$

where $\kappa_m = \mathrm{vol}_m(\mathrm{B}_m)$ is again the volume of the m-dimensional Euclidean unit ball and and $0 < \mathfrak{z}_0$ some absolute constant.

This pair of inequalities can also be directly extended to the adelic setting.

Proposition 4.3.3. *Let C be an adelic convex body. Then*

$$\frac{1}{(6d)^{nd}} \le \frac{\big|C \cap K^n\big|\cdot\big(\sqrt{|\Delta_K|}\big)^n}{\big|C^\star \cap K^n\big|\cdot \mathrm{vol}_{\mathbb{A}}(C)} \le \frac{6^{nd}}{\mathfrak{z}^{nd}\kappa_{nd}^2},$$

where $\frac{1}{2} \le \mathfrak{z} \le \frac{2}{\pi}$ is an absolute constant.

Proof. First, we remark that by (1.15), for any $C \in \mathscr{K}_0^m$ and $\Lambda \in \mathscr{L}^m$, $\Lambda = A\mathbb{Z}^m$,

$$\left|C \cap \Lambda\right| = \left|A^{-1}C \cap \mathbb{Z}^m\right| \quad \text{and} \quad \left|C^\star \cap \Lambda^\star\right| = \left|(A^{-1}C)^\star \cap \mathbb{Z}^m\right|$$

and on account of $\mathrm{vol}(A^{-1}C) = \mathrm{vol}(C)/\det\Lambda$ we can replace (4.8) by

$$\frac{1}{6^m} \leq \frac{\left|C \cap \Lambda\right| \cdot \det\Lambda}{\left|C^\star \cap \Lambda^\star\right| \cdot \mathrm{vol}_m(C)} \leq \frac{6^m}{\mathfrak{z}_0^m \kappa_m^2}. \tag{4.9}$$

Further, the constant $\mathfrak{z}_0$ of Gillet and Soulé is such that $\mathrm{vol}_m(C)\,\mathrm{vol}_m(C^\star) \geq \mathfrak{z}_0 \kappa_m^2$ and Kuperberg [Kup08] showed that it can be replaced by $\frac{1}{2} \leq \mathfrak{z} \leq \frac{2}{\pi}$.

With the notations as always,

$$\left|C \cap K^n\right| = \left|\rho(C_\infty) \cap \rho(\iota(\mathfrak{M}))\right| \geq \left|\left(\tfrac{1}{d}\rho(C_\infty)\right) \cap \rho(\iota(\mathfrak{M}))\right|$$

and

$$\left|C^\star \cap K^n\right| = \left|\rho(C_\infty^\star) \cap \rho(\bar{\iota}(\mathfrak{M}^\star))\right|.$$

With Proposition 4.1.7 we get

$$\left|\rho(C_\infty)^\star \cap \rho(\bar{\iota}(\mathfrak{M}^\star))\right| \leq \left|\rho(C_\infty^\star) \cap \rho(\bar{\iota}(\mathfrak{M}^\star))\right| \leq \left|d\rho(C_\infty)^\star \cap \rho(\bar{\iota}(\mathfrak{M}^\star))\right|.$$

Now these combined with (4.9) give

$$\frac{\left|\rho(C_\infty) \cap \rho(\iota(\mathfrak{M}))\right|}{\left|\rho(C_\infty^\star) \cap \rho(\bar{\iota}(\mathfrak{M}^\star))\right|} \leq \frac{\left|\rho(C_\infty) \cap \rho(\iota(\mathfrak{M}))\right|}{\left|\rho(C_\infty)^\star \cap \rho(\bar{\iota}(\mathfrak{M}))^\star\right|} \leq \frac{6^{nd}\,\mathrm{vol}_{nd}\big(\rho(C_\infty)\big)}{\mathfrak{z}^{nd}\kappa_{nd}^2 \,\det\big(\rho(\iota(\mathfrak{M}))\big)}$$

and

$$\frac{\left|\rho(C_\infty) \cap \rho(\iota(\mathfrak{M}))\right|}{\left|\rho(C_\infty^\star) \cap \rho(\bar{\iota}(\mathfrak{M}^\star))\right|} \geq \frac{\left|\left(\frac{1}{d}\rho(C_\infty)\right) \cap \rho(\iota(\mathfrak{M}))\right|}{\left|\left(\frac{1}{d}\rho(C_\infty)\right)^\star \cap \rho(\bar{\iota}(\mathfrak{M}))^\star\right|} \geq \frac{\mathrm{vol}_{nd}\left(\frac{1}{d}\rho(C_\infty)\right)}{6^{nd}\,\det\big(\rho(\iota(\mathfrak{M}))\big)}.$$

Thus

$$\frac{1}{(6d)^{nd}} \frac{\mathrm{vol}_{nd}\big(\rho(C_\infty)\big)}{\det\big(\rho(\iota(\mathfrak{M}))\big)} \leq \frac{\left|\rho(C_\infty) \cap \rho(\iota(\mathfrak{M}))\right|}{\left|\rho(C_\infty^\star) \cap \rho(\bar{\iota}(\mathfrak{M}^\star))\right|} \leq \frac{6^{nd}}{\mathfrak{z}^{nd}\kappa_{nd}^2} \frac{\mathrm{vol}_{nd}\big(\rho(C_\infty)\big)}{\det\big(\rho(\iota(\mathfrak{M}))\big)}.$$

The statement follows with (3.2), i.e.,

$$\frac{\left|C \cap K^n\right|}{\left|C^\star \cap K^n\right|} = \frac{\left|\rho(C_\infty) \cap \rho(\iota(\mathfrak{M}))\right|}{\left|\rho(C_\infty^\star) \cap \rho(\bar{\iota}(\mathfrak{M}^\star))\right|},$$

and $\mathrm{vol}_{\mathbb{A}}(C)/\left(\sqrt{|\Delta_K|}\right)^n = \mathrm{vol}_{nd}\big(\rho(C_\infty)\big)/\det\big(\rho(\iota(\mathfrak{M}))\big)$ by (1.28). □

5

Further Adelic Generalisations

In this final chapter we generalise two problems we already discussed in the classical setting to the Adelic Geometry of Numbers.

In Section 5.1 we deal with the generalisation of the results by Kannan and Lovász on the covering minima (1.11). We therefore extend the definition to the adelic setting and prove an adelic version of Theorem 1.1.2. While in the classical case Kannan and Lovász proved a direct connection between the first covering minimum and the first successive minimum of the polar body and lattice, a full adelic analogue is not provided. This is due to the fact that in Euclidean space polar lattice points describe lattice hyperplanes for the primal lattice, which does not hold in $K_{\mathbb{A}}^n$. We will however prove a partial connection.

Section 5.2 then constitutes the adelic generalisation of our own results from Chapter 2 on restricted successive minima. Here, the restrictions will of course be strict linear subspaces of K^n that correspond to lower-dimensional discrete subsets of K^n in $K_{\mathbb{A}}^n$. This topic has originally been considered by Fukshansky [Fuk06b] and Gaudron [Gau09], both recently generalised by Fukshansky and Henshaw [FH13] and Gaudron and Rémond [GR12], respectively. In analogy to the classical case, our approach removes the dependence on the number of restrictions.

