# Lattice Point Inequalities and Face Numbers of Polytopes in View of Central Symmetry

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## Zusammenfassung

In der vorliegenden Arbeit studieren wir klassische Probleme aus der Diskreten Geometrie und der Konvexgeometrie unter dem Gesichtspunkt von Zentralsymmetrie.

Im ersten Teil interessieren wir uns für untere Schranken an das Volumen eines Konvexkörpers in Abhängigkeit von der Anzahl der Gitterpunkte, die er enthält. Wir beweisen eine Verfeinerung einer klassischen Schranke von Blichfeldt und erzielen in unserem Hauptresultat eine solche Abschätzung für zentralsymmetrische Konvexkörper, die asymptotisch nicht verbessert werden kann. Am Beispiel von speziellen Körperklassen, wie die der Gitterzonotope oder der Ellipsoide, zeigen wir, wie zusätzliche Symmetrieeigenschaften ausgenutzt werden können um stärkere Abschätzungen zu erhalten.

Im zweiten Teil unserer Arbeit befassen wir uns mit Beziehungen zwischen Minkowskis sukzessiven Minima und den Koeffizienten des Ehrhart Polynoms eines Gitterpolytops. Wir studieren, unter anderem, eine vorgeschlagene diskrete Verallgemeinerung des 2. Satzes von Minkowski, die auf Betke, Henk und Wills zurückgeht. Für Gitterzonotope und sogenannte Lattice-face-Polytope sind geometrische Beschreibungen der Ehrhartkoeffizienten bekannt, mit deren Hilfe wir positive Resultate für diese Körperklassen erzielen.

Schließlich untersuchen wir im letzten Kapitel kombinatorische Eigenschaften von zentralsymmetrischen Polytopen. Wir sind dabei sowohl von einer wohlbekannten Vermutung über die minimale Anzahl von Seitenflächen eines zentralsymmetrischen Polytops, als auch von der stetigen Suche nach konkreten berechenbaren Beispielen motiviert. Unser Hauptbeitrag in diesem Teil ist eine explizite Formel für die Seitenanzahl von zentralsymmetrischen Polytopen, die aus den stabilen Mengen eines Graphen konstruiert werden, dessen Knotenmenge in eine Clique und eine stabile Menge partitioniert werden kann. Als Konsequenz sehen wir, dass diese Polytope die vermutete untere Schranke an die Seitenanzahl nicht unterschreiten und wir erhalten eine reiche Klasse von Beispielen, die diese Schranke überraschend wenig übersteigt.

## Abstract

In this dissertation, we investigate classical problems in Discrete and Convex Geometry in the light of central symmetry.

First, we are interested in lower bounding the volume of a convex body in terms of the number of lattice points that it contains. We prove a refinement of a classical bound of Blichfeldt and, as our main result, we obtain an asymptotically sharp bound on the class of centrally symmetric convex bodies. On particular classes, like lattice zonotopes or ellipsoids, we show how even more symmetry properties can be exploited to derive stronger estimates.

In the second part, we focus on the relation of Minkowski's successive minima and the coefficients of the Ehrhart polynomial of a lattice polytope. Among other things, we study a proposed discrete generalization of Minkowski's 2nd Theorem on successive minima due to Betke, Henk and Wills. An approach that utilizes the available geometric descriptions of the Ehrhart coefficients of lattice zonotopes and lattice-face polytopes leads us to positive results on these families.

In the last part, we study combinatorics of centrally symmetric polytopes. Our motivation comes from a famous conjecture on the minimal number of faces of centrally symmetric polytopes and from the constant quest for concrete examples. We derive an exact count for the number of faces of polytopes that are produced from the stable set structure of graphs that can be split into a clique and a stable set. The results imply the validity of the conjectured lower bound for these particular polytopes and provide a rich class of examples that exceed the conjectured bound surprisingly little.

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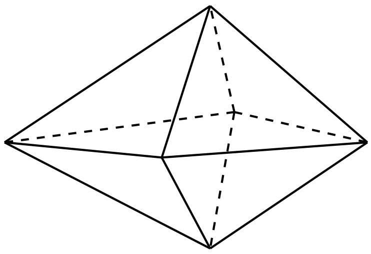
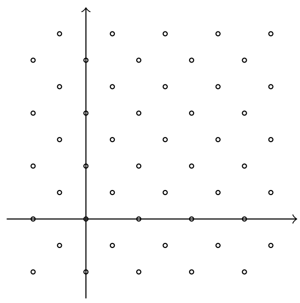
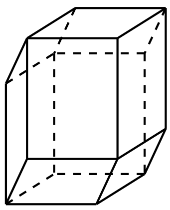
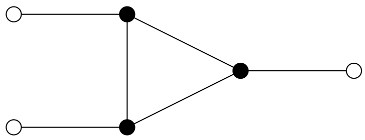
CHAPTER 1

# Introduction









This dissertation investigates problems from the realm of Discrete and Convex Geometry. These subjects study combinatorial and metric properties of convex compact sets in a Euclidean vector space. The combinatorial questions are mostly considered on the subclass of polytopes, which are defined as the convex hull of finitely many points.

We are mainly interested in utilizing the natural condition of central symmetry in questions on lattice point inequalities for convex bodies and on counting faces of polytopes. More specifically, we investigate convex bodies that are invariant under point reflection at the origin. Our motivation for such studies is taken from the common belief that central symmetry leads to stronger results in many classical problems that are solved in the general case. Let us discuss examples of such problems where the exploitation of central symmetry still remains a mystery. Precise definitions of the objects and magnitudes are given afterwards.

A popular exercise in classes on polytope theory is to show that every polytope has at least as many faces as a simplex of the same dimension. The same question restricted to the class of centrally symmetric polytopes is one of the most famous and still reluctant problems in polyhedral combinatorics. Kalai claims, in what is called his $3^d$-conjecture, that the cube has the minimal number of faces among all centrally symmetric polytopes.

An even more longstanding conjecture is that of Mahler. He asserts that the product of the volume of a centrally symmetric convex body and that of its polar body is minimized again by the cube. Polarity makes it an intricate task to define the corresponding functional for not necessarily symmetric bodies. Although, this was accomplished and many excellent partial results are available, the general problem remains open also in the nonsymmetric case.

A third problem concerns the interplay of continuous and discrete magnitudes, in particular, the estimation of the volume of a convex body by the number of lattice points that it contains. A classical result is due to Blichfeldt and provides a lower bound on the volume that is best possible for any potential number of lattice points. The analogous problem on the class of centrally symmetric convex bodies has not been addressed before and it is not even clear how extremal examples could look like.

Before we summarize our results that include contributions to the aforementioned $3^d$-conjecture and the Blichfeldt-type inequalities, we introduce the main players, some of their basic properties and the necessary notation.

**Convex Bodies and Polytopes.** We work in the $n$-dimensional Euclidean vector space $\mathbb{R}^n$. The interior $\operatorname{int} S$ and the boundary $\partial S$ of a subset $S$ of $\mathbb{R}^n$ are defined in the standard topology that is induced by the Euclidean norm $\|\cdot\|$. A *convex body* in $\mathbb{R}^n$ is defined to be a convex compact set of full dimension. The family $\mathcal{K}^n$ of all $n$-dimensional convex bodies contains the family $\mathcal{P}^n$ of *(convex) polytopes*. A polytope is the convex hull of finitely many points and the minimal set of points that span a polytope $P$ is called the set of *vertices* $\operatorname{vert} P$ of $P$. More generally, a *face* of a polytope $P \in \mathcal{P}^n$ is defined as the intersection of $P$ with a supporting hyperplane. Faces are themselves (possibly lower-dimensional) polytopes, and faces of dimension 1 and $n-1$ are called *edges* and *facets*, respectively. Vertices are 0-dimensional faces.

The *polar set* of some $K \in \mathcal{K}^n$ is defined by

$$K^\star = \{x \in \mathbb{R}^n : x^\mathsf{T} y \leq 1 \text{ for all } y \in K\}.$$

If $0 \in \operatorname{int} K$, then $K^\star$ is itself a convex body. Moreover, the polar operation is inclusion-reversing, which means that for $K_1, K_2 \in \mathcal{K}^n$ with $K_1 \subseteq K_2$, we have $K_2^\star \subseteq K_1^\star$. For a subspace $L$ of $\mathbb{R}^n$, we let $K|L$ be the orthogonal projection of $K$ onto $L$ with respect to the standard scalar product in $\mathbb{R}^n$. The orthogonal complement of $L$ is written as $L^\perp$. Intersections and projections are polar to each other in the sense that we have $K^\star \cap L = (K|L)^\star$, where $(K|L)^\star$ is meant with respect to the subspace $L$.

**Central Symmetry.** A convex body $K$ is said to be *centrally symmetric* if there exists an $x \in \mathbb{R}^n$ such that $K - x = -(K - x)$. Usually we think about central symmetry with respect to the origin $x = 0$, and we write $\mathcal{K}_0^n$ and $\mathcal{P}_0^n$ for the respective families of centrally symmetric convex bodies and polytopes. We make it clear when the center is a different one. For us, one of the most relevant magnitudes of a convex body $K \in \mathcal{K}^n$ is its *volume* (Lebesgue measure) $\operatorname{vol}_n(K)$. We omit the subscript when the ambient dimension of the body is clear from the context.

Considering the *difference body* $\mathcal{D}K = K - K$ of some $K \in \mathcal{K}^n$ is a natural symmetrization technique. It is a classical problem to study the relation between $K$ and $\mathcal{D}K$ and many useful results are available. Rogers and Shephard [RS57] studied the volume of difference bodies.

**Theorem 1.1** (Rogers and Shephard, 1957)**.** *Let $K \in \mathcal{K}^n$. Then*

$$2^n \operatorname{vol}(K) \leq \operatorname{vol}(\mathcal{D}K) \leq \binom{2n}{n} \operatorname{vol}(K). \tag{1.1}$$

*The lower bound is attained if and only if $K$ is centrally symmetric, the upper bound if and only if $K$ is a simplex.*

Let $c = \operatorname{cen}(K) = \frac{1}{\operatorname{vol}(K)} \int_K x \, dx$ be the *centroid* of the body $K \in \mathcal{K}^n$. In the book of Bonnesen and Fenchel [BF87, §7 (34.)], we find the inclusions

$$K - c \subseteq \frac{n}{n+1} \mathcal{D}K \subseteq n(K - c). \tag{1.2}$$

These inclusions are strict for every $K \in \mathcal{K}^n$ of dimension $n \geq 2$ but at the same time the factors $\frac{n}{n+1}$ and $n$ are best possible.

**Lattices.** The discrete structures in the problems of our interest are often *lattices*. A lattice is defined to be a discrete subgroup of $\mathbb{R}^n$ and the family of all lattices is denoted by $\mathcal{L}^n$. Every lattice $\Lambda \in \mathcal{L}^n$ can be written as $\Lambda = A\mathbb{Z}^n$. Here, $\mathbb{Z}^n$ is the *standard lattice* that consists of all vectors with integral coordinates, and $A \in \mathrm{GL}_n(\mathbb{R})$ is an invertible matrix of size $n$ whose column vectors are said to be a *basis* of the lattice. The *determinant* of $\Lambda$ is denoted by $\det(\Lambda) = |\det(A)|$ and is independent of the actual choice of $A$. For a linear subspace $L$ of $\mathbb{R}^n$, the intersection $\Lambda \cap L$ is called a *sublattice* of $\Lambda$. If $L$ and the affine hull of $\Lambda \cap L$ have the same dimension, then the determinant formula holds (cf. [**Mar03**, Prop. 1.2.9]):

$$\det(\Lambda) = \det(\Lambda \cap L)\det(\Lambda | L^{\perp}). \tag{1.3}$$

Also for lattices there is a notion of polarity. The *polar lattice* of $\Lambda \in \mathcal{L}^n$ is defined as

$$\Lambda^\star = \{x \in \mathbb{R}^n : x^\intercal y \in \mathbb{Z} \text{ for all } y \in \Lambda\}.$$

For example, the polar $(\mathbb{Z}^n)^\star$ of the standard lattice is $\mathbb{Z}^n$ itself. Analogously to polarity for convex bodies, sublattices and projected lattices are polar to each other. That is, for a subspace $L$ of $\mathbb{R}^n$ and a lattice $\Lambda \in \mathcal{L}^n$ such that $L$ and the affine hull of $\Lambda^\star \cap L$ have the same dimension, we have $\Lambda^\star \cap L = (\Lambda | L)^\star$, where again $(\Lambda | L)^\star$ is meant with respect to the subspace $L$. Together with the identity $\det(\Lambda^\star) = \det(\Lambda)^{-1}$, this gives (cf. [**Mar03**, Cor. 1.3.5]):

$$\det(\Lambda \cap L) = \det(\Lambda)\det(\Lambda^\star \cap L^{\perp}). \tag{1.4}$$

An important special case is that of an $(n-1)$-dimensional sublattice $\Lambda \cap L$. In fact, there exists an, up to the sign, uniquely determined vector $x^\star \in \Lambda^\star$ such that $\det(\Lambda \cap L) = \det(\Lambda)\|x^\star\|$.

**Counting Lattice Points in Convex Bodies.** The *lattice point enumerator* counts the lattice points from a lattice $\Lambda$ that are contained in a set $S$, in symbols, $\mathrm{G}(S, \Lambda) = \#(S \cap \Lambda)$. It can be understood as the discrete volume of $S$ with respect to $\Lambda$. We use the short notation $\mathrm{G}(S) = \mathrm{G}(S, \mathbb{Z}^n)$ for the case of the standard lattice. Observe, that $\mathrm{G}(S)$ is invariant under *unimodular transformations* of the set $S$, that is, affine transformations $x \mapsto Ux + t$ with $t \in \mathbb{Z}^n$ and $U \in \mathbb{Z}^{n \times n}$ being an integral matrix with determinant $\pm 1$.

In the late nineteenth century, Pick [**Pic99**] proved his famous formula which says that computing the volume of a lattice polygon is equivalent to counting its lattice points. In general, a polytope is said to be a *lattice polytope* if all its vertices are contained in a fixed lattice. When we do not particularly specify the lattice, we think about lattice polytopes with respect to the standard lattice.

**Theorem 1.2** (Pick, 1899)**.** *Let $P \in \mathcal{P}^2$ be a lattice polygon. Then*

$$\mathrm{G}(P) = \mathrm{vol}(P) + \frac{1}{2}\,\mathrm{G}(\partial P) + 1.$$

This formula holds in much greater generality, for example, for not necessarily convex polygonal regions whose vertices are lattice points. An analogous explicit identity for the number of lattice points in lattice polytopes of higher dimension is not available. Yet, a breakthrough was Ehrhart's [**Ehr62**] famous discovery that the number of lattice points in integral dilates of a general lattice polytope is a polynomial in the dilatation factor. This result initiated a whole theory of lattice point counting and is since then a very active area of research.

**Theorem 1.3** (Ehrhart, 1962). *Let $P \in \mathcal{P}^n$ be a lattice polytope. Then*

$$\mathrm{G}(kP) = \sum_{i=0}^{n} \mathrm{g}_i(P) k^i \quad \textit{for each} \quad k \in \mathbb{N}.$$

This polynomial is called the *Ehrhart polynomial* and the coefficients $\mathrm{g}_i(P)$ the *Ehrhart coefficients* of $P$. A direct consequence of this relation is that the $\mathrm{g}_i(P)$'s are *homogeneous functionals of degree $i$*, that is, $\mathrm{g}_i(tP) = t^i \, \mathrm{g}_i(P)$ for all $t \in \mathbb{N}$. The coefficient $\mathrm{g}_0(P)$ is the Euler characteristic of $P$ and therefore $\mathrm{g}_0(P) = 1$ for every lattice polytope $P$. Only two of the remaining coefficients admit transparent geometric descriptions. We denote the affine hull of a subset $S$ of $\mathbb{R}^n$ by aff $S$.

$$\mathrm{g}_n(P) = \mathrm{vol}(P) \quad \text{and} \quad \mathrm{g}_{n-1}(P) = \frac{1}{2} \sum_{F \text{ a facet of } P} \frac{\mathrm{vol}_{n-1}(F)}{\det(\operatorname{aff} F \cap \mathbb{Z}^n)}$$

We can extend the Ehrhart polynomial to all integers, and hence define $\mathrm{G}(-kP)$ for every $k \in \mathbb{N}$ in this way. A very useful and repeating phenomenon in counting problems are *reciprocity theorems*, which in our case means that $\mathrm{G}(-kP)$ is given a geometric meaning. Ehrhart [**Ehr68**], and independently MacDonald [**Mac71**], obtained the following description.

$$\mathrm{G}(-kP) = (-1)^n \, \mathrm{G}(\operatorname{int} kP) = \sum_{i=0}^{n} \mathrm{g}_i(P)(-k)^i \quad \text{for each} \quad k \in \mathbb{N}. \tag{1.5}$$

When we express the Ehrhart polynomial of a lattice polytope $P \in \mathcal{P}^n$ with respect to the polynomial basis $\left\{ \binom{k+n-i}{n} : i = 0, \dots, n \right\}$, we obtain another set of coefficients with very convenient properties. More precisely, we write

$$\mathrm{G}(kP) = \sum_{i=0}^{n} \binom{k+n-i}{n} \mathrm{a}_i(P) \quad \text{for each} \quad k \in \mathbb{N}_0, \tag{1.6}$$

and we say that $\mathrm{a}(P) = (\mathrm{a}_0(P), \dots, \mathrm{a}_n(P))$ is the a-*vector* of $P$. In the literature, one also finds the notation $\delta$-vector and $h^\star$-vector. From the definition and the reciprocity theorem (1.5), we get

$$\begin{aligned} &\mathrm{a}_0(P) = 1, \quad \mathrm{a}_1(P) = \mathrm{G}(P) - (n+1), \quad \mathrm{a}_n(P) = \mathrm{G}(\operatorname{int} P), \\ &\mathrm{a}_0(P) + \mathrm{a}_1(P) + \ldots + \mathrm{a}_n(P) = n! \, \mathrm{vol}(P). \end{aligned}$$

In contrast to the coefficients $\mathrm{g}_i(P)$, the functionals $\mathrm{a}_i(P)$ are not homogeneous.

For more information on Ehrhart theory, we refer the reader to [**BR07**]. Polytopes are extensively covered in [**Grü03, Zie95**], convex bodies and discrete geometry in [**Gru07**] and lattices in [**Mar03**].

Throughout this thesis, certain classes of convex bodies play a particular role. We introduce the reader to those classes that we often come across.

**Simplices.** The convex hull of $n+1$ affinely independent points in $\mathbb{R}^n$ is called a *simplex*. The *standard simplex* $S_n$, for example, is spanned by the origin and the coordinate unit vectors $e_1, \dots, e_n$. The volume of a simplex can be computed from its vertices $a_0, \dots, a_n$ as

$$\mathrm{vol}\left(\mathrm{conv}\{a_0, a_1, \dots, a_n\}\right) = \frac{1}{n!} \left|\det(a_1 - a_0, \dots, a_n - a_0)\right|.$$

In particular, lattice simplices always have a volume of at least $\frac{1}{n!}$.

**Crosspolytopes.** A *crosspolytope* in $\mathbb{R}^n$ is defined as the convex hull of $n$ linearly independent points and their reflections at the origin. For example, the *standard crosspolytope* is given by $C_n^\star = \operatorname{conv}\{\pm e_1, \ldots, \pm e_n\}$. Crosspolytopes are the centrally symmetric analogs of simplices in the sense that every polytope $P \in \mathcal{P}_0^n$ contains a crosspolytope whose vertices are vertices of $P$ itself. Again, the volume of a crosspolytope $C = \operatorname{conv}\{\pm a_1, \ldots, \pm a_n\}$ is not hard to compute since we can dissect it into $2^n$ simplices of equal volume by considering the pyramids with apex at the origin and the facets of $C$ as bases. In fact, we have

$$\operatorname{vol}(C) = 2^n \operatorname{vol}\left(\operatorname{conv}\{0, a_1, \ldots, a_n\}\right) = \frac{2^n}{n!} \left|\det(a_1, \ldots, a_n)\right|.$$

**Parallelepipeds.** Convex bodies in $\mathbb{R}^n$ that are affine images of the *unit cube* $C_n = [-1,1]^n$ are called *parallelepipeds*. Up to translations, parallelepipeds are the polar bodies of crosspolytopes. Another description of parallelepipeds is as the Minkowski sum of $n$ affinely independent line segments $[v, w] = \operatorname{conv}\{v, w\}$. The unit cube, for example, has the representation $C_n = [-e_1, e_1] + \ldots + [-e_n, e_n]$. The volume of a parallelepiped $P = [v_1, w_1] + \ldots + [v_n, w_n]$ can be computed from its edge directions as

$$\operatorname{vol}(P) = \left|\det(w_1 - v_1, \ldots, w_n - v_n)\right|.$$

Moreover, computing the volume of a lattice parallelepiped is equivalent to counting lattice points in its half-open counterpart. More precisely, for $P = [v_1, w_1] + \ldots + [v_n, w_n]$ with $v_1, w_1, \ldots, v_n, w_n \in \mathbb{Z}^n$, we have

$$\operatorname{vol}(P) = \#\left(\sum_{i=1}^n [v_i, w_i) \cap \mathbb{Z}^n\right). \tag{1.7}$$

For a proof of this well-known relation, we refer to [Bar08b, p. 89]. A lattice parallelepiped with vertices in a lattice $\Lambda$ and volume $\det \Lambda$ is called a *fundamental cell* of $\Lambda$.

**Zonotopes.** Zonotopes are generalizations of parallelepipeds. A *zonotope* is defined as the Minkowski sum of some set of line segments. The zonotopes are important instances of centrally symmetric polytopes which have the additional property that each of their faces is a zonotope itself. Moreover, Bolker [Bol69] identified zonotopes as those polytopes all of whose two-dimensional faces have a center of symmetry.

We have to be careful when we speak about centrally symmetric lattice zonotopes. In general, every lattice zonotope $Z \in \mathcal{P}^n$ can be written as $Z = [v_1, w_1] + \ldots + [v_m, w_m]$ for suitable lattice vectors $v_1, w_1, \ldots, v_m, w_m \in \mathbb{Z}^n$. Such a $Z$ is centrally symmetric with respect to the point $\sum_{i=1}^m \frac{v_i + w_i}{2}$, which is only half-integral and thus, in general, not a lattice point. As a consequence, there are centrally symmetric lattice zonotopes in $\mathcal{P}_0^n$, that is, with respect to the origin, that do not have a representation by a sum of line segments all of whose endpoints are lattice points. For example, the lattice parallelepiped $P = \operatorname{conv}\{\pm e_1, \pm e_2, \pm e_3, \pm(1,1,1)^\intercal\}$ has the representation $P = [-v_1, v_1] + [-v_2, v_2] + [-v_3, v_3]$ for $v_1 = (\frac{1}{2}, \frac{1}{2}, 0)^\intercal$, $v_2 = (\frac{1}{2}, 0, \frac{1}{2})^\intercal$ and $v_3 = (0, \frac{1}{2}, \frac{1}{2})^\intercal$. On the other hand, a centrally symmetric zonotope of the form $Z = [-v_1, v_1] + [-v_m, v_m]$ with $v_1, \dots, v_m \in \mathbb{Z}^n$ has the additional property that every face is itself centrally symmetric with respect to a lattice point.

**Lattice-face Polytopes.** This particular class of polytopes was introduced by Liu [**Liu08**]. For its definition, let $\pi^{(i)} : \mathbb{R}^n \to \mathbb{R}^{n-i}$ be the projection that forgets the last $i$ coordinates; we let $\mathbb{R}^0 = \{0\}$.

**Definition 1.4** (Liu, 2008)**.** *A polytope $P \in \mathcal{P}^n$ is said to be a* lattice-face polytope *if for every $k \in \{0, \dots, n-1\}$ and every subset $U$ of the vertices of $P$ that spans a $k$-dimensional affine space, we have $\pi^{(n-k)}(\operatorname{aff} U \cap \mathbb{Z}^n) = \mathbb{Z}^k$.*

Famous instances of lattice-face polytopes are the cyclic polytopes with integral vertices (cf. [**Zie95**, Sect. 8.4]). A *cyclic polytope* is defined by the condition that all its vertices lie on the moment curve $t \mapsto \nu(t) = (t, t^2, \dots, t^n)^\intercal$. The *standard cyclic polytope* with $m$ vertices is given by $C_n(m) = \operatorname{conv}\{\nu(1), \dots, \nu(m)\}$. Unexpectedly, the class of lattice-face polytopes is much richer. As Liu [**Liu09**, Thm. 2] shows, every lattice polytope is affinely equivalent to a lattice-face polytope.

**Reflexive Polytopes.** A polytope $P \in \mathcal{P}^n$ with $0 \in \operatorname{int} P$ is called *reflexive* if both $P$ and its polar $P^\star$ are lattice polytopes. There are many characterizations available, for instance, a lattice polytope $P$ is reflexive if and only if $\mathrm{a}_i(P) = \mathrm{a}_{n-i}(P)$ for all $i = 0, \dots, n$, and if and only if $\mathrm{g}_{n-1}(P) = \frac{n}{2}\operatorname{vol}(P)$ (cf. [**Hib92**, **BHW07**]). Examples of reflexive polytopes include $C_n$, $C_n^\star$ and the simplex $S = \operatorname{conv}\{e_1, \dots, e_n, -(e_1 + \dots + e_n)\}$.

Let us now summarize the results that are presented in the individual chapters.

**Overview of the thesis.** Chapters 2 and 3 concentrate on estimates on the volume in terms of the number of lattice points of convex bodies. Our studies are centered around a classical lower bound. Therein, $\dim(S)$ denotes the dimension of the affine hull of the set $S$.

**Theorem 1.5** (Blichfeldt, 1921)**.** *Let $K \in \mathcal{K}^n$ and $\dim(K \cap \mathbb{Z}^n) = n$. Then*

$$\operatorname{vol}(K) \geq \frac{\mathrm{G}(K) - n}{n!}.$$

After an introduction to the topic and a review of the relevant literature, we discuss generalizations and refinements of Blichfeldt's result in Chapter 2. Many of our results are based on an inequality of Hibi on the a-vector of a lattice polytope. Among our findings is the following:

**Theorem 1.6.** *Let $P \in \mathcal{P}^n$ be a lattice polytope. Then*

$$\operatorname{vol}(P) \geq \frac{\mathrm{G}(\partial P) + n\,\mathrm{G}(\operatorname{int} P) - n}{n!}.$$

We further provide a necessary condition for an integral vector to be the a-vector of a centrally symmetric lattice polytope, and investigate Blichfeldt's inequality more closely on the class of lattice-face polytopes.

Chapter 3 deals with the same problem, but now focussing on the class of centrally symmetric convex bodies. A corresponding result to Blichfeldt's inequality in which the symmetry condition is reflected in a stronger bound was missing so far. In our main theorem in this part, we prove such an inequality, which is moreover asymptotically best possible.

**Theorem 1.7.** *For every $\varepsilon \in (0,1]$ there exists an $n(\varepsilon) \in \mathbb{N}$ such that for every $n \geq n(\varepsilon)$ and every $K \in \mathcal{K}_0^n$ with $\dim(K \cap \mathbb{Z}^n) = n$, we have*

$$\mathrm{vol}(K) \geq \frac{(2-\varepsilon)^n}{n!} \, \mathrm{G}(K).$$

*The inequality is asymptotically sharp in the sense that the constant 2 in the expression on the right hand side cannot be replaced by a bigger one.*

Concerning special classes of convex bodies, we obtain, in collaboration with Martin Henk and Jörg M. Wills [**HHW11**], sharp inequalities of the preceding type for lattice crosspolytopes and lattice zonotopes. Furthermore, we discuss ellipsoids and the dimensions two and three with respect to this problem. Based on ideas of Gillet and Soulé, we can apply our inequalities to derive estimates involving the number of lattice points of a centrally symmetric convex body and its polar body.

**Theorem 1.8.** *For every $\varepsilon > 0$ there exists an $n(\varepsilon) \in \mathbb{N}$ such that for every $n \geq n(\varepsilon)$ and every $K \in \mathcal{K}_0^n$ with $\dim(K \cap \mathbb{Z}^n) = n$, we have*

$$(\pi + \varepsilon)^{-n} \leq \frac{\mathrm{G}(K)}{\mathrm{G}(K^\star)\,\mathrm{vol}(K)} \leq \frac{(\pi+\varepsilon)^n n!}{c^n},$$

*where $c \leq 4$ is an absolute constant.*

In Chapter 4, we investigate relations between the Ehrhart coefficients of a lattice polytope and Minkowski's successive minima. These magnitudes capture the size of a convex body $K \in \mathcal{K}_0^n$ with respect to a given lattice $\Lambda \in \mathcal{L}^n$. More precisely, the *ith successive minimum* $\lambda_i(K, \Lambda)$ is defined as the smallest positive scalar $\lambda$ such that $\lambda K$ contains $i$ linearly independent lattice points from $\Lambda$. Minkowski's main result on the successive minima is his famous *2nd Theorem on Convex Bodies*.

**Theorem 1.9** (Minkowski, 1896)**.** *Let $K \in \mathcal{K}_0^n$ and $\Lambda \in \mathcal{L}^n$. Then*

$$\frac{2^n}{n!} \det \Lambda \leq \lambda_1(K,\Lambda) \cdot \ldots \cdot \lambda_n(K,\Lambda) \,\mathrm{vol}(K) \leq 2^n \det \Lambda.$$

Motivated by a conjectured discrete generalization of the upper bound, which is due to Betke, Henk and Wills, we examine the validity of the following even stronger relations:

$$\mathrm{g}_i(P) \leq \sum_{\substack{J \subseteq \{1,\ldots,n\} \\ \#J = i}} \prod_{j \in J} \frac{2}{\lambda_j(P)} \quad \text{for each} \quad i = 1, \ldots, n. \tag{1.8}$$

Here, $P \in \mathcal{P}_0^n$ is a lattice polytope and the successive minima $\lambda_i(P)$ are understood with respect to the standard lattice $\mathbb{Z}^n$. These inequalities turn out to be too restrictive in general. In fact, we find that the polytope $\mathrm{conv}\{lC_{n-1} \times \{0\}, e_n\}$ does not satisfy all these inequalities if $l \in \mathbb{N}$ is large enough. In special cases though, they lead to a fruitful approach: We exploit explicit geometric descriptions of the Ehrhart coefficients of lattice-face polytopes and lattice parallelepipeds to give affirmative answers for these classes of convex bodies. For lattice zonotopes $Z$ that are generated by $m$ lattice vectors in general position, we prove the Inequalities (1.8) up to the factor $\frac{\binom{m}{i}}{\binom{n}{i}}$ for each $i = 1, \ldots, n$. These results appeared in a joint paper with Christian Bey, Martin Henk and Eva Linke [**BHHL11**].

In the second part of Chapter 4, we study possible extensions of a theorem by Henk, Schürmann and Wills on the second highest Ehrhart coefficient. More precisely, we are interested whether their inequality

$$\frac{\mathrm{g}_{n-1}(P)}{\mathrm{vol}(P)} \leq \sum_{i=1}^{n} \frac{\lambda_i(P)}{2} \quad \text{for each} \quad P \in \mathcal{P}_0^n$$

also holds for not necessarily symmetric lattice polytopes $P$. Using the difference body $\mathcal{D}P$ to extend the definition of the successive minima, we prove the best possible analog in dimension two and with respect to $\lambda_n(\frac{1}{2}\mathcal{D}P)$.

**Theorem 1.10.** *For lattice polygons $P \in \mathcal{P}^2$, we have*

$$\frac{\mathrm{g}_1(P)}{\mathrm{vol}(P)} \leq \frac{3}{2}\left(\frac{\lambda_1(\frac{1}{2}\mathcal{D}P)}{2} + \frac{\lambda_2(\frac{1}{2}\mathcal{D}P)}{2}\right).$$

*For lattice polytopes $P \in \mathcal{P}^n$, we have*

$$\frac{\mathrm{g}_{n-1}(P)}{\mathrm{vol}(P)} \leq \binom{n+1}{2} \frac{\lambda_n(\frac{1}{2}\mathcal{D}P)}{2}.$$

Furthermore, we obtain an estimate that respects all the minima and is best possible up to the multiplicative factor $1.45^{n+1}$. We also study the problem using an alternative definition of the successive minima that takes the centroid of the body as the dilatation center.

We finish Chapter 4 by exhibiting counterexamples to a long standing conjecture of Wills that proposed a generalization of Minkowski's 1st theorem on convex bodies: Among all centrally symmetric lattice polytopes $P \in \mathcal{P}_0^n$ with $\mathrm{int}\, P \cap \mathbb{Z}^n = \{0\}$, the unit cube $C_n$ has the biggest Ehrhart coefficients, more precisely, $\mathrm{g}_i(P) \leq \mathrm{g}_i(C_n)$ for all $i = 1, \ldots, n$. On the positive side, we confirm Wills' conjecture for lattice polytopes whose Ehrhart polynomial, considered as a polynomial on the whole complex plane, has only roots with real part equal to $-\frac{1}{2}$.

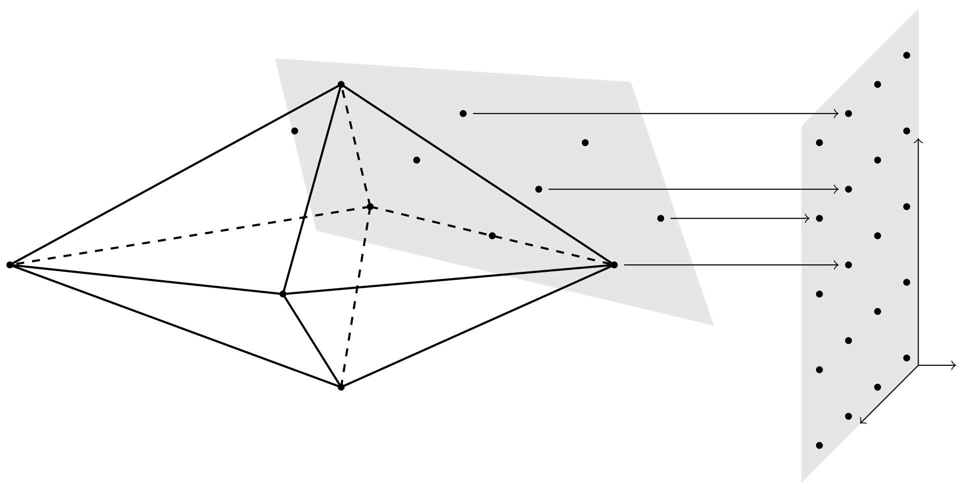
In Chapter 5, we investigate the combinatorics of certain centrally symmetric polytopes. In particular, we are interested in providing examples of centrally symmetric polytopes whose number of faces are close to that of the cube. This problem is interesting in view of Kalai's aforementioned $3^d$-*conjecture* that asserts that any centrally symmetric polytope should have as least as many faces as *Hanner polytopes* in the same dimension. The objects of our studies are *Hansen polytopes*, which are constructed out of the stable set structure of simple graphs. We refer to Section 5.2 for the precise definitions of Hanner and Hansen polytopes. As our main results, we characterize the Hanner polytopes that can be realized as a Hansen polytope and, more importantly, we give an exact count for the number of faces of Hansen polytopes of graphs whose node set can be partitioned into a stable set and a clique.

