Extended Formulations for Combinatorial Polytopes

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Abstract

Typically polytopes arising from real world problems have a lot of facets. In some cases even no linear descriptions for them are known. On the other hand many of these polytopes can be described much nicer and with less facets using extended formulations, i.e. as a projection of simpler higher dimensional polytopes. The presented work studies extended formulations for polytopes: the possibilities to construct extended formulations and limitations of them.

In the first part, some known techniques for constructions of extended formulations are reviewed and the new framework of polyhedral relations (see Kaibel and Pashkovich [2011]) is presented. We in particular elaborate on the special case of reflection relations. Reflection relations provide extended formulations for several polytopes that can be constructed by iteratively taking convex hulls of polytopes and their refelections at hyperplanes. Using this framework we are able to derive small extended formulations for the G-permutahedra of all finite reflection groups G.

The second part deals with extended formulations which use special structures of graphs involved in combinatorial problems. Here we present some known extended formulations and apply a few changes to the extended formulation of Gerards for the perfect matching polytope in graphs with small genus in order to reduce its size. Furthermore a new compact proof of an extended formulation of Rivin for the subtour elimination polytope is provided.

The third part (partly based on joint work with Volker Kaibel, Samuel Fiorini and Dirk Oliver Theis, see Fiorini, Kaibel, Pashkovich, and Theis [2011a]) involves general questions on the extended formulations of polytopes. The primal interest here are lower bounds for extended formulations. We study different techniques to obtain these lower bounds, all of which could be derived from so called non-negative factorizations of the slack matrix of the initial polytope. The minimal such factorization provides the minimal number of inequalities needed in an extended formulation. We compare different techniques, find their limitations and provide examples of the polytopes for which they give tight lower bounds on the complexity of extensions.

The fourth part studies the impact of symmetry on the sizes of extended formulations. In joint work with Volker Kaibel and Dirk Oliver Theis we showed that for certain constrained cardinality matching and cycle polytopes there exist no polynomial symmetric extended formulations, but there are polynomial non-symmetric ones (for further details see Kaibel, Pashkovich, and Theis [2010]). Beyond these results the thesis also contains a proof showing that the well known symmetric extended formulation for the permutahedron via the Birkhoff polytope is the best (up to a constant factor) one among symmetric extended formulations (see Pashkovich [2009]).

Zusammenfassung

Viele kombinatorische Polytope, die ihre Anwendung in praktischen Problemen finden, haben eine große Anzahl von Facetten. In manchen Beispielen ist nicht einmal eine lineare Beschreibung dieser Polytope bekannt. Andererseits lassen viele Polytope eine kompakte und schönere Darstellung mit Hilfe von erweiterten Formulierungen (d.h. als Projektion einfacherer höher-dimensionaler Polytope) zu. Die vorgelegte Arbeit untersucht Erweiterungen von Polytopen: Möglichkeiten, eine kompakte Erweiterung zu finden, und Einschränkungen dieses Ansatzes.

Im ersten Teil, werden einige bekannte Konstruktionen für Erweiterungen dargestellt und das neue Framework der Polyedrischen Relationen eingeführt, welches Teil einer gemeinsamen Arbeit mit Volker Kaibel ist (siehe Kaibel and Pashkovich [2011]). Insbesondere arbeiten wir den Fall von Spiegelungsrelationen aus. Spiegelungsrelationen liefern erweiterte Formulierungen für Polytope, die durch iterierte Bildung konvexer Hüllen von Polytopen und ihrer Spiegelungen an Hyperebenen konstruiert werden können. Mit Hilfe dieses Frameworks können wir kompakte Erweiterungen von G-Permutaedern für alle endliche Spiegelungsgruppen G konstruieren.

Der zweite Teil beschäftigt sich mit Erweiterungen, welche spezielle Eigenschaften von den Graphen ausnutzen, die in kombinatorischen Problemen auftauchen. Hier präsentieren wir einige bekannte Erweiterungen, und nehmen kleine Änderungen in der erweiterten Formulierung von Gerards für Perfekte Matching Polytope in Graphen mit kleinem Geschlecht vor, um die Grösse der Erweiterung zu reduzieren. Dazu präsentieren wir einen einfachen Beweis für die Erweiterung von Rivin des Subtour Elimination Polytops.

Der dritte Teil untersucht prinzipielle Grenzen des Konzepts der erweiterten Formulierungen. Das Hauptziel des dritten Teils ist es, untere Schranken für die Größe erweiterter Formulierung herzustellen. Hier stellen wir einige gemeinsame Ergebnisse mit Volker Kaibel, Samuel Fiorini und Dirk Oliver Theis dar (siehe Fiorini, Kaibel, Pashkovich, and Theis [2011a]). Wir vergleichen verschiedene Methoden, um untere Schranken zu bekommen, und schauen uns verschiedene Beispiele von Polytopen an (für manche Polytope sind die Schranken optimal).

Im vierten Teil präsentieren wir eine weitere gemeinsame Arbeit mit Volker Kaibel und Dirk Oliver Theis, die sich mit Symmetrien in Erweiterungen beschäftigt (sehe Kaibel, Pashkovich, and Theis [2010]). Hier haben wir Matching und Cycle Polytope gefunden, welche keine symmetrischen Erweiterungen haben, aber sich trotzdem mit Hilfe von Erweiterungen kompakt darstellen lassen. Über gemeinsame Arbeit hinaus, beweisen wir Ergebnisse bezüglich Erweiterungen von quadratischer Grösse, welche zum Beispiel zeigen, dass das Birkhoff Polytop eine asymptotisch minimale symmetrische Erweiterung des Permutaeders ist (sehe Pashkovich [2009]).

Contents

CHAPTER 1

Introduction

Combinatorial optimization problems in many cases can be transformed into linear optimization problems, where one identifies with every solution of the given combinatorial problem a point, and where the objective function can be understood as a linear function over the constructed points. Via such a transformation one obtains access to the complete machinery of linear programming, since optimization of a linear function over a set of points is equivalent to optimization of the linear function over the convex hull of these points.

However, algorithms for linear programming require a linear description of the problem, what can cause difficulties, since the polytopes associated with combinatorial problems usually do not admit a *compact linear description*, i.e. a linear description of polynomial size with respect to the size of the combinatorial problem.

Even though combinatorial polytopes may not possess a compact linear description, they may allow a *compact extended formulation*, i.e. such a polytope may be represented as a linear projection of a higher-dimensional polytope of polynomial size. And an optimization problem over the initial polytope can be transformed into an optimization problem over the extension.

Indeed, a lot of combinatorial polytopes do admit compact extended formulations, where for an extensive overview on extended formulations for combinatorial polytopes we recommend Conforti et al. [2010]. The power of this phenomenon relies on introducing additional variables, reflecting characteristics of combinatorial objects, such that these characteristics were "out of reach" for the linear programming using initial variables only.

Since linear programming is solvable in polynomial time, there were a lot of attempts to approach the famous complexity theoretical conjecture that $\mathcal{NP}$ is not equal $\mathcal{P}$, providing an easy to construct compact extended formulation for the travelling salesman polytope. Inspired by the request to review several of such papers with claimed compact extended formulations for the travelling salesman polytope Yannakakis tried to get an understanding of what can be achieved using extensions. In a seminal paper (see Yannakakis [1991]) he then showed that there is at least no symmetric extended formulation of polynomial size for the perfect matching polytope, where "symmetric" means that the formulation is invariant under permuting the nodes of the complete graph. As a corollary in this paper it was shown that there is no compact symmetric extended formulation for the travelling salesman polytope. This ruled out a lot of these constructions, since the majority of the proposed extended formulations for the travelling salesman polytope had been symmetric or were easy to symmetrise, retaining the polynomial size.

Yannakakis also conjectured that the symmetry requirement would just be a technical condition for the proof: "We do not think that asymmetry helps much. Thus, prove that the matching and TSP polytopes cannot be expressed by polynomial size LP's without the asymmetry assumption." Indeed, it turned out recently (see Fiorini et al. [2011b]) that there is no compact extended formulation for the travelling salesman polytope. However, until now it is unclear whether there is a compact asymmetric extension for the perfect matching polytope.

One part of this thesis studies the impact of symmetry requirements on the size of an extended formulation for matching and cycles polytopes. We disprove the conjecture of

Yannakakis in general case in Chapter 5, where we show that for some cardinality restricted matching and cardinality restricted cycle polytopes there exist no polynomial symmetric extended formulations, but there are polynomial non-symmetric ones. The results presented in that chapter have been published in Kaibel, Pashkovich, and Theis [2010].

Furthermore, we study the role of symmetry requirements for extensions of the cardinality indicating polytope and the permutahedron. For these polytopes we prove in Chapter 5 that the well-known symmetric extensions of them are asymptotically the best extensions, which one can get preserving the symmetry of extensions (see Pashkovich [2009]).

Actually, the interest for symmetric extended formulations of the permutahedron arose, since Goemans gave an elegant formulation of size $O(n \log n)$ for the permutahedron, where the best known symmetric extension was of size $O(n^2)$ via the Birkhoff polytope. In his construction, Goemans used a novel approach, which we generalize in Chapter 2 to the framework of reflection relations in order to produce extended formulations. Using this framework we obtain well-known extended formulations. Besides that we give a compact extended formulation for the Huffman polytope, for which no linear description up to now is known, as well as for G-permutahedron of finite reflection groups G. The results presented in that chapter have been published in Kaibel and Pashkovich [2011].

Goemans also showed that the size of his extended formulation is asymptotically minimal among all extended formulations for the permutahedron. His way to estimate the minimum size of an extended formulation for a polytope proves that the extended formulation for the Huffman polytope, which we construct in Chapter 2 is asymptotically minimal as well. This motivated us to systematize methods to estimate the minimum size of general extended formulations. In Chapter 4 we study the limitations of these approaches and provide several examples of their usage (see Fiorini, Kaibel, Pashkovich, and Theis [2011a]).

We also found an extended formulation for the spanning tree polytope for planar graphs, where the size of the extension is linear in the number of edges in the graph. However, it turned out that the extension was already provided by Williams [2002]. We nevertheless describe the construction in Chapter 3, and present a modified extension of Gerards [1991] for the perfect matching polytope, which is compact for graphs with sufficiently small genus, where our modifications were made in order to reduce the size of the extension.

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1.1. Preliminaries

Here, we introduce some definitions and notions used in the presented work.

1.1.1. Polytopes. A *polytope* $P \subseteq \mathbb{R}^m$ is defined as the convex hull of a finite set of points $X \subseteq \mathbb{R}^m$, i.e.

$$P = \operatorname{conv}(X) = \{\sum_{x \in X} \lambda_x x : \sum_{x \in X} \lambda_x = 1,\, \lambda \geq 0\}\,.$$

In turn, a *polyhedron* $P \subseteq \mathbb{R}^m$ is the Minkowski sum of the convex hull of a finite set of points $X \subseteq \mathbb{R}^m$ and the convex cone of a finite set of vectors $R \subseteq \mathbb{R}^m$, i.e.

$$P = \operatorname{conv}(X) + \operatorname{cone}(R)\,,$$

where the cone

$$\operatorname{rec}(P) = \operatorname{cone}(R) = \{\sum_{r \in R} \lambda_r r : \lambda \geq 0\}$$

is called the *recession cone* of the polyhedron P, and

$$\operatorname{lineal}(P) = -\operatorname{cone}(R) \cap \operatorname{cone}(R)$$

is called the *lineality space* of the polyhedron P.

A *face* $F \subseteq \mathbb{R}^m$ of a polyhedron $P \subseteq \mathbb{R}^m$ is defined as the intersection

$$F = H \cap P\,,$$

where H is a hyperplane, such that the polyhedron P lies in one of the closed halfspaces, defined by the hyperplane H. Additionally, the empty set $\varnothing$ and the polyhedron P are understood as faces of the polyhedron $P \subseteq \mathbb{R}^m$ as well. A *facet* and a *vertex* of a polyhedron $P \subseteq \mathbb{R}^m$ is a face of dimension $\dim(P) - 1$ and zero, respectively. The set of all faces ordered by inclusion forms the *face lattice* $\mathcal{L}(P)$ of the polyhedron P.

The Weyl-Minkowski Theorem states that every polyhedron $P \subseteq \mathbb{R}^m$ can be described as the solution set for a linear system, i.e.

$$P = \{x \in \mathbb{R}^m : A^{\leq}\, x \leq b^{\leq},\, A^{=}\, x \leq b^{=}\}\,,$$

where $A^{\leq} \in \mathbb{R}^{f \times m}$, $b^{\leq} \in \mathbb{R}^f$, $A^{=} \in \mathbb{R}^{r \times m}$, $b^{=} \in \mathbb{R}^r$. The minimum number f of inequalities, such that there exists a corresponding linear system, is equal to the number of facets of the polyhedron P. A linear description with the minimum number of inequalities is called a *minimal linear description* of the polyhedron P. Since every polytope is a polyhedron, it is easy to see that every polytope is the solution set for a system of linear inequalities, where the solution set is bounded. In turn, every linear system for which the set of solutions is bounded defines a polytope.

The Farkas Lemma has diverse equivalent formulations, and here we present the one below.

Lemma 1.1. *For a polyhedron $P \subseteq \mathbb{R}^m$, defined by the linear system*

$$P = \{x \in \mathbb{R}^m : A\, x \leq b\}\,,$$

where $A \in \mathbb{R}^{f \times m}$, $b \in \mathbb{R}^f$, and $a \in \mathbb{R}^m$, $\beta \in \mathbb{R}$, the inequality $\langle a, x\rangle \leq \beta$ is valid for P if and only if P is empty or there exists a non-negative vector $c \in \mathbb{R}^f$, such that

$$c\, A = a \quad \textit{and} \quad c\, b \leq \beta\,.$$

For all notions and results from polyhedral theory, mentioned in the presented work, we refer to Ziegler [1995] and Grünbaum [2003].

1.1.2. Extended Formulations, Extensions. An *extension* of a polytope $P \subseteq \mathbb{R}^m$ is a polyhedron $Q \subseteq \mathbb{R}^d$ together with an affine map $p : \mathbb{R}^d \to \mathbb{R}^m$ satisfying

$$p(Q) = P\,.$$

A description of Q by linear equations and inequalities (together with p) is called an *extended formulation* of P [1].

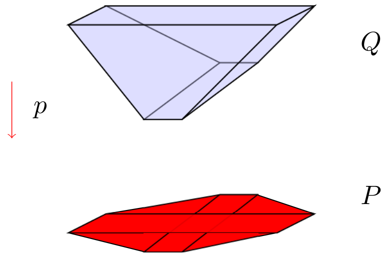

FIGURE 1. Example of an extension.

The *size* of an extension is the number of its facets. The *size* of an extended formulation is its number of inequalities, not including equations. Clearly, the size of an extended formulation is at least as large as the size of the extension it describes. Conversely, every extension is described by an extended formulation of at most the same size [2]. In this work, the notion of size does not involve the encoding length of the coefficients in extended formulation. Thus, all lower bounds, obtained in Chapters 4 and 5 provide lower bounds on the number of inequalities in extended formulations. Nevertheless, all extended formulations constructed in this work involve coefficients of polynomial size only (except for an extended formulation for regular polygons).

In this work, we are interested in *minimal extended formulations* and *extensions* for polytopes. In fact, we can assume that a minimal extension of a non-empty polytope P is given by a full-dimensional polytope $Q \subseteq \mathbb{R}^d$ and an affine map $p : \mathbb{R}^d \to \mathbb{R}^m$. Indeed, for every vector $r \in \mathbb{R}^d$ from the recession cone of the polyhedron Q and every point $z \in \mathbb{R}^d$, we have

$$p(z + r) = p(z)\,,$$

because Q, p form an extension of the bounded polyhedron P. Thus, if we consider a polyhedron Q^* defined as $Q - \operatorname{rec}(Q)$, and the affine map $p^* = p : \mathbb{R}^d \to \mathbb{R}^m$, we have

$$p^*(Q^*) = p(Q - \operatorname{rec}(Q)) = p(Q) = P\,,$$

what shows that Q^*, p^* form an extension of the polytope P. The recession cone of the polyhedron Q^* is equal to $\operatorname{rec}(Q) - \operatorname{rec}(Q)$, i.e. the recession cone of the polyhedron Q^* coincides with its lineality space. The size of the extension Q^*, p^* is bounded from above by the size of the extension Q, p for the polytope P (Appendix: Lemma 6.3). Now, let us consider the polyhedron Q^{**} equal to $Q^* \cap \operatorname{rec}(Q^*)^{\perp}$ and the affine map $p^{**} = p : \mathbb{R}^d \to \mathbb{R}^m$, which form an extension of the polytope P

$$p^{**}(Q^{**}) = p^*(Q^* \cap \operatorname{rec}(Q^*)^{\perp}) = p^*(\operatorname{proj}_{\operatorname{rec}(Q^*)^{\perp}}(Q^*)) = p(Q) = P\,,$$

[1] Analogously, an extension and extended formulation for a polyhedron can be defined. Even if this is not an object of the current work, this may be useful in the case when an extended formulation is constructed via some polyhedron, which has a compact extended formulation.

[2] For symmetric extensions and symmetric extended formulations, defined in Chapter 5, the same equivalence is shown, i.e. it is shown that for every symmetric extension there exists a symmetric extended formulation of the same size, and every symmetric extended formulation defines a symmetric extension of the same size.

whose size is less than or equal to the size of the extension Q^*, p^*. The recession cone of the polyhedron Q^* is equal to $\mathrm{proj}_{\mathrm{rec}(Q^*)^\perp}(\mathrm{rec}(Q^*))$, i.e. is equal to the zero vector, and thus the polyhedron Q^{**} is a polytope.

Finally, if the polytope $Q^{**} \subseteq \mathbb{R}^d$ is not full-dimensional, then we consider an extension given by the full-dimensional polytope $Q' = q(Q^{**}) \subseteq \mathbb{R}^{d'}$ and an affine map $p' = p^{**} \circ q^{-1} : \mathbb{R}^{d'} \to \mathbb{R}^m$, where the map $q : \mathrm{aff}(Q^{**}) \to \mathbb{R}^{d'}$ is an affine embedding of the affine hull of Q^{**} into the space $\mathbb{R}^{d'}$, with $d' = \dim(Q^{**})$.

For an extension $Q \subseteq \mathbb{R}^d$, $p : \mathbb{R}^d \to \mathbb{R}^m$ of a polytope $P \subseteq \mathbb{R}^m$, we define a *section map* $s : \mathrm{vert}(P) \to Q$, such that for every vertex x of the polytope P

$$p(s(x)) = x\,,$$

where $\mathrm{vert}(P)$ denotes the vertex set of P.

Note that if the section $s : \mathbb{R}^m \to \mathbb{R}^d$ is an affine map then the size of the extension via Q is at least as big as the size of the trivial extension via the polytope P itself. Indeed, the dimension of the polyhedron $Q \cap \mathrm{aff}(s(\mathrm{vert}(P)))$ is less than or equal to the dimension of the polytope P, if $s : \mathbb{R}^m \to \mathbb{R}^d$ is an affine map. On the other hand the polyhedron $Q \cap \mathrm{aff}(s(\mathrm{vert}(P)))$ with the affine map $p : \mathbb{R}^d \to \mathbb{R}^m$ is an extension of the polytope $P \subseteq \mathbb{R}^m$. Thus the polyhedron $Q \cap \mathrm{aff}(s(\mathrm{vert}(P)))$ is isomorphic to the polytope P, what shows that the number of facets of the polytope P is equal to the number of facets of the polytope $Q \cap \mathrm{aff}(s(\mathrm{vert}(P)))$ which is at most the size of the extension Q, p.

Of course, having an extension $Q \subseteq \mathbb{R}^d$, $p : \mathbb{R}^d \to \mathbb{R}^m$ of a polytope $P \subseteq \mathbb{R}^m$, the polyhedron

$$Q \cap \{z \in \mathbb{R}^d : \langle a, p(z)\rangle = b\}$$

together with $p : \mathbb{R}^d \to \mathbb{R}^m$ is an extension of the polytope

$$P \cap \{x \in \mathbb{R}^m : \langle a, x\rangle = b\}$$

of at most the same size. Thus, every extension of a polytope provides an extension of any of its faces, where the last extension has at most the same size.

1.1.3. Combinatorial Polytopes. Here, we define three combinatorial polytopes with a central role in the theory of extended formulations.

1.1.3.1. *Spanning Tree Polytope.* A *tree* in the graph $G = (V, E)$ is a connected subgraph of G, which does not contain any cycle. The set of trees in a graph G is defined by $\mathcal{T}(G)$, or $\mathcal{T}(n)$ if we deal with the complete graph K_n.

The *spanning tree polytope* $\mathrm{P}_{\mathrm{spt}}(G)$ for a graph G is defined as follows

$$\mathrm{P}_{\mathrm{spt}}(G) = \mathrm{conv}(\{\chi(T) \in \mathbb{R}^E : T \in \mathcal{T}(G)\})\,.$$

The following linear system describes the spanning tree polytope for a graph $G = (V, E)$ (see Edmonds [1971])

$$x(E(S)) \le |S| - 1 \quad \text{for} \quad \varnothing \neq S \subsetneq V$$
$$x(E) = |V| - 1 \quad \text{and} \quad 0 \le x\,.$$

Here and later, for every $x \in \mathbb{R}^d$ and $I \subseteq [d]$ the expression $x(I)$ denotes the sum $\sum_{i \in I} x_i$.

1.1.3.2. *Matching Polytope.* A *matching* in a graph $G = (V, E)$ is a set of disjoint edges in the graph G. The set of all matchings in the graph G is denoted by $\mathcal{M}(G)$, and by $\mathcal{M}(n)$, if we deal with the complete graph K_n. The set of all matchings with ℓ edges in the graph G is denoted by $\mathcal{M}^\ell(G)$, and by $\mathcal{M}^\ell(n)$, if G is the complete graph with n vertices. A *perfect matching* is a matching, which covers all vertices of the graph G.

The *perfect matching polytope* $\mathrm{P}^{\frac{n}{2}}_{\mathrm{match}}(G)$ for a graph G with n vertices (n is even) is the convex hull of characteristic vectors for all perfect matchings in G, i.e.

$$\mathrm{P}^{\frac{n}{2}}_{\mathrm{match}}(G) = \mathrm{conv}(\{\chi(M) \in \mathbb{R}^E : M \in \mathcal{M}^n(G)\})\,.$$

Edmonds [1965] gave a linear description of the perfect matching polytope by $x \geq 0$ and

$$\begin{aligned} x(\delta(S)) &\geq 1 && \text{for} \quad S \subseteq V,\ 1 \leq |S| \text{ is odd} \\ x(\delta(v)) &= 1 && \text{for} \quad v \in V\,. \end{aligned}$$

For the complete graph K_n, this system defines a minimal linear description of $\mathrm{P}^{\frac{n}{2}}_{\mathrm{match}}(n)$.

The *matching polytope* $\mathrm{P}_{\mathrm{match}}(G)$ for a graph G is the convex hull of characteristic vectors for all matchings in G, i.e.

$$\mathrm{P}_{\mathrm{match}}(G) = \mathrm{conv}(\{\chi(M) \in \mathbb{R}^E : M \in \mathcal{M}(G)\})\,.$$

Edmonds [1965] gave a linear description of the matching polytope by $x \geq 0$ and

$$\begin{aligned} x(E(S)) &\leq |S| - 1 && \text{for} \quad S \subseteq V,\ 1 \leq |S| \text{ is odd} \\ x(\delta(v)) &\leq 1 && \text{for} \quad v \in V\,. \end{aligned}$$

For the complete graph K_n, this system defines a minimal linear description of $\mathrm{P}_{\mathrm{match}}(n)$.

The *cardinality constrained matching polytope* $\mathrm{P}^{\ell}_{\mathrm{match}}(G)$ is the convex hull of all characteristic vectors for $\mathcal{M}^{\ell}(G)$, i.e.

$$\mathrm{P}^{\ell}_{\mathrm{match}}(G) = \mathrm{conv}(\{\chi(M) \in \mathbb{R}^E : M \in \mathcal{M}^{\ell}(G)\})\,.$$

Hence, for the cardinality equal to the halved number of vertices in G, the cardinality constrained matching polytope is the perfect matching polytope.

The cardinality constrained matching polytope $\mathrm{P}^{\ell}_{\mathrm{match}}(G)$ can be described as $x \geq 0$ and

$$\begin{aligned} x(E(S)) &\leq |S| - 1 && \text{for} \quad S \subseteq V,\ |S| \text{ is odd} \\ x(\delta(v)) &\leq 1 && \text{for} \quad v \in V \\ x(E) &= \ell\,, \end{aligned}$$

since the cardinalities of matchings, corresponding to any two adjacent vertices of the matching polytope, differ at most by one (see Schrijver [2003a]).

1.1.3.3. *Cycle Polytope.* Let $\mathcal{C}^{\ell}(G)$ denote the set of cycles in a graph $G = (V, E)$ of length ℓ.

The *cardinality constrained cycle polytope* $\mathrm{P}^{\ell}_{\mathrm{cycl}}(G)$ is defined as the convex hull of characteristic vectors of all cycles $\mathcal{C}^{\ell}(G)$, i.e.

$$\mathrm{P}^{\ell}_{\mathrm{cycl}}(G) = \mathrm{conv}(\{\chi(C) \in \mathbb{R}^E : C \in \mathcal{C}^{\ell}(G)\})\,.$$

If the cardinality of cycles is equal to the number of vertices of the whole graph and $G = K_n$, the cardinality constrained cycle polytope is the *travelling salesman polytope*. In contrast to the preceding two examples, we do not expect that there is a "reasonable" linear description of the travelling salesman polytope, as the associated optimization problem is $\mathcal{NP}$-hard.

For all notions and results from polyhedral combinatorics, mentioned in the presented work, we refer to Schrijver [2003a], Schrijver [2003b], Schrijver [2003c].

1.2. Extensions of Combinatorial Polytopes

The three mentioned types of combinatorial polytopes are important for our further considerations.

For the spanning tree polytope Martin [1991] constructed an extended formulation, defined by $z \geq 0$ and

$$\begin{aligned} x_{v,u} - z_{v,u,w} - z_{u,v,w} &= 0 && \text{for} \ \ v, u, w \in [n] \\ x_{v,u} + \sum_{w \in V \setminus \{v,u\}} z_{v,w,u} &= 1 && \text{for} \ \ v, u \in [n]\,, \end{aligned}$$

what shows that the spanning tree polytope $\mathrm{P}_{\mathrm{spt}}(n)$ admits an extension of size $O(n^3)$. The section of the mentioned extension can be defined as follows: $z_{v,u,w}$ is equal to one if the tree T contains the edge $\{v, u\}$ and the path from u to w in the tree T does not involve the vertex v, and $z_{v,u,w}$ is equal to zero, otherwise.

As mentioned above, the travelling salesman polytope does not admit a compact extended formulation, what was shown by Fiorini et al. [2011b].

For the perfect matching polytope and the matching polytope it is not known whether there exists an extended formulation of polynomial size. But Yannakakis [1991] showed that there exists no compact symmetric extension of these polytopes of polynomial size. Thus, it is still an open problem to construct a compact extension for the matching polytope or to show that no such extension exists.

CHAPTER 2

Balas Extensions, Flow Extensions and Polyhedral Relations

In this chapter, two central frameworks for the construction of extended formulations are presented: disjunctive and dynamic programming.

The ideas of disjunctive programming can be implemented in extended formulations via the Balas techniques (see Balas [1998]).The Balas method constructs an extended formulation for the convex hull of some set of polytopes, having at hands an extended formulation for each of them. Hence, this approach is effective for combinatorial polytopes, whenever one is able to partition the combinatorial objects, inducing the polytope, into tractable subclasses, i.e. for which small extended formulations are known.

In turn, the dynamic programming approach encodes mostly the way to optimize over the combinatorial objects, which induce the polytope. Flow polytopes play a crucial role in these extensions, since usually, the possible scenarios of the corresponding dynamic algorithm are encoded as a path in an acyclic network. Here, we also present some extended formulations constructed by Fiorini, Kaibel, Pashkovich, and Theis [2011a] and Kaibel, Pashkovich, and Theis [2010].

In the end of the chapter, we develop the polyhedral relations framework, and in particular, reflection relations (see Kaibel and Pashkovich [2011]). The reflection relations construct an extension for the convex hull of a polytope and its image under the reflection map, with respect to a hyperplane. Note that the Balas approach does not have any restrictions concerning the polytopes in the construction. But in comparison with reflection relations, disjunctive programming produces extensions of a bigger size, what results in the significant size of extension, constructed iteratively via the Balas method. With the help of reflection relations, we reproved a series of results concerning extended formulations of regular polygons (see Ben-Tal and Nemirovski [2001]), the permutahedron (see Goemans), the parity polytope (see Carr and Konjevod [2004]). Moreover, we obtained asymptotically minimal extensions for the cardinality indicating polytope and the Huffman polytope (currently, no linear description for the Huffman polytope is known).

2.1. Balas Extensions

One of the most important frameworks for the construction of extended formulations is *disjunctive programming* (see Balas [1998]). In this framework, an extended formulation of a polytope $P \subseteq \mathbb{R}^m$ is constructed, using already known extended formulations for a set of other non-empty polytopes $P_i \subseteq \mathbb{R}^m$, $i \in [k]$, such that

$$P = \operatorname{conv}\Big(\bigcup_{i\in[k]} P_i\Big)\,.$$

Theorem 2.1 (Balas [1998])**.** *If for each of the non-empty polytopes* $P_i \subseteq \mathbb{R}^m$*,* $i \in [k]$*, there exists an extended formulation, described by the linear system*

$$A^i z \le b^i\,, \tag{2.1.1}$$

where $A^i \in \mathbb{R}^{f_i \times d_i}$*,* $b^i \in \mathbb{R}^{f_i}$*, together with an affine map* $p^i : \mathbb{R}^{d_i} \to \mathbb{R}^m$*, such that*

$$p^i(z) = g^i(z) + \gamma^i\,,$$

where $g^i : \mathbb{R}^{d_i} \to \mathbb{R}^m$ is a linear map and $\gamma^i \in \mathbb{R}^m$, then the linear system

$$A^i y^i \le b^i \lambda_i \quad \textit{for } i \in [k]$$
$$\sum_{i=1}^k \lambda_i = 1 \quad \textit{and} \quad 0 \le \lambda \tag{2.1.2}$$
$$x = \sum_{i=1}^k g^i(y^i) + \sum_{i=1}^k \gamma^i \lambda_i\,,$$

together with the projection on x variables, forms an extended formulation of size at most $k + \sum_{i=1}^k f_i$ for the polytope

$$P = \operatorname{conv}(\bigcup_{i\in[k]} P_i)\,. \tag{2.1.3}$$

PROOF. It is necessary to prove that the polyhedron Q, defined by the linear system (2.1.2), together with the projection on x variables, forms an extension of the polytope P.

First, it is necessary to show that for every point $x \in \mathbb{R}^m$ from the polytope P, there are y and λ variables, which satisfy the linear system (2.1.2). Let the point x be written as the convex combination

$$x = \sum_{i=1}^k \lambda_i x^i$$
$$\sum_{i=1}^k \lambda_i = 1 \quad \text{and} \quad 0 \le \lambda\,,$$

where $x^i \in \mathbb{R}^m$ lies in the polytope P_i. For every point x^i, define $z^i \in \mathbb{R}^{d_i}$ to be a point, such that

$$A^i z^i \le b^i \quad \text{and} \quad p^i(z^i) = x^i\,.$$

To finish the construction, define the vector y^i to be equal to $\lambda_i z^i$ for all $i \in [k]$. Clearly, the defined values x, y and λ satisfy the linear system (2.1.2).

Second, assume that for some point $x \in \mathbb{R}^{d_i}$, there are λ and y variables, satisfying the constructed extended formulation. In the case $\lambda_i = 0$, the vector y^i lies in the recession cone of the corresponding extension for the polytope P_i. And thus, lies in the kernel of the linear map g^i (P_i is a non-empty polytope), i.e. the vector $g^i(y^i)$ is equal $\mathbf{0}_m$, whenever $\lambda_i = 0$. In the case $\lambda_i > 0$, the point

$$g^i(\lambda_i^{-1} y^i) + \gamma_i = p^i(\lambda_i^{-1} y^i)$$

belongs to the polytope P_i, due to $A^i(\lambda_i^{-1} y^i) \le b^i$. Consequently, the point x lies in the polytope P, because x satisfies

$$x = \sum_{i=1}^k g^i(y^i) + \sum_{i=1}^k \gamma_i \lambda_i = \sum_{\substack{i\in[k]\\ \lambda_i>0}} g^i(y^i) + \sum_{\substack{i\in[k]\\ \lambda_i>0}} \gamma_i \lambda_i = \sum_{\substack{i\in[k]\\ \lambda_i>0}} \lambda_i (g^i(\lambda_i^{-1} y^i) + \gamma_i)$$

and

$$\sum_{i=1}^k \lambda_i = 1 \quad \text{and} \quad 0 \le \lambda\,.$$

□

The *vertex extension* of a polytope $P \subseteq \mathbb{R}^m$ can be seen as a construction via the Balas method, where the set of polytopes P_i is the set of vertices of the polytope P. Thus, the size of the vertex extension for a polytope P is equal to the number of vertices of the polytope P.

Lemma 2.1. *For every polytope $P \subseteq \mathbb{R}^m$, there exists an extended formulation of size equal to the number of vertices of the polytope P.*

2.2. Dynamic Programming Extensions

Another important approach to construct extended formulations is *dynamic programming* (see Martin et al. [1990]). One of the possibilities to exploit dynamic programming is to solve an optimization problem as the shortest path problem in an acyclic network. Thus, the extended formulations, constructed via the dynamic programming method, usually are the path polytopes in some acyclic network.

Of course, for every polytope one is able to construct a network, such that the shortest path problem in the network is equivalent to the optimization problem over the polytope. For this, let the network to consist of the source s and sink t, and an arc for every vertex of the polytope with capacity one. However, such way to construct an extension, gives us the vertex extension of a polytope, what usually is not compact.

In this framework, a flow polyhedron for a network N plays a crucial role. Recall that a network $N = (V, A, c)$ is given by the set of nodes V, containing the source s and sink t, and by a set of arcs $A \subseteq V \times V$. Typically, the capacities $c \in \mathbb{R}^A$ of the arcs are assumed to be one if nothing else is stated, in this situation we omit the capacities in the definition of the network $N = (V, A)$.

The s-t *flow polyhedron* $\mathrm{P}^{\ell}_{s-t \text{ flow}}(N) \subseteq \mathbb{R}^A$, which is the set of all s-t flows in the network N of value ℓ. The flow polyhedron $\mathrm{P}^{\ell}_{s-t \text{ flow}}(N)$ is described as (see Schrijver [2003a])

$$
\begin{aligned}
&x(\delta^{out}(v)) = x(\delta^{in}(v)) \qquad \text{for} \quad v \in V \setminus \{s, t\}\\
&x(\delta^{out}(s)) = x(\delta^{in}(s)) + \ell\\
&0 \le x \le c\,.
\end{aligned}
$$

Clearly, the size of this linear formulation for the flow polyhedron $\mathrm{P}^{\ell}_{s-t \text{ flow}}(N)$ is equal to twice the number of arcs in the network N.

Moreover, whenever the network $N = (V, A)$ is acyclic and the flow value ℓ is equal to one, the flow polytope $\mathrm{P}^{\ell}_{s-t \text{ flow}}(N) \subseteq \mathbb{R}^A$ is equal to the convex hull of the characteristic vectors of all possible s-t paths in the network N.

Recently, Kaibel and Loos [2010] developed a powerful generalization of dynamic programming, so called polyhedral branching systems, which generalize the dynamic programming framework of Martin et al. [1990]. One of the most elegant applications of polyhedral branching systems is a compact extended formulation of full orbitopes, i.e. the convex hull of zero-one matrices with lexicographically ordered columns.

In this chapter, all dynamic extended formulations can be verified without a formal proof. Namely, we state an acyclic network and define the projection and section maps. It is left to show that all source-sink paths in the network are projected inside of the considered polytope, and that the section map defines a source sink-path in the provided network.

2.3. Flow Extensions

It is worth to mention that not all extended formulations, constructed via flow polyhedra, are considered to be dynamic programming extensions. Particularly, the extended formulation for the corner polyhedra of the perfect matching polytope, which was provided by Ventura and Eisenbrand [2003], and the extended formulation for the spanning tree polytope, constructed by Padberg and Wolsey [1983], Cunningham [1985].

For example, the polyhedron $P \subseteq \mathbb{R}^E$ for a graph $G = (V, E)$, described by $0 \le y$ and

$$y(\delta(S)) \ge \ell \quad \text{for all} \quad S \subseteq V,\ s \in S,\ t \notin S\,, \tag{2.3.1}$$

has an extension via the flow polyhedron $\mathrm{P}^{\ell}_{s-t\ \mathrm{flow}}(N)$, where the arcs capacities $c \in \mathbb{R}^A$ are treated as variables, and where the network $N = (V, A, c)$ has the set of arcs

$$A = \{(v, u) \in V \times V : \{v, u\} \in E\} .$$

and

$$c_{(v,u)} = y_{v,u} \quad \text{for all } v, u \in V .$$

Indeed, due to the Minimum Cut Maximum Flow Theorem, there exists an s-t flow of value ℓ in the network $N = (V, A, c)$ if and only if the point y belongs to the polyhedron P, what provides us with an extended formulation for the polyhedron P of size less than or equal to $4|E|$.

If the polyhedron $P \subseteq \mathbb{R}^E$ is described by $0 \leq y$ and

$$y(\delta(S)) \geq \ell \quad \text{for} \quad \varnothing \neq S \subsetneq V ,$$

then the polyhedron P is the intersection of the polyhedra, described by $0 \leq y$ and (2.3.1), where the vertex s is fixed and the vertex t ranges among the vertices $V \setminus \{s\}$.

For example, consider the *subtour elimination polytope* $\mathrm{P}_{\mathrm{ste}}(G) \subseteq \mathbb{R}^E$ for the graph $G = (V, E)$, defined as

$$\begin{aligned} &x(\delta(S)) \geq 2 && \text{for} \quad \varnothing \neq S \subsetneq V \\ &x(\delta(v)) = 2 && \text{for} \quad v \in V \\ &0 \leq x . \end{aligned}$$

From the discussion above, the following result can be obtained.

Proposition 2.1 (Yannakakis [1991]). *There is an extended formulation of size $O(|V||E|)$ for the subtour elimination polytope $\mathrm{P}_{\mathrm{ste}}(G)$, where $G = (V, E)$.*

2.4. Cardinality Indicating Polytope

The *cardinality indicating polytope* $\mathrm{P}^n_{\mathrm{card}} \subseteq \mathbb{R}^{2n+1}$ is defined as the convex hull of the points

$$\{(x, z) \in \{0, 1\}^n \times \{0, 1\}^{n+1} : z_k = 1 \text{ and } z_j = 0 \text{ if } j \neq k , \text{ where } k = \sum_{i=1}^{n} x_i + 1\} .$$

Obviously, these points define the set of vertices of the cardinality indicating polytope $\mathrm{P}^n_{\mathrm{card}}$. For every vertex of $\mathrm{P}^n_{\mathrm{card}}$, the first n coordinates represent the characteristic vector of a subset of the set $[n]$, while the last $n + 1$ coordinates encode the cardinality of this subset.

A minimal linear description of the cardinality indicating polytope $\mathrm{P}^n_{\mathrm{card}}$ was given by Köppe et al. [2008] and looks as follows

$$\begin{aligned} &\sum_{i \in S} x_i \leq \sum_{j=0}^{|S|} j z_{j+1} + |S| \sum_{j=|S|+1}^{n} z_{j+1} \quad \text{for } \varnothing \neq S \subsetneq [n] \\ &\sum_{i=1}^{n} x_i = \sum_{j=0}^{n} j z_{j+1} \\ &\sum_{j=0}^{n} z_{j+1} = 1 \\ &0 \leq x \quad \text{and} \quad 0 \leq z . \end{aligned}$$

Hence, the cardinality indicating polytope $\mathrm{P}^n_{\mathrm{card}}$ has exponentially many facets.

We construct an extended formulation for the cardinality indicating polytope $\mathrm{P}^n_{\mathrm{card}}$ of size $O(n^2)$. Namely, we apply the Balas techniques to the set of polytopes P_k, $k \in [n+1]$,

$$P_k = \{(x,z) \in \mathbb{R}^n \times \mathbb{R}^{n+1} : \sum_{i=1}^{n} x_i = k-1\,, 0 \le x \le 1\,, z_k = 1 \text{ and } z_j = 0 \text{ if } j \neq k\}\,.$$

The polytope P_k is integral, since the matrix of the linear system, defining the polytope P_k, is totally unimodular. Consequently, the polytope P_k is the convex hull of all $(2n+1)$-dimensional zero-one points, where the first n coordinates involve exactly $k-1$ ones and the last $n+1$ coordinates are zeros except for the k-th coordinate. Hence, the vertices of the polytopes P_k partition the vertices of the cardinality indicating polytope $\mathrm{P}^n_{\mathrm{card}}$ according to the sum of the first n coordinates, i.e.

$$\mathrm{vert}(\mathrm{P}^n_{\mathrm{card}}) = \bigcup_{k \in [n+1]} \mathrm{vert}(P_k)\,,$$

what shows that P is equal to the convex hull $\mathrm{conv}(\bigcup_{k\in[n+1]} P_k)$.

There is also an extended formulation of size $O(n^2)$, which is constructed using the dynamic programming approach for the network $N = (V, A)$, where

$$V = \{s\} \cup \{t\} \cup \{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 \le j \le i \le n\}$$

and

$$\begin{aligned} A = \{(s,(i,j)) \in V \times V : i = 1\} \cup \{((i,j),t) \in V \times V : i = n\} \cup \\ \{((i',j'),(i'',j'')) \in V \times V : j' \le j'' \le j'+1, i'' = i'+1\}\,. \end{aligned}$$

Considering the polytope $\mathrm{P}^1_{s-t\text{ flow}}(N) \subseteq \mathbb{R}^A$, associated with s-t paths in the network $N = (V, A)$, we get an extended formulation of the cardinality indicating polytope $\mathrm{P}^n_{\mathrm{card}}$, where the projection is given by the affine map $p : \mathbb{R}^A \to \mathbb{R}^n \times \mathbb{R}^{n+1}$

$$p_i(y) = \begin{cases} y_{(s,(1,1))} & \text{if } i = 1 \\ \sum_{j=1}^{i} y_{((i-1,j-1),(i,j))} & \text{if } 2 \le i \le n \\ y_{((n,i-n-1),t)} & \text{if } n+1 \le i \le 2n+1\,. \end{cases}$$

This network imitates the process of scanning the vector from the first till the last coordinate, saving the number of scanned ones and the current position.

Define the corresponding section $s : \mathrm{vert}(\mathrm{P}^n_{\mathrm{card}}) \to P$

$$s_a(x,z) = \begin{cases} x_1 = 0 & \text{if } a = (s,(1,0)) \\ x_1 = 1 & \text{if } a = (s,(1,1)) \\ x_{i+1} = 0 \wedge \sum_{t=1}^{i} x_t = j & \text{if } a = ((i,j),(i+1,j)) \\ x_{i+1} = 1 \wedge \sum_{t=1}^{i} x_t = j & \text{if } a = ((i,j),(i+1,j+1)) \\ \sum_{i=1}^{n} x_i = j & \text{if } a = ((n,j),t)\,. \end{cases}$$

The expressions in the section map are understood as logic formulas, which evaluate to one if the formula is satisfied, and to zero, otherwise.

Thus, both approaches lead to extended formulations of size $O(n^2)$ for the cardinality indicating polytope $\mathrm{P}^n_{\mathrm{card}}$.

Proposition 2.2. *For the cardinality indicating polytope* $\mathrm{P}^n_{\mathrm{card}}$*, there exists an extended formulation of size* $O(n^2)$.

2.5. Parity Polytope

The *parity polytope* $\mathrm{P}^n_{\mathrm{even}} \subseteq \mathbb{R}^n$ is defined as the convex hull of all n-dimensional zero-one vectors, which have an even number of coordinates equal to one. Analogously, the parity polytope $\mathrm{P}^n_{\mathrm{odd}} \subseteq \mathbb{R}^n$ is defined, with the vertices involving odd number of ones. Whenever nothing else is not stated, speaking about the parity polytope, we refer to $\mathrm{P}^n_{\mathrm{even}}$.

Jeroslow [1975] provided a minimal description of the parity polytope $\mathrm{P}^n_{\mathrm{even}} \subseteq \mathbb{R}^n$, which is given by $0 \le x \le 1$ and

$$\sum_{i\in S} x_i - \sum_{i\in[n]\setminus S} x_i \le |S| - 1 \qquad \text{for} \quad S \subseteq [n]\,, |S| \text{ is odd.}$$

Thus, every linear description of the parity polytope in the initial space involves $\Omega(2^n)$ inequalities.

Obviously, the face of the cardinality indicating polytope

$$\mathrm{P}^n_{\mathrm{card}} \cap \{(x,z) \in \mathbb{R}^n \times \mathbb{R}^{n+1} : z_{j+1} = 0, j \in [n]_{\mathrm{odd}}\}\,,$$

together with the projection on x variables, provides an extended formulation of the parity polytope $\mathrm{P}^n_{\mathrm{even}}$. Hence, there exists an extended formulation for the parity polytope $\mathrm{P}^n_{\mathrm{even}}$ of size $O(n^2)$, due to Proposition 2.2.

In addition, there is an extended formulation for the parity polytope $\mathrm{P}^n_{\mathrm{even}}$, constructed via the Balas techniques and the polytopes $P_k \subseteq \mathbb{R}^n$, $k \in [n+1]_{\mathrm{odd}}$, defined as

$$P_k = \{x \in \mathbb{R}^n : \sum_{i\in[n]} x_i = k-1,\ 0 \le x \le 1\}\,.$$

Because the linear system above is totally unimodular, the polytope P_k is the convex hull of n-dimensional zero-one vectors with k ones. Every polytope P_k has a linear description of size $O(n)$, what results in an extended formulation of size $O(n^2)$, constructed by Yannakakis [1991].

Carr and Konjevod [2004] provided a smaller extended formulation, using the dynamic programming approach. Analogously to the cardinality indicating polytope, construct the acyclic network $N = (V, A)$, where

$$V = \{s\} \cup \{t\} \cup \{(i,j) \in \mathbb{N} \times \mathbb{Z}_2 : 1 \le i \le n\}$$

and

$$\begin{aligned} A = \{(s,(i,j)) \in V \times V : i = 1\} \cup \\ \{((i,j),t) \in V \times V : i = n, j = 0\} \cup \\ \{((i',j'),(i'',j'')) \in V \times V : i'' = i' + 1\}\,. \end{aligned}$$

The polytope $\mathrm{P}^1_{s-t\ \mathrm{flow}}(N) \subseteq \mathbb{R}^A$, associated with s-t paths in the network N, together with the affine map $p : \mathbb{R}^A \to \mathbb{R}^n$

$$p_i(y) = \begin{cases} y_{(s,(1,1))} & \text{if } i = 1 \\ y_{((i-1,0),(i,1))} + y_{((i-1,1),(i,0))} & \text{if } 2 \le i \le n\,, \end{cases}$$

defines an extension of the parity polytope $\mathrm{P}^n_{\mathrm{even}}$. Moreover, define the section map s : $\mathrm{vert}(\mathrm{P}^n_{\mathrm{card}}) \to P$ as

$$s_a(x,z) = \begin{cases} x_1 = 0 & \text{if } a = (s,(1,0)) \\ x_1 = 1 & \text{if } a = (s,(1,1)) \\ x_{i+1} = 0 \wedge \sum_{t=1}^{i} x_t = j \mod (2) & \text{if } a = ((i,j),(i+1,j)) \\ x_{i+1} = 1 \wedge \sum_{t=1}^{i} x_t = j \mod (2) & \text{if } a = ((i,j),(i+1,j+1)) \\ 1 & \text{if } a = ((n,0),t) \\ 0 & \text{if } a = ((n,1),t)\,. \end{cases}$$

This network imitates the scanning process from the first till the last coordinate, storing the parity of the scanned number of ones and the current position [1].