5.1 Adelic Covering Minima

Let $W \subseteq K^n$ be a subspace of dimension ℓ and let $u_1, \dots, u_\ell$ be a basis. We think of W as an adelic lattice in $K_{\mathbb{A}}^n$. Let $U = (u_1 \dots u_\ell)$ be the matrix with columns u_i. For a set $I \subset \{1, \dots, n\}$ with $|I| = \ell$, denote by ${}_I U$ the $\ell \times \ell$-submatrix of U with rows indexed by I. For $v \nmid \infty$ fix $J_v \subset \{1, \dots, n\}$, $|J_v| = \ell$, such that

$$\left|\det {}_{J_v} U\right|_v = \max_{|I|=\ell} \left|\det {}_I U\right|_v .$$

We now define the map P_v as

$$P_v = U({}_{J_v}U)^{-1}{}_{J_v}(1_n) \qquad v \nmid \infty,$$
$$P_v = U(U^*U)^{-1}U^* \qquad v \mid \infty,$$

where U^* is the complex conjugate transpose of U and 1_n is the identity. By construction, P_v is a projection of K_v^n onto $W_v = \text{lin}_{K_v}(W) \subseteq K_v^n$ for all $v \in M(K)$, hence $P_v(x) \in W_v$ for all $x \in K_v^n$ and $P_v(y) = y$ for $y \in W_v$. If $v \mid \infty$ is a real place, this is just the orthogonal projection onto W_v as we saw before, page 13. See [Vaa87, Section 4] for more details on P_v for all $v \in M(K)$.

Finally let $\overline{P}_v = (1_n - P_v)$ for all $v \in M(K)$ and we define $P_W = \prod_v P_v$ and $\overline{P}_W = \prod_v \overline{P}_v$. Then P_W is a projection of $K_{\mathbb{A}}^n$ onto the adelic lattice plane $\text{lin}_{K_{\mathbb{A}}}(W)$ and $\overline{P}_W$ is the complementary projection.

Our adelic version of the covering minima introduced by Kannan and Lovász (1.11) is the following.

Definition 5.1.1. Let C be an adelic convex body, $L \subseteq K^n$ an adelic lattice. The i-th *adelic covering minimum* is

$$\mu_i(C, L) = \sup\left\{\mu(C|W, L|W) \,\middle|\, W \subseteq L,\ \dim W = n - i\right\}, \quad 1 \le i \le \dim L,$$

where $C|W = \overline{P}_W(C)$ and $L|W = \overline{P}_W(L)$ are the images of C and L under the projection $\overline{P}_W$ as defined above. We write $\mu_i(C) = \mu_i(C, K^n)$ for short.

This allows us to give the following adelic generalisation of Theorem 1.1.2.

Theorem 5.1.2. *Let C be a symmetric adelic convex body. Then for* $1 \le j < n$

$$\mu_{j+1}(C) \le \mu_j(C) + \nu(K)\lambda_{n-j}(C),$$

where $\nu(K)$ is the adelic field constant (1.23).

Proof. We mirror the proof of Theorem 1.1.2 and start with the case $j = n - 1$. Let $x \in K_{\mathbb{A}}^n$ be arbitrary.

Let $w \in \lambda_1(C) \cdot C \cap K^n$ and denote $W = \text{lin}_K(w)$ and $V = \overline{P}_W\big(K_{\mathbb{A}}^n\big)$. Then we have $x = \overline{P}_W(x) + P_W(x)$ with $\overline{P}_W(x) \in V$ and $P_W(x) = \alpha w$ for some $\alpha \in K_{\mathbb{A}}^n$ as well as $\overline{P}_W(x) \in \mu_{n-1}(C) \cdot C + K^n$ by definition.

By definition of $\nu(K)$, there exists $\lceil \alpha \rfloor \in K$, such that

$$|\lceil \alpha \rfloor - \alpha|_v \le \nu(K) \qquad v \mid \infty,$$
$$|\lceil \alpha \rfloor - \alpha|_v \le 1 \qquad v \nmid \infty.$$

Then

$$x = \overline{P}_W(x) + \alpha w = \underbrace{\overline{P}_W(x) + \lceil \alpha \rfloor w}_{\in \mu_{n-1}(C)\cdot C + K^n} + \underbrace{(\alpha - \lceil \alpha \rfloor) w}_{\in \nu(K)\lambda_1(C)\cdot C} \in \big(\mu_{n-1}(C) + \nu(K)\lambda_1(C)\big)C + K^n,$$

which proofs this case.

For general j let W be a $(j+1)$-dimensional adelic lattice, i.e. $W \subseteq K^n$ with $\dim_K W = j+1$. Then by the first part

$$\mu_{j+1}(C|W, K^n|W) \leq \mu_j(C|W, K^n|W) + \nu(K)\lambda_1(C|W, K^n|W).$$

But since $\mu_j(C|W, K^n|W) \leq \mu_j(C)$ and $\lambda_1(C|W, K^n|W) \leq \lambda_{n-j}(C)$, we have

$$\mu_{j+1}(C|W, K^n|W) \leq \mu_j(C) + \nu(K)\lambda_{n-j}(C).$$

for any W and so the statement follows. □

Finally, we prove one part of the adelic equivalent of (1.13).

Theorem 5.1.3. *Let C be an adelic convex body and $C^\star$ its polar. Then*

$$\mu_1(C)\lambda_1(C^\star) \leq \mathfrak{z}\binom{d+1}{2},$$

where $\mathfrak{z}$ is some absolute constant.

For the opposite inequality, in the classical case Kannan and Lovász use the fact that a point from $\Lambda^\star$, the polar of the lattice Λ, naturally defines a linear function $\mathbb{R}^m \to \mathbb{R}$, whose kernel is a $(n-1)$-dimensional lattice plane of Λ, see also the discussion following (1.13). No such relation exists in the adelic case and we were not able to find an alternative approach.

Proof. Let W be a 1-dimensional subspace of K^n. Then $\overline{W} = \operatorname{lin}_{\mathbb{R}}\big(\rho(\iota(W))\big)$ is a d-dimensional subspace of $\mathbb{R}^{nd}$ and

$$\mu\big(C|W, K^n|W\big) = \mu\big(\rho(C_\infty)\big|\overline{W}, \rho(\iota(\mathfrak{M}))\big|\overline{W}\big)$$

by (1.31). Thus by definition of μ_d,

$$\mu\big(\rho(C_\infty)\big|\overline{W}, \rho(\iota(\mathfrak{M}))\big|\overline{W}\big) \leq \mu_d\big(\rho(C_\infty), \rho(\iota(\mathfrak{M}))\big).$$

And since the inequality holds for all W, it is also true for the supremum,

$$\mu_1(C) \leq \mu_d\big(\rho(C_\infty), \rho(\iota(\mathfrak{M}))\big).$$

Further, by [KL88, Theorem 2.7],

$$\mu_d\big(\rho(C_\infty),\rho(\iota(\mathfrak{M}))\big) \le \mathfrak{z}'\binom{d+1}{2}\mu_1\big(\rho(C_\infty),\rho(\iota(\mathfrak{M}))\big)$$

for some absolute constant $\mathfrak{z}'$ and, by (1.13),

$$\mu_1\big(\rho(C_\infty),\rho(\iota(\mathfrak{M}))\big) = \frac{1}{2\lambda_1\big(\rho(C_\infty)^\star,\rho(\iota(\mathfrak{M}))^\star\big)}\,.$$

Now by Corollary 4.1.6 and (4.5)

$$\lambda_1\big(\rho(C_\infty)^\star,\rho(\iota(\mathfrak{M}))^\star\big) \ge \lambda_1\big(\rho(C_\infty^\star),\rho(\bar\iota(\mathfrak{M}^\star))\big) = \lambda_1(C^\star)\,.$$

The assertion follows with $\mathfrak{z} = \mathfrak{z}'/2$. □

5.2 Adelic Restricted Successive Minima

In this Section we will generalise the results of Chapter 2 to the adelic setting.