**Theorem 1.11.** *Let $G$ be a split graph and $H \in \mathcal{P}^n$ be the Hansen polytope of $G$. Then the number of nonempty faces of $H$ is given by $3^n + p_G$, where $p_G$ is a nonnegative number that can be read off from $G$ only.*

Based on this theorem, we construct a family of centrally symmetric polytopes that exceed Kalai's conjectured lower bound by only 16. The results of this chapter originate from a joint work with Ragnar Freij, Moritz W. Schmitt and Günter M. Ziegler [**FHSZ12**].

CHAPTER 2

# Blichfeldt-type Results and a-vector Inequalities



## 2.1. The Classical Blichfeldt Inequality

In the following, $K$ denotes a convex body and $\Lambda$ a lattice in $\mathbb{R}^n$. The first two chapters of the work at hand are devoted to the problem of estimating the discrete magnitude $\mathrm{G}(K,\Lambda)$ in terms of the continuous magnitude $\mathrm{vol}(K)$. There are many results in the literature that relate the lattice point enumerator to other geometric quantities, like the intrinsic volumes or the circumradius for example. The reader may consult the survey of Gritzmann and Wills [GW93] and the references therein.

Observe that, since $\Lambda = A\mathbb{Z}^n$, for some $A \in \mathrm{GL}_n(\mathbb{R})$, we have $\mathrm{G}(K,\Lambda) = \mathrm{G}(A^{-1}K, \mathbb{Z}^n)$ and $\frac{\mathrm{vol}(K)}{\det \Lambda} = \mathrm{vol}(A^{-1}K)$ and therefore it is no restriction to consider only the case $\Lambda = \mathbb{Z}^n$.

For large bodies, the two magnitudes almost coincide. More precisely, by properties of the Lebesgue measure, we have $\mathrm{vol}(K) = \lim_{s\to\infty} \frac{\mathrm{G}(sK)}{s^n}$ and thus $\mathrm{vol}(sK) \approx \mathrm{G}(sK)$ for large $s$. In general though, they can differ considerably, which raises the problem of finding sharp inequalities between the two.

In 1921, Blichfeldt proved the first general upper bound on the lattice point enumerator in terms of the volume.

**Theorem 2.1** (Blichfeldt [Bli21])**.** *Let $K \in \mathcal{K}^n$ and $\dim(K \cap \mathbb{Z}^n) = n$. Then*

$$\mathrm{vol}(K) \geq \frac{\mathrm{G}(K) - n}{n!}.$$

*Equality is attained for the simplices* $\mathrm{conv}\{0, le_1, e_2, \ldots, e_n\}$*, $l \in \mathbb{N}$.*

Throughout our work, we call results of this kind *Blichfeldt-type inequalities*. Note that, for such inequalities, we can restrict to lattice polytopes because $\mathrm{vol}(K) \geq \mathrm{vol}(\mathrm{conv}\{K \cap \mathbb{Z}^n\})$ and $\mathrm{G}(K) = \mathrm{G}(\mathrm{conv}\{K \cap \mathbb{Z}^n\})$. Moreover the condition $\dim(K \cap \mathbb{Z}^n) = n$ is necessary. To see this, consider very thin boxes around a fixed number of lattice points on a line (see Figure 2.1).

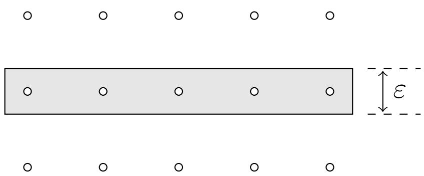



FIGURE 2.1. A box with five lattice points but arbitrary small volume

For later use we reserve a name for this condition.

**Definition 2.2** (lattice spanning)**.** *A convex body $K \in \mathcal{K}^n$ is said to be* lattice spanning *if its lattice points affinely span the whole space, that is,* $\dim(K \cap \mathbb{Z}^n) = n$.

Bey, Henk and Wills related the volume to the number of interior lattice points of a given lattice polytope $P \in \mathcal{P}^n$, obtaining an inequality similar to Blichfeldt's. They also characterized the equality case if $\mathrm{G}(\mathrm{int}\, P) = 1$ and $n \geq 3$. The full characterization was later obtained by Duong. Two polytopes $P$ and $Q$ are called unimodular equivalent, in symbols $P \simeq Q$, if there exists a unimodular transformation that maps one onto the other.

**Theorem 2.3** (Bey, Henk and Wills [BHW07], Duong [Duo08])**.** *Let $P \in \mathcal{P}^n$ be a lattice polytope. Then*

$$\mathrm{vol}(P) \geq \frac{n\,\mathrm{G}(\mathrm{int}\, P) + 1}{n!}.$$

*Equality holds for $n = 2$ if and only if $P$ is a triangle with $\partial P \cap \mathbb{Z}^2 = \mathrm{vert}\, P$, and for $n \geq 3$ if and only if $P \simeq S_n(l) = \mathrm{conv}\{e_1, \ldots, e_n, -l(e_1 + \ldots + e_n)\}$ for some $l \in \mathbb{N}_0$.*

The first results for lower bounds on the lattice point enumerator date back to works of Ehrhart and Scott for planar lattice polygons.

**Theorem 2.4** (Ehrhart [Ehr55b], Scott [Sco76])**.** *Let $P \in \mathcal{P}^2$ be a lattice polygon. Then*

$$\mathrm{vol}(P) \leq \begin{cases} \frac{9}{2} & \text{if } \mathrm{G}(\mathrm{int}\, P) = 1, \\ 2(\mathrm{G}(\mathrm{int}\, P) + 1) & \text{if } \mathrm{G}(\mathrm{int}\, P) \geq 2. \end{cases}$$

*Both inequalities are sharp.*

Hensley [Hen83] was the first to give such bounds for higher dimensions. Lagarias and Ziegler [LZ91], and later Pikhurko, refined this approach. Pikhurko's bound is the current state of the art and the first one that depends linearly on $\mathrm{G}(\mathrm{int}\, P)$.

**Theorem 2.5** (Pikhurko [Pik01])**.** *Let $P \in \mathcal{P}^n$ be a lattice polytope having at least one interior lattice point. Then*

$$\mathrm{vol}(P) \leq (8n)^n 15^{n 2^{2n+1}}\,\mathrm{G}(\mathrm{int}\, P).$$

Zaks, Perles and Wills [ZPW82] described lattice simplices $S$ that show that the dimensional constant in front of $\mathrm{G}(\mathrm{int}\, P)$ must be doubly exponential in $n$, in particular $\mathrm{vol}(S) \geq \frac{2^{2^{n-1}}}{n!}\,(\mathrm{G}(\mathrm{int}\, S) + 1)$. Note that the condition $\mathrm{G}(\mathrm{int}\, P) \geq 1$ cannot be dropped in order to upper bound $\mathrm{vol}(P)$ by $\mathrm{G}(\mathrm{int}\, P)$. The well-known Reeve simplices

$$R_n(l) = \mathrm{conv}\{0, e_1, \ldots, e_{n-1}, e_1 + \ldots + e_{n-1} + l e_n\},\ l \in \mathbb{N},$$

introduced by Reeve [Ree57], do not contain interior lattice points, but their volume equals $\frac{l}{n!}$.

**2.1.1. A Common Generalization of Theorem 2.1 and Theorem 2.3.** A detailed investigation of the proof of Theorem 2.3 that is given in [BHW07, Thm. 1.2] leads us to a common generalization of that inequality and Blichfeldt's classical result.

**Theorem 2.6.** *Let $P \in \mathcal{P}^n$ be a lattice polytope. Then*

$$\mathrm{vol}(P) \geq \frac{\mathrm{G}(\partial P) + n\,\mathrm{G}(\mathrm{int}\, P) - n}{n!}.$$

*Equality holds if $n = 2$ and in dimensions $n \geq 3$, e.g., for the simplex $S_n(l)$.*

**Remark 2.7.**

*i) By $\mathrm{G}(\partial P) + n\,\mathrm{G}(\mathrm{int}\, P) \geq \mathrm{G}(P)$ and $\mathrm{G}(\partial P) \geq n + 1$, our Theorem 2.6 implies Theorem 2.1 and Theorem 2.3, respectively.*

*ii)* *For $n = 2$, this inequality is actually an identity that is equivalent to Pick's Theorem 1.2.*

For the proof of Theorem 2.6, we need to introduce the concept of a triangulation of a polytope.

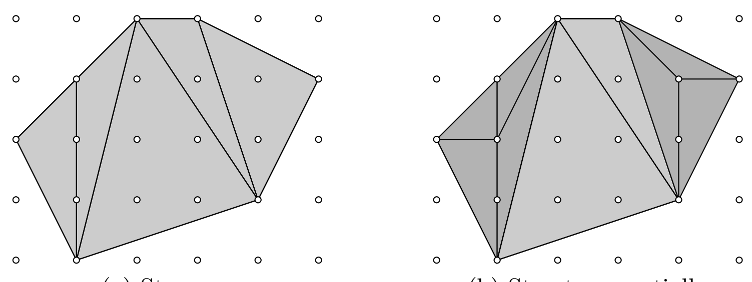
**Definition 2.8** (decomposition, triangulation). *Let $P \in \mathcal{P}^n$ be a polytope. A family of subpolytopes $\mathcal{T} = \{S_1, \ldots, S_k\}$ of $P$ is called a* decomposition *of $P$ if the following conditions hold*

*i)* $P = \bigcup_{i=1}^{k} S_i$,

*ii)* *$S_i \cap S_j$ is a common face of both $S_i$ and $S_j$, for all $i, j$.*

*If all $S_i$ are simplices, then the decomposition $\mathcal{T}$ is said to be a* triangulation.

Proof of Theorem 2.6. Every lattice simplex in $\mathbb{R}^n$ has volume at least $\frac{1}{n!}$. The result thereby follows, if we find a triangulation of $P$ into at least $\mathrm{G}(\partial P) + n\,\mathrm{G}(\operatorname{int} P) - n$ lattice simplices. As a first step, we triangulate $P$ into lattice simplices all of whose vertices are boundary lattice points of $P$ (see Figure 2.2a). To this end, let $\partial P \cap \mathbb{Z}^n = \{y_1, \ldots, y_l\}$ for some $l \geq n+1$, and without loss of generality let $y_1, \ldots, y_{n+1}$ be affinely independent. Using induction on $k \in \{n+1, \ldots, l\}$, we show that the polytopes $Q_k = \operatorname{conv}\{y_1, \ldots, y_k\}$ can be triangulated into at least $k - n$ lattice simplices. The case $k = n+1$ is immediate as $Q_{n+1}$ is a lattice simplex by assumption. Let $k < l$ and let $Q_k$ be triangulated into lattice simplices $S_1, \ldots, S_m$ for some $m \geq k-n$. There are two possibilities for $y_{k+1}$. If $y_{k+1}$ is not contained in $Q_k$, then it lies beyond[1] at least one facet of $Q_k$, which itself is triangulated by the triangulation of $Q_k$, and therefore gives at least one new simplex outside $Q_k$. If $y_{k+1} \in Q_k$, then it lies in the relative interior of a $j$-face, $j \geq 1$, of at least one simplex $S$ that is part of the triangulation of $Q_k$. Since a $j$-face of a simplex is contained in exactly $n - j$ many facets, we can triangulate $S$ into $j+1$ simplices by taking the convex hull of $y_{k+1}$ with any facet of $S$ that does not contain $y_{k+1}$. Therefore, by the induction hypothesis, we obtain a triangulation of $Q_{k+1} = Q_k$ into at least $m + j + 1 - 1 \geq (k+1) - n$ lattice simplices as desired.



(a) Step one (b) Step two, partially

Figure 2.2. Triangulating a lattice polygon

In the second step, we refine the triangulation from above with the help of the interior lattice points $\operatorname{int} P \cap \mathbb{Z}^n = \{z_1, \ldots, z_t\}$ for some $t \geq 0$ (see

[1] We say that $y_{k+1}$ lies *beyond* a facet of $Q_k = \{x \in \mathbb{R}^n : a_i^{\intercal} x \leq b_i,\ i = 1, \ldots, m\}$ (irredundant facet description) if there is an index $i$ with $a_i^{\intercal} y_{k+1} > b_i$.

Figure 2.2b). We show that after refining with $z_s$ for $s \in \{0, \dots, t\}$, there are at least $\mathrm{G}(\partial P) - n + n \cdot s$ lattice simplices in the triangulation. Suppose this is already done for $z_s$ and $s < t$. Then $z_{s+1}$ is contained in the relative interior of some $j$-face $F$ of a simplex, say $T_1$, of the triangulation that we have so far. As before, $F$ is contained in exactly $n - j$ facets of $T_1$, and since $z_{s+1}$ is an interior point of $P$, these facets cannot lie in the boundary of $P$. Because each such facet is a common facet of exactly two simplices from the current triangulation, $F$ is a $j$-face of at least $n - j$ further simplices $T_2, \dots, T_r$ for some $r \geq n - j + 1$. Again we triangulate $T_i$ for each $i = 1, \dots, r$ by taking the convex hull of $z_{s+1}$ and all facets of $T_i$ that do not contain that point. In this way, we dissect $T_i$ into $j + 1$ simplices and obtain a triangulation of $P$ with at least

$$\mathrm{G}(\partial P) - n + n \cdot s + r \cdot j \geq \mathrm{G}(\partial P) - n + n \cdot s + (n - j + 1) \cdot j \geq \mathrm{G}(\partial P) - n + n \cdot (s + 1)$$

lattice simplices. □

## 2.2. Applications of an Inequality by Hibi

Recall from the introduction, that the a-vector of a lattice polytope $P \in \mathcal{P}^n$ is defined by the identity

$$\mathrm{G}(kP) = \sum_{i=0}^{n} \binom{k+n-i}{n} \mathrm{a}_i(P) \quad \text{for each} \quad k \in \mathbb{N}_0.$$

In this section, we study applications of two important results on the coefficients $\mathrm{a}_i(P)$. We start with Stanley's celebrated nonnegativity and monotonicity theorem.

**Theorem 2.9** (Stanley [**Sta80**, **Sta93**]). *The* a*-vector of a lattice polytope $P \in \mathcal{P}^n$ is a nonnegative integral vector, that is, $\mathrm{a}_i(P) \in \mathbb{N}_0$ for all $i = 0, \dots, n$. Moreover, if $Q \in \mathcal{P}^n$ is a lattice polytope containing $P$, then $\mathrm{a}_i(P) \leq \mathrm{a}_i(Q)$ for all $i = 1, \dots, n$.*

Hibi strengthened Stanley's nonnegativity theorem for lattice polytopes which contain interior lattice points.

**Theorem 2.10** (Hibi [**Hib94**]). *Let $P \in \mathcal{P}^n$ be a lattice polytope such that $\operatorname{int} P \cap \mathbb{Z}^n \neq \emptyset$. Then $\mathrm{a}_i(P) \geq \mathrm{a}_1(P)$ for each $i = 1, \dots, n-1$.*

Both theorems above rely on methods from commutative algebra. Stapledon [**Sta09**] found purely combinatorial proofs of these and other linear relations among the a-vector entries of a lattice polytope. His paper also surveys the state of the art concerning such results. The characterization of all possible a-vectors, and thus Ehrhart polynomials, of lattice polytopes is an intricate problem and so far only solved in particular cases (see for example [**BN07**, **HHN11**]).

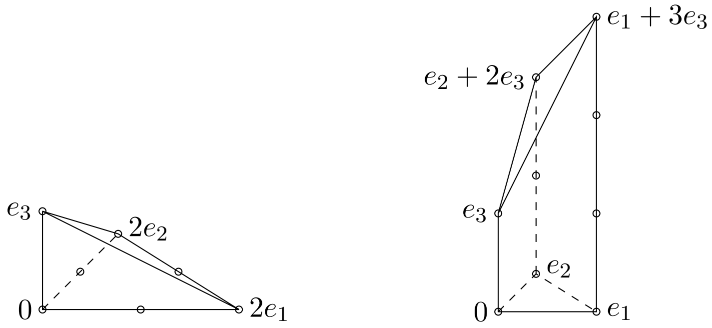
As a first consequence of the inequalities of Stanley and Hibi, we obtain a different and concise proof of Theorem 2.6. Moreover, this proof allows to characterize the lattice polytopes that attain equality in Theorem 2.6 or Theorem 2.1. We need to introduce two classes of lattice polytopes whose a-vectors were studied by Batyrev and Nill [**BN07**]. For $n \geq 2$, let

$$E_n = \operatorname{conv}\{0, 2e_1, 2e_2, e_3, \dots, e_n\},$$

and, for $h_1, \ldots, h_n \in \mathbb{N}_0$ and $n \geq 1$, let

$$L(h_1, \ldots, h_n) = \operatorname{conv}\{0, h_1 e_n, e_1, e_1 + h_2 e_n, \ldots, e_{n-1}, e_{n-1} + h_n e_n\}.$$

Some instances of these polytopes are drawn in Figure 2.3.




(a) $E_3$ (b) $L(1, 3, 2)$

FIGURE 2.3

**Proposition 2.11.** *Let $P \in \mathcal{P}^n$ be a lattice polytope. Then*

*i)* $\operatorname{vol}(P) = \dfrac{\mathrm{G}(\partial P) + n\,\mathrm{G}(\operatorname{int} P) - n}{n!}$ *if and only if $n = 2$, or*
 *a) $P \simeq E_n$, or*
 *b) $P \simeq L(h_1, \ldots, h_n)$ for some nonnegative integers $h_1, \ldots, h_n$, or*
 *c) $P \simeq S_n(l)$ for some $l \in \mathbb{N}$.*

*ii)* $\operatorname{vol}(P) = \dfrac{\mathrm{G}(P) - n}{n!}$ *if and only if*
 *a) $n = 2$ and $\mathrm{G}(\operatorname{int} P) = 0$, or*
 *b) $P \simeq E_n$, or*
 *c) $P \simeq L(h_1, \ldots, h_n)$ for some nonnegative integers $h_1, \ldots, h_n$.*

PROOF. i): Let us abbreviate $\mathrm{a}_i = \mathrm{a}_i(P)$ for all $i = 0, \ldots, n$. In order to reprove the inequality in Theorem 2.6, we distinguish two cases. First, assume that $\mathrm{a}_n = \mathrm{G}(\operatorname{int} P) = 0$. Then, by Theorem 2.9, we have $\mathrm{a}_i \geq 0$ and thus

$$\begin{aligned} n! \operatorname{vol}(P) &= 1 + \mathrm{a}_1 + \ldots + \mathrm{a}_n \geq 1 + \mathrm{a}_1 = \mathrm{G}(P) - n \\ &= \mathrm{G}(\partial P) + n\,\mathrm{G}(\operatorname{int} P) - n. \end{aligned}$$

Second, let $\mathrm{a}_n \neq 0$. By $\mathrm{a}_1 = \mathrm{G}(P) - n - 1 \geq \mathrm{G}(\operatorname{int} P) = \mathrm{a}_n$ and Hibi's Theorem 2.10, we get

$$\begin{aligned} n! \operatorname{vol}(P) &= 1 + \mathrm{a}_1 + \ldots + \mathrm{a}_n \geq 1 + \mathrm{a}_1 + (n-1)\,\mathrm{a}_n \\ &= \mathrm{G}(P) - n + (n-1)\,\mathrm{G}(\operatorname{int} P) = \mathrm{G}(\partial P) + n\,\mathrm{G}(\operatorname{int} P) - n. \end{aligned}$$

Now, let us assume that $P$ attains equality. From the above, we see that in the case $\mathrm{a}_n = 0$ this is equivalent to $\mathrm{a}(P) = (1, \mathrm{a}_1, 0, \ldots, 0)$. Batyrev and Nill [BN07] characterized such lattice polytopes $P$ as being unimodularly equivalent to either $E_n$ or to some $L(h_1, \ldots, h_n)$. In the case $\mathrm{a}_n \neq 0$, equality is attained if and only if $\mathrm{a}(P) = (1, \mathrm{a}_1, \mathrm{a}_n, \ldots, \mathrm{a}_n)$. By Hibi's inequalities, this means that for $n \geq 3$ we actually have $\mathrm{a}_n \geq \mathrm{a}_1 \geq \mathrm{a}_n$ and thus $\mathrm{a}(P) =$

$(1, \mathrm{a}_n, \ldots, \mathrm{a}_n)$. Duong [**Duo08**] shows that this is equivalent to $P \simeq S_n(l)$ for some $l \in \mathbb{N}$.

ii): As discussed in Remark 2.7, the classical Blichfeldt inequality in Theorem 2.1 follows from the above inequality by $\mathrm{G}(\partial P) + n\,\mathrm{G}(\operatorname{int} P) \geq \mathrm{G}(P)$. For $n \geq 2$, we therefore have equality if and only if $\mathrm{G}(\operatorname{int} P) = 0$ and equality holds in i). Hence, the second case in the first part cannot occur, and $P \simeq E_n$ or $P \simeq L(h_1, \ldots, h_n)$ as claimed. □

The mere existence of interior lattice points implies an improvement of Theorem 2.6 by a linear factor in front of $\mathrm{G}(\partial P)$. A weaker version of this result can already be found in a work by Hegedüs and Kasprzyk [**HK12**, Cor. 3.3]. Moreover, Duong [**Duo08**, Thm. 7.2.1] gives a proof for dimension $n = 3$ that uses triangulations. It would be interesting to find such an argument for general $n$.

**Proposition 2.12.** *Let $P \in \mathcal{P}^n$ be a lattice polytope such that $\mathrm{G}(\operatorname{int} P) \neq 0$. Then*

$$\operatorname{vol}(P) \geq \frac{(n-1)\,\mathrm{G}(\partial P) + n\,\mathrm{G}(\operatorname{int} P) - n^2 + 2}{n!}.$$

*Equality holds if and only if $\mathrm{a}(P) = (1, k, \ldots, k, l)$ for some $k, l \in \mathbb{N}$.*

Proof. We use the abbreviation $\mathrm{a}_i = \mathrm{a}_i(P)$ again. Since $\mathrm{a}_n = \mathrm{G}(\operatorname{int} P) \neq 0$, Hibi's Theorem 2.10 gives $\mathrm{a}_i \geq \mathrm{a}_1$ for all $i = 1, \ldots, n-1$ and thus

$$\begin{aligned} n! \operatorname{vol}(P) &= 1 + \mathrm{a}_1 + \ldots + \mathrm{a}_n \geq 1 + (n-1)\,\mathrm{a}_1 + \mathrm{a}_n \\ &= 1 + (n-1)\,(\mathrm{G}(P) - (n+1)) + \mathrm{G}(\operatorname{int} P) \\ &= (n-1)\,\mathrm{G}(\partial P) + n\,\mathrm{G}(\operatorname{int} P) - n^2 + 2. \end{aligned}$$

The equality characterization follows immediately. □

**Remark 2.13.** *In view of the equality case characterization in Proposition 2.12, it would be interesting to determine all lattice polytopes $P \in \mathcal{P}^n$ whose $\mathrm{a}$-vector is of the form*

$$\mathrm{a}(P) = (1, k, \ldots, k, l) \quad \textit{for some} \quad k, l \in \mathbb{N}_0.$$

*Examples of such $\mathrm{a}$-vectors are*

*i)* $\mathrm{a}(S_n(l)) = (1, l, \ldots, l)$ *for all* $l \in \mathbb{N}_0$,
*ii)* $\mathrm{a}(C_3^\star) = (1, 3, 3, 1)$, *and*
*iii)* $\mathrm{a}(\operatorname{conv}\{S_{n-1}(l), e_n\}) = (1, l, \ldots, l, 0)$ *for all* $l \in \mathbb{N}$ *(cf. [**HHN11**]).*

A little more a-vector yoga leads to a lower estimate on the number of interior lattice points in integral dilates of a lattice polytope.

**Proposition 2.14.** *Let $P \in \mathcal{P}^n$ be a lattice polytope and let $k \in \mathbb{N}$. Then*

$$\mathrm{G}(\operatorname{int} kP) \geq \left( \binom{n+k}{n+1} - \binom{k}{n+1} \right) \mathrm{G}(\operatorname{int} P) + \binom{k-1}{n}.$$

*Equality holds for $n = 2$ if and only if $P$ is a triangle with $\partial P \cap \mathbb{Z}^2 = \operatorname{vert} P$, and for $n \geq 3$ if and only if $P \simeq S_n(l)$ for some $l \in \mathbb{N}_0$.*

Proof. By the reciprocity theorem (1.5), we have

$$\mathrm{G}(\operatorname{int} kP) = (-1)^n\,\mathrm{G}(-kP) = \sum_{i=0}^{n} (-1)^n \binom{-k+n-i}{n} \mathrm{a}_i(P)$$

$$= \sum_{i=0}^{n} (-1)^n \binom{-(k+i-1)+n-1}{n} \mathrm{a}_i(P)$$

$$= \sum_{i=0}^{n} \binom{k+i-1}{n} \mathrm{a}_i(P). \tag{2.1}$$

The last equality follows from an identity for binomial coefficients for which we refer to the handbook [**AS92**].

Write $\mathrm{a}_i = \mathrm{a}_i(P)$ for all $i = 0, \dots, n$. By assumption, we have $\mathrm{a}_n = \mathrm{G}(\operatorname{int} P) \neq 0$ and therefore Hibi's Theorem 2.10 tells us that $\mathrm{a}_i \geq \mathrm{a}_1 \geq \mathrm{a}_n$ for all $i = 1, \dots, n-1$. With the equation above, we get

$$\mathrm{G}(\operatorname{int} kP) = \sum_{i=0}^{n} \binom{k+i-1}{n} \mathrm{a}_i(P) \geq \sum_{i=1}^{n} \binom{k+i-1}{n} \mathrm{a}_n(P) + \binom{k-1}{n}.$$

Finally, with the recurrence relation $\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}$ one can show inductively that

$$\sum_{i=1}^{n} \binom{k+i-1}{n} = \binom{n+k}{n+1} - \binom{k}{n+1}$$

and the claimed inequality follows.

Equality holds if and only if $\mathrm{a}(P) = (1, \mathrm{a}_n, \dots, \mathrm{a}_n)$. By Duong's characterization [**Duo08**], this is equivalent to having equality in Theorem 2.3. $\square$

In the flavor of Proposition 2.14, we can ask for lower bounds on the volume in terms of the number of lattice points in integral multiples $kP$ of a lattice polytope $P \in \mathcal{P}^n$. We provide an answer for the case $k = 2$.

**Proposition 2.15.** *Let $P \in \mathcal{P}^n$ be a lattice polytope. Then*

$$\operatorname{vol}(P) \geq \frac{1}{n!} \left( \frac{1}{n+1} \mathrm{G}(\partial\, 2P) + \frac{n}{n+2} \mathrm{G}(\operatorname{int} 2P) - \frac{n}{2} \right).$$

*Equality holds for $n = 2$ if and only if $P$ is a triangle with $\partial P \cap \mathbb{Z}^2 = \operatorname{vert} P$, and for $n \geq 3$ if and only if $P$ is in the list of Proposition 2.11 i).*

Proof. The case $n = 2$ follows from Pick's Theorem 1.2 and the fact that $\mathrm{G}(\partial\, 2P) \geq 6$ for all lattice polygons $P \in \mathcal{P}^2$. Indeed, the vertices of $2P$ lie in $2\mathbb{Z}^2$ and hence, the midpoint of each edge of $2P$ is an integral point as well. Equality holds if and only if $P$ is a triangle with $\partial P \cap \mathbb{Z}^2 = \operatorname{vert} P$.

Let $n \geq 3$ and let $\mathrm{a}_i = \mathrm{a}_i(P)$ for all $i = 0, \dots, n$. The identity (2.1) gives

$$\mathrm{G}(\operatorname{int} 2P) = \mathrm{a}_{n-1} + (n+1)\, \mathrm{a}_n$$

and

$$\mathrm{G}(\partial\, 2P) = \binom{n+2}{2} + (n+1)\, \mathrm{a}_1 + \mathrm{a}_2 - \mathrm{a}_{n-1} - (n+1)\, \mathrm{a}_n\,.$$

Thus, by virtue of $n! \operatorname{vol}(P) = \sum_{i=0}^{n} \mathrm{a}_i$, the claimed inequality is equivalent to

$$\sum_{i=0}^{n} \mathrm{a}_i \geq 1 + \mathrm{a}_1 + \frac{1}{n+1} \mathrm{a}_2 + \frac{n^2-2}{(n+1)(n+2)} \mathrm{a}_{n-1} + \frac{n^2-2}{n+2} \mathrm{a}_n\,.$$

For $n = 3$, this reduces to $\mathrm{a}_2 \geq \mathrm{a}_3$, which holds by Hibi's inequality. For $n \geq 4$, we have to prove

$$\frac{n}{n+1}\,\mathrm{a}_2 + \sum_{i=3}^{n-2} \mathrm{a}_i + \frac{3n+4}{(n+1)(n+2)}\,\mathrm{a}_{n-1} + \frac{n+4-n^2}{n+2}\,\mathrm{a}_n \geq 0.$$

If $\mathrm{a}_n = 0$, then this certainly holds because $\mathrm{a}_i \geq 0$ for each $i = 2, \ldots, n-1$. In the case $\mathrm{a}_n \neq 0$, we use Hibi's Theorem 2.10, that is $\mathrm{a}_i \geq \mathrm{a}_1 \geq \mathrm{a}_n$ for each $i = 2, \ldots, n-1$, and get

$$\begin{aligned}
&\frac{n}{n+1}\,\mathrm{a}_2 + \sum_{i=3}^{n-2} \mathrm{a}_i + \frac{3n+4}{(n+1)(n+2)}\,\mathrm{a}_{n-1} + \frac{n+4-n^2}{n+2}\,\mathrm{a}_n \\
&\geq \mathrm{a}_n \left( \frac{n}{n+1} + (n-4) + \frac{3n+4}{(n+1)(n+2)} + \frac{n+4-n^2}{n+2} \right) = 0.
\end{aligned}$$

Tracing back the inequalities, we have equality in the claimed estimate for $n \geq 3$ if and only if the a-vector of $P$ is given by $(1, \mathrm{a}_1, \mathrm{a}_n, \ldots, \mathrm{a}_n)$. Lattice polytopes having this a-vector are characterized in Proposition 2.11 i). $\square$

We conclude this section by adding a point to the list of necessary conditions on a nonnegative integer vector to be an a-vector of a centrally symmetric lattice polytope.

**Proposition 2.16.** *Let $P \in \mathcal{P}_0^n$ be a centrally symmetric lattice polytope and let $i \in \{0, \ldots, n\}$. Furthermore, let $n = \sum_{j \geq 0} n_j 2^j$ and $i = \sum_{j \geq 0} i_j 2^j$ be the binary expansions of $n$ and $i$, respectively. Then*

*$\mathrm{a}_i(P)$ is even if and only if there exists a $j \geq 0$ such that $n_j < i_j$.*

Proof. We will show that the parity of $\mathrm{a}_i(P)$ depends only on $n$ and $i$, in particular not on the polytope. To this end we observe that by central symmetry of $P$ the number of lattice points in $kP$ is odd for any $k \in \mathbb{N}$. We proceed by induction on $i$. The cases $i = 0$ and $i = 1$ are easy, because $\mathrm{a}_0(P) = 1$ and $\mathrm{a}_1(P) = \mathrm{G}(P) - (n+1)$. Now assume that $i \geq 2$. Since

$$\mathrm{G}(iP) = \sum_{j=0}^{n} \binom{i+n-j}{n} \mathrm{a}_j(P) = \sum_{j=0}^{i} \binom{i+n-j}{n} \mathrm{a}_j(P),$$

we have $\mathrm{a}_i(P) = \mathrm{G}(iP) - \sum_{j=0}^{i-1} \binom{i+n-j}{n} \mathrm{a}_j(P)$. By induction hypothesis the parity of all numbers on the right hand side only depend on $n$ and $i$, and so also $\mathrm{a}_i(P)$ does.

Therefore, we can decide the parity of $\mathrm{a}_i(P)$ by considering the standard crosspolytope $C_n^\star$. We have $\mathrm{a}_i(C_n^\star) = \binom{n}{i}$ for all $i = 0, \ldots, n$. We are done since Lucas' Theorem [Luc78, Sect. XXI] characterizes the parity of binomial coefficients by the claimed condition. $\square$

## 2.3. A Blichfeldt-type Inequality for Lattice-face Polytopes

In this section, we derive a Blichfeldt-type inequality for the class of lattice-face polytopes. On the way, we introduce the broader class of *weakly lattice-face polytopes*, which are suitable for inductive arguments.

We prove the following

**Theorem 2.17.** *Let $P \in \mathcal{P}^n$ be a lattice-face polytope. Then*

$$\mathrm{vol}(P) \geq \mathrm{G}(\mathrm{int}\, P) + (n-1)!\,.$$

This inequality is best possible in the dependence on $\mathrm{G}(\mathrm{int}\, P)$. Though, we do not know whether the summand $(n-1)!$ is best possible for $n \geq 4$. The best examples that we found, in the sense that they demand for the smallest additive constant on the right hand side, are the cyclic standard simplices $P_n = C_n(n+1)$. We obtained the following table with the help of `polymake` [GJ00] and `latte` [dLHTY04]. It compares our result with the data of $P_n$.

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| $(n-1)!$ | 1 | 2 | 6 | 24 | 120 | 720 |
| $\mathrm{vol}(P_n) - \mathrm{G}(\mathrm{int}\, P_n)$ | 1 | 2 | 9 | 138 | 7898 | 2086310 |

One part of the proof of Theorem 2.17 is a direct consequence of results by Liu [Liu09]. Recall that $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ denotes the projection that forgets the last coordinate.

**Lemma 2.18.** *Let $P \in \mathcal{P}^n$ be a lattice-face polytope. Then*

$$\mathrm{vol}(P) = \mathrm{G}(\mathrm{int}\, P) + \mathrm{G}(\mathrm{int}\, \pi(P)).$$

PROOF. By [Liu09, Thm. 1.1], we have

$$\mathrm{G}(P) = \mathrm{vol}(P) + \mathrm{G}(\pi(P)). \tag{2.2}$$

As $P$ is lattice-face, for each lattice point $z \in \mathrm{int}\, \pi(P)$ the two intersection points in $\pi^{-1}(z) \cap \partial P$ are integral. Furthermore, by [Liu09, Cor. 4.5], for each $z \in \partial\pi(P)$ there is exactly one point in the preimage $\pi^{-1}(z) \cap \partial P$, which is again integral. Therefore we have $\mathrm{G}(\partial P) = \mathrm{G}(\partial \pi(P)) + 2\,\mathrm{G}(\mathrm{int}\, \pi(P))$ and thus, with Equation (2.2), we get

$$\begin{aligned}\mathrm{vol}(P) &= \mathrm{G}(P) - \mathrm{G}(\pi(P)) = \mathrm{G}(\mathrm{int}\, P) + \mathrm{G}(\partial P) - \mathrm{G}(\pi(P))\\ &= \mathrm{G}(\mathrm{int}\, P) + \mathrm{G}(\mathrm{int}\, \pi(P)).\end{aligned}$$

□

In the remainder of this section, we show that $\mathrm{G}(\mathrm{int}\, \pi(P)) \geq (n-1)!$ for all lattice-face polytopes $P \in \mathcal{P}^n$, which in view of Lemma 2.18 gives Theorem 2.17.