Proposition 2.3 (Carr and Konjevod [2004])**.** *For the parity polytope* $\mathrm{P}^n_{\mathrm{even}}$*, there exists an extended formulations of size* $O(n)$*.*

2.6. Birkhoff Polytope and Perfect Matchings in Bipartite Graphs

Here, the Birkhoff polytope, i.e. the perfect matching polytope in bipartite graphs, is presented, which is used later as an extension for other polytopes.

Recall that the perfect matching polytope for a graph $G = (V, E)$ with $2n$ vertices is defined as the convex hull of the characteristic vectors for perfect matchings in G, i.e.

$$\mathrm{P}^n_{\mathrm{match}}(G) = \mathrm{conv}(\{\chi(M) : M \in \mathcal{M}^n(G)\})\,.$$

When G is bipartite, with the bipartition V_*, $V^* \subseteq V$, such that $|V_*| = |V^*| = n$, the perfect matching polytope $\mathrm{P}^n_{\mathrm{match}}(G)$ has a compact linear description (see Schrijver [2003a]), given by the non-negativity constraints $0 \le x$ and the equations

$$x(\delta(v)) = 1 \quad \text{for all } v \in V\,.$$

Thus, there is a linear description of the perfect matching polytope $\mathrm{P}^n_{\mathrm{match}}(G)$ for a bipartite graph G, where the size of the linear description is equal to n^2.

Proposition 2.4 (Birkhoff [1946])**.** *For the perfect matching polytope* $\mathrm{P}^n_{\mathrm{match}}(G)$*,* $G = K_{n,n}$*, there exists a linear description of size* n^2*.*

The *Birkhoff polytope* $\mathrm{P}^n_{\mathrm{birk}} \subseteq \mathbb{R}^{n\times n}$ is the convex hull of all zero-one $n \times n$ matrices, such that every row and every column contains $n-1$ zeros and one one. A minimal linear description (see Schrijver [2003a]) of the Birkhoff polytope consists of the non-negativity constraints $0 \le x$ and

$$\sum_{t=1}^{n} x_{i,t} = 1 \ \text{ for } i \in [n] \qquad \text{and} \qquad \sum_{t=1}^{n} x_{t,j} = 1 \ \text{ for } j \in [n]\,.$$

Proposition 2.5 (Birkhoff [1946])**.** *For the Birkhoff polytope* $\mathrm{P}^n_{\mathrm{birk}}$*, there exists a linear description of size* n^2*.*

Clearly, the Birkhoff polytope $\mathrm{P}^n_{\mathrm{birk}}$ is affinely isomorphic to the perfect matching polytope $\mathrm{P}^n_{\mathrm{match}}(G)$, where G is the complete bipartite graph $K_{n,n}$. An affine isomorphism can be defined by the map $p : \mathbb{R}^{E(V_*:V^*)} \to \mathbb{R}^{n\times n}$

$$p_{i,j}(x) = x_{v^*{}_i, v_{*j}} \qquad \text{for} \quad (i,j) \in [n] \times [n]\,,$$

where the bipartition of the graph G is given as two vertex sets

$$V_* = \{v_{*1}, \dots, v_{*n}\} \quad \text{and} \quad V^* = \{v^*{}_1, \dots, v^*{}_n\}\,.$$

[1] Of course, similar networks can be designed for the polytopes, which are convex hulls of all n-dimensional zero-one vectors, where the remainder from the division of the total number of ones in the vector by some number k belongs to a specified set of remainders. In this case, the dynamic programming approach provides us with an extended formulation of size $O(kn)$. Moreover, these ideas could be generalized to the variations, when the vertices are not zero-one vectors, but general integer vectors with coordinate values from some given set of numbers.

2.7. Permutahedron

The *permutahedron* $\Pi_n \subseteq \mathbb{R}^n$ is defined as the convex hull of the points

$$\{(\sigma(1), \ldots, \sigma(n)) : \sigma \in \mathfrak{S}(n)\} ,$$

which are the vertices of the permutahedron. A minimal description of Π_n in the space $\mathbb{R}^n$ looks as follows (see Rado [1952], Conforti et al. [2010])

$$\sum_{i=1}^{n} x_i = \frac{n(n+1)}{2}$$

$$\sum_{i \in S} x_i \geq \frac{|S|(|S|+1)}{2} \qquad \text{for } \varnothing \neq S \subsetneq [n] .$$

The Birkhoff polytope $\mathrm{P}^n_{\mathrm{birk}}$, together with the affine map $p : \mathbb{R}^{n \times n} \to \mathbb{R}^n$

$$p_i(x) = \sum_{j=1}^{n} j\, x_{i,j} \quad \text{for } i \in [n] ,$$

forms an extended formulation of the permutahedron Π_n (see Conforti et al. [2010]), where the section map $s : \mathrm{vert}(\Pi_n) \to \mathrm{P}^n_{\mathrm{birk}}$ looks as follows

$$s_{i,j}(x) = \begin{cases} 1 & \text{if} \quad x_i = j \\ 0 & \text{otherwise} . \end{cases}$$

Proposition 2.6 (see Conforti et al. [2010])**.** *For the permutahedron Π_n, there exists an extended formulation of size $O(n^2)$.*

2.8. Edge Polytope

The *edge polytope* $\mathrm{P}_{\mathrm{edge}}(G) \subseteq \mathbb{R}^V$ for a graph $G = (V, E)$ is defined as the convex hull of the points

$$\{\chi(e) \in \mathbb{R}^V : e \in E\} .$$

Hence, every vertex of the edge polytope corresponds to an edge of the graph G and indicates two vertices, connected by the chosen edge. It is easy to see that the linear system

$$x(S) - x(N(S)) \leq 0 \quad \text{for all stable sets } S \subseteq V$$

$$x(V(G)) = 2 \quad \text{and} \quad 0 \leq x$$

is valid for the edge polytope $\mathrm{P}_{\mathrm{edge}}(G)$. Kaibel and Loos [2011], Janssen and Kilakos [1999] showed that the above linear system describes the edge polytope $\mathrm{P}_{\mathrm{edge}}(G)$ [1].

Clearly, there exists a vertex extension of the edge polytope $\mathrm{P}_{\mathrm{edge}}(G)$ of size $|E|$, what can be bounded from above by $O(|V|^2)$.

Observation 2.1. *For the edge polytope $\mathrm{P}_{\mathrm{edge}}(G)$, $G = (V, E)$, there exists an extended formulation of size $O(|E|)$.*

But on the other hand, we are able to construct another extended formulation, using the following theorem, which is due to Tuza [1984], Erdős and Pyber [1997].

Theorem 2.2 (Tuza [1984])**.** *For every graph $G = (V, E)$, $|V| = n$, there exists a covering of the edges E with a total cost at most $\frac{n^2}{\log n}$ by complete bipartite subgraphs, where the cost of a complete bipartite subgraph is the number of its vertices.*

[1] Kaibel and Loos [2011] provided conditions, under which the inequalities of the linear system define facets of the edge polytope.

Having a complete bipartite subgraph with bipartition V_*, V^*, we define the polytope

$$P_{V^*,V_*} = \{x \in \mathbb{R}^V : x(V_*) = x(V^*) = 1,\, x(V) = 2,\, 0 \leq x\}\,.$$

Thus, the vertices of the polytope P_{V^*,V_*} are the characteristic vectors of the edges $E(V_* : V^*)$. Applying the Balas technique to the polytopes P_{V^*,V_*}, corresponding to the complete bipartite graphs participating in the edge covering from Theorem 2.2, we show the next result [1].

Proposition 2.7. *For the edge polytope $\mathrm{P}_{\mathrm{edge}}(G)$, $G = (V, E)$, $|V| = n$, there exists an extended formulation of size $O(\frac{n^2}{\log n})$.*

2.9. Cardinality Restricted Matching Polytopes

In this section, we provide extensions for the cardinality restricted matching polytopes. To construct an extended formulation of $\mathrm{P}^{\ell}_{\mathrm{match}}(n)$, we need the following result on the existence of small families of *perfect-hash functions* from Alon et al. [1995], where results from Fredman et al. [1984], Schmidt and Siegel [1990] are used.

Theorem 2.3 (Alon et al. [1995])**.** *There are maps $\phi_1, \dots, \phi_{q(n,r)} : [n] \to [r]$, such that for every $W \subseteq [n]$ with $|W| = r$, there is some $i \in [q(n,r)]$, for which the map ϕ_i is bijective on W and the inequality $q(n,r) \leq 2^{O(r)} \log n$ holds*[2].

Let $\phi_1, \dots, \phi_q$ be maps as guaranteed to exist by Theorem 2.3 with $r = 2\ell$ and $q = q(n, 2\ell) \leq 2^{O(\ell)} \log n$, and denote

$$\mathcal{M}_i = \{M \in \mathcal{M}^{\ell}(n) : \phi_i \text{ is bijective on } V(M)\}$$

for each $i \in [q]$. By Theorem 2.3, we have

$$\mathcal{M}^{\ell}(n) = \mathcal{M}_1 \cup \dots \cup \mathcal{M}_q\,.$$

Consequently, we construct an extended formulation, using Balas techniques for the polytopes P_i, where $i \in [q]$

$$P_i = \mathrm{conv}(\{\chi(M) : M \in \mathcal{M}_i\})\,.$$

To finish the construction, we have to provide extended formulations for the polytopes P_i. From the linear description of the perfect matching polytope (Schrijver [2003a]) and Lemma 6.2, we obtain

$$\begin{aligned} P_i = \{x \in \mathbb{R}^E : \quad & x\big(E(\phi_i^{-1}(s))\big) = 0 && \text{for} \quad s \in [2\ell]\,,\\ & x\big(E(\phi_i^{-1}(s) : \phi_i^{-1}(t))\big) = y_{s,t} && \text{for} \quad s,\, t \in [2\ell], s \neq t\,,\\ & 0 \leq x\,,\\ & y(\delta(S)) \geq 1 && \text{for} \quad S \subseteq [2\ell], |S| \text{ is odd}\}\,. \end{aligned}$$

As the number of inequalities in the description of P_i is bounded by $2^{O(\ell)} + n^2$, and the number of different P_i is bounded by $2^{O(\ell)} \log n$, we obtain the following theorem.

Theorem 2.4. *For all n and ℓ, there is an extended formulation for $\mathrm{P}^{\ell}_{\mathrm{match}}(n)$ of size $2^{O(\ell)} n^2 \log n$.*

---

[1]Note that the complexity of the construction of the mentioned extension is not clear for us. Erdős and Pyber [1997] proved a stronger result, namely that there are $n/\log n$ bicliques, covering the edges of the graph G. But, an approximation of a minimum biclique cover (minimum number of bicliques) within $n^{1/3-\epsilon}$ seems to be a hard problem, unless $\mathcal{P}$ is equal to $\mathcal{NP}$ (see Gruber and Holzer [2007]). The proof of Erdős and Pyber [1997] is constructive, but the "bottleneck" of the construction is finding a biclique of size $\log n$, what seems to be a hard problem as well (see Chen et al. [2006]).

[2]Moreover, the functions ϕ_i, $i \in [q(n,r)]$ are $O(1)$-time computable, i.e. having an index $i \in [q(n,r)]$ and $x \in [n]$, the value $\phi_i(x)$ can be calculated in $O(1)$ running time in the uniform cost model.

2.10. Cardinality Restricted Cycle Polytope

In this section, we construct an extended formulation of the cardinality restricted cycle polytope $\mathrm{P}^{\ell}_{\mathrm{cycl}}(n)$, size of which is bounded by $2^{O(\ell)} n^3 \log n$. Starting with the maps $\phi_1, \ldots, \phi_q$ as guaranteed to exist by Theorem 2.3 with $r = \ell$ and $q = q(n, \ell) \leq 2^{O(\ell)} \log n$, we define

$$\mathcal{C}_i = \{C \in \mathcal{C}^{\ell}(n) : \phi_i \text{ is bijective on } V(C)\}$$

for each $i \in [q]$. Thus, we have

$$\mathcal{C}^{\ell}(n) = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_q \,,$$

and denote

$$P_i = \mathrm{conv}(\{\chi(C) : C \in \mathcal{C}_i\})$$

for all $i \in [q]$.

For the Balas method, it suffices to exhibit, for each $i \in [q]$, an extension of P_i of size bounded by $O(2^{\ell} n^3)$. Towards this, let us design the following network $N_i = (W_i, A_i)$, where

$$W_i = \{s\} \cup \{t\} \cup \{(v, u, S) \in V \times V \times 2^{[\ell]} : \phi_i(v) \in S, \phi_i(u) = 1\}$$

and

$$\begin{aligned} A_i = \{(s, (v, u, S)) \in W_i \times W_i : S = \{1\}, u = v\} \cup \\ \{((v, u, S), t) \in W_i \times W_i : S = [\ell]\} \cup \\ \{((v', u', S'), (v'', u'', S'')) \in W_i \times W_i : S'' = S' \cup \{\phi_i(v'')\}, \phi_i(v'') \notin S', u' = u''\} \,. \end{aligned}$$

Consider the polytope $\mathrm{P}^1_{s-t \text{ flow}}(N_i)$ and define the projection map $p^i : \mathbb{R}^A \to \mathbb{R}^E$ by its coordinate maps $p_{v',v''} : \mathbb{R}^A \to \mathbb{R}$ as

$$\sum_{S \subseteq [\ell]} y_{((v', v', \{1\}), (v'', v', S))} + y_{((v'', v', [\ell]), t)} \,,$$

when $\phi_i(v')$ is equal to one, and

$$\sum_{\substack{u \in V \\ S', S'' \in 2^{[\ell]}}} y_{((v', u, S'), (v'', u, S''))} + \sum_{\substack{u \in V \\ S', S'' \in 2^{[\ell]}}} y_{((v'', u, S'), (v', u, S''))}$$

when nor $\phi_i(v')$ neither $\phi_i(v'')$ is equal to one.

The idea, of the network is the scanning process of vertices from the cycle, starting from the vertex, which is mapped to one by ϕ_i, in any direction of the cycle. The stored information consists of the last scanned vertex, of the start vertex and of the set of images of the vertices for ϕ_i, which are scanned so far. This perspective helps to construct a section map, in a straight-forward manner.

Theorem 2.5. *For all n and ℓ, there is an extended formulation for $\mathrm{P}^{\ell}_{\mathrm{cycl}}(n)$ of size $2^{O(\ell)} n^3 \log n$.*

2.11. Polyhedral Relations

In the rest of the chapter, we deal with the framework of polyhedral relations, developed by Kaibel and Pashkovich [2011]. This framework heavily exploits the structure of polyhedra, which are in its scope. Due to this fact, the framework keeps the size of the constructed extensions small, even when polyhedral relations are applied iteratively.

A *polyhedral relation* of *type* (n, m) is a non-empty polyhedron $R \subseteq \mathbb{R}^n \times \mathbb{R}^m$. The *image* of a subset $X \subseteq \mathbb{R}^n$ under such a polyhedral relation R is denoted by

$$R(X) = \{y \in \mathbb{R}^m : (x, y) \in R \text{ for some } x \in X\} \,.$$

Clearly, the images of polyhedra and convex sets under polyhedral relations are polyhedra and convex sets, respectively, since $R(X)$ is a linear projection of $R \cap (X \times \mathbb{R}^m)$.

A particularly simple class of polyhedral relations is defined by polyhedra $R \subseteq \mathbb{R}^n \times \mathbb{R}^m$ with

$$R = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : y = \varrho(x)\}$$

for some affine map $\varrho : \mathbb{R}^n \to \mathbb{R}^m$. For these polyhedral relations, a linear description of a polyhedron $P \subseteq \mathbb{R}^n$ forms an extended formulation of the polyhedron $R(P)$ via the projection ϱ.

The *domain* of a polyhedral relation $R \subseteq \mathbb{R}^n \times \mathbb{R}^m$ is the polyhedron

$$\operatorname{dom}(R) = \{x \in \mathbb{R}^n : (x, y) \in R \text{ for some } y \in \mathbb{R}^m\} .$$

Clearly, we have

$$R(X) = \bigcup_{x \in X \cap \operatorname{dom}(R)} R(x)$$

for all $X \subseteq \mathbb{R}^n$. Note that, in general, for a polytope $P = \operatorname{conv}(X)$ with a finite set $X \subseteq \mathbb{R}^n$ and a polyhedral relation $R \subseteq \mathbb{R}^n \times \mathbb{R}^m$, the inclusion

$$\operatorname{conv} \bigcup_{x \in X} R(x) \subseteq R(P) \tag{2.11.1}$$

holds without equality, even in case of $P \subseteq \operatorname{dom}(R)$ [1]. In Section 2.13, the equality in (2.11.1) is guaranteed for an important class of polyhedral relations.

2.12. Sequential Polyhedral Relations

A sequence of polyhedral relations $R_1, \ldots, R_r$, such that R_i is a polyhedral relation of type (d_{i-1}, d_i) for each $i \in [r]$, is called a *sequential polyhedral relation* of *type* $(d_0, \ldots, d_r)$ and *length* r. For this sequential polyhedral relation, we denote by

$$\mathcal{R} = R_r \circ \cdots \circ R_1$$

the set of all $(z^0, z^r) \in \mathbb{R}^{d_0} \times \mathbb{R}^{d_r}$ for which there is some $(z^1, \ldots, z^{r-1})$ with

$$(z^{i-1}, z^i) \in R_i \quad \text{for all} \quad i \in [r] .$$

Since $\mathcal{R}$ is a linear projection of a polyhedron, $\mathcal{R}$ is a polyhedron, and thus, a polyhedral relation of type (d_0, d_r) with

$$R_r \circ \ldots \circ R_1(X) = R_r(\ldots R_1(X) \ldots)$$

for all $X \subseteq \mathbb{R}^{d_0}$. We call $\mathcal{R} = R_r \circ \cdots \circ R_1$ the polyhedral relation that is *induced* by the sequential polyhedral relation $R_1, \ldots, R_r$.

For a polyhedron $P \subseteq \mathbb{R}^{d_0}$, the polyhedron Q defined by

$$z^0 \in P \quad \text{and} \quad (z^{i-1}, z^i) \in R_i \quad \text{for all } i \in [r] ,$$

together with the projection map on to the z^r variables, forms an extension of $\mathcal{R}(P)$. Thus, there is an extended formulation of the polyhedron $\mathcal{R}(P)$ with $d_0 + \cdots + d_r$ variables and $f_0 + \cdots + f_r$ constraints, whenever we have linear descriptions of the polyhedra P, R_1, $\ldots$, R_r with f_0, f_1, $\ldots$, f_r constraints, respectively. Of course, one can reduce the number of variables in this extended formulation to the dimension of the polyhedron Q.

In order to obtain useful upper bounds on this number by means of the polyhedral relations R_1, $\ldots$, R_r, let us denote, for any polyhedral relation $R \subseteq \mathbb{R}^n \times \mathbb{R}^m$, by $\delta_1(R)$ and $\delta_2(R)$ the dimension of the non-empty fibers of the orthogonal projection of $\operatorname{aff}(R)$ to the first and second factor of $\mathbb{R}^n \times \mathbb{R}^m$, respectively. Having

$$\operatorname{aff}(R) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m : Ax + By = c\} ,$$

[1] For example, we may consider $P = \operatorname{conv}\{0, 2\} \subseteq \mathbb{R}$ and $R = \operatorname{conv}\{(0, 0), (1, 1), (2, 0)\}$.

we get $\delta_1(R) = \dim(\ker(B))$ and $\delta_2(R) = \dim(\ker(A))$. With these parameters, we can estimate

$$\dim(Q) \le \min\{d_0 + \sum_{i=1}^{r} \delta_1(R_i), d_r + \sum_{i=1}^{r} \delta_2(R_i)\} .$$

Lemma 2.2. *Let $R_1, \ldots, R_r$ be a sequential polyhedral relation of type $(d_0, \ldots, d_r)$ with the induced polyhedral relation $\mathcal{R}$, and let f_i be the number of facets of R_i. If the polyhedron $P \subseteq \mathbb{R}^{d_0}$ has an extended formulation with d' variables and f' inequalities, then we can construct an extended formulation for $\mathcal{R}(P)$ with*

$$\min\{d' + \sum_{i=1}^{r} \delta_1(R_i), d_r + \sum_{i=1}^{r} \delta_2(R_i)\}$$

variables and $f' + f_1 + \cdots + f_r$ constraints.

2.13. Affinely Generated Polyhedral Relations

We call a relation $R \subseteq \mathbb{R}^n \times \mathbb{R}^m$ *affinely generated* by the family ϱ^j, $j \in J$, if the set J is finite and every $\varrho^j : \mathbb{R}^n \to \mathbb{R}^m$ is an affine map, such that

$$R(x) = \operatorname{conv} \bigcup_{j \in J} \varrho^j(x)$$

holds for all $x \in \operatorname{dom}(R)$.

The maps ϱ^j, $j \in J$ are called *affine generators* of R in this case. For such a polyhedral relation R and a polytope $P \subseteq \mathbb{R}^n$ with

$$P \cap \operatorname{dom}(R) = \operatorname{conv}(X)$$

for some $X \subseteq \mathbb{R}^n$, we find

$$\begin{aligned} R(P) = \bigcup_{x \in P \cap \operatorname{dom}(R)} R(x) &= \bigcup_{x \in P \cap \operatorname{dom}(R)} \operatorname{conv} \bigcup_{j \in J} \varrho^j(x) \\ &\subseteq \operatorname{conv} \bigcup_{x \in P \cap \operatorname{dom}(R)} \bigcup_{j \in J} \varrho^j(x) = \operatorname{conv} \bigcup_{x \in X} \bigcup_{j \in J} \varrho^j(x) \subseteq \operatorname{conv} \bigcup_{x \in X} R(x) , \end{aligned}$$

where, due to (2.11.1), all inclusions are equations. In particular, we have established the following result.

Proposition 2.8. *For every polyhedral relation $R \subseteq \mathbb{R}^n \times \mathbb{R}^m$ that is affinely generated by a finite family ϱ^j, $j \in J$, and for every polytope $P \subseteq \mathbb{R}^n$, we have*

$$R(P) = \operatorname{conv} \bigcup_{j \in J} \varrho^j(P \cap \operatorname{dom}(R)) . \tag{2.13.1}$$

As we will often deal with polyhedral relations $\mathcal{R} = R_r \circ \cdots \circ R_1$ that are induced by a sequential polyhedral relation $R_1, \ldots, R_r$, it is convenient to be able to derive affine generators for $\mathcal{R}$ from affine generators for $R_1,\ldots,R_r$. This, however, seems impossible in general, where the difficulties arise from the interplay between images and domains in a sequence of polyhedral relations. However, one still can derive a very useful analogue of one of the inclusions in (2.13.1).

Lemma 2.3. *If we have $\mathcal{R} = R_r \circ \cdots \circ R_1$ and for each $i \in [r]$ the relation R_i is affinely generated by the finite family ϱ^{j_i}, $j_i \in J_i$, then the inclusion*

$$\mathcal{R}(P) \subseteq \operatorname{conv} \bigcup_{j \in J} \varrho^j(P \cap \operatorname{dom}(\mathcal{R}))$$

holds for every polyhedron $P \subseteq \mathbb{R}^n$, where $J = J_1 \times \cdots \times J_r$ and $\varrho^j = \varrho^{j_r} \circ \cdots \circ \varrho^{j_1}$ for each $j = (j_1, \ldots, j_r) \in J$.

PROOF. Trivially, if $\mathcal{R}(P)$ is empty, then the statement holds. Otherwise, for every $x^r \in \mathcal{R}(P)$ there is $(x^0, x^1, \ldots, x^r)$, such that $x^0 \in P \cap \operatorname{dom}(R)$ and $(x^{i-1}, x^i) \in R_i$ for all $i \in [r]$. Since every relation R_i is generated by the affine maps ϱ^{j_i}, $j_i \in J_i$, we conclude that for every $i \in [r]$, we have $x^i = \sum_{j_i \in J_i} \mu_{i,j_i} \varrho^{j_i}(x^{i-1})$ with some $\mu_{i,j_i} \geq 0$ for all $j_i \in J_i$, satisfying $\sum_{j_i \in J_i} \mu_{i,j_i} = 1$. Applying this iteratively, we are able to represent x^r as

$$x^r = \sum_{(j_1,\ldots,j_r) \in J} \mu_{1,j_1} \cdots \mu_{r,j_r} \varrho^{(j_1,\ldots,j_r)}(x^0)\,,$$

where all products $\mu_{1,j_1} \cdots \mu_{r,j_r}$ are non-negative, satisfying

$$\sum_{(j_1,\cdots,j_r) \in J} \mu_{1,j_1} \cdots \mu_{r,j_r} = \Big(\sum_{j_1 \in J_1} \mu_{1,j_1}\Big) \cdots \Big(\sum_{j_r \in J_r} \mu_{r,j_r}\Big) = 1.$$

This shows that x^r belongs to $\operatorname{conv} \bigcup_{j \in J} \varrho^j(x^0)$. □

2.14. Affine Generators and Domains from Polyhedral Relation

In this section, we study, what polyhedral relations $R \subseteq \mathbb{R}^n \times \mathbb{R}^m$ are affinely generated [1]. To do this, we consider the map $p : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, which is the projection on the first factor of $\mathbb{R}^n \times \mathbb{R}^m$. Moreover, we can assume that $p(R) = \operatorname{dom}(R)$ is full-dimensional.

Let us assume that R is an affinely generated polyhedral relation with affine generators ϱ^j, $j \in J$. Clearly, for every face $F \in \mathcal{L}(R)$ and for every point x from $p(F)$, there exists an index $j \in J$, such that $(x, \varrho^j(x))$ lies in F, since

$$\operatorname{conv}(\{(x, \varrho^j(x)) : j \in J\}) = R \cap \{z \in \mathbb{R}^n \times \mathbb{R}^m : p(z) = x\}\,.$$

Consider a face $F \in \mathcal{L}(R)$, which is defined by

$$F = \{z \in R : \langle a, z \rangle = b\}\,,$$

such that $p(F) \subseteq \mathbb{R}^n$ is full-dimensional. If for every $j \in J$, the affine space

$$\{x \in \mathbb{R}^n : \langle a, z \rangle = b, z = (x, \varrho^j(x))\}\,, \tag{2.14.1}$$

is not full-dimensional, then there exists $x \in p(F)$, such that for every $j \in J$ the point $(x, \varrho^j(x))$ does not lie in F. Thus, there exists $j \in J$ for which the affine space (2.14.1) is full-dimensional, what implies that for every $x \in p(R)$ the equation $\langle a, (x, \varrho^j(x)) \rangle = b$ holds. Consequently, $p(R)$ is equal to $p(F)$, since $(x, \varrho^j(x))$ belongs to R for all $x \in p(R)$.

On the other hand, let us assume that $R(x)$ is a polytope for every x from $p(R)$, and for every face $F \in \mathcal{L}(R)$, such that $p(F)$ is full-dimensional, we have $p(F) = p(R)$. For every face $F \in \mathcal{L}(F)$, we define the set I_F

$$I_F = \{i \in I : \langle a^i, z \rangle = b_i \text{ for all } z \in F\}\,,$$

where R is described by the linear system

$$\langle a^i, z \rangle \leq b_i \quad \text{for } i \in I\,.$$

Denote by $q : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ the projection on the second factor of $\mathbb{R}^n \times \mathbb{R}^m$. Note that for every $x \in \mathbb{R}^n$ the polytope $R(x)$ is defined by the linear system

$$\langle q(a^i), y \rangle \leq b_i - \langle p(a^i), x \rangle \quad \text{for } i \in I\,.$$

Clearly, every vertex of the polytope $R(x)$ corresponds to the solution of the linear system

$$\langle q(a^i), y \rangle = b_i - \langle p(a^i), x \rangle \quad \text{for } i \in I_F \tag{2.14.2}$$

[1] Actually, the results of this section admit an elegant representation via chamber complexes (see Rambau [1996] for more on such complexes). A polyhedral relation $R \subseteq \mathbb{R}^n \times \mathbb{R}^m$ is affinely generated if and only if $R \subseteq \mathbb{R}^n \times \mathbb{R}^m$, $\operatorname{proj}_{\mathbb{R}^n} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ induce one maximal chamber and every fiber is bounded.

for some face $F \in \mathcal{L}(R)$, with $x \in p(F)$, where the constraint matrix for y has full column rank. For a fixed face F, the solution of such a linear system is the image of an affine map $\varrho^F : \mathbb{R}^n \to \mathbb{R}^m$, i.e. $y = \varrho^F(x)$ (multiplication of $b_i - \langle p(a^i), x\rangle$, $i \in I_F$ by a matrix with constant coefficients). And thus for all points x from $p(R)$, except for some set of measure zero (union of the sets $p(F)$, with $F \in \mathcal{L}(R)$, where $p(F)$ is not full-dimensional), we get

$$R(x) = \operatorname{conv}(\{\varrho^F(x) : F \in \mathcal{F}\}) ,$$

where $\mathcal{F}$ denotes the faces $F \in \mathcal{L}(R)$, where $p(F)$ is full-dimensional, and the linear system (2.14.2) has full column rank. From continuity reasons, the above representation of $R(x)$ holds for all $x \in p(R)$.

Proposition 2.9. *A polyhedral relation $R \subseteq \mathbb{R}^n \times \mathbb{R}^m$ is affinely generated if and only if $R(x)$ is a polytope for every $x \in \operatorname{dom}(R)$, and for all faces $F \in \mathcal{L}(R)$, such that the dimension of $p(F)$ is equal to the dimension of $\operatorname{dom}(R)$, the image $p(F)$ is equal $\operatorname{dom}(R)$, where $p : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the projection on the first factor of $\mathbb{R}^n \times \mathbb{R}^m$*

2.15. Polyhedral Relations from Affine Generators and Domains

A particular task is to analyze, for what affine maps ϱ^j, $j \in J$, there exists a polyhedral relation $R \subseteq \mathbb{R}^n \times \mathbb{R}^m$, which is affinely generated by ϱ^j, $j \in J$. A more special question is, for what affine map ϱ, there is a polyhedral relation affinely generated by the identity map and the map ϱ. For both these questions, the domain plays a crucial role, since we can choose arbitrary affine maps ϱ^j, $j \in J$, whenever the domain consists of one point.

For a non-zero vector $a \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$, we denote the corresponding hyperplane by

$$\mathrm{H}^{=}(a, \beta) = \{x \in \mathbb{R}^n : \langle a, x\rangle = \beta\} ,$$

and by

$$\mathrm{H}^{\leq}(a, \beta) = \{x \in \mathbb{R}^n : \langle a, x\rangle \leq \beta\}$$

one of the corresponding halfspaces.

Lemma 2.4. *If for an affine map $\varrho : \mathbb{R}^n \to \mathbb{R}^n$ there exists a polyhedral relation $R \subseteq \mathbb{R}^n \times \mathbb{R}^n$, which is affinely generated by ϱ and the identity map, then ϱ is equivalent to a translation map on the domain of R, or there exists a hyperplane $\mathrm{H}^{=}(a, \beta) \subseteq \mathbb{R}^n$ and a vector $c \in R^n$ such that* [1]

(1) *the domain of R lies in one of the closed halfspaces, defined by the hyperplane $\mathrm{H}^{=}(a, \beta)$, i.e.*

$$\operatorname{dom}(R) \subseteq \mathrm{H}^{\leq}(a, \beta) \quad \textit{or} \quad \operatorname{dom}(R) \subseteq \mathrm{H}^{\leq}(-a, -\beta)$$

(2) *for every point $x \in \operatorname{dom}(R)$, the vector $\varrho(x) - x$ is parallel to the vector c and*

$$(\langle a, x\rangle - \beta)c = \varrho(x) - x . \tag{2.15.1}$$

PROOF. First of all, we show that the vectors $\varrho(x) - x$, $\varrho(y) - y$ are parallel for every x, y from $\operatorname{dom}(R)$. Indeed, the points (x, x), $(x, \varrho(x))$, (y, y) and $(y, \varrho(y))$ belong to the polyhedral relation R. Hence, we have

$$\frac{x + y}{2}, \ \frac{\varrho(x) + y}{2}, \ \frac{x + \varrho(y)}{2} \quad \text{belong to} \quad R(\frac{x + y}{2}) ,$$

The polytope $R(\frac{x+y}{2})$ is one-dimensional, since the polyhedral relation R is generated by two affine maps. And thus, the vectors $\varrho(x) - x$, $\varrho(y) - y$ are parallel.

Let us denote by c the non-zero vector $\varrho(x) - x$ for some x from $\operatorname{dom}(R)$. If no such non-zero vector c exists, then the map $\varrho : \mathbb{R}^n \to \mathbb{R}^n$ is equivalent to the identity map on

[1] Note that the range of the affine maps described by Lemma 2.4 is bride: translations, reflections with respect to a hyperplane, shearing transformations etc.

$\mathrm{dom}(R)$, i.e. a translation map. For simplicity of representation, we may assume that the vector c is equal to $\mathbb{e}_n$. Thus, for every point x from $\mathrm{dom}(R)$, we have

$$\varrho_i(x) = \begin{cases} x_i & \text{if } 1 \le i \le n-1 \\ \sum_{i\in[n]} \alpha_i x_i - \beta & \text{if } i = n \end{cases},$$

for some numbers $\alpha_i \in \mathbb{R}$, $i \in [n]$ and $\beta \in \mathbb{R}$.

The affine map ϱ is equivalent to a translation map, if $\alpha_i = 0$ for $i \in [n-1]$ and $\alpha_n = 1$.

Otherwise, denote by $a \in \mathbb{R}^n$ the vector with $a_i = \alpha_i$, $i \in [n-1]$ and $a_n = \alpha_n - 1$, satisfying the equation (2.15.1). Additionally, if there exist two points x, y from $\mathrm{dom}(R)$, such that $\langle a, x\rangle > \beta$ and $\langle a, y\rangle < \beta$, then there exists z from $\mathrm{dom}(R)$, which is a convex combination of x and y and which lies on the hyperplane $\mathrm{H}^{=}(a,\beta)$, where $R(z)$ is one-dimensional. But the hyperplane $\mathrm{H}^{=}(a,\beta)$ defines the set of the invariant points for ϱ, and thus $\varrho(z)$ is equal to z.

□

In the next section, we can see that Lemma 2.4 provides a characterization of affine maps ϱ, for which there exists an affinely generated polyhedral relation by the identity map and the affine map ϱ.

2.16. Reflection Relations

The *reflection* at the hyperplane $H = \mathrm{H}^{=}(a,\beta)$ is the affine map $\varrho^H : \mathbb{R}^n \to \mathbb{R}^n$, where $\varrho^H(x)$ is the point, such that $\varrho^H(x) - x$ lies in the one-dimensional linear subspace

$$H^{\perp} = \{\lambda a : \lambda \in \mathbb{R}\}$$

that is orthogonal to H, and $\langle a, \varrho^H(x)\rangle = 2\beta - \langle a, x\rangle$.

The *reflection relation*, defined by a vector $a \in \mathbb{R}^n$ and a number $\beta \in \mathbb{R}$, is

$$\mathrm{R}_{a,\beta} = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : y - x \in \mathrm{H}^{=}(a,\beta)^{\perp}, \langle a,x\rangle \le \langle a,y\rangle \le 2\beta - \langle a,x\rangle\}.$$

For the halfspace $H^{\le}$ equal to $\mathrm{H}^{\le}(a,\beta)$, we also denote $\mathrm{R}_{a,\beta}$ by $\mathrm{R}_{H^{\le}}$. The domain of the reflection relation is

$$\mathrm{dom}(\mathrm{R}_{a,\beta}) = H^{\le},$$

because if (x,y) lies in $\mathrm{R}_{a,\beta}$, then $\langle a,x\rangle \le 2\beta - \langle a,x\rangle$, and thus $\langle a,x\rangle \le \beta$. Furthermore, for each x from $\mathrm{H}^{\le}(a,\beta)$, the point (x,x) belongs to the polyhedral relation $\mathrm{R}_{a,\beta}$ [1]. From the fact that the vector $y - x$ lies in $\mathrm{H}^{=}(a,\beta)^{\perp}$, it follows $\delta_1(\mathrm{R}_{a,\beta}) = 1$, what together with Lemma 2.2 leads us to the next result.

Remark 2.1. *If $\mathcal{R} \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is induced by a sequential polyhedral relation of length r, consisting of reflection relations only, then for a polyhedron $P \subseteq \mathbb{R}^n$, an extended formulation of $\mathcal{R}(P)$ with $n' + r$ variables and $f' + 2r$ inequalities can be constructed, provided one has at hands an extended formulation for P with n' variables and f' inequalities.*

Proposition 2.10. *For every non-zero vector $a \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$, the reflection relation $\mathrm{R}_{a,\beta}$ is affinely generated by the identity map and the reflection ϱ^H, where H denotes the hyperplane $\mathrm{H}^{=}(a,\beta)$.*

PROOF. We have to show that for every $x \in \mathrm{dom}(\mathrm{R}_{a,\beta})$

$$\mathrm{R}_{a,\beta}(x) = \mathrm{conv}\{x, \varrho^H(x)\}.$$

Obviously, for every x from $\mathrm{dom}(\mathrm{R}_{a,\beta})$, we have that

$$(x,x),\ (x,\varrho^H(x)) \quad \text{belong to} \quad \mathrm{R}_{a,\beta}(x).$$

[1] Note that, although (a,β) and $(-a,-\beta)$ define the same reflection, the reflection relations $\mathrm{R}_{a,\beta}$ and $\mathrm{R}_{-a,-\beta}$ have different domains.

On the other hand, let y be an arbitrary point in $\mathrm{R}_{a,\beta}(x)$. Indeed, since both x and $\varrho^H(x)$ belong to the line $y + H^{\perp}$, and since

$$\langle a, x\rangle \leq \langle a, y\rangle \leq 2\beta - \langle a, x\rangle = \langle a, \varrho^H(x)\rangle \,,$$

we conclude that y is a convex combination of x and $\varrho^H(x)$. □

Note that for all affine maps, described in Lemma 2.4, there is an extended formulation of the corresponding polyhedral relation, whose construction is similar to the above extended formulation for the reflection relation. From Proposition 2.8 and Proposition 2.10, one obtains the following result.

Lemma 2.5. *If $P \subseteq \mathbb{R}^n$ is a polytope, then for every non-zero vector $a \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$, defining the hyperplane $H = \mathrm{H}^{=}(a,\beta)$ and the halfspace $H^{\leq} = \mathrm{H}^{\leq}(a,\beta)$, we have*

$$\mathrm{R}_{a,\beta}(P) = \mathrm{conv}\left((P \cap H^{\leq}) \cup \varrho^H(P \cap H^{\leq})\right).$$

2.17. Sequential Reflection Relations

Lemma 2.5 describes images under single reflection relations, but for analyses of the images under sequences of reflection relations we need additional results. For each non-zero vector $a \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ the map $\varrho^{\star(H^{\leq})} : \mathbb{R}^n \to \mathbb{R}^n$, which assigns a *canonical preimage* to every $y \in \mathbb{R}^n$, is defined

$$\varrho^{\star(H^{\leq})}(y) = \begin{cases} y & \text{if } y \in H^{\leq} \\ \varrho^H(y) & \text{otherwise} \end{cases} ,$$

where $H^{\leq} = \mathrm{H}^{\leq}(a,\beta)$, and $H = \mathrm{H}^{=}(a,\beta)$. For all $y \in \mathbb{R}^n$, we have

$$y \in \mathcal{R}(\varrho^{\star(H_1^{\leq})} \circ \cdots \circ \varrho^{\star(H_r^{\leq})}(y)) \,, \tag{2.17.1}$$

where $\mathcal{R}$ denotes the polyhedral relation $\mathrm{R}_{H_r^{\leq}} \circ \ldots \circ \mathrm{R}_{H_1^{\leq}}$.

Theorem 2.6. *For the polyhedral relation*

$$\mathcal{R} = \mathrm{R}_{H_r^{\leq}} \circ \ldots \circ \mathrm{R}_{H_1^{\leq}} \,,$$

with halfspaces $H_1^{\leq}, \ldots, H_r^{\leq} \subseteq \mathbb{R}^n$ and boundary hyperplanes $H_1, \ldots, H_r$, as well as polytopes $P, Q \subseteq \mathbb{R}^n$ where $Q = \mathrm{conv}(X)$, $X \subseteq \mathbb{R}^n$, we have $Q = \mathcal{R}(P)$, whenever the following two conditions are satisfied:

(1) *$P \subseteq Q$ and $\varrho^{H_i}(Q) \subseteq Q$ for all $i \in [r]$.*
(2) *$\varrho^{\star(H_1^{\leq})} \circ \cdots \circ \varrho^{\star(H_r^{\leq})}(x) \in P$ for all $x \in X$.*

PROOF. From the first condition it follows that the image of P under every combination of maps ϱ^{H_i} lies in Q. And from Lemma 2.3, this leads to the inclusion $\mathcal{R}(P) \subseteq Q$. By the second condition and (2.17.1), we have $X \subseteq \mathcal{R}(P)$, and hence $Q = \mathrm{conv}(X) \subseteq \mathcal{R}(P)$, due to the convexity of $\mathcal{R}(P)$. □

2.18. Signing of Polytopes

In order to provide simple examples of extended formulations, obtained from reflection relations, let us define the *signing* of a polyhedron $P \subseteq \mathbb{R}^n$ to be

$$\mathrm{sign}(P) = \mathrm{conv} \bigcup_{\epsilon \in \{-,+\}^n} \epsilon.P \,,$$

where $\epsilon.x$ is the vector, obtained from $x \in \mathbb{R}^n$ by changing the signs of all coordinates i with ϵ_i being minus. For $x \in \mathbb{R}^n$, we denote by $x^{(\mathrm{abs})} \in \mathbb{R}^n$ the vector that is obtained from x by changing every component to its absolute value.

For the construction below, we use the reflection relations $\mathrm{R}_{-\mathbb{e}_k,0}$, denoted by S_k, where $k \in [n]$. The corresponding reflection $\sigma_k : \mathbb{R}^n \to \mathbb{R}^n$ is the sign change of the k-th coordinate, given by

$$\sigma_k(x)_i = \begin{cases} -x_i & \text{if } i = k \\ x_i & \text{otherwise} . \end{cases}$$

The map, which defines the canonical preimage with respect to the relation S_k is given by

$$\sigma_k^\star(y)_i = \begin{cases} |y_i| & \text{if } i = k \\ y_i & \text{otherwise} . \end{cases}$$

Proposition 2.11. *If $\mathcal{R}$ is the polyhedral relation $\mathrm{S}_n \circ \ldots \circ \mathrm{S}_1$, and $P \subseteq \mathbb{R}^n$ is a polytope, such that $v^{(abs)} \in P$ for each vertex v of P, then we have*

$$\mathcal{R}(P) = \operatorname{sign}(P) .$$

PROOF. With $Q = \operatorname{sign}(P)$, the first condition of Theorem 2.6 is satisfied. Furthermore, we have $Q = \operatorname{conv}(X)$ with $X = \{\epsilon.v : \epsilon \in \{-,+\}^n, v \text{ vertex of } P\}$. As, for every $x \in X$ with $x = \epsilon.v$ for some vertex x of P and $\epsilon \in \{-,+\}^n$ we have $\sigma_1^\star \circ \cdots \circ \sigma_n^\star(x) = x^{(\mathrm{abs})} = v^{(\mathrm{abs})} \in P$, also the second condition of Theorem 2.6 is satisfied. Hence, the claim follows. □

The next result follows from Proposition 2.11 and Remark 2.1.

Theorem 2.7. *For every polytope $P \subseteq \mathbb{R}^n$, such that $v^{(abs)} \in P$ for each vertex v of P, there is an extended formulation of $\operatorname{sign}(P)$ with $n' + n$ variables and $f' + 2n$ inequalities, whenever the polytope P admits an extended formulation with n' variables and f' inequalities.*

2.19. Reflection Groups

A *finite reflection group* is a group G of finite cardinality that is generated by a finite family $\varrho^{H_i} : \mathbb{R}^n \to \mathbb{R}^n$, $i \in I$ of reflections at hyperplanes $H_i \subseteq \mathbb{R}^n$, containing the origin. We refer to Humphreys [1990], Fomin and Reading [2007] for all results on reflection groups that will be mentioned below. The set of *reflection hyperplanes* $H \subseteq \mathbb{R}^n$, where $\varrho^H \in G$, is called the *Coxeter arrangement* of G. Every Coxeter arrangement cuts $\mathbb{R}^n$ into open connected components, which are called the *regions* corresponding to G. The group G is in bijection with the set of its regions, and it acts transitively on these regions. We distinguish the topological closure of one of them as the *fundamental domain* Φ_G of G. Additionally, for every point $x \in \mathbb{R}^n$, there is a unique point $x^{(\Phi_G)} \in \Phi_G$ that belongs to the orbit of x under the action of the group G on $\mathbb{R}^n$.

A finite reflection group G is called *irreducible*, if the set of reflection hyperplanes cannot be partitioned into two sets $\mathcal{H}_1$ and $\mathcal{H}_2$, such that the normal vectors of all hyperplanes in $\mathcal{H}_1$ are orthogonal to the normal vectors of all hyperplanes from $\mathcal{H}_2$. According to a central classification result, up to linear transformations, the family of irreducible finite reflection groups consists of the four infinite subfamilies $I_2(m)$ on $\mathbb{R}^2$, A_{n-1}, B_n, and D_n on $\mathbb{R}^n$, as well as six special groups.

For a finite reflection group G on $\mathbb{R}^n$ and a polytope $P \subseteq \mathbb{R}^n$, the *G-permutahedron* $\Pi_G(P)$ of P is the convex hull of the union of the orbit of P under the action of G, i.e.

$$\Pi_G(P) = \operatorname{conv} \bigcup_{\varrho \in G} \varrho(P) .$$

In the next sections, we construct an extended formulation for $\Pi_G(P)$ from an extended formulation for P, if G is one of $I_2(m)$, A_{n-1}, B_n, or D_n. The number of inequalities in the constructed extended formulations will be bounded by $f' + \mathrm{O}(\log m)$, in the case of $G = I_2(m)$, and by $f' + \mathrm{O}(n \log n)$ in the other cases, provided that we have at hands an extended formulation of P with f' inequalities.

By the decomposition into irreducible finite reflection groups, one can extend these constructions to arbitrary finite reflection groups G on $\mathbb{R}^n$, where the resulting extended formulations have $f'+\mathrm{O}(n\log m)+\mathrm{O}(n\log n)$ inequalities, where m is the largest number such that $I_2(m)$ appears in the decomposition of G into irreducible finite reflection groups.

To see this, let us assume that the set of reflection hyperplanes $\mathcal{H}$ can be partitioned into two sets $\mathcal{H}_1$ and $\mathcal{H}_2$, such that the normal vectors of all hyperplanes in $\mathcal{H}_1$ are orthogonal to the normal vectors of all hyperplanes from $\mathcal{H}_2$. Let $\mathcal{H}_1$, $\mathcal{H}_2$ induce two reflection groups G_1, G_2. Then, we can represent the G-permutahedron as

$$\Pi_G(P) = \Pi_{G_1}(\Pi_{G_2}(P))\,.$$

Moreover, for every reflection map ϱ^{H_2}, $H_2 \in \mathcal{H}_2$, and for $a \in \mathbb{R}^n$, $b \in \mathbb{R}$, such that $H_1 = \mathrm{H}^{=}(a,b)$, $H_1 \in \mathcal{H}_1$, we have $\langle a, x\rangle = \langle a, \varrho^{H_2}(x)\rangle$ for all $x \in \mathbb{R}^n$. Hence, we can apply Theorem 2.6 for the polytope $\Pi_{G_2}(P)$ and the group G_1, whenever the conditions of Theorem 2.6 hold for the polytope P and for both groups G_1 and G_2.