Analogously to Definition 2.0.1 we define the adelic restricted minima.

Definition 5.2.1. Let K be algebraic number field and let $L_1,\ldots,L_s \subsetneq K^n$ be linear subspaces. Let C be an adelic convex body. Then

$$\lambda_i\big(C,K^n\setminus\bigcup_{j=1}^s L_j\big) = \min\big\{\lambda>0 \,\big|\, \dim_K\big(\lambda C\cap K^n\setminus\bigcup_{j=1}^s L_j\big)\ge i\big\},\quad 1\le i\le n,$$

is the *i-th restricted successive minimum* of C with respect to $K^n\setminus\bigcup_{j=1}^s L_j$.

We start with the following adelic version of Theorem 2.1.1.

Theorem 5.2.2. *Let C be an adelic convex body and $L_1,\ldots,L_s\subset K^n$ linear subspaces with $n_j = \dim_K L_j < n$. Then*

$$\lambda_1\big(C,K^n\setminus\bigcup_{j=1}^s L_j\big) \le \frac{6^{nd-1}\big(\sqrt{|\Delta_K|}\big)^n}{3^{d-1}\,\lambda_1(C)^{nd-2}\,\mathrm{vol}_{\mathbb{A}}(C)}\left(\sum_{j=1}^s\frac{1}{\lambda_1(C,L_j)}\right)+2\,\frac{\big(\sqrt{|\Delta_K|}\big)^{1/d}}{\sqrt[nd]{\mathrm{vol}_{\mathbb{A}}(C)}}\,.$$

Proof. The proof follows the same argument as that of Theorem 2.1.1. In particular, we can assume $\lambda_1(C)=1$. We need to find $\gamma>0$ such that there exists $x\in\gamma C\cap\big(K^n\setminus(L_1\cup\ldots\cup L_s)\big)$.

We apply our adelic version of Theorem 1.1.6, Proposition 3.3.10, in the same way as in (2.4). So for $\gamma \geq 1$ we again get

$$\left|\gamma C \setminus \{0\} \cap L_j\right| \leq \gamma^{n_j d} 3^{n_j d} \frac{1}{\lambda_1(C, L_j)} \leq \gamma^{(n-1)d} 3^{(n-1)d} \frac{1}{\lambda_1(C, L_j)} \tag{5.1}$$

for all j.

For the lower bound we use our van der Corput-type result of Theorem 3.2.1.

This, combined with (5.1), gives us for $\gamma \geq 1$

$$\begin{aligned}
\left|\gamma C \setminus \{0\} \cap K^n \setminus \bigcup_{j=1}^{s} L_j\right| &\geq \left|\gamma C \setminus \{0\} \cap K^n\right| - \sum_{j=1}^{s} \left|\gamma C \setminus \{0\} \cap L_j\right| \\
&\geq \gamma^{nd} \frac{\mathrm{vol}_{\mathbb{A}}(C)}{2^{nd-1}\left(\sqrt{|\Delta_K|}\right)^n} - 2 - \gamma^{(n-1)d} 3^{(n-1)d} \sum_{j=1}^{s} \frac{1}{\lambda_1(C, L_j)} \\
&= \frac{\mathrm{vol}_{\mathbb{A}}(C)}{2^{nd-1}\left(\sqrt{|\Delta_K|}\right)^n} \left(\gamma^{nd} - \gamma^{(n-1)d} \beta - \rho\right)
\end{aligned}$$

where

$$\beta = \frac{6^{nd-1}\left(\sqrt{|\Delta_K|}\right)^n}{3^{d-1}\ \mathrm{vol}_{\mathbb{A}}(C)} \left(\sum_{j=1}^{s} \frac{1}{\lambda_1(C, L_j)}\right), \qquad \rho = 2^{nd} \frac{\left(\sqrt{|\Delta_K|}\right)^n}{\mathrm{vol}_{\mathbb{A}}(C)}.$$

By Bombieri-Vaaler's Theorem 1.2.14 and our assumption, $\rho \geq \left(\sqrt{|\Delta_K|}\right)^n \geq 1$.

Similarly to the proof of Theorem 2.1.1, for $\overline{\gamma} = \beta + \rho^{1/nd}$, we have

$$\begin{aligned}
\overline{\gamma}^{nd} - \overline{\gamma}^{(n-1)d} \beta &= \left(\beta + \rho^{1/nd}\right)^{nd} - \left(\beta + \rho^{1/nd}\right)^{(n-1)d} \beta \\
&= \left(\left(\beta + \rho^{1/nd}\right)^{d} - \beta\right) \cdot \left(\beta + \rho^{1/nd}\right)^{(n-1)d} \\
&> \rho^{d/nd} \cdot \rho^{(n-1)d/nd} \\
&= \rho\,.
\end{aligned}$$

and thus $\left|\overline{\gamma} C \setminus \{0\} \cap K^n \setminus \bigcup_{j=1}^{s} L_j\right| \geq 1$, establishing the Theorem for our special case $\lambda_1(C) = 1$. □

We can also extend Corollary 2.1.2 to the adelic setting, using the same line of argument as in the classical case. We then get the following statement.