We say that an affine space is *tangent* to a polytope $P$ if it is contained in the affine hull of some face of $P$. Compare our next definition with the Definition 1.4 of a lattice-face polytope.

**Definition 2.19** (weakly lattice-face polytope)**.** *A polytope $P \in \mathcal{P}^n$ is said to be a* weakly lattice-face polytope *if for every $k \in \{0, \ldots, n-1\}$ and every subset $U \subset \mathrm{vert}\, P$ that spans a $k$-dimensional affine space that is tangent to $P$, we have $\pi^{(n-k)}(\mathrm{aff}\, U \cap \mathbb{Z}^n) = \mathbb{Z}^k$.*

Roughly speaking, for a weakly lattice-face polytope the lattice-face property is only required in the boundary. Figure 2.4 illustrates the condition $\pi^{(n-k)}(\mathrm{aff}\, U \cap \mathbb{Z}^n) = \mathbb{Z}^k$ on a two-dimensional tangent subspace. Clearly, any lattice-face polytope is weakly lattice-face and on simplices these notions coincide. Conversely, the class of weakly lattice-face polytopes is richer than the class of lattice-face polytopes. Examples include the polytopes $\operatorname{conv}\{\binom{0}{0}, \binom{1}{1}, \binom{1}{-1}, \binom{2}{0}\}$ and $\operatorname{conv}\{0, (1, \pm 2, 0)^\intercal, (2, 0, \pm 4)^\intercal, (3, 0, 0)^\intercal\}$.

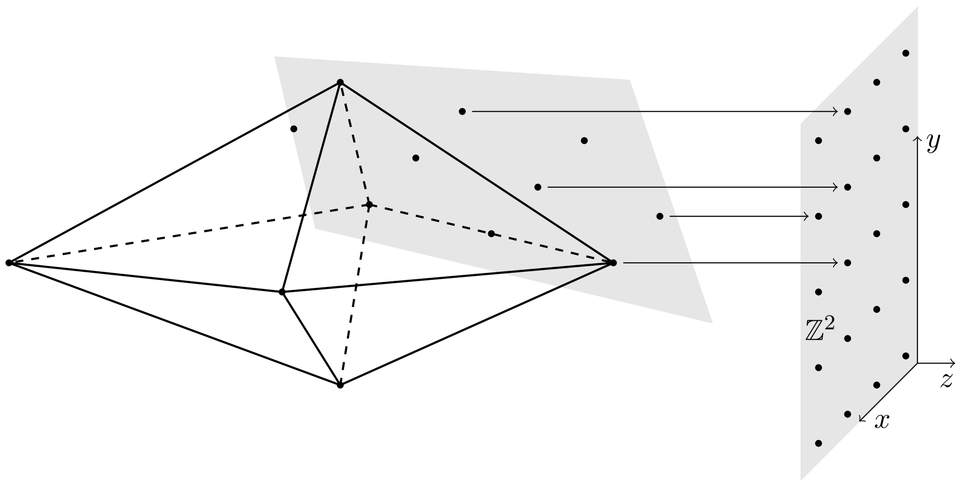



FIGURE 2.4. A weakly lattice-face polytope together with a tangent two-dimensional subspace. The projection of the sublattice of $\mathbb{Z}^3$ that is contained in that subspace is $\mathbb{Z}^2$.

Analogously to [**Liu09**, Def. 3.1], we can also think about weakly lattice-face polytopes in a recursive way.

**Proposition 2.20.** *A polytope $P \in \mathcal{P}^n$ is weakly lattice-face if and only if, either $P$ is a one-dimensional lattice polytope, or $n \geq 2$, and for every subset $U \subset \operatorname{vert} P$ that spans an $(n-1)$-dimensional affine space that is tangent to $P$, we have*

*i)* $\pi(\operatorname{conv} U)$ *is a weakly lattice-face polytope and*
*ii)* $\pi(\operatorname{aff} U \cap \mathbb{Z}^n) = \mathbb{Z}^{n-1}$.

PROOF. For the necessity of the condition we proceed by induction on $n$. For $n = 1$, there is nothing to show. Let $n \geq 2$ and let $U \subset \operatorname{vert} P$ span an $(n-1)$-dimensional affine space that is tangent to $P$. Since $P$ is weakly lattice-face, we have $\pi(\operatorname{aff} U \cap \mathbb{Z}^n) = \mathbb{Z}^{n-1}$ and we must prove that $\pi(\operatorname{conv} U)$ is weakly lattice-face. To this end, let $k \in \{0, \ldots, n-2\}$ and let $W \subset \operatorname{vert}(\pi(\operatorname{conv} U))$ span a $k$-dimensional affine space that is tangent to $\pi(\operatorname{conv} U)$. Then there exists a subset $W' \subseteq \operatorname{vert}(\operatorname{conv} U) = U$ such that $W = \pi(W')$ and $\dim(\operatorname{aff} W') = \dim(\operatorname{aff} W) = k$. The subset $W'$ spans a tangent space of $P$ since $\operatorname{aff} W' \subset \operatorname{aff} U$ and the latter of which is tangent to $P$ by assumption. Therefore

$$\begin{aligned}\pi^{(n-1-k)}(\operatorname{aff} W \cap \mathbb{Z}^{n-1}) &= \pi^{(n-1-k)}(\pi(\operatorname{aff} W') \cap \pi(\mathbb{Z}^n)) \\ &= \pi^{(n-k)}(\operatorname{aff} W' \cap \mathbb{Z}^n) = \mathbb{Z}^k.\end{aligned}$$

For the converse direction, let $k \in \{0, \ldots, n-1\}$ and let $V \subset \operatorname{vert} P$ span a $k$-dimensional affine space that is tangent to $P$. We have to show that $\pi^{(n-k)}(\operatorname{aff} V \cap \mathbb{Z}^n) = \mathbb{Z}^k$. If $k = n-1$, this holds right by the definition. So let $k < n-1$ and assume that the claim is proven for all such $V$ with dimension bigger than $k$. There exists a $W \subseteq V$ consisting of $k+1$ affinely independent points. Since $\operatorname{aff} V$ is tangent to $P$, there is a face $F$ of $P$ of dimension at least $k+1$ such that $\operatorname{aff} W = \operatorname{aff} V \subset \operatorname{aff} F$. Let $v$ be a vertex of $F$ that is affinely independent of $W$ and write $W' = W \cup \{v\}$. We get that $W'$ spans a $(k+1)$-simplex whose affine hull is tangent to $P$. By the induction hypothesis, we have that $\pi^{(n-(k+1))}(\operatorname{conv} W')$ is a weakly lattice-face polytope and furthermore $\pi^{(n-(k+1))}(\operatorname{aff} W)$ is tangent to $\pi^{(n-(k+1))}(\operatorname{conv} W')$ because $\operatorname{conv} W'$ is a simplex. Therefore, we get

$$\pi^{(n-k)}(\operatorname{aff} V \cap \mathbb{Z}^n) = \pi\left(\pi^{(n-(k+1))}(\operatorname{aff} W) \cap \mathbb{Z}^{k+1}\right) = \mathbb{Z}^k,$$

and thus $P$ is a weakly lattice-face polytope. □

Note that the first part of the above proof shows that for $U \subset \operatorname{vert} P$ spanning an $(n-1)$-dimensional affine space that is tangent to a weakly lattice-face polytope $P \in \mathcal{P}^n$, the projection $\pi(\operatorname{conv} U)$ is a lattice-face polytope. Therefore, we can triangulate the negative boundary of $P$, that is, the union of those facets of $P$ with an outer normal vector having negative last coordinate, which induces a triangulation of $\pi(P)$ into lattice-face simplices. This yields that the projection $\pi(P)$ is again a weakly lattice-face polytope.

A priori it is not clear that weakly lattice-face polytopes have integral vertices. The proof that they indeed are lattice polytopes goes along the same lines as [**Liu08**, Lem. 3.3], which is the respective statement for lattice-face polytopes.

**Proposition 2.21.** *Every weakly lattice-face polytope is a lattice polytope.*

Proof. We use Proposition 2.20 to induct on the dimension $n$ of a weakly lattice-face polytope $P$. For $n = 1$, $P$ is a lattice polytope, so let $n \geq 2$. Let $v$ be a vertex of $P$ and let $U$ be an $n$-element subset of $\operatorname{vert} P$ containing $v$ and spanning an $(n-1)$-dimensional affine space that is tangent to $P$. Then $\pi(\operatorname{conv} U)$ is a weakly lattice-face $(n-1)$-simplex. The induction hypothesis yields the integrality of $\pi(\operatorname{conv} U)$ and so $\pi(v) \in \mathbb{Z}^{n-1}$. Since $\pi(\operatorname{aff} U \cap \mathbb{Z}^n) = \mathbb{Z}^{n-1}$, we get that $v = \pi^{-1}(\pi(v)) \cap \operatorname{aff} U \in \mathbb{Z}^n$. □

**Remark 2.22.** *It would be interesting to find out if Liu's relation* (2.2) *holds on the broader class of weakly lattice-face polytopes. All the examples that we came across fulfill this identity. Liu's arguments are based on a triangulation of lattice-face polytopes into lattice-face simplices. Weakly lattice-face polytopes do not in general admit such a triangulation.*

A crucial property for us is that certain intersections of weakly lattice-face polytopes are weakly lattice-face once again. Let $s$ and $l \in \mathbb{Z}$ be the smallest, respectively the largest, first coordinate of the vertices of a weakly lattice-face polytope $P \in \mathcal{P}^n$. For the integers $j \in \{s, \ldots, l\}$ we consider the intersections $S_j = P \cap H_j$. Here, $H_j = \{x \in \mathbb{R}^n : x_1 = j\}$ is affinely equivalent to $\mathbb{R}^{n-1}$, in symbols $H_j \cong \mathbb{R}^{n-1}$.

**Proposition 2.23.** *Let $n \geq 2$ and let $P \in \mathcal{P}^n$ be weakly lattice-face. Then, for every $j \in \{s+1, \ldots, l-1\}$, the intersections $S_j$ are themselves weakly lattice-face $(n-1)$-polytopes in the space $H_j \cong \mathbb{R}^{n-1}$ of the last $n-1$ coordinates.*

PROOF. Let $U \subset \operatorname{vert} S_j$ be with $\dim(\operatorname{aff} U) = k \in \{0, \ldots, n-2\}$ and $\operatorname{aff} U$ tangent to $S_j$. We have to show that

$$\pi^{(n-1-k)}(\operatorname{aff} U \cap \mathbb{Z}^n) = \pi^{(n-(k+1))}(H_j) \cap \mathbb{Z}^{k+1} \cong \mathbb{Z}^k.$$

By the definition of $S_j$, we have $\partial S_j \subset \partial P$ and therefore $\operatorname{aff} U$ is contained in some $(k+1)$-dimensional affine space of $\mathbb{R}^n$ that is spanned by a set $\bar{U}$ of vertices of $P$. Hence, $\operatorname{aff} U = \operatorname{aff} \bar{U} \cap H_j$. Since $P$ is a weakly lattice-face polytope, this yields the desired equality as

$$\begin{aligned}\pi^{(n-1-k)}(\operatorname{aff} U \cap \mathbb{Z}^n) &= \pi^{(n-1-k)}(\operatorname{aff} \bar{U} \cap H_j \cap \mathbb{Z}^n)\\ &= \pi^{(n-(k+1))}(H_j) \cap \pi^{(n-(k+1))}(\operatorname{aff} \bar{U} \cap \mathbb{Z}^n)\\ &= \pi^{(n-(k+1))}(H_j) \cap \mathbb{Z}^{k+1}. \qquad \square\end{aligned}$$

**Remark 2.24.** *For a lattice-face polytope, the intersections $S_j$ from Proposition 2.23 are in general not themselves lattice-face polytopes. As an example, consider the cyclic standard simplex*

$$P = C_4(5) = \operatorname{conv}\left\{(t, t^2, t^3, t^4)^\intercal : t = 0, \ldots, 4\right\}$$

*and the hyperplane $H_2 = \{x \in \mathbb{R}^4 : x_1 = 2\}$. We have*

$$P \cap H_2 \cong \operatorname{conv}\left\{\begin{pmatrix}4\\8\\16\end{pmatrix}, \begin{pmatrix}5\\14\\41\end{pmatrix}, \begin{pmatrix}6\\18\\54\end{pmatrix}, \begin{pmatrix}6\\22\\86\end{pmatrix}, \begin{pmatrix}8\\32\\128\end{pmatrix}\right\}.$$

*This is not a lattice-face polytope since the vertices with first coordinate equal to 6 span a line $L$ that is projected to one point, and thus $\pi^{(2)}(L \cap \mathbb{Z}^3) \neq \mathbb{Z}$.*

A second crucial property is the fatness of weakly lattice-face polytopes in the direction of the first unit vector.

**Proposition 2.25.** *For a weakly lattice-face polytope $P \in \mathcal{P}^n$, we have*

$$\max\{v_1 : v \in \operatorname{vert} P\} - \min\{v_1 : v \in \operatorname{vert} P\} \geq n.$$

PROOF. First of all, we observe that any line segment connecting two vertices of $P$ that have the same first coordinate has to pass through the interior of $P$. Otherwise it would span a one-dimensional affine space contradicting the assumptions in Definition 2.19. This means, that there is a unique vertex $w$ of $P$ with smallest first coordinate and a unique vertex $z$ of $P$ with largest first coordinate.

By the above observation, no two vertices in a facet of $P$ have the same first coordinate. So, if $P$ has a facet with at least $n+1$ vertices, then our assertion is shown by the integrality of $P$, which was proven in Proposition 2.21. In general, this already leads to the bound

$$\max\{v_1 : v \in \operatorname{vert} P\} - \min\{v_1 : v \in \operatorname{vert} P\} = z_1 - w_1 \geq n-1. \tag{2.3}$$

Assume that $P$ is simplicial, that is, all facets are simplices and moreover assume that we have equality in (2.3). This implies that the vertices $w$ and $z$ are contained in every facet of $P = \{x \in \mathbb{R}^n : a_i^\intercal x \leq b_i,\ i = 1, \ldots, m\}$. So $w - z$ is a nontrivial solution of $a_i^\intercal(w - z) = 0$ for all $i = 1, \ldots, m$. This is a contradiction since $P$ is full-dimensional. $\square$

Now we have all the ingredients to prove the desired lower bound on the number of interior lattice points in $\pi(P)$.

**Lemma 2.26.** *Let $n \geq 2$ and let $P \in \mathcal{P}^n$ be a weakly lattice-face polytope. Then*

$$\mathrm{G}(\operatorname{int} \pi(P)) \geq (n-1)!\,.$$

Proof. We proceed by induction on $n$. If $n = 2$, then Proposition 2.25 says that the integral interval $\pi(P)$ has length at least two and thus $\pi(P)$ contains an interior lattice point.

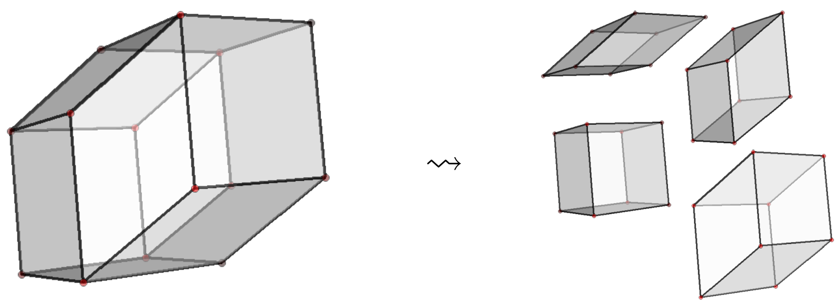
Now assume that $n \geq 3$. Again by Proposition 2.25, there are at least $n-1$ intersections $S_j = P \cap H_j$, which by Proposition 2.23 are weakly lattice-face $(n-1)$-polytopes in their respective hyperplane $H_j$. Thus, by the induction hypothesis, we get

$$\begin{aligned} \mathrm{G}(\operatorname{int} \pi(P)) &= \mathrm{G}(\operatorname{int} \pi(S_{s+1})) + \ldots + \mathrm{G}(\operatorname{int} \pi(S_{l-1})) \\ &\geq (n-1) \cdot (n-2)! = (n-1)!\,. \end{aligned}$$

□

CHAPTER 3

# Blichfeldt-type Inequalities for Centrally Symmetric Bodies



## 3.1. A Blichfeldt-type Inequality for Centrally Symmetric Bodies

Blichfeldt's inequality in Theorem 2.1 provides a sharp lower bound on the volume of lattice spanning convex bodies in terms of the number of lattice points. In Section 2.1, we saw that, despite the efforts of various mathematicians which culminated in Theorem 2.5, a sharp upper bound of this type is still to be determined. In contrast to this situation, a best possible upper bound on the volume of centrally symmetric convex bodies is known since works of Blichfeldt and van der Corput.

**Theorem 3.1** (Blichfeldt [Bli21], van der Corput [vdC36])**.** *Let $K \in \mathcal{K}_0^n$. Then*

$$\mathrm{vol}(K) \leq 2^{n-1}\left(\mathrm{G}(\mathrm{int}\, K) + 1\right) \quad \textit{and} \quad \mathrm{vol}(K) < 2^{n-1}\left(\mathrm{G}(K) + 1\right).$$

The parallelepipeds $C_{n,l} = \{x \in \mathbb{R}^n : |x_1| \leq l, |x_i| \leq 1, i = 2, \ldots, n\}$ for $l \in \mathbb{N}$, and $\varepsilon \cdot C_{n,l}$ for $\varepsilon \in (0,1)$ close to 1, respectively, show that these inequalities are best possible. For each $l \in \mathbb{N}$, we have

$$\mathrm{vol}(C_{n,l}) = 2^n l = 2^{n-1}\left((2l-1)+1\right) = 2^{n-1}\left(\mathrm{G}(\mathrm{int}\, C_{n,l}) + 1\right),$$

and, for $\varepsilon \to 1$,

$$\mathrm{vol}(\varepsilon \cdot C_{n,l}) = \varepsilon^n 2^n l \to 2^{n-1}\left((2l-1)+1\right) = 2^{n-1}\left(\mathrm{G}(\varepsilon \cdot C_{n,l}) + 1\right).$$

For centrally symmetric convex bodies though, the respective lower bound has not been settled yet. Bey, Henk and Wills [BHW07, Prop. 1.4] prove the following estimate on the class of lattice crosspolytopes and conjecture that it holds throughout.

**Conjecture 3.2** (Bey, Henk, and Wills [BHW07])**.** *Let $P \in \mathcal{P}_0^n$ be a lattice polytope. Then*

$$\mathrm{vol}(P) \geq \frac{2^{n-1}}{n!}\left(\mathrm{G}(\mathrm{int}\, P) + 1\right).$$

This proposed inequality would be tight, as shown by the crosspolytopes $C_{n,l}^\star = \mathrm{conv}\{\pm l e_1, \pm e_2, \ldots, \pm e_n\}$. For all $l \in \mathbb{N}$, we have

$$\mathrm{vol}(C_{n,l}^\star) = \frac{2^n}{n!} l = \frac{2^{n-1}}{n!}\left((2l-1)+1\right) = \frac{2^{n-1}}{n!}\left(\mathrm{G}(\mathrm{int}\, C_{n,l}^\star) + 1\right).$$

It is natural to ask if these crosspolytopes also serve as minimal examples for a Blichfeldt-type inequality with respect to the number of all lattice points in the body. One calculates that $\mathrm{vol}(C_{n,l}^\star) = \frac{2^{n-1}}{n!}\big(\mathrm{G}(C_{n,l}^\star) - (2n-1)\big)$ for all $l \in \mathbb{N}$.

The constant $2n-1$ that appears on the right hand side could be explained by comparing sharp Blichfeldt-type inequalities as Theorem 2.1, Proposition 3.6 and Theorem 3.19. In all of these cases, the respective constant is the minimal number of lattice points contained in an $(n-1)$-dimensional body from the considered class. Our quest for examples of lattice spanning $K \in \mathcal{K}_0^n$ with a small value of $c_n$ in an inequality of the form

$$\mathrm{vol}(K) \geq c_n \cdot \left(\mathrm{G}(K) - (2n-1)\right) \tag{3.1}$$

revealed the following polytopes which, surprisingly, are smaller in this sense than the crosspolytopes $C_{n,l}^\star$.

For $n = 3, 4, 5$, let $Q_n = \mathrm{conv}\{C_3, \pm e_4, \ldots, \pm e_n\}$ where $C_3 = [-1, 1]^3$. We have $\mathrm{vol}(Q_n) = 3 \cdot \frac{2^{n+1}}{n!}$ and $\mathrm{G}(Q_n) = 2n + 21$, and thus

$$\mathrm{vol}(Q_n) = \frac{6}{11} \cdot \frac{2^{n-1}}{n!} \left(\mathrm{G}(Q_n) - (2n-1)\right).$$

For $n \geq 6$, our extremal examples are $P_{n,k} = C^\star_{n-1} \times [-ke_n, ke_n]$ where $C^\star_n$ is the standard crosspolytope. We have $\mathrm{vol}(P_{n,k}) = \frac{2^n}{(n-1)!}k$ and $\mathrm{G}(P_{n,k}) = (2k+1)(2n-1)$, and therefore

$$\mathrm{vol}(P_{n,k}) = \frac{n}{2n-1} \cdot \frac{2^{n-1}}{n!} \left(\mathrm{G}(P_{n,k}) - (2n-1)\right).$$

The subject of this section is to obtain an asymptotically sharp value for $c_n$ in the Inequality (3.1). More precisely, we show:

**Theorem 3.3.** *For every $\varepsilon \in (0, 1]$ there exists an $n(\varepsilon) \in \mathbb{N}$ such that for every $n \geq n(\varepsilon)$ and every lattice spanning $K \in \mathcal{K}^n_0$, we have*

$$\mathrm{vol}(K) \geq \frac{(2-\varepsilon)^n}{n!}\, \mathrm{G}(K).$$

*The inequality is asymptotically sharp in the sense that the constant 2 in the expression on the right hand side cannot be replaced by a bigger one.*

Our argument is based on two lemmas, which are interesting in their own right. Before we can state them, we have to fix some notation. A convex body $T \in \mathcal{K}^n$ is said to be a *lattice tile with respect to the lattice* $\Lambda \in \mathcal{L}^n$, if $T$ tiles $\mathbb{R}^n$ by vectors in $\Lambda$, that is, $\mathbb{R}^n = \Lambda + T$ and $(x + \mathrm{int}\, T) \cap (y + \mathrm{int}\, T) = \emptyset$ for all different $x, y \in \Lambda$. In other words, at the same time $\Lambda + T$ is a covering of $\mathbb{R}^n$ and $\Lambda$ is a packing set of $T$. It is well-known that lattice tiles are polytopes, and thus we can assume that every lattice tile has the origin as a vertex. For a survey on tilings and references to the relevant literature, we refer the reader to [**Sch93b**].

Betke and Wills [**BW79**] (see also [**GW93**, Sect. 3]) showed that, for every convex body $K \in \mathcal{K}^n$, the number of lattice points in $K$ is bounded by $\mathrm{G}(K) \leq \mathrm{vol}(K + L)$, where $L$ is a fundamental cell of $\mathbb{Z}^n$. They asked to determine all bodies $L$ that admit such an inequality. With the following lemma, we identify lattice tiles as bodies with this property. The set of all lattice parallelepipeds in $\mathcal{K}^n$ whose edges are parallel to a given lattice parallelepiped $P \in \mathcal{K}^n$ is denoted by $\mathcal{Q}(P)$.

**Lemma 3.4.** *Let $K \in \mathcal{K}^n$ and let $T$ be a lattice tile with respect to a sublattice $\Lambda$ of $\mathbb{Z}^n$. Then*

$$\mathrm{G}(K) \leq \mathrm{vol}(K + T).$$

*If $T$ is a lattice parallelepiped $P$, then equality holds if and only if $\Lambda = \mathbb{Z}^n$ and $K \in \mathcal{Q}(P)$.*

Proof. Since for all $x, y \in \Lambda$ we have $(x + \mathrm{int}\, T) \cap (y + \mathrm{int}\, T) = \emptyset$, unless $x = y$, every residue class modulo $\Lambda$ is a packing set of $T$. Let $\{r_1, \ldots, r_m\} \subset \mathbb{Z}^n$ be a maximal subset of different representatives of residue classes modulo $\Lambda$. Writing $\Lambda_j = r_j + \Lambda$, we have for every $j = 1, \ldots, m$ that

$$\#(K \cap \Lambda_j) = \frac{\mathrm{vol}\left((K \cap \Lambda_j) + T\right)}{\mathrm{vol}(T)}.$$

Since $T$ is a lattice tile, we have $\mathrm{vol}(T) = \det \Lambda = m$, and therefore

$$\mathrm{G}(K) = \sum_{j=1}^{m} \#(K \cap \Lambda_j) = \frac{1}{m} \sum_{j=1}^{m} \mathrm{vol}\left((K \cap \Lambda_j) + T\right) \leq \mathrm{vol}(K + T).$$

By the compactness of the involved sets, equality is attained if and only if $(K \cap \Lambda_j) + T = K + T$ for all $j = 1, \ldots, m$. In particular, there can only be one residue class and thus $m = \det \Lambda = 1$, which means $\Lambda = \mathbb{Z}^n$. In the case that the lattice tile is a lattice parallelepiped $P = \sum_{i=1}^{n} [0, a_i]$, every hyperplane supporting a facet of the convex polytope $K + P = (K \cap \mathbb{Z}^n) + P$ is parallel to a hyperplane supporting a facet of $P$. Therefore, $K + P$ is a lattice translate of $\sum_{i=1}^{n} [0, t_i a_i]$ for some $t_i \in \mathbb{N}$, and so $K$ is a lattice translate of $\sum_{i=1}^{n} [0, (t_i - 1) a_i] \in \mathcal{Q}(P)$.

Conversely, if $P$ is a fundamental cell of $\mathbb{Z}^n$, then we find lattice vectors $v_1, \ldots, v_n \in \mathbb{Z}^n$ such that, up to a lattice translation, $P = \sum_{i=1}^{n} [0, v_i]$. Again, up to a lattice translation, every $K \in \mathcal{Q}(P)$ is of the form $K = \sum_{i=1}^{n} [0, l_i v_i]$ for some $l_1, \ldots, l_n \in \mathbb{N}$. Since $P$ is a fundamental cell, we have $\mathrm{vol}(P) = 1 = \#\big(\sum_{i=1}^{n} [0, v_i) \cap \mathbb{Z}^n\big)$ and thus, by the volume formula (1.7) for lattice parallelepipeds,

$$\mathrm{G}(K) = \# \left( \sum_{i=1}^{n} [0, (l_i + 1) v_i) \cap \mathbb{Z}^n \right) = \mathrm{vol}(K + P). \qquad \square$$

A result of Davenport [**Dav51**] states that the number of lattice points in a convex body $K$ is bounded from above by the sum of the volumes of the projections of $K$ onto coordinate hyperplanes. Our next lemma is a generalization of Davenport's result, which allows to choose a lattice parallelepiped $P$ that determines the subspaces that $K$ is projected onto. Choosing $P = [0, 1]^n$, gives the classical bound. Recall that $K|L$ denotes the orthogonal projection of $K$ onto the subspace $L$ and $\binom{[n]}{i}$ is the set of all $i$-element subsets of $[n] = \{1, \ldots, n\}$.

**Lemma 3.5.** *Let $K \in \mathcal{K}^n$ and let $P = \sum_{j=1}^{n} [0, z_j]$ be a lattice parallelepiped. Then*

$$\mathrm{G}(K) \leq \sum_{i=0}^{n} \sum_{J \in \binom{[n]}{i}} \mathrm{vol}_{n-i}(K|L_J^{\perp}) \, \mathrm{vol}_i(P_J),$$

*where $L_J = \mathrm{lin}\{z_j : j \in J\}$ and $P_J = \sum_{j \in J} [0, z_j]$ for each $J \in \binom{[n]}{i}$. Equality holds if and only if $P$ is a fundamental cell of $\mathbb{Z}^n$ and $K \in \mathcal{Q}(P)$.*

Proof. The lattice parallelepiped $P = \sum_{j=1}^{n} [0, z_j]$ is a lattice tile with respect to the sublattice of $\mathbb{Z}^n$ that is spanned by $z_1, \ldots, z_n$. Based on an alternative way by Ulrich Betke to prove the aforementioned inequality of Davenport, we use Lemma 3.4 and develop the volume of $K + P$ into a sum of the mixed volumes $\mathrm{V}(K, n-i; P, i)$ of $K$ and $P$:

$$\mathrm{G}(K) \leq \mathrm{vol}(K + P) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{V}(K, n-i; P, i). \tag{3.2}$$

We refer to the books of Gardner [**Gar95**] and Schneider [**Sch93a**] for details and properties on mixed volumes. The linearity and nonnegativity of the mixed volumes give

$$\begin{aligned}\mathrm{V}(K,n-i;P,i) &= \sum_{j_1=1}^{n}\cdots\sum_{j_i=1}^{n}\mathrm{V}(K,n-i;[0,z_{j_1}],\ldots,[0,z_{j_i}])\\ &= \sum_{J\in\binom{[n]}{i}} i!\,\mathrm{V}(K,n-i;[0,z_j],j\in J). \qquad (3.3)\end{aligned}$$

Moreover, the Equation (A.41) in [**Gar95**, App. A.5] yields

$$\binom{n}{i} i!\,\mathrm{V}(K,n-i;[0,z_j],j\in J) = \mathrm{vol}_{n-i}(K|L_J^{\perp})\,\mathrm{vol}_i(P_J), \qquad (3.4)$$

for all $J\in\binom{[n]}{i}$. Combining (3.2), (3.3) and (3.4) gives the desired result.

The characterization of the equality case is inherited from Lemma 3.4 since (3.2) is the only step where there could be an inequality. $\square$

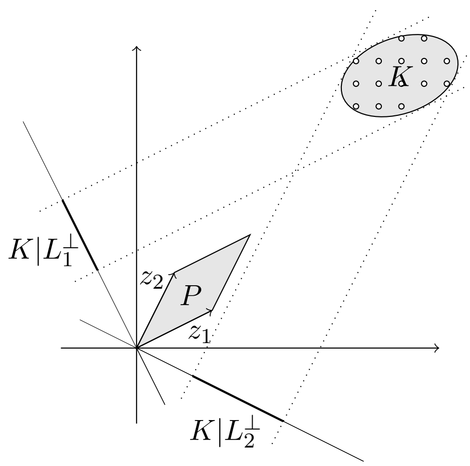



FIGURE 3.1. An example for Lemma 3.5.

We illustrate the planar situation of Lemma 3.5 in Figure 3.1. The inequality here reads $\mathrm{G}(K)\le \mathrm{vol}(P)+\|K|L_1^{\perp}\|\cdot\|z_1\|+\|K|L_2^{\perp}\|\cdot\|z_2\|+\mathrm{vol}(K)$.

Now we are ready to prove the main theorem of this chapter.

PROOF OF THEOREM 3.3. By assumption, we find $n$ linearly independent lattice points $z_1,\ldots,z_n$ inside $K$. Applying Lemma 3.5 with respect to the lattice parallelepiped $P=\sum_{j=1}^{n}[0,z_j]$ gives

$$\mathrm{G}(K)\le\sum_{i=0}^{n}\sum_{J\in\binom{[n]}{i}}\mathrm{vol}_{n-i}(K|L_J^{\perp})\,\mathrm{vol}_i(P_J).$$

By the construction of the subspaces $L_J$, we have

$$\mathrm{vol}_i(K\cap L_J)\ge\mathrm{vol}_i\left(\mathrm{conv}\{\pm z_j : j\in J\}\right)=\frac{2^i}{i!}\mathrm{vol}_i(P_J).$$

For centrally symmetric $K$ and $i$-dimensional subspaces $L$, the estimate

$$\mathrm{vol}(K)\le\mathrm{vol}_{n-i}(K|L^{\perp})\,\mathrm{vol}_i(K\cap L)\le\binom{n}{i}\mathrm{vol}(K) \qquad (3.5)$$

holds (see [RS58, Thm. 1] or [BM87, Lem. 3.1]). Therefore, we get

$$\begin{aligned} \mathrm{G}(K) &\leq \sum_{i=0}^{n} \sum_{J \in \binom{[n]}{i}} \frac{i!}{2^i} \operatorname{vol}_{n-i}(K|L_J^{\perp}) \operatorname{vol}_i(K \cap L_J) \\ &\leq \operatorname{vol}(K) \sum_{i=0}^{n} \binom{n}{i}^2 \frac{i!}{2^i} = \operatorname{vol}(K) \frac{n!}{2^n} L_n(2), \end{aligned} \tag{3.6}$$

where $L_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{x^k}{k!}$ denotes the $n$th Laguerre polynomial. For two functions $f, g : \mathbb{N} \to \mathbb{R}$, we denote by $f(n) \approx g(n)$ that $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$. In Szegő's book [Sze75, p. 199] one finds the approximation

$$L_n(x) \approx \frac{n^{-\frac{1}{4}}}{2\sqrt{\pi}} \frac{e^{-\frac{x}{2}}}{x^{\frac{1}{4}}} e^{2\sqrt{x(n+\frac{1}{2})}} \quad \text{for all fixed} \quad x > 0.$$

Therefore, by $\lim_{n\to\infty} e^{\frac{2\sqrt{2n+1}}{n}} = 1$, we have

$$\frac{L_n(2)}{2^n} \approx \frac{1}{2e\sqrt{\pi}\sqrt[4]{2n}} \frac{e^{2\sqrt{2n+1}}}{2^n} < \frac{e^{2\sqrt{2n+1}}}{2^n} \leq \frac{1}{(2-\varepsilon)^n} \tag{3.7}$$

for every $\varepsilon \in (0, 1]$ and large enough $n \in \mathbb{N}$. Hence, for large enough $n$, we arrive at

$$\mathrm{G}(K) \leq \operatorname{vol}(K) \frac{n!}{2^n} L_n(2) \leq \operatorname{vol}(K) \frac{n!}{(2-\varepsilon)^n}.$$

In order to see that this inequality is asymptotically sharp, we consider the crosspolytope $C_{n,l}^{\star} = \operatorname{conv}\{\pm l e_1, \pm e_2, \ldots, \pm e_n\}$ again. We have $\mathrm{G}(C_{n,l}^{\star}) = 2(n+l) - 1$ and $\operatorname{vol}(C_{n,l}^{\star}) = \frac{2^n}{n!} l$. Therefore, $\sqrt[n]{\frac{n! \operatorname{vol}(C_{n,l}^{\star})}{\mathrm{G}(C_{n,l}^{\star})}} = \sqrt[n]{\frac{2^n l}{2(n+l)-1}}$ tends to 2 when $l$ and $n$ tend to infinity. On the other hand, the above inequality shows that for every $\varepsilon \in (0, 1]$ we have $\sqrt[n]{\frac{n! \operatorname{vol}(K)}{\mathrm{G}(K)}} \geq 2 - \varepsilon$ for $n \to \infty$. $\square$

### 3.2. Dimensions Two and Three

Pick's Theorem 1.2 makes life easy in dimension two. It implies, for example, that for every planar centrally symmetric lattice polygon $P$, we have

$$\operatorname{vol}(P) = \mathrm{G}(\operatorname{int} P) + \frac{1}{2} \mathrm{G}(\partial P) - 1 \geq \mathrm{G}(\operatorname{int} P) + 1.$$

This is an affirmative answer to Conjecture 3.2. Equality holds if and only if $P$ is a quadrilateral with no other boundary lattice points besides its vertices. Similarly, for every lattice polygon $P \in \mathcal{P}_0^2$, we have

$$\operatorname{vol}(P) = \mathrm{G}(P) - \frac{1}{2} \mathrm{G}(\partial P) - 1 \geq \frac{1}{2} \left( \mathrm{G}(P) - 1 \right),$$

and equality holds if and only if $\mathrm{G}(\operatorname{int} P) = 1$.