2.20. Reflection group $I_2(m)$

Let us denote by H_φ, $\varphi \in \mathbb{R}$ the hyperplane $\mathrm{H}^{=}((-\sin\varphi, \cos\varphi), 0)$ and by $H_\varphi^{\leq}$ the halfspace $\mathrm{H}^{\leq}((-\sin\varphi, \cos\varphi), 0)$. The group $I_2(m)$ is generated by the reflections at H_0 and $H_{\pi/m}$, what is the symmetry group of the regular m-gon. The group $I_2(m)$ consists of the finite set of all reflections $\varrho^{H_{k\pi/m}}$, $k \in \mathbb{Z}$, and the finite set of all rotations around the origin by angles $2k\pi/m$, $k \in \mathbb{Z}$. Here, we choose the fundamental region

$$\Phi_{I_2(m)} = \{x \in \mathbb{R}^2 : x_2 \geq 0, x \in H_{\pi/m}^{\leq}\}\,.$$

Proposition 2.12. *If $\mathcal{R}$ is the polyhedral relation*

$$\mathrm{R}_{H^{\leq}_{2^r\pi/m}} \circ \cdots \circ \mathrm{R}_{H^{\leq}_{2\pi/m}} \circ \mathrm{R}_{H^{\leq}_{\pi/m}}$$

with $r = \lceil \log m \rceil$ and $P \subseteq \mathbb{R}^2$ is a polytope, such that $v^{(\Phi_{I_2(m)})} \in P$ for each vertex v of P, then we have

$$\mathcal{R}(P) = \Pi_{I_2(m)}(P)\,.$$

PROOF. The first condition of Theorem 2.6 is satisfied for $Q = \Pi_{I_2(m)}(P)$. Furthermore, we have $Q = \mathrm{conv}(X)$ with $X = \{\gamma.v : \gamma \in I_2(m), v \text{ vertex of } P\}$. Let $x \in X$ be some point with $x = \gamma.v$ for a vertex v of P and $\gamma \in I_2(m)$. Observing that

$$\varrho^{\star(H^{\leq}_{\pi/m})} \circ \varrho^{\star(H^{\leq}_{2\pi/m})} \circ \cdots \circ \varrho^{\star(H^{\leq}_{2^r\pi/m})}(x)$$

is contained in $\Phi_{I_2(m)}$, we conclude that it equals $x^{(\Phi_{I_2(m)})} = v^{(\Phi_{I_2(m)})} \in P$. Therefore, also the second condition of Theorem 2.6 is satisfied. □

From Proposition 2.12 and Remark 2.1, we can conclude the following theorem.

Theorem 2.8. *For each polytope $P \subseteq \mathbb{R}^2$, such that $v^{(\Phi_{I_2(m)})} \in P$ for every vertex v of P, there is an extended formulation of $\Pi_{I_2(m)}(P)$ with $n' + \lceil \log m \rceil + 1$ variables and $f' + 2\lceil \log m \rceil + 2$ inequalities, whenever P admits an extended formulation with n' variables and f' inequalities.*

In particular, we obtain an extended formulation of a regular m-gon with $\lceil \log m \rceil + 1$ variables and $2\lceil \log m \rceil + 2$ inequalities, by choosing $P = \{(1,0)\}$ in Theorem 2.8, what reproves the following result Ben-Tal and Nemirovski [2001].

Proposition 2.13 (Ben-Tal and Nemirovski [2001])**.** *For every regular m-gon, there exists an extended formulation of size $\lceil \log m \rceil + 1$.*

2.21. Reflection group A_{n-1}

The group A_{n-1} is generated by the reflections in $\mathbb{R}^n$ at $\mathrm{H}^{=}(\mathbb{e}_k - \mathbb{e}_\ell, 0)$ for all pairwise distinct $k, \ell \in [n]$. It is the symmetry group of the $(n-1)$-dimensional simplex[1] $\operatorname{conv}\{\mathbb{e}_1, \dots, \mathbb{e}_n\} \subseteq \mathbb{R}^n$. We choose

$$\Phi_{A_{n-1}} = \{x \in \mathbb{R}^n : x_1 \leq \cdots \leq x_n\}$$

as the fundamental domain. The orbit of a point $x \in \mathbb{R}^n$ under the action of A_{n-1} consists of all points, which can be obtained from x by permuting coordinates. Thus, the A_{n-1}-permutahedron of a polytope $P \subseteq \mathbb{R}^n$ is

$$\Pi_{A_{n-1}}(P) = \operatorname{conv} \bigcup_{\gamma \in \mathfrak{S}(n)} \gamma.P\,,$$

where $\gamma.x$ is the vector, obtained from $x \in \mathbb{R}^n$ by permuting the coordinates according to γ.

Let us consider more closely the reflection relation $\mathrm{T}_{k,\ell} \subseteq \mathbb{R}^n \times \mathbb{R}^n$, given as $\mathrm{R}_{\mathbb{e}_k - \mathbb{e}_\ell, 0}$. The corresponding reflection $\tau_{k,\ell} = \varrho^{H_{k,\ell}} : \mathbb{R}^n \to \mathbb{R}^n$, where $H_{k,\ell}$ is the hyperplane $\mathrm{H}^{=}(\mathbb{e}_k - \mathbb{e}_\ell, 0)$ is the transposition of coordinates k and ℓ, i.e.

$$\tau_{k,\ell}(x)_i = \begin{cases} x_\ell & \text{if } i = k \\ x_k & \text{if } i = \ell \\ x_i & \text{otherwise} \end{cases}\,.$$

The map $\tau^\star_{k,\ell} = \varrho^{\star(H_{k,\ell})} : \mathbb{R}^n \to \mathbb{R}^n$, assigning canonical preimages, is given by

$$\tau^\star_{k,\ell}(y) = \begin{cases} \tau_{k,\ell}(y) & \text{if } y_k > y_\ell \\ y & \text{otherwise} \end{cases}\,.$$

A sequence $(k_1, \ell_1), \dots, (k_r, \ell_r)$ from $[n] \times [n]$, where numbers in every pair are distinct, is called a *sorting network* if

$$\tau^\star_{k_1,\ell_1} \circ \cdots \circ \tau^\star_{k_r,\ell_r}(y) = y^{(\text{sort})}$$

holds for all $y \in \mathbb{R}^n$, where we denote by $y^{(\text{sort})} \in \mathbb{R}^n$ the vector that is obtained from y by sorting the components in non-decreasing order. Note that for every $y \in \mathbb{R}^n$ we have

$$y^{(\Phi_{A_{n-1}})} = y^{(\text{sort})}\,.$$

Proposition 2.14. *If $\mathcal{R}$ is the polyhedral relation*

$$\mathrm{T}_{k_r,\ell_r} \circ \ldots \circ \mathrm{T}_{k_1,\ell_1}\,,$$

where the sequence $(k_1, \ell_1), \dots, (k_r, \ell_r)$ is a sorting network, and $P \subseteq \mathbb{R}^n$ is a polytope, such that $v^{(\text{sort})} \in P$ for every vertex v of P, then we have

$$\mathcal{R}(P) = \Pi_{A_{n-1}}(P)\,.$$

PROOF. With $Q = \Pi_{A_{n-1}}(P)$, the first condition of Theorem 2.6 is satisfied. Furthermore, we have $Q = \operatorname{conv}(X)$ with $X = \{\gamma.v : \gamma \in \mathfrak{S}(n), v \in \operatorname{vert}(P)\}$. As, for every $x \in X$ with $x = \gamma.v$ for some vertex v of P and $\gamma \in \mathfrak{S}(n)$, we have

$$\tau^\star_{k_1,\ell_1} \circ \cdots \circ \tau^\star_{k_r,\ell_r}(x) = x^{(\text{sort})} = v^{(\text{sort})} \in P\,,$$

also the second condition of Theorem 2.6 is satisfied. Hence the claim follows. □

Since due to Ajtai et al. [1983], there are sorting networks of size $r = \mathrm{O}(n \log n)$, from Proposition 2.14 and Remark 2.1 we can conclude the following theorem.

[1] This explains the index in the notation A_{n-1}.

Theorem 2.9. *For each polytope $P \subseteq \mathbb{R}^n$, with $v^{(sort)} \in P$ for each vertex v of P, there is an extended formulation of $\Pi_{A_{n-1}}(P)$ with $n' + \mathrm{O}(n \log n)$ variables and $f' + \mathrm{O}(n \log n)$ inequalities, whenever P admits an extended formulation with n' variables and f' inequalities.*

Note that the sorting networks described in Ajtai et al. [1983] can be constructed in time that is bounded polynomially in n.

For the polytope $P = \{(1, 2, \dots, n)\} \subseteq \mathbb{R}^n$, Theorem 2.9 yields the same extended formulation of the permutahedron

$$\Pi_n = \Pi_{A_{n-1}}(P)\,,$$

that has been constructed in Goemans, where the extended formulation involves $O(n \log n)$ variables and inequalities.

Proposition 2.15 (Goemans)**.** *For the permutahedron $\Pi_n \subseteq \mathbb{R}^n$, there exists an extended formulation of size $O(n \log n)$.*

If we take the vertex extension for the polytope $P \subseteq \mathbb{R}^n \times \mathbb{R}^{n+1}$, which is the convex hull of $n + 1$ points $(\mathbf{0}_{n-i+1}, \mathbf{1}_{i-1}, \mathbb{e}_i) \in \mathbb{R}^n \times \mathbb{R}^{n+1}$, where $i \in [n + 1]$. Theorem 2.9 yields an extended formulation with $\mathrm{O}(n \log n)$ variables and inequalities of the cardinality indicating polytope

$$\Pi_{A_{n-1}}(P)\,.$$

Proposition 2.16. *For the cardinality indicating polytope $\mathrm{P}^n_{\mathrm{card}} \subseteq \mathbb{R}^n \times \mathbb{R}^{n+1}$, there exists an extended formulation of size $O(n \log n)$.*

2.22. Reflection group B_n

The group B_n is generated by the reflections in $\mathbb{R}^n$ at the hyperplanes $\mathrm{H}^=(\mathbb{e}_k + \mathbb{e}_\ell, 0)$, $\mathrm{H}^=(\mathbb{e}_k - \mathbb{e}_\ell, 0)$ and $\mathrm{H}^=(\mathbb{e}_k, 0)$ for all pairwise distinct $k, \ell \in [n]$. It is the symmetry group of both the n-dimensional cube $\mathrm{conv}\{-1, +1\}^n$ and the n-dimensional cross-polytope $\mathrm{conv}\{\pm\mathbb{e}_1, \dots, \pm\mathbb{e}_n\}$. We choose

$$\Phi_{B_n} = \{x \in \mathbb{R}^n : 0 \le x_1 \le \cdots \le x_n\}$$

as the fundamental domain. The orbit of a point $x \in \mathbb{R}^n$ under the action of B_n consists of all points, which can be obtained from x by permuting its coordinates and changing the signs of some subset of coordinates. Note that we have $y^{(\Phi_{B_n})} = y^{(\text{sort-abs})}$ for all $y \in \mathbb{R}^n$, where $y^{(\text{sort-abs})} = v^{(\text{sort})}$, where $v = y^{(\text{abs})}$.

Proposition 2.17. *If $\mathcal{R}$ is a polyhedral relation*

$$S_n \circ \ldots \circ S_1 \circ \mathrm{T}_{k_r,\ell_r} \circ \ldots \circ \mathrm{T}_{k_1,\ell_1}\,,$$

where $(k_1, \ell_1), \dots, (k_r, \ell_r)$ is a sorting network, and S_i are defined as at the end of Section 2.18, and $P \subseteq \mathbb{R}^n$ is a polytope, such that $v^{(\text{sort-abs})} \in P$ for each vertex v of P, then we have

$$\mathcal{R}(P) = \Pi_{B_n}(P)\,.$$

Proof. With $Q = \Pi_{B_n}(P)$, the first condition of Theorem 2.6 is satisfied. Furthermore, we have $Q = \mathrm{conv}(X)$ with $X = \{\gamma.\epsilon.v : \gamma \in \mathfrak{S}(n), \epsilon \in \{-,+\}^n, v \in \mathrm{vert}(P)\}$. As, for every $x \in X$ with $x = \gamma.\epsilon.v$ for some vertex v of P and $\gamma \in \mathfrak{S}(n)$, $\epsilon \in \{-,+\}^n$, we have

$$\tau^\star_{k_1,\ell_1} \circ \cdots \circ \tau^\star_{k_r,\ell_r} \circ \sigma^\star_1 \circ \cdots \circ \sigma^\star_n(x) = x^{(\text{sort-abs})} = v^{(\text{sort-abs})} \in P\,,$$

also the second condition of Theorem 2.6 is satisfied. Hence, the claim follows. □

As for A_{n-1}, we thus can conclude the following from Proposition 2.17 and Remark 2.1.

Theorem 2.10. *For every polytope $P \subseteq \mathbb{R}^n$, such that $v^{(\text{sort-abs})} \in P$ for every vertex v of P, there is an extended formulation of $\Pi_{B_n}(P)$ with $n' + \mathrm{O}(n \log n)$ variables and $f' + \mathrm{O}(n \log n)$ inequalities, whenever P admits an extended formulation with n' variables and f' inequalities.*

2.23. Reflection group D_n

The group D_n is generated by the reflections in $\mathbb{R}^n$ at the hyperplanes $\mathrm{H}^{=}(\mathbb{e}_k + \mathbb{e}_\ell, 0)$ and $\mathrm{H}^{=}(\mathbb{e}_k - \mathbb{e}_\ell, 0)$ for all pairwise distinct $k, \ell \in [n]$. Thus, D_n is a proper subgroup of B_n, but it is not the symmetry group of a polytope. We choose

$$\Phi_{D_n} = \{x \in \mathbb{R}^n : |x_1| \leq x_2 \leq \cdots \leq x_n\}$$

as the fundamental domain. The orbit of a point $x \in \mathbb{R}^n$ under the action of D_n consists of all points, which can be obtained from x by permuting its coordinates and changing the signs of an even number of its coordinates. For every x, the point $x^{(\Phi_{D_n})}$ arises from $x^{(\text{sort-abs})}$ by changing the sign of the first component, if x has an odd number of negative components. For distinct $k, \ell \in [n]$, we denote by $E_{k,\ell}$ the polyhedral relation $\mathrm{R}_{-\mathbb{e}_k - \mathbb{e}_\ell, 0} \circ \mathrm{R}_{\mathbb{e}_k - \mathbb{e}_\ell, 0}$.

Proposition 2.18. *If $\mathcal{R}$ is the polyhedral relation*

$$E_{n-1,n} \circ \cdots \circ E_{1,2} \circ \mathrm{T}_{k_r,\ell_r} \circ \ldots \circ \mathrm{T}_{k_1,\ell_1} ,$$

where $(k_1, \ell_1), \ldots, (k_r, \ell_r)$ is a sorting network, and $P \subseteq \mathbb{R}^n$ is a polytope, such that $x^{(\Phi_{D_n})} \in P$ for each vertex v of P, then we have

$$\mathcal{R}(P) = \Pi_{D_n}(P) .$$

PROOF. With $Q = \Pi_{D_n}(P)$, the first condition of Theorem 2.6 is satisfied. Let us denote by $\{-,+\}^n_{\text{even}}$ the set of all $\epsilon \in \{-,+\}^n$ with an even number of components equal to minus. Then, we have $Q = \mathrm{conv}(X)$ with

$$X = \{\gamma.\epsilon.v : \gamma \in \mathfrak{S}(n), \epsilon \in \{-,+\}^n_{\text{even}}, v \in \mathrm{vert}(P)\} .$$

For distinct $k, \ell \in [n]$, we define

$$\eta^\star_{k,\ell} = \varrho^{\star(\mathrm{H}^{\leq}(\mathbb{e}_k - \mathbb{e}_\ell, 0))} \circ \varrho^{\star(\mathrm{H}^{\leq}(-\mathbb{e}_k - \mathbb{e}_\ell, 0))} .$$

For every $y \in \mathbb{R}^n$, the vector $\eta^\star_{k,\ell}(y)$ is the vector $y' \in \{y, \tau_{k,\ell}(y), \rho_{k,\ell}(y), \rho_{k,\ell}(\tau_{k,\ell}(y))\}$ with $|y'_k| \leq y'_\ell$, where $\rho_{k,\ell}(y)$ arises from y by changing the sign of both components k and ℓ. As, for every $x \in X$ with $x = \gamma.\epsilon.v$ for some vertex v of P and $\gamma \in \mathfrak{S}(n)$, $\epsilon \in \{-,+\}^n_{\text{even}}$, we have

$$\tau^\star_{k_1,\ell_1} \circ \cdots \circ \tau^\star_{k_r,\ell_r} \circ \eta^\star_{1,2} \circ \cdots \circ \eta^\star_{n-1,n}(x) = x^{(\Phi_{D_n})} = v^{(\Phi_{D_n})} \in P ,$$

also the second condition of Theorem 2.6 is satisfied. Hence, the claim follows. □

And again, similarly to the cases A_{n-1} and B_n, we derive the following result from Proposition 2.18 and Remark 2.1.

Theorem 2.11. *For every polytope $P \subseteq \mathbb{R}^n$, such that $v^{(\Phi_{D_n})}(v) \in P$ for every vertex v of P, there is an extended formulation of $\Pi_{D_n}(P)$ with $n' + \mathrm{O}(n \log n)$ variables and $f' + \mathrm{O}(n \log n)$ inequalities, whenever P admits an extended formulation with n' variables and f' inequalities.*

Restricting our attention to the polytopes

$$P = \{(-1, 1, \ldots, 1)\} \subseteq \mathbb{R}^n \quad \text{or} \quad P = \{(1, 1, \ldots, 1)\} \subseteq \mathbb{R}^n ,$$

we can remove the reflection relations $T_{i_1,j_1},\dots,T_{i_r,j_r}$ from the construction in Proposition 2.18. Thus, we obtain extended formulations with $2(n-1)$ variables and $4(n-1)$ inequalities of the convex hulls of all vectors in $\{-1,+1\}^n$ with odd, respectively even number of minus ones. Thus, applying the affine transformation of $\mathbb{R}^n$ given by $q:\mathbb{R}^n\to\mathbb{R}^n$

$$q(y)=\frac{1}{2}(\mathbf{1}_n-y)\,,$$

we derive extended formulations with $2(n-1)$ variables and $4(n-1)$ inequalities for the parity polytopes, what reproves Proposition 2.3.

2.24. Huffman Polytopes

A *Huffman-vector* is a vector $v\in\mathbb{R}^n$, such that there is a rooted binary tree with n leaves, which are labeled by the numbers from $[n]$, and for every $i\in[n]$, the number of arcs on the path from the root to the i-th leaf equals v_i. We denote by $\mathrm{V}^n_{\mathrm{huff}}$ the set of all Huffman-vectors in $\mathbb{R}^n$, and by

$$\mathrm{P}^n_{\mathrm{huff}}=\mathrm{conv}(\mathrm{V}^n_{\mathrm{huff}})$$

the *Huffman polytope*. Note that currently no linear description of the Huffman polytope $\mathrm{P}^n_{\mathrm{huff}}$ in $\mathbb{R}^n$ is known [1].

Nevertheless, the properties of Huffman vectors and Huffman polytopes listed below can be easily verified. Moreover, these properties appear to be useful for our further discussion.

Observation 2.2.

(1) *For every* $\gamma\in\mathfrak{S}(n)$
$$\gamma.\,\mathrm{V}^n_{\mathrm{huff}}=\mathrm{V}^n_{\mathrm{huff}}\ .$$
(2) *For every* $v\in\mathrm{V}^n_{\mathrm{huff}}$*, there are at least two components of* v *equal to*
$$\max_{k\in[n]}v_k\ .$$
(3) *For every* $v\in\mathrm{V}^n_{\mathrm{huff}}$ *and*
$$v_i=v_j=\max_{k\in[n]}v_k$$
for some pair of distinct i, j*, the point*
$$(v_1,\dots,v_{i-1},v_i-1,v_{i+1},\dots,v_{j-1},v_{j+1},\dots,v_n)$$
lies in $\mathrm{V}^{n-1}_{\mathrm{huff}}$.

(4) *For every* $x\in\mathrm{V}^{n-1}_{\mathrm{huff}}$*, the point*
$$(x_1,\dots,x_{n-2},x_{n-1}+1,x_{n-1}+1)$$
lies in $\mathrm{V}^n_{\mathrm{huff}}$.

To construct an extended formulation of the Huffman polytope, we need to define the embedding

$$P^{n-1}=\{(x_1,\dots,x_{n-2},x_{n-1}+1,x_{n-1}+1):(x_1,\dots,x_{n-1})\in\mathrm{P}^{n-1}_{\mathrm{huff}}\}$$

of $\mathrm{P}^{n-1}_{\mathrm{huff}}$ into $\mathbb{R}^n$.

Proposition 2.19. *If* $\mathcal{R}\subseteq\mathbb{R}^n\times\mathbb{R}^n$*, where* $3\le n$*, is the polyhedral relation*

$$\mathrm{T}_{1,2}\circ\mathrm{T}_{2,3}\circ\cdots\circ\mathrm{T}_{n-2,n-1}\circ\mathrm{T}_{n-1,n}\circ \mathrm{T}_{1,2}\circ\mathrm{T}_{2,3}\circ\cdots\circ\mathrm{T}_{n-3,n-2}\circ\mathrm{T}_{n-2,n-1}\,, \tag{2.24.1}$$

then we have $\mathcal{R}(P^{n-1})=\mathrm{P}^n_{\mathrm{huff}}$.

[1] In fact, it seems that such descriptions are extremely complicated. For instance, it was proved that $\mathrm{P}^n_{\mathrm{huff}}$ has $\Omega(n!)$ facets Nguyen et al. [2010].

PROOF. With $P = P^{n-1}$ and $Q = \mathrm{P}^n_{\mathrm{huff}}$, the first condition of Theorem 2.6 is obviously satisfied, what is due to parts (1) and (4) of Observation 2.2. We have $Q = \mathrm{conv}(X)$ with $X = \mathrm{V}^n_{\mathrm{huff}}$. Furthermore, for every $x \in X$ and $y = \tau^\star(x)$ with

$$\tau^\star = \tau^\star_{n-2,n-1} \circ \tau^\star_{n-3,n-2} \circ \cdots \circ \tau^\star_{2,3} \circ \tau^\star_{1,2} \circ \tau^\star_{n-1,n} \circ \tau^\star_{n-2,n-1} \circ \cdots \circ \tau^\star_{2,3} \circ \tau^\star_{1,2}\,, \tag{2.24.2}$$

we have

$$y_n = y_{n-1} = \max_{i \in [n]} x_i\,,$$

the part (3) of Observation 2.2 implies $\tau^\star(x) \in P^{n-1}$. Therefore, the claim follows by Theorem 2.6. □

Thus, from Remark 2.1, we obtain an extended formulation for $\mathrm{P}^n_{\mathrm{huff}}$ with $n' + 2n - 3$ variables and $f' + 4n - 6$ inequalities, provided we have an extended formulation for $\mathrm{P}^{n-1}_{\mathrm{huff}}$ with n' variables and f' inequalities. Since the Huffman polytope $\mathrm{P}^2_{\mathrm{huff}}$ is a single point, inductive application of this approach leads to the following result.

Proposition 2.20. *For the Huffman polytope* $\mathrm{P}^n_{\mathrm{huff}}$*, there is an extended formulation of size* $O(n^2)$.

Actually, the Huffman polytope $\mathrm{P}^n_{\mathrm{huff}}$ has an extended formulation of size $O(n \log n)$, but this demands another sorting approach. In order to indicate the necessary modifications, let us denote by Θ_k the sequence

$$(k-2, k-1), (k-3, k-2), \ldots, (1,2), (k-1, k), (k-2, k-1), \ldots, (1,2)$$

of index pairs, which are used in (2.24.1) and (2.24.2). For every sequence

$$\Theta = ((i_1, j_1), \ldots, (i_r, j_r))$$

of pairs of distinct indices, we define

$$\tau^\star_\Theta = \tau^\star_{i_1,j_1} \circ \cdots \circ \tau^\star_{i_r,j_r}\,,$$

thus $\tau^\star_{\Theta_n}$ is denoted by $\tau^\star$ in (2.24.2). Furthermore, let $\pi_k : \mathbb{R}^k \to \mathbb{R}^{k-1}$ to be the linear map defined via

$$\pi_k(y) = (y_1, \ldots, y_{k-2}, y_{k-1} - 1)$$

for all $y \in \mathbb{R}^k$. For the above construction, we need that for every $v \in \mathrm{V}^n_{\mathrm{huff}}$ and every $k \geq 3$ the vector

$$x^k = \tau^\star_{\Theta_k} \circ \pi_{k+1} \circ \tau^\star_{\Theta_{k+1}} \circ \cdots \circ \pi_n \circ \tau^\star_{\Theta_n}(v) \tag{2.24.3}$$

satisfies

$$x^k_{k-1} = x^k_k = \max_{i \in [k]} x^k_i\,.$$

It turns out that this property is preserved, when replacing the sequence Θ_n by an arbitrary sorting network, and for every $k \geq 3$, the sequence Θ_k by the sequence

$$(i^k_2, i^k_1), (i^k_3, i^k_2), \ldots, (i^k_{r_k}, i^k_{r_k-1}), (i^k_{r_k-1}, i^k_{r_k-2}), \ldots, (i^k_3, i^k_2), (i^k_2, i^k_1)$$

with

$$i^k_t = \begin{cases} k & \text{if } t = 1 \\ k-1 & \text{if } t = 2 \\ i^k_{t-1} - 2^{t-3} & \text{otherwise} \end{cases}\,,$$

and where r_k is the maximal t, such that i^k_t is greater than zero. Denote by J_k the set of indices, involved in this sorting transformation Θ_k, i.e.

$$J_k = \{i^k_t : t \in [r_k]\}\,.$$

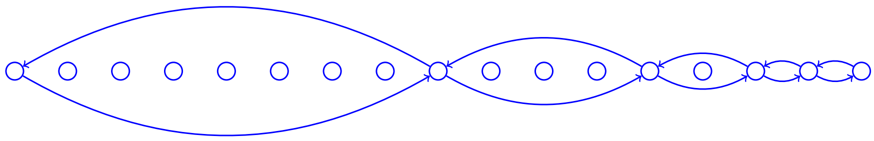

FIGURE 1. The sorting procedure: first the comparators on the path above are applied, and then the comparators on the path below (the smallest of two elements moves always into the left side).

Proposition 2.21. *For every $2 \le k \le n$, the Huffman vector x^k, defined by* (2.24.3)*, is sorted or it has the following form*

$$\begin{array}{llll} x^k_k & = \cdots & = x^k_{k-p_k+1} & = \max\limits_{i \in [k]} x^k_i \\ x^k_{k-p_k} & = \cdots & = x^k_{k-p_k-q_k+1} & = \max\limits_{i \in [k]} x^k_i - 1 \\ x^k_{k-p_k-q_k} & = \cdots & = x^k_{k-p_k-q_k-\ell_k+1} & = \max\limits_{i \in [k]} x^k_i \\ x^k_1 & \le \cdots & \le x^k_{k-p_k-q_k-\ell_k} & \le \max\limits_{i \in [k]} x^k_i - 1\,, \end{array}$$

where the index $k - p_k - q_k + 1$ belongs to J^k and p_k is strictly greater than ℓ_k.

PROOF. The proof is by induction on the number n, i.e. we assume that if a vector $x^k \in \mathbb{R}^k$ satisfies

$$x^k = \tau^\star_{\Theta_k} \circ \pi_{k+1} \circ \tau^\star_{\Theta_{k+1}} \circ \cdots \circ \pi_m(x)$$

for a sorted Huffman vector $x \in \mathbb{R}^m$, where $m < n$, then the vector x^k satisfies the claim above.

If the Huffman vector

$$y^{n-1} = \pi_n \circ \tau^\star_{\Theta_n}(v)$$

is sorted, then we can apply the induction assumption for $m = n - 1$ and the Huffman vector

$$x = \tau^\star_{\Theta_{n-1}} \circ \pi_n \circ \tau^\star_{\Theta_n}(v)\,.$$

Otherwise, for the Huffman vector y^{n-1}, we have

$$y^{n-1}_{n-1} = u - 1 \quad \text{and} \quad y^{n-1}_1 \le \cdots \le y^{n-1}_{n-2} = u\,,$$

where u is the maximum value among the coordinates of the Huffman vector x^n. After application of the sorting transformation Θ_{n-1} to y^{n-1}, we get the Huffman vector x^{n-1} with

$$\begin{array}{llll} x^{n-1}_{n-1} & = \cdots & = x^{n-1}_{(n-1)-p_{n-1}+1} & = u \\ x^{n-1}_{(n-1)-p_{n-1}} & & & = u - 1 \\ x^{n-1}_{(n-1)-p_{n-1}-1} & = \cdots & = x^{n-1}_{(n-1)-p_{n-1}-\ell_{n-1}} & = u \\ x^{n-1}_1 & \le \cdots & \le x^{n-1}_{(n-1)-p_{n-1}-\ell_{n-1}-1} & \le u - 1\,, \end{array}$$

where $p_{n-1} = 2^{i-1}$ and $\ell_{n-1} < 2^{i-1}$. If the Huffman vector

$$x^{n-1} = \tau^\star_{\Theta_{n-1}} \circ \pi_n \circ \tau^\star_{\Theta_n}(v)$$

is sorted, i.e. $l_{n-1} = 0$, then the induction assumption for $m = n - 1$ and $x = x^{n-1}$ finishes the proof. Otherwise, the index $(n-1) - p_{n-1}$ belongs to J_{n-1}, thus the assumption of the proposition holds for $k = n - 1$.

Let us assume that for the Huffman vector

$$x^k = \tau^\star_{\Theta_k} \circ \cdots \circ \pi_n \circ \tau^\star_{\Theta_n}(v)$$

the claim holds. Then, the Huffman vector

$$y^{k-1} = \pi_k \circ \tau^{\star}_{\Theta_k} \circ \cdots \circ \pi_n \circ \tau^{\star}_{\Theta_n}(v)$$

reads

$$\begin{array}{llll} y^{k-1}_{k-1} & & & = u-1 \\ y^{k-1}_{k-2} & = \cdots & = y^{k-1}_{k-p_k+1} & = u \\ y^{k-1}_{k-p_k} & = \cdots & = y^{k-1}_{k-p_k-q_k+1} & = u-1 \\ y^{k-1}_{k-p_k-q_k} & = \cdots & = y^{k-1}_{k-p_k-q_k-\ell_k+1} & = u \\ x^k_1 & \leq \cdots & \leq x^k_{k-p_k-q_k-\ell_k} & \leq u-1\,, \end{array}$$

Obviously, the set of indices J_{k-1} is obtained from the set of indices J_k, decreasing every element by one and excluding the index zero. Hence, the index $(k-1)-(p_k-1)-q_k$ belongs to the index set J_{k-1}.

Let us consider the coordinates of y^{k-1} with indices in J_{k-1}, i.e. the coordinates participating in Θ_{k-1}. Note that there exists just one u in this sequence before the $u-1$ block, since $(k-1)-(p_k-1)-q_k$ belongs to the indices set J_{k-1} and $p_k > l_k$. Clearly, the action of $\tau^{\star}_{\Theta_{k-1}}$ is equivalent to swapping of the first u-value with the last $(u-1)$-value in this sequence of coordinates. Thus after the sorting transformation $\tau^{\star}_{\Theta_{k-1}}$ the Huffman vector

$$x^{k-1} = \tau^{\star}_{\Theta_{k-1}} \circ \pi_k \circ \tau^{\star}_{\Theta_k} \circ \cdots \circ \pi_n \circ \tau^{\star}_{\Theta_n}(v)$$

has the desired form. Additionally, we have $l_{k-1} < p_{k-1}$ and $(k-1) - p_{k-1} - q_{k-1} \in J^{k-1}$.

□

To finish the construction, we have to verify that

$$x^k_{k-1} = x^k_k = \max_{i \in [k]} x^k_i\,.$$

for the Huffman vector x^k, $k \geq 3$. Obviously, this follows from Proposition 2.21, because the inequality $p_k > \ell_k$ implies $p_k \geq 2$, since every Huffman vector has even number of maximal elements, i.e. $p_k + \ell_k$ has to be even. We obtain the following theorem, since the number r_k is bounded by $O(\log k)$ and since there are sorting networks of size $O(n \log n)$, as in Section 2.21.

Theorem 2.12. *For the Huffman polytope* $\mathrm{P}^n_{\mathrm{huff}}$, *there is an extended formulation of size* $O(n \log n)$.

CHAPTER 3

Planar Graphs

In this chapter, we consider extended formulations for polytopes, associated with combinatorial objects in planar graphs. Indeed, for a lot of polytopes, for which no compact extended formulation is known in the general case, there are compact extended formulations, whenever we restrict our attention to planar graphs.

For the perfect matching polytope, which is one of the central polytopes for the theory of extended formulations, there exist compact extensions in the case of planar graphs (see Barahona [1993], Gerards [1991]). Moreover, the cut polytope for planar graphs has a compact extension (see Barahona [1993]). This is an illustrative example, since for the cut polytope of the complete graph no linear description in the initial space is known, and no compact extended formulation exists (see Fiorini et al. [2011b]).

In Section 3.2, we construct a compact extended formulation for the perfect matching polytope in graphs, with the genus not greater than the logarithm of the number of vertices in the graph. This construction is based on the extension given by Gerards [1991], which is produced via the T-join polyhedron. The modifications we undertake in the extended formulation lead to a size reduction.

In Section 3.3, compact extended formulations for the cut polytope and T-join polytope of Barahona [1993] are presented, what gives a compact extended formulation of the perfect matching polytopes of planar graphs.

In Sections 3.4 and 3.5, extended formulations for the spanning tree polytope, which is due to Williams [2002], and the subtour elimination polytope, due to Rivin [1996], Rivin [2003], Cheung [2003], are presented. The initial extended formulation for the subtour elimination polytope is constructed for planar graphs, where every face involves three vertices, i.e. for graphs, defining triangulations. In this work, we provide another extension and a simple proof for the validity of the extended formulation without the restriction to triangulations. On the other hand, for the graphs, defining triangulations, the presented extension and the extension in Rivin [1996], Rivin [2003], Cheung [2003] coincide.

3.1. Graph Embeddings

The *genus* $\gamma(G)$ of a graph $G = (V, E)$ is the minimum genus of a closed orientable surface $\mathcal{S}$, such that the graph G can be embedded on the surface $\mathcal{S}$ without crossing edges. A graph G is called $planar$ if it is embeddable on the plane, i.e. on the closed orientable surface with genus zero. We refer to White [1973], Schrijver [2003c] for all relevant facts about graphs and embeddings of graphs on surfaces.

Having an embedding of a graph G on a surface $\mathcal{S}$ without crossings, define the *dual graph* $G = (V^*, E^*)$, where V^* is the set of faces, induced by the graph G on the surface $\mathcal{S}$, and the edges E^* correspond to the edges E, where each edge from E^* connects two neighbor faces. The *Euler characteristic* $\chi(G)$ of a graph $G = (V, E)$ is equal $2 - 2\gamma(G)$. Moreover, if a connected graph G is embedded into a surface $\mathcal{S}$, where $\chi(G) = \chi(\mathcal{S})$, the Euler Formula states

$$|V| - |E| + |F| = \chi(G)\,,$$

where F denotes the set of faces induced by the embedding of the graph G on the surface $\mathcal{S}$.

Note that an embedding of a graph $G = (V, E)$ on a surface $\mathcal{S}$ can be obtained in $O(|V|^{O(\gamma(\mathcal{S}))})$ running time (see Filotti et al. [1979]) In fact, we should not expect an algorithm with a running time, which is polynomial in both $|V|$ and $\gamma(\mathcal{S})$, since Thomassen [1989] showed that it is an $\mathcal{NP}$-hard problem to determine the genus of a graph $G = (V, E)$.

3.2. Extended Formulation of T-join Polyhedron

Given a graph $G = (V, E)$, define the *T-join polytope* $\mathrm{P}^T_{\mathrm{join}}(G) \subseteq \mathbb{R}^E$ for some even subset of vertices $T \subseteq V$, as

$$\mathrm{P}^T_{\mathrm{join}}(G) = \mathrm{conv}(\{\chi(T) \in \mathbb{R}^E : T \in \mathcal{J}^T(G)\}) \,,$$

where $\mathcal{J}^T(G)$ denotes the set of all *T-joins* in the graph G, i.e. the set of all $R \subseteq E$, such that

$$T = \{v \in V : |\delta(v) \cap R| \text{ is odd}\} \,.$$

The T-join polytope $\mathrm{P}^T_{\mathrm{join}}(G)$ can be described by $0 \leq x \leq 1$ and the following linear inequalities (see Edmonds and Johnson [1973], Schrijver [2003a])

$$x(\delta(S) \setminus F) - x(F) \geq 1 - |F| \quad \text{for} \quad S \subseteq V,\ F \subseteq \delta(S),\ |S \cap T| + |F| \text{ is odd} \,.$$

Note that the perfect matching polytope for the graph $G = (V, E)$ is a face of the V-join polytope $\mathrm{P}^V_{\mathrm{join}}(G)$, where the face is defined by the equations

$$x(\delta(v)) = 1 \quad \text{for } v \in V \,.$$

The *T-join polyhedron* is the polyhedron

$$\mathrm{P}^T_{\mathrm{join}}(G) + \mathbb{R}^E_+ \,.$$

Moreover, the linear inequalities

$$x(\delta(S)) \geq 1 \qquad \text{for } S \subseteq V,\ |S \cap T| \text{ is odd} \,, \tag{3.2.1}$$

together with the non-negativity constraints $x \geq 0$, describe the T-join polyhedron (see Edmonds and Johnson [1973]).

Similarly, the perfect matching polytope is the face of the V-join polyhedron defined by

$$x(\delta(v)) = 1 \quad \text{for } v \in V \,.$$

Thus, every extension $Q \subseteq \mathbb{R}^d$, $p : \mathbb{R}^d \to \mathbb{R}^E$ of the V-join polyhedron (or the V-join polytope) provides an extension of the perfect matching polytope via the polyhedron

$$Q \cap \{z \in \mathbb{R}^d : x = p(z)\,, x(\delta(v)) = 1 \text{ for } v \in V\}$$

and the affine map $p : \mathbb{R}^d \to \mathbb{R}^E$. Thus, in order to construct an extended formulation for the perfect matching polytope, one can focus on extended formulations for the T-join polyhedron or T-join polytope.

3.2.1. Vector Spaces. Consider an embedding of a connected graph $G = (V, E)$ on a surface $\mathcal{S}$, such that [1]

$$\chi(G) = \chi(\mathcal{S}) \,,$$

and the corresponding dual graph $G^* = (V^*, E^*)$, where G^* may have loops and parallel edges. Here and later, identify the edges of the graph G with the edges of the graph G^*.

Consider two vector spaces, defined over the Galois field $\mathrm{GF}(2)$ as

$$\mathcal{V} = \{y \in \mathrm{GF}(2)^{E^*} : y = \chi(C),\ C \subseteq E^*,\ |C \cap \delta(v)| \text{ is even for all } v \in V^*\}$$

and

$$\mathcal{W} = \{y \in \mathrm{GF}(2)^E : y = \chi(\delta(S)),\ S \subseteq V\} \,.$$

[1] In this chapter, $\chi(G)$ stands for the Euler characteristic of a graph G, what applies also to a surface $\mathcal{S}$. In other cases, $\chi(S)$ denotes the characteristic vector of S, as it was in the previous chapters.

Here adding a loop $\{v, v\} \in E^*$ to $C \subseteq E^*$ does not affect the parity of $|C \cap \delta(v)|$ for every $v \in V^*$.

Indeed, it is not hard to see that $\mathcal{V}$ forms a vector space over $\mathrm{GF}(2)$. On the other hand, $\mathcal{W}$ is a vector space over $\mathrm{GF}(2)$, since the sum of every two vectors from $\mathcal{W}$ belongs to $\mathcal{W}$, what is a fundamental property of cuts in a graph.

Clearly, the vector space $\mathcal{W}$ is a subspace of the vector space $\mathcal{V}$, because the cardinality of the intersection of every cycle with a cut is even. And, the dimensions of the vector spaces $\mathcal{V}$ and $\mathcal{W}$ over $\mathrm{GF}(2)$ satisfy

$$\dim(\mathcal{V}) = |E^*| - |V^*| + 1 = |E| - |F| + 1$$

and

$$\dim(\mathcal{W}) = |V| - 1\,,$$

since the graph G is connected. To verify this, one can construct basis vectors for $\mathcal{V}$ and $\mathcal{W}$. For the vector space $\mathcal{W}$, take $\chi(\delta(v))$ as basis vectors, where the vertex v ranges over V except a fixed vertex v_{inf}. For the vector space $\mathcal{V}$, take $\chi(C_e)$, where the edge e ranges over E^* except the edges of a fixed spanning tree T for the graph G^*, and C_e is the cycle in G^*, defined by the edge e and the tree T (loops are considered to be cycles).

From the Euler Formula, for the dimensions of the vector spaces $\mathcal{W}$ and $\mathcal{V}$, one has

$$\dim(\mathcal{V}) - \dim(\mathcal{W}) = 2 - (|F| - |E| + |V|) = 2 - \chi(\mathcal{S})\,, \tag{3.2.2}$$

what, for simplicity, is denoted as

$$k = 2 - \chi(\mathcal{S}) = 2\gamma(\mathcal{S})\,.$$

Consequently, there exists a linear function $\phi : \mathrm{GF}(2)^E \to \mathrm{GF}(2)^k$, such that

$$\mathcal{W} = \{y \in \mathcal{V} : \phi(y) = \mathbf{0}_k\}\,. \tag{3.2.3}$$

Moreover, there exists another linear function $\psi : \mathrm{GF}(2)^E \to \mathrm{GF}(2)$, such that

$$\{y \in \mathrm{GF}(2)^E : y = \chi(\delta(S)),\ S \subseteq V,\ |S \cap T| \text{ is odd}\} = \{y \in \mathcal{V} : \phi(y) = \mathbf{0}_k, \psi(y) = 1\}\,. \tag{3.2.4}$$

For example, the linear function $\psi : \mathrm{GF}(2)^E \to \mathrm{GF}(2)$, defined as

$$\psi(y) = y(R)\,,$$

where $R \subseteq E$ is a T-join in the graph G, satisfies (3.2.4). Indeed, for a vector $y \in \mathrm{GF}(2)^E$, given as

$$y = \chi(\delta(S))\,,$$

where $S \subseteq V$, over $\mathrm{GF}(2)$ the following holds

$$y(R) = \sum_{v \in S} |\delta(v) \cap R| = \sum_{v \in T \cap S} |\delta(v) \cap R| = |T \cap S|\,,$$

because R is a T-join in the graph G.

3.2.2. Extended Formulation of T-join Polyhedron. Now, the modified extended formulation of Gerards [1991] for the T-join polyhedron is presented [1]. The variables in the extended formulation of the T-join polyhedron are indexed by triples

$$\{(\varnothing, f, g) : f \in \mathrm{GF}(2)^k, g \in \mathrm{GF}(2)\}$$

and

$$\{(\{v, u\}, f, g) : v, u \in V^*, f \in \mathrm{GF}(2)^k, g \in \mathrm{GF}(2)\}\,.$$

[1] In Gerards [1991] , the extended formulation is divided into a set of linear systems, what results in a formulation of size $O(2^{k^2} + 2^k|V^*||E|)$, in comparison to the size $O(2^k|V^*||E|)$ of the presented formulation. Nevertheless, the fundamental ideas of both formulations are similar.

Consider the following system of linear inequalities[1]

$$z_{\{v,u\},f+\phi(\{v,u\}),g+\psi(\{v,u\})} - z_{\varnothing,f,g} \leq x_{\{v,u\}} \tag{3.2.5}$$

$$z_{\{v,w\},f+\phi(\{v,u\}),g+\psi(\{v,u\})} - z_{\{u,w\},f,g} \leq x_{\{v,u\}} \tag{3.2.6}$$

$$z_{\varnothing,f+\phi(\{v,u\}),g+\psi(\{v,u\})} - z_{\{u,v\},f,g} \leq x_{\{v,u\}}\,, \tag{3.2.7}$$

for all $f \in \mathrm{GF}(2)^k$, $g \in \mathrm{GF}(2)$ and for all distinct vertices v, w, $u \in V^*$, such that $\{v,u\} \in E^*$, and

$$z_{\varnothing,f+\phi(\{v,v\}),g+\psi(\{v,v\})} - z_{\varnothing,f,g} \leq x_{\{v,v\}} \tag{3.2.8}$$

$$z_{\{v,u\},f+\phi(\{v,v\}),g+\psi(\{v,v\})} - z_{\{v,u\},f,g} \leq x_{\{v,v\}}\,, \tag{3.2.9}$$

for all $f \in \mathrm{GF}(2)^k$, $g \in \mathrm{GF}(2)$ and for all distinct vertices v, $u \in V^*$, such that $\{v,v\} \in E^*$, and

$$z_{\varnothing,\mathbf{0}_k,0} = 0 \quad \text{and} \quad z_{\varnothing,\mathbf{0}_k,1} \geq 1 \tag{3.2.10}$$

$$0 \leq x\,. \tag{3.2.11}$$

In the linear system above, the expressions $\phi(\{v,u\})$ and $\psi(\{v,u\})$ are used as shortcuts for $\phi(\chi(\{v,u\}))$ and $\psi(\chi(\{v,u\})$, respectively.

Theorem 3.1. *For every connected graph $G = (V,E)$, the linear system, described by* (3.2.5) – (3.2.11)*, together with the projection on the x variables, forms an extended formulation of the T-join polyhedron.*

PROOF. First, prove that for every point $x \in \mathbb{R}^E$ from the T-join polyhedron in the graph G, there are z variables to satisfy (3.2.5) – (3.2.11). For this, define z variables as follows

$$z_{\varnothing,f,g} = \min_{\substack{R \subseteq E \\ \chi(R) \in \mathcal{V} \\ \phi(\chi(R))=f,\psi(\chi(R))=g}} x(R)\,,$$

for all $f \in \mathrm{GF}(2)^k$, $g \in \mathrm{GF}(2)$, and

$$z_{\{v,u\},f,g} = \min_{\substack{R \in \mathcal{J}^{\{u,v\}}(G^*) \\ \phi(\chi(R))=f,\psi(\chi(R))=g}} x(R)\,,$$

for all distinct $u, v \in V^*$ and for all $f \in \mathrm{GF}(2)^k$, $g \in \mathrm{GF}(2)$, where adding a loop $\{w,w\} \in E^*$ to $R \subseteq E^*$ does not affect the parity of $|R \cap \delta(w)|$ for every $w \in V^*$ (the value is zero, if the set over which the minimum is defined is empty). Obviously, the constraints (3.2.5)– (3.2.9) are satisfied for the defined z values. And the constraint (3.2.10)

$$z_{\varnothing,\mathbf{0}_k,1} \geq 1$$

holds, since

$$z_{\varnothing,\mathbf{0}_k,1} = \min_{\substack{R \subseteq E \\ \chi(R) \in \mathcal{V} \\ \phi(\chi(R))=\mathbf{0}_k,\psi(\chi(R))=1}} x(R) = \min_{\substack{S \subseteq V \\ R=\delta(S) \\ |S \cap T| \text{ is odd}}} x(R) \geq 1\,,$$

due to (3.2.1) and (3.2.4). Moreover, the variable $z_{\varnothing,\mathbf{0}_k,0}$ is equal to zero, due to the non-negativity of the vector x.

On the other hand, for every $x \in \mathbb{R}^E$, such that there are z values, satisfying the linear system (3.2.5) – (3.2.11), the inequalities (3.2.1) hold, i.e.

$$x(\delta(S)) \geq 1$$

---

[1] This linear system imitates a network, with the vertex set equal to the index set of the additional variables in the extended formulation, where the source s and sink t in the network are defined as

$$s = (\varnothing, \mathbf{0}_k, 0) \quad \text{and} \quad t = (\varnothing, \mathbf{0}_k, 1)\,.$$

The extended formulation, describes upper bounds on the lengths of a subset of arcs in the network, and demands that the shortest path between the source s and the sink t is not less than one.

for every $S \subseteq V$, such that $|S \cap T|$ is odd. Indeed, the characteristic vector

$$y = \chi(\delta(S))$$

belongs to

$$\{r \in \mathcal{V} : \phi(r) = \mathbf{0}_k, \psi(r) = 1\}.$$

The vector y defines a set of walks in the graph G^*, since this is true for all vectors from $\mathcal{V}$. Ordering these edge disjoint walks $(e_1^1, \dots, e_{\ell_1}^1)$, ..., $(e_1^t, \dots, e_{\ell_t}^t)$, let us write these edges in one sequence

$$(e_1, \dots, e_\ell) = (e_1^1, \dots, e_{\ell_1}^1, e_1^2, \dots, e_{\ell_{t-1}}^{t-1}, e_1^t, \dots, e_{\ell_t}^t)$$

in the same order as they appear in the walk. Summing the inequalities (3.2.5) – (3.2.10), depending on the values

$$f = \phi\big(\chi(\cup_{i \le j} e_i)\big) \quad \text{and} \quad g = \psi\big(\chi(\cup_{i \le j} e_i)\big)$$

and at most two vertices u, $v \in V^*$, with odd degrees in $\cup_{i \le j} e_i$, the desired inequality

$$1 \le z_{\varnothing, \mathbf{0}_k, 1} - z_{\varnothing, \mathbf{0}_k, 0} \le x(\delta(S))$$

is obtained. □

The constructed extension of the T-join polyhedron has size $O(2^k |E^*||V^*|)$. This, due to (3.2.2), leads us to the next generalization of the theorem of Gerards.