Corollary 5.2.3. *Let C be an adelic convex body and $L_1,\dots,L_s \subset K^n$ linear subspaces with $n_j = \dim_K L_j < n$. Then for all $i = 0,\dots,n-1$*

$$\lambda_{i+1}\big(C, K^n \setminus \bigcup_{j=1}^{s} L_j\big) \le \frac{6^{nd-1}\left(\sqrt{|\Delta_K|}\right)^n}{3^{d-1}\,\lambda_1(C)^{nd-2}\,\mathrm{vol}_{\mathbb{A}}(C)}\left(\sum_{j=1}^{s}\frac{1}{\lambda_1(C,L_j)}\right) + \left(2^{nd-1}\frac{\left(\sqrt{|\Delta_K|}\right)^n}{\mathrm{vol}_{\mathbb{A}}(C)}\frac{3^{id}}{\lambda_1(C,\overline{L})^{id}} + \left(2^{nd}\frac{\left(\sqrt{|\Delta_K|}\right)^n}{\mathrm{vol}_{\mathbb{A}}(C)}\right)^{\frac{(n-i)d}{nd}}\right)^{\frac{1}{(n-i)d}}.$$

Proof. As in the classical case, we proceed by induction. The case $i = 0$ follows directly from Theorem 5.2.2. Let $z_k \in \lambda_k\left(C, K^n \setminus \bigcup_{j=1}^{s} L_j\right) C \cap K^n$ for $1 \le k \le i$ be linearly independent, and let $\overline{L} = \mathrm{lin}_K\{z_1,\dots,z_i\}$. Then

$$\lambda_{i+1}\Big(C, K^n \setminus \bigcup_{j=1}^{s} L_j\Big) = \lambda_1\Big(C, K^n \setminus \Big(\bigcup_{j=1}^{s} L_j \cup \overline{L}\Big)\Big) \tag{5.2}$$

and we follow the proof of Theorem 5.2.2 just as we did in the proof of Corollary 2.1.2. We assume $\lambda_1(C) = 1$, and for $\gamma \ge 1$ in addition to the inequalities of (5.1) we use for $\gamma \ge \lambda_1(C,\overline{L}) \ge \lambda_1(C){=}1$ the bound from Proposition 3.3.10

$$\left|\gamma C \setminus \{0\} \cap \overline{L}\right| < \left(\frac{2\gamma}{\lambda_1(C,\overline{L})} + 1\right)^{id} \le \frac{\gamma^{id}\,3^{id}}{\lambda_1(C,\overline{L})^{id}}.$$

This, combined with the adelic van der Corput inequality of Theorem 3.2.1, gives for $\gamma \ge \lambda_1(C,\overline{L})$ the estimate

$$\begin{aligned}
\left|\gamma C \setminus \{0\} \cap K^n \setminus \big(\bigcup_{j=1}^{s} L_j \cup \overline{L}\big)\right| &\ge \left|\gamma C \setminus \{0\} \cap K^n\right| - \sum_{j=1}^{s}\left|\gamma C \setminus \{0\} \cap L_j\right| - \left|\gamma C \setminus \{0\} \cap \overline{L}\right| \\
&\ge \gamma^{nd}\frac{\mathrm{vol}_{\mathbb{A}}(C)}{2^{nd-1}\left(\sqrt{|\Delta_K|}\right)^n} - 2 \qquad (5.3)\\
&\quad - \gamma^{(n-1)d}3^{(n-1)d}\sum_{j=1}^{s}\frac{1}{\lambda_1(C,L_j)} - \frac{\gamma^{id}\,3^{id}}{\lambda_1(C,\overline{L})^{id}} \\
&= \frac{\mathrm{vol}_{\mathbb{A}}(C)}{2^{nd-1}\left(\sqrt{|\Delta_K|}\right)^n}\left(\gamma^{nd} - \gamma^{(n-1)d}\beta - \gamma^{id}\alpha - \rho\right),
\end{aligned}$$

where

$$\beta = \frac{6^{nd-1}\left(\sqrt{|\Delta_K|}\right)^n}{3^{d-1}\,\mathrm{vol}_{\mathbb{A}}(C)}\left(\sum_{j=1}^{s}\frac{1}{\lambda_1(C,L_j)}\right), \qquad \alpha = 2^{nd-1}\frac{\left(\sqrt{|\Delta_K|}\right)^n}{\mathrm{vol}_{\mathbb{A}}(C)}\frac{3^{id}}{\lambda_1(C,\overline{L})^{id}}$$

and

$$\rho = 2^{nd} \frac{\left(\sqrt{|\Delta_K|}\right)^n}{\mathrm{vol}_{\mathbb{A}}(C)}.$$

Setting now $\overline{\gamma} = \beta + \left(\alpha + \rho^{\frac{(n-i)d}{nd}}\right)^{\frac{1}{(n-i)d}} \geq \rho^{\frac{1}{nd}} \geq 1$, we observe

$$\begin{aligned}
\text{(5.4)} \qquad \overline{\gamma}^{nd} - \overline{\gamma}^{(n-1)d}\beta - \overline{\gamma}^{id}\alpha - \rho &= \overline{\gamma}^{id}\left(\overline{\gamma}^{(n-i)d} - \beta\overline{\gamma}^{(n-i-1)d} - \alpha\right) - \rho \\
&\geq \overline{\gamma}^{id}\left(\overline{\gamma}^{(n-i)d} - \beta\overline{\gamma}^{(n-i)d-1} - \alpha\right) - \rho \\
&= \overline{\gamma}^{id}\rho^{\frac{(n-i)d}{nd}} - \rho,
\end{aligned}$$

since

$$\begin{aligned}
&\overline{\gamma}^{(n-i)d} - \beta\overline{\gamma}^{(n-i)d-1} - \alpha \\
&\quad = \left(\beta + \left(\alpha + \rho^{\frac{(n-i)d}{nd}}\right)^{\frac{1}{(n-i)d}}\right)^{(n-i)d} - \beta\left(\beta + \left(\alpha + \rho^{\frac{(n-i)d}{nd}}\right)^{\frac{1}{(n-i)d}}\right)^{(n-i)d-1} - \alpha \\
&\quad = \sum_{k=0}^{(n-i)d}\left(\alpha + \rho^{\frac{(n-i)}{nd}}\right)^{\frac{(n-i)d-k}{(n-i)d}}\beta^k - \beta\sum_{k=0}^{(n-i)d-1}\left(\alpha + \rho^{\frac{(n-i)d}{nd}}\right)^{\frac{(n-i)d-1-k}{(n-i)d}}\beta^k - \alpha \\
&\quad = \left(\alpha + \rho^{\frac{(n-i)d}{nd}}\right) + \sum_{k=1}^{(n-i)d}\left(\alpha + \rho^{\frac{(n-i)d}{nd}}\right)^{\frac{(n-i)d-k}{(n-i)d}}\beta^k \\
&\qquad\qquad - \sum_{k=0}^{(n-i)d-1}\left(\alpha + \rho^{\frac{(n-i)d}{nd}}\right)^{\frac{(n-i)-(k+1)}{(n-i)d}}\beta^{k+1} - \alpha \\
&\quad = \rho^{\frac{(n-i)d}{nd}}.
\end{aligned}$$

Now $\overline{\gamma}^{id} \geq \rho^{\frac{id}{nd}}$, and thus

$$\overline{\gamma}^{id}\rho^{\frac{(n-i)d}{nd}} - \rho > \rho^{\frac{id}{nd}}\rho^{\frac{(n-i)d}{nd}} - \rho = 0.$$

Therefore, the left-hand side of (5.4) is strictly positive and thus also the left-hand side of (5.3) if $\overline{\gamma} \geq \lambda_1(C, \overline{L})$ holds. But since $\overline{\gamma} > \beta + \rho^{1/nd}$, which we established in the proof of Theorem 5.2.2 to be an upper bound on $\lambda_1\left(C, K^n \setminus \bigcup_{j=1}^{s} L_j\right)$, this is true by construction of $\overline{L}$. Thus $\overline{\gamma}$ is the required upper bound with respect to the normalisation $\lambda_1(C) = 1$. □

We now also give an adelic variant of our Theorem 2.1.3.