However, we are interested in a bound with a maximal factor in front of $\mathrm{G}(P)$ while maintaining positivity on the right hand side for all centrally symmetric lattice polygons $P$. We start with explaining such a bound that in fact holds on a much broader class.

**Proposition 3.6.** *Let $P \in \mathcal{P}^2$ be a lattice polygon that is not unimodularly equivalent to the triangle $T = \operatorname{conv}\{0, 3e_1, 3e_2\}$ and contains at least one interior lattice point. Then*

$$\operatorname{vol}(P) \geq \frac{2}{3}\left(\mathrm{G}(P) - 3\right).$$

*The rectangle $\{x \in \mathbb{R}^2 : |x_1| \leq 1, 0 \leq x_2 \leq l\}$ attains equality for every $l \geq 2$.*

Proof. Scott [**Sco76**] showed that, for lattice polygons $P$ with interior lattice points, we have

$$\mathrm{G}(\partial P) \leq 2\,\mathrm{G}(\operatorname{int} P) + 7. \tag{3.8}$$

Moreover, equality holds if and only if $P$ is unimodularly equivalent to $T$. Since we want to exclude $T$, we get $2\,\mathrm{G}(\operatorname{int} P) \geq \mathrm{G}(\partial P) - 6$ and thus $\mathrm{G}(\operatorname{int} P) \geq \frac{1}{3}\left(\mathrm{G}(P) - 6\right)$.

Combining this inequality with Pick's Theorem yields

$$2\operatorname{vol}(P) = \mathrm{G}(P) + \mathrm{G}(\operatorname{int} P) - 2 \geq \frac{4}{3}\,\mathrm{G}(P) - 4,$$

which is equivalent to the claimed inequality. □

In dimension two, the central symmetry condition on a polygon is equivalent to being a zonotope (see Bolker [**Bol69**] or [**Zie95**, Sect. 7.3]). We can say considerably more than Proposition 3.6 when we exploit this structural property. Our result has a similar flavor as results of Bárány [**Bár08a**] in the sense that it quantifies the intuition that, if a lattice polygon has a lot of vertices, then its volume is well-approximated by its number of lattice points. Note that at this point we think about central symmetry of $P$ with respect to some center $x \in \mathbb{R}^2$, that is, $P - x = -(P - x)$. Moreover, a lattice vector $a \in \mathbb{Z}^n$ is said to be *primitive*, if the greatest common divisor $\gcd(a)$ of its entries equals one. Equivalently, the line segment $[0, a]$ contains exactly two lattice points.

**Theorem 3.7.** *Let $P \in \mathcal{P}^2$ be a centrally symmetric lattice polygon with $2m$ vertices. Then*

$$\operatorname{vol}(P) \geq \frac{m-1}{m}\left(\mathrm{G}(P) - \frac{3m-4}{m-1}\right).$$

*The rectangle $\{x \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, 0 \leq x_2 \leq l\}$ attains equality for every $l \in \mathbb{N}$.*

Proof. Since $P$ is a zonotope, we can write it, up to a lattice translation, as $P = \sum_{i=1}^m [0, a_i]$ for suitable pairwise nonparallel $a_1, \ldots, a_m \in \mathbb{Z}^2 \setminus \{0\}$. A result of Shephard [**She74**, Thm. (54)] shows that $P$ has a decomposition into parallelograms that are lattice translations of $P_{i,j} = [0, a_i] + [0, a_j]$, where $\{i, j\}$ runs over all two-element subsets of $[m]$. Moreover, we get that the number of lattice points in a half-open edge of $P$ equals the greatest common divisor $\gcd(a_i)$ of the entries of some $a_i$. Starting from Pick's Theorem 1.2, yields

$$\mathrm{G}(P) = \operatorname{vol}(P) + \frac{1}{2}\,\mathrm{G}(\partial P) + 1 = \operatorname{vol}(P) + \sum_{i=1}^{m} \gcd(a_i) + 1$$

$$= \sum_{\{i,j\}\in\binom{[m]}{2}} \mathrm{vol}(P_{i,j}) + \sum_{i=1}^{m} \gcd(a_i) + 1.$$

As we know from Equation (1.7) in the introduction, the volume of a lattice parallelogram equals the number of lattice points in its half-open counterpart. Thus, for all $\{i,j\} \in \binom{[m]}{2}$, we have

$$\mathrm{vol}(P_{i,j}) = \mathrm{G}([0,a_i) + [0,a_j)) \geq \gcd(a_i) + \gcd(a_j) - 1,$$

and equality holds if and only if $\mathrm{int}\, P_{i,j} \cap \mathbb{Z}^2 = \emptyset$. Writing

$$\gamma(P) = \#\left\{\{i,j\} \in \tbinom{[m]}{2} : \mathrm{int}\, P_{i,j} \cap \mathbb{Z}^2 = \emptyset\right\},$$

we obtain

$$\begin{aligned}\mathrm{G}(P) &= \sum_{\{i,j\}\in\binom{[m]}{2}} \mathrm{G}([0,a_i) + [0,a_j)) + \sum_{i=1}^{m} \gcd(a_i) + 1 \\ &\geq m \sum_{i=1}^{m} \gcd(a_i) + 1 - \gamma(P) = \frac{m}{2}\, \mathrm{G}(\partial P) + 1 - \gamma(P).\end{aligned}$$

Having this inequality at hand, we again use Pick's Theorem to derive

$$\begin{aligned}\mathrm{vol}(P) &= \mathrm{G}(P) - \frac{1}{2}\, \mathrm{G}(\partial P) - 1 \geq \frac{m-1}{m}\, \mathrm{G}(P) + \frac{1-\gamma(P)-m}{m} \\ &= \frac{m-1}{m}\left(\mathrm{G}(P) - \frac{m+\gamma(P)-1}{m-1}\right).\end{aligned}$$

To finish the proof, we need the estimate $\gamma(P) \leq 2m-3$. For $\{i,j\} \in \binom{[m]}{2}$, we have $\mathrm{int}\, P_{i,j} \cap \mathbb{Z}^2 = \emptyset$ if and only if one of the generators, say $a_i$, is primitive, and $\{a_i, \frac{a_j}{\gcd(a_j)}\}$ is a basis of $\mathbb{Z}^2$. Therefore, $\gamma(P)$ is at most the number of pairs $\{a_i, a_j\}$, $i \neq j$, such that $\{\frac{a_i}{\gcd(a_i)}, \frac{a_j}{\gcd(a_j)}\}$ is a basis of $\mathbb{Z}^2$. In Theorem 3.10 below, we show that this number is bounded from above by the desired $2m-3$. □

Kołodziejczyk and Olszewska [**KO07**] obtained a sharpening of Scott's inequality (3.8). They showed that, for lattice polygons $P$ with interior lattice points, we have $\mathrm{G}(\partial P) \leq 2\,\mathrm{G}(\mathrm{int}\, P) - v(P) + 10$, where $v(P)$ is the number of vertices of $P$. They also propose to further improve upon this bound when $v(P)$ is large. As a corollary to Theorem 3.7, we obtain such an improvement for centrally symmetric lattice polygons with at least six vertices.

**Corollary 3.8.** *Let $P \in \mathcal{P}^2$ be a centrally symmetric lattice polygon with $2m \geq 6$ vertices. Then*

$$\mathrm{G}(\partial P) \leq \frac{2}{m-2}\, \mathrm{G}(\mathrm{int}\, P) + 4.$$

Proof. Using Pick's Theorem together with Theorem 3.7 gives

$$\mathrm{vol}(P) = \mathrm{G}(\mathrm{int}\, P) + \frac{1}{2}\, \mathrm{G}(\partial P) - 1 \geq \frac{m-1}{m}\left(\mathrm{G}(P) - \frac{3m-4}{m-1}\right).$$

Equivalently, $\mathrm{G}(\partial P) \leq \frac{2}{m-2}\, \mathrm{G}(\mathrm{int}\, P) + 4$. □

**3.2.1. Fundamental Cells Among a Set of Lattice Vectors.** In this subsection, we give the details for a result that was used in the proof of Theorem 3.7 above. More precisely, we give an upper bound on the number of bases of the integer lattice that are contained in a given set of $m$ lattice vectors. The underlying question is very similar in flavor to questions that are termed "problems on repeated subconfigurations" in [**BMP05**, Ch. 6]. Indeed, we can also formulate it as follows: Given $m$ points in $\mathbb{Z}^n$, how many simplices of volume $\frac{1}{n!}$ can be built by these vectors using the origin as a common vertex?

We need the following elementary lemma that we only use for $n = 2$, but that holds in full generality.

**Lemma 3.9.** *Let $a \in \mathbb{Z}^n$ be a primitive lattice vector and $M = \|a\|_\infty \geq 2$. Write $\eta(a, n) = \#\{x \in [-M, M]^n \cap \mathbb{Z}^n : a^\intercal x = 1\}$. Then*

$$\eta(a, 2) = 2 \quad \textit{and} \quad \eta(a, n) \leq 2\,(2M + 1)^{n-2} \quad \textit{for all} \quad n \geq 3.$$

*Equality holds, for example, for $a = (M, 1, 0, \ldots, 0) \in \mathbb{Z}^n$.*

Proof. We proceed by induction on $n$. First, we consider the case $n = 2$. Assume, that $M = |a_1| \geq |a_2|$. By Bézout's identity and the reversed Euclidean algorithm, there are integers $s, t \in \mathbb{Z}$ such that $a_1 s + a_2 t = 1$ and any other solution $(x_1, x_2)$ to the equation $a^\intercal x = 1$ has the form $x_1 = s + k a_2, x_2 = t - k a_1$ for some $k \in \mathbb{Z}$. Since $x_2 \equiv t \not\equiv 0 \bmod a_1$, there are exactly two such $x_2$ with $|x_2| \leq M$, and actually it is $|x_2| < M$ in every case. It remains to check that the corresponding values $x_1$ also lie in that range. By $a_1 x_1 + a_2 x_2 = 1$, $a_1 \neq 0$ and $|a_1| \geq |a_2|$, we in fact get

$$|x_1| = \frac{|1 - a_2 x_2|}{|a_1|} \leq \frac{1 + |a_2 x_2|}{|a_1|} \leq \frac{1 + |a_2|(M-1)}{|a_1|} \leq \frac{1 + M(M-1)}{M} < M.$$

Now let $n \geq 3$. Again we can assume that $M = |a_1| \geq |a_2| \geq \ldots \geq |a_n|$. Since $\gcd(a_1, \ldots, a_{n-2}, \gcd(a_{n-1}, a_n)) = \gcd(a_1, \ldots, a_n) = 1$, we can reduce the situation to the $(n-1)$-dimensional case.

Let $s, t \in \mathbb{Z}$ be such that $g = \gcd(a_{n-1}, a_n) = a_{n-1}s + a_n t$. Then, for all $y, z \in \mathbb{Z}$, we have

$$a_{n-1}y + a_n z = g \underbrace{\left(\frac{a_{n-1}}{g} y + \frac{a_n}{g} z\right)}_{\bar{x} \in \mathbb{Z}} = a_{n-1} s \bar{x} + a_n t \bar{x}.$$

So, we can write a solution $x \in \mathbb{Z}^n$ of $a^\intercal x = 1$ as $(\bar{x}_1, \ldots, \bar{x}_{n-2}, s\bar{x}_{n-1}, t\bar{x}_{n-1})$, where $a_1 \bar{x}_1 + \cdots + a_{n-2}\bar{x}_{n-2} + g\bar{x}_{n-1} = 1$. This yields

$$\eta(a, n) \leq \eta(\bar{a}, n-1) \cdot \#\left\{(s, t) \in [-M, M]^2 \cap \mathbb{Z}^2 : g = a_{n-1}s + a_n t\right\},$$

where $\bar{a} = (a_1, \ldots, a_{n-2}, \gcd(a_{n-1}, a_n))$. As soon as we found a solution $(s, t) \in \mathbb{Z}^2$ of the equation $g = a_{n-1}s + a_n t$, every other solution is given by $(s - k\frac{a_n}{g}, t + k\frac{a_{n-1}}{g})$ for some $k \in \mathbb{Z}$. This implies

$$\#\left\{(s, t) \in [-M, M]^2 \cap \mathbb{Z}^2 : g = a_{n-1}s + a_n t\right\} \leq 2M + 1,$$

and therefore

$$\eta(a, n) \leq \eta(\bar{a}, n-1) \cdot (2M + 1) \leq 2\,(2M + 1)^{n-2},$$

by the induction hypothesis.

In order to see that equality is attained for the vector $a = (M, 1, 0, \ldots, 0)$, we note that $a^\intercal x = 1$ is equivalent to $Mx_1 + x_2 = 1$, which has exactly two solutions $(x_1, x_2) \in [-M, M]^2 \cap \mathbb{Z}^2$ by the arguments in the case $n = 2$. The remaining $n-2$ coordinates of a solution $x$ can be chosen independently in $[-M, M] \cap \mathbb{Z}$ and therefore $\eta(a, n) = 2\,(2M+1)^{n-2}$. □

**Theorem 3.10.** *Let $S = \{z_1, \ldots, z_m\} \subset \mathbb{Z}^2$ be a set of $m \geq 2$ nonparallel lattice vectors and let*

$$\beta(S) = \#\left\{\{i,j\} \in \tbinom{[m]}{2} : |\det(z_i, z_j)| = 1\right\}.$$

*Then, we have $\beta(S) \leq 2m-3$, and equality holds, for example, for $S = \{(0,1), (1,0), (1,1), \ldots, (1, m-2)\}$.*

Proof. By multiplying some of the vectors $z_j \in S$ by $-1$, it is no restriction to assume $S \subset \{(x,y) \in \mathbb{Z}^2 : x \geq 0\}$. Let $M = \max\{\|z_1\|_\infty, \ldots, \|z_m\|_\infty\}$ be the maximal absolute value of the entries among the vectors in $S$. Before we can use induction on $M$, we have to deal with some special cases:

- $M = 1$
  This case can be checked by hand. Note that $S$ can contain at most four elements since not both $(0,1)$ and $(0,-1)$ are allowed.
- $m = 2$
  This follows since among two vectors there is only one possible pair and $2m-3 = 1$.
- $S \subset \{(x,y) \in \mathbb{Z}^2 : |x| = M \text{ or } |y| = M\}$ and $M \geq 2$
  By $\det\left(\binom{a}{\pm M}, \binom{b}{\pm M}\right) \in M\mathbb{Z}$ and $\det\left(\binom{M}{a}, \binom{M}{b}\right) \in M\mathbb{Z}$ for all $a, b \in \mathbb{Z}$, a determinant-one-pair of vectors must differ in absolute value in either the first or the second coordinate. On the other hand,
  $$\left|\det\left(\begin{pmatrix}\pm M\\ a\end{pmatrix}, \begin{pmatrix} b \\ \pm M\end{pmatrix}\right)\right| = \left|\pm M^2 - ab\right| \geq \left|M^2 - |ab|\right| \geq 2M-1 > 1$$
  for all $a, b \in \mathbb{Z}$ with $|a|, |b| \leq M-1$. Therefore, we have $\beta(S) = 0$.
- $M \geq 2$ and $S \setminus \{z_i\} \subset \{(x,y) \in \mathbb{Z}^2 : |x| = M \text{ or } |y| = M\}$
  By the previous observation, determinant-one-pairs must contain $z_i$ and therefore $\beta(S) \leq m-1$.
- If $z_i$ has both coordinates positive and $z_j$ has one positive and one negative coordinate, then $|\det(z_i, z_j)| = 1$ can only hold if $z_i, z_j \in \{(x,y) \in \mathbb{Z}^2 : |x| \leq 1, |y| \leq 1\}$.

Now, by these five observations, we can assume that $M \geq 2$, $m \geq 3$ and we can relabel $S = \{z_1, \ldots, z_t, z_{t+1}, \ldots, z_m\}$ such that $t \geq 2$ and $\{z_1, \ldots, z_t\} \subset \{(x,y) \in \mathbb{Z}^2 : |x| \leq M-1, |y| \leq M-1\}$ and $\{z_{t+1}, \ldots, z_m\} \subset \{(x,y) \in \mathbb{Z}^2 : |x| = M \text{ or } |y| = M\}$. Thus $\beta(\{z_{t+1}, \ldots, z_m\}) = 0$ and by the induction hypothesis, we have $\beta(\{z_1, \ldots, z_t\}) \leq 2t-3$. Due to Lemma 3.9, every vector $z_j$, $j = t+1, \ldots, m$, can have determinant $\pm 1$ with at most two other $z_i$'s. This yields $\beta(S) \leq \beta(\{z_1, \ldots, z_t\}) + 2(m-t) \leq 2m-3$, and we are done. □

We need some preparation to investigate the same question in higher dimensions. For lattice vectors $a_1, \ldots, a_{n-1} \in \mathbb{Z}^n$ and an arbitrary $x \in \mathbb{Z}^n$, we consider the matrix $A(x) = (a_1, \ldots, a_{n-1}, x)$. Let $A_{ij}(x)$ be the

$(n-1)\times(n-1)$ submatrix obtained from $A(x)$ by deleting the $i$th row and the $j$th column. Furthermore, for every $j \in \{1,\dots,n\}$, we write

$$v_j(x) = \big((-1)^{1+j}\det A_{1j}(x), \dots, (-1)^{n+j}\det A_{nj}(x)\big) \in \mathbb{Z}^n.$$

We stress that $v_n(x)$ is independent of $x$ by writing $v_n = v_n(x)$.

**Lemma 3.11.** *Let $a_1,\dots,a_{n-1} \in \mathbb{Z}^n$ and, for all $x \in \mathbb{Z}^n$ and $j \in \{1,\dots,n\}$, let $v_j(x)$ be defined as above. Then the equation $\det(a_1,\dots,a_{n-1},x) = 1$ has at most $3^{n-1}$ solutions $x \in \mathbb{Z}^n$ with the property that $\|v_j(x)\|_\infty \le \|v_n\|_\infty$ for all $j = 1,\dots,n-1$.*

Proof. When the $a_i$ are chosen such that there is no solution to the equation, the claim of the lemma clearly holds. So we assume that there is at least one solution $x \in \mathbb{Z}^n$ of $\det(a_1,\dots,a_{n-1},x) = 1$. In particular, this means that $\{a_1,\dots,a_{n-1},x\}$ is a basis of $\mathbb{Z}^n$. By the Laplace expansion formula, we have

$$\begin{aligned}1 &= \det(a_1,\dots,a_{n-1},x)\\ &= (-1)^{n+1}\det A_{1n}(x)\,x_1 + \ldots + (-1)^{n+n}\det A_{nn}(x)\,x_n\\ &= v_n(x)^\intercal x = v_n^\intercal x.\end{aligned}$$

Therefore, for every other solution $y \in \mathbb{Z}^n$, we have $v_n^\intercal(y-x) = 0$. By the definition $v_n^\intercal a_i = 0$ for each $i = 1,\dots,n-1$, and thus $\operatorname{lin}\{v_n\}^\perp = \operatorname{lin}\{a_1,\dots,a_{n-1}\}$. Since $\{a_1,\dots,a_{n-1}\}$ is part of a basis of $\mathbb{Z}^n$, we have $\operatorname{lin}\{a_1,\dots,a_{n-1}\} \cap \mathbb{Z}^n = \mathbb{Z}\,a_1 + \dots + \mathbb{Z}\,a_{n-1}$. We conclude that $y = x + k_1a_1 + \dots + k_{n-1}a_{n-1}$ for suitable $k_i \in \mathbb{Z}$.

From its definition, we see that $v_j(x)$ is a linear function in $x \in \mathbb{R}^n$ and that $v_j(a_i) = 0$ for all $i,j = 1,\dots,n-1$ with $i \ne j$. Thus $v_j(y) = v_j(x) + k_jv_j(a_j)$ for all $j = 1,\dots,n-1$, and by $\|v_j(a_j)\|_\infty = \|v_n\|_\infty$ we see that there are at most 3 possibilities for $k_j \in \mathbb{Z}$ such that $\|v_j(y)\|_\infty \le \|v_n\|_\infty$. This leaves us with at most $3^{n-1}$ possible solutions $y = x + k_1a_1 + \ldots + k_{n-1}a_{n-1}$ that fulfill the required conditions. □

**Theorem 3.12.** *Let $S = \{z_1,\dots,z_m\} \subset \mathbb{Z}^n$ with $m \ge n$ and write*

$$\beta(S) = \#\left\{I \in \tbinom{[m]}{n} : |\det(z_i : i \in I)| = 1\right\}.$$

*Then $\beta(S) \le 3^{n-1}\left(\binom{m}{n-1} - 1\right) \in O(m^{n-1})$ and the order of magnitude is best possible in the dependence on $m$.*

Proof. For a subset $J \in \binom{[m]}{n-1}$ let $M_J = \|\,(\det Z_1(J),\dots,\det Z_n(J))\,\|_\infty$ where $Z_i(J)$ is the $(n-1)\times(n-1)$ matrix obtained from the matrix $(z_j)_{j\in J}$ by deleting the $i$th row. Let $\binom{[m]}{n-1} = \{J_1,\dots,J_{\binom{m}{n-1}}\}$ be indexed such that $M_{J_1} \le M_{J_2} \le \ldots \le M_{J_{\binom{m}{n-1}}}$. For $k = 1,\dots,\binom{m}{n-1}$, we write $S_k = \bigcup_{t=1}^k \{z_j : j \in J_t\}$. Let $k \ge 2$. Then, by Lemma 3.11, there are at most $3^{n-1}$ elements $x \in S_{k-1}$ such that $|\det(z_{j_1},\dots,z_{j_{n-1}},x)| = 1$, where $J_k = \{j_1,\dots,j_{n-1}\}$. Every such solution gives rise to an $n$-tuple which is counted by $\beta(S)$.

Conversely, every $n$-tuple that is counted by $\beta(S)$ is “found” by this procedure. To see this, we assume that $|\det(z_{i_1},\dots,z_{i_n})| = 1$, and, for every

$j = 1, \ldots, n$, we define $I_j = \{i_1, \ldots, i_n\} \setminus \{i_j\} \in \binom{[m]}{n-1}$. We further assume that we labeled the lattice vectors $z_{i_j}$ such that $M_{I_1} \leq \ldots \leq M_{I_n}$. The described procedure counts the $n$-tuple $\{z_{i_1}, \ldots, z_{i_n}\}$ as soon as $I_n$ is considered.

Conclusively, since $S = S_{\binom{m}{n-1}}$, we get

$$\beta(S) \leq 3^{n-1}\left(\binom{m}{n-1} - 1\right) \in O(m^{n-1}).$$

Let us see why the given order of magnitude is best possible. For $m \geq 1$, let the set $S_m^n \subset \mathbb{Z}^n$ be given by

$$S_m^n = \left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ m-1 \\ \vdots \\ m-1 \\ m-1 \\ m \end{pmatrix}, \begin{pmatrix} 1 \\ m-1 \\ \vdots \\ m-1 \\ m \\ m \end{pmatrix}, \ldots, \begin{pmatrix} 1 \\ m \\ \vdots \\ m \\ m \\ m \end{pmatrix} \right\}.$$

For fixed $k = 1, \ldots, m$, we compute the determinant

$$\begin{aligned} D_k^{k_2,\ldots,k_n} &= \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ k & k_2 & k_3 - 1 & \ldots & k_n - 1 \\ k & k_2 & k_3 & \ldots & k_n - 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k & k_2 & k_3 & \ldots & k_n - 1 \\ k & k_2 & k_3 & \ldots & k_n \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & 0 & -1 & \ldots & -1 \\ 0 & 0 & 0 & \ldots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -1 \\ k & k_2 & k_3 & \ldots & k_n \end{vmatrix} \\ &= (-1)^{n+1}k \begin{vmatrix} 1 & 1 & \ldots & 1 \\ 0 & -1 & \ldots & -1 \\ 0 & 0 & \ldots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -1 \end{vmatrix} + (-1)^{n+2}k_2 \begin{vmatrix} 1 & 1 & \ldots & 1 \\ 0 & -1 & \ldots & -1 \\ 0 & 0 & \ldots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -1 \end{vmatrix} \\ &= -k + k_2. \end{aligned}$$

Therefore, $D_k^{k-1,k_3,\ldots,k_n} = -1$ for all $k_3, \ldots, k_n \in \{1, \ldots, m\}$. This means that $\beta(S_m^n) \geq \sum_{k=1}^m m^{n-2} = m^{n-1}$, which is of the order $|S_m^n|^{n-1}$ since $|S_m^n| = (n-1)m + 1$. $\square$

**3.2.2. A Special Case of Conjecture 3.2 in $\mathbb{R}^3$.** After this excursus, we come back to the study of Blichfeldt-type inequalities in small dimensions. McMullen [McM79] proved that the first Ehrhart coefficient $\mathrm{g}_1$ is a linear functional. This property enables us to derive a useful bound on $\mathrm{g}_1$ in terms of the volume for difference bodies $\mathcal{D}P = P - P$ of lattice polytopes $P$.

**Proposition 3.13.** *Let $P \in \mathcal{P}^n$ be a lattice polytope. Then*

$$\mathrm{g}_1(\mathcal{D}P) \leq \frac{(n-1)!}{2^{n-1}} \operatorname{vol}(\mathcal{D}P) + 2H_{n-1},$$

*where $H_k = \sum_{i=1}^k \frac{1}{i}$ denotes the $k$th harmonic number.*

Proof. Betke and McMullen [BM85] showed that

$$\mathrm{g}_1(P) \le (-1)^{n-1} s(n,1)\,\mathrm{vol}(P) + (-1)^{n-2}\frac{s(n,2)}{(n-1)!}.$$

Here, $s(n,j)$ is the $j$th Stirling number of the first kind, that is, the coefficient in front of $t^j$ in $\prod_{i=0}^{n}(t-i)$ (see [Sta97] for details on Stirling numbers). By $s(n,1) = (-1)^{n-1}(n-1)!$ and $s(n,2) = (-1)^n(n-1)!H_{n-1}$, we get $\mathrm{g}_1(P) \le (n-1)!\,\mathrm{vol}(P) + H_{n-1}$. McMullen [McM79] proved that $\mathrm{g}_1$ is a linear functional on the class of lattice polytopes, and therefore we have $\mathrm{g}_1(\mathcal{D}P) = \mathrm{g}_1(P) + \mathrm{g}_1(-P) = 2\,\mathrm{g}_1(P)$. By the Inequality (1.1), we have $\mathrm{vol}(P) \le \frac{1}{2^n}\,\mathrm{vol}(\mathcal{D}P)$, and thus

$$\begin{aligned}\mathrm{g}_1(\mathcal{D}P) &= 2\,\mathrm{g}_1(P) \le 2(n-1)!\,\mathrm{vol}(P) + 2H_{n-1}\\ &\le \frac{(n-1)!}{2^{n-1}}\,\mathrm{vol}(\mathcal{D}P) + 2H_{n-1}.\end{aligned}$$

□

In $\mathbb{R}^3$ this inequality leads to a confirmation of Conjecture 3.2 on the class of difference bodies of lattice polytopes.

**Proposition 3.14.** *Let $P \in \mathcal{P}^3$ be a lattice polytope. Then*

$$\mathrm{vol}(\mathcal{D}P) \ge \frac{2}{3}\left(\mathrm{G}(\operatorname{int}\mathcal{D}P) + 1\right).$$

Proof. Comparing coefficients in

$$\mathrm{G}(kP) = \sum_{i=0}^{n}\binom{k+n-i}{n}\mathrm{a}_i(P) = \sum_{i=0}^{n}\mathrm{g}_i(P)k^i,$$

for $n = 3$, yields that for every lattice polytope $P \in \mathcal{P}^3$, we have

$$\mathrm{g}_1(P) = \mathrm{a}_3(P) + \mathrm{g}_2(P) - \mathrm{g}_3(P) + 1 \text{ and } \mathrm{g}_2(P) = 1 + \frac{1}{2}\,\mathrm{a}_1(P) - \frac{1}{2}\,\mathrm{a}_3(P).$$

Proposition 3.13 says that

$$\mathrm{g}_1(\mathcal{D}P) \le \frac{1}{2}\,\mathrm{vol}(\mathcal{D}P) + 3,$$

which in view of the above relations and $\mathrm{a}_1(\mathcal{D}P) = \mathrm{G}(\mathcal{D}P) - 4, \mathrm{a}_3(\mathcal{D}P) = \mathrm{G}(\operatorname{int}\mathcal{D}P), \mathrm{g}_3(\mathcal{D}P) = \mathrm{vol}(\mathcal{D}P)$, leads to $\mathrm{a}_1(\mathcal{D}P) + \mathrm{a}_3(\mathcal{D}P) \le 3\,\mathrm{g}_3(\mathcal{D}P) + 2$. Equivalently,

$$2\,\mathrm{G}(\operatorname{int}\mathcal{D}P) + 2 \le 3\,\mathrm{vol}(\mathcal{D}P) + 8 - \mathrm{G}(\partial\mathcal{D}P). \tag{3.9}$$

Now we show that $\mathrm{G}(\partial\mathcal{D}P) \ge 8$ for every lattice polytope $P \in \mathcal{P}^3$. Assume to the contrary, that $\mathrm{G}(\partial\mathcal{D}P) = 6$. Then $\mathcal{D}P = P - P$ is a crosspolytope. Studies by Grünbaum [Grü03, Sect. 15.1] on the decomposability of convex polytopes with respect to Minkowski addition imply that $P$ itself is a lattice crosspolytope. Thus, $P = x + \mathrm{conv}\{\pm v_1, \pm v_2, \pm v_3\}$ for some $x, v_1, v_2, v_3 \in \mathbb{R}^3$ with $x \pm v_i \in \mathbb{Z}^3$ for all $i = 1,2,3$. Therefore, for every $j = 1,2,3$, the $j$th coordinate of $v_1, v_2$ and $v_3$ is either half-integral for all the three or integral for all the three. So not only the vertices of $\mathcal{D}P = \mathrm{conv}\{\pm 2v_1, \pm 2v_2, \pm 2v_3\}$ are lattice vectors, but also the midpoints $\frac{\pm 2v_i \pm 2v_j}{2}$ of its edges. This contradicts the assumption $\mathrm{G}(\partial\mathcal{D}P) = 6$.

Conclusively, with (3.9) we arrive at $2\,\mathrm{G}(\operatorname{int}\mathcal{D}P) + 2 \le 3\,\mathrm{vol}(\mathcal{D}P)$, which is equivalent to the claimed inequality. □

**Remark 3.15.** *The above proof shows that not every centrally symmetric lattice polytope in $\mathcal{P}^3$ is covered by Proposition 3.14. The crosspolytopes $C^\star_{3,l} = \operatorname{conv}\{\pm le_1, \pm e_2, \pm e_3\}$ for example have exactly 6 boundary lattice points for each $l \in \mathbb{N}$. But we have seen that $\mathrm{G}(\partial \mathcal{D}P) \geq 8$ for all lattice polytopes $P \in \mathcal{P}^3$.*

## 3.3. Blichfeldt-type Inequalities on Special Classes

In this section, we investigate Blichfeldt-type inequalities on special classes of centrally symmetric convex bodies. We derive best possible results for lattice crosspolytopes and lattice zonotopes in each case with respect to both the number of interior lattice points and the number of all lattice points. For the class of lattice spanning ellipsoids, we find useful yet nonsharp bounds.

**3.3.1. Lattice Crosspolytopes.** Let us start with the class of lattice crosspolytopes. These polytopes are minimal in $\mathcal{P}^n_0$ with respect to containment and the number of vertices and therefore it is desirable to understand them thoroughly.

We aim at proving the following theorem. The first part provides an alternative argument for the original result by Bey, Henk and Wills [BHW07, Prop. 1.4] who used a method of Beck and Sottile [BS07] for the description of the a-vector of a lattice polytope.

**Theorem 3.16** ([HHW11]). *Let $C \in \mathcal{P}^n_0$ be a lattice crosspolytope. Then*

*i)* $\operatorname{vol}(C) \geq \frac{2^{n-1}}{n!}\left(\mathrm{G}(\operatorname{int} C) + 1\right)$ *and*

*ii)* $\operatorname{vol}(C) \geq \frac{2^{n-2}}{n!}\left(\mathrm{G}(C) - (2n-3)\right)$.

*The standard crosspolytope $C^\star_n = \operatorname{conv}\{\pm e_1, \ldots, \pm e_n\}$ shows that both inequalities are tight.*

In the following, $C = \operatorname{conv}\{\pm a_1, \ldots, \pm a_n\}$ denotes a lattice crosspolytope, so the $a_i \in \mathbb{Z}^n$ are linearly independent lattice points. Our arguments are based on a decomposition of $C$ into handy lattice simplices, for which we have to introduce some notation.