Theorem 3.2. *For every graph $G = (V, E)$, $|V| = n$, and $T \subseteq V$, there exists a compact extended formulation for the T-join polyhedron, whenever the genus $\gamma(G)$ is equal to $O(\log n)$.*

Here, one can get rid of the connectivity condition, since an extended formulation can be constructed for every connected component of the graph G, separately.

Recall that the perfect matching polytope for a graph $G = (V, E)$ is a face of the V-join polyhedron, what implies the next result.

Proposition 3.1. *For every graph $G = (V, E)$, $|V| = n$, there exists a compact extended formulation for the perfect matching polytope, whenever the genus $\gamma(G)$ is equal to $O(\log n)$.*

Moreover, since the inequality $|E| \le 3|V| - 6$ holds for every planar graph $G = (V, E)$, $|V| \ge 3$, the next proposition follows from the Euler Formula.

Proposition 3.2 (Gerards [1991]). *For every planar graph $G = (V, E)$, $|V| = n$, there exists an extended formulation for the perfect matching polytope of size $O(n^3)$.*

3.2.3. Construction of Extended Formulation. In the presented construction, it is necessary to obtain the dual graph G^* from an embedding of the graph G on a surface $\mathcal{S}$, where the genus $\gamma(\mathcal{S})$ is equal to $\gamma(G)$, what can be done in $O(|V|^{O(\gamma(G))})$ time, due to Filotti et al. [1979]. The map $\phi : \mathrm{GF}(2)^E \to \mathrm{GF}(2)^k$ can be obtained in polynomial running time, using the basis vectors for $\mathcal{V}$, $\mathcal{W}$, and the map $\psi : \mathrm{GF}(2)^E \to \mathrm{GF}(2)$ can be obtained in polynomial running time, since it is enough to find a T-join in the graph G, what can be done in $O(|V|^3)$ running time (see Schrijver [2003a]).

3.3. Extended Formulation of Cut Polytope in Planar Graphs

The next extended formulation of the perfect matching polytope in planar graphs, is due to Barahona [1993], and is obtained via an extended formulation for the *cut polytope* $\mathrm{P}_{\mathrm{cut}}(G)$, $G = (V, E)$, which is defined as

$$\mathrm{P}_{\mathrm{cut}}(G) = \mathrm{conv}(\{\chi(\delta(S)) \in \mathbb{R}^E : S \subseteq V\})$$

Clearly, the inequalities

(3.3.1) $x(F) - x(C \setminus F) \le |F| - 1 \quad$ for $C \in \mathcal{C}(G),\ F \subseteq C,\ |F|$ is odd

(3.3.2) $0 \le x \le 1$

are valid for every point x from the cut polytope $\mathrm{P}_{\mathrm{cut}}(G)$, since the cardinality of the intersection of a cut and a cycle is even. But, the linear system (3.3.1), (3.3.2) defines the cut polytope $\mathrm{P}_{\mathrm{cut}}(G)$ if and only if G is not contractible to K_5 (see Barahona and Mahjoub [1986]).

For further discussion, define $Q(G) \subseteq \mathbb{R}^E$ to be the polytope, described by the linear system (3.3.1), (3.3.2).

3.3.1. Projecting Linear System. The relation between the linear systems (3.3.1), (3.3.2) for a graph and its subgraph is studied here.

Lemma 3.1. *For every graph $G = (V, E)$, the polytope $Q(G') \subseteq \mathbb{R}^E$, where $G' = (V, E')$, $E' = E \setminus \{e\}$ for some edge $e \in E$, is obtained from the polytope $Q(G)$, by the projection to the variables E'.*

PROOF. The projection of the polytope $Q(G)$ to the variables E' satisfies the linear system (3.3.1), (3.3.2) for the graph G', since the set of cycles $\mathcal{C}(G')$ is a subset of the set of cycles $\mathcal{C}(G)$. To finish the proof of the lemma, it is necessary to show that all inequalities, which are valid for the projection of $Q(G)$ to the variables E', follow from the linear system (3.3.1), (3.3.2) for the graph G'. For this, use the Fourier-Motzkin elimination method, what leads to three possible cases.

For $x_e \le 1$ and $x(F) - x(C \setminus F) \le |F| - 1$, where $e \in C \setminus F$, $C \in \mathcal{C}(G)$, $F \subseteq C$, $|F|$ is odd, one gets

$$x(F) - x(C \setminus (F \cup \{e\})) \le |F|\,,$$

what follows from (3.3.2) for G'.

For $-x_e \le 0$ and $x(F) - x(C \setminus F) \le |F| - 1$, where $e \in F$, $C \in \mathcal{C}(G)$, $F \subseteq C$, $|F|$ is odd, one gets

$$x(F \setminus \{e\}) - x(C \setminus F) \le |F| - 1\,,$$

what follows from (3.3.2) for G'.

Considering $x(F_1) - x(C_1 \setminus F_1) \le |F_1| - 1$ and $x(F_2) - x(C_2 \setminus F_2) \le |F_2| - 1$, where $e \in F_1$, $e \in C_2 \setminus F_2$, C_1, $C_2 \in \mathcal{C}(G)$, $F_1 \subseteq C_1$, $F_2 \subseteq C_2$, $|F_1|$ and $|F_2|$ are odd, one gets

$$x(F_1) - x(C_1 \setminus F_1) + x(F_2) - x(C_2 \setminus F_2) \le |F_1| + |F_2| - 2\,,$$

which can be transformed into the inequality

$$x(F_1 \setminus \{e\}) + x(F_2) - x(C_1 \setminus F_1) - x(C_2 \setminus (F_2 \cup \{e\})) \le |F_1 \setminus \{e\}| + |F_2| - 1\,,$$

what follows from (3.3.1), (3.3.2) for G'. Indeed, in the case of

$$F_1 \cap (C_2 \setminus F_2) \ne \{e\} \quad \text{or} \quad F_2 \cap (C_1 \setminus F_1) \ne \varnothing\,,$$

the above inequality follows from (3.3.2) for G'. Otherwise,

$$(F_1 \setminus \{e\}) \bigtriangleup F_2 \subseteq C_1 \bigtriangleup C_2\,,$$

and the desired inequality follows from (3.3.2) and (3.3.1), taken for a cycle C from $C_1 \bigtriangleup C_2$, such that the cardinality of $F = C \cap ((F_1 \setminus \{e\}) \bigtriangleup F_2)$ is odd. Note that $C_1 \bigtriangleup C_2$ is a union of edge disjoint cycles, and the cardinality of $(F_1 \setminus \{e\}) \bigtriangleup F_2$ is odd. □

3.3.2. Redundant Inequalities.

Lemma 3.2. *For every cycle $C \in \mathcal{C}(G)$, which has a chord $e \in E$, the inequalities* (3.3.1) *for the cycle C*

$$x(F) - x(C \setminus F) \le |F| - 1 \quad \text{for } F \subseteq C,\ |F| \text{ is odd}$$

are implied by (3.3.1) *for two cycles C_1, $C_2 \in \mathcal{C}(G)$ such that*

$$C_1 \cup C_2 = C \cup \{e\} \quad \text{and} \quad C_1 \cap C_2 = \{e\}\,.$$

PROOF. Consider $F \subseteq C$, such that the cardinality of F is odd. Assume that the cardinality of $F_1 = C_1 \cap F$ is odd. Define $F_2 = (F \cap C_2) \cup \{e\}$, the cardinality of which is odd, too. Adding two inequalities

$$x(F_1) - x(C_1 \setminus F_1) \le |F_1| - 1 \quad \text{and} \quad x(F_2) - x(C_2 \setminus F_2) \le |F_2| - 1\,,$$

the inequality (3.3.1) for the cycle C and the set F is obtained. □

Lemma 3.3. *For every cycle $C \in \mathcal{C}(G)$, $|C| = 3$, the inequalities*

$$0 \le x_e \le 1 \quad \text{for } e \in C$$

are implied by the inequalities (3.3.1) *for the cycle C.*

PROOF. Let the cycle C be given as $\{v, u, w\}$, $v, u, w \in V$, then there are four inequalities (3.3.1), associated with the cycle C

$$x_{v,u} + x_{v,w} + x_{u,w} \le 2 \tag{3.3.3}$$
$$x_{v,u} - x_{v,w} - x_{u,w} \le 0 \tag{3.3.4}$$
$$x_{v,w} - x_{v,u} - x_{u,w} \le 0 \tag{3.3.5}$$
$$x_{u,w} - x_{v,u} - x_{v,w} \le 0\,. \tag{3.3.6}$$

From (3.3.3), (3.3.4), get $x_{v,u} \le 1$. On the other hand, the inequality $x_{v,u} \ge 0$ follows from (3.3.5), (3.3.6). □

3.3.3. Extended Formulation of Cut Polytope. Due to Lemma 3.1, for every graph $G = (V, E)$, $|V| = n$, the polytope $Q(G)$ is obtained from the polytope $Q(K_n)$ by the projection on the variables E.

From Lemmas 3.2 and 3.3, the polytope $Q(K_n)$ is described by

$$\begin{aligned} x_{v,u} + x_{v,w} + x_{u,w} &\le 2 && \text{for } v, u, w \in V \\ x_{v,u} - x_{v,w} - x_{u,w} &\le 0 && \text{for } v, u, w \in V\,, \end{aligned}$$

since in the complete graph K_n every cycle with more than three edges has a chord. Thus, the above linear system, together with the projection on the variables E, forms an extended formulation of the polytope $Q(G)$, what for graphs not contractible to K_5 is equal to the cut polytope (see Barahona and Mahjoub [1986]).

Theorem 3.3 (Barahona [1993])**.** *For every graph $G = (V, E)$, $|V| = n$, which is not contractible to K_5, there exists an extended formulation of the cut polytope $\mathrm{P}_{\mathrm{cut}}(G)$ of size $O(n^3)$.*

And since every planar graph is not contractible to K_5, an extended formulation of the cut polytope for planar graphs is obtained.

Proposition 3.3 (Barahona [1993])**.** *For every planar graph $G = (V, E)$, $|V| = n$, there exists an extended formulation of the cut polytope $\mathrm{P}_{\mathrm{cut}}(G)$ of size $O(n^3)$.*

3.3.4. Extended Formulation for T-join Polytope and Perfect Matching Polytope. Let us consider a planar graph $G = (V, E)$ and its dual $G^* = (V^*, E^*)$, which may have loops and parallel edges.

Every T-join in G can be obtained from any other T-join as the symmetric difference with a union of edge disjoint cycles. On the other hand, the symmetric difference of a T-join and a union of edge disjoint cycles is again a T-join. Additionally, notice that the set of unions of edge disjoint cycles in G corresponds to the set of cuts in the dual graph G^*.

Thus, having an extended formulation $Q \subseteq \mathbb{R}^d$, $p : \mathbb{R}^d \to \mathbb{R}^{E^*}$ of the cut polytope $\mathrm{P}_{\mathrm{cut}}(G^*)$, the polyhedron $Q \subseteq \mathbb{R}^d$ and the affine map $p' : \mathbb{R}^d \to \mathbb{R}^E$

$$p'_e(z) = \begin{cases} 1 - p_e(z) & \text{if } e \in R \\ p_e(z) & \text{otherwise} \end{cases}$$

form an extended formulation of the T-join polytope $\mathrm{P}^T_{\mathrm{join}}(G)$, where R is a T-join in the graph G. Due to Proposition 3.3, an extended formulation of the T-join polytope is obtained[1].

Proposition 3.4 (Barahona [1993])**.** *For every planar graph $G = (V, E)$, $|V| = n$, and $T \subseteq V$, there exists an extended formulation for the T-join polytope $\mathrm{P}^T_{\mathrm{join}}(G)$ of size $O(n^3)$.*

And since the perfect matching polytope for a graph G is a face of $\mathrm{P}^V_{\mathrm{join}}(G)$, Proposition 3.2 is reproved.

3.3.5. Construction of Extended Formulation. Note that for the construction of the extended formulation for the cut polytope in planar graphs it is not necessary to consider a dual graph of G. On, the other hand in the case of the T-join polytope and the perfect matching polytope, a dual graph of a planar graph is needed, which can be constructed in polynomial time (see Filotti et al. [1979]). Additionally, for the T-join polytope and the perfect matching polytope, it is necessary to find a T-join in the graph G, what can be done in polynomial running time as well (see Schrijver [2003a]).

3.4. Spanning Tree Polytope

In this section, an extended formulation of the spanning tree polytope for planar graphs is presented, which was proved by Williams [2002] and independently reproved by us.

Consider a planar graph $G = (V, E)$ and its dual graph $G^* = (V^*, E^*)$. Fix a face $v^*_{\inf} \in V^*$ and a vertex $v_{\inf} \in V$, which belongs to the face $v^*_{\inf}$, and define the following linear system [2]

$$x_e + z_{e,v} + z_{e,u} = 1 \quad \text{for } v, u \in V^*,\ e = \{v, u\} \in E^* \tag{3.4.1}$$

$$(1 - x_e) + y_{e,v} + y_{e,u} = 1 \quad \text{for } v, u \in V,\ e = \{v, u\} \in E \tag{3.4.2}$$

---

[1]Note that in Proposition 3.3 the graph G is a simple graph, i.e. G does not have loops and parallel edges. On the other hand, if G has loops or parallel edges, the cut polytope $\mathrm{P}_{\mathrm{cut}}(G)$ arises from the cut polytope for the simple graph, obtained from G by deleting all loops and leaving one edge for every set of parallel edges. Indeed, the variables corresponding to loops are equal to zero for all points from $\mathrm{P}_{\mathrm{cut}}(G)$, and the variables, corresponding to a set of parallel edges are equal for every point from $\mathrm{P}_{\mathrm{cut}}(G)$.

[2]For constraints (3.4.3), (3.4.5), every loop appears once in the sum over $\delta(v)$, $v \in V^*$. The same is true for the constraints (3.4.1). Actually, it is not critical, since the system gives an extended formulation for the spanning tree polytope, even if every loop is written twice, but in this case, we can not use the totally unimodularity of the constraint matrix. Moreover, for every loop the variable z is equal to zero in the provided extension of the spanning tree polytope, since every loop corresponds to some bridge, which participates in every spanning tree.

$$\sum_{e\in\delta(v)} z_{e,v} = 1 \quad \text{for} \quad v \in V^* \setminus \{v^*_{\text{inf}}\} \tag{3.4.3}$$

$$\sum_{e\in\delta(v)} y_{e,v} = 1 \quad \text{for} \quad v \in V \setminus \{v_{\text{inf}}\} \tag{3.4.4}$$

$$\sum_{e\in\delta(v)} z_{e,v} = 0 \quad \text{for} \quad v = v^*_{\text{inf}} \tag{3.4.5}$$

$$\sum_{e\in\delta(v)} y_{e,v} = 0 \quad \text{for} \quad v = v_{\text{inf}} \tag{3.4.6}$$

and the non-negativity constraints

$$z \geq 0 \quad \text{and} \quad y \geq 0\,. \tag{3.4.7}$$

In this section it is shown that the linear system (3.4.1) – (3.4.7) and the projection on x variables define an extended formulation of the spanning tree polytope $\mathrm{P}_{\text{spt}}(G)$.

To prove this, use the fact that for every planar graph G and its dual G^* spanning trees in G are associated with spanning trees in G^* in the following manner

$$\mathcal{T}(G) = \{E \setminus T : T \in \mathcal{T}(G^*)\}\,.$$

Lemma 3.4. *For every planar graph $G = (V, E)$ and its dual $G^* = (V^*, E^*)$, the linear system* (3.4.1) – (3.4.7)*, together with the projection map on x variables, forms an extended formulation of the spanning tree polytope* $\mathrm{P}_{\text{spt}}(G)$.

PROOF. First, show that for every vertex x of the spanning tree polytope $\mathrm{P}_{\text{spt}}(G)$, where $x = \chi(T)$, $T \in \mathcal{T}(G)$, there are z, y variables, satisfying the above linear system. To define these variables, consider arborescences N, $N^* \subseteq E$, defined by the tree T in G and the tree $E \setminus T$ in G^*, rooted at the vertices v_{inf} and v^*_{inf}, respectively. The variable $y_{e,v}$, where $v \in V$ and $e = \{v, u\}$, is defined to be equal one, if the arc (u, v) belongs to the arborescence N. Analogously, define z variables for the arborescence N^*.

On the other hand, it is necessary to show that for every vertex (x, y, z) of the polytope, defined by the linear system (3.4.1) – (3.4.7), the point x belongs to the spanning tree polytope $\mathrm{P}_{\text{spt}}(G)$. For this, note that the linear system is totally unimodular, due to the Ghouila-Houri characterization of totally unimodular matrices (z, y variables participate once in (3.4.1), (3.4.2), and once in (3.4.3) – (3.4.6), and x variables participate once in (3.4.1) and once in (3.4.2), but with opposite coefficients). And thus, the vertices of the polytope, defined by the system (3.4.1) – (3.4.7), have zero-one components, i.e. $x = \chi(T)$ for some $T \subseteq E$.

Thus, it is necessary to show that T is a tree in the graph G. For this, define N, $N^* \subseteq E$ as follows

$$N = \{(u, v) \in V \times V : y_{e,v} = 1, e = \{u, v\}, e \in E\}$$

and

$$N^* = \{(u, v) \in V^* \times V^* : z_{e,v} = 1, e = \{u, v\}, e \in E\}\,.$$

Due to (3.4.3), (3.4.4), for every vertex from $V \setminus \{v_{\text{inf}}\}$, $V^* \setminus \{v^*_{\text{inf}}\}$ there is exactly one ingoing arc in N, N^*, respectively. And due to (3.4.5), (3.4.6), in N, N^* there exists no ingoing arc for v_{inf}, v^*_{inf}, respectively.

Hence, it is enough to prove that N does not contain a directed cycle (every cycle in N or N^* is a directed). For this, define the *interior for a cycle* in G, G^*, as that one of the two regions, defined by the cycle, which does not contain v^*_{inf}, v_{inf}, respectively. To make the definition consistent, fix an embedding of the graph G on a surface $\mathcal{S}$, what induces the dual graph G^*.

Whenever N contains a directed cycle C_1, then due to (3.4.1), (3.4.2), all variables z, involving the edges of the cycle C_1 are equal to zero. Hence, there exists no arcs in N^*, between the faces from the interior of the cycle C_1 in the graph G and the faces from the exterior of the cycle C_1. And since v^*_{inf} does not lie in the interior of the cycle C_1, there exists a directed cycle C^*_1 in N^*, lying in the interior of the cycle C_1 in G, because for every vertex $v \in V^*$ from the interior of C_1 there is an ingoing arc. Due to (3.4.1), (3.4.2), all variables y, involving the edges of the cycle C^*_1 are equal to zero. Analogously, there exists a cycle C_2 in N, which lies in the interior of the cycle C^*_1. Iterating this, one obtains a set of distinct cycles $C_i \in \mathcal{C}(G)$, $i \in \mathbb{N}$, since every cycle from the sequence lies strictly inside the preceding cycles (no two cycles have a common edge). But, the number of different cycles in the graph G is finite. □

Theorem 3.4 (Williams [2002])**.** *For every planar graph $G = (V, E)$, there exists an extended formulation for the spanning tree polytope $\mathrm{P}_{\mathrm{spt}}(G)$ of size $4|E|$.*

3.5. Subtour Elimination Polytope

In this section, the subtour elimination polytope $\mathrm{P}_{\mathrm{ste}}(G) \subseteq \mathbb{R}^E$, $G = (V, E)$, $|V| \geq 3$, is considered. Recall the linear description of the subtour elimination polytope

$$x(E(S)) \leq |S| - 1 \quad \text{for all} \quad \varnothing \neq S \subsetneq V \tag{3.5.1}$$

$$x(\delta(v)) = 2 \quad \text{for all} \quad v \in V \tag{3.5.2}$$

$$0 \leq x\,. \tag{3.5.3}$$

As we saw in Section 2.3, there exists an extended formulation for the subtour elimination polytope $\mathrm{P}_{\mathrm{ste}}(G)$ of size $O(|V||E|)$ (see Yannakakis [1991]). Here, a more compact extended formulation is constructed, exploiting the planarity. For this, fix a vertex $v_{\mathrm{inf}} \in V$ and a face v^*_{inf}, such that the face v^*_{inf} contains the vertex $v_{\mathrm{inf}} \in V$. Note that in Sections 3.5.1, 3.5.2, graphs are not restricted to be planar.

3.5.1. Redundant Inequalities.

Lemma 3.5. *For every graph $G = (V, E)$, the inequalities* (3.5.2), (3.5.3)*, and the inequalities* (3.5.1) *for vertex sets $S \subseteq V$, such that the induced subgraphs $G(S)$, $G(V \setminus S)$ are connected and the vertex set S does not contain the vertex v_{inf}, form a linear description of the subtour elimination polytope* $\mathrm{P}_{\mathrm{ste}}(G)$.

PROOF. In the above system, every inequality (3.5.1), indexed by a set S is equivalent to the inequality $x(\delta(S)) \geq 2$, since

$$x(\delta(S)) = \sum_{v \in S} x(\delta(v)) - 2x(E(S)) = 2|S| - 2x(E(S))\,.$$

Thus, these constraints can be excluded, where the set S contains the vertex v_{inf}, because the constraints (3.5.1) for the set S and the set $V \setminus S$ are equivalent.

Now, if $G(S)$ is not connected, there are two sets of vertices S_1, $S_2 \subseteq V$

$$S = S_1 \cup S_2 \qquad \text{where} \quad S_1, S_2 \neq \varnothing,\ S_1 \cap S_2 = \varnothing, \delta(S_1) \cap \delta(S_2) = \varnothing\,,$$

then the inequalities (3.5.1) for S_1, S_2 imply the inequality (3.5.1) for the set S, since

$$x(E(S)) = x(E(S_1)) + x(E(S_2)) \leq |S_1| - 1 + |S_2| - 1 = |S| - 2 < |S| - 1\,.$$

Thus, the inequality (3.5.1) for the set S is not tight for the subtour elimination polytope $\mathrm{P}_{\mathrm{ste}}(G)$. Analogously, one treats the case, when $G(V \setminus S)$ is not connected. □

3.5.2. Extended Formulation via Spanning Tree Polytope. Here, an extended formulation of the subtour elimination polytope via the spanning tree polytope is presented. From Lemma 3.5 and from the linear description of Edmonds [1971] for the spanning tree polytope $\mathrm{P}_{\mathrm{spt}}(G)$

$$x(E(S)) \leq |S| - 1 \quad \text{for all} \quad \varnothing \neq S \subseteq V$$

$$0 \leq x \quad \text{and} \quad x(E) = V - 1\,,$$

the next result follows[1].

Lemma 3.6 (Schrijver [2003b])**.** *For every graph $G = (V, E)$, the following linear system*

$$\operatorname{proj}_{E'} x \in \mathrm{P}_{\mathrm{spt}}(G') \quad \textit{and} \qquad 0 \le x$$
$$x(\delta(v)) = 2 \quad \textit{for} \quad v \in V\,,$$

where $G' = (V', E')$, $V' = V \setminus \{v_{\inf}\}$, $E' = E \setminus \delta(v_{\inf})$, defines the subtour elimination polytope $\mathrm{P}_{\mathrm{ste}}(G)$.

For planar graphs, Lemmas 3.4 and 3.6 provide us with an extended formulation for the polytope $\mathrm{P}_{\mathrm{ste}}(G)$ of size at most $4|E|$. But there is a more compact extended formulation of the polytope $\mathrm{P}_{\mathrm{ste}}(G)$, which is constructed in the next section.

Lemma 3.7. *For every planar graph $G = (V, E)$, there exists an extended formulation for the subtour elimination polytope $\mathrm{P}_{\mathrm{ste}}(G)$ of size $4|E|$.*

3.5.3. Extended Formulation for Subtour Elimination Polytope. Now, we construct an extended formulation of the subtour elimination polytope $\mathrm{P}_{\mathrm{ste}}(G)$, using additional variables $z_{e,v}$ for every edge $e \in E$, $v_{\inf} \notin e$, and face $v \in V^*$, such that $v \in e$. Consider the following linear system

$$x_e + z_{e,v} + z_{e,u} = 1 \qquad \text{for} \quad v_{\inf} \notin e = \{v, u\},\ v, u \in V^* \tag{3.5.4}$$

$$\sum_{e \in \delta(v)} z_{e,v} = 1 \qquad \text{for} \quad v \in V^*,\ v_{\inf} \notin v \tag{3.5.5}$$

and

$$x(\delta(v)) = 2 \quad \text{for all} \quad v \in V \qquad \text{and} \qquad z \ge 0\,, \quad x \ge 0 \tag{3.5.6}$$

together with the projection map on x variables [2].

The presented extended formulation generalizes the extended formulation of Rivin [1996] (see also Rivin [2003], Cheung [2003]). But whenever the graph defines a triangulation, our extension is identical to the extension of Rivin [1996].

Lemma 3.8. *For every planar graph $G = (V, E)$, the linear system* (3.5.4) – (3.5.6), *together with the projection map on x variables, defines an extended formulation of the subtour elimination polytope $\mathrm{P}_{\mathrm{ste}}(G)$.*

PROOF. We want, to show that every point $x \in \mathbb{R}^E$, for which there are z variables, satisfying (3.5.4) – (3.5.6), belongs to the subtour elimination polytope, i.e. to show that x satisfies (3.5.1) – (3.5.3). Fix the embedding of the graph G on the plane.

Due to Lemma 3.5, the vertex $v_{\inf}$ is assumed to be not in the vertex set S for the inequalities (3.5.1) and the induced graphs $G(S)$, $G(V \setminus S)$ are connected. We sum the equations (3.5.4), indexed by the edges $E(S)$

$$|E(S)| = \sum_{e \in E(S)} (x_e + \sum_{v \in V^*,\, v \in e} z_{e,v}) = x(E(S)) + \sum_{e \in E(S)} \sum_{v \in V^*,\, v \in e} z_{e,v}\,,$$

and consider the induced subgraph $G(S) = (S, E(S))$ with the embedding, inherited from the graph G. Let the set F' denote the faces, defined by the embedding of $G(S)$ into the

---

[1] This may be be also proved using the notion of $v_{\inf}$-tree (see Schrijver [2003b]).

[2] For constraints (3.5.5), every loop appears once in the sum over $\delta(v)$, $v \in V^*$. The same is true for the constraints (3.5.4). Note that if the dual graph has loops then the subtour elimination polytope is empty. The same holds, if there exists only one face in the graph G, i.e. G is a forest.

plane, which are also faces for the graph G. Thus,

$$\sum_{e\in E(S)}\sum_{v\in V^*,\, v\in e} z_{e,v} = \sum_{e\in E(S)}\sum_{v\in F',\, v\in e} z_{e,v} + \sum_{e\in E(S)}\sum_{v\in V^*\setminus F',\, v\in e} z_{e,v} \geq \sum_{e\in E(S)}\sum_{v\in F',\, v\in e} z_{e,v} = \sum_{v\in F'}\sum_{e\in\delta(v)} z_{e,v}\,,$$

since $\delta(v) \subseteq E(S)$ for every face $v \in F'$. From (3.5.5), we get the equation

$$\sum_{v\in F'}\sum_{e\in\delta(v)} z_{e,v} = |F'|\,,$$

because the vertex v_{inf} does not belong to S. But note that there can be at most one face among the faces $F(S)$ of the graph $G(S)$, which is not a face of the graph G, otherwise the graph $G(V \setminus S)$ is not connected. Thus,

$$|E(S)| \geq x(E(S)) + |F'| \geq x(E(S)) + |F(S)| - 1\,,$$

and since the graph $G(S)$ is connected, apply the Euler Formula to get the desired inequality $|S| - 1 \geq x(E(S))$.

On the other hand, it is necessary to prove that for every x from $\text{P}_{\text{ste}}(G)$, there exist z variables, satisfying (3.5.4) – (3.5.6). Fix the embedding of the graph G on the plane.

Due to Lemma 3.4 and Lemma 3.6, we can use $z'_{e,v}$ variables from the extended formulation for the subtour elimination polytope, constructed via the extended formulation for the spanning tree polytope. Recall that in this case, we consider the spanning tree polytope $\text{P}_{\text{spt}}(G')$, $G' = G(V \setminus \{v_{\text{inf}}\})$. In the extension of the spanning tree polytope $\text{P}_{\text{spt}}(G')$, choose $v^{*'}_{\text{inf}}$ to be the face of G', defining the region with the vertex v_{inf} inside. Now, set the variables $z_{e,v}$ to the variable $z'_{e,v^{*'}_{\text{inf}}}$ from the extension for the spanning tree polytope $\text{P}_{\text{spt}}(G')$, whenever the vertex v_{inf} belongs to the face v, otherwise set the variable $z_{e,v}$ to the variable $z'_{e,v}$. □

Thus, the linear system (3.5.4) – (3.5.6) provides an extended formulation of the subtour elimination polytope $\text{P}_{\text{ste}}(G)$ of size $3|E|$.

Theorem 3.5. *For every planar graph $G = (V, E)$, there exists an extended formulation for the subtour elimination polytope $\text{P}_{\text{ste}}(G)$ of size $3|E|$.*

CHAPTER 4

Bounds on General Extended Formulations for Polytopes

In this chapter, we describe lower bounds for sizes of extended formulations of polytopes. In 1991, Yannakakis showed that the size of a minimal extended formulation for a polytope is essentially equal to the non-negative rank of a slack matrix for this polytope (see Yannakakis [1991]), what we tighten to the statement that the extension complexity of a polytope is equal to the non-negative rank of a slack matrix (the one-point polytope is an exception). This result allows to establish a lower bound on the size of an extended formulation for a polytope as the minimum number of monochromatic non-zero combinatorial rectangles of the entries in the slack matrix, which are needed to cover all non-zero entries of a slack matrix. The lower bounds presented in this section are coming from this rectangle covering problem and provide lower bounds on the rectangle covering number of the slack matrix. The results presented in this chapter are partly based on the joint work with Fiorini et al. [2011a].

The rectangle covering problem is well known as the non-deterministic communication complexity problem. Not so many techniques are known to establish a lower bound on the rectangle covering number. In Section 4.14, we give an overview of the most used ones. And in later sections, we give examples of their use in order to establish lower bounds for extended formulations.

But we have to mention that the rectangle covering lower bound for extensions is, in a certain sense, weak, since it takes into consideration only the combinatorial structure of a polytope. In his paper, Yannakakis showed that the rectangle covering bound is equal to $O(n^4)$ for the perfect matching polytope of a complete graph with n vertices, for which up to now no extended formulation of polynomial size is known. Moreover, there are polygons with n vertices, which do not admit any extension with size less than $\sqrt{2n}$ (see Fiorini et al. [2011c]), but with the covering number $O(\log n)$ for their slack matrix. Thus, there is a size gap between extended formulations for polytopes with the same combinatorial structure already in dimension two, since for regular n-gons there is an extension of size $O(\log n)$ (see Proposition 2.13 due to Ben-Tal and Nemirovski [2001]).

Another question, which stays outside of our consideration, is the coefficients in the extended formulation for a polytope. From counting reasons, one can conclude that there are n-dimensional zero-one polytopes, which do not admit a compact extended formulation, when the coefficients in the extended formulation have polynomial size. Recently, it was shown that there are matroid polytopes, which do not admit a compact extended formulation, even if no restrictions on the coefficients are posed (see Rothvoß [2011]).

Results concerning the non-negative factorization can be generalized for extended formulations, which are defined by other cones, not necessary a polyhedral cone (for further details see Gouveia et al. [2011], Fiorini et al. [2011b]). Here, we generalized the notion of extended formulation to the notion of extended relaxation, making properties and ideas, used in the proofs, more evident. Moreover, even these notions can be generalized to extended relaxations of a convex set up to a convex set, using work of Gouveia et al. [2011].

4.1. Minimal Extended Relaxation

In the beginning of this chapter, we consider a generalization of extended formulations, so called extended relaxations. Indeed, a lot of results proved in this chapter can be generalized, using this notion. Moreover, the notion of extended relaxations reveals the nature of the used argumentation.

A polyhedron $Q \subseteq \mathbb{R}^d$ and an affine map $p : \mathbb{R}^d \to \mathbb{R}^m$ form an *extended relaxation* of a polyhedron $P_* \subseteq \mathbb{R}^m$ up to a polyhedron $P^* \subseteq \mathbb{R}^m$, where $P_* \subseteq P^* \subseteq \mathbb{R}^m$, if the inclusion

$$P_* \subseteq p(Q) \subseteq P^*$$

holds.

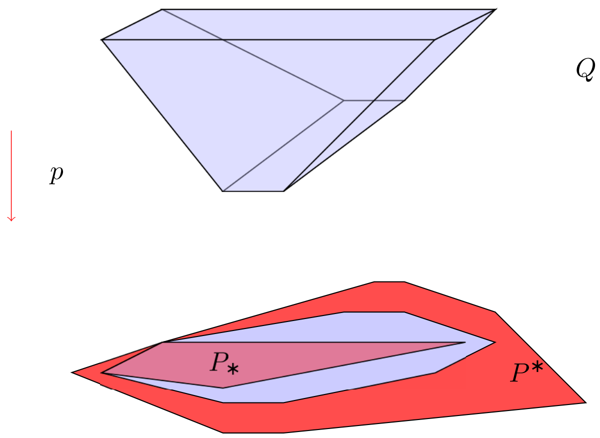

FIGURE 1. Example of an extended relaxation for polyhedra P_*, P^*.

We call a relaxation via a polyhedron $Q \subseteq \mathbb{R}^d$ and an affine map $p : \mathbb{R}^d \to \mathbb{R}^m$ a *minimal extended relaxation* of a polyhedron $P_* \subseteq \mathbb{R}^m$ up to a polyhedron $P^* \subseteq \mathbb{R}^m$, if the size of the extended relaxation $Q \subseteq \mathbb{R}^d$, $p : \mathbb{R}^d \to \mathbb{R}^m$, i.e. the number of facets of the polyhedron $Q \subseteq \mathbb{R}^d$, is equal to the minimum size among all extended relaxations of the polyhedron $P_* \subseteq \mathbb{R}^m$ up to the polyhedron $P^* \subseteq \mathbb{R}^m$.

Apparently, when a polyhedron $P_* \subseteq \mathbb{R}^m$ is equal to a polyhedron $P^* \subseteq \mathbb{R}^m$, an extended relaxation of the polyhedron $P_* \subseteq \mathbb{R}^m$ up to the polyhedron $P^* \subseteq \mathbb{R}^m$ defines an extension of the polyhedron $P \subseteq \mathbb{R}^m$, where

$$P = P^* = P_* \,.$$

4.2. Slack Matrices of Polyhedra

Here, we define a central notion of this chapter. For a polyhedron

$$P^* = \{x : \langle a^i, x\rangle \le b_i \,,\ i \in I^*\}\,,$$

given by a linear system with finite number of inequalities, and a polyhedron

$$P_* = \operatorname{conv}(X_*)\,,$$

given as the convex hull of points $X_* \subseteq P^* \subseteq \mathbb{R}^m$, which is allowed to be infinite, a *slack matrix* $\mathbf{M}_{\text{slack}}(P^*, P_*) \subseteq \mathbb{R}_+^{I^* \times X_*}$ is defined as follows

$$\mathbf{M}_{\text{slack}}(P^*, P_*)_{i,x} = b_i - \langle a^i, x\rangle$$

for $x \in X_*$ and $i \in I^*$. Clearly, the entries of the slack matrix $\mathbf{M}_{\text{slack}}(P^*, P_*)$ are nonnegative, since each inequality $\langle a^i, x\rangle \le b_i$, $i \in I$ is valid for all points x in X_*. Despite our notation, the slack matrix $\mathbf{M}_{\text{slack}}(P^*, P_*)$ depends on the choice of the linear system $\langle a^i, x\rangle \le b_i$, $i \in I^*$ and the set X_*, rather than on polyhedra P^* and P_*.

Obviously, the rank of the slack matrix $\mathbf{M}_{\text{slack}}(P^*, P_*)$ is at most $m+1$, for every choice of the linear system for a polyhedron $P^* \subseteq \mathbb{R}^m$ and the set of points for a polyhedron $P_* \subseteq \mathbb{R}^m$, since the matrix $\mathbf{M}_{\text{slack}}(P^*, P_*)$ is the product

$$\mathbf{M}_{\text{slack}}(P^*, P_*) = \begin{pmatrix} -A \,, b \end{pmatrix} \begin{pmatrix} X_* \\ \mathbf{1}_{X_*} \end{pmatrix}$$

of matrices with dimensions $|I^*| \times (m+1)$ and $(m+1) \times |X_*|$, where the linear system $Ax \leq b$ denotes the linear system, consisting of the inequalities $\langle a^i, x \rangle \leq b_i$, $i \in I^*$.

Whenever polyhedra P_*, $P^* \subseteq \mathbb{R}^m$, $m \geq 1$, are full-dimensional polytopes, the rank of every slack matrix $\mathbf{M}_{\text{slack}}(P^*, P_*)$ is equal to $m+1$, for every choice of the linear system for the polyhedron $P^* \subseteq \mathbb{R}^m$ and the set of points for the polyhedron $P_* \subseteq \mathbb{R}^m$. Indeed, in this case, there are $m+1$ affinely independent points $x^1, \ldots, x^{m+1}$ in X_*. And hence, the matrix

$$\begin{pmatrix} x^1 & \ldots & x^{m+1} \\ 1 & \ldots & 1 \end{pmatrix}$$

is non-singular. Additionally, there are $m+1$ linearly independent vectors (a^{i_t}, b_{i_t}), $i_t \in I^*$, $t \in [m+1]$, and thus, the matrix

$$\begin{pmatrix} -a^{i_1} & b_{i_1} \\ \ldots & \ldots \\ -a^{i_{m+1}} & b_{i_{m+1}} \end{pmatrix}$$

is non-singular. Finally, the product of these two matrices is a submatrix of the slack $\mathbf{M}_{\text{slack}}(P^*, P_*)$, what shows that the rank of the slack matrix is at least $m+1$.

As previously mentioned, a slack matrix $\mathbf{M}_{\text{slack}}(P^*, P_*)$ for polyhedra P_*, $P^* \subseteq \mathbb{R}^m$ is not unique, due to possible reorderings of I^* and X_*, and due to possible introducing in I^* some redundant inequalities for the polyhedron P^*, and into X_* some additional points from the polyhedron P_*. Nevertheless, later it is shown that every slack matrix $\mathbf{M}_{\text{slack}}(P^*, P_*)$ can be used to determine the minimum size of an extended relaxation for the polyhedron $P_* \subseteq \mathbb{R}^m$ up to the polyhedron $P^* \subseteq \mathbb{R}^m$.

4.3. Non-Negative Factorization, Non-Negative Rank

A *non-negative factorization* of a matrix $M \in \mathbb{R}_+^{I \times X}$ is a representation of the matrix M as the product

$$M = TS$$

of two matrices $T \in \mathbb{R}_+^{I \times r}$, $S \in \mathbb{R}_+^{r \times X}$, where the number r is called the *size of the non-negative factorization*.

The *non-negative rank* $\operatorname{rank}_+(M)$ of a matrix $M \in \mathbb{R}_+^{I \times X}$ is the minimum r, such that there is a non-negative factorization for the matrix M of size r.

Clearly, dropping the restrictions on the matrices $T \in \mathbb{R}_+^{I \times r}$, $S \in \mathbb{R}_+^{r \times X}$ to have non-negative entries, the definition of the non-negative rank transforms into the definition of the rank for the matrix M. Thus, the obvious lower bound

$$\operatorname{rank}(M) \leq \operatorname{rank}_+(M)$$

for the non-negative rank of a matrix $M \in \mathbb{R}^{I \times X}$ is obtained.

Later, we establish a direct connection between non-negative factorization of a matrix and finding extended relaxations for a pair of polyhedra.

4.4. Extended Relaxations from Non-Negative Factorizations

Now, we transform a non-negative factorization of a slack matrix for a pair of polyhedra into an extended relaxation for these polyhedra of the same size, using the notation from Sections 4.2 and 4.3.

Lemma 4.1. *For a non-negative factorization of size r for $\mathbf{M}_{\text{slack}}(P^*, P_*)$ for a pair of polyhedra $P_* \subseteq P^* \subseteq \mathbb{R}^m$, there is an extended relaxation of the polyhedron $P_* \subseteq \mathbb{R}^m$ up to the polyhedron $P^* \subseteq \mathbb{R}^m$ of size r.*

PROOF. Having a non-negative factorization

$$\mathbf{M}_{\text{slack}}(P^*, P_*) = TS$$

of a slack matrix $\mathbf{M}_{\text{slack}}(P^*, P_*)$, where $T \in \mathbb{R}_+^{I^* \times r}$, $S \in \mathbb{R}_+^{r \times X_*}$, define the polyhedron $Q \subseteq \mathbb{R}^{m+r}$ by the following linear system

$$b_i - \langle a^i, x\rangle = \langle T_{i,*}, z\rangle \text{ for } i \in I^* \qquad \text{and} \qquad z \geq 0\,. \tag{4.4.1}$$

The polyhedron Q, together with the orthogonal projection on x variables, forms an extended relaxation of the polyhedron $P_* \subseteq \mathbb{R}^m$ up to the polyhedron $P^* \subseteq \mathbb{R}^m$.

Indeed, the inclusion $p(Q) \subseteq P^*$ holds, i.e. for every point x from $\text{proj}_x(Q)$ all inequalities $\langle a^i, x\rangle \leq b_i$, $i \in I^*$ are satisfied, because the vectors $T_{i,*}$ and z are non-negative. On the other hand, for every $x \in X_*$, define z to be equal to $S_{*,x}$, what satisfies the linear system (4.4.1), since $S_{*,x}$ is non-negative and

$$b_i - \langle a^i, x\rangle = \mathbf{M}_{\text{slack}}(P)_{i,x} = \langle T_{i,*}, S_{*,x}\rangle$$

holds for every $i \in I^*$. □

The next observation strengthens Lemma 4.1 for extensions of polytopes, and follows from the fact that minimal extensions for polytopes are given by polytopes.

Observation 4.1. *For a non-negative factorization of a slack matrix $\mathbf{M}_{\text{slack}}(P)$ of size r for a polytope $P \subseteq \mathbb{R}^m$, there is an extension of the polytope $P \subseteq \mathbb{R}^m$ of size at most r via a polyhedron $Q \subseteq \mathbb{R}^d$ that is bounded, i.e. that is a polytope itself* [1].

4.5. Non-Negative Factorizations from Extended Relaxations

For every extended relaxation $Q \subseteq \mathbb{R}^d$, $p : \mathbb{R}^d \to \mathbb{R}^m$ of a polyhedron $P_* \subseteq \mathbb{R}^m$, where $P_* = \text{conv}(X_*)$, $X_* \subseteq \mathbb{R}^m$, up to a polyhedron $P^* \subseteq \mathbb{R}^m$, a *section* $s : X_* \to Q$ is a map satisfying

$$p(s(x)) = x\,,$$

for every $x \in X_*$. Moreover, if the polyhedron Q is given as

$$Q = \{z : \langle c^j, z\rangle \leq k_j,\ j \in [r]\}\,, \tag{4.5.1}$$

where $c^j \in \mathbb{R}^d$, $k_j \in \mathbb{R}$, the corresponding *slack covectors* are the vectors v^1 ,..., $v^r \in \mathbb{R}^{X_*}$, such that

$$v^j_x = k_j - \langle c^j, s(x)\rangle\,, \tag{4.5.2}$$

for all $j \in [r]$ and $x \in X_*$.

Lemma 4.2. *For an extended relaxation of size r of a polyhedron $P_* \subseteq \mathbb{R}^m$ up to a polyhedron $P^* \subseteq \mathbb{R}^m$, there is a non-negative factorization of size $r+1$ for every slack matrix $\mathbf{M}_{\text{slack}}(P^*, P_*)$.*

PROOF. Having an extended relaxation $Q \subseteq \mathbb{R}^d$, $p : \mathbb{R}^d \to \mathbb{R}^m$, we fix a minimal linear description of Q

$$Q = \{z : \langle c^j, z\rangle \leq k_j,\ j \in [r]\}\,. \tag{4.5.3}$$

For the polyhedron $P^* \subseteq \mathbb{R}^m$, we fix its linear description $\langle a^i, x\rangle \leq b_i$, $i \in I^*$, and for the polyhedron $P_* \subseteq \mathbb{R}^m$ a set of points X_*, $P = \text{conv}(X_*)$.

From the Farkas Lemma, every linear inequality, which is valid for the polyhedron Q, is a non-negative combination of the inequalities from the linear description (4.5.3) and the

[1] Observation 4.1 holds also for extended relaxations, when the polyhedron P^* is bounded, i.e. a polytope. In this case, Observation 4.1 can be shown via the proof of Lemma 4.1.

trivial inequality $0 \le 1$. Hence, for every $i \in I^*$, there exists a vector $t^i \in \mathbb{R}_+^{r+1}$, such that for all $z \in \mathbb{R}^d$ the equation

$$b_i - \langle a^i, p(z)\rangle = t^i_{r+1} + \sum_{j\in[r]} t^i_j (k_j - \langle c^j, z\rangle)\,,$$

holds, since $p(Q) \subseteq P^*$.

Let $s : X_* \to Q$ be a section for the extended relaxation Q, p. We obtain a non-negative factorization of the slack matrix $\mathbf{M}_{\text{slack}}(P^*, P_*)$, defined by two matrices $T \in \mathbb{R}_+^{I^*\times(r+1)}$, $S \in \mathbb{R}_+^{(r+1)\times X_*}$, where

$$T_{i,j} = t^i_j \qquad \text{for } i \in I^*$$
$$S_{j,x} = k_j - \langle c^j, s(x)\rangle \qquad \text{for } x \in X_*$$

in the case $j \in [r]$, and

$$T_{i,j} = t^i_j \qquad \text{for } i \in I^*$$
$$S_{j,x} = 1\,, \qquad \text{for } x \in X_*,$$

in the case $j = r+1$, which corresponds to the slack of the inequality $0 \le 1$. □

Observation 4.2. *For every extension $Q \subseteq \mathbb{R}^d$, $p : \mathbb{R}^d \to \mathbb{R}^m$ with size r of a non-trivial polytope P, there is a non-negative factorization of size r for every slack matrix $\mathbf{M}_{\text{slack}}(P)$.*

Moreover, rows of the right matrix in the non-negative factorization can be chosen as the slack covectors, corresponding to any linear description of the polyhedron Q.

PROOF. Here, the proof of Lemma 4.2 can be modified, exploiting the observation that the inequality $0 \le 1$ is a non-negative combination of the inequalities from the linear system (4.5.3) (see Appendix: Lemma 6.1). □

4.6. Non-Negative Factorizations, Extensions of Polytopes

Observations 4.1 and 4.2, lead to the next result.

Theorem 4.1 (Yannakakis [1991])**.** *For a polytope $P \subseteq \mathbb{R}^m$, $|\text{vert}(P)| \ge 2$, the size of a minimal extension for P is equal to the non-negative rank of any slack matrix $\mathbf{M}_{\text{slack}}(P)$.*

Theorem 4.1 gives us the equivalence between finding the minimal extension of a polytope and determining the non-negative rank of its slack matrix. For example, the next observation, which does not appear to be trivial, initially, can be derived from Theorem 4.1.