Theorem 5.2.4. *Let C be an adelic convex body and $L_1,\dots,L_s \subset K^n$ linear subspaces with $n_j = \dim_K L_j < n$, then*

$$\lambda_1\big(C, K^n \setminus \bigcup_{j=1}^{s} L_j\big) \le \frac{(10n)^{nd-1}}{\lambda_1(C)^{nd-1}\operatorname{vol}_{\mathbb{A}}(C)}\left(\sum_{j=1}^{s}\max_{k=1,\dots,n_j}(5k)^{kd}\lambda_1(C)^k\operatorname{vol}_{\mathbb{A}}^{(k)}\big(C_k^{(j)}\big)\right) + 2\frac{\big(\sqrt{|\Delta_K|}\big)^{1/d}}{\sqrt[nd]{\operatorname{vol}_{\mathbb{A}}(C)}},$$

where for each j

$$\big\{x_{j,k} \in \lambda_k(C\cap L_j)\cdot C \,\big|\, 1 \le k \le n_j\big\} \quad \text{is a basis of } L_j$$

and

$$\big(C_v\big)_k^{(j)} = C_v \cap \operatorname{lin}_{K_v}\{x_{j,1},\dots,x_{j,k}\} \quad \text{and} \quad C_k^{(j)} = \prod_v \big(C_v\big)_k^{(j)}.$$

Here $\operatorname{vol}_{\mathbb{A}}^{(k)}$ denotes the appropriate adelic measure on the k-dimensional adelic lattice plane containing $C_k^{(j)}$.

Proof. The proof combines the arguments of the proof of Theorem 5.2.2 with those of Theorem 2.1.3. Again we assume $\lambda_1(C) = 1$.

We will use the adelic variant on the Blichfeldt-bound by Gaudron, (3.9). For a fixed L_j we have for all $1 \le k \le n_j$

$$\big|C_k^{(j)} \setminus \{0\} \cap K^n\big| < (5k)^{kd}\operatorname{vol}_{\mathbb{A}}^{(k)}\big(C_k^{(j)}\big).$$

For the lower bound we again use our van der Corput-type result of Theorem 3.2.1.

This gives for $\gamma \ge 1$

$$\begin{aligned}
\big|\gamma C \setminus \{0\} \cap K^n \setminus \bigcup_{j=1}^{s} L_j\big| &\ge \big|\gamma C \setminus \{0\} \cap K^n\big| - \sum_{j=1}^{s}\big|\gamma C \setminus \{0\} \cap L_j\big| \\
&\ge \gamma^{nd}\frac{\operatorname{vol}_{\mathbb{A}}(C)}{2^{nd-1}\big(\sqrt{|\Delta_K|}\big)^n} - 2 \\
&\qquad - \sum_{j=1}^{s}\gamma^{(n-1)d}\max_{k=1,\dots,n_j}(5k)^{kd}\operatorname{vol}_{\mathbb{A}}^{(k)}\big(C_k^{(j)}\big) \\
&= \frac{\operatorname{vol}_{\mathbb{A}}(C)}{2^{nd-1}\big(\sqrt{|\Delta_K|}\big)^n}\big(\gamma^{nd} - \gamma^{(n-1)d}\beta - \rho\big),
\end{aligned}$$

where

$$\beta = \frac{2^{nd-1}\big(\sqrt{|\Delta_K|}\big)^n}{\mathrm{vol}_{\mathbb{A}}(C)} \left(\sum_{j=1}^{s} \max_{k=1,\dots,n_j} (5k)^{kd} \, \mathrm{vol}_{\mathbb{A}}^{(k)}\big(C_k^{(j)}\big) \right), \quad \rho = 2^{nd} \frac{\big(\sqrt{|\Delta_K|}\big)^n}{\mathrm{vol}_{\mathbb{A}}(C)}.$$

We finish the proof by the same argument as in the proof of Theorem 5.2.2. We have $\rho \geq \big(\sqrt{|\Delta_K|}\big)^n \geq 1$, and for $\overline{\gamma} = \beta + \rho^{1/nd}$, with

$$\overline{\gamma}^{nd} - \overline{\gamma}^{(n-1)d}\beta - \rho > 0\,,$$

we establish the bound for $\lambda_1(C) = 1$. □

We also have an adelic version of Proposition 2.1.4.

Proposition 5.2.5. *Let C be an adelic convex body and $L_1, \dots, L_s \subset K^n$ linear subspaces with $n_j = \dim_K L_j < n$. Then*

$$\lambda_1\Big(C, K^n \setminus \bigcup_{j=1}^{s} L_j\Big) \leq (s+1)\,\mu(C)$$

and hence, $\lambda_i\big(C, K^n \setminus \bigcup_{j=1}^{s} L_j\big) \leq (s+2)\,\mu(C)$ for $2 \leq i \leq n$.

Proof. As in the classical case, the bound for $i \geq 2$ follows from the first one on account of (5.2).

The bound on $\lambda_1\Big(C, K^n \setminus \bigcup_{j=1}^{s} L_j\Big)$ itself can be inferred directly from the classical result. Denote as always

$$\mathfrak{M} = \bigcap_{v\nmid\infty} \big(C_v \cap K^n\big) \quad \text{and} \quad \mathfrak{L}_j = \bigcap_{v\nmid\infty} \big(C_v \cap L_j\big) \quad \text{as well as} \quad C_\infty = \prod_{v\mid\infty} C_v\,.$$

Using our standard embedding $\rho \circ \iota : K^n \hookrightarrow \mathbb{R}^{nd}$, cf. (1.25) and thereafter, the forbidden submodules $\mathfrak{L}_j$ map to sublattices of $\rho(\iota(\mathfrak{M}))$ of rank at most $(n-1)d$. Then, by Proposition 2.1.4, we have

$$\widehat{\lambda}_1\Big(\rho\big(C_\infty\big), \rho\big(\iota\big(\mathfrak{M}\big)\big) \setminus \bigcup_{j=1}^{s} \rho\big(\iota\big(\mathfrak{L}_j\big)\big)\Big) \leq (s+1)\,\widehat{\mu}\big(\rho(C_\infty), \rho(\iota(\mathfrak{M}))\big)\,.$$

By the injectivity of the embedding, any point corresponding to this restricted minimum corresponds also to the adelic restricted minimum. The statement follows on account of (1.31). □

Comparison to Previous Results

Similar to Definition 1.2.20 we can also define restricted successive minima in terms of heights for $A \in \mathrm{GL}_n(K)$ and linear subspaces $L_1, \dots, L_s \subsetneq K^n$ as

$$\tilde{\lambda}_i\Big(A, K^n \setminus \bigcup_{j=1}^{s} L_j\Big) = \inf\Big\{\lambda > 0 \,\Big|\, \exists\, x_1, \dots, x_i \in K^n \setminus \bigcup_{j=1}^{s} L_j \text{ lin. indep. over } K \\ \text{s.t. } H_A(x_j) \le \lambda \text{ for all } j\Big\}.$$

We get the following immediate connection between the two notions.