For $\delta \in \{0, \pm 1\}^n$, let $\operatorname{supp}(\delta) = \{i \in [n] : \delta_i \neq 0\}$. We consider the simplex $S_\delta = \operatorname{conv}\{0, \delta_1 a_1, \ldots, \delta_n a_n\}$ and its face $F_\delta = \operatorname{conv}\{\delta_i a_i : \delta_i \neq 0\}$. The simplices $\{S_\varepsilon : \varepsilon \in \{\pm 1\}^n\}$ define a triangulation of the crosspolytope $C$ and, by definition, $S_\delta$ is a face of $S_\varepsilon$ if and only if $\operatorname{supp}(\delta) \subseteq \operatorname{supp}(\varepsilon)$ and $\delta_i = \varepsilon_i$ for all $i \in \operatorname{supp}(\delta)$. Therefore, every relative interior point of $S_\delta$ and $F_\delta$ for some $\delta \in \{0, \pm 1\}^n$ with $\# \operatorname{supp}(\delta) = i$, is contained in exactly $2^{n-i}$ full-dimensional simplices $S_\varepsilon$. We write $\tilde{S}_\delta = \operatorname{relint} S_\delta \cup \operatorname{relint} F_\delta$,

$$\mathcal{S}_i = \bigcup_{\substack{\delta \in \{0,\pm 1\}^n \\ \# \operatorname{supp}(\delta) = i}} \operatorname{relint} S_\delta \quad \text{and} \quad \tilde{\mathcal{S}}_i = \bigcup_{\substack{\delta \in \{0,\pm 1\}^n \\ \# \operatorname{supp}(\delta) = i}} \tilde{S}_\delta.$$

With this notation, we have $\mathrm{G}(\operatorname{int} C) = \sum_{i=0}^n \mathrm{G}(\mathcal{S}_i)$, $\mathrm{G}(C) = \sum_{i=0}^n \mathrm{G}(\tilde{\mathcal{S}}_i)$. Moreover, writing

$$\Delta_i(\varepsilon) = \{\delta \in \{0, \pm 1\}^n : \# \operatorname{supp}(\delta) = i,\ S_\delta \text{ a face of } S_\varepsilon\}$$

for all $i = 0, \dots, n$ and $\varepsilon \in \{\pm 1\}^n$, we have

$$\mathrm{G}(\mathcal{S}_i) = \frac{1}{2^{n-i}} \sum_{\varepsilon \in \{\pm 1\}^n} \sum_{\delta \in \Delta_i(\varepsilon)} \mathrm{G}(\operatorname{relint} S_\delta) \quad \text{and} \tag{3.10}$$

$$\mathrm{G}(\tilde{\mathcal{S}}_i) = \frac{1}{2^{n-i}} \sum_{\varepsilon \in \{\pm 1\}^n} \sum_{\delta \in \Delta_i(\varepsilon)} \mathrm{G}(\tilde{S}_\delta). \tag{3.11}$$

The method of our proof is to attach the simplices $S_\delta$ to the vertices of the parallelepiped $P_C = \sum_{i=1}^n [0, a_i]$. To this end, let $v_\gamma = \sum_{i=1}^n \gamma_i a_i$ for each $\gamma \in \{0,1\}^n$. Afterwards we apply a formula for the lattice points in $P_C$ and then cautiously identify lattice points in $C$ and $P_C$.

More precisely, we define $f : \mathbb{R}^n \to \mathbb{R}^n$ to be the map $f_i(x) = \frac{1-x_i}{2}$ for all $x \in \mathbb{R}^n$ and $i = 1, \dots, n$. For every $\varepsilon \in \{\pm 1\}^n$, we have $v_{f(\varepsilon)} + S_\varepsilon \subset P_C$. Indeed, the vertices of the simplex $v_{f(\varepsilon)} + S_\varepsilon$ are given by $\sum_{j=1}^n \frac{1-\varepsilon_j}{2} a_j$ and $\sum_{j=1}^n \frac{1-\varepsilon_j}{2} a_j + \varepsilon_i a_i$ for $i = 1, \dots, n$, which can be seen to be vertices of $P_C$. Thus, the simplices in $\mathcal{T}(C) = \left\{ v_{f(\varepsilon)} + S_\varepsilon : \varepsilon \in \{\pm 1\}^n \right\}$ are spanned by vertices of $P_C$. Figure 3.2 illustrates this rearrangement of the simplices $S_\varepsilon$ in the planar case.

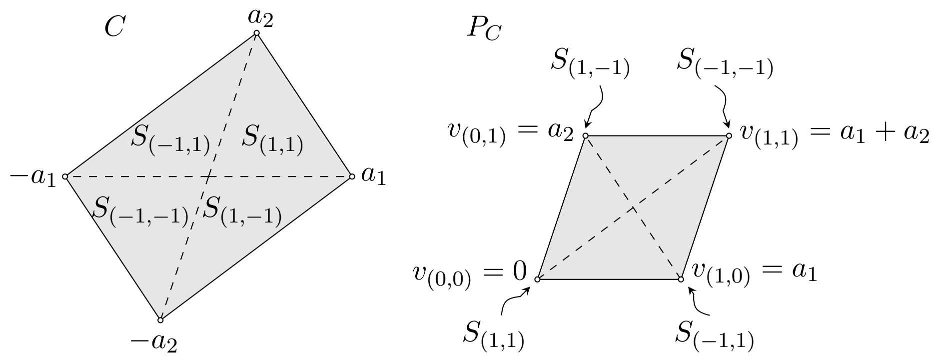



FIGURE 3.2. Rearrangement of the dissection of $C$ in $P_C$.

The following lemma shows how relative interior points of $P_C$ are covered by relative interior points of $v_{f(\varepsilon)} + S_\varepsilon$ and $v_{f(\varepsilon)} + F_\varepsilon$.

**Lemma 3.17.** *Let $\varepsilon, \varepsilon' \in \{\pm 1\}^n$ and, for some $i = 1, \dots, n$, let $\delta \in \Delta_i(\varepsilon)$ and $\delta' \in \Delta_i(\varepsilon')$.*

*i) If $\operatorname{relint}\left(v_{f(\varepsilon)} + S_\delta\right) \cap \operatorname{relint}\left(v_{f(\varepsilon')} + S_{\delta'}\right) \neq \emptyset$, then $\varepsilon$ and $\varepsilon'$ differ in at most one coordinate.*

*ii) If $\operatorname{relint}\left(v_{f(\varepsilon)} + S_\delta\right) \cap \operatorname{relint}\left(v_{f(\varepsilon')} + F_{\delta'}\right) \neq \emptyset$, then $\varepsilon$ and $\varepsilon'$ differ in at most one coordinate.*

*iii) If $\operatorname{relint}\left(v_{f(\varepsilon)} + F_\delta\right) \cap \operatorname{relint}\left(v_{f(\varepsilon')} + F_{\delta'}\right) \neq \emptyset$ and $i \geq 3$, then $\varepsilon$ and $\varepsilon'$ differ in at most one coordinate. For $i \leq 2$, there is no restriction on $\varepsilon$ and $\varepsilon'$.*

PROOF. First of all, it is no restriction to only consider the case $i = n$, because if there is an intersection in part i), ii) or iii), then this happens in a fixed $i$-face of $P_C$ which forces already $n - i$ coordinates of $\varepsilon$ and $\varepsilon'$ to be the same. Moreover, we only give the arguments for i), since parts ii) and iii) are analogous. By assumption, there is an $x \in \operatorname{int}\left(v_{f(\varepsilon)} + S_\varepsilon\right) \cap \operatorname{int}\left(v_{f(\varepsilon')} + S_{\varepsilon'}\right)$. This point has two representations with barycentric coordinates of the vertices of $v_{f(\varepsilon)} + S_\varepsilon$ and $v_{f(\varepsilon')} + S_{\varepsilon'}$, respectively. That is, there are $\alpha, \beta \in (0,1)^{n+1}$ with $\sum_{i=0}^n \alpha_i = \sum_{i=0}^n \beta_i = 1$ such that

$$\begin{aligned} x &= \alpha_0 \sum_{j=1}^n \frac{1-\varepsilon_j}{2} a_j + \sum_{i=1}^n \alpha_i \left( \sum_{j=1}^n \frac{1-\varepsilon_j}{2} a_j + \varepsilon_i a_i \right) \\ &= \beta_0 \sum_{j=1}^n \frac{1-\varepsilon'_j}{2} a_j + \sum_{i=1}^n \beta_i \left( \sum_{j=1}^n \frac{1-\varepsilon'_j}{2} a_j + \varepsilon'_i a_i \right). \end{aligned}$$

Collecting for the $a_i$'s and using $\sum_{i=0}^n \alpha_i = \sum_{i=0}^n \beta_i = 1$ gives

$$\sum_{i=1}^n \left( \frac{1-\varepsilon_i}{2} + \alpha_i \varepsilon_i \right) a_i = \sum_{i=1}^n \left( \frac{1-\varepsilon'_i}{2} + \beta_i \varepsilon'_i \right) a_i.$$

Since the $a_i$'s are linearly independent, these representations coincide and we get $(2\alpha_i - 1)\varepsilon_i = (2\beta_i - 1)\varepsilon'_i$ for all $i = 1, \ldots, n$. The coordinates of $\varepsilon$ and $\varepsilon'$ are either 1 or $-1$, and so $\alpha_i = \beta_i$ whenever $\varepsilon_i = \varepsilon'_i$, and $\alpha_j = 1 - \beta_j$ whenever $\varepsilon_j = -\varepsilon'_j$. We now relabel the indices such that $\varepsilon_i = \varepsilon'_i$ for $i = 1, \ldots, k$, and $\varepsilon_j = -\varepsilon'_j$ for $j = k+1, \ldots, n$. This means that $k$ is the number of coordinates where $\varepsilon$ and $\varepsilon'$ agree. Exploiting $\sum_{i=0}^n \alpha_i = \sum_{i=0}^n \beta_i = 1$ and $\alpha_i, \beta_i > 0$, we obtain

$$\begin{aligned} 1 &= \sum_{i=0}^n \alpha_i = \alpha_0 + \sum_{i=1}^k \beta_i + \sum_{j=k+1}^n (1 - \beta_j) \\ &= \alpha_0 + \sum_{i=1}^k \beta_i - \sum_{j=k+1}^n \beta_j + n - k = \alpha_0 + \beta_0 - 1 + 2\sum_{i=1}^k \beta_i + n - k \\ &> n - k - 1. \end{aligned}$$

Because of $k \geq n-1$, the vectors $\varepsilon$ and $\varepsilon'$ differ in at most one coordinate. $\square$

**Lemma 3.18.** *Let $P \in \mathcal{P}^n$ be a lattice parallelepiped and let $\mathcal{F}_i(P)$ be the union of all $i$-faces of $P$ for each $i \in \{0, \ldots, n\}$. Then*

$$2^n \operatorname{vol}(P) = \sum_{i=0}^n 2^i \operatorname{G}(\operatorname{relint} \mathcal{F}_i(P)).$$

Proof. Since the claimed equality is invariant under lattice translations of $P$, we can assume that $P = \sum_{i=1}^n [0, v_i]$ for some linearly independent $v_1, \ldots, v_n \in \mathbb{Z}^n$. By the volume formula (1.7) for lattice parallelepipeds, we have $\operatorname{vol}(P) = \#\left(\sum_{i=1}^n [0, v_i) \cap \mathbb{Z}^n\right)$. At any vertex of $P$, we now place a copy of $P$ which is generated by the edges that emanate from that vertex. In this way, we count the lattice points in $P$ with a multiplicity that we can determine. Indeed, the lattice points in the relative interior of an $i$-face $F$ of $P$ are covered exactly $2^i$ times, since $F$ has that many vertices and a copy of $P$ contributes relative interior lattice points to $F$ if and only if it was placed at a vertex of $F$. Furthermore, we get $2^n$ times the volume of $P$ in this way and the desired formula follows. $\square$

Similarly to the definition of $\mathcal{F}_i(P)$ in the preceding lemma, we write

$$\mathcal{T}_i(C) = \bigcup_{\varepsilon\in\{\pm1\}^n} \bigcup_{\delta\in\Delta_i(\varepsilon)} \operatorname{relint}(v_{f(\varepsilon)} + S_\delta) \quad \text{and}$$
$$\tilde{\mathcal{T}}_i(C) = \bigcup_{\varepsilon\in\{\pm1\}^n} \bigcup_{\delta\in\Delta_i(\varepsilon)} (v_{f(\varepsilon)} + \tilde{S}_\delta).$$

By construction, we have $\mathcal{T}_i(C) \subset \tilde{\mathcal{T}}_i(C) \subseteq \operatorname{relint} \mathcal{F}_i(P_C)$ for all $i = 2, \dots, n$. Furthermore,

$$\mathcal{T}_0(C) = \tilde{\mathcal{T}}_0(C) = \operatorname{relint} \mathcal{F}_0(P_C) = \operatorname{vert} P_C,$$
$$\mathcal{T}_1(C) = \operatorname{relint} \mathcal{F}_1(P_C) \quad \text{and} \quad \tilde{\mathcal{T}}_1(C) = \operatorname{relint} \mathcal{F}_1(P_C) \cup \operatorname{relint} \mathcal{F}_0(P_C).$$

In the following figure, we depict the regions $\mathcal{T}_i(C)$ for a planar crosspolytope.

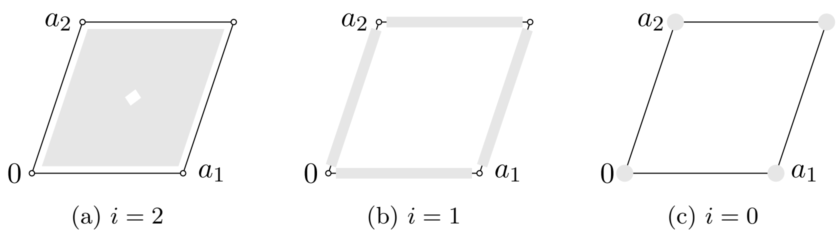



FIGURE 3.3. The gray regions are $\mathcal{T}_i(C)$ for $C = \operatorname{conv}\{\pm a_1, \pm a_2\}$.

Now we are prepared to prove the main result of this part.

PROOF OF THEOREM 3.16. The volume of $C = \operatorname{conv}\{\pm a_1, \dots, \pm a_n\}$ is

$$\operatorname{vol}(C) = \frac{2^n}{n!} |\det(a_1, \dots, a_n)| = \frac{2^n}{n!} \operatorname{vol}(P_C).$$

Starting from Lemma 3.18, the first statement i) follows as

$$n! \operatorname{vol}(C) = \sum_{i=0}^{n} 2^i \operatorname{G}(\operatorname{relint} \mathcal{F}_i(P_C))$$

$$\geq \sum_{i=0}^{n} 2^i \operatorname{G}(\mathcal{T}_i(C)) \tag{3.12}$$

$$\geq \sum_{i=1}^{n} 2^i \left( \frac{1}{2} \sum_{\varepsilon\in\{\pm1\}^n} \sum_{\delta\in\Delta_i(\varepsilon)} \operatorname{G}(\operatorname{relint} S_\delta) \right) + 2^n \tag{3.13}$$

$$= \sum_{i=1}^{n} 2^{i-1} 2^{n-i} \operatorname{G}(\mathcal{S}_i) + 2^n = 2^{n-1} \left( \operatorname{G}(\operatorname{int} C) + 1 \right). \tag{3.14}$$

The line (3.12) is an inequality, because not every lattice point in $P_C$ has to be covered by some $v_{f(\varepsilon)} + S_\delta$. Inequality (3.13) follows from Lemma 3.17 i) and (3.14) comes from Equality (3.10).

For the second part, we start in the same manner as above and get

$$n!\operatorname{vol}(C) = 2^n \operatorname{vol}(P_C) = \sum_{i=0}^{n} 2^i \operatorname{G}(\operatorname{relint}\mathcal{F}_i(P_C))$$

$$\geq \sum_{i=2}^{n} 2^i \operatorname{G}(\tilde{\mathcal{T}}_i(C)) + 2\operatorname{G}(\operatorname{relint}\mathcal{F}_1(P_C)) + 2^n. \tag{3.15}$$

Lemma 3.17 implies that, for all $i \geq 2$, every lattice point in $\operatorname{relint}\mathcal{F}_i(P_C)$ is covered at most four times by simplices $v_{f(\varepsilon)} + \tilde{S}_\delta$ with $\#\operatorname{supp}(\delta) = i$. Therefore, by virtue of Equation (3.11), we have for all $i \geq 2$, that

$$\operatorname{G}(\tilde{\mathcal{T}}_i(C)) \geq \frac{1}{4} \sum_{\varepsilon\in\{\pm 1\}^n} \sum_{\delta\in\Delta_i(\varepsilon)} \operatorname{G}(\tilde{S}_\delta) = 2^{n-i-2}\operatorname{G}(\tilde{\mathcal{S}}_i).$$

Next, we need to consider the lattice points in the edges of $P_C$. We have

$$\begin{aligned}2\operatorname{G}(\operatorname{relint}\mathcal{F}_1(P_C)) &= \sum_{\varepsilon\in\{\pm 1\}^n} \sum_{\delta\in\Delta_1(\varepsilon)} \operatorname{G}(\operatorname{relint} S_\delta)\\ &= \sum_{\varepsilon\in\{\pm 1\}^n} \sum_{\delta\in\Delta_1(\varepsilon)} \Big(\operatorname{G}(\tilde{S}_\delta) - 1\Big) = 2^{n-1}\operatorname{G}(\tilde{\mathcal{S}}_1) - n2^n.\end{aligned}$$

With these two relations we can now continue Inequality (3.15) by

$$\begin{aligned}n!\operatorname{vol}(C) &\geq 2^{n-2}\sum_{i=2}^{n}\operatorname{G}(\tilde{\mathcal{S}}_i) + 2^{n-1}\operatorname{G}(\tilde{\mathcal{S}}_1) - n2^n + 2^n\\ &= 2^{n-2}\Big(\operatorname{G}(C) - \operatorname{G}(\tilde{\mathcal{S}}_1) - \operatorname{G}(\tilde{\mathcal{S}}_0)\Big) + 2^{n-1}\operatorname{G}(\tilde{\mathcal{S}}_1) - n2^n + 2^n\\ &\geq 2^{n-2}\left(\operatorname{G}(C) - (2n-3)\right).\end{aligned}$$

The last inequality follows since, by the definition of $\tilde{S}_\delta$, we have $\operatorname{G}(\tilde{S}_\delta) = 1$ for the case of $\#\operatorname{supp}(\delta) = 0$, and $\operatorname{G}(\tilde{S}_\delta) \geq 1$ for the case of $\#\operatorname{supp}(\delta) = 1$. Therefore $\operatorname{G}(\tilde{\mathcal{S}}_0) = 1$ and $\operatorname{G}(\tilde{\mathcal{S}}_1) \geq 2n$. ☐

**3.3.2. Lattice Zonotopes.** In the preceding part, we used the nice decomposition property of crosspolytopes to derive a sharp Blichfeldt-type inequality on that class. Also our method for lattice zonotopes relies on a decomposition into polytopes that can be easily handled. The result we want to prove is the following:

**Theorem 3.19** ([HHW11])**.** *Let $Z \in \mathcal{P}^n$ be a lattice zonotope.*

*i) Then*

$$\operatorname{vol}(Z) \geq \left(\frac{1}{2}\right)^{n-1}\left(\operatorname{G}(Z) - 2^{n-1}\right).$$

*Equality holds if and only if* $Z \simeq \sum_{i=1}^{n-1}[0, e_i] + [0, le_n]$ *for some* $l \in \mathbb{N}$.

*ii) If* $Z = \sum_{i=1}^{m}[-a_i, a_i]$*, for some* $a_1, \ldots, a_m \in \mathbb{Z}^n$*, then*

$$\operatorname{vol}(Z) \geq \left(\frac{2}{3}\right)^{n-1}\left(\operatorname{G}(Z) - 3^{n-1}\right).$$

*Equality holds if and only if* $Z \simeq \sum_{i=1}^{n-1}[-e_i, e_i]+[-le_n, le_n]$ *for some* $l \in \mathbb{N}$.

First, we consider lattice parallelepipeds. These are the building blocks to obtain the result for arbitrary lattice zonotopes.

**Lemma 3.20.** *Let* $P \in \mathcal{P}^n$ *be a lattice parallelepiped. Then*

$$\mathrm{vol}(P) \geq \left(\frac{1}{2}\right)^{n-1} \left(\mathrm{G}(P) - 2^{n-1}\right).$$

*Equality holds if and only if* $P \simeq \sum_{i=1}^{n-1}[0, e_i] + [0, le_n]$ *for some* $l \in \mathbb{N}$.

Proof. Using the notation from Lemma 3.18, we clearly have

$$\mathrm{G}(P) = \sum_{i=0}^{n} \mathrm{G}(\mathrm{relint}\, \mathcal{F}_i(P)) \quad \text{and} \quad \mathrm{G}(\mathrm{relint}\, \mathcal{F}_0(P)) = 2^n.$$

Thus, Lemma 3.18 yields

$$\begin{aligned} \mathrm{vol}(P) &= \sum_{i=0}^{n} 2^{i-n}\, \mathrm{G}(\mathrm{relint}\, \mathcal{F}_i(P)) \\ &\geq \left(\frac{1}{2}\right)^{n-1} \sum_{i=1}^{n} \mathrm{G}(\mathrm{relint}\, \mathcal{F}_i(P)) + 1 = \left(\frac{1}{2}\right)^{n-1} \left(\mathrm{G}(P) - 2^{n-1}\right). \end{aligned}$$

We also see that equality holds if and only if $\mathrm{G}(\mathrm{relint}\, \mathcal{F}_i(P)) = 0$ for all $i = 2, \dots, n$, which means that all lattice points of $P$ are contained in its edges. By a suitable lattice translation, we can write $P = \sum_{i=1}^{n}[0, a_i]$ for some linearly independent generators $a_i \in \mathbb{Z}^n$. Then there is at most one nonprimitive generator, say $a_n$, among the $a_i$'s. Indeed, if we assume that $a_1$ is also nonprimitive, then the interior lattice points in the segments $[0, a_1]$ and $[0, a_n]$ yield an interior lattice point in the 2-face $[0, a_1] + [0, a_n]$ of $P$, which has just been excluded.

So, there is a $k \in \mathbb{N}$ such that $a_n' = \frac{1}{k} a_n \in \mathbb{Z}^n$ is primitive and the half-open lattice parallelepiped $P' = \sum_{i=1}^{n-1}[0, a_i) + [0, a_n')$ contains exactly one lattice point. This means that the generators $a_1, \dots, a_{n-1}, a_n'$ span a basis of $\mathbb{Z}^n$, giving the claimed equality case characterization. □

The analogous lemma for parallelepipeds of the form $P = \sum_{i=1}^{n}[-a_i, a_i]$, with generators $a_1, \dots, a_n \in \Lambda \in \mathcal{L}^n$, can be proved by similar arguments. However, we present another method of proof that is also applicable to obtain Lemma 3.20. We formulate the lemma for arbitrary lattices $\Lambda \in \mathcal{L}^n$, clearing the path for an inductive argument.

**Lemma 3.21.** *Let* $\Lambda \in \mathcal{L}^n$ *and let* $P = \sum_{i=1}^{n}[-a_i, a_i]$ *be a lattice parallelepiped with generators* $a_1, \dots, a_n \in \Lambda$. *Then*

$$\frac{\mathrm{vol}(P)}{\det \Lambda} \geq \left(\frac{2}{3}\right)^{n-1} \left(\mathrm{G}(P, \Lambda) - 3^{n-1}\right).$$

*Equality holds if and only if* $P \simeq \sum_{i=1}^{n-1}[-b_i, b_i]+[-kb_n, kb_n]$ *for some* $k \in \mathbb{N}$, *where* $\{b_1, \dots, b_n\}$ *is a basis of* $\Lambda$.

Proof. In order to use an inductive argument, we consider, for $l = 1, \dots, n$, the $l$-dimensional lattice parallelepiped $P_l = \sum_{i=1}^{l}[-a_i, a_i]$ and the sublattice $\Lambda_l = \Lambda \cap \operatorname{lin} P_l$ of $\Lambda$. For the equality case characterization, let $\{b_1, \dots, b_l\}$ be a basis of $\Lambda_l$ and let $R_k^l = \sum_{i=1}^{l-1}[-b_i, b_i] + [-kb_l, kb_l]$ for some $k \in \mathbb{N}$.

For $l = 1$, it is straightforward that $\frac{\operatorname{vol}_1(P_1)}{\det \Lambda_1} = \mathrm{G}(P_1, \Lambda_1) - 1$ and that there is a $k \in \mathbb{N}$ such that $P_1 = R_k^1$. So let $l \geq 2$. Then, by the geometry of $P_l$, we have

$$\operatorname{vol}_l(P_l) = 2\, \|a_l|(\operatorname{lin} P_{l-1})^{\perp}\| \operatorname{vol}_{l-1}(P_{l-1}), \tag{3.16}$$

where $a_l|(\operatorname{lin} P_{l-1})^{\perp}$ is the orthogonal projection of $a_l$ onto the orthogonal complement $(\operatorname{lin} P_{l-1})^{\perp}$ of $\operatorname{lin} P_{l-1}$ that is taken in $\operatorname{lin} P_l$. Let $w_l \in \mathbb{N}$ be such that $2w_l + 1$ is the number of lattice hyperplanes in $\Lambda_l$ that are parallel to $\Lambda_{l-1}$ and intersect $P_l$. If we consider $\Lambda_{l-1} \subset \Lambda_l$ as a sublattice, then we can use Equation (1.3) and obtain

$$\det \Lambda_l = \det \Lambda_{l-1} \cdot \det(\Lambda_l|(\operatorname{lin} P_{l-1})^{\perp}) = \det \Lambda_{l-1} \cdot \frac{\|a_l|(\operatorname{lin} P_{l-1})^{\perp}\|}{w_l}. \tag{3.17}$$

By the relations (3.16), (3.17) and the induction hypothesis, we get

$$\begin{aligned} \frac{\operatorname{vol}_l(P_l)}{\det \Lambda_l} &= \frac{2\, \|a_l|(\operatorname{lin} P_{l-1})^{\perp}\| \operatorname{vol}_{l-1}(P_{l-1})}{\det \Lambda_{l-1} \cdot \frac{\|a_l|(\operatorname{lin} P_{l-1})^{\perp}\|}{w_l}} \\ &\geq 2w_l \left(\frac{2}{3}\right)^{l-2} \left(\mathrm{G}(P_{l-1}, \Lambda_{l-1}) - 3^{l-2}\right). \end{aligned}$$

This is greater than or equal to $\left(\frac{2}{3}\right)^{l-1}\left(\mathrm{G}(P_l, \Lambda_l) - 3^{l-1}\right)$ if and only if

$$3w_l\, \mathrm{G}(P_{l-1}, \Lambda_{l-1}) \geq \mathrm{G}(P_l, \Lambda_l) + 3^{l-1}(w_l - 1).$$

This inequality holds by $\mathrm{G}(P_{l-1}, \Lambda_{l-1}) \geq 3^{l-1}$ and $(2w_l + 1)\, \mathrm{G}(P_{l-1}, \Lambda_{l-1}) \geq \mathrm{G}(P_l, \Lambda_l)$, where the latter comes from counting lattice points in $P_l$ by the lattice hyperplanes that are parallel to $\Lambda_{l-1}$ and the relation

$$\mathrm{G}(t + P_{l-1}, \Lambda_{l-1}) \leq \mathrm{G}(P_{l-1}, \Lambda_{l-1}) \quad \text{for all} \quad t \in \operatorname{lin} P_{l-1}.$$

This last inequality follows by partitioning $P_{l-1}$ into half-open "subparallelepipeds" and using the well-known fact that $\mathrm{G}(t + Q) \in \{0, \mathrm{G}(Q)\}$ for every $t \in \mathbb{R}^n$ and every possibly lower-dimensional half-open lattice parallelepiped $Q$ (cf. [**HW08a**, Sect. 2]).

Equality is attained if and only if there is equality in the induction hypothesis and the equations $\mathrm{G}(P_{l-1}, \Lambda_{l-1}) = 3^{l-1}$ and $(2w_l + 1)\, \mathrm{G}(P_{l-1}, \Lambda_{l-1}) = \mathrm{G}(P_l, \Lambda_l)$ hold. This means that the generators $\{a_1, \dots, a_{l-1}\}$ of $P_{l-1}$ span a basis of $\Lambda_{l-1}$ and $a_l$ is an integral multiple of some $b_l \in \Lambda_l$ that completes that basis to a basis of $\Lambda_l$. In other words, $P_l$ is unimodularly equivalent to $R_k^l$ for some $k \in \mathbb{N}$. $\square$

**Remark 3.22.** *For lattice parallelepipeds whose center of symmetry is a lattice point, but which are not of the form $\sum_{i=1}^{n}[-a_i, a_i]$ for some $a_i \in \mathbb{Z}^n$, the inequality in Lemma 3.21 does not hold in general.*

*As an example, consider $P_k = [-kv_1, kv_1] + [-v_2, v_2] + [-v_3, v_3]$ with $v_1 = (\frac{1}{2}, \frac{1}{2}, 0)^{\intercal}, v_2 = (\frac{1}{2}, 0, \frac{1}{2})^{\intercal}$ and $v_3 = (0, \frac{1}{2}, \frac{1}{2})^{\intercal}$. $P_k$ is a lattice parallelepiped for odd $k \in \mathbb{N}$, and we have $\operatorname{vol}(P_k) = \frac{9}{10} \cdot \frac{k}{k-1}\left(\frac{2}{3}\right)^2 (\mathrm{G}(P_k) - 9)$.*

*Examples in arbitrary dimension are obtained by successively taking prisms over $P_k$. More precisely, for $P_k^n = P_k + \sum_{i=4}^n [-e_i, e_i]$, we have*

$$\mathrm{vol}(P_k^n) = \frac{9}{10} \cdot \frac{k}{k-1} \left(\frac{2}{3}\right)^{n-1} \left(\mathrm{G}(P_k^n) - 3^{n-1}\right) \quad \textit{for all odd} \quad k \geq 3.$$

Now that we familiarized ourselves with lattice parallelepipeds, we can use a dissection property of zonotopes in order to transfer the inequalities to arbitrary lattice zonotopes.

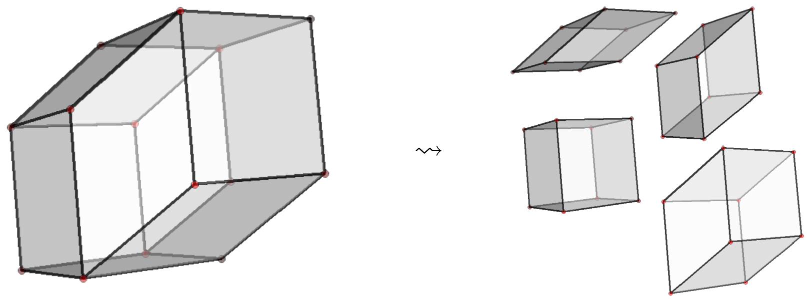


FIGURE 3.4. A dissection of a three-dimensional zonotope.

PROOF OF THEOREM 3.19. Let us prove part i). First of all, up to a lattice translation, every lattice zonotope has the form $Z = \sum_{i=1}^m [0, a_i]$ for some $a_1, \ldots, a_m \in \mathbb{Z}^n$ and $m \geq n$. Analogously to the planar case (cf. Theorem 3.7), Shephard's result [She74, Thm. (54)] gives a dissection $\mathcal{C}(Z)$ of $Z$ into parallelepipeds that are lattice translates of $\sum_{j=1}^n [0, a_{i_j}]$. Such a dissection can be obtained by starting with $\mathcal{C}_n(Z) = \left\{ \sum_{i=1}^n [0, a_i] \right\}$ and then successively processing the generators $\{a_j : j > n\}$ in order to obtain collections $\mathcal{C}_j(Z)$ of parallelepipeds in the following way: for a generator $a_j$, consider all the facets of parallelepipeds in $\mathcal{C}_{j-1}(Z)$ that can be "seen" by $a_j$, that is to say that there is a point on the ray $\{\lambda a_j : \lambda \geq 0\}$ that lies beyond the facet. These facets together with the segment $[0, a_j]$ generate the new parallelepipeds in $\mathcal{C}_j(Z) \setminus \mathcal{C}_{j-1}(Z)$. After having processed all the generators in this way, we let $\mathcal{C}(Z) = \mathcal{C}_m(Z)$. This process naturally induces an ordering $P_1, \ldots, P_t$ of the parallelepipeds in $\mathcal{C}(Z)$ such that for all $j \in \{2, \ldots, t\}$ the intersection $P_j \cap \left(\bigcup_{i=1}^{j-1} P_i\right)$ contains at least a facet of $P_j$.

In order to prove the claimed inequality for $Z = \bigcup_{i=1}^t P_i$, we proceed by induction on $t$. The case $t = 1$ is precisely Lemma 3.20. For $t \geq 2$, we write $Q_{t-1} = \bigcup_{i=1}^{t-1} P_i$. By the induction hypothesis and Lemma 3.20, we get

$$\begin{aligned} \mathrm{vol}(Z) &= \mathrm{vol}(P_t \cup Q_{t-1}) = \mathrm{vol}(P_t) + \mathrm{vol}(Q_{t-1}) \\ &\geq \left(\frac{1}{2}\right)^{n-1} \left(\mathrm{G}(P_t) - 2^{n-1} + \mathrm{G}(Q_{t-1}) - 2^{n-1}\right) \\ &\geq \left(\frac{1}{2}\right)^{n-1} \left(\mathrm{G}(Z) - 2^{n-1}\right). \end{aligned}$$

The last inequality holds since, as noted above, $P_t \cap Q_{t-1}$ contains at least a facet of $P_t$ and thus at least $2^{n-1}$ lattice points, which then are counted twice.

In order to derive the equality case characterization, let us assume, without loss of generality, that $t > 1$ and that no two of the generators of $Z$ are parallel. The argumentation above shows that equality can only hold if the intersection $P_t \cap Q_{t-1}$ is precisely a facet, say $F$, of $P_t$. By the construction of $\mathcal{C}(Z)$, there must be some $j \in \{1, \dots, t-1\}$ such that $P_t \cap P_j = F$. The zonotope $Z = P_t \cup Q_{t-1}$ is convex, which means that $P_j$ must be contained in the intersection of the half-spaces corresponding to the facets of $P_t$ that are not equal to $F$. But this is a contradiction since the set of generators of $P_t$ and $P_j$ must be different. Thus, equality can only hold for $t = 1$ and, by Lemma 3.20, $Z$ is unimodularly equivalent to the claimed parallelepiped.

Part ii) follows from Lemma 3.21 in a similar way as part i) above followed from Lemma 3.20. □

**Remark 3.23.** *The above proof shows that Theorem 3.19 holds in a stronger way. Let $Z \in \mathcal{P}^n$ be a zonotope with vertices in $l\mathbb{Z}^n$ for some $l \in \mathbb{N}$.*

*i) Then*

$$\mathrm{vol}(Z) \geq \left(\frac{l}{l+1}\right)^{n-1} \left(\mathrm{G}(Z) - (l+1)^{n-1}\right).$$

*Equality holds if and only if* $Z \simeq \sum_{i=1}^{n-1} [0, le_i] + [0, kle_n]$ *for some* $k \in \mathbb{N}$.

*ii) If $Z = \sum_{i=1}^{m} [-a_i, a_i]$, for some $a_1, \dots, a_m \in l\mathbb{Z}^n$, then*

$$\mathrm{vol}(Z) \geq \left(\frac{2l}{2l+1}\right)^{n-1} \left(\mathrm{G}(Z) - (2l+1)^{n-1}\right).$$

*Equality holds if and only if* $Z \simeq \sum_{i=1}^{n-1} [-le_i, le_i] + [-kle_n, kle_n]$ *for* $k \in \mathbb{N}$.

We conclude our investigations of Blichfeldt-type inequalities for lattice zonotopes with a result concerning the number of interior lattice points.