Observation 4.3. *For every full-dimensional polytope $P \subseteq \mathbb{R}^m$, containing the origin in its interior, the minimum size of an extension of the polytope P is equal to the minimum size of an extension of the polytope that is polar to the polytope P.*

PROOF. The proof follows from the fact that a slack matrix $\mathbf{M}_{\text{slack}}(P)$ of the polytope $P \subseteq \mathbb{R}^m$ is also a transposed slack matrix of the polar polytope. □

Nevertheless, even if the non-negative rank of a slack matrix provides a minimal number of facets that an extension of a polytope can have, this characterization of the extension complexity of a polytope is hard to use, since even to determine, whether the non-negative rank of a matrix is equal to its rank (the trivial lower bound) is $\mathcal{NP}$-hard (see Vavasis [2009]).

4.7. Extended Relaxation Problem from Non-Negative Rank Problem

In this section, we will see that not only the problems of minimal extended relaxation and minimal extended formulation can be reduced to the non-negative factorization problem, but the non-negative factorization problem can be transformed into the problem of finding a minimal extended relaxation. Due to Lemma 4.2 to do this, it is enough to show that every matrix with non-negative entries is a slack matrix for some pair of polyhedra[1].

Theorem 4.2. *Every non-zero matrix $M \in \mathbb{R}_+^{I\times X}$ can be transformed, via deleting zero columns and scaling columns by non-negative constants, into a slack matrix M' for a pair of polyhedra from $\mathbb{R}^m$, where $m = \operatorname{rank}(M) - 1$, such that*

$$\operatorname{rank}_+(M) = \operatorname{rank}_+(M')\,.$$

PROOF. Delete zero columns from the matrix $M \in \mathbb{R}_+^{I\times X}$, what changes neither its rank nor its non-negative rank. Analogously, scaling columns, such that the columnwise sums of elements are equal, changes neither its rank nor its non-negative rank. Denote the resulting matrix by $M' \in \mathbb{R}_+^{I'\times X'}$.

There exists a factorization (not necessary non-negative) for M' of the form

$$M' = \begin{pmatrix} -A\,, & b \end{pmatrix} \begin{pmatrix} S \\ \mathbf{1}_X \end{pmatrix}\,,$$

where $A \in \mathbb{R}^{I'\times m}$, $b \in \mathbb{R}^{I'}$ and $m = \operatorname{rank}(M) + 1$, since the columnwise sums of elements are equal.

Define the set of points $X_* \subseteq \mathbb{R}^m$

$$X_* = \{S_{*,x} \in \mathbb{R}^m : x \in X'\}\,,$$

and the corresponding polyhedron $P_* = \operatorname{conv}(X_*) \subseteq \mathbb{R}^m$, and define the polyhedron $P^* \subseteq \mathbb{R}^m$ by the linear system $A\,x \leq b$. Hence, the matrix M' is a slack matrix for the polyhedra $P_* \subseteq P^* \subseteq \mathbb{R}^m$. □

4.8. Lattice Embedding

Recall that the *face lattice* $\mathcal{L}(Q)$ of a polyhedron $Q \subseteq \mathbb{R}^d$ is the set of all faces of the polyhedron $Q \subseteq \mathbb{R}^d$, including $\varnothing$ and Q, ordered by inclusion.

Speaking about the *face poset* $\mathcal{L}(P^*, P_*)$ of polyhedra $P^* \subseteq \mathbb{R}^m$ and $P_* \subseteq \mathbb{R}^m$, $P_* \subseteq P^*$, we refer to all sets $F^* \cap P_*$, ordered by inclusion, where F^* is a face of the polyhedron $P^* \subseteq \mathbb{R}^m$. Note that the face poset $\mathcal{L}(P^*, P_*)$ is a subposet of the face lattice $\mathcal{L}(P_*)$.

Lemma 4.3. *For every extended relaxation of a polyhedron $P_* \subseteq \mathbb{R}^m$ up to a polyhedron $P^* \subseteq \mathbb{R}^m$, given by a polyhedron $Q \subseteq \mathbb{R}^d$ and an affine map $p : \mathbb{R}^d \to \mathbb{R}^m$, there is an embedding, i.e. injective and order preserving map, of the face poset $\mathcal{L}(P^*, P_*)$ into the face lattice $\mathcal{L}(Q)$ of the polyhedron Q.*

PROOF. Let us denote by $P \subseteq \mathbb{R}^m$ the polyhedron $p(Q)$. We define the desired embedding as a combination of two embeddings: an embedding of the face poset $\mathcal{L}(P^*, P_*)$ into the face lattice $\mathcal{L}(P)$, and an embedding of the face lattice $\mathcal{L}(P)$ into the face lattice $\mathcal{L}(Q)$.

For every face F^*_* from $\mathcal{L}(P^*, P_*)$, define a map $j^* : \mathcal{L}(P^*, P_*) \to \mathcal{L}(P)$. For every face $F^*_* \in \mathcal{L}(P^*, P_*)$, choose the inclusion minimal face F^* from $\mathcal{L}(P^*)$, such that $F^*_* \subseteq F^*$, i.e. the inclusion minimal face of P^*, such that $F^*_* = F^* \cap P_*$, and set

$$j^*(F^*_*) = P \cap F^*\,,$$

what is a face of the polyhedron P, because $F^* \in \mathcal{L}(P^*)$ and $P \subseteq P^*$.

[1] A similar construction was used in Gillis and Glineur [2010], where so called restricted non-negative rank was studied.

The map j^* is inclusion preserving, what follows from its definition. Moreover, the map j^* is injective, since for every $F^*_* \in \mathcal{L}(P^*, P_*)$

$$j^*(F^*_*) \cap P_* = (F^* \cap P) \cap P_* = F^* \cap P_* = F^*_* \,,$$

where F^* is the inclusion minimal face of P^*, such that $F^*_* = F^* \cap P_*$.

Define the map $j : \mathcal{L}(P) \to \mathcal{L}(Q)$, such that

$$j(F) = p^{-1}(F) \cap Q \,,$$

for each face F of the polyhedron P. The map j defines an embedding from the face lattice $\mathcal{L}(P)$ into the face lattice $\mathcal{L}(Q)$. Indeed, for the face $F \in \mathcal{L}(P)$, defined by an inequality $\langle a, x\rangle \leq b$, the image $j(F)$ is the face of the polyhedron Q, defined by the inequality

$$\langle a, p(z)\rangle \leq b \,.$$

Obviously, the map j is inclusion preserving. Moreover, for every face F of the polyhedron P, we have $p(j(F)) = F$, what implies that the map $j(F)$, $F \in \mathcal{L}(P)$ is injective. □

Corollary 4.1. *For every pair of polyhedra P_*, P^*, $P_* \subseteq P^* \subseteq \mathbb{R}^m$, the minimum number of facets of a polyhedron Q, such that there exists an embedding of the face poset $\mathcal{L}(P^*, P_*)$ into the face lattice $\mathcal{L}(Q)$, defines a lower bound on the size of an extended relaxation of the polyhedron P_* up to the polyhedron P^*.*

Corollary 4.2. *For every polytope $P \subseteq \mathbb{R}^m$, the minimum number of facets of a polytope Q, such that there exists an embedding of the face lattice $\mathcal{L}(P)$ into the face lattice $\mathcal{L}(Q)$, defines a lower bound on the size of an extension for the polytope P.*

4.9. Relaxations of Lattice Embeddings

Thus, the embedding of the poset $\mathcal{L}(P^*, P_*)$ provides a lower bound on the size of extended relaxations for a polyhedron $P_* \subseteq \mathbb{R}^m$ up to a polyhedron $P^* \subseteq \mathbb{R}^m$. But the restriction that the face poset $\mathcal{L}(P^*, P_*)$ has to be embedded into the face lattice of some other polyhedron is hard to handle. Because of that, one considers different relaxations of the conditions on the lattice, into which $\mathcal{L}(P^*, P_*)$ has to be embedded.

For every lattice Λ, let us denote Λ^0 to be the poset, obtained from the lattice Λ by deleting the maximum and minimum of the lattice Λ. In this setting, consider the following embedding of the face poset $\mathcal{L}(P^*, P_*)$ into a lattice Λ, where

(4.9.1) for all $G_1, G_2 \in \Lambda^0$ with $G_1 \not\leq G_2$ there is a maximal element $G \in \Lambda^0$ such that $G_1 \not\leq G$ and $G_2 \leq G$.

The face lattice $\mathcal{L}(Q)$ of every polyhedron $Q \subseteq \mathbb{R}^d$ satisfies the condition (4.9.1). Thus, the minimum number of maximal elements in the poset Λ^0, corresponding to the lattice Λ that satisfies (4.9.1), and in which the face poset $\mathcal{L}(P^*, P_*)$ can be embedded, is a lower bound on the size of an extended relaxation for the polyhedron P_* up to the polyhedron P^*.

In fact, we will show that the condition (4.9.1) is a reformulation of the lattice embedding bound. Nevertheless, this reformulation reveals the properties of the lattice Λ, that we use later, to prove the rectangle covering bound in the next section.

Observation 4.4. *Every lattice Λ, satisfying the condition* (4.9.1), *can be embedded into the face lattice of a simplex with the number of facets equal to the number of maximal elements in the poset Λ^0 plus one, if the poset Λ^0 contains the minimum element, and equal to the number of maximal elements in Λ^0, otherwise* [1].

[1] Note that Λ^0 can have the maximum element, even if Λ is the face lattice of some polyhedron Q. For example, if the polyhedron Q is given as a polyhedral cone, the poset Λ^0 possesses the minimum element. But it is not hard to see that Λ^0 has the minimum element just in case, when Q is a polyhedral cone. Since we are

PROOF. Due to to the property (4.9.1), every element in the poset Λ^0 can be uniquely identified with the set of maximal elements bigger than it. Moreover, for two elements G_1, $G_2 \in \Lambda^0$, we have $G_1 \leq G_2$ if and only if the set of maximal elements in Λ^0, which are bigger than G_2, is a subset of the corresponding set for G_1.

Thus, the lattice Λ can be embedded into the face lattice of the simplex with the number of facets equal to the number of maximal elements in Λ^0, or to the number of maximal elements in Λ^0 plus one, depending on the existence of the minimum element in Λ_0.

Indeed, whenever the poset Λ_0 has the maximum or minimum element, there may no be an element in the simplex face lattice (the simplex with the number of facets equal to the number of maximal elements in Λ^0) for the maximum or minimum elements of the lattice Λ, respectively. In the case, when the poset Λ_0 has the maximum element, then from (4.9.1) it consists from one element only, and thus, the maximum is equal to the minimum in Λ_0. In the case, when the poset Λ_0 contains the minimum element, the dimension of the simplex is increased, to embed the minimum element of Λ. □

4.10. Rectangle Coverings from Lattice Embeddings

The set of *non-zero rectangles* $\mathcal{R}(M)$ for a matrix $M \in \mathbb{R}^{I \times X}$ is defined as follows

$$\mathcal{R}(M) = \{I' \times X' \subseteq \operatorname{supp}(M) : I' \subseteq I,\, X' \subseteq X\}\,.$$

A *rectangle covering* of the matrix $M \in \mathbb{R}^{I \times X}$ is a set $\mathcal{R} \subseteq \mathcal{R}(M)$, such that for every $(i, x) \in \operatorname{supp}(M)$, $i \in I$, $x \in X$ there is a rectangle $R \in \mathcal{R}$, such that the rectangle R contains (i, x), i.e.

$$\operatorname{supp}(M) = \bigcup_{R \in \mathcal{R}} R\,.$$

The *rectangle covering number* for a matrix M is the minimum number of rectangles in a rectangle covering for the matrix M.

Lemma 4.4. *Having an embedding of the face poset $\mathcal{L}(P^*, P_*)$ of a polyhedron $P_* \subseteq \mathbb{R}^m$ and a polyhedron $P^* \subseteq \mathbb{R}^m$, $P_* \subseteq P^*$, into a lattice Λ, which satisfies the condition* (4.9.1)*, there is a rectangle cover for every slack matrix for the polyhedra P_*, P^*, with the size equal to the number of maximal elements in the poset Λ^0 plus one.* [1].

PROOF. Here, we use the notation from Section 4.2. There are the three following cases, for every point $x_* \in X_*$ and every inequality $\langle a^i, x\rangle \leq b_i$, $i \in I^*$, such that the face F^* of the polyhedron P^*, induced by the inequality $\langle a^i, x\rangle \leq b_i$, does not contain the point x_*.

First, the polytope P_* is the minimal face in $\mathcal{L}(P^*, P_*)$, containing the point x_*. Second, the intersection $F^* \cap P_*$ is empty.

Third, we can assume

$$\varnothing \neq F^*_* \neq P_* \quad \text{and} \quad \varnothing \neq F^* \cap P_* \neq P_*\,,$$

interested in extended formulations of polytopes, we have to note that only one-point polytopes admit extensions given by polyhedral cones.

[1]The size of the rectangle covering, given in Lemma 4.4, is tight. Note that an additional rectangle must be taken in some cases, even if P_* and P^* are polytopes. For example, the face poset $\mathcal{L}(P^*, P_*)$ for $P_* = \operatorname{conv}(\{(0,0),(0,1),(1,0)\})$ and $P^* = \{x \in \mathbb{R}^2 : 0 \leq x_1 \leq 2, 0 \leq x_2 \leq 2\}$ can be embedded into a lattice Λ, satisfying (4.9.1), and where Λ^0 has two maximal elements. But, the slack matrix $\mathbf{M}_{\text{slack}}(P^*, P_*)$

$$\begin{pmatrix} 0 & 0 & 1 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & 2 \end{pmatrix}$$

needs at least three monochromatic rectangles to be covered.

where the face $F^*_* \in \mathcal{L}(P^*, P_*)$ is the minimal face in $\mathcal{L}(P^*, P_*)$, containing the point x_*. Due to the condition (4.9.1), there have to exist a maximal element $G \in \Lambda^0$, such that

$$G \geq j(F^* \cap P_*) \quad \text{and} \quad G \ngeq j(F^*_*)\,.$$

In turn, for a face $F^* \in \mathcal{L}(P^*)$ and a point $x_* \in X_*$, whenever there exists a maximal element G in the lattice Λ^0, such that $G \geq j(F^* \cap P_*)$ and $G \ngeq j(F^*_*)$, where the face $F^*_* \in \mathcal{L}(P^*, P_*)$ is the minimal face in $\mathcal{L}(P^*, P_*)$, containing the point x_*, the point x_* does not belong to the face F^*.

It is not hard to verify that the rectangles

$$\{i \in I^* : j(F^* \cap P_*) \leq G \text{ and } P_* \subsetneq F^* \text{ where } F^* = \{y \in P^* : \langle a^i, y\rangle = b_i\}\} \times \\ \{x \in X_* : F^*_* = P_* \text{ or } j(F^*_*) \nleq G \text{ where } F^*_* = \cap_{\substack{F \in \mathcal{L}(P^*, P_*) \\ x \in F}} F\}\,,$$

indexed by maximal elements G in Λ^0, together with one additional rectangle to cover the entries consisting of the columns with $F^* \cap P_* = \varnothing$, form a rectangle covering for the slack matrix of P_*, P^*. □

Observation 4.5. *For every embedding of the face lattice $\mathcal{L}(P)$ of a polytope $P \subseteq \mathbb{R}^m$, $|\mathrm{vert}(P)| \geq 2$, into a lattice Λ, satisfying the condition* (4.9.1)*, there is a rectangle cover for every slack matrix of the polytope P, whose size is equal to the number of maximal elements in the poset Λ^0.*

PROOF. The proof of Lemma 4.4 can be modified for the claim of the above observation, using Lemma 6.1. □

4.11. Lattice Embeddings from Rectangle Coverings

Lemma 4.5. *For a rectangle cover $\mathcal{R}$, $|\mathcal{R}| \geq 1$, of a slack matrix for polyhedra $P_* \subseteq P^* \subseteq \mathbb{R}^m$, there is an embedding of the face poset $\mathcal{L}(P^*, P_*)$ into the face lattice $\mathcal{L}(Q)$ of a simplex Q with $|\mathcal{R}| + 1$ facets* [1].

PROOF. Associate the facets of a $d-1$-dimensional simplex $Q \subseteq \mathbb{R}^d$ with the rectangles from the rectangle cover $\mathcal{R}$. Thus, the face lattice $\mathcal{L}(Q)$ is associated with all possible subsets of the rectangles from the cover $\mathcal{R}$.

Define an embedding $j : \mathcal{L}(P^*, P_*) \to \mathcal{L}(Q)$, taking for every face $F^*_* \in \mathcal{L}(P^*, P_*)$ an element from the lattice $\mathcal{L}(Q)$, corresponding to the set

$$\mathcal{R}_{F^*_*} = \{I' \times X' \in \mathcal{R} : F^*_* \subseteq F_i \text{ for some } i \in I'\}$$

of rectangles from $\mathcal{R}$, where $F_i \in \mathcal{L}(P^*)$ denotes the face, induced by the inequality $\langle a^i, x\rangle \leq b_i$.

Obviously, the map $j : \mathcal{L}(P^*, P_*) \to \mathcal{L}(Q)$ is inclusion preserving. To prove that the map $j : \mathcal{L}(P^*, P_*) \to \mathcal{L}(Q)$ is injective, consider a non-empty face $F \in \mathcal{L}(P^*, P_*)$, and thus,

$$F = P_* \cap (\bigcap_{\substack{i \in I^* \\ F \subseteq F_i}} F_i)$$

[1]The number of the facets of the simplex, given in Lemma 4.5, is tight. Note that an additional facet must be taken in some cases. For example, the face poset $\mathcal{L}(P^*, P_*)$ for $P_* = \mathrm{conv}(\{(0,0),(0,1),(1,0)\})$ and $P^* = \{x \in \mathbb{R}^2 : 0 \leq x_1, 0 \leq x_2\}$ can not be embedded into the face lattice of the one-dimensional simplex. But, the slack matrix $\mathbf{M}_{\mathrm{slack}}(P^*, P_*)$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

can be covered by two non-zero rectangles. But, an additional rectangle is not needed, whenever P_* and P^* are not trivial polytopes.

and

$$F \cap X_* = X_* \cap (\bigcap_{\substack{i \in I^* \\ F \subseteq F_i}} F_i) = \bigcap_{\substack{i \in I^* \\ F \subseteq F_i}} (X_* \cap F_i) = \bigcap_{\substack{R = I' \times X' \\ R \in \mathcal{R}_F}} (X_* \setminus X') .$$

And since every face $F \in \mathcal{L}(P^*, P_*)$ can be uniquely determined from the set $X_* \cap F$, the map j is injective on $\mathcal{L}(P^*, P_*) \setminus (\{\varnothing\} \cap \{P_*\})$.

The face $\varnothing \in \mathcal{L}(P^*, P_*)$, can be embedded in the simplex face lattice $\mathcal{L}(Q)$, unless

$$P_* \cap (\bigcap_{i \in I^*} F_i) \neq \varnothing ,$$

and the face $P_* \in \mathcal{L}(P^*, P_*)$ can be embedded in the simplex face lattice $\mathcal{L}(Q)$, unless $|\mathcal{R}| = 1$. In both these cases, the increasement of the number of facets of the simplex allows to define the desired embedding. □

Observation 4.6. *For a rectangle cover $\mathcal{R}$ of a slack matrix for a non-trivial polytope $P \subseteq \mathbb{R}^m$, there is an embedding of the face lattice $\mathcal{L}(P)$ into the face lattice $\mathcal{L}(Q)$ of a simplex Q with $|\mathcal{R}|$ facets.*

PROOF. The proof of Lemma 4.5 can be modified, since for a non-trivial polytope there is no point, which belongs to every facet of the polytope. And for every slack matrix, there is no a rectangle cover with less than two rectangles. □

The lower bound on sizes of extensions for a polytope $P \subseteq \mathbb{R}^m$, which arises from the minimum size of a rectangle cover for a slack matrix, is called *rectangle covering bound* and is denoted by $\mathrm{rc}(P)$. From Observations 4.5 and 4.6, the rectangle covering lower bound does not depend on the choice of the slack matrix, whenever the polytope P is not a one-point polytope.

4.12. Communication Complexity

To determine the rectangle covering number of the support for some matrix is a non-trivial task. This is an object of study in communication complexity theory, known as *non-deterministic communication complexity* (see Kushilevitz and Nisan [1997]).

For example, the proof of the fact that the cut polytope $\mathrm{P_{cut}}(n)$ does not have an extended formulation of size less than $2^{\Omega(n)}$, due to Fiorini et al. [2011b], is conducted, by determining a submatrix of a slack matrix, for which the non-deterministic complexity states the lower bound $2^{\Omega(n)}$ on the rectangle covering number, due to de Wolf [2003].

On the other hand, the *deterministic communication complexity*, which produces a partitioning of the support of a matrix via non-zero rectangles[1], can produce extended formulations (in the case, when the corresponding slack matrix has zero-one entries). In Yannakakis [1991], for perfect graphs with n vertices, an extended formulation of the stable set polytope with size $n^{O(\log n)}$ was constructed, using a deterministic protocol. Additionally in Faenza et al. [2011], for claw-free perfect graphs with n vertices, an extended formulation with size $O(n^3)$ was constructed from a deterministic communication protocol[2].

Moreover, there is a reformulation of the non-negative factorization problem for a matrix as a communication complexity protocol, which calculates matrix elements in expectation Faenza et al. [2011].

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[1]In this case, not every partitioning of the support via rectangles defines a deterministic communication protocol, as it is with rectangle coverings and non-deterministic protocols. Nevertheless, since the constructed extensions do not use the fact that they are obtained from a deterministic protocol, but rather, the fact that we deal with partitionings, one can construct an extension from every partitioning of the same size.

[2]Actually, this deterministic communication protocol produces an extended formulation of size $O(n^k)$ for the stable set polytope of perfect graphs with n vertices, where no vertex has k pairwise non-adjacent neighbors.

4.13. Upper Bounds on Rectangle Covering Number

Here, coverings of slack matrices for certain polytopes are provided, what shows that the lower bounds, obtained via rectangle coverings, can not be better than the sizes of the given coverings. Due to Observations 4.5 and 4.6, every linear description and inner description of a polytope can be taken, to provide an upper bound on the rectangle covering bound.

4.13.1. Matching Polytope. Yannakakis [1991] showed that the rectangle covering bound can not give a superpolynomial lower bound on the size of a minimal extension for the perfect matching polytope. Namely, in the case of the perfect matching polytope for the complete graph K_n, there exists a rectangle covering of a slack matrix of size $O(n^4)$, due to Yannakakis [1991]. Nevertheless, there is no proof that for the perfect matching polytope the covering bound can not be asymptotically better than the trivial bound, as the rank of a slack matrix.

Now, a rectangle cover for the slack matrix, given for the vertices of the perfect matching polytope and for the linear description $x \geq 0$ and

$$\begin{aligned} x(\delta(S)) &\geq 1 && \text{for} \quad S \subseteq [n],\ |S| \quad \text{is odd} \\ x(\delta(v)) &= 1 && \text{for} \quad v \in [n]\,, \end{aligned}$$

is constructed.

The non-zero entries in the slack matrix involving the non-negativity constraints $x \geq 0$ are easy to cover by $\binom{n}{2}$ rectangles, i.e. one rectangle for every of the corresponding rows.

For the non-zero entries involving odd cut inequalities $x(\delta(S)) \geq 1$, $S \subseteq [n]$, $|S| \in [n]_{\text{odd}}$, consider

$$R_{e_1,e_2} = \{S \subseteq [n] : |S| \in [n]_{\text{odd}},\ e_1, e_2 \in \delta(S)\} \times \{M \in \mathcal{M}^{\frac{n}{2}}(n) : e_1, e_2 \in M\}\,,$$

where e_1, $e_2 \in \binom{n}{2}$ is a pair of disjoint edges. Obviously, the rectangles R_{e_1,e_2}, e_1, $e_2 \in \binom{n}{2}$, $e_1 \cap e_2 = \varnothing$, form a rectangle covering for the rest of the slack matrix, since an entry (S, M), where $M \in \mathcal{M}^{\frac{n}{2}}(n)$ and $S \subseteq [n]$, $|S| \in [n]_{\text{odd}}$, is non-zero if and only if there are at least two edges in the set $\delta(S) \cap M$.

Proposition 4.1 (Yannakakis [1991])**.** *The rectangle covering bound* $\mathrm{rc}(\mathrm{P}^{\frac{n}{2}}_{\text{match}}(n))$ *is bounded from above by* $O(n^4)$ *for every perfect matching polytope* $\mathrm{P}^{\frac{n}{2}}_{\text{match}}(n)$.

4.13.2. Polytopes with Few Vertices on Every Facet. For a polytope $P \subseteq \mathbb{R}^m$ with few vertices on every facet, Lemma 6.4 provides an upper bound on the rectangle covering number by associating with every facet the set of vertices belonging to it. In the setting of Lemma 6.4, define k_1 to be the maximal number of vertices of the polytope $P \subseteq \mathbb{R}^m$, belonging to the same facet, and k_2 to be equal one.

Observation 4.7. *The rectangle covering bound* $\mathrm{rc}(P)$ *is bounded by* $O(k^2 \log n)$*, for every polytope* $P \subseteq \mathbb{R}^m$*, with* $|\mathrm{vert}(P)| = n$*, such that the maximal number of vertices of the polytope* P*, lying on the same facet, does not exceed* k.

The most natural application of the above observation are simplicial polytopes, what leads to the following observation.

Observation 4.8. *The rectangle covering bound* $\mathrm{rc}(P)$ *is equal* $O(m^2 \log n)$*, for every simplicial polytope* $P \subseteq \mathbb{R}^m$*, with* $|\mathrm{vert}(P)| = n$.

4.13.3. Edge Polytopes. Proposition 2.7 states that there exists an extended formulation of size $O(\frac{n^2}{\log n})$ for every edge polytope $\mathrm{P}_{\mathrm{edge}}(G)$, $G = (V, E)$, $V = [n]$.

Corollary 4.3. *The rectangle covering bound* $\mathrm{rc}(\mathrm{P}_{\mathrm{edge}}(G))$ *is bounded from above by* $O(\frac{n^2}{\log n})$ *for every edge polytope* $\mathrm{P}_{\mathrm{edge}}(G) \subseteq \mathbb{R}^n$*,* $G = (V, E)$*,* $|V| = n$*.*

Recall that the edge polytope $\mathrm{P}_{\mathrm{edge}}(G) \subseteq \mathbb{R}^n$ (see Kaibel and Loos [2011]) is described by $x \geq 0$ and

$$x(S) - x(N(S)) \leq 0 \quad \text{for all stable sets} \quad S \subseteq V\,.$$

The slack entries, associated with the non-negativity constraints, can be covered by n rectangles. Thus, the entries are left, which correspond to the inequalities $x(S) - x(N(S)) \leq 0$, indexed by stable sets $S \subseteq [n]$ of the graph G.

Consider the matrix M, indexed by pairs of stable sets S and edges of the graph $e \in E$, where an entry $M_{S,e}$ is non-zero if and only if S and e are disjoint, but $N(S)$ and e are not.

Thus, it is left to construct a rectangle covering for the matrix M. For this, define two matrices M', M'', indexed by pairs of a stable set $S \subseteq [n]$ and an edge $e \in E$, where

$$M'_{S,e} = \begin{cases} 1 & \text{if} \quad S \cap e = \varnothing \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \quad M''_{S,e} = \begin{cases} 0 & \text{if} \quad N(S) \cap e = \varnothing \\ 1 & \text{otherwise} \end{cases} \qquad .$$

Due to Lemma 6.4, for M' there exists a rectangle cover $\mathcal{R}'$ of size $O(\alpha^3 \log n)$, where α denotes the maximal size of a stable set in the graph G. For the matrix M'', there exists a trivial rectangle cover $\mathcal{R}''$ by n rectangles, indexed by the vertices of the graph G

$$R''_v = \{S \subseteq [n] : v \in N(S)\} \times \{e \in E : v \in e\}\,.$$

Since the entry $M_{S,e}$ is non-zero if and only if both entries $M'_{S,e}$, $M''_{S,e}$ are non-zero, the rectangle covers $\mathcal{R}'$, $\mathcal{R}''$ of the matrices M', M'' induce a rectangle cover of the matrix M of size $O(\alpha^3 n \log n)$ using the rectangles $R' \cap R''$, where $R' \in \mathcal{R}'$, $R'' \in \mathcal{R}''$.

Proposition 4.2. *The rectangle covering bound* $\mathrm{rc}(\mathrm{P}_{\mathrm{edge}}(G))$ *equals* $O(\alpha^3 n \log n)$ *for every edge polytope* $\mathrm{P}_{\mathrm{edge}}(G) \subseteq \mathbb{R}^n$*, where* $G = (V, E)$*,* $|V| = n$ *and* α *is the maximal size of a stable set in the graph* G*.*

From the Turan's Theorem, which states

$$|E| \geq \frac{|V|^2}{2\alpha(G)}\,,$$

and the above proposition, a non-trivial class of edge polytopes is obtained, for which the vertex extension could not be proved to be optimal via rectangle covering techniques.

Proposition 4.3. *The rectangle covering bound* $\mathrm{rc}(\mathrm{P}_{\mathrm{edge}}(G))$ *is equal* $o(|E|)$ *for every edge polytope* $\mathrm{P}_{\mathrm{edge}}(G) \subseteq \mathbb{R}^n$*,* $G = (V, E)$*,* $|V| = n$*, such that*

$$\alpha = o\big((\frac{n}{\log n})^{\frac{1}{4}}\big)\,,$$

where α *is the maximal size of a stable set in the graph* G*.*

4.14. Lower Bound on Rectangle Covering

In this section different sorts of lower bounds on the size of rectangle covering are presented.

4.14.1. Fooling Sets. A *fooling set* for a matrix is a set of entries from the matrix support, such that there is no non-zero rectangle, which covers more than one element from this set. Clearly, this implies that for every matrix, the cardinality of a fooling set is a lower bound on the rectangle covering number.

But, the fooling set technique is quite limited. For every matrix $M \in \mathbb{R}^{I \times X}$, the cardinality of a fooling set does not exceed $\operatorname{rank}(M)^2$. Thus, in the case of extensions for a polytope $P \subseteq \mathbb{R}^m$, the fooling set bound is at most $(m+1)^2$ (see Dietzfelbinger et al. [1996], Fiorini et al. [2011a]).

Nevertheless, there are examples of zero-one matrices with rank 3^n, for which there are fooling sets of cardinality at least 4^n Dietzfelbinger et al. [1996][1]. For combinatorial polytopes, one of the most successful applications of the fooling set technique is the stable set polytope, due to Huang and Sudakov [2010]. There a family of graphs were given, such that a slack matrix for the corresponding stable set polytope has a fooling set of cardinality $n^{\frac{6}{5}}$, where n denotes the number of vertices.

A notion of a *fooling set of order* k was proposed in Dietzfelbinger et al. [1996], what is defined as a set of entries from the matrix support, such that every $k+1$ elements of this set span a rectangle, containing at least one element with zero value. Thus, having a fooling set of order k and cardinality r, the value $\frac{r}{k}$ is a lower bound on the rectangle covering number, since no non-zero rectangle contains more than k elements from the fooling set of order k. Moreover, it was shown Dietzfelbinger et al. [1996] that for every matrix $M \in \mathbb{R}^{I \times X}$ the rectangle covering number $\operatorname{rc}(M)$ is equal to $O(\max_k(\frac{r_k}{k})\log(|\operatorname{supp}(M)|))$, where r_k is the maximum cardinality of a fooling set of order k for the matrix M [2].

4.14.2. Linear Relaxation. The next approach to bound the rectangle covering number for a matrix $M \in \mathbb{R}^{I \times X}$ is a linear relaxation of the corresponding set cover problem

$$\begin{aligned} \gamma = \min \sum_{R \in \mathcal{R}(M)} t_R & \\ \sum_{R : (i,x) \in R} t_R \geq 1 & \qquad \text{for} \quad (i,x) \in \operatorname{supp}(M) \\ t_R \geq 0 & \qquad \text{for} \quad R \in \mathcal{R}(M)\,, \end{aligned}$$

(see Karchmer et al. [1995]). Obviously, the optimal value γ for this problem is a lower bound on the rectangle covering number $\operatorname{rc}(M)$. Moreover, for every matrix M, the rectangle covering number $\operatorname{rc}(M)$ does not exceed $(1 + \log(|\operatorname{supp}(M)|))\gamma$ (see Lovász [1975]).

It is not hard to see that every fooling set of cardinality r and order k provides a lower bound $\frac{r}{k}$ on the optimal value γ. For this, sum the inequalities, indexed by the elements of a fooling set of order k, and obtain

$$\sum_{R \in \mathcal{R}(M)} \gamma_R t_R \geq r\,,$$

where γ_R denotes the number of elements from the fooling set covered by R, what gives us the lower bound

$$\gamma = \sum_{R \in \mathcal{R}(M)} t_R \geq \frac{r}{k}\,,$$

since for all $R \in \mathcal{R}(M)$, γ_R is at most k.

[1] In Dietzfelbinger et al. [1996] a matrix, having the rank r and a fooling set of cardinality t, is transformed into another matrix, having the rank r^n and a fooling set of cardinality t^n, $n \geq 0$. Thus, to prove that the upper bound $\operatorname{rank}(M)^2$ for the cardinality of a maximal fooling in matrix M is asymptotically tight, it is enough to construct a matrix with a fooling set of cardinality equal to the squared rank.

[2] In the original paper of Dietzfelbinger et al. [1996], the bound $O(\max_k(\frac{r_k}{k})\log(|I \times X|))$ was stated, but the proof of this bound implies also the bound $O(\max_k(\frac{r_k}{k})\log(|\operatorname{supp}(M)|))$.

4.14.3. Measure of Rectangles. The next bound comes from the dual problem, corresponding to the fractional relaxation of the set cover problem, i.e.

$$\gamma = \max \sum_{(i,x)\in \mathrm{supp}(M)} y_{i,x}$$

$$\sum_{(i,x):(i,x)\in R} y_{i,x} \le 1 \qquad \text{for} \quad R \in \mathcal{R}(M)$$

$$y_{i,x} \ge 0 \qquad \text{for} \quad (i,x) \in \mathrm{supp}(M)\,.$$

This provides a well known lower bound, described in terms of measures. Introducing an atomic measure $\mu : I \times X \to \mathbb{R}$, such that $\mu(i,x) = 0$, for all $i \in I$, $x \in X$, where the element, corresponding to (i,x) is equal zero, gives a lower bound $\mu(M)$ on the rectangle covering number, whenever the measure of every non-zero rectangle $R \in \mathcal{R}(M)$ does not exceed one.

Note that in the original definition of the fooling set of order k it is not allowed to use any of the entries from the support more than once. Without this constraint, the linear relaxation bound is achieved by some generalized fooling set with multiple usage of elements. Considering an optimal dual solution

$$y_{i,x} = \frac{y'_{i,x}}{Y}\,,$$

where both $y'_{i,x}$ and Y are integer, construct a fooling set of order Y, taking $y'_{i,x}$ copies of every element (i,x).

4.14.4. Number of Different Sign Patterns in Columns. Whenever a matrix M with non-negative entries, has n different sign patterns of its rows, the rectangle covering number is at least $\lceil \log_2 n \rceil$. Otherwise, fixing a rectangle cover with less than $\lceil \log_2 n \rceil$ rectangles and associating with each row the set of rectangles from the cover, which involve this row, there are two rows among these n rows with the same set of rectangles, and thus, with the same sign pattern.

Choosing for polyhedra $P_* \subseteq \mathbb{R}^m$, $P^* \subseteq \mathbb{R}^m$ a slack matrix $\mathbf{M}_{\mathrm{slack}}(P^*, P_*) \in \mathbb{R}^{I^* \times X_*}$, such that the set I^* contains an inequality for every face of $\mathcal{L}(P^*, P_*)$, Goemans obtained the lower bound

$$\lceil \log_2(|\mathcal{L}(P^*, P_*)|) \rceil$$

on the rectangle covering number $\mathrm{rc}(M)$, since every face in $\mathcal{L}(P^*, P_*)$ is uniquely determined by the set of incident points from X_*, and thus, rows corresponding to different faces have different sign patterns.

Note that for a zero-one polytope $P \subseteq \mathbb{R}^m$ the face counting bound is $O(m \log m)$, since the number of faces of the polytope P is equal $2^{O(m \log m)}$ (see Fleiner et al. [1999]). If the coordinates of the vertices of a polytope $P \subseteq \mathbb{R}^m$ are from the set $[k]$, then the face counting bound is equal $O(m \log km)$ (see Fleiner et al. [1999]). Actually, both these bounds can be tight for the extension complexity of a polytope (see Section 4.18).

4.15. Rectangle Covering: Graph Point of View

Consider a graph $G(M) = (V, E)$, $V = \mathrm{supp}(M)$, and for every two vertices (i_1, x^1), (i_2, x^2) there exists an edge between them if and only if at least one of the entries(i_1, x^2), (i_2, x^1) does not belong to the support of the matrix M.

Lemma 4.6. *For every matrix $M \in \mathbb{R}^{I \times X}$, the coloring number $\chi(G(M))$ is equal to the rectangle covering number of the matrix $M \in \mathbb{R}^{I \times X}$.*

Moreover, every maximal stable set in the graph $G(M)$ defines a rectangle from $\mathcal{R}(M)$, which is spanned by the corresponding entries in the matrix M. In turn, every rectangle from $\mathcal{R}(M)$ defines a stable set in the graph $G(M)$, vertices of which correspond to the elements from the rectangle.

PROOF. Clearly, every rectangle $R \in \mathcal{R}(M)$ is a stable set in the graph $G(M)$. Thus, having a rectangle covering, we color every vertex of $G(M)$ by a color, corresponding to some rectangle from the cover, which contains the vertex, what establishes the inequality $\chi(G(M)) \le \mathrm{rc}(M)$.

On the other hand, for some coloring of the graph $G(M)$, consider the rectangles, spanned in the matrix M by coloring classes. These rectangles are non-zero, since every coloring class defines a stable set in the graph $G(M)$. Finally, this set of rectangles forms a rectangle covering of the entries $\mathrm{supp}(M)$, and thus, the inequality $\chi(G(M)) \ge \mathrm{rc}(M)$ holds. □

Additionally, fooling sets for a matrix $M \in \mathbb{R}^{I\times X}$ correspond to cliques in the graph $G(M)$. Thus, the lower bound, given by the clique number $\omega(G(M))$, i.e.

$$\omega(G(M)) \le \chi(G(M))\,,$$

is the fooling set bound.

For every matrix M, $G(M) = (V, E)$ and $W \subseteq V$defined as , we have

$$\frac{|W|}{\alpha(G(M)_W)} \le \chi(G(M))\,,$$

where the graph $G(M)_W$ is the subgraph of $G(M)$, induced by W. But, every set of vertices $W \subseteq V$ induces a fooling set of cardinality $|W|$ and of order $\alpha(G(M)_W)$. On the other hand, every fooling set $W \subseteq \mathrm{supp}(M)$ of order k defines a set of vertices $W \subseteq V$, where $\alpha(G(M)_W) \le k$. And thus, the lower bound $\frac{|W|}{\alpha(G(M)_W)}$, where $W \subseteq V$, is the generalized fooling set bound. It is well known (see Lovász [1975]) that the coloring number $\chi(G)$ is bounded by

$$O(\max_{W\subseteq V} \frac{|W|}{\alpha(G(M)_W)} \log |V|)\,,$$

what was reproved by Dietzfelbinger et al. [1996], for the case $G = G(M)$.

The linear relaxation bounds can be obtained in this setting as well, since the coloring number $\chi(G(M))$ is equal to the covering number of the vertices of the graph $G(M)$ by stable sets (inclusion maximal stable sets in the graph $G(M)$ induce non-zero rectangles in the matrix M).

4.16. Lower Bounds on Rectangle Covering Number: Rectangle Measures

Proposition 4.4. *The rectangle covering number* $\mathrm{rc}(P)$ *is equal to* $2m$ *for the cube* $P = [0, 1]^m$.

PROOF. Let us fix the index set I' and let R be a non-zero rectangle of the form $I' \times X'$, with the maximum number of elements, in the slack matrix $\mathbf{M}_{\mathrm{slack}}(P)$, defined by

$$X = \{0, 1\}^m \quad \text{and} \quad P = \{x \in \mathbb{R}^m : 0 \le x_i \le 1\}\,.$$

Thus, the rectangle R is empty, if the index set I' contains both inequalities $0 \le x_i$ and $x_i \le 1$ for some $i \in [m]$. Otherwise, $|X'| = 2^{m-k}$, since every inequality in I' fixes one of the coordinates. Consequently, the rectangle R involves $2^{m-k}k$ entries, where $|I'| = k$.

It is easy to verify that the function $k2^{m-k}$ achieves maximum 2^{m-1} at $k = 1$ or $k = 2$. Moreover, the slack matrix $\mathbf{M}_{\mathrm{slack}}(P)$ has $2m2^{m-1}$ non-zero entries, what proves the statement of the proposition. □

Observation 4.9. *The* m*-dimensional cube* $P = [0, 1]^m$ *forms a minimal extension of itself.*

From Observations 4.9 and 4.3, the vertex extension of the m-dimensional *cross-polytope*, which is defined as

$$\mathrm{conv}(\bigcup_{i=1}^{m} \mathbb{e}_i \cup \bigcup_{i=1}^{m} -\mathbb{e}_i)\,,$$

states a minimal extension of the cross-polytope.

4.17. Lower Bounds on Rectangle Covering Number: Fooling Set

In this section, applications of the fooling set technique are presented. For all examples here, the fooling set technique gives a tight estimation of the rectangle covering bound. Moreover, it is shown that the listed polytopes are minimal extensions of themself.

4.17.1. Combinatorial Cube. Here, Proposition 4.4 is reproved, providing a fooling set of the proper cardinality.

Proposition 4.5. *The rectangle covering bound is equal* $2m$ *for the* m*-dimensional cube* $P = [0,1]^m$.

PROOF. Associate to every inequality $0 \le x_i$, $x_i \le 1$, $i \in [m]$ a vertex of the cube P, such that the resulting pairs form a fooling set. The vertex $x^{0,i} \in \mathbb{R}^m$, corresponding to the inequality $0 \le x_i$, $i \in [m]$, is defined by its coordinates

$$x_j^{0,i} = \begin{cases} 1 & \text{if} \quad 1 \le j \le i \\ 0 & \text{if} \quad i < j \le m\,. \end{cases}$$

And for the inequality $x_i \le 1$, $i \in [m]$, we define the vertex $x^{1,i} \in \mathbb{R}^m$

$$x_j^{1,i} = \begin{cases} 0 & \text{if} \quad 1 \le j \le i \\ 1 & \text{if} \quad i < j \le m\,. \end{cases}$$

Clearly, the defined set of vertex-facet pairs forms a fooling set of cardinality $2m$. □

4.17.2. Birkhoff Polytope.

Proposition 4.6. *The rectangle covering bound* $\mathrm{rc}(\mathrm{P}^n_{\mathrm{birk}})$ *equals* n^2 *for the Birkhoff polytope* $\mathrm{P}^n_{\mathrm{birk}} \subseteq \mathbb{R}^{n^2}$, $n \ge 5$.

PROOF. Recall that the Birkhoff polytope $\mathrm{rc}(\mathrm{P}^n_{\mathrm{birk}})$ is described as $0 \le x$ and

$$\sum_{t=1}^{n} x_{i,t} = 1 \;\text{ for all } i \in [n] \qquad \text{and} \qquad \sum_{t=1}^{n} x_{t,j} = 1 \;\text{ for all } j \in [n]\,.$$

For every inequality $x_{i,j} \ge 0$, $i \in [n]$, $j \in [n]$, define a vertex $x^{i,j} \in \mathbb{R}^{n^2}$, giving a permutation $\pi \in \mathfrak{S}(n)$, i.e. $x^{i,j}_{k,t}$ is equal one if and only if $\pi(k)$ is equal t. Let $\pi(i) = j$ and $\pi(i+1) = j+1$ (indices are understood modulo n). Moreover, set $\pi(t) = i+j+1-t$ for all t not equal to i or $i+1$.

Let us assume that for two different inequalities $0 \le x_{i',j'}$, $0 \le x_{i'',j''}$ (the pair (i',j') is not equal (i'',j'')), where π', $\pi'' \in \mathfrak{S}(n)$ are the corresponding permutations, the equations $\pi'(i'') = j''$ and $\pi''(i') = j'$ hold. Thus, $i''+j''-i'-j'$ is equal 1 or 2 (modulo n), due to $\pi'(i'') = j''$. Similarly, $i'+j'-i''-j''$ is equal 1 or 2 (modulo n). But this is impossible because $n \ge 5$, what shows that the constructed vertex-facet set is a fooling set. □

Observation 4.10. *The Birkhoff polytope* $\mathrm{P}^n_{\mathrm{birk}} \subseteq \mathbb{R}^{n^2}$, $n \ge 5$, *is a minimal extension of itself.*

4.17.3. Matching Polytope in Full Bipartite Graph. From Section 4.17.2 and linear isomorphism of the Birkhoff polytope $\mathrm{P}^n_{\mathrm{birk}}$ and the perfect matching polytope $\mathrm{P}^n_{\mathrm{match}}(G)$, where G is the complete bipartite graph $K_{n,n}$, one obtains the following result.

Proposition 4.7. *The rectangle covering bound* $\mathrm{rc}(\mathrm{P}^n_{\mathrm{match}}(G))$, $G = K_{n,n}$, *is equal* n^2 *for the perfect matching polytope* $\mathrm{P}^n_{\mathrm{match}}(G) \subseteq \mathbb{R}^{n^2}$, $n \geq 5$.

Actually, the fooling set constructed in the proof of Proposition 4.6 can be extended, to show the next result.

Proposition 4.8. *The rectangle covering bound* $\mathrm{rc}(\mathrm{P}_{\mathrm{match}}(G))$, $G = K(n,n)$, *is equal* $n^2 + 2n$ *for the matching polytope* $\mathrm{P}_{\mathrm{match}}(G) \subseteq \mathbb{R}^{n^2}$, $n \geq 5$.

PROOF. Considering general matchings, there are $2n$ additional inequalities

$$\sum_{t=1}^{n} x_{i,t} \leq 1 \text{ for all } i \in [n] \qquad \text{and} \qquad \sum_{t=1}^{n} x_{t,j} \leq 1 \text{ for all } j \in [n]\,,$$

indexed by vertices $V^* = [n]$, $V_* = [n]$, where V^*, V_* define the bipartition of G.

For the non-negativity constraints $x_{v^*,v_*} \geq 0$, $v^* \in V^* = [n]$, $v_* \in V_* = [n]$, take the matchings, associated with the corresponding permutations from Proposition 4.6.

Additionally, for the inequalities $\sum_{v^*=1}^{n} x_{v^*,v_*} \leq 1$, $v_* \in V_*$, take the matching

$$\{(w^*, w_*) : w^* = j+1, w^* \in V^* \text{ and } w_* = j, w_* \in V_*\,, j \neq v_*\}$$

of cardinality $n-1$. In the same way, define the matching

$$\{(w^*, w_*) : w^* = j\,, w^* \in V^* \text{ and } w_* = j+1, w_* \in V_*\,, j \neq v^*\}$$

for the inequality $\sum_{v_*=1}^{n} x_{v^*,v_*} \leq 1$, $v^* \in V^*$, to finish the construction of the fooling set. □

Observation 4.11. *The matching polytope* $\mathrm{P}_{\mathrm{match}}(G) \subseteq \mathbb{R}^{n^2}$, $G = K_{n,n}$, $n \geq 5$, *is a minimal extension of itself.*

4.18. Lower Bounds on Rectangle Covering Number: Face Counting

4.18.1. Permutahedron.

Proposition 4.9 (Goemans)**.** *The rectangle covering bound* $\mathrm{rc}(\Pi_n)$ *for the permutahedron* $\Pi_n \subseteq \mathbb{R}^n$ *is equal* $\Omega(n \log n)$.

PROOF. The number of vertices of the permutahedron Π_n is equal $n!$, what gives us the lower bound $\log(n!) = \Omega(n \log n)$. □

Observation 4.12. *The extended formulation in Section 2.21 is an asymptotically minimal extension for the permutahedron* $\Pi_n \subseteq \mathbb{R}^n$.

4.18.2. Huffman Polytope.

Proposition 4.10. *The rectangle covering bound* $\mathrm{rc}(\mathrm{P}^n_{\mathrm{huff}})$ *is equal* $\Omega(n \log n)$ *for the Huffman polytope* $\mathrm{P}^n_{\mathrm{huff}} \subseteq \mathbb{R}^n$.