Corollary 5.2.6. *We have*

$$\tilde{\lambda}_i\Big(A, K^n \setminus \bigcup_{j=1}^{s} L_j\Big) \le \lambda_i\Big(A^{-1}\mathrm{B}_{\mathbb{A}}, K^n \setminus \bigcup_{j=1}^{s} L_j\Big)$$

for every $A \in GL_n(K_{\mathbb{A}})$.

Proof. In the proof of Lemma 1.2.21 we showed that $x \in \overline{\lambda} \cdot A^{-1}\mathrm{B}_{\mathbb{A}}$ always implies $H_A(x) \le \overline{\lambda}$. □

Fukshansky also proved an adelic extension of his Euclidean result (2.2). In a simplified form, where we assume $\max_j\big(\dim_K L_j\big) = n-1$, [Fuk06b, Theorem 1.2] (see also [Fuk10, Theorem 1.3]) states

$$\tilde{\lambda}_1\Big(I, K^n \setminus \bigcup_{j=1}^{s} L_j\Big) \le C_{K,n}\left(\left(\sum_{j=1}^{s} \frac{1}{H(L_j)^d}\right)^{\frac{1}{d}} + s^{\frac{1}{d+1}}\right), \tag{5.5}$$

where

$$C_{K,n} = 2^n(d+1)\big(\sqrt{|\Delta_K|}\big)^n\left((nd)^n\binom{nd}{\ell d}^{\frac{1}{2d}}\right)$$

for the identity $I \in \mathrm{GL}_n(K_{\mathbb{A}})$ and $\ell = \lfloor \frac{n}{2} \rfloor$. This has most recently been extended to higher restricted minima in [CFH13, Appendix A], with a refined dimensional dependence but introducing several further algebraic properties of K.

Fukshansky's proof uses a similar technique, employing a counting argument, [Fuk06b, Lemma 3.2], similar to our van der Corput-type bound of Theorem 3.2.1. However, on account of Corollary 5.2.6, our bound of Theorem 5.2.2 yields an improved dimensional dependence compared to (5.5).

Fukshansky's result has also been improved and generalised to arbitrary bodies by Gaudron [Gau09, Theorem 1.1], which, again for $\max_j\{n_j\} = n-1$, where $n_j = \dim_K L_j$, states

$$\lambda_1\Big(C, K^n \setminus \bigcup_{j=1}^{s} L_j\Big) \le \nu \max_{1\le j\le s}\left\{1\ ,\ \left(\frac{\nu^{n_i}}{H(L_j)}\right)^{\frac{1}{n-n_j}}\ ,\ \left(\frac{\nu}{\lambda_1(C, L_j)}\right)^{\frac{n_j-1}{n-n_j+1}}\right\} \tag{5.6}$$

where $\nu = (41n)^{d/2}\sqrt{|\Delta_K|}\, s^{1/n}$ and the fractions in the curly brackets are replaced by 1 for $n_j = 0$. Recently, Gaudron and Rémond [GR12, Corollary 3.3] gave a more refined bound, which is however not explicit, as it is expressed in terms of a minimum over all subfields $K' \subseteq K$. We therefore exclude it from the discussion here.

Remark 5.2.7. *Observe that for vanishing restrictions, i.e. all $\lambda_1(C, L_j) \to \infty$, just as in the classical case of Section 2.1, see the remarks on page 44, our bound reduces to*

$$\lambda_1(C) \le \frac{2\big(\sqrt{|\Delta_K|}\big)^{1/d}}{\sqrt[nd]{\mathrm{vol}_{\mathbb{A}}(C)}}\,.$$

This can be seen as the adelic version of Minkowski's first theorem and is a weaker variant of the upper bound in the Bombieri-Vaaler Theorem 1.2.14. Fukshansky's bound (5.5)*, however, still has the dependence on $s^{1/(d+1)}$ if all $H(L_j) \to \infty$. The same is true for Gaudron's bound* (5.6) *with a dependence on $s^{1/n}$ if all $H(L_j) \to \infty$ and all $\lambda_1(C, L_j) \to \infty$.*

Full-dimensional Restrictions

To conclude the section, we turn to the adelic generalisation of full-dimensional restrictions as in Section 2.2. Since there are no full-dimensional subspaces of K^n apart from K^n itself, see Remark 1.2.6, a direct adelic analogue of Theorem 2.2.5 is not feasible. In fact, by Definition 5.2.1, the adelic restrictions are automatically of lower dimension, on account of the requirement $L_j \ne K^n$ for all j.

We can apply the results of Section 2.2 to get the following variant, however.

Proposition 5.2.8. *Let C, D be adelic convex bodies with $D \subsetneq C$ and $C_\infty = D_\infty$. Then for*

$$\gamma \ge \frac{2^{nd}\big(\sqrt{|\Delta_K|}\big)^n}{\lambda_1(D)^{nd-1}\,\mathrm{vol}_{\mathbb{A}}(C)} + \lambda_1(C)$$

there exists

$$x \in (\gamma C \setminus \gamma D) \cap K^n\,.$$

Proof. Denote

$$\mathfrak{M} = \bigcap_{v\nmid\infty} \left(C_v \cap K^n\right) \quad \text{and} \quad \mathfrak{N} = \bigcap_{v\nmid\infty} \left(D_v \cap K^n\right).$$

Then by construction, $\rho\big(\iota(\mathfrak{N})\big) \subsetneq \rho\big(\iota(\mathfrak{M})\big) \subset \mathbb{R}^{nd}$ are lattices of full rank, and we have the convex body $\rho(D_\infty) = \rho(C_\infty) \subset \mathbb{R}^{nd}$. So by Corollary 2.2.6

$$\begin{aligned}\lambda_1\big(\rho(C_\infty), \rho(\iota(\mathfrak{M})) \setminus \rho(\iota(\mathfrak{N}))\big) \leq{}& \frac{2^{nd} \det\big(\rho(\iota(\mathfrak{M})\big)}{\lambda_1\big(\rho(C_\infty), \rho(\iota(\mathfrak{N})\big)^{nd-1} \operatorname{vol}_{nd}\big(\rho(C_\infty)\big)} \\ &+ \lambda_1\big(\rho(C_\infty), \rho(\iota(\mathfrak{M}))\big).\end{aligned}$$

Using (1.30), which is an equality for $\ell = 0$, we can express the upper bound in terms of the adelic minima and adelic volume, (1.28), as

$$\lambda_1\big(\rho(C_\infty), \rho(\iota(\mathfrak{M})) \setminus \rho(\iota(\mathfrak{N}))\big) \leq \frac{2^{nd}\left(\sqrt{|\Delta_K|}\right)^n}{\lambda_1(D)^{nd-1} \operatorname{vol}_{\mathbb{A}}(C)} + \lambda_1(C).$$

Finally, any point corresponding to this restricted minimum has the desired property. □

Bibliography

[Ali08] Iskander Aliev. *"Siegel's lemma and sum-distinct sets"*. In: *Discrete Comput. Geom.* 39.1-3 (2008), pp. 59–66.

[Bal91] Keith Ball. *"The plank problem for symmetric bodies"*. In: *Invent. Math.* 104.3 (1991), pp. 535–543.