**Proposition 3.24.** *Let $Z \in \mathcal{P}^n$ be a lattice zonotope with $m \geq n$ pairwise nonparallel generators. Then*

$$\mathrm{vol}(Z) \geq \mathrm{G}(\mathrm{int}\, Z) + m - n + 1.$$

Proof. Since we use an inductive argument, we shall consider the claimed inequality stated for arbitrary lattices $\Lambda \in \mathcal{L}^n$, that is,

$$\frac{\mathrm{vol}(Z)}{\det \Lambda} \geq \mathrm{G}(\mathrm{int}\, Z, \Lambda) + m - n + 1.$$

After a suitable lattice translation, we find generators $a_1, \dots, a_m \in \Lambda$ such that $Z = \sum_{i=1}^{m} [0, a_j]$. We proceed by induction both on the dimension $n$ and on the number of generators $m$ of $Z$. In the case $n = 1$, there is nothing to show because there can only be one generator and $\frac{\mathrm{vol}(Z)}{\det \Lambda} = \mathrm{G}(\mathrm{int}\, Z, \Lambda) + 1$. When $m = n$, the zonotope $Z$ is a parallelepiped and the assertion holds by

$$\frac{\mathrm{vol}(Z)}{\det \Lambda} = \#\left(\sum_{i=1}^{n} [0, a_j) \cap \Lambda\right)$$

$$= \mathrm{G}(\operatorname{int} Z, \Lambda) + \sum_{i=1}^{n-1} \sum_{J \in \binom{[n]}{i}} \# \left( \sum_{j \in J} (0, a_j) \cap \Lambda \right) + 1$$
$$\geq \mathrm{G}(\operatorname{int} Z, \Lambda) + 1.$$

We have equality if and only if $\partial Z \cap \Lambda = \operatorname{vert} Z$.

Now, assume that $m \geq n+1$. We can find a generator, say $a_m$, of $Z$ such that $Z' = \sum_{i=1}^{m-1} [0, a_i]$ is full-dimensional. Writing $P_J = \sum_{j \in J} [0, a_j]$ for $J \in \binom{[m]}{n}$, we get, by the volume formula for zonotopes (cf. [She74, Equ. (57)]) and the induction hypothesis, that

$$\frac{\operatorname{vol}(Z)}{\det \Lambda} = \sum_{J \in \binom{[m]}{n}} \frac{\operatorname{vol}(P_J)}{\det \Lambda} = \frac{\operatorname{vol}(Z')}{\det \Lambda} + \sum_{J \in \binom{[m]}{n},\, m \in J} \frac{\operatorname{vol}(P_J)}{\det \Lambda}$$
$$\geq \mathrm{G}(\operatorname{int} Z', \Lambda) + (m-1) - n + 1 + \frac{\operatorname{vol}(Z|a_m^{\perp}) \cdot \|a_m\|}{\det \Lambda}. \tag{3.18}$$

Here, $a_m^{\perp}$ denotes the subspace orthogonal to $a_m$. Let $m' \geq n-1$ be the number of nonparallel generators of the projected zonotope $Z|a_m^{\perp}$. Then, by the determinant formula (1.3) for projected lattices and the induction hypothesis, we obtain

$$\frac{\operatorname{vol}(Z|a_m^{\perp}) \cdot \|a_m\|}{\det \Lambda} = \frac{\operatorname{vol}(Z|a_m^{\perp}) \cdot \|a_m\|}{\det(\Lambda|a_m^{\perp}) \det(\operatorname{lin}\{a_m\} \cap \Lambda)}$$
$$\geq \#([0, a_m) \cap \Lambda) \Big( \mathrm{G}(\operatorname{int} Z|a_m^{\perp}, \Lambda|a_m^{\perp}) + m' - (n-1) + 1 \Big)$$
$$\geq \mathrm{G}(\operatorname{int} Z \setminus \operatorname{int} Z', \Lambda) + \#([0, a_m) \cap \Lambda)$$
$$\geq \mathrm{G}(\operatorname{int} Z \setminus \operatorname{int} Z', \Lambda) + 1.$$

Figure 3.5 demonstrates this line of inequalities.

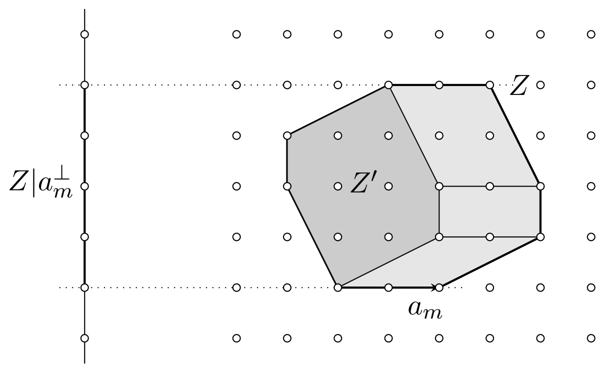



FIGURE 3.5

Together with (3.18), this yields the desired inequality as

$$\frac{\operatorname{vol}(Z)}{\det \Lambda} \geq \mathrm{G}(\operatorname{int} Z', \Lambda) + \mathrm{G}(\operatorname{int} Z \setminus \operatorname{int} Z', \Lambda) + m - n + 1$$
$$= \mathrm{G}(\operatorname{int} Z, \Lambda) + m - n + 1.$$

□

The above proof shows that, if $Z$ is a lattice parallelepiped with $\mathrm{vol}(Z) = \mathrm{G}(\mathrm{int}\, Z) + 1$, then $\partial Z \cap \mathbb{Z}^n = \mathrm{vert}\, Z$. A lattice polytope with this property is called *clean*. This condition seems to be quite restrictive for lattice zonotopes. It is therefore natural to ask how many generators a clean lattice zonotope can have. Together with Christian Bey and Gohar Kyureghyan (personal communication) we are able to give an almost complete answer.

**Proposition 3.25.** *Let $Z \in \mathcal{P}^n$ be a clean lattice zonotope with $m$ generators. Then*

$$m \leq \begin{cases} \infty & \text{for } n = 2, \\ 7 & \text{for } n = 3, \\ 8 & \text{for } n = 4, \\ n+1 & \text{for } n \geq 5. \end{cases}$$

*Moreover, the following sets of generators give clean lattice zonotopes*

$$n = 2: \quad \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 \\ m \end{pmatrix} \right\} \text{ for all } m \in \mathbb{N},$$

$$n = 3: \quad \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix} \right\},$$

$$n = 4: \quad \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -2 \\ -1 \end{pmatrix} \right\},$$

$$n \geq 5: \quad \{e_1, \dots, e_n, e_1 + \dots + e_n\}.$$

Proof. There exist exactly 8 different three-dimensional lattice vectors modulo 2. So, if a lattice zonotope $Z \in \mathcal{P}^3$ has at least 8 generators, then either one of them is divisible by 2, which leads to an interior lattice point in an edge of $Z$, or two of them are equal modulo 2, which means that the midpoint of the parallelogram that is spanned by these two generators is an interior lattice point of a 2-face of $Z$. In both cases $Z$ cannot be clean.

Let $n = 4$ and let $Z \in \mathcal{P}^4$ be clean with generators $\{a_1, \dots, a_m\}$. Then, for every $j \leq n-1$ of the generators $a_{i_1}, \dots, a_{i_j}$, the vector $a_{i_1} + \dots + a_{i_j}$ is not divisible by 2. Equivalently, every subset of $j \leq n-1$ generators of $Z$ is linearly independent over $\mathbb{F}_2 = \{0, 1\}$. We write $S = \{s_1, \dots, s_m\} \subseteq \mathbb{F}_2^n$ where $s_i = a_i \mod 2$ for $i = 1, \dots, m$. Now, if $m \geq 9$, then there exists an $\bar{s} \in S$ such that $|\{\bar{s} + s : s \in S \setminus \{\bar{s}\}\}| = |S \setminus \{\bar{s}\}| \geq 8$. This is because, by assumption, $\bar{s} + s \neq 0 \mod 2$ and $s \neq s' \mod 2$ for all $s, s' \in S \setminus \{\bar{s}\}$. Therefore, there exists a nonzero $s' \in S \setminus \{\bar{s}\} \cap \{\bar{s} + s : s \in S \setminus \{\bar{s}\}\}$ yielding a sum of three generators divisible by 2. This contradicts the assumption that $Z$ is clean, and thus $m \leq 8$.

Now, let $n \geq 5$ and let $Z \in \mathcal{P}^n$ be clean and let $S = \{s_1, \dots, s_m\} \subseteq \mathbb{F}_2^n$ be as above. For every invertible matrix $M \in \mathbb{F}_2^{n \times n}$ the set $M \cdot S$ shares the property of $S$ that every $j \leq n-1$ elements are linearly independent. Therefore, we can assume that $s_i = e_i$ for $i = 1, \dots, n-1$, since every $(n-1)$-element subset of $S$ is linearly independent. If there is an $s_j$, for some $j \geq n$, with last coordinate equal to zero, then it must be $s_j = (1, \dots, 1, 0)$. This is because otherwise, together with $s_1, \dots, s_{n-1}$, we could construct a linearly dependent subset of size at most $n-1$. So, by $|S| \geq n+2$, there must be a vector in $S$ with last coordinate 1. This gives a linearly independent subset of size $n$ in $S$, and we can again apply a suitable invertible transformation and assume that $s_n = e_n$. Now every other $s \in S \setminus \{s_1, \ldots, s_n\}$ can have at most one zero coordinate in order to avoid linearly dependent subsets of size at most $n-1$. A case distinction will now give the result.

- case 1: Besides the unit vectors, there is no other element in $S$ with a zero coordinate. Clearly, the only possibility is $S = \{e_1, \ldots, e_n, (1, \ldots, 1)^\intercal\}$, a contradiction.
- case 2: Besides the unit vectors, there is exactly one other element in $S$, say $s_{n+1}$, with $j$th coordinate equal to zero. By $|S| \geq n+2$, there also must be $s_{n+2} = (1, \ldots, 1)^\intercal \in S$. Then $\{s_j, s_{n+1}, s_{n+2}\}$ is linearly dependent of size $3 \leq n-1$.
- case 3: Besides the unit vectors, there are at least two other elements in $S$ with a zero coordinate. Without loss of generality let $s_{n+1} = (1, \ldots, 1, 0)^\intercal$ and $s_{n+2} = (1, \ldots, 1, 0, 1)^\intercal \in S$. Then $\{s_{n-1}, s_n, s_{n+1}, s_{n+2}\}$ is linearly dependent and of size $4 \leq n-1$, again a contradiction. □

Note that the considerations in the above proof relate to the theory of *maximum distance separable codes*. It follows, for example, from a recent paper by Ball [**Bal12**, Lem. 1.2] that a set $S \subseteq \mathbb{F}_2^n$ contains at most $n+1$ elements, if every $n$-element subset of $S$ is linearly independent.

**3.3.3. Ellipsoids.** Every ellipsoid $E \in \mathcal{K}^n$ has a representation $E = AB_n + t$ for a suitable $A \in \mathrm{GL}_n(\mathbb{R})$ and a translation vector $t \in \mathbb{R}^n$. So by $\mathrm{vol}(E) = |\det A| \mathrm{vol}(B_n)$, $\det(A^{-1}\mathbb{Z}^n) = |\det A|^{-1}$ and $\mathrm{G}(E) = \mathrm{G}(B_n, A^{-1}(\mathbb{Z}^n - t))$, we can think about Blichfeldt-type inequalities either by considering an arbitrary ellipsoid $E$ with respect to the standard lattice $\mathbb{Z}^n$ or by considering the unit ball $B_n$ with respect to an arbitrary lattice $\Lambda \in \mathcal{L}^n$. To us it seems more convenient to consider the latter and we do so throughout this part.

We aim at proving a Blichfeldt-type inequality for centrally symmetric ellipsoids $E \in \mathcal{K}_0^n$. In order to introduce our method, we first investigate this problem, more generally, for ellipsoids whose center is not necessarily the origin, deriving a weaker bound. The technical refinements needed for our target result are presented thereafter.

The following definition is essential:

**Definition 3.26** (maximal lattice plane)**.** *Let $\Lambda \in \mathcal{L}^n$ and $K \in \mathcal{K}^n$. We say that an $i$-dimensional affine subspace $L$ of $\mathbb{R}^n$ is a* maximal lattice $i$-plane (of $\Lambda$ with respect to $K$)*, if*

$$\mathrm{G}(K, \Lambda \cap L) = \max \{\mathrm{G}(K, \Lambda \cap S) : S \textit{ an } i\textit{-dimensional affine subspace}\}.$$

If $K$ is lattice spanning with respect to $\Lambda$, that is, $\dim(K \cap \Lambda) = n$, we have that $\dim(K \cap (\Lambda \cap L)) = i$ for every maximal lattice $i$-plane $L$. Indeed, if the intersection would have smaller dimension, we could extend the lattice plane $\mathrm{aff}(K \cap (\Lambda \cap L))$ with lattice points of $K$ that lie outside of $L$ to a lattice $i$-plane having at least one more lattice point of $K$ than $L$.

The volume of the Euclidean unit ball $B_n$ is denoted by

$$\kappa_n = \mathrm{vol}(B_n) = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)},$$

where $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ is the gamma function (cf. [Gar95, p. 13]). In order to compare our first result with the classical Blichfeldt bound (see Theorem 2.1) and Theorem 3.3, we note that the Stirling approximation gives $\frac{\kappa_n}{4^n} \approx \frac{\sqrt{2}\left(\frac{\pi}{8e}n\right)^{\frac{n}{2}}}{n!}$.

**Proposition 3.27.** *Let $\Lambda \in \mathcal{L}^n$ and $t \in \mathbb{R}^n$ such that $\dim(B_n \cap (\Lambda + t)) = n$. Then*

$$\frac{\mathrm{vol}(B_n)}{\det \Lambda} \geq \frac{\kappa_n}{4^n}\, \mathrm{G}(B_n, \Lambda + t).$$

Proof. For $n = 1$, we have $\frac{\mathrm{vol}(B_1)}{\det \Lambda} \geq \frac{1}{2}\, \mathrm{G}(B_1, \Lambda + t) = \frac{\kappa_1}{4}\, \mathrm{G}(B_1, \Lambda + t)$. Let $n \geq 2$ and let $L$ be a maximal lattice $(n-1)$-plane of $\Lambda + t$ with respect to $B_n$. Furthermore, let $w$ be the number of lattice planes of $\Lambda$ that are parallel to $\Lambda \cap L$ and intersect $B_n$. Since $\dim(B_n \cap (\Lambda + t)) = n$, we have $w \geq 2$. Because the determinant of the projected lattice $\Lambda | L^\perp$ is the minimal distance of two distinct lattice planes parallel to $\Lambda \cap L$, we get that $(w-1)\det(\Lambda | L^\perp)$ is at most the diameter of $B_n$, which is 2. In view of Equation (1.3), we get

$$\det \Lambda = \det(\Lambda \cap L)\det(\Lambda | L^\perp) \leq \det(\Lambda \cap L)\frac{2}{w-1}.$$

Since $L$ is a maximal lattice plane, we have $\dim(B_n \cap (\Lambda \cap L)) = n - 1$ and moreover $B_n \cap L \cong r B_{n-1}$ for some $r \in (0, 1]$. These observations allow us to argue inductively and we obtain

$$\frac{\mathrm{vol}(B_n)}{\det \Lambda} \geq \frac{\kappa_n(w-1)}{2\kappa_{n-1}} \frac{\mathrm{vol}_{n-1}(B_n \cap L)}{\det(\Lambda \cap L)} \geq \frac{\kappa_n(w-1)}{2 \cdot 4^{n-1}}\, \mathrm{G}(B_n \cap L, \Lambda \cap L).$$

Because of $w \geq 2$ and $w\, \mathrm{G}(B_n \cap L, \Lambda \cap L) \geq \mathrm{G}(B_n, \Lambda + t)$, the right hand side is now greater than or equal to the claimed $\frac{\kappa_n}{4^n}\, \mathrm{G}(B_n, \Lambda + t)$. □

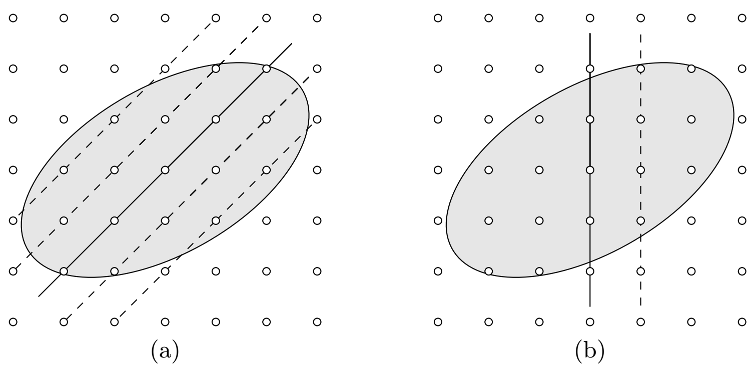


Figure 3.6. A lattice line through the origin that (a) has the property of Lemma 3.28 and (b) one that does not.

In the above proof, we reduced the situation to a maximal lattice $(n-1)$-plane and then applied induction. The following lemma helps us to refine this approach in order to reduce to suitable lattice lines that are maximal among all their parallel lattice lines (see Figure 3.6).

**Lemma 3.28.** *Let $K \in \mathcal{K}_0^n$ and $\Lambda \in \mathcal{L}^n$. Then there exists a line $l$ through the origin with $\mathrm{G}(K, \Lambda \cap l) > \mathrm{G}(K, \Lambda \cap l')$ for every other line $l'$ parallel to $l$.*

Proof. If $K \cap \Lambda = \{0\}$, then every line through the origin meets the requirements. Let $K \cap \Lambda \neq \{0\}$ and let $v \in K \cap \Lambda$ be of maximal length, that is,

$$\|v\| = \max\{\|x\| : x \in K \cap \Lambda\}.$$

We claim that the line $l_v = \mathrm{lin}\{v\}$ has the desired property. The definition of the greatest common divisor extends naturally to arbitrary lattice vectors via $\gcd(v) = \max\{k \in \mathbb{N} : \frac{1}{k} v \in \Lambda\}$. We can write the determinant of the sublattice $\Lambda_v = \Lambda \cap l_v$ of $\Lambda$ as $\det \Lambda_v = \frac{\|v\|}{\gcd(v)}$. This is also the minimal distance of two distinct lattice points in $\Lambda_v$. By the central symmetry of $K$, we have $\mathrm{G}(K, \Lambda_v) = 2\gcd(v) + 1$.

Assume that there is a line $l'$ parallel to $l_v$ with $\mathrm{G}(K, \Lambda \cap l') \geq \mathrm{G}(K, \Lambda_v)$. Let $s, t \in K \cap \Lambda \cap l'$ be such that $\mathrm{conv}\{s, t\} = \mathrm{conv}\{K \cap \Lambda \cap l'\}$. If $l' \neq l_v$, then $s$ and $t$ are linearly independent. Therefore,

$$\begin{aligned} 2\gcd(v)\det\Lambda_v &\leq (\mathrm{G}(K, \Lambda \cap l') - 1)\det\Lambda_v \leq \mathrm{vol}_1(\mathrm{conv}\{K \cap \Lambda \cap l'\}) \\ &= \|s - t\| < \|s\| + \|t\| \leq 2\,\|v\| \\ &= 2\gcd(v)\det\Lambda_v, \end{aligned}$$

which is a contradiction. ☐

Let $K \in \mathcal{K}_0^n$ and let $l$ be a lattice line of a lattice $\Lambda \in \mathcal{L}^n$ with the property of the above lemma. Then

$$\begin{aligned} \mathrm{G}(K, \Lambda) &= \sum_{x \in K|l^\perp \cap \Lambda|l^\perp} \mathrm{G}(K, \Lambda \cap (l + x)) \\ &\leq \mathrm{G}(K, \Lambda \cap l) + (\mathrm{G}(K|l^\perp, \Lambda|l^\perp) - 1)(\mathrm{G}(K, \Lambda \cap l) - 1), \end{aligned}$$

or equivalently,

$$\mathrm{G}(K, \Lambda) - 1 \leq \mathrm{G}(K|l^\perp, \Lambda|l^\perp)(\mathrm{G}(K, \Lambda \cap l) - 1). \tag{3.19}$$

In the case that $K$ is lattice spanning with respect to $\Lambda$, we want to refine this inequality by considering lattice lines in projections. We denote the linear hull of some $v \in \mathbb{R}^n$ by $l_v = \mathrm{lin}\{v\}$. Let $K_1 = K, \Lambda_1 = \Lambda$ and, according to Lemma 3.28, let $v_1 \in K \cap \Lambda$ be of maximal length. We write $K_2 = K|l_{v_1}^\perp$, and $\Lambda_2 = \Lambda|l_{v_1}^\perp$, and we let $v_2$ be a vector of maximal length in $K_2 \cap \Lambda_2$. Continuing in this way, we define $K_i = K_{i-1}|l_{v_{i-1}}^\perp$, and $\Lambda_i = \Lambda_{i-1}|l_{v_{i-1}}^\perp$, and we let $v_i$ to be of maximal length in $K_i \cap \Lambda_i$ for all $i = 3, \ldots, n$. Note that all the $v_i$'s are nonzero since $\dim(K \cap \Lambda) = n$ assures that $\dim(K_i \cap \Lambda_i) = n - i + 1$ for all $i = 1, \ldots, n$.

**Lemma 3.29.** *Let $K \in \mathcal{K}_0^n$ be lattice spanning with respect to $\Lambda \in \mathcal{L}^n$. Then*

$$\mathrm{G}(K, \Lambda) - 1 \leq 2\left(1 - \frac{1}{2^n}\right)\prod_{i=1}^n (\mathrm{G}(K_i, \Lambda_i \cap l_{v_i}) - 1).$$

PROOF. We apply Inequality (3.19) to $K_i$, $\Lambda_i$ and $v_i$. For sake of brevity, write $\mathrm{G}_i = \mathrm{G}(K_i, \Lambda_i \cap l_{v_i})$ for all $i = 1, \dots, n$. Then

$$\begin{aligned}\mathrm{G}(K,\Lambda) &\le 1 + \mathrm{G}(K_2,\Lambda_2)(\mathrm{G}(K,\Lambda\cap l_{v_1}) - 1)\\ &\le 1 + (1+\mathrm{G}(K_3,\Lambda_3)(\mathrm{G}_2 - 1))(\mathrm{G}_1 - 1)\\ &= 1 + (\mathrm{G}_1 - 1) + \mathrm{G}(K_3,\Lambda_3)(\mathrm{G}_2 - 1)(\mathrm{G}_1 - 1)\\ &\le 1 + \sum_{j=1}^{n-2}\prod_{i=1}^{j}(\mathrm{G}_i - 1) + \mathrm{G}(K_n,\Lambda_n)\prod_{i=1}^{n-1}(\mathrm{G}_i - 1).\end{aligned}$$

By the central symmetry of $K$, we now have $\mathrm{G}_i \ge 3$ for all $i = 1, \dots, n-1$, and also $\mathrm{G}(K_n, \Lambda_n) = \mathrm{G}(K_n, \Lambda_n \cap l_{v_n}) \ge 3$. Therefore, $\prod_{i=1}^{j}(\mathrm{G}_i - 1) \le \frac{1}{2^{n-j-1}} \prod_{i=1}^{n-1}(\mathrm{G}_i - 1)$, which implies, in view of $\mathrm{G}_n \le \frac{3}{2}(\mathrm{G}_n - 1)$, that

$$\begin{aligned}\mathrm{G}(K,\Lambda) - 1 &\le \left(\mathrm{G}_n + \sum_{j=1}^{n-2}\frac{1}{2^{n-j-1}}\right)\prod_{i=1}^{n-1}(\mathrm{G}_i - 1)\\ &\le \left(\frac{3}{2} + \sum_{j=1}^{n-2}\frac{1}{2^{n-j}}\right)\prod_{i=1}^{n}(\mathrm{G}_i - 1).\end{aligned}$$

The claim follows by $\frac{3}{2} + \sum_{j=1}^{n-2}\frac{1}{2^{n-j}} = 2\left(1 - \frac{1}{2^n}\right)$. □

**Remark 3.30.** *In the above inequality the factor* $2\left(1 - \frac{1}{2^n}\right)$ *cannot be replaced by* $1$ *in general. In fact, for* $l \in \mathbb{N}$ *the pair* $K_l = lB_2 \cap \{x \in \mathbb{R}^2 : |x_2| \le 1\}$ *and* $\Lambda = \mathbb{Z}^2$ *shows that the inequality is best possible for* $n = 2$. *Here, we have* $\mathrm{G}(K_l) = 2l + 1 + 2(2l-1) = 6l - 1$ *and* $v_1 = le_1$, $v_2 = e_2$. *Thus,* $\mathrm{G}_1 = \mathrm{G}(K_l, \mathbb{Z}^2 \cap l_{v_1}) = 2l + 1$, $\mathrm{G}_2 = \mathrm{G}(K_l | l_{v_1}^{\perp}, \mathbb{Z}^2 | l_{v_1}^{\perp}) = 3$ *and*

$$\mathrm{G}(K_l) - 1 = 6l - 2 \quad \textit{and} \quad \frac{3}{2}(\mathrm{G}_1 - 1)(\mathrm{G}_2 - 1) = 6l \quad \textit{for all} \quad l \in \mathbb{N}.$$

We are now ready to prove our Blichfeldt-type inequality for centrally symmetric ellipsoids.

**Theorem 3.31.** *Let* $\Lambda \in \mathcal{L}^n$ *be a lattice such that* $\dim(B_n \cap \Lambda) = n$. *Then*

$$\frac{\mathrm{vol}(B_n)}{\det\Lambda} \ge \frac{\kappa_n}{2^{n+1} - 2}\left(\mathrm{G}(B_n, \Lambda) - 1\right).$$

PROOF. We use the notation that we fixed before Lemma 3.29, thus $(B_n)_i$ are the projected bodies that we deal with. By construction and Equation (1.3), we have $\det \Lambda_i = \det \Lambda_{i+1} \cdot \det(\Lambda_i \cap l_{v_i})$ for all $i = 1, \dots, n-1$, and $\det \Lambda_n = \det(\Lambda_n \cap l_{v_n})$. Therefore,

$$\det\Lambda = \prod_{i=1}^{n}\det(\Lambda_i \cap l_{v_i}).$$

Since projections of $B_n$ onto linear subspaces are lower dimensional unit balls, we have $\mathrm{vol}_1((B_n)_i \cap l_{v_i}) = 2$ for all $i = 1, \dots, n$. Thus, by Lemma 3.29, we obtain

$$\frac{\mathrm{vol}(B_n)}{\det\Lambda} = \frac{\kappa_n}{2^n}\prod_{i=1}^{n}\frac{\mathrm{vol}_1((B_n)_i \cap l_{v_i})}{\det(\Lambda_i \cap l_{v_i})}$$

$$\geq \frac{\kappa_n}{2^n} \prod_{i=1}^{n} (\mathrm{G}((B_n)_i, \Lambda_i \cap l_{v_i}) - 1)$$
$$\geq \frac{\kappa_n}{2^n} \cdot \frac{\mathrm{G}(B_n, \Lambda) - 1}{2\left(1 - \frac{1}{2^n}\right)} = \frac{\kappa_n}{2^{n+1} - 2} \left(\mathrm{G}(B_n, \Lambda) - 1\right). \qquad \square$$

**Remark 3.32.** *The method above does not apply to arbitrary $K \in \mathcal{K}_0^n$. We would need an inequality of the form*

$$\mathrm{vol}(K) \geq c_n \prod_{i=1}^{n} \mathrm{vol}_1(K_i \cap l_{v_i}),$$

*for a "good" value of $c_n$. But the standard crosspolytope $C_n^\star$ shows that $c_n \leq \frac{1}{n!}$, which is too weak in order to get an improvement upon Theorem 2.1.*

Since $\mathrm{G}(B_n) = 2n + 1$, the integer lattice $\mathbb{Z}^n$ shows that the dimensional constant on the right hand side of Theorem 3.31 is in general at most $\frac{\kappa_n}{2n}$. Although we could not come up with an example that supports it, we conjecture that the best possible factor is of the form $\frac{\kappa_n}{c^n}$ for some $c > 1$.

**Problem 3.33.** *Determine the optimal factor $c_n$ such that*

$$\frac{\mathrm{vol}(B_n)}{\det \Lambda} \geq c_n \left(\mathrm{G}(B_n, \Lambda) - d\right)$$

*for some $d \leq 2n$ and all $\Lambda \in \mathcal{L}^n$ with $\dim(B_n \cap \Lambda) = n$.*

If the answer to this problem is "good" enough, it would yield another argument for a Blichfeldt-type inequality for all lattice spanning $K \in \mathcal{K}_0^n$ (cf. Theorem 3.3). More precisely, by John's Theorem (cf. [**Gru07**, Ch. 11]), there exists an ellipsoid $E \subseteq K$ such that $K \subset \sqrt{n}E$. Thus, we would get

$$\mathrm{vol}(K) \geq \sqrt{n}^{-n} \mathrm{vol}(\sqrt{n}E) \geq \frac{c_n}{\sqrt{n}^n} \left(\mathrm{G}(\sqrt{n}E) - d\right) \geq \frac{c_n}{\sqrt{n}^n} \left(\mathrm{G}(K) - d\right).$$

Stirling's formula shows, that a factor $c_n = \left(c\sqrt{\frac{e}{2\pi}}\right)^n \kappa_n$ for some $c > 1$, would imply $\mathrm{vol}(K) \geq \frac{c^n}{n!} \left(\mathrm{G}(K) - d\right)$.

## 3.4. Application to a Functional Introduced by Gillet and Soulé

As an application of our Blichfeldt-type inequalities from the preceding sections, we bound the magnitude $\frac{\mathrm{G}(K)}{\mathrm{G}(K^\star)\,\mathrm{vol}(K)}$ by constants that depend on the dimension $n$ but not on the body $K$. Recall that

$$K^\star = \{x \in \mathbb{R}^n : x^\intercal y \leq 1 \text{ for all } y \in K\}$$

denotes the polar body of $K \in \mathcal{K}_0^n$. Estimates of such kind were first studied and applied by Gillet and Soulé [**GS91**] who showed that

$$6^{-n} \leq \frac{\mathrm{G}(K)}{\mathrm{G}(K^\star)\,\mathrm{vol}(K)} \leq \frac{6^n n!}{c^n} \quad \text{for some} \quad c \leq 4.$$

We improve upon these bounds and derive the following inequalities.

**Theorem 3.34.** *For every $\varepsilon > 0$ there exists an $n(\varepsilon) \in \mathbb{N}$ such that for every $n \geq n(\varepsilon)$ and every lattice spanning $K \in \mathcal{K}_0^n$, we have*

$$(\pi + \varepsilon)^{-n} \leq \frac{\mathrm{G}(K)}{\mathrm{G}(K^\star)\,\mathrm{vol}(K)} \leq \frac{(\pi + \varepsilon)^n n!}{c^n},$$

*where $c \le 4$ is an absolute constant.*

We follow the ideas of Gillet and Soulé whose arguments are based on the inequality [**GS91**, Prop. 3]

$$\mathrm{G}(K)\,\mathrm{vol}(K^\star) \le 6^n \quad \text{for every lattice spanning} \quad K \in \mathcal{K}_0^n.$$

Therefore, we first study the functional $K \mapsto \mathrm{G}(K)\,\mathrm{vol}(K^\star)$ in more detail.

**Corollary 3.35.** *For every $\varepsilon > 0$ there exists an $n(\varepsilon) \in \mathbb{N}$ such that for every $n \ge n(\varepsilon)$ and every lattice spanning $K \in \mathcal{K}_0^n$, we have*

$$\frac{c^n}{n!} \le \mathrm{G}(K)\,\mathrm{vol}(K^\star) \le (\pi + \varepsilon)^n.$$

*Here, $c \le 2$ is an absolute constant and the lower bound holds for every $n \in \mathbb{N}$ and arbitrary $K \in \mathcal{K}_0^n$.*

Proof. For the lower bound we combine Theorem 3.1 and the estimate

$$\mathrm{vol}(K)\,\mathrm{vol}(K^\star) \ge \frac{C^n}{n!}, \tag{3.20}$$

which is due to Bourgain and Milman [**BM87**] and holds for some universal constant $C \le 4$. Indeed, we have

$$\mathrm{G}(K)\,\mathrm{vol}(K^\star) \ge \frac{1}{2^n}\,\mathrm{vol}(K)\,\mathrm{vol}(K^\star) \ge \frac{(C/2)^n}{n!}.$$

Now we restrict to $K \in \mathcal{K}_0^n$ with $\dim(K \cap \mathbb{Z}^n) = n$. The exact inequality (3.6) in the proof of Theorem 3.3 and the Blaschke-Santaló [**San49**] inequality imply that

$$\mathrm{G}(K)\,\mathrm{vol}(K^\star) \le \frac{n!\,L_n(2)}{2^n}\,\mathrm{vol}(K)\,\mathrm{vol}(K^\star) \le \frac{n!\,\kappa_n^2\,L_n(2)}{2^n}. \tag{3.21}$$

Hence, by (3.7), we have that for every $\varepsilon' \in (0,1]$ and large enough $n \in \mathbb{N}$

$$\mathrm{G}(K)\,\mathrm{vol}(K^\star) \le \frac{n!\,\kappa_n^2}{(2-\varepsilon')^n}.$$

Stirling approximation and $\kappa_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ give

$$\frac{n!\,\kappa_n^2}{(2-\varepsilon')^n} \approx \frac{\sqrt{2\pi n}(\frac{n}{e})^n\pi^n}{(2-\varepsilon')^n\Gamma(\frac{n}{2}+1)^2} \approx \frac{\sqrt{2\pi n}(\frac{n}{e})^n\pi^n}{(2-\varepsilon')^n\pi n(\frac{n}{2e})^n} \le \left(\frac{2\pi}{2-\varepsilon'}\right)^n.$$

For every $\varepsilon \in (0,1)$ there exists an $\varepsilon' \in (0,1)$ such that $\frac{2\pi}{2-\varepsilon'} \le \pi + \varepsilon$, and we conclude that for large $n$, the inequality $\mathrm{G}(K)\,\mathrm{vol}(K^\star) \le (\pi+\varepsilon)^n$ holds. □

In the planar case, we are able to give sharp bounds. Recall that two sets are unimodularly equivalent if there is a lattice preserving affine transformation that maps one onto the other.