PROOF. Nguyen et al. [2010] showed that the number of facets of the Huffman polytope $\mathrm{P}^n_{\mathrm{huff}}$ is equal $\Omega(n!)$, what gives us the lower bound $\Omega(n \log n)$. □

Observation 4.13. *The extended formulation in Section 2.24 is an asymptotically minimal extension for the Huffman polytope* $\mathrm{P}^n_{\mathrm{huff}} \subseteq \mathbb{R}^n$.

4.18.3. Cardinality Indicating Polytope.

Proposition 4.11. *The rectangle covering bound* $\mathrm{rc}(\mathrm{P}^n_{\mathrm{card}})$ *is equal* $\Omega(n \log n)$ *for the cardinality indicating* $\mathrm{P}^n_{\mathrm{card}} \subseteq \mathbb{R}^{2n+1}$.

PROOF. The cardinality indicating polytope $\mathrm{P}^n_{\mathrm{card}}$ has $n!$ different faces, what proves the rectangle covering lower bound $\Omega(n \log n)$.

Indeed, define a non-trivial face of the cardinality indicating polytope $\mathrm{P}^n_{\mathrm{card}}$, which is indexed by a permutations $\mu \in \mathfrak{S}(n)$, as the intersection of $n-1$ facets

$$\sum_{v=1}^{q} x_{\mu^{-1}(v)} - \sum_{k=0}^{q} k z_{k+1} - q \sum_{k=q+1}^{n} z_{k+1} = 0 \quad \text{for} \quad 1 \le q \le n-1.$$

Two such faces are different, whenever they correspond to different permutations μ', $\mu'' \in \mathfrak{S}(n)$. Namely, there is q, $1 \le q \le n-1$, such that $\mu'^{-1}([q])$ is not equal to $\mu''^{-1}([q])$, and thus, the vertex of the cardinality indicating polytope $\mathrm{P}^n_{\mathrm{card}}$, defined as

$$\begin{aligned} x_i &= 1 \quad \text{if } \mu'(i) \in [q] \\ x_i &= 0 \quad \text{otherwise} \\ z_j &= 1 \quad \text{if } j = q+1 \\ z_j &= 0 \quad \text{otherwise}\ , \end{aligned}$$

belongs to the face, indexed by the permutation μ', but does not belong to the face, indexed by the permutation μ''. □

Observation 4.14. *The extended formulation in Section 2.21 is an asymptotically minimal extension for the cardinality indicating polytope* $\mathrm{P}^n_{\mathrm{card}} \subseteq \mathbb{R}^{2n+1}$.

4.19. Lower Bounds on Rectangle Covering Number: Direct Application

Sometimes, one has to study the possible rectangle coverings directly, what, for example, is done in Lemma 6.5. This lemma provides us a lower bound for k-neighborly polytopes with n vertices.

Proposition 4.12. *The rectangle covering bound* $\mathrm{rc}(P)$ *is equal to*

$$\min(n-k, \frac{(k+1)(k+2)}{2} - 1)$$

for every k-neighborly polytope $P \subseteq \mathbb{R}^m$, $\mathrm{vert}(P) = n$.

Thus, the above proposition provides an asymptotically tight bound on the extension complexity when $n = \Theta(\sqrt{k})$. In this case, the vertex extension provides an asymptotically minimal extension for every k-neighborly polytope $P \subseteq \mathbb{R}^m$, $|\mathrm{vert}(P)| = n = \Theta(\sqrt{k})$.

CHAPTER 5

Bounds on Symmetric Extended Formulations of Polytopes

A special type of extended formulations are extended formulations which preserve symmetries of the initial polytope. Combinatorial polytopes are a natural field to study symmetric extended formulations, since many objects which induce combinatorial polytopes are highly symmetric, what is inherited by the polytopes themself. In some sense it could appear natural to regard extended formulations, which respect the symmetries of the initial polytope, and indeed, a lot of extended formulations are symmetric. But such a restriction to symmetric extended formulations, as we will show, could be quite expensive in terms of the size of the obtained formulations.

For many combinatorial polytopes we can provide strong lower bounds on the sizes of symmetric extended formulations in contrast to general lower bounds. The first result in this area was given by Yannakakis [1991] in his pathbreaking paper, where he showed that for the perfect matching polytope for the complete graph K_n a compact symmetric extended formulation does not exist. As a corollary the non-existence of a compact symmetric extended formulation for the travelling salesman polytope in the complete graph K_n was obtained.

In his paper Yannakakis also conjectured that the symmetry requirement is not more than a technical condition for the proof: "We do not think that asymmetry helps much. Thus, prove that the matching and TSP polytopes cannot be expressed by polynomial size LP's without the asymmetry assumption". Indeed, the travelling salesman polytope does not admit a compact extension, what was shown by Fiorini et al. [2011b].

Even though there is no known compact extended formulation for the perfect matching polytope, we will show examples of other related polytopes where no compact symmetric extended formulation exists, but nevertheless we will provide a compact extended formulation, what establishes a significant size gap between symmetric and non-symmetric extended formulations in general (see Kaibel, Pashkovich, and Theis [2010]).

Moreover, we will use the techniques, which were invented by Yannakakis, to handle such subtle cases as the permutahedron and the cardinality indicating polytope, where the symmetric extended formulations have size $\Omega(n^2)$, but one can provide an extended formulation of size $\Theta(n \log n)$. These examples are interesting, since the bounds established for the symmetric and non-symmetric extensions are tight. For the parity polytope we will prove that the symmetric extended formulations are of size $\Omega(n \log n)$, but the minimal known symmetric extended formulation has size $\Theta(n^2)$.

5.1. Symmetric Extensions

Consider a polytope $P \subseteq \mathbb{R}^m$ with an extension, given by a polyhedron $Q \subseteq \mathbb{R}^d$ and an affine map $p : \mathbb{R}^d \to \mathbb{R}^m$. The size of this extension, as in the previous chapters, is defined as the number of facets of the polyhedron Q. Moreover, we assume that a finite group of affine maps G acts on the polytope P. The extension Q, p is called *symmetric* with respect to the symmetry group G, if for every $\pi \in G$, there exists an affine map $\kappa_\pi : \mathbb{R}^d \to \mathbb{R}^d$, such that

$$\kappa_\pi.Q = Q \tag{5.1.1}$$

and the map κ_π is compatible with the affine map $p : \mathbb{R}^d \to \mathbb{R}^m$ in the following way

$$p(\kappa_\pi.y) = \pi.p(y) \quad \text{for every} \quad y \in \mathbb{R}^d\,. \tag{5.1.2}$$

From (5.1.1), it follows that for each $\pi \in G$ the affine map κ_π maps the affine hull $\mathrm{aff}(Q)$ to itself. Moreover, the linear map μ_π, associated with κ_π, maps the recession cone $\mathrm{rec}(Q)$ and the lineality space $\mathrm{lineal}(Q)$ of Q to themselves.

As in the case of general extended formulations, we can show that symmetric extended formulations of polytopes can be assumed to be realized by polytopes without any loss in terms of size.

Lemma 5.1. *For every symmetric extension $Q \subseteq \mathbb{R}^d$, $p : \mathbb{R}^d \to \mathbb{R}^m$ of a polytope $P \subseteq \mathbb{R}^m$ with respect to a symmetry group G, there exists a symmetric extension of smaller or the same size, defined via a full-dimensional polytope $Q' \subseteq \mathbb{R}^{d'}$ and an affine map $p' : \mathbb{R}^{d'} \to \mathbb{R}^m$.*

PROOF. The polyhedron $Q \subseteq \mathbb{R}^d$ can be assumed to be full-dimensional. Otherwise, consider an extension $Q^* = q(Q) \subseteq \mathbb{R}^{d^*}$, $p^* = p \circ q^{-1} : \mathbb{R}^{d^*} \to \mathbb{R}^m$ of the polytope $P \subseteq \mathbb{R}^m$, where d^* is the dimension of the affine space $\mathrm{aff}(Q)$, and the map $q : \mathrm{aff}(Q) \to \mathbb{R}^{d^*}$ defines an affine isomorphism between $\mathrm{aff}(Q)$ and $\mathbb{R}^{d^*}$. To show that the constructed extension is symmetric, define the affine map $\kappa^*_\pi = q \circ \kappa_\pi \circ q^{-1} : \mathbb{R}^{d^*} \to \mathbb{R}^{d^*}$. Obviously, the conditions (5.1.1) and (5.1.2) are satisfied for the extension $Q^* \subseteq \mathbb{R}^{d^*}$, $p^* : \mathbb{R}^{d^*} \to \mathbb{R}^m$ and the symmetry group G.

Moreover, it can be assumed that the lineality space $\mathrm{lineal}(Q)$ coincides with the recession cone $\mathrm{rec}(Q)$ of the full-dimensional polyhedron Q. Otherwise, we can transform the extension $Q \subseteq \mathbb{R}^d$, $p : \mathbb{R}^d \to \mathbb{R}^m$ into another extension $Q^* \subseteq \mathbb{R}^d$, $p^* : \mathbb{R}^d \to \mathbb{R}^m$ of the same or smaller size, such that the recession cone $\mathrm{rec}(Q^*)$ coincides with the lineality space $\mathrm{lineal}(Q^*)$. Indeed, since P is a polytope, i.e. the recession cone $\mathrm{rec}(P)$ is the zero vector, the recession cone $\mathrm{rec}(Q)$ is contained in the kernel of the linear map, associated with p. Namely, for every vector $r \in \mathbb{R}^d$ from the recession cone $\mathrm{rec}(Q)$ of Q and $y \in \mathbb{R}^d$

$$p(y + r) = p(y) \tag{5.1.3}$$

holds, and thus the polyhedron $Q^* = Q - \mathrm{rec}(Q) \subseteq \mathbb{R}^d$, together with the affine map $p^* = p : \mathbb{R}^d \to \mathbb{R}^m$, forms a symmetric extension of the polytope P, where the maps $\kappa^*_\pi = \kappa_\pi$, $\pi \in G$, are defined as for the extension Q, p. Due to the equation (5.1.3)

$$p^*(Q^*) = p(Q - \mathrm{rec}(Q)) = p(Q) = P$$

and since the linear map μ_π associated to κ_π, $\pi \in G$, maps $\mathrm{rec}(Q)$ on itself, it follows that

$$\kappa^*_\pi(Q^*) = \kappa_\pi.(Q - \mathrm{rec}(Q)) = \kappa_\pi(Q) - \mu_\pi(\mathrm{rec}(Q)) = Q - \mathrm{rec}(Q) = Q^*$$

holds for every $\pi \in G$. Finally, the number of facets of the polyhedron $Q^* = Q - \mathrm{rec}(Q)$ is less or equal to the number of facets of the initial polyhedron Q (Appendix: Lemma 6.3).

If the symmetric extension $Q \subseteq \mathbb{R}^d$, $p : \mathbb{R}^d \to \mathbb{R}^m$ is, such that the polyhedron $Q \subseteq \mathbb{R}^d$ is full-dimensional and the recession cone $\mathrm{rec}(Q)$ coincides with the lineality space $\mathrm{lineal}(Q)$, we can construct another symmetric extension via the polytope $Q^* = Q \cap \mathrm{lineal}(Q)^\perp \subseteq \mathbb{R}^d$ and the affine map $p^* = p : \mathbb{R}^d \to \mathbb{R}^m$.

Indeed, from (5.1.3) for the polyhedron Q^*, the equation

$$p^*(Q^*) = p(Q \cap \mathrm{lineal}(Q)^\perp) = p(Q) = P$$

holds, since the intersection $Q \cap \mathrm{lineal}(Q)^\perp$ is the orthogonal projection $\mathrm{proj}_{\mathrm{lineal}(Q)^\perp}(Q)$ of the polyhedron Q on the affine space $\mathrm{lineal}(Q)^\perp$. The recession cone of the polyhedron $Q^* \subseteq \mathbb{R}^d$ is equal to $\mathrm{lineal}(Q) \cap \mathrm{lineal}(Q)^\perp = \{\mathbf{0}_d\}$, what implies that Q^* is a polytope. The size of the extension Q^*, p^* is less or equal to the size of the extension Q, p, since the number of facets of the polytope $Q^* = Q \cap \mathrm{lineal}(Q)^\perp$ is equal to the number of facets of the polyhedron Q.

To show that the obtained extension Q^*, p^* is symmetric, define the affine maps κ^*_π, $\pi \in G$ to be equal $\mathrm{proj}_{\mathrm{lineal}(Q)^\perp} \circ \kappa_\pi$. Due to (5.1.3), we have

$$p^*(\kappa^*_\pi.y) = p(\mathrm{proj}_{\mathrm{lineal}(Q)^\perp}(\kappa_\pi.y)) = p(\kappa_\pi.y) = \pi.p(y) = \pi.p^*(y)\,,$$

and since the maps κ_π are non-degenerate (due to (5.1.1) and since the polyhedron Q is full-dimensional)

$$\begin{aligned}\kappa^*_\pi.Q^* =\, & \mathrm{proj}_{\mathrm{lineal}(Q)^\perp}(\kappa_\pi.(Q \cap \mathrm{lineal}(Q)^\perp)) = \\ & \mathrm{proj}_{\mathrm{lineal}(Q)^\perp}(\kappa_\pi.Q \cap \kappa_\pi.\,\mathrm{lineal}(Q)^\perp) = \\ & \mathrm{proj}_{\mathrm{lineal}(Q)^\perp}(Q \cap \kappa_\pi.\,\mathrm{lineal}(Q)^\perp) = \\ & \mathrm{proj}_{\mathrm{lineal}(Q)^\perp}(Q) \cap \mathrm{proj}_{\mathrm{lineal}(Q)^\perp}(\kappa_\pi.\,\mathrm{lineal}(Q)^\perp) = \\ & Q^* \cap \mathrm{lineal}(Q)^\perp = Q^*\,.\end{aligned}$$

For the fourth equation, we used the equation $Q = Q + \mathrm{lineal}(Q)$, and thus

$$\mathrm{proj}_{\mathrm{lineal}(Q)^\perp}(Q \cap U) = \mathrm{proj}_{\mathrm{lineal}(Q)^\perp}(Q) \cap \mathrm{proj}_{\mathrm{lineal}(Q)^\perp}(U)$$

holds for every set $U \subseteq \mathbb{R}^d$. The fifth equation is based on the fact that the non-degenerate linear map μ_π, associated with the affine map κ_π, maps $\mathrm{lineal}(Q)$ on itself. And thus, $\mathrm{proj}_{\mathrm{lineal}(Q)^\perp}(\mu_\pi.\,\mathrm{lineal}(Q)^\perp)$ is equal to $\mathrm{lineal}(Q)^\perp$, what implies that the affine space $\mathrm{proj}_{\mathrm{lineal}(Q)^\perp}(\kappa_\pi.\,\mathrm{lineal}(Q)^\perp)$ is equal to the affine space $\mathrm{lineal}(Q)^\perp$.

As in the beginning of the proof, we transform the extension $Q^* = Q \cap \mathrm{lineal}(Q)^\perp \subseteq \mathbb{R}^d$, $p^* = p : \mathbb{R}^d \to \mathbb{R}^m$ into a symmetric extension of the polytope P via a full-dimensional polytope $Q' \subseteq \mathbb{R}^{d'}$ and an affine map $p' : \mathbb{R}^{d'} \to \mathbb{R}^m$. ☐

Lemma 5.2. *For every symmetric extension $Q \subseteq \mathbb{R}^d$, $p : \mathbb{R}^d \to \mathbb{R}^m$ of a polytope $P \subseteq \mathbb{R}^m$ with respect to the symmetry group G, there exists a symmetric extension of smaller or the same size, defined via a full-dimensional polytope $Q' \subseteq \mathbb{R}^{d'}$ and an affine map $p' : \mathbb{R}^{d'} \to \mathbb{R}^m$, such that for each $\pi \in G$ the affine maps $\kappa'_\pi : \mathbb{R}^{d'} \to \mathbb{R}^{d'}$, $\pi \in G$ are isometries.*

PROOF. Due to Lemma 5.1, there is an extension $Q^* \subseteq \mathbb{R}^{d^*}$, $p^* : \mathbb{R}^{d^*} \to \mathbb{R}^m$ of the polytope $P \subseteq \mathbb{R}^m$, where Q^* is a full-dimensional polytope. The size of the extension Q^*, p^* is less or equal to the size of the extension Q, p.

The group G^*, generated by the affine maps κ^*_π, $\pi \in G$ is finite, since every element of this group can be uniquely identified with some permutation of the vertices $\mathrm{vert}(Q^*)$ of the polytope Q^*. Thus, the group H^*, which consists of the linear maps, corresponding to the affine maps in G^*, is also finite (in particular we have $|H^*| = |G^*|$). Thus, we are able to define a new scalar product on $\mathbb{R}^{d^*}$

$$\langle x, y\rangle^* = \sum_{\mu^* \in H^*} \frac{\langle \mu^*.x, \mu^*.y\rangle}{|H^*|}\,.$$

With respect to this new scalar product, every affine map $\kappa^{**} \in G^*$ acts as an isometry, since for every $\mu^{**} \in H^*$

$$\langle \mu^{**}.x, \mu^{**}.y\rangle^* = \sum_{\mu^* \in H^*} \frac{\langle \mu^*\mu^{**}.x, \mu^*\mu^{**}.y\rangle}{|H^*|} = \sum_{\mu^{***} \in H^*} \frac{\langle \mu^{***}.x, \mu^{***}.y\rangle}{|H^*|} = \langle x, y\rangle^*\,.$$

To preserve the standard form of the scalar product as the sum $\sum_{i=1}^{d^*} x_i y_i$, consider another symmetric extension of the polytope $P \subseteq \mathbb{R}^m$, given by the polytope $Q' = q(Q^*) \subseteq \mathbb{R}^{d^*}$ and the affine map $p' = p^* \circ q^{-1} : \mathbb{R}^{d^*} \to \mathbb{R}^m$, where the affine map $q : \mathbb{R}^{d^*} \to \mathbb{R}^{d^*}$ is defined as a transformation from the standard orthonormal basis for the scalar product $\langle x, y\rangle$ to an orthonormal basis for the scalar product $\langle x, y\rangle^*$. The resulting extension is symmetric, where the affine maps κ'_π, $\pi \in G$ are defined as $q \circ \kappa^*_\pi \circ q^{-1}$. Additionally, we have

$$\langle \mu'_\pi.x, \mu'_\pi.y\rangle = \langle q\mu^*_\pi q^{-1}.x, q\mu^*_\pi q^{-1}.y\rangle = \langle \mu^*_\pi q^{-1}.x, \mu^*_\pi q^{-1}.y\rangle^* = \langle q^{-1}.x, q^{-1}.y\rangle^* = \langle x, y\rangle$$

for all $x, y \in \mathbb{R}^{d^*}$. □

5.2. Symmetric Extended Formulations

An extended formulation of a polytope $P \subseteq \mathbb{R}^m$, given by the linear system

$$A^{\leq} y \leq b^{\leq} \quad \text{and} \quad A^{=} y = b^{=}\,, \tag{5.2.1}$$

where $A^{\leq} \in \mathbb{R}^{f\times d}$, $b^{\leq} \in \mathbb{R}^f$, $A^{=} \in \mathbb{R}^{r\times d}$, $b^{=} \in \mathbb{R}^r$ and an affine map $p : \mathbb{R}^d \to \mathbb{R}^m$, is called *symmetric* with respect to the action of a group G on the polytope P, if for every $\pi \in G$ there exists an affine map $\zeta_\pi : \mathbb{R}^d \to \mathbb{R}^d$, such that it satisfies (5.1.2), i.e. $p(\zeta_\pi.y) = \pi.p(y)$ for every $y \in \mathbb{R}^d$, and the linear system

$$A^{\leq} \zeta_\pi.y \leq b^{\leq} \quad \text{and} \quad A^{=} \zeta_\pi.y = b^{=}\,, \tag{5.2.2}$$

is the linear system (5.2.1) with reordered constraints (after collecting the coefficients). The size of the extended formulation is defined as the number of inequalities in the linear system (5.2.1).

The following lemma is a trivial observation from the definition of symmetric extended formulation.

Lemma 5.3. *For every symmetric extended formulation of a polytope $P \subseteq \mathbb{R}^m$, there exists a symmetric extension of a smaller or the same size.*

We call an extension $Q \subseteq \mathbb{R}^d$, $p : \mathbb{R}^d \to \mathbb{R}^m$ of a polytope $P \subseteq \mathbb{R}^m$ a *subspace extension*, if the polyhedron Q is an intersection of the first orthant $\mathbb{R}^d_+$ with some affine subspace. Analogously, an extended formulation, given by a system $A^{=}y = b^{=}$, $y \in \mathbb{R}^d_+$ and an affine map $p : \mathbb{R}^d \to \mathbb{R}^m$, is called a *subspace extended formulation*. Obviously, the size of a subspace extension is less or equal to the dimension d of the ambient space $\mathbb{R}^d$, the same holds for subspace extended formulations.

The next lemma shows that every symmetric extension induces a symmetric subspace extended formulation of a smaller or the same size. Additionally the group action of G on the ambient space of the extended formulation, could be restricted to coordinate permutations.

Lemma 5.4. *For every symmetric extension $Q \subseteq \mathbb{R}^d$, $p : \mathbb{R}^d \to \mathbb{R}^m$ of a polytope $P \subseteq \mathbb{R}^m$ with respect to a group G, there exists a symmetric subspace extended formulation of a smaller or the same size, such that for every $\pi \in G$ the affine map ζ_π is a coordinate permutation.*

PROOF. Due to Lemma 5.2, the symmetric extension can be assumed to be given by a full-dimensional polytope $Q \subseteq \mathbb{R}^d$ with f facets and an affine map $p : \mathbb{R}^d \to \mathbb{R}^m$, such that the affine maps κ_π, $\pi \in G$ are isometries.

The polytope Q is defined uniquely (up to reordering of constraints) by a minimal system of linear inequalities

$$0 \leq b^{\leq} - A^{\leq} y\,, \tag{5.2.3}$$

such that $A^{\leq} \in \mathbb{R}^{f\times d}$, $\|A^{\leq}_{i,*}\| = 1$ for all $i \in [f]$. Collecting coefficients in the system $A^{\leq} \kappa_\pi.y \leq b^{\leq}$, we obtain another system of linear inequalities

$$0 \leq b^{\leq} - A^{\leq} \kappa_\pi.y = b^{\leq^*} - A^{\leq^*} y\,, \tag{5.2.4}$$

such that $\|A^{\leq^*}_{i,*}\| = 1$, because for every $\pi \in G$ the affine map κ_π is an isometry. The linear system $A^{\leq^*} y \leq b^{\leq^*}$ describes the polytope $\kappa_\pi^{-1}.Q = Q$ (affine maps κ_π, $\pi \in G$

are non-degenerate). Thus, since the system (5.2.3) for the polytope $Q \subseteq \mathbb{R}^d$ is unique (up to reordering of constraints), the linear system $A^{\leq^*} y \leq b^{\leq^*}$ is obtained from the system $A^{\leq} y \leq b^{\leq}$ by a constraint permutation $\sigma_\pi \in \mathfrak{S}(f)$.

Consider the following extended formulation of the polytope $P \subseteq \mathbb{R}^m$ given by the linear system

$$A^{\leq} y + z = b^{\leq}, \quad z \geq 0 \tag{5.2.5}$$

and the affine map p^* defined as $p \circ \mathrm{proj}_y$.

The system (5.2.5), together with the map p^*, defines an extended formulation of the polytope P, since the projection $\mathrm{proj}_y(Q^*)$ of the polyhedron Q^* (actually, Q^* is a polytope), defined by the system (5.2.5), is equal to the polytope Q.

Moreover, this extended formulation is symmetric with the following affine maps ζ^*_π for every $\pi \in G$

$$\zeta^*_\pi.(y,\, z) = (\kappa_\pi.y,\, \sigma_\pi.z)\,,$$

since the condition (5.1.2) follows from

$$p^*(\zeta^*_\pi.(y\,, z)) = p(\mathrm{proj}_y(\zeta^*_\pi.(y\,, z))) = p(\kappa_\pi.y) = \pi.p(y) = \pi.p^*(y, z)\,,$$

and (5.2.1) is satisfied from the construction of the extended formulation (5.2.5).

Moreover, since the matrix $A^{\leq} \in \mathbb{R}^{f \times d}$ has full column rank (Q is a polytope), there exists an affine map $q : \mathbb{R}^f \to \mathbb{R}^d$, such that $y = q(z)$ for all (y, z), satisfying the linear system (5.2.5). This shows,that the projection $Q' = \mathrm{proj}_z(Q^*)$ of the polyhedron Q^* on z variables, together with the affine map $p' = p \circ q : \mathbb{R}^f \to \mathbb{R}^m$ defines a symmetric extension of the polytope P, where the affine maps κ'_π, $\pi \in G$ are defined as coordinate permutations σ_π. Indeed,

$$\kappa'_\pi.Q' = \sigma_\pi.Q' = \sigma_\pi.\,\mathrm{proj}_z(Q^*) = \mathrm{proj}_z(\zeta_\pi.Q^*) = \mathrm{proj}_z(Q^*) = Q'$$

and

$$p'(\kappa'_\pi.z) = p(q(\sigma_\pi.z)) = p(\kappa_\pi.q(z)) = \pi.p(q(z)) = \pi.p'(z)\,,$$

since $y = q(z)$ for all points (y, z) from Q^* and $\zeta^*_\pi.Q^* = Q^*$.

The projection $Q' = \mathrm{proj}_z(Q^*) \subseteq \mathbb{R}^f$ is defined by a linear system of the form $A'z = b'$, $z \geq 0$, where $A' \in \mathbb{R}^{r' \times q}$ and no two rows of A' are equal. We can assume that containing an equation $\langle a_i, z\rangle = b_i$, the linear equations $A'z = b'$ contain also the equation $\langle \sigma_\pi.a_i, z\rangle = b_i$ for every coordinate permutation σ_π, $\pi \in G$, since the group generated by σ_π, $\pi \in G$ is finite (σ_π, $\pi \in G$ are coordinate permutations). □

Further, we can assume that the affine transformations in the ambient space of symmetric extensions, corresponding to elements of the symmetry group G, are given as coordinate permutations.

Observation 5.1. *Every symmetric extension of a polytope induces a symmetric subspace extension of smaller or the same size. Moreover, for every $\pi \in G$ the affine map $\kappa_\pi : \mathbb{R}^d \to \mathbb{R}^d$ for the induced symmetric extension is a coordinate permutation.*

5.3. Symmetric Section

Through the proofs of lower bounds, we do not use the symmetry of extensions directly, but the existence of so called symmetric section, which is a weaker condition.

A map $s : \mathrm{vert}(P) \to Q$ is called a *section* for an extension $Q \subseteq \mathbb{R}^d$, $p : \mathbb{R}^d \to \mathbb{R}^m$ of a polytope $P \subseteq \mathbb{R}^m$, if for every $x \in \mathrm{vert}(P)$

$$p(s(x)) = x\,. \tag{5.3.1}$$

Thus section s assigns to every vertex $x \in \mathrm{vert}(P)$ a point $s(x) \in Q$ from the fiber

$$p^{-1}(x) = \{y \in \mathbb{R}^d : p(y) = x\}\,.$$

The section s induces a bijection between $\operatorname{vert}(P)$ and the set $s(\operatorname{vert}(P)) \subseteq Q$, whose inverse map is defined by p.

A section is called *symmetric* with respect to the action of a group G on a polytope P, if for every $\pi \in G$, there exists an affine map $\eta_\pi : \mathbb{R}^d \to \mathbb{R}^d$, such that

$$s(\pi.x) = \eta_\pi.s(x) \tag{5.3.2}$$

for every $x \in \operatorname{vert}(P)$.

Note that the maps η_π, $\pi \in G$ do not have to satisfy the conditions (5.1.1) from the definition for symmetric extensions, i.e. $\eta_\pi.Q$ does not have to coincide with the polyhedron Q. However, the equation $p(\eta_\pi.y) = \pi.p(y)$ is satisfied automatically for the maps η_π, $\pi \in G$ and points from

$$\operatorname{aff}(\{s(x) : x \in \operatorname{vert}(P)\})\,.$$

Due to Observation 5.1, we can restrict our attention to symmetric subspace extensions $Q \subseteq \mathbb{R}^d$, $p : \mathbb{R}^d \to \mathbb{R}^m$, where affine maps κ_π, $\pi \in G$ are coordinate permutations $\mathfrak{S}(d)$. Thus, for further considerations we can assume that the conditions of the following lemma are satisfied.

Lemma 5.5. *For every symmetric extension via a polyhedron $Q \subseteq \mathbb{R}^d$ and an affine map $p : \mathbb{R}^d \to \mathbb{R}^m$ of a polytope P with respect to a group G, such that the affine map κ_π for every $\pi \in G$ is a coordinate permutation, there exists a symmetric section, and the affine maps η_π, $\pi \in G$ can be chosen as the maps κ_π.*

PROOF. First, observe that a symmetric extension satisfies

$$\kappa_\pi.p^{-1}(x) = p^{-1}(\pi.x) \tag{5.3.3}$$

for all $\pi \in G$. The inclusion $\kappa_\pi.p^{-1}(x) \subseteq p^{-1}(\pi.x)$ follows already from (5.1.2), what leads to the equality since both affine subspaces $\kappa_\pi.p^{-1}(x)$, $p^{-1}(\pi.x)$ have the same dimension.

We assume that the group G acts transitively on the set of vertices $\operatorname{vert}(P)$, otherwise consider each orbit under the action of G on $\operatorname{vert}(P)$ separately. Fix a vertex $x^* \in \operatorname{vert}(P)$ with a point $y^* \in Q$, such that $p(y^*) = x^*$, and define

$$s(x^*) = \frac{\sum_{\kappa \in S^*} \kappa.y^*}{|S^*|}\,, \tag{5.3.4}$$

where S^* is a subroup of the finite group G^*, generated by κ_π, $\pi \in G$, such that

$$S^* = \{\kappa \in G^* : \kappa.p^{-1}(x^*) = p^{-1}(x^*)\}\,. \tag{5.3.5}$$

The point $s(x^*)$ lies in the polyhedron Q and its projection $p(s(x^*))$ is equal to x^*, since $s(x^*)$ is a convex combination of points with these properties.

For every $x \in \operatorname{vert}(P)$, we choose $\pi_x \in G$, such that $\pi_x.x^* = x$, using the transitivity of the action of G on $\operatorname{vert}(P)$, and define

$$s(x) = \kappa_{\pi_x}.s(x^*)\,, \tag{5.3.6}$$

and thus, the point $s(x)$ lies in $Q \cap p^{-1}(x)$ due to (5.1.1) and (5.1.2).

To finish the proof, it suffices to show $\kappa_\sigma.s(x) = s(\sigma.x)$ for every $\sigma \in G$ and $x \in \operatorname{vert}(P)$. Let us show that the map $\kappa^{-1}_{\pi_{\sigma.x}}\kappa_\sigma\kappa_{\pi_x}$ belongs to S^*, i.e.

$$\kappa^{-1}_{\pi_{\sigma.x}}\kappa_\sigma\kappa_{\pi_x}.p^{-1}(x^*) = \kappa^{-1}_{\pi_{\sigma.x}}\kappa_\sigma.p^{-1}(x) = \kappa^{-1}_{\pi_{\sigma.x}}.p^{-1}(\sigma.x) = p^{-1}(x^*)\,.$$

Therefore,

$$\kappa^{-1}_{\pi_{\sigma.x}}\kappa_\sigma\kappa_{\pi_x}.s(x^*) = \frac{\sum_{\kappa \in S^*} \kappa^{-1}_{\pi_{\sigma.x}}\kappa_\sigma\kappa_{\pi_x}\kappa.y^*}{|S^*|} = s(x^*)\,, \tag{5.3.7}$$

what implies the equation

$$\kappa_\sigma.s(x) = \kappa_\sigma\kappa_{\pi_x}.s(x^*) = \kappa_{\pi_{\sigma.x}}.s(x^*) = s(\sigma.x)\,.$$

□

5.4. Examples: Symmetric Extension, Symmetric Section

The spanning tree polytope $\mathrm{P}_{\mathrm{spt}}(n) \subseteq \mathbb{R}^E$ in the complete graph K_n on n vertices, possesses a certain symmetry group $G \subseteq \mathfrak{S}(n)$. Namely, for every $\pi \in \mathfrak{S}(n)$

$$(\pi.x)_{v,w} = x_{\pi^{-1}(v),\pi^{-1}(w)} \quad \text{for} \quad v, w \in [n]$$

maps the spanning tree polytope $\mathrm{P}_{\mathrm{spt}}(n)$ on itself. Indeed, the action of the group $\mathfrak{S}(n)$ on the vertices of the spanning tree polytope is induced by permutations of the vertices in the graph K_n. And every vertex permutation for the graph K_n maps the set of all spanning trees $\mathcal{T}(n)$ on itself.

The extended formulation of the spanning tree polytope $\mathrm{P}_{\mathrm{spt}}(n)$ (see Martin [1991]), defined by $z \geq 0$ and

$$x_{v,u} + \sum_{w \in V \setminus \{v,u\}} z_{v,w,u} = 1 \qquad \text{for distinct } v, u \in [n]$$

$$x_{v,u} - z_{v,u,w} - z_{u,v,w} = 0 \qquad \text{for distinct } v, u, w \in [n]\,,$$

where the affine map p, associated with the linear system, is the orthogonal projection on x variables. The polytope $\mathrm{Q}_{\mathrm{spt}}(n)$, described by the linear system, together with the affine map p, defines an extension of the spanning tree polytope $\mathrm{P}_{\mathrm{spt}}(n)$.

This extended formulation of the spanning tree polytope is symmetric, since for every $\pi \in \mathfrak{S}(n)$ and for all vectors x, z, there exists an affine map $\zeta_\pi.(x, z)$, which can be defined as $(\pi.x, \pi.z)$, where

$$(\pi.z)_{v,w,u} = z_{\pi^{-1}(v),\pi^{-1}(w),\pi^{-1}(u)} \quad \text{for distinct} \quad v, w, u \in [n]\,. \tag{5.4.1}$$

Obviously, every affine map ζ_π leads to a permutation of the constraints in the linear system above. Moreover, affine maps ζ_π, $\pi \in \mathfrak{S}(n)$ are compatible with the projection p, since for all $\pi \in \mathfrak{S}(n)$

$$p(\zeta_\pi.(x, z)) = p(\pi.x, \pi.z) = \pi.x\,.$$

The corresponding polytope, together with the affine map p, forms a symmetric extension of the spanning tree polytope $\mathrm{P}_{\mathrm{spt}}(n)$, where the affine maps κ_π, $\pi \in \mathfrak{S}(n)$ are the affine maps ζ_π, $\pi \in \mathfrak{S}(n)$.

Note that the obtained extension (extended formulation) is a symmetric subspace extension (symmetric subspace extended formulation), where the affine maps κ_π, $\pi \in G$ (ζ_π, $\pi \in G$) are coordinate permutations, i.e. it satisfies the statement of Observation 5.1 (Lemma 5.4, respectively).

A section map s is defined uniquely, and thus due to Lemma 5.5 is symmetric. The image $s(x)$ is equal to (x, z) for each vertex $x = \chi(T)$, $T \in \mathcal{T}(n)$, where $z_{v,u,w}$ is equal to one if the tree T contains the edge $\{v, u\}$ and the path from u to w in the tree T does not involve the vertex v, and $z_{v,u,w}$ is equal to zero, otherwise. It is straightforward to check that the defined section s is symmetric with the affine maps η_π, $\pi \in \mathfrak{S}(n)$, defined as the affine maps ζ_π, $\pi \in \mathfrak{S}(n)$.

5.5. Faces of a Symmetric Extensions

Lemma 5.6. *Let $Q \subseteq \mathbb{R}^d$ be an extension of a polytope $P \subseteq \mathbb{R}^m$ with projection $p : \mathbb{R}^d \to \mathbb{R}^m$, and let a face P' of P be an extension of a polytope $R \subseteq \mathbb{R}^k$ with projection $q : \mathbb{R}^m \to \mathbb{R}^k$. Then the face $Q' = p^{-1}(P') \cap Q \subseteq \mathbb{R}^d$ of Q is an extension of R via the composed projection $q \circ p : \mathbb{R}^d \to \mathbb{R}^k$.*

If the extension Q of P is symmetric with respect to the action of a group G on $\mathbb{R}^m$ (with $\pi.P = P$ for all $\pi \in G$), and a group H acts on $\mathbb{R}^k$ such that, for every $\tau \in H$, we have $\tau.R = R$, and there is some $\pi_\tau \in G$ with $\pi_\tau.P' = P'$ and $q(\pi_\tau.x) = \tau.q(x)$ for all $x \in \mathbb{R}^m$, then the extension Q' of R is symmetric with respect to the action of the group H.

PROOF. Due to $q(p(Q')) = q(P') = R$, the polyhedron Q', together with the projection $q \circ p$ is an extension of R.

In order to prove the statement on the symmetry of this extension, let $\tau \in H$ be an arbitrary element of H with $\pi_\tau \in G$ as guaranteed to exist for τ in the statement of the lemma, and let $\kappa_{\pi_\tau} \in \mathfrak{S}(d)$ be a permutation, as guaranteed to exist by the symmetry of the extension Q of P. Since we have

$$q(p(\kappa_{\pi_\tau}.y)) = q(\pi_\tau.p(y)) = \tau.(q(p(y)))\,,$$

it suffices to show $\kappa_{\pi_\tau}.Q' = Q'$. As $y \mapsto \kappa_{\pi_\tau}.y$ defines an automorphism of Q (mapping faces of Q to faces of the same dimension), it suffices to show $\kappa_{\pi_\tau}.Q' \subseteq Q'$. Due to $\kappa_{\pi_\tau}.Q = Q$, this relation is implied by $\kappa_{\pi_\tau}.p^{-1}(P') \subseteq p^{-1}(P')$, which follows from

$$p(\kappa_{\pi_\tau}.p^{-1}(P')) = \pi_\tau.p(p^{-1}(P')) = \pi_\tau.P' = P'\,.$$

□

Thus, from every lower bound on size of symmetric extensions for the polytope $R \subseteq \mathbb{R}^k$ with respect to the action of the group H, we automatically obtain the same lower bound on size of symmetric extensions for the polytope $P \subseteq \mathbb{R}^m$ with respect to the action of the group G. This is due to the fact that the polyhedron $Q' = p^{-1}(P') \cap Q \subseteq \mathbb{R}^d$, together with the map $q \circ p : \mathbb{R}^d \to \mathbb{R}^k$, providing a symmetric extension of the polytope $R \subseteq \mathbb{R}^k$ with respect to the action of the group H, is a face of Q, and thus has not more facets than the polyhedron Q.

5.6. Yannakakis' Method

This section describes the modified method, where the original method was used to prove a lower bound on size of symmetric extensions for the perfect matching polytope by Yannakakis [1991].

5.6.1. Action of Group G. Due to Observation 5.1 and Lemma 5.5, we can assume that a symmetric extension for a polytope $P \subseteq \mathbb{R}^m$ with the minimum size is a subspace extension, given by a polyhedron $Q \subseteq \mathbb{R}^d$ and an affine map $p : \mathbb{R}^d \to \mathbb{R}^m$, with a symmetric section $s : \mathrm{vert}(P) \to Q$. Moreover, the affine maps κ_π, $\pi \in G$ are defined as coordinate permutations.

In this setting, we define an action of the group G on the component functions of the section s

$$\mathcal{S} = \{s_1, \ldots, s_d\}\,,$$

where the component functions do not have to be pairwise distinct functions, via

$$\pi.s_j = s_{\kappa^{-1}_{\pi^{-1}}(j)}\,.$$

The action of the symmetry group G on the component function is well-defined and yields a group action. To show this, consider the following equation

$$(\pi.s_j)(x) = s_{\kappa^{-1}_{\pi^{-1}}(j)}(x) = (\kappa_{\pi^{-1}}.s(x))_j = s_j(\pi^{-1}.x) \tag{5.6.1}$$

for every $\pi \in G$, $j \in [d]$ and $x \in \mathrm{vert}(P)$, what implies that $1_G.s_j = s_j$ and $\pi\sigma.s_j = \pi.(\sigma.s_j)$ for every $\pi, \sigma \in G$.

The *isotropy* group of $s_j \in \mathcal{S}$ under the action of G is defined as

$$\mathrm{iso}_G(s_j) = \{\pi \in G : \pi.s_j = s_j\}\,.$$

The component function $s_j : \mathrm{vert}(P) \to \mathbb{R}$ has the same value on every orbit of the action $\mathrm{iso}_G(s_j)$ on $\mathrm{vert}(P)$, since due to (5.6.1), the equation $s_j(\pi.x) = s_j(x)$ holds for every $x \in \mathrm{vert}(P)$ and every $\pi \in \mathrm{iso}_G(s_j)$.

Obviously, in general settings it is not possible to identify the isotropy group $\mathrm{iso}_G(s_j)$, but we are able to estimate the index of the isotropy group $\mathrm{iso}_G(s_j)$ in the group G. The index $G : \mathrm{iso}_G(s_j)$ is equal to the cardinality of the orbit for the component function s_j under the action of G on $\mathcal{S}$, and thus

$$G : \mathrm{iso}_G(s_j) \leq |\mathcal{S}| \leq d\,, \tag{5.6.2}$$

since the cardinality of the orbit can not exceed the cardinality of $\mathcal{S}$.

5.6.2. Action of Group $G = \mathfrak{S}(n)$. For many combinatorial polytopes the symmetry group G is given as the symmetric group $\mathfrak{S}(n)$ for some n. In this case, to study the structure of subgroups of the group G with small indices, we apply the theorem below Yannakakis [1991] (Appendix: Theorem 6.1)

Theorem 5.1. *For each subgroup U of $\mathfrak{S}(n)$ with $(\mathfrak{S}(n) : U) \leq \binom{n}{k}$, $1 \leq k < \frac{n}{4}$, there is some $W \subseteq [n]$, $|W| \leq k$, such that*

$$\{\pi \in \mathfrak{A}(n) : \pi(v) = v \textit{ for all } v \in W\} \subseteq U$$

holds.

Thus, whenever we have $d \leq \binom{n}{k}$, $1 \leq k < \frac{n}{4}$, there exists a set $V_j \subseteq [n]$ with $|V_j| \leq k$ for every $s_j \in \mathcal{S}$, such that

$$\{\pi \in \mathfrak{A}(n) : \pi(v) = v \text{ for all } v \in V_j\} \subseteq \mathrm{iso}_G(s_j)\,. \tag{5.6.3}$$

Moreover, if for every $x \in \mathrm{vert}(P)$ and every $V \subseteq [n]$, $|V| \leq k$, there exists an odd permutation $\sigma \in \mathfrak{S}(n)$, such that

$$\sigma.x = x \quad \text{and} \quad \sigma.v = v\,, \text{for } v \in V\,,$$

then the following inclusion holds

$$\{\pi \in \mathfrak{S}(n) : \pi(v) = v \text{ for all } v \in V_j\} \subseteq \mathrm{iso}_G(s_j)\,. \tag{5.6.4}$$

Indeed, for every vertex $x \in \mathrm{vert}(P)$ and an odd permutation $\pi \in \mathfrak{S}(n)$, such that $\pi.v = v$, $v \in V_j$, we let $\sigma \in \mathfrak{S}(n)$ to be an odd permutation, such that $\sigma.x = x$ and $\sigma.v = v$, $v \in V_j$. Then the permutation $\pi\sigma$ is an even permutation, such that $\pi\sigma.v = v$, $v \in V_j$, and thus, the permutation $\pi\sigma$ lies in the isotropy group $\mathrm{iso}_G(s_j)$. And thus, the equation

$$s_j(x) = \pi\sigma.s_j(x) = \pi.s_j(\sigma^{-1}.x) = \pi.s_j(x)\,.$$

holds for every vertex $x \in \mathrm{vert}(P)$, what shows that $\pi \in \mathrm{iso}_G(s_j)$, and thus finishes the proof.

An information about some subgroup H of the isotropy group $\mathrm{iso}_G(s_j)$, $s_j \in \mathcal{S}$ enables us to consider orbits under the action of this subgroup H on the set of vertices $\mathrm{vert}(P)$. The component function s_j has the same value on such orbits, since these orbits are subsets of the orbits under the action of $\mathrm{iso}_G(s_j)$ on $\mathrm{vert}(P)$.

5.6.3. Section Slack Covectors. The extension $Q \subseteq \mathbb{R}^d$, $p : \mathbb{R}^d \to \mathbb{R}^m$ is a subspace extension and thus all facet defining inequalities for the polyhedron Q are nonnegativity constraints. Here, we assume that the polytope P is not trivial, i.e. not a one-point polytope.

We obtain a contradiction to the fact that Q, p is a subspace extension of the polytope P, if there exist numbers λ_x, $x \in \mathrm{vert}(P)$ and an inequality $\langle a, x\rangle \leq b$, which is valid for P, such that

$$\sum_{x \in \mathrm{vert}(P)} s_j(x)\lambda_x \geq 0 \quad \text{for all } j \in [d] \tag{5.6.5}$$

$$\sum_{x \in \mathrm{vert}(P)} (b - \langle a, x\rangle)\lambda_x < 0\,. \tag{5.6.6}$$

Indeed, if the sum $\sum_{x\in\mathrm{vert}(P)} \lambda_x$ is equal zero, the vector

$$r = \sum_{x\in\mathrm{vert}(P)} \lambda_x s(x)\,,$$

belongs to the recession cone $\mathrm{rec}(Q)$, since Q, p is a subspace extension. And the vector

$$\mu(r) = \sum_{x\in\mathrm{vert}(P)} \lambda_x \mu(s(x)) + \beta \sum_{x\in\mathrm{vert}(P)} \lambda_x = \sum_{x\in\mathrm{vert}(P)} \lambda_x p(s(x)) = \sum_{x\in\mathrm{vert}(P)} \lambda_x x$$

is not equal to zero, since

$$\sum_{x\in\mathrm{vert}(P)} (b - \langle a, x\rangle)\lambda_x = \sum_{x\in\mathrm{vert}(P)} \langle a, \lambda_x x\rangle = \langle a, \sum_{x\in\mathrm{vert}(P)} \lambda_x x\rangle < 0\,,$$

here the projection $p(y)$ is represented as $\mu(y) + \beta$, where $\mu : \mathbb{R}^d \to \mathbb{R}^m$ is a linear map and $\beta \in \mathbb{R}$. This contradicts the fact that P is a polytope.

If the sum $\sum_{x\in\mathrm{vert}(P)} \lambda_x$ is not equal zero, the point

$$\frac{1}{\sum_{x\in\mathrm{vert}(P)} \lambda_x} \sum_{x\in\mathrm{vert}(P)} \lambda_x s(x)\,,$$

belongs to the polyhedron Q, since Q, p is a subspace extension, but the projection of this point does not satisfy the constraint $\langle a, x\rangle \le b$. Note that $\sum_{x\in\mathrm{vert}(P)} \lambda_x \ge 0$, since from the Farkas Lemma, for every extension Q, whose recession cone does not have the dimension equal to the dimension of Q (in this case Q can be an extension of trivial polytopes only), the function $\phi : \mathrm{vert}(P) \to \mathbb{R}$, where $\phi(x) = 1$ for all $x \in \mathrm{vert}(P)$, can be obtained as a non-negative combination of the section component functions s_j, $j \in [d]$.

5.7. Matching Polytope

In this section, we prove the following theorem, which gives us a lower bound on size of symmetric extended formulations for the cardinality restricted matching polytope $\mathrm{P}^{\ell}_{\mathrm{match}}(n) \subseteq \mathbb{R}^E$.