[Ban93] Wojciech Banaszczyk. *"New bounds in some transference theorems in the geometry of numbers"*. In: *Math. Ann.* 296.4 (1993), pp. 625–635.

[Ban96] Wojciech Banaszczyk. *"Inequalities for convex bodies and polar reciprocal lattices in* $\mathbb{R}^n$*. II. Application of* K*-convexity"*. In: *Discrete Comput. Geom.* 16.3 (1996), pp. 305–311.

[BG06] Enrico Bombieri and Walter Gubler. *"Heights in Diophantine geometry"*. Vol. 4. New Mathematical Monographs. Cambridge: Cambridge University Press, 2006.

[BH94] Károly Bezdek and Tamás Hausel. *"On the number of lattice hyperplanes which are needed to cover the lattice points of a convex body"*. In: *Intuitive geometry (Szeged, 1991)*. Vol. 63. Colloq. Math. Soc. János Bolyai. Amsterdam: North-Holland, 1994, pp. 27–31.

[BHPT01] Imre Bárány, Gergely Harcos, János Pach, and Gábor Tardos. *"Covering lattice points by subspaces"*. In: *Period. Math. Hungar.* 43.1-2 (2001), pp. 93–103.

[BHW93] Ulrich Betke, Martin Henk, and Jörg M. Wills. *"Successive-minima-type inequalities"*. In: *Discrete Comput. Geom.* 9.2 (1993), pp. 165–175.

[BL09] Károly Bezdek and Alexander E. Litvak. *"Covering convex bodies by cylinders and lattice points by flats"*. In: *J. Geom. Anal.* 19.2 (2009), pp. 233–243.

[BL78] Peter E. Blanksby and John H. Loxton. *"A note on the characterization of CM-fields"*. In: *J. Austral. Math. Soc. Ser. A* 26.1 (1978), pp. 26–30.

[BM87] Jean Bourgain and Vitali D. Milman. *"New volume ratio properties for convex symmetric bodies in $\mathbb{R}^n$"*. In: *Invent. Math.* 88.2 (1987), pp. 319–340.

[BN09] Johannes Blömer and Stefanie Naewe. *"Sampling methods for shortest vectors, closest vectors and successive minima"*. In: *Theoret. Comput. Sci.* 410.18 (2009), pp. 1648–1665.

[BR07] Matthias Beck and Sinai Robins. *"Computing the continuous discretely"*. Undergraduate Texts in Mathematics. Integer-point enumeration in polyhedra. New York: Springer, 2007.

[Bur92] Edward B. Burger. *"Homogeneous Diophantine approximation in S-integers"*. In: *Pacific J. Math.* 152.2 (1992), pp. 211–253.

[BV83] Enrico Bombieri and Jeffrey D. Vaaler. *"On Siegel's lemma"*. In: *Invent. Math.* 73.1 (1983), pp. 11–32.

[BV84] Enrico Bombieri and Jeffrey D. Vaaler. *"Addendum to: 'On Siegel's lemma'"*. In: *Invent. Math.* 75.2 (1984), p. 377.

[BV93] Edward B. Burger and Jeffrey D. Vaaler. *"On the decomposition of vectors over number fields"*. In: *J. Reine Angew. Math.* 435 (1993), pp. 197–219.

[CFH13] Wai Kiu Chan, Lenny Fukshansky, and Glenn Henshaw. *"Small zeros of quadratic forms missing a union of varieties"*. In: *Trans. Amer. Math. Soc.* (2013). to appear.

[DLRS10] Jesús A. De Loera, Jörg Rambau, and Francisco Santos. *"Triangulations"*. Vol. 25. Algorithms and Computation in Mathematics. Structures for algorithms and applications. Berlin: Springer-Verlag, 2010.

[Ehr62] Eugène Ehrhart. *"Sur les polyèdres rationnels homothétiques à n dimensions"*. In: *C. R. Acad. Sci. Paris* 254 (1962), pp. 616–618.

[Fal91] Gerd Faltings. *"Diophantine approximation on abelian varieties"*. In: *Ann. of Math. (2)* 133.3 (1991), pp. 549–576.

[FH13] Lenny Fukshansky and Glenn Henshaw. *"Lattice point counting and height bounds over number fields and quaternion algebras"*. In: *Online Journal of Analytic Combinatorics* 8 (2013).

[Fuk06a] Lenny Fukshansky. *"Integral points of small height outside of a hypersurface"*. In: *Monatsh. Math.* 147.1 (2006), pp. 25–41.

[Fuk06b] Lenny Fukshansky. *"Siegel's lemma with additional conditions"*. In: *J. Number Theory* 120.1 (2006), pp. 13–25.

[Fuk10] Lenny Fukshansky. *"Algebraic points of small height missing a union of varieties"*. In: *J. Number Theory* 130.10 (2010), pp. 2099–2118.

[Gau09] Éric Gaudron. *"Géométrie des nombres adélique et lemmes de Siegel généralisés"*. In: *Manuscripta Math.* 130.2 (2009), pp. 159–182.

[GL87] Peter M. Gruber and C. Gerrit Lekkerkerker. *"Geometry of numbers"*. Second Edition. Vol. 37. North-Holland Mathematical Library. Amsterdam: North-Holland Publishing Co., 1987.

[GLS93] Martin Grötschel, László Lovász, and Alexander Schrijver. *"Geometric algorithms and combinatorial optimization"*. Second Edition. Vol. 2. Algorithms and Combinatorics. Berlin: Springer-Verlag, 1993.

[Gou97] Fernando Q. Gouvêa. *"p-adic numbers"*. Second Edition. Universitext. An introduction. Berlin: Springer-Verlag, 1997.

[GR12] Éric Gaudron and Gaël Rémond. *"Lemmes de Siegel d'évitement"*. In: *Acta Arith.* 154.2 (2012), pp. 125–136.

[GR13] Éric Gaudron and Gaël Rémond. *"Polarisations et isogénies"*. preprint. 2013.

[Gru07] Peter M. Gruber. *"Convex and discrete geometry"*. Vol. 336. Grundlehren der Mathematischen Wissenschaften. Berlin: Springer, 2007.

[GS09] Henri Gillet and Christophe Soulé. *"Erratum for: 'On the number of lattice points in convex symmetric bodies and their duals'"*. In: *Israel J. Math.* 171 (2009), pp. 443–444.

[GS91] Henri Gillet and Christophe Soulé. *"On the number of lattice points in convex symmetric bodies and their duals"*. In: *Israel J. Math.* 74.2-3 (1991), pp. 347–357.

[GW93] Peter Gritzmann and Jörg M. Wills. *"Lattice points"*. In: *Handbook of convex geometry, Vol. A, B.* Amsterdam: North-Holland, 1993, pp. 765–797.

[Hen02] Martin Henk. *"Successive minima and lattice points"*. In: *Rend. Circ. Mat. Palermo (2) Suppl.* 70, part I (2002). IV International Conference in "Stochastic Geometry, Convex Bodies, Empirical Measures & Applications to Engineering Science", Vol. I (Tropea, 2001), pp. 377–384.