**Proposition 3.36.** *Let $K \in \mathcal{K}_0^2$ be lattice spanning. Then*

$$2 \le \mathrm{G}(K)\,\mathrm{vol}(K^\star) \le 21.$$

*In the upper bound, equality holds if and only if $K$ is unimodularly equivalent to the hexagon $H = \mathrm{conv}\{\pm e_1, \pm e_2, \pm(e_1+e_2)\}$. The lower bound is also best possible and holds for arbitrary $K \in \mathcal{K}_0^2$.*

Proof. For the upper bound, we can restrict to lattice polygons $P \in \mathcal{P}_0^2$ since for $P_K = \mathrm{conv}\{K \cap \mathbb{Z}^2\}$ we have $\mathrm{G}(K)\,\mathrm{vol}(K^\star) \leq \mathrm{G}(P_K)\,\mathrm{vol}(P_K^\star)$. From Pick's Theorem 1.2, we get

$$\begin{aligned}\mathrm{G}(P)\,\mathrm{vol}(P^\star) &= \left(\mathrm{vol}(P) + \frac{1}{2}\,\mathrm{G}(\partial P) + 1\right)\mathrm{vol}(P^\star)\\ &= \left(\mathrm{vol}(P) + \frac{1}{2}\,\mathrm{G}(P) - \frac{1}{2}\,\mathrm{G}(\mathrm{int}\,P) + 1\right)\mathrm{vol}(P^\star)\end{aligned}$$

and therefore

$$\mathrm{G}(P)\,\mathrm{vol}(P^\star) = 2\,\mathrm{vol}(P)\,\mathrm{vol}(P^\star) + \mathrm{vol}(P^\star)\left(2 - \mathrm{G}(\mathrm{int}\,P)\right).$$

Using the Blaschke-Santaló inequality [San49] in the plane, gives

$$\mathrm{G}(P)\,\mathrm{vol}(P^\star) \leq 2\,\mathrm{vol}(P)\,\mathrm{vol}(P^\star) \leq 2\pi^2 < 21,$$

whenever $\mathrm{G}(\mathrm{int}\,P) > 1$. Up to unimodular equivalence, there are only three centrally symmetric lattice polygons with exactly one interior lattice point: the square $[-1,1]^2$, the diamond $\mathrm{conv}\{\pm e_1, \pm e_2\}$ and the hexagon $H$ (see for example [Nil05, Prop. 2.1]). Among these three, the hexagon is the only maximizer of $\mathrm{G}(P)\,\mathrm{vol}(P^\star)$.

For the lower bound, we use Mahler's inequality $\mathrm{vol}(K)\,\mathrm{vol}(K^\star) \geq 8$ for every $K \in \mathcal{K}_0^2$ (see [Mah38]). By the same lines as in the proof of Corollary 3.35, this yields $\mathrm{G}(K)\,\mathrm{vol}(K^\star) \geq 2$. For small $\varepsilon > 0$, the square $K_\varepsilon = (1-\varepsilon)[-1,1]^2$ shows that this is best possible. Indeed, we have $\mathrm{G}(K_\varepsilon) = 1$ and $\mathrm{vol}(K_\varepsilon^\star) = \frac{2}{(1-\varepsilon)^2}$ for every $\varepsilon \in (0,1)$. □

Our Blichfeldt-type inequalities for lattice zonotopes and ellipsoids lead to good estimates for $\mathrm{G}(K)\,\mathrm{vol}(K^\star)$ on these classes.

**Corollary 3.37.**

*i) Let $Z \in \mathcal{K}_0^n$ be a lattice zonotope. Then*

$$\mathrm{G}(Z)\,\mathrm{vol}(Z^\star) \leq 2^n \kappa_n^2.$$

*ii) Let $E \in \mathcal{K}_0^n$ be a lattice spanning ellipsoid. Then*

$$\mathrm{G}(E)\,\mathrm{vol}(E^\star) \leq (2^n - 1)(2n+1)\frac{\kappa_n}{n}.$$

Proof. Just like in the proof of Corollary 3.35, we use the Blaschke-Santaló inequality [San49] together with Theorem 3.19 for i) and with Theorem 3.31 for part ii). □

Based on these bounds and computer experiments in small dimensions, we conjecture that the hexagon is an exception and that, for $n \geq 3$, a maximizing example is the standard crosspolytope $C_n^\star = \mathrm{conv}\{\pm e_1, \ldots, \pm e_n\}$.

**Conjecture 3.38.** *Let $n \geq 3$ and let $K \in \mathcal{K}_0^n$ be lattice spanning. Then*

$$\mathrm{G}(K)\,\mathrm{vol}(K^\star) \leq \mathrm{G}(C_n^\star)\,\mathrm{vol}(C_n) = (2n+1)2^n.$$

Finally we are ready to prove Theorem 3.34. The arguments are similar to those given by Gillet and Soulé [GS91, Sect. 1.6]. We repeat them here with the necessary adjustments to our bounds and for the sake of completeness.

Proof of Theorem 3.34. The upper bound can be derived by applying the lower bound to $K^\star$ and by using Inequality (3.20) of Bourgain and Milman. In fact, there is an absolute constant $c \le 4$ with

$$\frac{\mathrm{G}(K)}{\mathrm{G}(K^\star)\,\mathrm{vol}(K)} = \frac{\mathrm{G}(K)\,\mathrm{vol}(K^\star)}{\mathrm{G}(K^\star)\,\mathrm{vol}(K)\,\mathrm{vol}(K^\star)} \le \frac{(\pi+\varepsilon)^n}{\mathrm{vol}(K)\,\mathrm{vol}(K^\star)} \le \frac{(\pi+\varepsilon)^n n!}{c^n}.$$

Here, $\varepsilon > 0$ and $n \in \mathbb{N}$ is large enough.

For the lower bound, let $L = \mathrm{lin}(K^\star \cap \mathbb{Z}^n)$ and $k = \dim L$. Let us abbreviate $L_n = L_n(2)$. We use the exact inequality (3.21) from the proof of Corollary 3.35 in the sublattice $\mathbb{Z}^n \cap L$ and obtain

(3.22)
$$\mathrm{G}(K^\star) = \mathrm{G}(K^\star \cap L) \le \frac{k!\,\kappa_k^2\,L_k}{2^k}\frac{\det((\mathbb{Z}^n \cap L)^\star)}{\mathrm{vol}_k((K^\star \cap L)^\star)} = \frac{k!\,\kappa_k^2\,L_k}{2^k}\frac{\det(\mathbb{Z}^n|L)}{\mathrm{vol}_k(K|L)}.$$

By $K = (K^\star)^\star$, we have $\mathrm{int}\,K = \{x \in K : |x^\intercal y| < 1 \text{ for all } y \in K^\star\}$ and thus $x^\intercal y = 0$ for all $x \in \mathrm{int}\,K \cap \mathbb{Z}^n$ and $y \in K^\star \cap \mathbb{Z}^n$. Therefore, using Theorem 3.1 in the sublattice $\mathbb{Z}^n \cap L^\perp$, we get

$$\begin{aligned} \mathrm{G}(K) &\ge \mathrm{G}(\mathrm{int}\,K) = \mathrm{G}(\mathrm{int}\,K \cap L^\perp) \\ &\ge \frac{1}{2^{n-k}}\frac{\mathrm{vol}_{n-k}(\mathrm{int}\,K \cap L^\perp)}{\det(\mathbb{Z}^n \cap L^\perp)} = \frac{1}{2^{n-k}}\frac{\mathrm{vol}_{n-k}(K \cap L^\perp)}{\det(\mathbb{Z}^n \cap L^\perp)}. \end{aligned} \tag{3.23}$$

Combining (3.22), (3.23) and the lower bound in Inequality (3.5) gives

$$\frac{\mathrm{G}(K)}{\mathrm{G}(K^\star)} \ge \frac{4^k}{2^n\,k!\,\kappa_k^2\,L_k}\frac{\mathrm{vol}_{n-k}(K \cap L^\perp)\,\mathrm{vol}_k(K|L)}{\det(\mathbb{Z}^n \cap L^\perp)\det(\mathbb{Z}^n|L)} \ge \frac{4^k\,\mathrm{vol}(K)}{2^n\,k!\,\kappa_k^2\,L_k}.$$

Note that we also used $\det(\mathbb{Z}^n \cap L^\perp)\det(\mathbb{Z}^n|L) = \det(\mathbb{Z}^n) = 1$ here (cf. (1.3)). What remains is to show that the function $g(k) = \frac{4^k}{k!\,\kappa_k^2\,L_k}$ is nonincreasing for $k \ge 0$, because together with the previous inequality we then arrive at

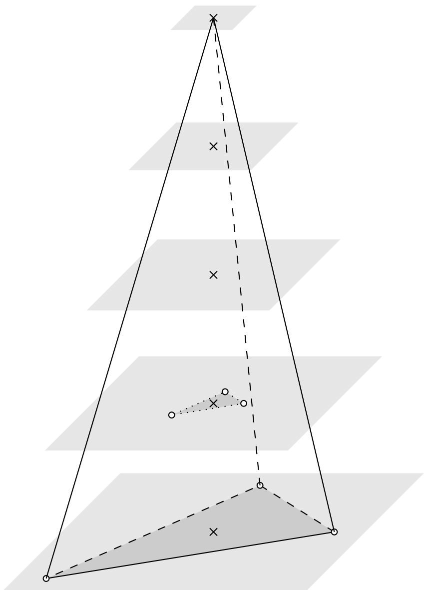
$$\frac{\mathrm{G}(K)}{\mathrm{G}(K^\star)\,\mathrm{vol}(K)} \ge \frac{4^k}{2^n\,k!\,\kappa_k^2\,L_k} \ge \frac{2^n}{n!\,\kappa_n^2\,L_n}.$$

In the proof of Corollary 3.35, we have seen that for every $\varepsilon > 0$ the right hand side of the above inequality is at least $(\pi+\varepsilon)^{-n}$ for large enough $n \in \mathbb{N}$.

Now $g(k) \ge g(k+1)$ if and only if $\frac{k+1}{4} \ge \frac{\kappa_k^2}{\kappa_{k+1}^2}\frac{L_k}{L_{k+1}}$. By the estimate $\frac{\kappa_k^2}{\kappa_{k+1}^2} \le \frac{k+2}{2\pi}$ (see [**BGW82**, Lem. 1]), it is therefore enough to prove $\frac{k+1}{4} \ge \frac{k+2}{2\pi}\frac{L_k}{L_{k+1}}$. After an elementary calculation, we see that this follows from the recurrence relation $L_{k+1} = \sum_{i=0}^{k+1}\binom{k+1}{i}\frac{2^i}{i!} = 2L_k - \frac{2^k(k-1)}{(k+1)!}$. $\square$

CHAPTER 4

# Successive Minima Type Inequalities



## 4.1. Introduction to Minkowski's Successive Minima

In this chapter, we study the *successive minima* that are functionals measuring the size of a convex body with respect to a lattice. They were introduced by Minkowski in his seminal monograph [Min96].

**Definition 4.1** (successive minimum). *Let $K \in \mathcal{K}_0^n$ and $\Lambda \in \mathcal{L}^n$. The* $i$th successive minimum of $K$ with respect to $\Lambda$ *is defined as*

$$\lambda_i(K, \Lambda) = \min\{\lambda > 0 : \dim(\lambda K \cap \Lambda) \geq i\} \quad \textit{for all} \quad i = 1, \ldots, n.$$

In particular, $\lambda_1(K, \Lambda)$ is the smallest dilatation factor $\lambda$ such that $\lambda K$ contains nontrivial lattice points. Immediate yet important properties of the successive minima are the following:

$$\begin{aligned} (\textit{monotonicity}) \quad & \lambda_1(K, \Lambda) \leq \lambda_2(K, \Lambda) \leq \ldots \leq \lambda_n(K, \Lambda) \\ (\textit{homogeneity}) \quad & \lambda_i(tK, \Lambda) = \tfrac{1}{t}\lambda_i(K, \Lambda) \text{ and } \lambda_i(K, t\Lambda) = t\lambda_i(K, \Lambda), t > 0 \\ (\textit{linear invariance}) \quad & \lambda_i(AK, A\Lambda) = \lambda_i(K, \Lambda), A \in \mathrm{GL}_n(\mathbb{R}) \end{aligned}$$

Using these functionals, Minkowski [Min96] proved a fundamental and effective criterion for a centrally symmetric convex body to contain nontrivial lattice points. His result opened the way to many applications in the geometry of numbers and especially in existence results for solutions of Diophantine equations (cf. [GL87, Gru93]).

**Theorem 4.2** (Minkowski's 1st convex body theorem). *Let $K \in \mathcal{K}_0^n$ and $\Lambda \in \mathcal{L}^n$. Then*

$$\lambda_1(K, \Lambda)^n \operatorname{vol}(K) \leq 2^n \det \Lambda.$$

A nontrivial lower bound on $\lambda_1(K, \Lambda)^n \operatorname{vol}(K)$ does not exist as the boxes $B_\varepsilon = C_{n-1} \times [-\varepsilon, \varepsilon]$ show. Indeed, we have $\lambda_1(B_\varepsilon, \mathbb{Z}^n)^n \operatorname{vol}(B_\varepsilon) = 2^n\varepsilon$, which tends to zero when $\varepsilon$ does. Yet a lower bound exists if the whole series of successive minima is taken into account. More importantly, the following deep result by Minkowski provides a strengthening of his 1st theorem above.

**Theorem 4.3** (Minkowski's 2nd convex body theorem). *Let $K \in \mathcal{K}_0^n$ and $\Lambda \in \mathcal{L}^n$. Then*

$$\frac{2^n}{n!} \det \Lambda \leq \lambda_1(K, \Lambda) \cdot \ldots \cdot \lambda_n(K, \Lambda) \operatorname{vol}(K) \leq 2^n \det \Lambda.$$

The literature contains various proofs of this fundamental inequality. A clear and modern presentation of Minkowski's arguments and a collection of references to other proofs were given in [Hen02]. The survey of Gruber [Gru93] offers background information on the impact of Minkowski's theorems in the geometry of numbers.

Already Gruber and Lekkerkerker [GL87, Ch. 2, §9] proposed to extend the definition of the successive minima to arbitrary not necessarily symmetric convex bodies $K \in \mathcal{K}^n$. In fact, via the difference body $\mathcal{D}K = K - K$, one can define

$$\lambda_i(K, \Lambda) \coloneqq \lambda_i(\tfrac{1}{2}\mathcal{D}K, \Lambda) \quad \text{for all} \quad i = 1, \ldots, n.$$

By $\operatorname{vol}(K) \leq \operatorname{vol}(\frac{1}{2}\mathcal{D}K)$, the upper bound in Minkowski's 2nd Theorem holds with respect to every $K \in \mathcal{K}^n$. Also the lower bound can be extended to the general case.

**Proposition 4.4.** *Let $K \in \mathcal{K}^n$ and $\Lambda \in \mathcal{L}^n$. Then*

$$\frac{2^n}{n!} \det \Lambda \leq \lambda_1(K, \Lambda) \cdot \ldots \cdot \lambda_n(K, \Lambda) \operatorname{vol}(K).$$

PROOF. For abbreviation, we write $\lambda_i = \lambda_i(K, \Lambda)$ for all $i = 1, \ldots, n$. We choose linearly independent lattice vectors $z_1, \ldots, z_n \in \Lambda$ with $z_i \in \lambda_i \frac{1}{2}\mathcal{D}K$ for each $i = 1, \ldots, n$. Therefore, there exist $x_1, y_1, \ldots, x_n, y_n \in K$ such that $\frac{2}{\lambda_i} z_i = y_i - x_i$ for each $i = 1, \ldots, n$. Since the $z_i$'s are linearly independent, the convex hull of $x_1, y_1, \ldots, x_n, y_n$ is a full-dimensional polytope contained in $K$. Using [**BH93**, Thm. 2], we obtain the desired inequality by

$$\begin{aligned}
\operatorname{vol}(K) &\geq \operatorname{vol}(\operatorname{conv}\{x_1, y_1, \ldots, x_n, y_n\}) \\
&\geq \frac{|\det(y_1 - x_1, \ldots, y_n - x_n)|}{n!} = \frac{2^n}{n!} \frac{|\det(z_1, \ldots, z_n)|}{\lambda_1 \cdot \ldots \cdot \lambda_n} \\
&\geq \frac{2^n}{n!} \frac{\det \Lambda}{\lambda_1 \cdot \ldots \cdot \lambda_n}.
\end{aligned}$$

The last inequality holds since the $z_i$'s span a sublattice of $\Lambda$. □

The lower bound in Minkowski's 2nd Theorem can only be attained by crosspolytopes. For certain applications it is useful to have stronger inequalities of this kind on particular subclasses of convex bodies. Henk, Linke and Wills obtained such a result on the class of zonotopes.

**Theorem 4.5** (Henk, Linke and Wills [**HLW10**])**.** *Let $Z \in \mathcal{K}^n$ be a zonotope and $\Lambda \in \mathcal{L}^n$. Then*

$$\frac{2^n}{n^{\frac{n}{2}}} \det \Lambda \leq \lambda_1(Z, \Lambda) \cdot \ldots \cdot \lambda_n(Z, \Lambda) \operatorname{vol}(Z).$$

## 4.2. On a Discrete Version of Minkowski's 2nd Theorem

The wide range of applications and the appealing nature of Minkowski's results on successive minima motivated the investigation of possible strengthenings and generalizations (see [**HW08b**] for a survey). In this part, we discuss a possible discrete version of Theorem 4.3 that was proposed by Betke, Henk and Wills. The symbol $\lfloor x \rfloor$ denotes the biggest integer smaller than or equal to $x$.

**Conjecture 4.6** (Betke, Henk and Wills [**BHW93**])**.** *Let $K \in \mathcal{K}_0^n$ and $\Lambda \in \mathcal{L}^n$ be a lattice. Then*

$$\mathrm{G}(K, \Lambda) \leq \prod_{i=1}^n \left\lfloor \frac{2}{\lambda_i(K, \Lambda)} + 1 \right\rfloor.$$

In fact, by virtue of

$$\frac{\operatorname{vol}(K)}{\det \Lambda} = \lim_{s \to 0} s^n \, \mathrm{G}(K, s\Lambda) \leq \lim_{s \to 0} \prod_{i=1}^n s \left\lfloor \frac{2}{\lambda_i(K, s\Lambda)} + 1 \right\rfloor = \prod_{i=1}^n \frac{2}{\lambda_i(K, \Lambda)},$$

this inequality is stronger than Minkowski's 2nd Theorem.

Betke, Henk and Wills [**BHW93**] proved the analogous discrete version

$$\mathrm{G}(K, \Lambda) \leq \left\lfloor \frac{2}{\lambda_1(K, \Lambda)} + 1 \right\rfloor^n \tag{4.1}$$

of Minkowski's 1st Theorem and they verified Conjecture 4.6 for $n = 2$. Malikiosis [**Mal10a**] proposed an inductive approach to this problem and managed to settle the case $n = 3$. In fact, he obtained his result for general $K \in \mathcal{K}^n$ and suggested that in Conjecture 4.6 the symmetry assumption on the body is not necessary. In a second paper, Malikiosis [**Mal10b**] confirms Conjecture 4.6 up to a factor of roughly $1.644^n$ on the right hand side, and recently, he proved the inequality for the class of ellipsoids [**Mal12**].

**4.2.1. A Coefficient-wise Approach.** In the following, we investigate the somewhat weaker inequality

$$\mathrm{G}(K,\Lambda) \leq \prod_{i=1}^{n} \left( \frac{2}{\lambda_i(K,\Lambda)} + 1 \right). \tag{4.2}$$

It has the advantage of being amenable to Ehrhart theory as we see below. First, we note that it suffices to prove this inequality for lattice polytopes. Indeed, writing $P_K = \operatorname{conv}\{K \cap \Lambda\}$, for $K \in \mathcal{K}^n$, we have $\mathrm{G}(K,\Lambda) = \mathrm{G}(P_K,\Lambda)$ and since $\mathcal{D}P_K \subseteq \mathcal{D}K$, the monotonicity of the successive minima implies $\lambda_i(P_K,\Lambda) \geq \lambda_i(K,\Lambda)$ for all $i = 1,\ldots,\dim P_K$. Moreover, by the linear invariance of the successive minima and $\Lambda = A\mathbb{Z}^n \in \mathcal{L}^n$ for some $A \in \mathrm{GL}_n(\mathbb{R})$, it is no restriction to consider only the case $\Lambda = \mathbb{Z}^n$. In this case, we use the short notation $\lambda_i(K) = \lambda_i(K,\mathbb{Z}^n)$.

Henk and Wills [**HW08b**] proposed to consider both sides of Inequality (4.2) as polynomials, that is, for lattice polytopes $P \in \mathcal{P}^n$ and $k \in \mathbb{N}$,

$$\mathrm{G}(kP) = \sum_{i=0}^{n} \mathrm{g}_i(P)k^i \leq \prod_{i=1}^{n} \left( \frac{2}{\lambda_i(P)}k + 1 \right) = \sum_{i=0}^{n} \sigma_i(P)k^i = \mathrm{L}(kP),$$

where

$$\sigma_i(P) = \sum_{J \in \binom{[n]}{i}} \prod_{j \in J} \frac{2}{\lambda_j(P)} \tag{4.3}$$

is the $i$th elementary symmetric polynomial evaluated at the numbers $\frac{2}{\lambda_i(P)}$. One can now ask whether the coefficient-wise inequalities

$$\mathrm{g}_i(P) \leq \sigma_i(P) \quad \text{for all} \quad i = 0,\ldots,n \tag{4.4}$$

hold. This question is supported by the valid cases:

- $\mathrm{g}_0(P) = \sigma_0(P) = 1$,
- $\mathrm{g}_n(P) \leq \sigma_n(P)$, which is a reformulation of Minkowski's 2nd Theorem,
- $\mathrm{g}_{n-1}(P) \leq \sigma_{n-1}(P)$, which holds for $P \in \mathcal{P}_0^n$ and is proved in [**HSW05**].

Later on, we see how this approach leads to positive results for lattice zonotopes and lattice-face polytopes. However, in general the coefficient-wise inequalities do not hold as the following examples show.

**Proposition 4.7.** *Let $Q_l^n = \operatorname{conv}\{lC_{n-1} \times \{0\}, \pm e_n\}$, where $C_n = [-1,1]^n$ is the standard cube. For any constant $c > 0$ and large enough $l \in \mathbb{N}$,*

*i)* $\mathrm{g}_{n-2}(Q_l^n) > c\,\sigma_{n-2}(Q_l^n)$, *if $n \geq 3$, and*
*ii)* $\mathrm{g}_{n-3}(Q_l^n) > c\,\sigma_{n-3}(Q_l^n)$, *if $n \geq 4$.*

PROOF. By cutting $kQ_l^n$ by lattice planes orthogonal to $e_n$, we find that the Ehrhart polynomial of $Q_l^n$ is given by

$$\begin{aligned}
\mathrm{G}(kQ_l^n) &= (2kl+1)^{n-1} + 2\sum_{j=0}^{k-1}(2jl+1)^{n-1} \\
&= (2kl+1)^{n-1} + 2\sum_{j=0}^{k-1}\sum_{i=0}^{n-1}\binom{n-1}{i}(2jl)^i \\
&= \sum_{i=0}^{n-1}\binom{n-1}{i}(2l)^i k^i + 2\sum_{i=0}^{n-1}\binom{n-1}{i}(2l)^i\left(\sum_{j=0}^{k-1} j^i\right).
\end{aligned}$$

Faulhaber's formula (see [**AS92**, §23.1]) expresses the sum $\sum_{j=0}^{k-1} j^i$ as a polynomial in $k$, more precisely

$$\sum_{j=0}^{k-1} j^i = \frac{1}{i+1}\sum_{j=0}^{i}\binom{i+1}{j}B_j k^{i-j+1} = \frac{1}{i+1}\sum_{j=1}^{i+1}\binom{i+1}{j}B_{i-j+1}k^j,$$

where $B_j$ is the $j$th Bernoulli number. Therefore, we continue as

$$\begin{aligned}
\mathrm{G}(kQ_l^n) &= \sum_{i=0}^{n-1}\binom{n-1}{i}(2l)^i k^i + 2\sum_{i=0}^{n-1}\binom{n-1}{i}\frac{(2l)^i}{i+1}\sum_{j=1}^{i+1}\binom{i+1}{j}B_{i-j+1}k^j \\
&= \sum_{i=0}^{n-1}\binom{n-1}{i}(2l)^i k^i + \frac{2}{n}\sum_{j=1}^{n}\sum_{i=j-1}^{n-1}\binom{n}{i+1}(2l)^i\binom{i+1}{j}B_{i-j+1}k^j.
\end{aligned}$$

Thus, the Ehrhart coefficients of $Q_l^n$ are given by

$$\mathrm{g}_i(Q_l^n) = \binom{n-1}{i}(2l)^i + \frac{2}{n}\sum_{j=i-1}^{n-1}\binom{n}{j+1}(2l)^j\binom{j+1}{i}B_{j-i+1}, \tag{4.5}$$

for all $i = 1, \ldots, n$. In view of $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}$ and $B_3 = 0$, this gives

$$\begin{aligned}
\mathrm{g}_{n-2}(Q_l^n) &= (n-1)(2l)^{n-3}\left(\frac{2}{3}l^2+1\right) \quad \text{and} \\
\mathrm{g}_{n-3}(Q_l^n) &= \frac{2}{3}\binom{n-1}{2}(2l)^{n-4}\left(2l^2+1\right).
\end{aligned}$$

The successive minima of $Q_l^n$ are $\lambda_1(Q_l^n) = \ldots = \lambda_{n-1}(Q_l^n) = \frac{1}{l}$ and $\lambda_n(Q_l^n) = 1$, from which we get

$$\sigma_i(Q_l^n) = \binom{n-1}{i}(2l)^i + 2\binom{n-1}{i-1}(2l)^{i-1} \quad \text{for all} \quad i = 1, \ldots, n-1.$$

Seen as polynomials in $l$, the $\sigma_i(Q_l^n)$ have degree $i$, whereas $\mathrm{g}_{n-2}(Q_l^n)$ and $\mathrm{g}_{n-3}(Q_l^n)$ have degree $n-1$ and $n-2$, respectively, with positive leading coefficients. Thus, for indices $i \in \{n-2, n-3\}$ and every fixed constant $c$, there exists an $l \in \mathbb{N}$ such that $\mathrm{g}_i(Q_l^n) > c\,\sigma_i(Q_l^n)$. □

Note that Conjecture 4.6 nevertheless holds for the polytopes $Q_l^n$. In fact,

$$\mathrm{G}(Q_l^n) = (2l+1)^{n-1} + 2 \leq 3(2l+1)^{n-1} = \prod_{i=1}^{n} \left\lfloor \frac{2}{\lambda_i(Q_l^n)} + 1 \right\rfloor .$$

Moreover, we can extract from the above proof that

$$\mathrm{G}(kQ_l^3) = \frac{8}{3} l^2 k^3 + 4lk^2 + \left(\frac{4}{3} l^2 + 2\right) k + 1 \quad \text{and}$$
$$\mathrm{L}(kQ_l^3) = 8l^2 k^3 + (4l^2 + 8l)k^2 + (4l+2)k + 1.$$

In particular, the coefficient-wise approach does not work on the class of lattice polytopes with nonnegative Ehrhart coefficients. We see now that it does work when we have even more information on the $\mathrm{g}_i$'s.

**4.2.2. Coefficient-wise Approach for Lattice Zonotopes.** In this part, we investigate the proposed coefficient-wise inequalities (4.4) on the class of lattice zonotopes. Our goal is to prove the following theorem where we require the generators of the zonotope $Z \in \mathcal{P}^n$ to be in *general position*, that is, every $n$ of them are linearly independent.

**Theorem 4.8** ([BHHL11])**.** *Let $Z = \sum_{i=1}^m [0, v_i]$ be a lattice zonotope whose generators $v_1, \ldots, v_m \in \mathbb{Z}^n$ are in general position. Then*

$$\mathrm{g}_i(Z) \leq \frac{\binom{m}{i}}{\binom{n}{i}} \sigma_i(Z) \quad \textit{for all} \quad i = 1, \ldots, n.$$

The case $m = n$ gives an affirmative answer to the coefficient-wise approach to Conjecture 4.6 for the class of lattice parallelepipeds.

**Corollary 4.9.** *Let $Z \in \mathcal{P}^n$ be a lattice parallelepiped. Then*

$$\mathrm{g}_i(Z) \leq \sigma_i(Z) \quad \textit{for all} \quad i = 1, \ldots, n.$$

We note that these inequalities are best possible. For instance, consider the cube $Z = [0,1]^n = \sum_{i=1}^n [0, e_i]$. We have $\lambda_i(\mathcal{D}Z) = \lambda_i([-1,1]^n) = 1$, and $\mathrm{G}(kZ) = (k+1)^n$ for every integer $k \in \mathbb{N}$. Hence $\mathrm{g}_i(Z) = \binom{n}{i} = \sigma_i(Z)$.

The arguments for Theorem 4.8 are based on the particularly nice description of the Ehrhart coefficients of lattice zonotopes. For the proof of these identities we refer to Linke's thesis [Lin11] and to [BHHL11].

**Lemma 4.10.** *Let $Z = \sum_{i=1}^m [0, v_i]$ be a lattice zonotope and for $J \in \binom{[m]}{i}$ let $P_J = \sum_{j \in J} [0, v_j]$ be the $i$-dimensional parallelepiped generated by the vectors that are indexed by $J$. Then*

$$\mathrm{g}_i(Z) = \sum_{J \in \binom{[m]}{i}} \frac{\mathrm{vol}_i(P_J)}{\det(\mathrm{lin}\, P_J \cap \mathbb{Z}^n)} \quad \textit{for all} \quad i = 1, \ldots, n.$$

Moreover, we have to provide two auxiliary lemmas.

**Lemma 4.11.** *Let $\{b_1, \ldots, b_n\}$ and $\{a_1, \ldots, a_n\}$ be two bases of an $n$-dimensional vector space $V$ and let $i \in \{1, \ldots, n-1\}$. Then there exists a bijection $\phi : \binom{[n]}{i} \to \binom{[n]}{n-i}$ such that $\{b_k : k \in I\} \cup \{a_j : j \in \phi(I)\}$ is a basis of $V$ for all $I \in \binom{[n]}{i}$.*

Proof. We use a standard linear algebra argument involving the exterior algebra $\Lambda(V)=\bigoplus_{i=0}^{n}\Lambda_i(V)$ of $V$ for whose definition and properties we refer to [MLB79, Ch. XVI]. For $I\in\binom{[n]}{i}$ and $J\in\binom{[n]}{n-i}$ let $b_I=\wedge_{k\in I}b_k\in\Lambda_i(V)$ and $a_J=\wedge_{j\in J}a_j\in\Lambda_{n-i}(V)$, respectively. Consider the square matrix $M$ with row index set $\binom{[n]}{i}$ and column index set $\binom{[n]}{n-i}$, whose $(I,J)$-entry is $b_I\wedge a_J$. First, we note that $\det M\neq 0$. Here we identify every entry $b_I\wedge a_J$ of $M$ with its coefficient with respect to a fixed basis vector of $\Lambda_n(V)$.

Assume the contrary and suppose that some nontrivial linear combination of the rows of $M$ is zero, say

$$0=\sum_{I\in\binom{[n]}{i}}c_I(b_I\wedge a_J)=\left(\sum_{I\in\binom{[n]}{i}}c_Ib_I\right)\wedge a_J,$$

for all $J\in\binom{[n]}{n-i}$, with scalars $c_I$, not all zero. Expanding the nonzero vector $\sum_{I\in\binom{[n]}{i}}c_Ib_I\in\Lambda_i(V)$ in terms of the basis $\{a_I : I\in\binom{[n]}{i}\}$ of $\Lambda_i(V)$ yields

$$0=\left(\sum_{I\in\binom{[n]}{i}}d_Ia_I\right)\wedge a_J=\sum_{I\in\binom{[n]}{i}}d_I(a_I\wedge a_J),$$

for all $J\in\binom{[n]}{n-i}$, with scalars $d_I$, not all zero. But since $a_I\wedge a_J\neq 0$ if and only if $I=[n]\setminus J$, we conclude that $d_I=0$ for all $I\in\binom{[n]}{i}$, a contradiction.