Theorem 5.2. *For every odd $0 \le \ell \le \frac{n}{2}$, $6 \le n$, there is no symmetric extended formulation for the matching polytope $\mathrm{P}^{l}_{\mathrm{match}}(n) \subseteq \mathbb{R}^E$ of size less than $\binom{n}{\frac{\ell-1}{2}}$.*

This theorem gives also a lower bound on the size of symmetric extensions for the polytope $\mathrm{P}^{\ell}_{\mathrm{match}}(n)$, when the number ℓ is not restricted to be odd. Because the face

$$\mathrm{P}^{\ell}_{\mathrm{match}}(n) \cap \{x \in \mathbb{R}^{\binom{n}{2}} : x_{n-1,n} = 1\}$$

of the polytope $\mathrm{P}^{\ell}_{\mathrm{match}}(n)$ provides a symmetric extension of the polytope $\mathrm{P}^{\ell-1}_{\mathrm{match}}(n-2)$ with respect to the action of the group $\mathfrak{S}(n-2)$. From Lemma 5.6, we obtain the lower bound

$$\binom{n-2}{\frac{\ell-2}{2}} \ge \frac{1}{4}\binom{n}{\lfloor\frac{\ell-1}{2}\rfloor}\,,$$

when $0 \le \ell \le \frac{n}{2}$ is even and $n \ge 6$.

Theorem 5.3. *For every $0 \le \ell \le \frac{n}{2}$, $6 \le n$, there is no symmetric extended formulation for the matching polytope $\mathrm{P}^{\ell}_{\mathrm{match}}(n) \subseteq \mathbb{R}^E$ of size less than $\frac{1}{4}\binom{n}{\lfloor\frac{\ell-1}{2}\rfloor}$.*

And from Theorems 2.4 and 5.3, we can conclude that for $\ell = \Theta(\log n)$ there exists a compact extended formulation for the matching polytope $\mathrm{P}^{\ell}_{\mathrm{match}}(n)$, but there is no compact symmetric extended formulation for the matching polytope $\mathrm{P}^{\ell}_{\mathrm{match}}(n)$, what establishes a gap between symmetric and non-symmetric extensions for the cardinality restricted matching polytopes.

Corollary 5.1. *For $\Omega(\log n) \le \ell \le n$, there is no compact extended formulation for $\mathrm{P}^{\ell}_{\mathrm{match}}(n)$, that is symmetric with respect to the group $\mathfrak{S}(n)$.*

5.7.1. Action of Group $G = \mathfrak{S}(n)$. Due to Observation 5.1, there exists a symmetric subset extension $Q \subseteq \mathbb{R}^d$, $p : \mathbb{R}^d \to \mathbb{R}^E$ with $d \le \binom{n}{k}$, $\frac{n}{2} \ge \ell = 2k+1$, such that the affine maps $\kappa_\pi : \mathbb{R}^d \to \mathbb{R}^d$, $\pi \in \mathfrak{S}(n)$ are coordinate permutations.

The results of Section 5.6.2 imply that under the assumption $d < \binom{n}{k}$ there is a subset $V_j \subseteq V$ of nodes with $|V_j| \le k$ for each $j \in [d]$, such that

$$H_j = \{\pi \in \mathfrak{S}(n) : \pi(v) = v \text{ for all } v \in V_j\} \subseteq \mathrm{iso}_{\mathfrak{S}(n)}(s_j)\,.$$

Indeed, for every vertex $x = \chi(M)$, $M \in \mathcal{M}^{2k+1}(n)$,and every set $W \subseteq V$, $|W| \le k$, there exists an edge $e = \{w, u\}$, $w, u \notin W$, in the matching M, which defines the transposition (w, u) as an odd permutation $\sigma \in \mathfrak{S}(n)$, such that $\sigma(v) = v$, $v \in W$ and $\sigma.x = x$.

Hence, two vertices $\chi(M_1)$ and $\chi(M_2)$, $M_1, M_2 \in \mathcal{M}^{2k+1}(n)$, are in the same orbit under the action of the group H_j if and only if we have

$$M_1 \cap E(V_j) = M_2 \cap E(V_j) \quad \text{and} \quad V_j \setminus M_1 = V_j \setminus M_2\,. \tag{5.7.1}$$

This implies that if

$$M_1 \cap \big(E(V_j) \cup \delta(V_j)\big) = M_2 \cap \big(E(V_j) \cup \delta(V_j)\big)$$

for two vertices $\chi(M_1)$, $\chi(M_2)$, then

$$s_j(\chi(M_1)) = s_j(\chi(M_2))\,,$$

due to (5.7.1).

5.7.2. Section Slack Covectors. Choose two disjoint sets $V_\star, V^\star \subseteq V$, such that $|V_\star| = |V^\star| = 2k+1$, recall the inequality $2k+1 \le \frac{n}{2}$. And define an extension of the polytope $\mathrm{P}^{2k+1}_{\mathrm{match}}(4k+2)$ by the polyhedron

$$Q' = Q \cap \{y \in \mathbb{R}^d : x = p(y),\, x_e = 0 \ \text{ for } \ e \in E \setminus E(V^\star \cup V_\star)\}$$

and the affine map $p' = \mathrm{proj}_{E(V^* \cup V_*)} \circ p$.

A section $s' : \mathrm{vert}(\mathrm{P}^{2k+1}_{\mathrm{match}}(4k+2)) \to \mathbb{R}^d$ is the restriction of the section s to the characteristic vectors of matchings $\mathcal{M}^{2k+1}(n)$, which cover the nodes of $V_\star \cup V^\star$.

From (5.7.1), we have $s'_j(\chi(M_1)) = s'_j(\chi(M_2))$ for $M_1, M_2 \in \mathcal{M}^{2k+1}(4k+2)$, whenever

$$M_1 \cap \big(E(V'_j) \cup \delta(V'_j)\big) = M_2 \cap \big(E(V'_j) \cup \delta(V'_j)\big)\,,$$

where V'_j is the set of nodes $V_j \cap (V_\star \cup V^\star)$.

Denote by $\mathcal{A}_j$ the set of all matching on $V_\star \cup V^\star$, such that $A \subseteq E(V'_j) \cup \delta(V'_j)$, hence $|A| \le |V_j| = k$. And denote by $s'_j(A)$, $A \in \mathcal{A}_j$, the value $s_j(\chi(M))$, where $M \in \mathcal{M}^{2k+1}(4k+2)$, such that $A = M \cap (E(V'_j) \cup \delta(V'_j))$.

Now, we find λ_x, $x \in \mathrm{vert}(\mathrm{P}^{2k+1}_{\mathrm{match}}(4k+2))$ as described in Section 5.6.3. For this, define

$$\mathcal{M}^\star_i = \{M \in \mathcal{M}^{2k+1}(4k+2) : |M \cap E(V_\star : V^\star)| = i\}\,.$$

Thus, $\mathcal{M}^\star_i$ is the set of perfect matchings on $K(V_\star \cup V^\star)$, with exactly i edges between $V_\star$ and $V^\star$. Note that the sets $\mathcal{M}^\star_i$, $i \in [2k+1]_{\mathrm{odd}}$ form a partition of the set $\mathcal{M}^{2k+1}(4k+2)$, since the cardinalities of the sets $V_\star$, $V^\star$ are odd.

Choose the numbers λ_x, $x \in \mathcal{M}_i^\star$ to be equal $\frac{\lambda_i^*}{|\mathcal{M}_i^\star|}$ for all $i \in [2k+1]_{\text{odd}}$. In turn, the numbers λ_i^*, $i \in [2k+1]_{\text{odd}}$ are chosen as a solution to the following linear system

$$\sum_{i\in[2k+1]_{\text{odd}}} \lambda_i^* = 1$$
$$\sum_{i\in[2k+1]_{\text{odd}}} i^t\lambda_i^* = 0 \quad \text{for all} \quad t \in [k]\,.$$

The matrix, defining the linear system, is a Vandermonde matrix, and thus non-singular. Hence, there exist such numbers λ_i^*, $i \in [2k+1]_{\text{odd}}$. Note that this definition of λ_i^*, $i \in [2k+1]_{\text{odd}}$ implies the equation

$$\sum_{i\in[2k+1]_{\text{odd}}} q(i)\lambda_i^* = q_0 \sum_{i\in[2k+1]_{\text{odd}}} \lambda_i^* + \sum_{t\in[k]} q_t \sum_{i\in[2k+1]_{\text{odd}}} i^t\lambda_i^* = q_0 = q(0) \tag{5.7.2}$$

for every polynom $q = q_0 + \sum_{j\in[k]} q_j x^j$, i.e. for every polynom with degree at most k.

First, we show that the inequality (5.6.5) holds for every $j \in [d]$, i.e.

$$0 \le \sum_{x\in\text{vert}(\mathrm{P}^{2k+1}_{\text{match}}(4k+2))} \lambda_x s'_j(x)$$

for every $j \in [d]$. For this, let us rewrite the left size of the inequality above

$$\sum_{M\in\mathcal{M}^{2k+1}(4k+2)} \lambda_{\chi(M)} s'_j(\chi(M)) = \sum_{A\in\mathcal{A}_j} \sum_{A\subseteq M\in\mathcal{M}^{2k+1}(4k+2)} \lambda_{\chi(M)} s'_j(A) =$$
$$\sum_{A\in\mathcal{A}_j} \sum_{i\in[2k+1]_{\text{odd}}} \frac{\lambda_i^*}{|\mathcal{M}_i^\star|} |\{M \in \mathcal{M}_i^\star : A \subseteq M\}| s'_j(A) =$$
$$\sum_{A\in\mathcal{A}_j} s'_j(A) \sum_{i\in[2k+1]_{\text{odd}}} \frac{\lambda_i^*}{|\mathcal{M}_i^\star|} |\{M \in \mathcal{M}_i^\star : A \subseteq M\}|\,.$$

From the symmetry in definition of λ_i^*, $i \in [2k+1]_{\text{odd}}$, it follows that the value

$$\sum_{i\in[2k+1]_{\text{odd}}} \frac{\lambda_i^*}{|\mathcal{M}_i^\star|} |\{M \in \mathcal{M}_i^\star : A \subseteq M\}|\,,$$

corresponding to a matching A, is equal for all matchings A from the set $\mathcal{A}_{a^\star,a_\star,a^\star_\star}$, consisting of all matchings $A' \subseteq E(V^\star \cap V_\star)$, where $a^\star = |A' \cap E(V^\star)|$, $a_\star = |A' \cap E(V_\star)|$, $a^\star_\star = |A' \cap E(V^\star : V_\star)|$. And, since $s_j(\chi(M))$ is non-negative for all matchings $M \in \mathcal{M}^{2k+1}(4k+2)$ and $j \in [d]$, it is enough to show that the inequality

$$\sum_{A\in\mathcal{A}_{a^\star,a_\star,a^\star_\star}} \sum_{i\in[2k+1]_{\text{odd}}} \frac{\lambda_i^*}{|\mathcal{M}_i^\star|} |\{M \in \mathcal{M}_i^\star : A \subseteq M\}| \ge 0$$

holds, what can be seen from the following chain of equations

$$\sum_{i\in[2k+1]_{\text{odd}}} \sum_{A\in\mathcal{A}_{a^\star,a_\star,a^\star_\star}} \frac{\lambda_i^*}{|\mathcal{M}_i^\star|} |\{M \in \mathcal{M}_i^\star : A \subseteq M\}| =$$
$$\sum_{i\in[2k+1]_{\text{odd}}} \sum_{M\in\mathcal{M}_i^\star} \frac{\lambda_i^*}{|\mathcal{M}_i^\star|} |\{A \in \mathcal{A}_{a^\star,a_\star,a^\star_\star} : A \subseteq M\}| =$$
$$\sum_{i\in[2k+1]_{\text{odd}}} \sum_{M\in\mathcal{M}_i^\star} \frac{\lambda_i^*}{|\mathcal{M}_i^\star|} \binom{\frac{2k+1-i}{2}}{a^\star} \binom{i}{a^\star_\star} \binom{\frac{2k+1-i}{2}}{a_\star} =$$
$$\sum_{i\in[2k+1]_{\text{odd}}} \lambda_i^* \binom{\frac{2k+1-i}{2}}{a^\star} \binom{i}{a^\star_\star} \binom{\frac{2k+1-i}{2}}{a_\star}\,,$$

where the last expression can be considered as a polynom of degree $|a^\star| + |a^\star_\star| + |a_\star| \le k$, which evaluated at the point 0 gives a non-negative value.

On the other hand, the inequality $x\big(E(V_\star : V^\star)\big) \ge 1$, which is valid for the polytope $\mathrm{P}^{2k+1}_{\mathrm{match}}(4k+2)$, can be used in the constraint (5.6.6)

$$\sum_{M\in\mathcal{M}^{2k+1}(4k+2)} \lambda_{\chi(M)}(|M\cap E(V_\star : V^\star)| - 1) = \sum_{i\in[2k+1]_{\mathrm{odd}}} \sum_{M\in\mathcal{M}^\star_i} \lambda_{\chi(M)}(i-1) =$$

$$\sum_{i\in[2k+1]_{\mathrm{odd}}} |\mathcal{M}^\star_i| \frac{\lambda^*_i}{|\mathcal{M}^\star_i|}(i-1) = \sum_{i\in[2k+1]_{\mathrm{odd}}} \lambda^*_i(i-1) = -1\,,$$

since $i-1$ is a polynom of degree smaller than k (in the case $k=0$, the lower bound is trivial).

5.8. Cycle Polytope

Theorem 5.4. *For $6 \le \ell \le n$, the size of every symmetric extension for $\mathrm{P}^\ell_{\mathrm{cycl}}(n)$, with respect to the group $\mathfrak{S}(n)$, is bounded from below by*

$$\frac{1}{16}\binom{\lfloor \frac{n}{3}\rfloor}{\lfloor(\lfloor\frac{\ell}{6}\rfloor - 1)/2\rfloor}\,.$$

PROOF. Let us define

$$\bar\ell = \ell \bmod 6\,,\quad n' = \lfloor\frac{n-\bar\ell}{3}\rfloor \quad\text{and}\quad k = \lfloor\frac{\ell}{6}\rfloor = \frac{\ell-\bar\ell}{6}\,.$$

Choose three pairwise disjoint subsets $V^\star$, $V_\star$, $V^\star_\star \subseteq V$ of nodes with the cardinality n' each. And denote the elements of these three sets as follows

$$V^\star = \{v^\star_1,\dots,v^\star_{n'}\}\quad V_\star = \{v_{\star 1},\dots,v_{\star n'}\}\quad V^\star_\star = \{v^\star_{\star 1},\dots,v^\star_{\star n'}\}\,.$$

Define the set of edges

$$E_0 = \{\{v^\star, v\}\in E : v^\star\in V^\star, v\in V\setminus(V^\star\cup V^\star_\star)\}\cup$$
$$\{\{v^\star_{\star i}, v\}\in E : i\in[n'], v\in V\setminus\{v^\star{}_i, v_{\star i}\}\,,$$

and let F be the following face of $\mathrm{P}^\ell_{\mathrm{cycl}}(n)$

$$F = \mathrm{P}^\ell_{\mathrm{cycl}}(n)\cap\{x\in\mathbb{R}^E : x_e = 0 \text{ for all } e\in E_0\}\,.$$

Every cycle $C\in\mathcal{C}^\ell(n)$, such that $C\cap E_0 = \varnothing$, satisfies the inequality

$$|V(C)\cap V^\star_\star| \le 2\lfloor\ell/6\rfloor\,,$$

because the cycle C goes through at least three edges between any two visits to $V^\star_\star$, and there has to be an even number of these visits, since after coming in $V^\star_\star$ from $V^\star$, the cycle goes into $V_\star$, and vice versa. Therefore, denoting

$$\mathcal{C}' = \{C\in\mathcal{C}^\ell(n) : C\cap E_0 = \varnothing, |V(C)\cap V^\star_\star| = 2\lfloor\ell/6\rfloor\}\,,$$

we define the following face of $\mathrm{P}^\ell_{\mathrm{cycl}}(n)$

$$F' = \mathrm{conv}(\{\chi(C) : C\in\mathcal{C}'\}) = \{x\in F : x(\delta(V^\star_\star)) = 4\lfloor\ell/6\rfloor\}\,.$$

Moreover, for every $C\in\mathcal{C}'$, we have

$$|C\cap E(V^\star)| \ge \lfloor\ell/6\rfloor\,.$$

Thus, if we denote

$$\mathcal{C}'' = \{C\in\mathcal{C}' : |C\cap E(V^\star)| = \lfloor\ell/6\rfloor\}\,,$$

we find that

$$F'' = \mathrm{conv}(\{\chi(C) : C\in\mathcal{C}''\} = \{x\in F' : x(E(V^\star)) = \lfloor\ell/6\rfloor\})$$

is a face of $\mathrm{P}^\ell_{\mathrm{cycl}}(n)$.

A cycle $C \in \mathcal{C}^\ell(n)$ is contained in $\mathcal{C}''$ only if $C \cap E(V^\star)$ is a matching of size $\ell' = \lfloor \ell/6 \rfloor$. And every matching M in $E(V^\star)$ of size $\ell' = \lfloor \ell/6 \rfloor$, can be extended to some cycle $C \in \mathcal{C}^\ell(n)$. Thus, for the orthogonal projection $q : \mathbb{R}^E \to \mathbb{R}^{E(V^\star)}$, the following equation holds

$$q(F'') = \mathrm{P}^{\ell'}_{\mathrm{match}}(n') \,,$$

after an identification of $V^\star$ with the node set of the complete graph $K_{n'}$.

Moreover, for every $\tau \in \mathfrak{S}(n')$ the permutation $\pi_\tau \in \mathfrak{S}(n)$ with

$$\pi_\tau(v^\star{}_i) = v^\star{}_{\tau^{-1}(i)}, \quad \pi_\tau(v_{\star i}) = v_{\star \tau^{-1}(i)}, \quad \pi_\tau(v^\star_{\star i}) = v^\star_{\star \tau^{-1}(i)}$$

for all $i \in [n']$, and $\pi_\tau(v) = v$ for all $v \notin V^\star, V_\star, V^\star_\star$, satisfies $\pi_\tau.F'' = F''$, and

$$q(\pi_\tau.x) = \tau.q(x) \quad \text{for all } x \in \mathbb{R}^E \,.$$

Due to Lemma 5.6, a symmetric extension of the restricted cycle polytope $\mathrm{P}^\ell_{\mathrm{cycl}}(n)$ yields a symmetric extension of the restricted matching polytope $\mathrm{P}^{\ell'}_{\mathrm{match}}(n')$ of at most the same size. From Theorem 5.3, we obtain the lower bound

$$\frac{1}{4}\binom{\lfloor \frac{n-\bar{\ell}}{3} \rfloor}{\lfloor (\lfloor \frac{\ell}{6} \rfloor - 1)/2 \rfloor} \geq \frac{1}{16}\binom{\lfloor \frac{n}{3} \rfloor}{\lfloor (\lfloor \frac{\ell}{6} \rfloor - 1)/2 \rfloor}$$

on size of symmetric extensions for the polytope $\mathrm{P}^\ell_{\mathrm{cycl}}(n)$.

□

Corollary 5.2. *For $\Omega(\log n) \leq \ell \leq n$, there is no compact extended formulation for $\mathrm{P}^\ell_{\mathrm{cycl}}(n)$, that is symmetric with respect to the group $\mathfrak{S}(n)$.*

From Theorems 2.5 and 5.2, we can conclude that for $\ell = \Theta(\log n)$, there exists a compact extended formulation for the cycle polytope $\mathrm{P}^\ell_{\mathrm{cycl}}(n)$, but there is no compact symmetric extended formulation for the cycle polytope $\mathrm{P}^\ell_{\mathrm{cycl}}(n)$.

5.9. Symmetric Subspace Extensions of Quadratic Size

In this section, we study symmetric subspace extensions of an arbitrary polytope $P \subseteq \mathbb{R}^{n+m}$, $n \geq 6$. The group $\mathfrak{S}(n)$ acts on the vertex set $\mathrm{vert}(P)$ by permuting the first n coordinates.

Due to Observation 5.1, we assume that a polyhedron $Q \subseteq \mathbb{R}^d$, $2d < n(n-1)$ and $p : \mathbb{R}^d \to \mathbb{R}^{m+n}$, which forms a minimal extension of the polytope P, is a symmetric subspace extension of the polytope P with respect to the action of $G = \mathfrak{S}(n)$, where affine maps κ_π, $\pi \in G$, are coordinate permutations. A symmetric section $s : \mathrm{vert}(P) \to Q$ is, such that the corresponding affine maps $\eta_\pi : \mathbb{R}^d \to \mathbb{R}^d$, $\pi \in G$, are given as the coordinate permutations $\kappa_\pi : \mathbb{R}^d \to \mathbb{R}^d$, $\pi \in G$.

The main result of this section is the following theorem, which describes the action of the group $\mathfrak{A}(n)$ on the component functions s_j.

Theorem 5.5. *There exists a partition of the set $[d]$ into sets $\mathcal{A}_i$ and $\mathcal{B}_j$, such that each set $\mathcal{B}_j$ contains exactly one element b_j, and each set $\mathcal{A}_i$ consists of n elements a^i_1, a^i_2,…, a^i_n with*

$$s_{a^i_t}(\pi.x) = s_{a^i_{\pi^{-1}(t)}}(x) \qquad s_{b_j}(\pi.x) = s_{b_j}(x) \tag{5.9.1}$$

for every vertex $x \in \mathrm{vert}(P)$ and all $\pi \in \mathfrak{A}(n)$.

Before proving Theorem 5.5, we would like to show the following result, which can be extremely useful in case of zero-one polytopes.

Theorem 5.6. *In the case, when the vertex set* $\mathrm{vert}(P)$ *belongs to* $\{0,1\}^n \times \mathbb{R}^m$*, there exists a partition of the set* $[d]$ *into sets* $\mathcal{A}_i$ *and* $\mathcal{B}_j$*, such that each set* $\mathcal{B}_j$ *contains exactly one element* b_j*, and each set* $\mathcal{A}_i$ *consists of* n *elements* a_1^i*,* a_2^i*,. . . ,*a_n^i *with*

$$s_{a_v^i}(x) = s_{a_w^i}(x) \qquad \text{if } x_v = x_w \tag{5.9.2}$$

$$s_{b_j}(x) = s_{b_j}(y) \qquad \text{if } x = \pi.y \quad \text{for some } \pi \in \mathfrak{S}(n) \tag{5.9.3}$$

for all vertices $x, y \in \mathrm{vert}(P)$.

PROOF. Consider a partition $\mathcal{A}_i$, $\mathcal{B}_j$, which is guaranteed to exist by Theorem 5.5. We prove that this partition satisfies the statement of the current theorem.

Let us assume that x_v and x_w are equal, but $s_{a_v^i}(x)$ and $s_{a_w^i}(x)$ are not. We can choose two distinct elements v', w' different from v, w such that $x_{v'}$ is equal to $x_{w'}$ (since $n \geq 6$). For the even permutation $\pi = (v, w)(v', w')$ from Theorem 5.5, the value $s(x)$ is not equal to $s(\pi.x)$. But this contradicts the definition of a section, since x and $\pi.x$ represent the same vertex.

By Theorem 5.5, the equation $s_{b_j}(\pi.x) = s_{b_j}(x)$ holds for all permutations $\pi \in \mathfrak{A}(n)$ and vertices $x \in \mathrm{vert}(P)$. Assume that this equation is not satisfied for some odd permutation π and vertex x. We can choose two elements v', w' such that $x_{v'}$ is equal to $x_{w'}$ (since $n \geq 6$). Consider the transposition $\tau = (v', w')$ and the corresponding even permutation $\pi' = \pi\tau$. For this even permutation π', the equation

$$\pi'.x = \pi\tau.x = \pi.x$$

holds. From Theorem 5.5, we can conclude

$$s_{b_j}(x) = s_{b_j}(\pi'.x) = s_{b_j}(\pi.x)\,,$$

what contradicts our assumption that $s_{b_j}(x) \neq s_{b_j}(\pi.x)$. □

The proof of Theorem 5.5, presented below, consists of a series of small lemmas[1].

5.9.1. Action of Group $\mathfrak{A}(n)$ on Component Functions. As in Section 5.6.2, for all $j \in [d]$, we are able to establish

$$\{\pi \in \mathfrak{A}(n) : \pi(v) = v \text{ for all } v \in V_j\} \subseteq \mathrm{iso}_G(s_j)$$

for some set $V_j \subset [n]$, $|V_j| \leq 2$, due to Lemma 5.1 and the assumption $d < \binom{n}{2}$. In the next lemma, we prove that the set V_j can be chosen to contain not more than one element.

Lemma 5.7. *For each* $j \in [d]$*, there is some* $v_j \in [n]$*, such that*

$$\{\pi \in \mathfrak{A}(n) : \pi(v_j) = v_j\} \subseteq \mathrm{iso}_G(s_j)\,.$$

This element v_j *is uniquely determined, unless* $\mathfrak{A}(n) \subseteq \mathrm{iso}_G(s_j)$

PROOF. Let us assume the set V_j to contain two elements v and w. If V_j is a fixed block for the group $\mathrm{iso}_G(s_j)$, then the following inequality

$$d < \frac{n(n-1)}{2} \leq (\mathfrak{S}(n) : \mathrm{iso}_G(s_j))$$

holds, but $(\mathfrak{S}(n) : \mathrm{iso}_G(s_j))$ is equal to the cardinality of the orbit for s_j under the action of the group $\mathfrak{S}(n)$. Thus, there is a permutation $\tau \in \mathrm{iso}_G(s_j)$, such that without loss of generality $\tau(v) \neq v$ and $\tau(v) \neq w$.

For convenience, we prove that the permutation τ can be chosen, such that $\tau(w) = w$ and $\tau \in \mathfrak{A}(n)$. Whenever $\tau(w) \neq w$ or $\tau \notin \mathfrak{A}(n)$, consider the permutation $\tau' = \tau^{-1}\beta\tau \in \mathfrak{A}(n)$, where $\beta \in \mathfrak{A}(n)$, such that

$$\beta(v) = v,\ \beta(w) = w,\ \beta\tau(w) = \tau(w) \text{ and } \beta\tau(v) \neq \tau(v)\,.$$

[1]Actually, the proof of Theorem 5.5 can be significantly reduced (see Braun and Pokutta [2011]), due to Lemma 5.7 and the fact that the isotropy groups for v_j and s_j, considering the action of $\mathfrak{A}(n)$, are equal.

Such a permutation β exists since $\tau(v)$ is not equal to any of the three elements v, w,$\tau(w)$ (note $n \geq 6$). The construction of τ' guarantees that

$$\tau'(w) = w,\ \tau'(v) \neq v \ \text{ and } \ \tau' \in \mathrm{iso}_G(s_j)\,.$$

Hence, we assume that $\tau(w) = w$ and $\tau \in \mathfrak{A}(n)$.

To prove that the element v_j described in the lemma exists, we show that the element w has the desired properties, i.e.

$$\{\pi \in \mathfrak{A}(n) : \pi(w) = w\} \subseteq \mathrm{iso}_G(s_j)\,. \tag{5.9.4}$$

Every permutation $\pi \in \mathfrak{A}(n)$, $\pi(w) = w$, $\pi(v) \neq v$ can be represented as

$$\pi = (\pi(\tau\alpha)^{-1})\tau\alpha$$

for any $\alpha \in \mathfrak{S}(n)$. Choose a permutation $\alpha \in \mathfrak{A}(n)$, such that

$$\alpha(v) = v,\ \alpha(w) = w \ \text{ and } \ \alpha\pi^{-1}(v) = \tau^{-1}(v)\,.$$

The existence of this α can be trivially proved, since $n \geq 6$. Thus, the permutation π belongs to $\mathrm{iso}_G(s_j)$, because all three permutations τ, α and $\pi(\tau\alpha)^{-1}$ belong to $\mathrm{iso}_G(s_j)$ (note that $\pi(\tau\alpha)^{-1}$ and α are even permutations, which fix elements v, w). Thus, every permutation $\pi \in \mathfrak{A}(n)$, $\pi(w) = w$, belongs to $\mathrm{iso}_G(s_j)$, whenever $\pi(v) = v$. Therefore, the inclusion (5.9.4) holds.

Having another element $u \in [n]$, $u \neq w$, such that

$$\{\pi \in \mathfrak{A}(n) : \pi(u) = u\} \subseteq \mathrm{iso}_G(s_j)\,, \tag{5.9.5}$$

one can prove that $\mathfrak{A}(n) \subseteq \mathrm{iso}_G(s_j)$, since every permutation $\pi \in \mathfrak{A}(n)$ is a composition of not more than four permutations, described by (5.9.4) and (5.9.5). □

5.9.2. Action of Cycles on Component Functions. To prove Theorem 5.5, define permutations ρ_v, $v \in [n-2]$, consisting of the cycle $(v, v+1, v+2)$, respectively. Initially, we find a partition $\mathcal{A}_i$, $\{b_j\}$, such that Theorem 5.5 is satisfied for the permutations ρ_v, $v \in [n-2]$. Finally, since every permutation $\pi \in \mathfrak{A}(n)$ is a product of permutations ρ_v, the proof of Theorem 5.5 follows.

Note that two permutation κ' and κ from $\mathfrak{S}(d)$ are equivalent in our discussion, if the equation $s_{\kappa'^{-1}(j)}(x) = s_{\kappa^{-1}(j)}(x)$ holds for all x from $\mathrm{vert}(P)$ and for all j from $[d]$. For example, we can take the identity permutation instead of κ, if $s_{\kappa^{-1}(j)}(x) = s_j(x)$ for all $x \in \mathrm{vert}(P)$ and all $j \in [d]$.

Lemma 5.8. *For each $\pi = (w_1, w_2, w_3) \in \mathfrak{A}(n)$, there exists a permutation κ in $\mathfrak{S}(d)$, which is equivalent to κ_π, such that all cycles in κ are of the form (j_1, j_2, j_3), with $v_{j_t} = w_t$ and $\mathfrak{A}(n) \not\subseteq \mathrm{iso}_G(s_{j_t})$ for all $t \in [3]$.*

PROOF. The permutation κ_π^3 is equivalent to the identity permutation, since the permutation π^3 is the identity permutation.

Thus, every cycle C of the permutation κ_π of length not divisible by three, permutes indices of the identical component functions of s. Hence, we can assume that the length of every cycle C in κ_π is divisible by three.

The same augmentation allows us to transform every cycle $(j_1, j_2, \cdots j_{3l})$ of the permutation κ_π into the following cycles $(j_1, j_2, j_3), \ldots, (j_{3l-2}, j_{3l-1}, j_{3l})$, offering an equivalent permutation to κ_π. Thus, we may assume that κ_π contains cycles of length three only.

Now, we consider one of the cycles (j_1, j_2, j_3) in the permutation κ_π. If the element v_{j_1} does not belong to $\{w_1, w_2, w_3\}$ or $\mathfrak{A}(n) \subseteq \mathrm{iso}_G(s_{j_1})$, then we have $\pi \in \mathrm{iso}_G(s_{j_1})$, and thus $\pi, \pi^2 \in \mathrm{iso}_G(s_{j_1})$, what yields

$$\begin{aligned} s_{j_1}(x) = \ & s_{j_1}(\pi.x) = & s_{\kappa_\pi{}^{-1}(j_1)}(x) = \ & s_{j_3}(x) \\ s_{j_1}(x) = \ & s_{j_1}(\pi^2.x) = & s_{\kappa_\pi{}^{-2}(j_1)}(x) = \ & s_{j_2}(x)\,. \end{aligned}$$

This shows that the component functions s_{j_1}, s_{j_2}, s_{j_3} are identical, and thus, the cycle (j_1, j_2, j_3) can be omitted.

Hence, we may assume $v_{j_1} = w_1$. For every permutation $\tau' \in \mathfrak{A}(n)$, $\tau'(w_3) = w_3$, and the permutation $\tau = \pi\tau'\pi^{-1} \in \mathfrak{A}(n)$, we have

$$\tau(w_1) = \pi\tau'\pi^{-1}(w_1) = \pi\tau'(w_3) = \pi(w_3) = w_1\,,$$

and thus, $\tau \in \mathrm{iso}_G(s_{j_1})$, since $\tau \in \mathfrak{A}(n)$ and $\tau(w_1) = w_1$.Therefore, the equation

$$\begin{aligned} s_{j_3}(\pi^{-1}\tau\pi.x) &= s_{\kappa_\pi^{-1}(j_1)}(\pi^{-1}\tau\pi.x) = s_{j_1}(\pi\pi^{-1}\tau\pi.x) \\ &= s_{j_1}(\tau\pi.x) = s_{j_1}(\tau.(\pi.x)) = s_{j_1}(\pi.x) = s_{\kappa_\pi^{-1}(j_1)}(x) = s_{j_3}(x) \end{aligned}$$

holds for all $x \in \mathrm{vert}(P)$, and thus, $\tau' \in \mathrm{iso}_G(s_{j_3})$. Hence, the element v_{j_3} is equal the element w_3, unless $\mathfrak{A}(n) \subseteq \mathrm{iso}_G(s_{j_3})$ (where $\mathfrak{A}(n) \subseteq \mathrm{iso}_G(s_{j_1})$ would allow us to remove the cycle (j_1, j_2, j_3)). Similarly, one can obtain that the elements v_{j_2}, w_2 are equal. □

5.9.3. Interaction of Two Cycles.

Lemma 5.9. *For every two permutations $\pi = (w_1, w_2, w_3)$ and $\sigma = (w_2, w_3, w_4)$, $w_1 \neq w_4$, and the corresponding permutations κ_π and κ_σ satisfying the conditions in Lemma 5.8 the following holds: if the permutation κ_π contains a cycle (j_1, j_2, j_3) with $v_{j_t} = w_t$ for all $t \in [3]$, then one of these statements is true:*

(1) *The permutation κ_σ contains a cycle (j_2, j_3, j_4) with $v_{j_4} = w_4$.*
(2) *The permutation κ_σ contains two cycles (j_2, j_3', j_4') and (j_2'', j_3, j_4'') with $v_{j_2''} = w_2$, $v_{j_3'} = w_3$ and $v_{j_4'} = v_{j_4''} = w_4$. Additionally, the component function $s_{j_2''}$ is identical to s_{j_2} and the component function $s_{j_3'}$ is identical to s_{j_3}.*

PROOF. Assume that the permutation κ_σ does not contain any cycle involving the index j_2. Every permutation $\mu \in \mathfrak{A}(n)$ can be represented as a combination $\tau'\sigma\tau$, where τ', τ are even permutations with $\tau'(w_2) = \tau(w_2) = w_2$. Thus, for every permutation $\mu \in \mathfrak{A}(n)$, we have

$$s_{j_2}(\mu.x) = s_{j_2}(\tau'\sigma\tau.x) = s_{j_2}(\sigma\tau.x) = s_{\kappa_\sigma^{-1}(j_2)}(\tau.x) = s_{j_2}(\tau.x) = s_{j_2}(x)\,.$$

This contradicts the conditions on κ_π in Lemma 5.8. Similarly, we treat the case, when no cycle in κ_σ involves the index j_3.

Let us assume that there are two different cycles (j_2, j_3', j_4') and (j_2'', j_3, j_4'') in the permutation κ_σ. And let us consider the permutation $\pi\sigma$ which could be written as a combination of two disjoint cycles $(w_1, w_2)(w_3, w_4)$. From this, conclude that $(\pi\sigma)^2$ is the identity permutation, what implies that $(\kappa_\pi\kappa_\sigma)^2$ is equivalent to the identity permutation.

For every vertex $x \in \mathrm{vert}(P)$, we have

$$\begin{aligned} s_{j_3}(x) &= s_{j_3}((\pi\sigma)^2.x) = s_{\kappa_\pi^{-1}(j_3)}(\sigma\pi\sigma.x) = s_{j_2}(\sigma\pi\sigma.x) = \\ &s_{\kappa_\sigma^{-1}(j_2)}(\pi\sigma.x) = s_{j_4'}(\pi\sigma.x) = s_{j_4'}(\sigma.x) = s_{\kappa_\sigma^{-1}(j_4')}(x) = s_{j_3'}(x)\,. \end{aligned}$$

Thus, the component functions s_{j_3} and $s_{j_3'}$ are identical. Considering $s_{j_2}((\pi\sigma)^2.x)$, we get that the component functions s_{j_2} and $s_{j_2''}$ are identical as well.

□

5.9.4. Construction of Partition $\mathcal{A}_i$, $\mathcal{B}_j$.

Lemma 5.10. *For every cycle (j_1, j_2, j_3) in the permutation κ_{ρ_1}, satisfying the conditions in Lemma 5.8, there is a set*

$$\mathcal{S}_{(j_1,j_2,j_3)} = \{j_1, j_2, \cdots, j_n\}\,,$$

such that, for every ρ_v there is a permutation equivalent to κ_{ρ_1}, which contains the cycle (j_v, j_{v+1}, j_{v+2}) and has the properties, required in Lemma 5.8.

PROOF. Let us construct the set $\mathcal{S}_{(j_1,j_2,j_3)}$ iteratively. We start with

$$\mathcal{S}_{(j_1,j_2,j_3)} = \{j_1, j_2, j_3\}\,, \tag{5.9.6}$$

which satisfies the claim of the lemma for v equal to one.

From Lemma 5.9 for $\pi = \rho_1$, $\sigma = \rho_2$, there are two possible cases, concerning the cycle (j_1, j_2, j_3). In the case (1) of Lemma 5.9, extend the set $\mathcal{S}_{(j_1,j_2,j_3)}$ to the set $\{j_1, j_2, j_3, j_4\}$, and thus, $\mathcal{S}_{(j_1,j_2,j_3)}$ satisfies the claim of the lemma for v equal one and two. In the case (2) of Lemma 5.9, update κ_{ρ_2} by changing cycles (j_2, j_3', j_4'), (j_2'', j_3, j_4'') to (j_2, j_3, j_4'), (j_2'', j_3', j_4''), what produces a permutation equivalent to κ_{ρ_2}, and choose the set $\mathcal{S}_{(j_1,j_2,j_3)}$ to be equal to $\{j_1, j_2, j_3, j_4'\}$.

Going from v equal to three till $n-2$, and setting π to be ρ_{v-1} and σ to be ρ_v, extend the set $\mathcal{S}_{(j_1,j_2,j_3)}$, and, if necessary, update the permutation κ_{ρ_v}. □

Due to Lemma 5.10, construct disjoint sets $\mathcal{S}_{(j_1,j_2,j_3)}$ indexed by cycles of κ_{ρ_1} . Moreover, there is no cycles in $\kappa_{\rho_2}, \cdots, \kappa_{\rho_{n-2}}$, which does not contain any index from the constructed sets $\mathcal{S}_{(j_1,j_2,j_3)}$, due to Lemma 5.9.

Now, we can choose the sets $\mathcal{A}_i$ to be the sets $\mathcal{S}_{(j_1,j_2,j_3)}$, where the singletones $\{b_j\}$ involve the rest of component functions. Lemma 5.10 guarantees equation (5.9.1), and thus, we finish the proof of Theorem 5.5.

5.10. Permutahedron

Here, we establish a lower bound on sizes of symmetric extensions for the permutahedron.

Theorem 5.7. *For every $n \geq 6$, there exists no symmetric extension of the permutahedron $P = \Pi_n$ of size less than $\frac{n(n-1)}{2}$, with respect to the group $G = \mathfrak{S}(n)$.*

Define the function $\Lambda : \mathfrak{S}(n) \to \mathbb{R}^n$ as

$$\Lambda(\zeta) = (\zeta^{-1}(1), \zeta^{-1}(2), \cdots, \zeta^{-1}(n))\,.$$

Thus, we have

$$\mathrm{vert}(\Pi_n) = \{\Lambda(\zeta) : \zeta \in \mathfrak{S}(n)\}\,,$$

and the action of $\mathfrak{S}(n)$ on the vertex set $\mathrm{vert}(\Pi_n)$ is defined as

$$(\pi.\Lambda(\zeta))_v = \Lambda(\zeta)_{\pi^{-1}(v)}$$

holds for all $\pi \in \mathfrak{S}(n)$, $\zeta \in \mathfrak{S}(n)$.

Theorem 5.5 provides an information about the action of $\mathfrak{A}(n)$ on the component functions s_j, $j \in [d]$, and we fix the provided partition $\mathcal{A}_i$, b_j of the component functions.

Lemma 5.11. *There exists an element w, $1 \leq w \leq n-1$, such that the statements*

$$\text{if} \quad s_{a^i_w}(\Lambda(1_{\mathfrak{S}(n)})) > 0 \quad \text{then} \quad \sum_{v>w} s_{a^i_v}(\Lambda(1_{\mathfrak{S}(n)})) > 0 \tag{5.10.1}$$

$$\text{if} \quad s_{a^i_{w+1}}(\Lambda(1_{\mathfrak{S}(n)})) > 0 \quad \text{then} \quad \sum_{v\leq w} s_{a^i_v}(\Lambda(1_{\mathfrak{S}(n)})) > 0 \tag{5.10.2}$$

hold for all sets $\mathcal{A}_i$.

PROOF. Each set $\mathcal{A}_i$ consists of n components, what implies that the number of different $\mathcal{A}_i$ is less than $\frac{n-1}{2}$, since $d < \frac{n(n-1)}{2}$.

For every set $\mathcal{A}_i$, there is at most one element w from $[n-1]$, which violates the statement (5.10.1), since it should be the maximal element from $[n-1]$ for which the value $s_{a^i_w}(\Lambda(1_{\mathfrak{S}(n)}))$ is positive. Analogously, for every set $\mathcal{A}_i$, there is at most one element w from $[n-1]$, which violates the statement (5.10.2).

Thus, for at least one element $w \in [n-1]$ both (5.10.1) and (5.10.2) are satisfied for all sets $\mathcal{A}_i$. □

Choose an element w, satisfying Lemma 5.11, and define the following subgroup of $\mathfrak{A}(n)$

$$H = \{\pi \in \mathfrak{A}(n) : \pi([w]) = [w]\}\,.$$

Now, to disprove the possibility of such a symmetric extension, we apply the results of Section 5.6.3. For this, choose λ_x, $x \in \mathrm{vert}(P)$, where $x = \Lambda(\zeta)$, as follows

$$\lambda_x = \begin{cases} 1 & \text{if} \quad \zeta \in H \\ -\ \epsilon & \text{if} \quad \zeta \in H\tau \\ 0 & \text{otherwise}\,, \end{cases}$$

where $\tau \in \mathfrak{A}(n)$ is given as the cycle $(1, w, w+1)$ or $(n, w+1, w)$, depending on whether w is equal to one, and $H\tau$ denotes the right coset for H and the element $\tau \in \mathfrak{A}(n)$.

We have to guarantee that the inequalities (5.6.5) hold for some $\epsilon > 0$, i.e.

$$\sum_{x \in \mathrm{vert}(P)} \lambda_x s_{b_j}(x) \geq 0 \quad \text{for every} \quad b_j \in \mathcal{B} \tag{5.10.3}$$

$$\sum_{x \in \mathrm{vert}(P)} \lambda_x s_{a_t^i}(x) \geq 0 \quad \text{for every} \quad a_t^i \in \mathcal{A}_i\,. \tag{5.10.4}$$

The left side of (5.10.3) can be rewritten as follows

$$\sum_{x \in \mathrm{vert}(P)} \lambda_x s_{b_j}(x) = \sum_{\pi \in \mathfrak{S}(n)} \lambda_{\Lambda(\pi)} s_{b_j}(\Lambda(\pi)) = \sum_{\pi \in \mathfrak{S}(n)} \lambda_{\Lambda(\pi)} s_{b_j}(\pi.\Lambda(1_{\mathfrak{S}(n)})) =$$
$$\sum_{\pi \in H} s_{b_j}(\pi.\Lambda(1_{\mathfrak{S}(n)})) - \sum_{\pi \in H\tau} \epsilon s_{b_j}(\pi.\Lambda(1_{\mathfrak{S}(n)})) = |H|(1-\epsilon) s_{b_j}(\Lambda(1_{\mathfrak{S}(n)}))\,,$$

what is non-negative for all ϵ, $\epsilon \leq 1$.

The left side of (5.10.4) can be rewritten as follows

$$\sum_{x \in \mathrm{vert}(P)} \lambda_x s_{a_t^i}(x) = \sum_{\pi \in H} s_{a_t^i}(\Lambda(\pi)) - \sum_{\pi \in H\tau} \epsilon s_{a_t^i}(\Lambda(\pi)) =$$
$$\sum_{\pi \in H} s_{a_t^i}(\pi.\Lambda(1_{\mathfrak{S}(n)})) - \epsilon \sum_{\pi \in H\tau} s_{a_t^i}(\pi.\Lambda(1_{\mathfrak{S}(n)}))\,.$$

For $t \leq w$ this expression is equal

$$\sum_{\pi \in H} s_{a_t^i}(\pi.\Lambda(1_{\mathfrak{S}(n)})) - \epsilon \sum_{\pi \in H\tau} s_{a_t^i}(\pi.\Lambda(1_{\mathfrak{S}(n)})) = \sum_{v \leq w} \sum_{\substack{\pi^{-1}(t)=v \\ \pi \in H}} s_{a_v^i}(\Lambda(1_{\mathfrak{S}(n)})) -$$
$$\sum_{v < w} \sum_{\substack{\pi^{-1}(t)=v \\ \pi \in H\tau}} \epsilon s_{a_v^i}(\Lambda(1_{\mathfrak{S}(n)})) - \sum_{\substack{\pi^{-1}(t)=w+1 \\ \pi \in H\tau}} \epsilon s_{a_v^i}(\Lambda(1_{\mathfrak{S}(n)})) =$$
$$\frac{|H|}{w}\Big(\sum_{v \leq w} s_{a_v^i}(\Lambda(1_{\mathfrak{S}(n)})) - \epsilon \sum_{v \leq w-1} s_{a_v^i}(\Lambda(1_{\mathfrak{S}(n)})) - \epsilon s_{a_{w+1}^i}(\Lambda(1_{\mathfrak{S}(n)})) \Big)\,.$$

For $t > w$ this expression is equal

$$\sum_{\pi \in H} s_{a_t^i}(\pi.\Lambda(1_{\mathfrak{S}(n)})) - \epsilon \sum_{\pi \in H\tau} s_{a_t^i}(\pi.\Lambda(1_{\mathfrak{S}(n)})) = \sum_{v \geq w+1} \sum_{\substack{\pi^{-1}(t)=v \\ \pi \in H}} s_{a_v^i}(\Lambda(1_{\mathfrak{S}(n)})) -$$
$$\sum_{v > w+1} \sum_{\substack{\pi^{-1}(t)=v \\ \pi \in H\tau}} \epsilon s_{a_v^i}(\Lambda(1_{\mathfrak{S}(n)})) - \sum_{\substack{\pi^{-1}(t)=w \\ \pi \in H\tau}} \epsilon s_{a_v^i}(\Lambda(1_{\mathfrak{S}(n)})) =$$
$$\frac{|H|}{n-w}\Big(\sum_{v \geq w+1} s_{a_v^i}(\Lambda(1_{\mathfrak{S}(n)})) - \epsilon \sum_{v \geq w+2} s_{a_v^i}(\Lambda(1_{\mathfrak{S}(n)})) - \epsilon s_{a_w^i}(\Lambda(1_{\mathfrak{S}(n)})) \Big)\,.$$

Since for the element w the conditions (5.10.1), (5.10.2) are satisfied, the above expressions are non-negative for some choice of $\epsilon > 0$.

But on the other hand, considering the inequality

$$\sum_{v\in[w]} x_v \geq \frac{w(w+1)}{2}\,,$$

which is valid for the permutahedron, the condition (5.6.6) looks as

$$\sum_{x\in\mathrm{vert}(P)} \lambda_x\Big(\sum_{v\in[w]} x_v - \frac{w(w+1)}{2}\Big) =$$
$$\sum_{\pi\in H}\sum_{v\in[w]}\Big((\Lambda(\pi))_v - \frac{w(w+1)}{2}\Big) - \epsilon\sum_{\pi\in H\tau}\sum_{v\in[w]}\Big((\Lambda(\pi))_v - \frac{w(w+1)}{2}\Big) =$$
$$-\sum_{\pi\in H\tau}\epsilon < 0\,,$$

what finishes the proof of Theorem 5.7.

In Section 2.7, a symmetric extended formulation for the permutahedron Π_n of size $O(n^2)$ was presented, and thus, Lemma 5.10 provides an asymptotically tight bound on the size of symmetric extensions for Π_n.

In turn, there is an extended formulation of size $O(n\log n)$ constructed by Goemans (Section 2.21), what is an asymptotically minimal extension for the permutahedron Π_n (Section 4.18.1). Hence, we established a gap between symmetric and non-symmetric extensions of the permutahedron.