[Hen12] Martin Henk. *"Löwner-John ellipsoids"*. In: *Doc. Math.* Extra volume: Optimization stories (2012), pp. 95–106.

[Hen13] Matthias Henze. *"A Blichfeldt-type inequality for centrally symmetric convex bodies"*. In: *Monatsh. Math.* 170.3-4 (2013), pp. 371–379.

[HT13] Martin Henk and Carsten Thiel. *"Restricted successive minima"*. In: *Pacific J. Math.* (2013). to appear.

[KL88] Ravi Kannan and László Lovász. *"Covering minima and lattice-point-free convex bodies"*. In: *Ann. of Math. (2)* 128.3 (1988), pp. 577–602.

[Kna06] Anthony W. Knapp. *"Basic algebra"*. Cornerstones. Boston, MA: Birkhäuser Boston Inc., 2006.

[Kna07] Anthony W. Knapp. *"Advanced algebra"*. Cornerstones. Boston, MA: Birkhäuser Boston Inc., 2007.

[Kup08] Greg Kuperberg. *"From the Mahler conjecture to Gauss linking integrals"*. In: *Geom. Funct. Anal.* 18.3 (2008), pp. 870–892.

[Lin11] Eva Linke. *"Rational Ehrhart quasi-polynomials"*. In: *J. Combin. Theory Ser. A* 118.7 (2011), pp. 1966–1978.

[Mah39] Kurt Mahler. *"Ein Übertragungsprinzip für konvexe Körper"*. In: *Časopis Pěst. Mat. Fys.* 68 (1939), pp. 93–102.

[Mal10] Romanos Malikiosis. *"An optimization problem related to Minkowski's successive minima"*. In: *Discrete Comput. Geom.* 43.4 (2010), pp. 784–797.

[Mal12a] Romanos Malikiosis. *"A Discrete Analogue for Minkowski's Second Theorem on Successive Minima"*. In: *Adv. Geom.* 12.2 (2012), pp. 365–380.

[Mal12b] Romanos Malikiosis. *"Lattice-point enumerators of ellipsoids,"* preprint. 2012. arXiv:1202.3876v1.

[McF71] R. B. McFeat. *"Geometry of numbers in adele spaces"*. In: *Dissertationes Math. Rozprawy Mat.* 88 (1971), pp. 1–49.

[McM79] Peter McMullen. *"Lattice invariant valuations on rational polytopes"*. In: *Arch. Math. (Basel)* 31.5 (1978/79), pp. 509–516.

[Neu92] Jürgen Neukirch. *"Algebraische Zahlentheorie"*. Berlin etc.: Springer-Verlag, 1992.

[Nor49] Douglas G. Northcott. *"An inequality in the theory of arithmetic on algebraic varieties"*. In: *Proc. Cambridge Philos. Soc.* 45 (1949), pp. 502–509.

[OV93] Robbin O'Leary and Jeffrey D. Vaaler. *"Small solutions to inhomogeneous linear equations over number fields"*. In: *Trans. Amer. Math. Soc.* 336.2 (1993), pp. 915–931.

[Rot10] Mark Peter Rothlisberger. *"An analogue of the Korkin-Zolotarev lattice reduction for vector spaces over number fields"*. PhD thesis. The University of Texas at Austin, 2010.

[RS57] C. Ambrose Rogers and Geoffrey C. Shephard. *"The difference body of a convex body"*. In: *Arch. Math. (Basel)* 8 (1957), pp. 220–233.

[RT96] Damien Roy and Jeffrey Lin Thunder. *"An absolute Siegel's lemma"*. In: *J. Reine Angew. Math.* 476 (1996), pp. 1–26.

[RT99] Damien Roy and Jeffrey Lin Thunder. *"Addendum and erratum to: 'An absolute Siegel's lemma'"*. In: *J. Reine Angew. Math.* 508 (1999), pp. 47–51.

[San49] Luis A. Santaló. *"An affine invariant for convex bodies of n-dimensional space"*. In: *Portugaliae Math.* 8 (1949), pp. 155–161.

[Sie29] Carl L. Siegel. *"Über einige Anwendungen diophantischer Approximationen."* Abhandlungen Akad. Berlin, Nr. 1, 70 p. 1929.

[Thi12] Carsten Thiel. *"Adelic geometry and polarity"*. In: *J. Number Theory* 132.8 (2012), pp. 1720–1730.

[Thu02] Jeffrey Lin Thunder. *"Remarks on adelic geometry of numbers"*. In: *Number theory for the millennium, III (Urbana, IL, 2000)*. Natick, MA, 2002, pp. 253–259.

[Thu09] Axel Thue. *"Über Annäherungswerte algebraischer Zahlen"*. In: *J. Reine Angew. Math.* 135 (1909), pp. 284–305.

[Vaa87] Jeffrey D. Vaaler. *"Small zeros of quadratic forms over number fields"*. In: *Trans. Amer. Math. Soc.* 302.1 (1987), pp. 281–296.

[Wei95] André Weil. *"Basic Number Theory"*. Classics in Mathematics. Reprint of the second edition. Berlin: Springer-Verlag, 1995.

Index

N

P

S

T

V

Notation

C	convex body in $\mathbb{R}^m$ or $K_{\mathbb{A}}^n$
$C^\star$	polar body of C
Λ	lattice in $\mathbb{R}^m$
$\Lambda^\star$	polar lattice of Λ
K	algebraic number field
$\mathscr{O}$	ring of integers of K
${}^\star\mathscr{O}$	codifferent
Δ_K	discriminant of K
$\mathrm{N}_{K/\mathbb{Q}}$	field norm of $K/\mathbb{Q}$
$\mathrm{Tr}_{K/\mathbb{Q}}$	field trace of $K/\mathbb{Q}$
$K_{\mathbb{A}}$	the adele ring of K
m	dimension of the Euclidean space
n	rank of the adele-module
d	degree of the field extension $K/\mathbb{Q}$
r_1, r_2	numbers of real and complex places of K
$\mathscr{K}_0^m$	set of m-dimensional origin-symmetric convex bodies in $\mathbb{R}^m$
$\mathscr{K}^m$	set of m-dimensional convex bodies in $\mathbb{R}^m$
$\mathscr{L}^m$	set of lattices of rank at most m
conv	convex hull
lin	linear hull
B_m	m-dimensional Euclidean unit ball
κ_m	volume of B_m
$\lambda_i(\cdot)$	i-th successive minimum
$\mu(\cdot)$	inhomogeneous minimum
$\mu_i(\cdot)$	i-th covering minimum
$\nu(K)$	adelic field constant, cf. (1.23)

$v \mid \infty$	archimedean place
$v \nmid \infty$	non-archimedean place
$M(K)$	set of all places of K
$\lvert\cdot\rvert_v$	absolute value at place v
K_v	completion of K with respect to $\lvert\cdot\rvert_v$
$\Im$	Imaginary part of a complex number
$\Re$	Real part of a complex number
C_∞	infinite part of the adelic convex body C
$\mathfrak{M}$	$\mathscr{O}$-submodule of K^n induced by finite part of adelic convex body C