5.11. Cardinality Indicating Polytope

Theorem 5.8. *For every $n \geq 6$, there exists no symmetric extension of the cardinality indicating polytope $P = \mathrm{P}^n_{\mathrm{card}}$ of size less than $\frac{n(n-1)}{2}$, with respect to the group $G = \mathfrak{S}(n)$.*

The operator $\Lambda(W)$ maps every set $W \subseteq [n]$ to the vector $(\chi(W), \mathbb{e}_{|W|+1})$. Thus, we have

$$\mathrm{vert}(P) = \{\Lambda(W) : W \subseteq [n]\}\,,$$

and for every permutation $\pi \in \mathfrak{S}(n)$ and set $W \subseteq [n]$

$$(\pi.\Lambda(W))_v = \Lambda(W)_{\pi^{-1}(v)} \qquad \text{for } 1 \leq v \leq n$$
$$(\pi.\Lambda(W))_k = \Lambda(W)_k \qquad \text{for } n+1 \leq k \leq 2n+1\,.$$

Note that for the cardinality indicating polytope the group $\mathfrak{S}(n)$ does not act transitively on the vertex set $\mathrm{vert}(P)$, i.e. all vertices are divided into orbits, corresponding to possible cardinalities.

From Theorem 5.6, we get that for every set $W \subseteq [n]$ the value $s_{a^i_v}(\Lambda(W))$ depends only on the cardinality of the set W, and whether v belongs to the set W. In the same way, the value s_{b_j} depends on the cardinality of the set W only. Introduce shortcuts for these values

$$c^0_i(k) = s_{a^i_v}(\Lambda(W)) \qquad \text{for } v \notin W \text{ and } |W| = k$$
$$c^1_i(k) = s_{a^i_v}(\Lambda(W)) \qquad \text{for } v \in W \text{ and } |W| = k$$
$$c_j(k) = s_{b_j}(\Lambda(W)) \qquad \text{for } |W| = k$$

Lemma 5.12. *There exists, a cardinality k^*, $1 \le k^* \le n-1$, such that the statement*

$$\text{(5.11.1)} \quad \textit{if} \quad c_i^0(k^*) > 0 \quad \textit{or} \quad c_i^1(k^*) > 0 \quad \textit{then} \quad \sum_{0 \le k < k^*} c_i^0(k) + \sum_{k^* < k \le n} c_i^1(k) > 0$$

holds for all sets $\mathcal{A}_i$.

PROOF. Each set $\mathcal{A}_i$ consists of n components, and thus the number of sets $\mathcal{A}_i$ is less than $\frac{n-1}{2}$, since $d < \frac{n(n-1)}{2}$.

For every set $\mathcal{A}_i$, there are at most two cardinalities, such that the condition (5.11.1) is not satisfied. To prove this, assign to $\mathcal{A}_i$ the minimum cardinality $k_{\min}$ and the maximum cardinality $k_{\max}$ for which the statement (5.11.1) is violated. From (5.11.1), both values $c_i^0(k)$ and $c_i^1(k)$ are equal to 0 for all cardinalities k, $k_{\min} < k < k_{\max}$. Thus, the statement (5.11.1) holds for the set $\mathcal{A}_i$ and for all cardinalities k, $k_{\min} < k < k_{\max}$.

Hence, there exists at least one cardinality from 1 till $n-1$, which satisfies the condition (5.11.1) for all $\mathcal{A}_i$. □

Applying the results of Section 5.6.3, we choose λ_x, $x \in \operatorname{vert}(P)$, for $x = \Lambda(W)$, $W \subseteq [n]$, as follows

$$\lambda_x = \begin{cases} 1 & \text{if} \quad W = [t],\ 0 \le t \le n,\ t \ne k^* \\ 1+\epsilon & \text{if} \quad W = [k^*] \\ -\epsilon & \text{if} \quad W = [k^*-1] \cup \{k^*+1\} \\ 0 & \text{otherwise}\,, \end{cases}$$

where k^*, $1 \le k^* \le n-1$ is a cardinality, satisfying Lemma 5.12.

We have to guarantee that the inequality (5.6.5) hold for some $\epsilon > 0$, i.e.

$$\text{(5.11.2)} \qquad \sum_{x \in \operatorname{vert}(P)} \lambda_x s_{b_j}(x) \ge 0 \quad \text{for every} \quad b_j \in \mathcal{B}$$

$$\text{(5.11.3)} \qquad \sum_{x \in \operatorname{vert}(P)} \lambda_x s_{a_t^i}(x) \ge 0 \quad \text{for every} \quad a_t^i \in \mathcal{A}_i\,.$$

The left side of (5.11.2) can be rewritten as follows

$$\sum_{x \in \operatorname{vert}(P)} \lambda_x s_{b_j}(x) = \sum_{0 \le k \le n} c_j(k) + \epsilon c_j(k^*) - \epsilon c_j(k^*) = \sum_{0 \le k \le n} c_j(k)\,,$$

what is non-negative for all ϵ.

The left side of (5.11.3) for $t \notin \{k^*, k^*+1\}$ is equal

$$\sum_{x \in \operatorname{vert}(P)} \lambda_x s_{a_t^i}(x) = \sum_{0 \le k \le t-1} c_i^0(k) + \sum_{t \le k \le n} c_i^1(k)\,,$$

what is non-negative.

And for $t = k^*$ is equal

$$\sum_{x \in \operatorname{vert}(P)} \lambda_x s_{a_t^i}(x) = \sum_{0 \le k \le k^*-1} c_i^0(k) + \sum_{k^* \le k \le n} c_i^1(k) - \epsilon c_i^0(k^*) + \epsilon c_i^1(k^*) =$$
$$\sum_{0 \le k < k^*} c_i^0(k) + \sum_{k^* < k \le n} c_i^1(k) - \epsilon c_i^0(k^*) + (1+\epsilon) c_i^1(k^*)\,,$$

and for $t = k^*+1$ is equal

$$\sum_{x \in \operatorname{vert}(P)} \lambda_x s_{a_t^i}(x) = \sum_{0 \le k \le k^*} c_i^0(k) + \sum_{k^*+1 \le k \le n} c_i^1(k) - \epsilon c_i^1(k^*) + \epsilon c_i^0(k^*) =$$
$$\sum_{0 \le k < k^*} c_i^0(k) + \sum_{k^* < k \le n} c_i^1(k) - \epsilon c_i^1(k^*) + (1+\epsilon) c_i^0(k^*)\,.$$

Due to the condition (5.11.1), there exists $\epsilon > 0$, such that all above expressions are non-negative.

Use the inequality

$$\sum_{1 \le v \le k^*} x_v - \sum_{1 \le k \le k^*} k z_k - \sum_{k^* < k \le n} k^* z_k \le 0\,, \tag{5.11.4}$$

which is valid for $\mathrm{P}^n_{\mathrm{card}}$, to guarantee the condition (5.6.6)

$$\sum_{x \in \mathrm{vert}(P)} \lambda_x (-x_v + \sum_{1 \le k \le k^*} k z_k + \sum_{k^* < k \le n} k^* z_k) = \lambda_{\Lambda([k^*-1] \cap \{k^*+1\})} = -\epsilon < 0\,,$$

since for all vertices $x \in \mathrm{vert}(P)$, except the vertex $\Lambda([k^*-1] \cap \{k^*+1\})$, the coefficient λ_x or the inequality (5.11.4) is satisfied at equality.

Since in Section 2.4, a symmetric extended formulation for the cardinality indicating polytope of size $O(n^2)$ was presented, Lemma 5.8 provides an asymptotically tight bound on size of symmetric extensions for $\mathrm{P}^n_{\mathrm{card}}$.

In turn, we constructed an extended formulation of size $O(n \log n)$ in Section 2.21, which is an asymptotically minimal extension for the cardinality indicating polytope (Section 4.18.3). And thus, we established a gap between symmetric and non-symmetric extensions for the cardinality indicating polytope.

5.12. Parity Polytope

Theorem 5.9. *For every $n \ge 6$, there exists no symmetric extension of the parity polytope $P = \mathrm{P}^n_{\mathrm{even}}$ of size less than $n \log(\frac{n}{4})$, with respect to the group $G = \mathfrak{S}(n)$.*

Due to Jeroslow [1975], the parity polytope $\mathrm{P}^n_{\mathrm{even}}$ can be described as $0 \le x \le 1$ and

$$\sum_{v \in S} x_v - \sum_{v \in [n] \setminus S} x_v \le |S| - 1 \qquad \text{for } S \subseteq [n],\ |S| \text{ is odd}\ . \tag{5.12.1}$$

5.12.1. Symmetric Non-Negative Factorization of Slack Matrix. Consider a symmetric subspace extension $Q \subseteq \mathbb{R}^d$, $p : \mathbb{R}^d \to \mathbb{R}^m$ of the parity polytope, where d is less than $n \log(\frac{n}{4})$, with a symmetric section $s : \mathrm{vert}(P) \to Q$.

From Observation 4.2, there exist vectors $t(S) \in \mathbb{R}^d_+$, $S \subseteq [n]$, $|S| \in [n]_{\mathrm{odd}}$, such that the equation

$$\langle t(S), s(x) \rangle = |S| - 1 - \sum_{v \in S} x_v + \sum_{v \in [n] \setminus S} x_v\,, \tag{5.12.2}$$

holds for every $x \in \mathrm{vert}(P)$.

Theorem 5.6 describes the structure of the section $s(x)$. But, we want to study also the structure of the vectors $t(S)$, where $S \subseteq [n]$, $|S| \in [n]_{\mathrm{odd}}$.

To do this, consider a permutation $\pi \in \mathfrak{S}(n)$, such that $\pi.S$ is equal S. Due to (5.12.2), the slack variable, corresponding to a vertex x and the inequality, indexed by the set S, is equal to

$$\sum_{v \in [n]} \sum_{\mathcal{A}_i} t_{a^i_v}(S) s_{a^i_v}(x) + \sum_{b_j} t_{b_j}(S) s_{b_j}(x)\,.$$

Since the slack variables, corresponding to the vertices x, $\pi.x$ and the inequality (5.12.1) for the set S, are equal, we have

$$\sum_{v\in[n]}\sum_{\mathcal{A}_i} t_{a^i_v}(S)s_{a^i_v}(x)+\sum_{b_j} t_{b_j}(S)s_{b_j}(x)=$$
$$\sum_{v\in[n]}\sum_{\mathcal{A}_i} t_{a^i_v}(\pi.S)s_{a^i_v}(\pi.x)+\sum_{b_j} t_{b_j}(\pi.S)s_{b_j}(\pi.x)=$$
$$\sum_{v\in[n]}\sum_{\mathcal{A}_i} t_{a^i_v}(S)s_{a^i_v}(\pi.x)+\sum_{b_j} t_{b_j}(S)s_{b_j}(\pi.x)=$$
$$\sum_{v\in[n]}\sum_{\mathcal{A}_i} t_{a^i_v}(S)s_{a^i_{\pi^{-1}(v)}}(x)+\sum_{b_j} t_{b_j}(S)s_{b_j}(x)=$$
$$\sum_{v\in[n]}\sum_{\mathcal{A}_i} t_{a^i_{\pi(v)}}(S)s_{a^i_v}(x)+\sum_{b_j} t_{b_j}(S)s_{b_j}(x)\,.$$

Thus, we have

$$\sum_{v\in[n]}\sum_{\mathcal{A}_i} t_{a^i_v}(\pi.S)s_{a^i_v}(\pi.x)+\sum_{b_j} t_{b_j}(\pi.S)s_{b_j}(\pi.x)=$$
$$\frac{1}{|G_S|}\sum_{\pi\in G_S}\sum_{v\in[n]}\sum_{\mathcal{A}_i} t_{a^i_{\pi(v)}}(S)s_{a^i_v}(x)+\sum_{b_j} t_{b_j}(S)s_{b_j}(x)=$$
$$\sum_{\mathcal{A}_i}\sum_{v\in S} s_{a^i_v}(x)\frac{\sum_{w\in S} t_{a^i_{\pi(w)}}(S)}{|G_S|}+\sum_{\mathcal{A}_i}\sum_{v\notin S} s_{a^i_v}(x)\frac{\sum_{w\notin S} t_{a^i_{\pi(w)}}(S)}{|G_S|}+\sum_{b_j} t_{b_j}(S)s_{b_j}(x)\,,$$

where the group G_S is defined as follows

$$G_S=\{\pi\in\mathfrak{S}(n):\pi.S=S\}\,.$$

This allows us to assume that the condition

$$t(S)_{a^i_v}=t(S)_{a^i_w}\qquad\text{if}\quad v,w\in S\quad\text{or}\quad v,w\notin S$$

holds.

Similarly, we can assume that for two sets S_1, $S_2\subseteq[n]$, cardinalities of which are equal and odd, we can assume

$$t(S_1)_{a^i_v}=t(S_2)_{a^i_{\pi(v)}}\quad\text{and}\quad t(S_1)_{b_j}=t(S_2)_{b_j}$$

for every permutation $\pi\in\mathfrak{S}(n)$, where the image $\pi.S_1$ is equal S_2.

Due to the structure of the vectors $s(x)$, $x\in\operatorname{vert}(P)$, and vectors $t(S)$, $S\subseteq[n]$, $|S|\in[n]_{\text{odd}}$, we may introduce the following notation

$$c_i^{0,k}=s_{a^i_v}(\chi(R))\qquad\text{for } R\subseteq[n],\ |R|=k\in[n]_{\text{even}}\text{ and } v\notin R$$
$$c_i^{1,k}=s_{a^i_v}(\chi(R))\qquad\text{for } R\subseteq[n],\ |R|=k\in[n]_{\text{even}}\text{ and } v\in R$$
$$c_j^{k}=s_{b_j}(\chi(R))\qquad\text{for } R\subseteq[n],\ |R|=k\in[n]_{\text{even}}$$

and

$$t_i^{0,\ell}=t_{a^i_v}(S)\qquad\text{for } S\subseteq[n],\ |S|=\ell\in[n]_{\text{odd}}\text{ and } v\notin S$$
$$t_i^{1,\ell}=t_{a^i_v}(S)\qquad\text{for } S\subseteq[n],\ |S|=\ell\in[n]_{\text{odd}}\text{ and } v\in S$$
$$t_j^{\ell}=t_{b_j}(S)\qquad\text{for } S\subseteq[n],\ |S|=\ell\in[n]_{\text{odd}}\,.$$

Additionally, let $c^{0,k}$, $c^{1,k}$, $k\in[n]_{\text{even}}$ and $t^{0,\ell}$, $t^{1,\ell}$, $\ell\in[n]_{\text{odd}}$ denote the vectors, indexed by the sets $\mathcal{A}_i$, which have the corresponding coordinates. In the same manner, define the vectors c^k, $k\in[n]_{\text{even}}$ and r^ℓ, $\ell\in[n]_{\text{odd}}$, that are indexed by elements from the sets $\mathcal{B}_j$

Due to (5.12.2), the slack variable corresponding to the inequality, indexed by a set S, $S \subseteq [n]$, $|S| = \ell \in [n]_{\text{odd}}$ and vertex $x = \chi(R)$, $R \subseteq [n]$, $|R| = k \in [n]_{\text{even}}$, is equal

$$|S \cap R|\, \langle t^{1,\ell}, c^{1,k}\rangle + |S \setminus R|\, \langle t^{1,\ell}, c^{0,k}\rangle + |R \setminus S|\, \langle t^{0,\ell}, c^{1,k}\rangle + (n - |(S \cup R)|)\, \langle t^{0,\ell}, c^{0,k}\rangle + \langle t^{\ell}, c^{k}\rangle \,. \tag{5.12.3}$$

5.12.2. Lower Bound on Symmetric Non-negative Factorizations. Now, consider a set $S \subseteq [n]$, $|S| = \ell \in [n-1]_{\text{odd}}$ and two sets R_1 and R_2, $|R_1| = |R_2| = k \in [n-1]_{\text{even}}$, such that the equation

$$|S \cap R_1| = |S \cap R_2| - 1$$

holds.

From (5.12.3), we get

$$\begin{aligned}
&|S \cap R_1|\langle t^{1,\ell}, c^{1,k}\rangle + |S \setminus R_1|\langle t^{1,\ell}, c^{0,k}\rangle + |R_1 \setminus S|\langle t^{0,\ell}, c^{1,k}\rangle + \\
&\qquad (n - |(S \cup R_1)|)\langle t^{0,\ell}, c^{0,k}\rangle + \langle t^{\ell}, c^{k}\rangle = \\
&(|S \cap R_2| - 1)\langle t^{1,\ell}, c^{1,k}\rangle + (|S \setminus R_2| + 1)\langle t^{1,\ell}, c^{0,k}\rangle + (|R_2 \setminus S| + 1)\langle t^{0,\ell}, c^{1,k}\rangle + \\
&\qquad (n - |(S \cup R_1)| - 1)\langle t^{0,\ell}, c^{0,k}\rangle + \langle t^{\ell}, c^{k}\rangle = \\
&|S \cap R_2|\langle t^{1,\ell}, c^{1,k}\rangle + |S \setminus R_2|\langle t^{1,\ell}, c^{0,k}\rangle + |R_2 \setminus S|\langle t^{0,\ell}, c^{1,k}\rangle + \\
&\qquad (n - |(S \cup R_2)|)\langle t^{0,\ell}, c^{0,k}\rangle + \langle t^{\ell}, c^{k}\rangle + \langle t^{1,\ell} - t^{0,\ell}, c^{0,k} - c^{1,k}\rangle \,,
\end{aligned}$$

and calculating the relation between the corresponding slack variables

$$|S| - 1 - |R_1 \cap S| + |R_1 \setminus S| = |S| - 1 - |R_2 \cap S| + |R_2 \setminus S| + 2 \,,$$

we get that the equation

$$2 + \langle t^{1,\ell} - t^{0,\ell}, c^{1,k} - c^{0,k}\rangle = 0 \tag{5.12.4}$$

holds for all $k \in [n-1]_{\text{even}}$ and $\ell \in [n-1]_{\text{odd}}$. From the non-negativity of the vectors $c^{0,k}$, $c^{1,k}$, $t^{0,\ell}$, $t^{1,\ell}$, we get

$$\langle t^{1,\ell}, c^{0,k}\rangle + \langle t^{0,\ell}, c^{1,k}\rangle \geq 2 \,. \tag{5.12.5}$$

Let us assume that there are $k_2 < k_1 \leq \lfloor \frac{n}{2} \rfloor$, k_1, k_2 are even, such that the vectors c^{1,k_1}, c^{1,k_2} have the same support[1].

First, consider the case, when a set $S \subseteq [n]$, $|S| = \ell = k_1 - 1$ is disjoint to a set $R \subseteq [n]$, $|R| = k_1$. From (5.12.3), we have

$$(n - \ell - k_1)\langle t^{0,\ell}, c^{0,k_1}\rangle + k_1\langle t^{0,\ell}, c^{1,k_1}\rangle + \ell\langle t^{1,\ell}, c^{0,k_1}\rangle + \langle t^{\ell}, c^{k}\rangle = k_1 + \ell - 1 \,,$$

what leads to

$$(\ell + 1)\langle t^{0,\ell}, c^{1,k_1}\rangle + \ell\langle t^{1,\ell}, c^{0,k_1}\rangle \leq 2\ell \,,$$

and thus, from the inequality (5.12.5)

$$\langle t^{0,\ell}, c^{1,k_1}\rangle = 0 \,. \tag{5.12.6}$$

On the other hand, consider the case, when a set $S \subseteq [n]$, $|S| = \ell = k_1 - 1$ is disjoint to a set $R \subseteq [n]$, $|R| = k_2$. From (5.12.3), we get

$$(n - \ell - k_2)\langle t^{0,\ell}, c^{0,k_2}\rangle + k_2\langle t^{0,\ell}, c^{1,k_2}\rangle + \ell\langle t^{1,\ell}, c^{0,k_2}\rangle + \langle t^{\ell}, c^{k_2}\rangle = k_2 + \ell - 1 \,,$$

and thus

$$k_2\langle t^{0,\ell}, c^{1,k_2}\rangle + \ell\langle t^{1,\ell}, c^{0,k_2}\rangle \leq k_2 + \ell - 1 \,,$$

but due to (5.12.4), the inequality

$$(k_2 - \ell)\langle t^{0,\ell}, c^{1,k_2}\rangle + 2\ell \leq k_2 + \ell - 1 \,,$$

[1] Here, we apply an argumentation similar to the argumentation in Section 4.14.4.

holds, what leads to (note $k_2 < \ell = k_1 - 1$)

$$\langle t^{0,\ell}, c^{1,k_2} \rangle \geq \frac{k_2 - \ell - 1}{k_2 - \ell} = \frac{k_1 - k_2}{k_1 - k_2 - 1} > 0\,. \tag{5.12.7}$$

Due to the equation (5.12.6) and inequality (5.12.7), the vectors c^{1,k_1}, c^{1,k_2} cannot have the same support. The vectors $c^1(k_1)$, $c^1(k_2)$ are indexed by the sets $\mathcal{A}_i$, and thus the number of sets $\mathcal{A}_i$ is at least $\log(\frac{n}{4})$, what states the lower bound $n\log(\frac{n}{4})$ on the number of variables in symmetric extensions of the parity polytope, since every set $\mathcal{A}_i$ contains n elements. This finishes the proof of Theorem 5.9.

And since in Section 2.5, an extended formulation of the parity polytope with size $O(n)$ was presented Carr and Konjevod [2004], we established a size gap between symmetric and non-symmetric extensions for the parity polytope.

But it is unknown, whether Theorem 5.9 provides an asymptotically tight bound on size of symmetric extensions for the parity polytope, since the best known symmetric extension Yannakakis [1991] has size $O(n^2)$ (Section 2.5, Balas approach).

CHAPTER 6

Appendix

In Appendix, we collected results, which were used in the chapters below, but were left out of the consideration, in order to focus the attention on the content of the corresponding chapter.

6.1. Polytopes, Extended Formulations, Extensions

Lemma 6.1. *For a polyhedron $P \subseteq \mathbb{R}^d$, the trivial inequality $0 \le 1$ can be obtained as a non-negative combination of the inequalities in a linear system, defining the polyhedron P, unless the dimension of the recession cone $\operatorname{rec}(P)$ is equal to the dimension of the polyhedron P.*

PROOF. Let us assume that the linear system

$$cA = \mathbf{0}_d \,, \langle c, b\rangle = 1 \text{ and } c \ge 0$$

does not have a solution, for a matrix $A \in \mathbb{R}^{f\times d}$ and $b \in \mathbb{R}^f$, such that

$$P = \{x \in \mathbb{R}^d : Ax \le b\}\,.$$

From the Farkas Lemma, the system

$$Ay \le -b$$

has a solution $y' \in \mathbb{R}^d$. Taking $\dim(P)+1$ affinely independent points $x^1, \ldots, x^{\dim(P)+1}$, obtain $\dim(P)+1$ affinely independent points $y'+x^1, \ldots, y'+x^{\dim(P)+1}$ in the recession cone $\operatorname{rec}(P)$. □

Lemma 6.2. *Whenever the linear system*

$$Ax \le b \quad \textit{and} \quad 0 \le x\,, \tag{6.1.1}$$

where $A \in \mathbb{R}^{f\times d}$, $b \in \mathbb{R}^d$, defines an integral polyhedron, the linear system

$$\begin{aligned} &Ax \le b \quad \textit{and} \quad 0 \le y \\ &x_i = \sum_{t\in d_i} y^i_t \quad \textit{for} \quad i \in [d] \end{aligned} \tag{6.1.2}$$

defines an integral polyhedron as well.

PROOF. It is enough, to show that for all integer vectors $c \in \mathbb{Z}^d$ and $c^i \in \mathbb{Z}^{d_i}$, $i \in [d]$, the maximum of the linear function $\langle c, x\rangle + \sum_{i\in[d]}\langle c^i, y^i\rangle$, with respect to the linear system (6.1.2), is integer or is infinite (see Schrijver [1986]). Consider the optimization problem

$$\max_{(t_1,\ldots,t_d)\in[d_1]\times\ldots\times[d_d]} \max_x (c_1 + c^1_{t_1})x_1 + \ldots + (c_d + c^d_{t_d})x_d$$

over the system of linear inequalities (6.1.2). It is not hard to see that the optimal values for both problems coincide, what finishes the proof. □

Lemma 6.3. *For every polyhedron $Q \subseteq \mathbb{R}^d$, described by*

$$\langle a^i, y\rangle \le b_i \quad \textit{for all} \quad i \in I\,, \tag{6.1.3}$$

the polyhedron $Q - \mathrm{rec}(Q) \subseteq \mathbb{R}^d$ *is described by the linear system*

$$\langle a^i, y\rangle \le b_i \quad \textit{for all} \quad i \in I'\,, \tag{6.1.4}$$

where I' *consists of all indices* $i \in I$*, such that* $\langle a^i, r\rangle = 0$ *is satisfied for all* $r \in \mathrm{rec}(Q)$*.*

PROOF. Let Q' be the polyhedron, described by the liner system (6.1.4). The inclusion $Q - \mathrm{rec}(Q) \subseteq Q'$ is trivial, since Q satisfies the linear system (6.1.4) and $-\mathrm{rec}(Q)$ belongs to the recession cone $\mathrm{rec}(Q')$.

Note that for each inequality $\langle a^i, y\rangle \le b_i$, $i \in I \setminus I'$, there exists a vector $r^i \in \mathrm{rec}(Q)$, such that $\langle a^i, r^i\rangle < 0$. Thus, for every point $y' \in Q'$, there exist non-negative numbers $\lambda_j \in \mathbb{R}$, $j \in I \setminus I'$, such that

$$\langle a^i, y' + \sum_{j\in I\setminus I'} \lambda_j r^j\rangle \le b_i$$

for $i \in I \setminus I'$, and obviously

$$\langle a^i, y' + \sum_{j\in I\setminus I'} \lambda_j r^j\rangle = \langle a^i, y'\rangle + \sum_{j\in I\setminus I'} \lambda_j \langle a^i, r^j\rangle \le \langle a^i, y'\rangle \le b_i$$

for $i \in I'$, i.e. the point $y' + \sum_{j\in I\setminus I'} \lambda_j r^j$ satisfies the linear system (6.1.3), and thus, $y' \in Q - \mathrm{rec}(Q)$. □

6.2. Rectangle Covering

Lemma 6.4. *For a matrix* $M \in \mathbb{R}^{I\times J}$*, where*

$$I = \{S_1 \subseteq [n] : |S_1| \le k_1\} \quad \textit{and} \quad J = \{S_2 \subseteq [n] : |S_2| \le k_2\}\,,$$

where $k_2 \le k_1 \le n$*, such that the entry* M_{S_1,S_2} *is non-zero if and only if the sets* S_1*,* S_2 *are disjoint, there exists a rectangle cover for the matrix* $M \in \mathbb{R}^{I\times J}$ *of size*

$$O((k_1 + k_2)(\frac{k_1 + k_2}{k_2})^{k_2} \log n)\,.$$

PROOF. Consider the rectangles R_U, $U \subseteq [n]$, defined in the following way

$$\{S_1 \subseteq [n] : S_1 \subseteq U\} \times \{S_2 \subseteq [n] : S_2 \cap U = \varnothing\}\,.$$

Obviously, if a pair (S_1, S_2) lies in the rectangle R_U, $U \subseteq [n]$, then the sets S_1, S_2 are disjoint, hence, every rectangle R_U is a non-zero rectangle.

Choose a set $U \subseteq [n]$, taking elements of $[n]$ independently with probability

$$p = \frac{k_1}{k_1 + k_2}\,.$$

Thus, for a fixed pair (S_1, S_2) of disjoint sets S_1, $S_2 \subseteq [n]$, the probability to be covered is at least

$$(1-p)^{k_2} p^{k_1} = \big(1 - \frac{k_1}{k_1 + k_2}\big)^{k_2} \big(\frac{k_1}{k_1 + k_2}\big)^{k_1} \ge \big(\frac{k_2}{k_1 + k_2}\big)^{k_2} e^{\frac{k_1 k_2}{k_1 + k_2}} \ge \big(\frac{k_2}{k_1 + k_2}\big)^{k_2} e^{\frac{1}{2}}\,.$$

Let us bound the logarithm of the expected number of entries from $\mathrm{supp}(M)$, which are not covered, if we choose independently r such rectangles

$$\log\Big(\binom{n + k_1}{k_1}\binom{n + k_2}{k_2}(1-q)^r\Big) < \log\big((2n)^{k_1}(2n)^{k_2}\big) + r\log(1-q) =$$

$$(k_1 + k_2)\log(2n) - r\big(\frac{k_2}{k_1 + k_2}\big)^{k_2} e^{\frac{1}{2}}\,.$$

Whenever the above upper bound for the logarithm of the expected number of uncovered entries from $\mathrm{supp}(M)$ is negative, we can conclude that there exists a rectangle cover for the matrix $M \in \mathbb{R}^{I\times J}$ of size r. Thus, there exists a rectangle cover of size $O((k_1 + k_2)\big(\frac{k_1+k_2}{k_2}\big)^{k_2} \log n\big)$.

□

Lemma 6.5. *For a matrix $M \in \mathbb{R}^{I\times X}$, where $I = \{S \subseteq [n] : |S| \le k\}$, $X = [n]$ for $k \le n$, such that the entry $M_{S,x}$ is non-zero if and only if $x \notin S$, the rectangle covering number of the support of the matrix $M \in \mathbb{R}^{I\times J}$ is at least* $\min(n-k, \frac{(k+1)(k+2)}{2} - 1)$.

PROOF. For every rectangle cover $\mathcal{R}$ for the matrix $M \in \mathbb{R}^{I\times X}$, we can assume that every rectangle $R \in \mathcal{R}$ is induced by some set $V \subseteq X$ in the following way

$$R = \{S \subseteq [n] : S \cap V = \varnothing\} \times V\,,$$

since the maximal non-zero rectangles have the above form. Thus, we are able to consider the set $\mathcal{V}$, consisting of these sets $V \subseteq [n]$, which induce the rectangle covering $\mathcal{R}$. Additionally, for each $x \in X$ denote by $\mathcal{V}_x$ the set of all sets $V \in \mathcal{V}$, such that $x \in V$. Moreover, assume that $\mathcal{V}_x = \{\{x\}\}$, whenever $\{x\} \in \mathcal{V}_x$, what can be achieved by excluding $x \in X$ from all other sets in $\mathcal{V}$.

Define X' as follows

$$X' = \{x \in X : \mathcal{V}_x = \{\{x\}\}\}\,.$$

Choose k distinct elements $x_1,\dots,x_k$ from the set $X \setminus X'$, which is possible, since otherwise $|\mathcal{R}| \ge n-k$. For $i \in [k]$ consider the set

$$\mathcal{V}'_i = \mathcal{V}_{x_i} \setminus \Big(\bigcup_{j\in[i-1]} \mathcal{V}_{x_j} \Big)\,,$$

i.e. all sets in $\mathcal{V}$, which contain x_i but do not contain any x_j, $j \in [i-1]$.

Assume that the cardinality of the set $\mathcal{V}'_i$ is smaller than $k+2-i$ for some $i \in [k]$. Construct a set V' by choosing an element from each set in $\mathcal{V}'_i$. Thus, we get that $|V'| \le k+1-i$ from the assumption on the cardinality $\mathcal{V}'_i$. Define the set $S \subseteq [n]$, $|S| \le k$ as

$$S = V' \cup \{v_1, \dots, v_{i-1}\}\,,$$

such that x_i does not belong S, but every set in $\mathcal{V}_{x_i}$ is not disjoint to the set S. Thus, there exists no rectangle in $\mathcal{R}$, which covers the entry (S, x_i).

So the cardinality of every set $\mathcal{V}'_i$, $i \in [k]$, is at least $k+2-i$, and since the sets $\mathcal{V}'_i$, $i \in [k]$ are disjoint

$$|\mathcal{R}| \ge \frac{(k+2)(k+1)}{2} - 1\,.$$

□

6.3. Groups

The next theorem is the central theorem in Chapter 5, and is due to Yannakakis [1991].

Theorem 6.1 (Yannakakis [1991])**.** *For every subgroup U of the group $\mathfrak{S}(n)$, where $\mathfrak{S}(n) : U$ is at most $\binom{n}{k}$, $k < \frac{n}{4}$, there exists $W \subseteq [n]$, such that*

$$\{\pi \in \mathfrak{A}(n) : \pi.w = w \quad \textit{for all} \quad w \in W\}\,.$$

PROOF. Let us assume that the group U is not transitive. Consider an orbit B of U with maximal cardinality. The cardinality of B is at least $n-k$, since otherwise the cardinality of the group U is less than $(n-k)!k!$, what contradicts the condition on the index of U in $\mathfrak{S}(n)$. If the action of the group U on B is not primitive, with $t \ge 2$ blocks of imprimitivity and ℓ elements in each, then the cardinality of U is at most

$$t!(\ell!)^t(n-t\ell)!\,,$$

where $t\ell$ is at least $n-k$. It is not hard to see that under these conditions $t!(\ell!)^t(n-t\ell)!$ is smaller than $k!(n-k)!$, and thus, the action of the group U on B is primitive.

Let us denote by U_1, U_2 the permutation groups defining the action of the group U on the sets B, W, respectively, where W denotes the set $[n] \setminus B$. And let U_1^* be the subgroup of U_1, which is defined by the action of the group

$$\{\pi \in U : \pi.w = w \quad \text{for all} \quad w \in W\}\,,$$

on the set B. Obviously, $|U|$ is equal to $|U_1^*||U_2|$ and U_1^* is a normal subgroup of U_1. Thus, U_1^* acts transitively on B, since U_1^* is a non-trivial normal subgroup of the primitive permutation group U_1. The group U_1^* acts primitively on B, due to the cardinality reasons above. Thus, the group U_1^* contains $\mathfrak{A}(B)$, because the index of every primitive subgroup of $\mathfrak{S}(B)$ is at least $\lfloor\frac{|B|+1}{2}\rfloor!$, unless it is $\mathfrak{S}(B)$ or $\mathfrak{A}(B)$ (see Wielandt [1964]), and because the inequality

$$(n-k)! > \frac{|B|!}{\lfloor\frac{|B|+1}{2}\rfloor!}$$

holds. □

6.4. Notation List

$G=(V,E)$	graph with vertices V and edges E
K_n	complete graph with n vertices
$K_{n,m}$	complete bipartite graph with n and m vertices in bipartition
$E(V:U)$	edges, having one vertex in V and one vertex in U
$\alpha(G)$	stable set number of G
$\gamma(G)$	genus of G
$\omega(G)$	clique number of G
$\chi(G)$	coloring number of G
$\delta^{in}(V)$	incoming edges for vertex set V
$\delta^{out}(V)$	outgoing edges for vertex set V
$\mathcal{C}^\ell(G)$	cycles of size ℓ in G
$\mathcal{J}^T(G)$	T-joins in G
$\mathcal{M}^\ell(G)$	matchings of size ℓ in G
$\mathcal{T}(G)$	spanning trees in G
$\mathrm{V}^n_{\mathrm{huff}}$	n-dimensional Huffman vectors
$\mathbf{M}_{\mathrm{slack}}(P^*,P_*)$	slack matrix for polyhedra P_* and P^*
$\mathbf{M}_{\mathrm{slack}}(P)$	slack matrix for polyhedron P
$\mathrm{supp}(M)$	support of M
$\mathcal{R}(M)$	non-zero rectangles of M
$\mathcal{L}(P^*,P_*)$	face poset for polyhedra P_* and P^*
$\mathcal{L}(P)$	face lattice for polyhedron P
$\mathrm{rank}_+ M$	non-negative rank of M
$\mathrm{rc}(P)$	rectangle covering bound for P
$\chi(S)$	characteristic vector of S with respect to the corresponding superset
$\langle a,b\rangle$	$\sum_{i=1}^d a_ib_i$ for a, $b\in\mathbb{R}^d$
$\mathfrak{S}(n)$	symmetric group on n elements
$\mathfrak{A}(n)$	alternating group on n elements
$\mathrm{iso}_G(s)$	isotropy group of s with respect to the action of G
$\mathrm{GF}(2)$	Galois field
$[n]$	set of numbers from one till n
$[n]_{\mathrm{odd}}$	set of odd numbers from one till n
$[n]_{\mathrm{even}}$	set of even numbers from one till n
$\mathbf{0}_d$	d-dimensional vector $(0,\dots,0)$
$\mathbf{1}_d$	d-dimensional vector $(1,\dots,1)$
$\mathrm{P}^\ell_{\mathrm{match}}(G)$	cardinality restricted matching polytope
$\mathrm{P}^\ell_{\mathrm{cycl}}(G)$	cardinality restricted cycle polytope
$\mathrm{P}^n_{\mathrm{card}}$	cardinality indicating polytope
$\mathrm{P}_{\mathrm{edge}}(G)$	edge polytope
$\mathrm{P}^n_{\mathrm{birk}}$	Birkhoff polytope
$\mathrm{P}^T_{\mathrm{join}}(G)$	T-joins polytope
$\mathrm{P}_{\mathrm{cut}}(G)$	cut polytope
$\mathrm{P}^n_{\mathrm{even}}$	parity polytope
$\mathrm{P}^n_{\mathrm{odd}}$	parity polytope
$\mathrm{P}^n_{\mathrm{huff}}$	Huffman polytope
$\mathrm{P}^\ell_{s-t\ \mathrm{flow}}(N)$	flow polyhedron
$\mathrm{P}_{\mathrm{ste}}(G)$	spanning tree polytope
$\mathrm{lineal}(P)$	lineality space of P
$\mathrm{rec}(P)$	recession cone of P
$\mathrm{vert}(P)$	vertices of P
$\mathrm{aff}(X)$	affine hull of X

$\mathrm{conv}(X)$	convex hull of X
$\mathrm{cone}(X)$	convex cone of X

Bibliography

M. Ajtai, J. Komlós, and E. Szemerédi. Sorting in $c \log n$ parallel steps. *Combinatorica*, 3 (1):1–19, 1983.

N. Alon, R. Yuster, and U. Zwick. Color-coding. *Journal of the Association for Computing Machinery*, 42(4):844–856, 1995.

E. Balas. Disjunctive programming: Properties of the convex hull of feasible points. *Discrete Applied Mathematics*, 89(1-3):3 – 44, 1998.

F. Barahona. On cuts and matchings in planar graphs. *Mathematical Programming*, 60(1, Ser. A):53–68, 1993.

F. Barahona and A. R. Mahjoub. On the cut polytope. *Mathematical Programming*, 36(2): 157–173, 1986.

A. Ben-Tal and A. Nemirovski. On polyhedral approximations of the second-order cone. *Mathematics of Operations Research*, 26(2):193–205, 2001.

G. Birkhoff. Three observations on linear algebra. *Univ. Nac. Tucumán. Revista A.*, 5: 147–151, 1946.

G. Braun and S. Pokutta. An algebraic view on symmetric extended formulations. 2011.

R. D. Carr and G. Konjevod. Polyhedral combinatorics. In *Tutorials on emerging methodologies and applications in Operations Research*, chapter 2. Springer, 2004.

J. Chen, X. Huang, I. A. Kanj, and G. Xia. Strong computational lower bounds via parameterized complexity. *Journal of Computer and System Sciences*, 72(8):1346–1367, 2006.

K. K. H. Cheung. *Subtour Elimination Polytopes and Graphs of Inscribable Type*. PhD thesis, University of Waterloo, 2003.

M. Conforti, G. Cornuéjols, and G. Zambelli. Extended formulations in combinatorial optimization. *A Quarterly Journal of Operations Research*, 8(1):1–48, 2010.

W. H. Cunningham. Minimum cuts, modular functions, and matroid polyhedra. *Networks*, 15(2):205–215, 1985.

R. de Wolf. Nondeterministic quantum query and communication complexities. *SIAM Journal on Computing*, 32(3):681–699, 2003.

M. Dietzfelbinger, J. Hromkovič, and G. Schnitger. A comparison of two lower-bound methods for communication complexity. *Theoretical Computer Science*, 168(1):39–51, 1996.

J. Edmonds. Maximum matching and a polyhedron with $0, 1$-vertices. *Journal of Research of the National Bureau of Standards*, 69B:125–130, 1965.

J. Edmonds. Matroids and the greedy algorithm. *Mathematical Programming*, 1:127–136, 1971.

J. Edmonds and E. L. Johnson. Matching, Euler tours and the Chinese postman. *Mathematical Programming*, 5:88–124, 1973.

P. Erdős and L. Pyber. Covering a graph by complete bipartite graphs. *Discrete Mathematics*, 170:249–251, 1997.

Y. Faenza, S. Fiorini, R. Grappe, and H. R. Tiwary. Extended formulations, non-negative factorizations and randomized communication protocols. *ArXiv e-prints*, May 2011.

I. S. Filotti, G. L. Miller, and J. Reif. On determining the genus of a graph in o(v o(g)) steps(preliminary report). In *Proceedings of the eleventh annual ACM symposium on*

Theory of computing, STOC '79, pages 27–37, New York, 1979.

S. Fiorini, V. Kaibel, K. Pashkovich, and D. O. Theis. Combinatorial Bounds on Nonnegative Rank and Extended Formulations. *ArXiv e-prints*, Nov. 2011a. submitted.

S. Fiorini, S. Massar, S. Pokutta, H. R. Tiwary, and R. de Wolf. Linear vs. Semidefinite Extended Formulations: Exponential Separation and Strong Lower Bounds. *ArXiv e-prints*, Nov. 2011b.

S. Fiorini, T. Rothvoß, and H. R. Tiwary. Extended formulations for polygons. *ArXiv e-prints*, July 2011c.

T. Fleiner, V. Kaibel, and G. Rote. Upper bounds on the maximal number of facets of 0/1-polytopes, 1999.

S. Fomin and N. Reading. Root systems and generalized associahedra. In *Geometric combinatorics*, volume 13 of *IAS/Park City Math. Ser.*, pages 63–131. Amer. Math. Soc., Providence, 2007.

M. L. Fredman, J. Komlós, and E. Szemerédi. Storing a sparse table with $O(1)$ worst case access time. *Journal of the Association for Computing Machinery*, 31(3):538–544, 1984.

A. M. H. Gerards. Compact systems for T-join and perfect matching polyhedra of graphs with bounded genus. *Operations Research Letters*, 10(7):377–382, 1991.

N. Gillis and F. Glineur. On the geometric interpretation of the nonnegative rank. *ArXiv e-prints*, Sept. 2010.

M. Goemans. Smallest compact formulation for the permutahedron.

J. Gouveia, P. A. Parrilo, and R. Thomas. Lifts of convex sets and cone factorizations. *ArXiv e-prints*, Nov. 2011.

H. Gruber and M. Holzer. Inapproximability of nondeterministic state and transition complexity assuming $\mathrm{P} \neq \mathrm{NP}$. In *Developments in language theory*, volume 4588 of *Lecture Notes in Comput. Sci.*, pages 205–216. Springer, Berlin, 2007.

B. Grünbaum. *Convex polytopes*, volume 221 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2003. Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler.

H. Huang and B. Sudakov. A counterexample to the Alon-Saks-Seymour conjecture and related problems. *ArXiv e-prints*, Feb. 2010.

J. E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

J. Janssen and K. Kilakos. Bounded stable sets: Polytopes and colorings. *SIAM Journal Discrete Mathematics*, 12(2):262–275, February 1999.

R. G. Jeroslow. On defining sets of vertices of the hypercube by linear inequalities. *Discrete Mathematics*, 11:119–124, 1975.

V. Kaibel and A. Loos. Branched polyhedral systems. In F. Eisenbrand and B. Shepherd, editors, *Integer Programming and Combinatorial Optimization (Proc. IPCO XIV)*, volume 6080 of *LNCS*, pages 177–190. Springer, 2010.

V. Kaibel and A. Loos. Finding descriptions of polytopes via extended formulations and liftings. 2011.

V. Kaibel and K. Pashkovich. Constructing extended formulations from reflection relations. In O. Günlük and G. Woeginger, editors, *Integer Programming and Combinatoral Optimization*, volume 6655 of *Lecture Notes in Computer Science*, pages 287–300. Springer Berlin / Heidelberg, 2011.

V. Kaibel, K. Pashkovich, and D. O. Theis. Symmetry matters for the sizes of extended formulations. In F. Eisenbrand and B. Shepherd, editors, *Integer Programming and Combinatorial Optimization (Proc. IPCO XIV)*, volume 6080 of *LNCS*, pages 135–148. Springer, 2010.

M. Karchmer, E. Kushilevitz, and N. Nisan. Fractional covers and communication complexity. *SIAM Journal on Discrete Mathematics*, 8(1):76–92, 1995.

M. Köppe, Q. Louveaux, and R. Weismantel. Intermediate integer programming representations using value disjunctions. *Discrete Optimization*, 5(2):293 – 313, 2008.

E. Kushilevitz and N. Nisan. *Communication complexity*. Cambridge University Press, Cambridge, 1997.

L. Lovász. On the ratio of optimal integral and fractional covers. *Discrete Mathematics*, 13(4):383–390, 1975.

R. Martin. Using separation algorithms to generate mixed integer model reformulations. *Operations Research Letters*, 10(3):119 – 128, 1991.

R. K. Martin, R. L. Rardin, and B. A. Campbell. Polyhedral characterization of discrete dynamic programming. *Operations Research*, 38:127–138, 1990.

V. H. Nguyen, T. H. Nguyen, and J.-F. Maurras. On the convex hull of Huffman trees. *Electronic Notes in Discrete Mathematics*, 36:1009–1016, 2010.

M. W. Padberg and L. A. Wolsey. Trees and cuts. In *Combinatorial mathematics (Marseille-Luminy, 1981)*, volume 75 of *North-Holland Math. Stud.*, pages 511–517. North-Holland, Amsterdam, 1983.

K. Pashkovich. Symmetry in Extended Formulations of the Permutahedron. *ArXiv e-prints*, Dec. 2009.

R. Rado. An inequality. *Journal of the London Mathematical Society*, 27:1–6, 1952.

J. Rambau. *Polyhedral Subdivisions and Projections of Polytopes*. PhD thesis, Technical University of Berlin, 1996.

I. Rivin. A characterization of ideal polyhedra in hyperbolic 3-space. *The Annals of Mathematics*, 143:51–70, 1996.

I. Rivin. Combinatorial optimization in geometry. *Advances in Applied Mathematics*, 31: 242–271, 2003.

T. Rothvoß. Some 0/1 polytopes need exponential size extended formulations. *ArXiv e-prints*, Apr. 2011.

J. P. Schmidt and A. Siegel. The spatial complexity of oblivious k-probe hash functions. *SIAM Journal on Computing*, 19(5):775–786, 1990.

A. Schrijver. *Theory of linear and integer programming*. Wiley-Interscience Series in Discrete Mathematics. John Wiley & Sons Ltd., Chichester, 1986.

A. Schrijver. *Combinatorial optimization. Polyhedra and efficiency. Vol. A*, volume 24 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2003a.

A. Schrijver. *Combinatorial optimization. Polyhedra and efficiency. Vol. B*, volume 24 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2003b. Matroids, trees, stable sets, Chapters 39–69.

A. Schrijver. *Combinatorial optimization. Polyhedra and efficiency. Vol. C*, volume 24 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2003c. Disjoint paths, hypergraphs, Chapters 70–83.

C. Thomassen. The graph genus problem is NP-complete. *Journal of Algorithms*, 10(4): 568–576, 1989.

Z. Tuza. Covering of graphs by complete bipartite subgraphs: complexity of 0-1 matrices. *Combinatorica*, 4:111–116, 1984.

S. A. Vavasis. On the complexity of nonnegative matrix factorization. *SIAM Journal on Optimization*, 20(3):1364–1377, 2009.

P. Ventura and F. Eisenbrand. A compact linear program for testing optimality of perfect matchings. *Operations Research Letters*, 31(6):429 – 434, 2003.

A. T. White. *Graphs, groups and surfaces*. North-Holland Publishing Co., Amsterdam, 1973. North-Holland Mathematics Studies, No. 8.

H. Wielandt. *Finite permutation groups*. Academic Press, New York, 1964.

J. C. Williams. A linear-size zero-one programming model for the minimum spanning tree problem in planar graphs. *Networks*, 39(1):53–60, 2002.

M. Yannakakis. Expressing combinatorial optimization problems by linear programs. *Journal of Computer and System Sciences*, 43(3):441–466, 1991.

G. M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.