Therefore $\det M\neq 0$, and by Leibniz' formula there exists a bijection $\phi:\binom{[n]}{i}\to\binom{[n]}{n-i}$ with $b_I\wedge a_{\phi(I)}\neq 0$ for all $I\in\binom{[n]}{i}$. Equivalently, the set $\{b_k : k\in I\}\cup\{a_j : j\in\phi(I)\}$ is a basis of $V$ for all $I\in\binom{[n]}{i}$ (cf. [MLB79, Thm. XVI.13]), which we wanted to show. □

**Lemma 4.12.** *Let $K\in\mathcal{K}_0^n$ and let $a_1,\dots,a_n\in\mathbb{Z}^n$ be linearly independent such that $a_j\in\lambda_j(K)\,K$ for all $j=1,\dots,n$. Moreover, for $i\in\{1,\dots,n-1\}$ let $\overline{L}$ be an $i$-dimensional linear subspace containing $i$ linearly independent points of $\mathbb{Z}^n$, and assume that $\operatorname{lin}\{a_{j_1},\dots,a_{j_{n-i}}\}\cap\overline{L}=\{0\}$. Then*

$$\prod_{j=1}^{i}\lambda_j(K\cap\overline{L},\mathbb{Z}^n\cap\overline{L})\geq\prod_{k\notin\{j_1,\dots,j_{n-i}\}}\lambda_k(K).$$

Proof. For abbreviation we let $\overline{\Lambda}=\mathbb{Z}^n\cap\overline{L}$, $\overline{K}=K\cap\overline{L}$, $\overline{\lambda}_j=\lambda_j(K\cap\overline{L},\mathbb{Z}^n\cap\overline{L})$ for all $j=1,\dots,i$, and $\lambda_j=\lambda_j(K)$ for all $j=1,\dots,n$. Moreover, let $\overline{w}_1,\dots,\overline{w}_i\in\overline{\Lambda}$ be linearly independent such that $\overline{w}_j\in\overline{\lambda}_j\,\overline{K}$. Let $j_1<j_2<\dots<j_{n-i}$ and let $k_1<k_2<\dots<k_i$ be the indices in $[n]\setminus\{j_1,\dots,j_{n-i}\}$. Suppose there exists an index $l\in\{1,\dots,i\}$ with

$$\overline{\lambda}_l<\lambda_{k_l}, \tag{4.6}$$

and let $m$ be the smallest index such that $\lambda_m=\lambda_{k_l}$. Since $\overline{K}\subset K$, $\overline{\Lambda}\subset\mathbb{Z}^n$, we get by (4.6), the choice of $m$ and the definition of the successive minima that

$$\{\overline{w}_1,\dots,\overline{w}_l\}\cup\{a_j : j\in[m-1]\cap\{j_1,\dots,j_{n-i}\}\}\subseteq\operatorname{int}(\lambda_m\,K)\cap\mathbb{Z}^n.$$

Since there are at most $l-1$ indices in $[m-1]$ belonging to $\{k_1,\dots,k_i\}$, we conclude that $\#\{j : j\in[m-1]\cap\{j_1,\dots,j_{n-i}\}\}\geq m-l$. Hence, on the left hand side of the inclusion above we have at least $m$ lattice vectors which by the assumption $\mathrm{lin}\{a_{j_1},\dots,a_{j_{n-i}}\}\cap\overline{L}=\{0\}$ are linearly independent. This, however, contradicts the definition of $\lambda_m$, and so we have shown $\overline{\lambda}_l\geq\lambda_{k_l}$ for all $l=1,\dots,i$, which implies the assertion. $\square$

Proof of Theorem 4.8. We abbreviate $\lambda_j=\lambda_j(\mathcal{D}Z)$ and for $J\in\binom{[m]}{i}$ we let $P_J=\sum_{j\in J}[0,v_j]$ and $\Lambda_J=\mathrm{lin}\{v_j:j\in J\}\cap\mathbb{Z}^n$. We have

$$\mathcal{D}P_J=P_J-P_J=\left\{\sum_{j\in J}\mu_jv_j:-1\leq\mu_j\leq 1\right\}.$$

In view of Lemma 4.10 and $\mathrm{vol}_i(P_J)=\frac{1}{2^i}\mathrm{vol}_i(\mathcal{D}P_J)$, we have to show

$$\mathrm{g}_i(Z)=\frac{1}{2^i}\sum_{J\in\binom{[m]}{i}}\frac{\mathrm{vol}_i(\mathcal{D}P_J)}{\det\Lambda_J}\leq\frac{\binom{m}{i}}{\binom{n}{i}}\sum_{I\in\binom{[n]}{i}}\frac{1}{\prod_{k\in I}\lambda_k}.$$

Applying Minkowski's 2nd Theorem to each $\mathcal{D}P_J$, we can estimate the summands on the left hand side and get

$$\mathrm{g}_i(Z)=\frac{1}{2^i}\sum_{J\in\binom{[m]}{i}}\frac{\mathrm{vol}_i(\mathcal{D}P_J)}{\det\Lambda_J}\leq\sum_{J\in\binom{[m]}{i}}\frac{1}{\prod_{j=1}^i\lambda_j(\mathcal{D}P_J,\Lambda_J)}.$$

Hence it suffices to prove

$$\sum_{J\in\binom{[m]}{i}}\frac{1}{\prod_{j=1}^i\lambda_j(\mathcal{D}P_J,\Lambda_J)}\leq\frac{\binom{m}{i}}{\binom{n}{i}}\sum_{I\in\binom{[n]}{i}}\frac{1}{\prod_{k\in I}\lambda_k}.\tag{4.7}$$

Since every $J\in\binom{[m]}{i}$ is contained in $\binom{m-i}{n-i}$ sets $I\in\binom{[m]}{n}$, we can replace the left hand side by

$$\frac{1}{\binom{m-i}{n-i}}\sum_{I\in\binom{[m]}{n}}\sum_{J\in\binom{I}{i}}\frac{1}{\prod_{j=1}^i\lambda_j(\mathcal{D}P_J,\Lambda_J)}.$$

and (4.7) becomes

$$\sum_{I\in\binom{[m]}{n}}\sum_{J\in\binom{I}{i}}\frac{1}{\prod_{j=1}^i\lambda_j(\mathcal{D}P_J,\Lambda_J)}\leq\binom{m}{n}\sum_{I\in\binom{[n]}{i}}\frac{1}{\prod_{k\in I}\lambda_k}.\tag{4.8}$$

Now, let $a_1,\dots,a_n\in\mathbb{Z}^n$ be linearly independent with $a_j\in\lambda_j\mathcal{D}Z$ for all $j=1,\dots,n$. By our assumption, every choice of $n$ generators $v_{i_1},\dots,v_{i_n}\in\mathbb{Z}^n$ is linearly independent and so we may apply Lemma 4.11 to every $n$-subset $I=\{i_1,\dots,i_n\}\subseteq[m]$. This means that there is a bijection $\phi:\binom{I}{i}\to\binom{[n]}{n-i}$ such that for all $J\in\binom{I}{i}$

$$\mathrm{lin}\{v_j:j\in J\}\cap\mathrm{lin}\{a_k:k\in\phi(J)\}=\{0\}.$$

Thus together with Lemma 4.12, we get

$$\prod_{j=1}^i\lambda_j(\mathcal{D}Z\cap\mathrm{lin}\{v_l:l\in J\},\mathbb{Z}^n\cap\mathrm{lin}\{v_l:l\in J\})\geq\prod_{k\notin\phi(J)}\lambda_k,$$

and by $\lambda_j(\mathcal{D}P_J, \Lambda_J) \geq \lambda_j(\mathcal{D}Z \cap \operatorname{lin}\{v_l : l \in J\}, \mathbb{Z}^n \cap \operatorname{lin}\{v_l : l \in J\})$, we get

$$\frac{1}{\prod_{j=1}^{i} \lambda_j(\mathcal{D}P_J, \Lambda_J)} \leq \frac{1}{\prod_{k \notin \phi(J)} \lambda_k}.$$

Since $\phi$ is a bijection, we have

$$\sum_{J \in \binom{I}{i}} \frac{1}{\prod_{j=1}^{i} \lambda_j(\mathcal{D}P_J, \Lambda_J)} \leq \sum_{T \in \binom{[n]}{i}} \frac{1}{\prod_{t \in T} \lambda_t},$$

which yields (4.8) as desired. □

A disadvantage of the bound in Theorem 4.8 is that it depends on the number of generators of the zonotope. A similar estimate, which moreover holds for arbitrary lattice zonotopes, appears in [**Lin11**] (cf. [**BHHL11**]) where the factor only depends on the dimension.

**Theorem 4.13.** *Let $Z \in \mathcal{P}^n$ be a lattice zonotope. Then*

$$\frac{\mathrm{g}_i(Z)}{\mathrm{vol}(Z)} \leq \frac{n!}{i!} \prod_{j=i+1}^{n} \lambda_j(\mathcal{D}Z) \quad \textit{for all} \quad i = 1, \ldots, n.$$

*In particular, $\mathrm{g}_i(Z) \leq \frac{n!}{i!}\sigma_i(Z)$ for all $i = 1, \ldots, n$.*

**4.2.3. Coefficient-wise Approach for Lattice-face Polytopes.** In the following, we study the coefficient-wise approach to Inequality (4.2) on the class of lattice-face polytopes. To this end, recall that for $i = 0, \ldots, n$ we denote by $\pi^{(i)}$ the projection that forgets the last $i$ coordinates. For sake of brevity, we write $\pi = \pi^{(1)}$.

Our considerations are based on the nice geometric description of the Ehrhart coefficients of lattice-face polytopes that is due to Liu.

**Theorem 4.14** (Liu [**Liu09**])**.** *Let $P \in \mathcal{P}^n$ be a lattice-face polytope. Then*

$$\mathrm{G}(kP) = \sum_{i=0}^{n} \mathrm{vol}_i(\pi^{(n-i)}(P))k^i,$$

*where* $\mathrm{vol}_0(\pi^{(n)}(P)) \coloneqq 1$.

Liu also provides a list of useful properties of these particular polytopes.

**Lemma 4.15** (Liu [**Liu09**])**.** *Let $P \in \mathcal{P}^n$ be a lattice-face polytope. Then*

*i) $\pi(P) \in \mathcal{P}^{n-1}$ is a lattice-face polytope.*
*ii) $mP$ is a lattice-face polytope, for any integer $m$.*
*iii) Let $H$ be an $(n-1)$-dimensional affine space spanned by some subset of* $\operatorname{vert} P$*. Then, for every lattice point $y \in \mathbb{Z}^{n-1}$, the preimage $\pi^{-1}(y) \cap H$ is also a lattice point.*
*iv) $P$ is a lattice polytope.*

The successive minima of lattice-face polytopes behave very nicely with respect to projections.

**Lemma 4.16.** *Let $P \in \mathcal{P}^n$ be a lattice-face polytope.*

*i) If $P$ is centrally symmetric, then*

$$\lambda_j(\pi^{(n-i)}(P), \mathbb{Z}^i) \geq \lambda_j(P) \quad \textit{for all} \quad 1 \leq j \leq i \leq n.$$

*ii) If* $0 \in \operatorname{vert} P$ *and* $\mathcal{S}P = \operatorname{conv}(P, -P)$*, then*

$$\lambda_j(\pi^{(n-i)}(\mathcal{S}P), \mathbb{Z}^i) \geq \lambda_j(\mathcal{S}P) \quad \text{for all} \quad 1 \leq j \leq i \leq n.$$

PROOF. i): It suffices to show that $\lambda_j = \lambda_j(\pi(P), \mathbb{Z}^{n-1}) \geq \lambda_j(P)$ for all $j = 1, \dots, n-1$. To this end, let $\{z_1, \dots, z_j\} \subset \mathbb{Z}^{n-1}$ be linearly independent lattice points in $\lambda_j \pi(P)$. Our first observation is that every set of vectors $\{\bar{z}_1, \dots, \bar{z}_j\} \subset \mathbb{R}^n$ with $z_i = \pi(\bar{z}_i)$ for $i = 1, \dots, j$ is also linearly independent. Indeed, any linear dependence would be preserved by the projection $\pi$. Therefore, we need to show that, for all $i = 1, \dots, j$, there is always a lattice point $\bar{z}_i \in \lambda_j P$ such that $z_i = \pi(\bar{z}_i)$.

In order to see this, we fix an $i$ and set $z = z_i$ and $\mu = \lambda_i > 0$. In particular, we have $z \in \mu\pi(P) \cap \mathbb{Z}^{n-1}$. Since, $0 \in \mu\pi(P)$, there are linearly independent $v_1, \dots, v_{n-1} \in \operatorname{vert} \pi(P)$ and $\gamma_1, \dots, \gamma_{n-1} \in [0, 1]$ with $\sum_{i=1}^{n-1} \gamma_i \leq 1$, such that $z = \mu \sum_{i=1}^{n-1} \gamma_i v_i$. For every $v_i$ there is a vertex $\bar{v}_i$ of $P$ in the preimage of $v_i$ under $\pi$, and these $\bar{v}_1, \dots, \bar{v}_{n-1}$ are linearly independent. This means, that the hyperplane $H = \operatorname{aff}\{0, \bar{v}_1, \dots, \bar{v}_{n-1}\} = \operatorname{aff}\{\pm\bar{v}_1, \dots, \pm\bar{v}_{n-1}\}$ is $(n-1)$-dimensional and spanned by vertices of $P$, because $P = -P$. Therefore, since $P$ is a lattice-face polytope, we have by Lemma 4.15 iii) that the point $\bar{z} = \pi^{-1}(z) \cap H$ has integral coordinates. It remains to show that $\bar{z}$ lies in $\mu P$. The containment of $\bar{z}$ in $H$ gives us $\beta_1, \dots, \beta_{n-1} \in \mathbb{R}$ such that $\bar{z} = \sum_{i=1}^{n-1} \beta_i \bar{v}_i$. Furthermore, it is

$$\mu \sum_{i=1}^{n-1} \gamma_i v_i = z = \pi(\bar{z}) = \sum_{i=1}^{n-1} \beta_i \pi(\bar{v}_i) = \sum_{i=1}^{n-1} \beta_i v_i,$$

which yields $\beta_i = \mu\gamma_i$ for all $i = 1, \dots, n-1$, because the $v_i$'s were chosen to be linearly independent. So, with $\sum_{i=1}^{n-1} \gamma_i \leq 1$, we get $\bar{z} = \mu \sum_{i=1}^{n-1} \gamma_i \bar{v}_i \in \mu P$ as claimed.

In conclusion, we found the point $\bar{z} \in \mu P \cap \mathbb{Z}^n$ for which $z = \pi(\bar{z})$ and we are done.

The proof of ii) follows the same lines as above. We only note that $\operatorname{vert} \mathcal{S}P \subseteq \{\pm v : v \in \operatorname{vert} P\}$ and the assumption $0 \in \operatorname{vert} P$ is used to simultaneously control the signs of the vertices that span $H$. $\square$

**Remark 4.17.** *Lemma 4.16 does not hold for general polytopes. For example, consider* $P_t = \operatorname{conv}\left\{\pm\binom{t-1}{1}, \pm\binom{t}{1}\right\}$ *for* $t \in \mathbb{N}$*. We have* $\lambda_1(P_t, \mathbb{Z}^2) = 1$ *and* $\lambda_1(P_t|e_2^\perp, \mathbb{Z}) = \frac{1}{t}$*. Therefore, there does not even exist a constant depending on the dimension such that the successive minima of the projection could be bounded from below, up to this constant, by those of the original polytope.*

We collected all ingredients to give a positive answer to the coefficient-wise approach for centrally symmetric lattice-face polytopes. For not necessarily symmetric lattice-face polytopes, we get an analogous result with respect to the somewhat weaker symmetrization $\mathcal{S}P$.

**Theorem 4.18.** *Let* $P \in \mathcal{P}^n$ *be a lattice-face polytope.*

*i) If* $P$ *is centrally symmetric, then*

$$\mathrm{g}_i(P) \leq \sigma_i(P) \quad \text{for all} \quad i = 1, \dots, n.$$

*ii) If* $0 \in \operatorname{vert} P$ *and* $\mathcal{S}P = \operatorname{conv}(P, -P)$*, then*

$$\mathrm{g}_i(P) \le \sigma_i(\mathcal{S}P) \quad \textit{for all} \quad i = 1, \dots, n.$$

Proof. i): By Theorem 4.14 and Minkowski's 2nd Theorem (cf. Theorem 4.3) applied to $\pi^{(n-i)}(P)$, we obtain

$$\mathrm{g}_i(P) = \operatorname{vol}_i(\pi^{(n-i)}(P)) \le \prod_{j=1}^{i} \frac{2}{\lambda_j(\pi^{(n-i)}(P), \mathbb{Z}^i)} \quad \text{for all} \quad i = 1, \dots, n.$$

Using Lemma 4.16 i), we continue this inequality to get

$$\mathrm{g}_i(P) \le \prod_{j=1}^{i} \frac{2}{\lambda_j(P)} \le \sigma_i(P).$$

Note that for $i \ne n$ the last inequality sign is actually a strict one.

ii): By $P \subset \mathcal{S}P$ we have $\operatorname{vol}_i(\pi^{(n-i)}(P)) \le \operatorname{vol}_i(\pi^{(n-i)}(\mathcal{S}P))$. Thus, using Lemma 4.16 ii), we can argue in the same way as in the first part. □

As an example for Theorem 4.18, we want to construct a three-dimensional centrally symmetric lattice-face polytope. To this end, we use Liu's proof for the fact that every lattice polytope is affinely equivalent to a lattice-face polytope (cf. [**Liu09**, Sect. 5]). In fact, she provides an algorithm to determine such a transformation explicitly. Let us begin with the crosspolytope

$$P = \operatorname{conv}\left\{ \pm\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}, \pm\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \pm\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

This crosspolytope is, in the notation of Liu, in $\pi$-*general position*, which means that for every subset $U \subseteq \operatorname{vert} P$ that spans a $k$-dimensional affine space, we have $\pi^{(n-k)}(\operatorname{aff} U) = \mathbb{R}^k$. This property is necessary for a lattice-face polytope and can be achieved for every lattice polytope by a suitable transformation that we do not discuss here (cf. [**Liu09**, Prop. 5.3]).

Liu's algorithm is based on the following observation. Let $H = \{x \in \mathbb{R}^n : a^{\intercal} x = b\}$ be an affine $(n-1)$-dimensional space with integral normal vector $a \in \mathbb{Z}^n$, $a_n \ne 0$ and $b \in \mathbb{Z}$. The lattice-face condition $\pi(H \cap \mathbb{Z}^n) = \mathbb{Z}^{n-1}$ is fulfilled when $a_n = 1$. Indeed, in this case we find a preimage $(y, y_n) \in H \cap \mathbb{Z}^n$ of $y \in \mathbb{Z}^{n-1}$ by setting $y_n = b - a_1 y_1 - \ldots - a_{n-1} y_{n-1}$. Now, if $a_n \ne 1$, then we can correct that by applying the transformation $\operatorname{diag}(1, \dots, 1, a_n)$ to $H$, where $\operatorname{diag}(c_1, \dots, c_n)$ denotes the diagonal matrix with diagonal entries $c_1, \dots, c_n$.

The first step in transforming $P$ into a lattice-face polytope is to consider the projection of $P$ onto the space of the first two coordinates and transform it into a lattice-face polygon. As noted above, the lattice-face condition holds if every line through every two vertices of $\pi(P)$ has a normal vector with last coordinate equal to 1. The line through the vertices $\binom{1}{4}$ and $\binom{-2}{-1}$ for example has $\binom{-5}{3}$ as a normal vector. Hence, multiplying $\pi(P)$ with the diagonal matrix $\operatorname{diag}(1,3)$ establishes the lattice-face condition for these two vertices. In order to achieve this simultaneously for all lines that are spanned by vertices of $\pi(P)$, we have to multiply $\pi(P)$ with $\operatorname{diag}(1, m)$, where $m \in \mathbb{Z}$ is a common multiple of the last coordinates of normal vectors of such lines.

In our concrete case, we can choose $m = 30$ and thus $\operatorname{diag}(1, 30)\,\pi(P) = \operatorname{conv}\left\{\pm\binom{1}{120}, \pm\binom{2}{30}, \pm\binom{3}{0}\right\}$ is a lattice-face polygon.

The second step is to consider two-dimensional affine subspaces that are spanned by vertices of $P' = \operatorname{diag}(1, 30, 1)\,P$. For instance, the affine space that is spanned by $\pm(1, 120, 0)^\intercal$ and $(2, 30, 1)^\intercal$ has $(-120, 1, 210)^\intercal$ as a normal vector. Again, by the above observation, this means that multiplying $P'$ with $\operatorname{diag}(1, 1, 210)$ establishes the lattice-face property for this affine space. Now, 36960 is a common multiple of the last coordinates of normal vectors of the two-dimensional affine spaces that are spanned by vertices of $P'$. Therefore, by Liu's arguments,

$$\bar{P} = \operatorname{diag}(1, 30, 36960) \cdot P = \operatorname{conv}\left\{\pm\begin{pmatrix}1\\120\\0\end{pmatrix}, \pm\begin{pmatrix}2\\30\\36960\end{pmatrix}, \pm\begin{pmatrix}3\\0\\0\end{pmatrix}\right\}$$

is a lattice-face polytope.

With the help of `polymake` [GJ00], we found that the Ehrhart polynomial of this polytope is given by $1+6k+720k^2+17\,740\,800k^3$. Moreover, we found lattice vectors in $\bar{P}$ that show $\lambda_1(\bar{P}) \leq \frac{1}{20000}$, $\lambda_2(\bar{P}) \leq \frac{1}{50}$ and $\lambda_3(\bar{P}) \leq \frac{1}{3}$. Therefore, abbreviating $\lambda_i = \lambda_i(\bar{P})$ for all $i = 1, 2, 3$, we have

$$\begin{aligned}
\mathrm{g}_1(\bar{P}) &= 6 < 40\,106 \leq \frac{2}{\lambda_1} + \frac{2}{\lambda_2} + \frac{2}{\lambda_3} = \sigma_1(\bar{P}),\\
\mathrm{g}_2(\bar{P}) &= 720 < 4\,240\,600 \leq \frac{4}{\lambda_1\lambda_2} + \frac{4}{\lambda_1\lambda_3} + \frac{4}{\lambda_2\lambda_3} = \sigma_2(\bar{P}), \quad \text{and}\\
\mathrm{g}_3(\bar{P}) &= 17\,740\,800 < 24\,000\,000 \leq \frac{8}{\lambda_1\lambda_2\lambda_3} = \sigma_3(\bar{P}).
\end{aligned}$$

## 4.3. On an Inequality of Henk, Schürmann and Wills

In 2005, Henk, Schürmann and Wills proved the following relation between $\mathrm{g}_{n-1}$ and the sum of the successive minima of a centrally symmetric lattice polytope.

**Theorem 4.19** (Henk, Schürmann and Wills [HSW05]). *Let $P \in \mathcal{P}_0^n$ be a lattice polytope. Then*

$$\frac{\mathrm{g}_{n-1}(P)}{\operatorname{vol}(P)} \leq \sum_{i=1}^{n} \frac{\lambda_i(P)}{2}.$$

Equality holds both for $C_n$ and $C_n^\star$, and more generally whenever $P$ is reflexive. This is because for reflexive polytopes, we have $\lambda_i(P) = 1$ for all $i = 1, \ldots, n$ and $\mathrm{g}_{n-1}(P) = \frac{n}{2}\operatorname{vol}(P)$ (cf. [BHW07, Lem. 3.1]).

As mentioned earlier, this result has a strong connection to the coefficient-wise approach to Conjecture 4.6 that we discussed in the previous section. Indeed, together with Minkowski's 2nd Theorem it implies (cf. (4.3) and (4.4))

$$\mathrm{g}_{n-1}(P) \leq \operatorname{vol}(P) \sum_{i=1}^{n} \frac{\lambda_i(P)}{2} \leq \sum_{J \in \binom{[n]}{n-1}} \prod_{j \in J} \frac{2}{\lambda_j(P)} = \sigma_{n-1}(P).$$

Here, we want to study possible generalizations of Theorem 4.19 – and thus also of the inequality $\mathrm{g}_{n-1}(P) \leq \sigma_{n-1}(P)$ – to the class of all (not necessarily centrally symmetric) lattice polytopes $P \in \mathcal{P}^n$. An advantage of the inequality in Theorem 4.19 is that it depends on the sum of the successive minima rather than on $\sigma_{n-1}(P)$ which consist of products of $n-1$ of them.

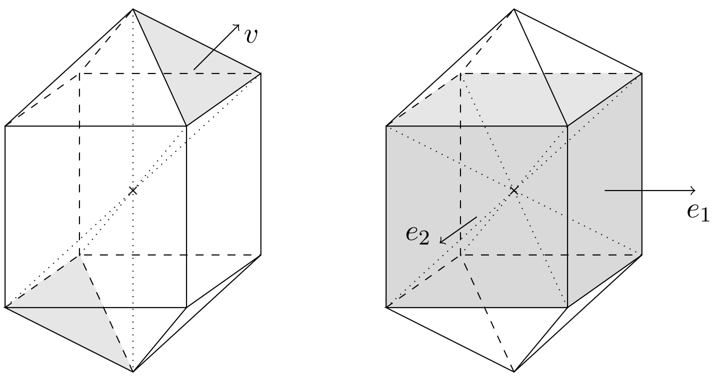
The argument of Henk, Schürmann and Wills is based on a generalization of the pyramid formula for the volume of polytopes, which we investigate first.

**4.3.1. On a Generalized Pyramid Formula.** The well-known pyramid formula states that the volume of a polytope can be obtained by summing the volumes of pyramids whose bases are the facets of the polytope and whose apices are a common interior point of the polytope. A generalization thereof is conjectured by Henk, Schürmann and Wills (personal communication, cf. [**HSW05**]). They propose to consider only those pyramids whose corresponding facets of the polytope are orthogonal to a given subspace in favor of an additional factor that reflects this selection. Recall that the centroid of a convex body $K \in \mathcal{K}^n$ is given by $\operatorname{cen}(K) = \frac{1}{\operatorname{vol}(K)} \int_K x\, dx$.

**Conjecture 4.20.** *Let $P = \{x \in \mathbb{R}^n : a_j^\intercal x \leq b_j,\ j = 1, \dots, m\}$ be a polytope with facets $F_j$ corresponding to the normal vector $a_j$. Further, let $L_k$ be a $k$-dimensional linear subspace. Then the centroid $c = \operatorname{cen}(P)$ of $P$ satisfies*

$$\operatorname{vol}(P) \geq \frac{n}{k} \sum_{a_j \in L_k} \operatorname{vol}(F_j^c),$$

*where $F_j^c = \operatorname{conv}\{c, F_j\}$.*




(a) $L_1 = \operatorname{lin}\{v\}$ (b) $L_2 = \operatorname{lin}\{e_1, e_2\}$

FIGURE 4.1. Selecting pyramids over facets of a polytope with respect to $L_k$.

The knowledge on this conjecture can be summarized as follows:

i) It holds for centrally symmetric polytopes $P \in \mathcal{P}_0^n$ as shown by Henk, Schürmann and Wills [**HSW05**, Lem. 3.1].
ii) Taking $P = C_n$ and $L_k = \operatorname{lin}\{e_1, \dots, e_k\}$ shows that the bound would be sharp.
iii) María A. Hernández Cifre (personal communication) showed it for arbitrary $P \in \mathcal{P}^n$ and subspaces $L_k$ such that $\#\{j : a_j \in L_k\} \leq 2k$. In particular, Conjecture 4.20 holds for $k = 1$.

iv) It holds for simplices, since $\#\{j : a_j \in L_k\} \leq k$ for all $k = 1, \ldots, n-1$ and all subspaces $L_k$.

Item ii) in this list can be generally explained as follows.

**Proposition 4.21.** *Let $L_k$ be a $k$-dimensional linear subspace of $\mathbb{R}^n$ and let $Q \subseteq L_k$ and $R \subseteq L_k^\perp$ be polytopes lying in these subspaces. Then the product polytope $P = Q \times R$ attains equality in Conjecture 4.20 with respect to $L_k$ and an arbitrary interior point $c \in \operatorname{int} P$. More precisely,*

$$\operatorname{vol}(P) = \frac{n}{k} \sum_{\substack{F \text{ a facet of } P \\ \text{with normal in } L_k}} \operatorname{vol}(F^c).$$

Proof. Since the claimed identity is invariant under rotations of $P$ and translations of $R$ by vectors in $L_k^\perp$, we can assume that $L_k = \operatorname{lin}\{e_1, \ldots, e_k\}$ and $0 \in \operatorname{int} R$. Let $G$ be a facet of $Q$ and write $h_G^q$ for the height of $G$ with respect to an interior point $q \in \operatorname{int} Q$. Using the pyramid formula in the subspace $L_k$, gives

$$\operatorname{vol}_k(Q) = \sum_{G \text{ a facet of } Q} \operatorname{vol}_k(G^q) = \sum_{G \text{ a facet of } Q} \frac{1}{k} \operatorname{vol}_{k-1}(G) h_G^q.$$

The facets $F$ of $P$ whose normal vectors are contained in $L_k$ are in one-to-one correspondence with products $G \times R$, where $G$ is a facet of $Q$. Therefore, for $c = q \times \{0\}^{n-k} \in \operatorname{int} P$, we get

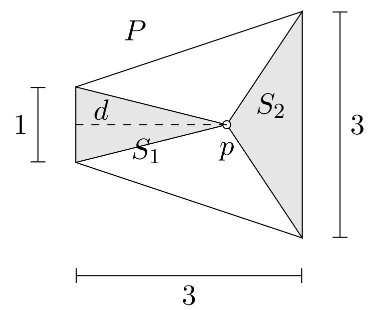
$$\operatorname{vol}(F^c) = \frac{1}{n} \operatorname{vol}_{n-1}(G \times R) h_G^q = \frac{1}{n} \operatorname{vol}_{k-1}(G) \operatorname{vol}_{n-k}(R) h_G^q.$$

Since $Q$ and $R$ live in orthogonal subspaces, we conclude by

$$\begin{aligned} \operatorname{vol}(P) = \operatorname{vol}_k(Q) \operatorname{vol}_{n-k}(R) &= \sum_{G \text{ a facet of } Q} \frac{1}{k} \operatorname{vol}_{k-1}(G) \operatorname{vol}_{n-k}(R) h_G^q \\ &= \frac{n}{k} \sum_{\substack{F \text{ a facet of } P \\ \text{with normal in } L_k}} \operatorname{vol}(F^c). \end{aligned}$$

□

Although the preceding proposition holds with respect to an arbitrary interior point of the polytope, the choice of the centroid as the apex of the pyramids in Conjecture 4.20 is, in general, essential. As an example, we consider the trapezoid $P$ in the following figure.



We check the claim of Conjecture 4.20 in dependence of the distance $d \in [0,3]$ of the point $p$ from the left vertical edge of $P$. We have $\mathrm{vol}(P) = 6$, $\mathrm{vol}(S_1) = \frac{d}{2}$ and $\mathrm{vol}(S_2) = \frac{3}{2}(3-d)$. Conjecture 4.20 claims that

$$\mathrm{vol}(P) \geq 2\,(\mathrm{vol}(S_1) + \mathrm{vol}(S_2)) = d + 3(3-d),$$

which is equivalent to $d \geq \frac{3}{2}$. Therefore, the claimed inequality does not hold, if the apex $p$ is closer to the short vertical edge of $P$ than to the longer one.

The proof by Henk, Schürmann and Wills of Conjecture 4.20 for centrally symmetric polytopes $P$ relies on the fact that the volume maximal section of $P$ parallel to $L_k$ contains the origin. With the help of existing bounds on the volume of such a maximal section in terms of the volume of the section through the centroid of $P$, we obtain a general estimate.

**Proposition 4.22.** *Let $P = \{x \in \mathbb{R}^n : a_j^\intercal x \leq b_j,\ j = 1, \ldots, m\}$ be a polytope with facets $F_j$ corresponding to the normal vector $a_j$. Further, let $L_k$ be a $k$-dimensional linear subspace. Then the centroid $c = \mathrm{cen}(P)$ of $P$ satisfies*

$$\mathrm{vol}(P) \geq \left(\frac{n-k+1}{n+1}\right)^{n-k} \frac{n}{k} \sum_{a_j \in L_k} \mathrm{vol}(F_j^c).$$

Proof. Fradelizi [Fra97] obtained the best possible inequality

$$\mathrm{vol}_{n-k}(P \cap (L_k^\perp + c)) \geq \left(\frac{n-k+1}{n+1}\right)^{n-k} \max_{x \in L_k} \mathrm{vol}_{n-k}(P \cap (L_k^\perp + x)).$$

Exchanging this for Equation (3.2) in the proof of [HSW05, Lem. 3.1] and following the remainder of the given arguments there yields the desired estimate. □

**4.3.2. Henk-Schürmann-Wills Inequality for Arbitrary Lattice Polytopes.** In this part, we investigate the inequality in Theorem 4.19 on the class of arbitrary lattice polytopes. Let us start with an example that illustrates the need of an additional factor in such a result.

We consider the standard simplex $S_n = \mathrm{conv}\{0, e_1, \ldots, e_n\}$. Its Ehrhart polynomial is given by (cf. [BR07, Sect. 2.3])

$$\mathrm{G}(kS_n) = \binom{k+n}{n} = \frac{1}{n!} \sum_{j=0}^{n} \left( \sum_{i=j}^{n} s(n,i) \binom{i}{j} n^{i-j} \right) k^j,$$

where $s(n,i)$ are the Stirling numbers of the 1st kind (cf. [Sta97, Sect. 4.3]). Hence, by $s(n,n) = 1$ and $s(n,n-1) = -\binom{n}{2}$, we get

$$\frac{\mathrm{g}_{n-1}(S_n)}{\mathrm{vol}(S_n)} = \frac{\frac{\binom{n+1}{2}}{n!}}{\frac{1}{n!}} = \frac{n(n+1)}{2}.$$

Now, any vertex of $S_n - S_n$ has coordinates either 0 or $\pm 1$, so $\lambda_1(\frac{1}{2}\mathcal{D}S_n) = \ldots = \lambda_n(\frac{1}{2}\mathcal{D}S_n) = 2$. Altogether this yields

$$\frac{\mathrm{g}_{n-1}(S_n)}{\mathrm{vol}(S_n)} = \frac{n(n+1)}{2} = \frac{n+1}{2} \sum_{i=1}^{n} \frac{\lambda_i(\frac{1}{2}\mathcal{D}S_n)}{2}. \tag{4.9}$$

On the class of lattice polygons this is extremal:

**Theorem 4.23.** *Let $P \in \mathcal{P}^2$ be a lattice polygon. Then*

$$\frac{\mathrm{g}_1(P)}{\mathrm{vol}(P)} \le \frac{3}{2}\left(\frac{\lambda_1(\frac{1}{2}\mathcal{D}P)}{2} + \frac{\lambda_2(\frac{1}{2}\mathcal{D}P)}{2}\right),$$

*and equality can only hold for triangles.*

Proof. McMullen [McM79] showed that $\mathrm{g}_1$ is a linear functional on the class of lattice polytopes – compare the proof of Proposition 3.13. Therefore, $\mathrm{g}_1(\mathcal{D}P) = 2\,\mathrm{g}_1(P)$, and by the Rogers-Shephard Inequality (1.1), we have $\mathrm{vol}(\mathcal{D}P) \le 6\,\mathrm{vol}(P)$, where equality holds if and only if $P$ is a triangle. Since $\mathcal{D}P$ is a centrally symmetric lattice polytope, we can apply Theorem 4.19 and obtain

$$\begin{aligned}\frac{\mathrm{g}_1(P)}{\mathrm{vol}(P)} \le 3\frac{\mathrm{g}_1(\mathcal{D}P)}{\mathrm{vol}(\mathcal{D}P)} &\le 3\left(\frac{\lambda_1(\mathcal{D}P)}{2} + \frac{\lambda_2(\mathcal{D}P)}{2}\right)\\ &= \frac{3}{2}\left(\frac{\lambda_1(\frac{1}{2}\mathcal{D}P)}{2} + \frac{\lambda_2(\frac{1}{2}\mathcal{D}P)}{2}\right).\end{aligned}$$ □

In the general case, Proposition 4.22 enables us to adjust the proof of Theorem 4.19 (see [HSW05, Thm. 1.2]) in order to get at least an estimate.

**Proposition 4.24.** *Let $P \in \mathcal{P}^n$ be a lattice polytope. Then*

$$\frac{\mathrm{g}_{n-1}(P)}{\mathrm{vol}(P)} < 1.45^{n+1}\frac{n+1}{2}\sum_{i=1}^{n}\frac{\lambda_i(\frac{1}{2}\mathcal{D}P)}{2}.$$

Proof. Due to the homogeneity of the Ehrhart coefficients and the successive minima, the desired inequality is invariant under dilatations by positive integers. Moreover, it is invariant under lattice translations. The centroid of the lattice polytope $P$ is a vector with rational entries. Indeed, for any triangulation of $P$ into lattice simplices $S_1, \dots, S_t$, we have

$$\mathrm{cen}(P) = \sum_{j=1}^{t}\frac{\mathrm{cen}(S_j)}{\mathrm{vol}(S_j)} \quad \text{and} \quad \mathrm{cen}(S_j) = \frac{1}{n+1}\sum_{v \in \mathrm{vert}\, S_j} v \in \mathbb{Q}^n.$$

So, by $\mathrm{vol}(S_j) \in \mathbb{Q}$ for all $j = 1, \dots, t$, we get $\mathrm{cen}(P) \in \mathbb{Q}^n$. Therefore, after a suitable integral dilatation and translation of $P$ we can assume that the centroid of $P$ is the origin.

Let $P = \{x \in \mathbb{R}^n : a_j^\intercal x \le b_j,\ j = 1, \dots, m\}$ with $b_j \in \mathbb{N}$ and primitive normal vectors $a_j \in \mathbb{Z}^n$. Writing $F_j$ for the corresponding facet, Equation (1.4) yields $\|a_j\| = \det(\mathrm{aff}\, F_j \cap \mathbb{Z}^n)$ for all $j = 1, \dots, m$. In view of the geometric description of $\mathrm{g}_{n-1}$, we thus get

$$\mathrm{g}_{n-1}(P) = \frac{1}{2}\sum_{j=1}^{m}\frac{\mathrm{vol}_{n-1}(F_j)}{\det(\mathrm{aff}\, F_j \cap \mathbb{Z}^n)} = \frac{1}{2}\sum_{j=1}^{m}\frac{\mathrm{vol}_{n-1}(F_j)}{\|a_j\|}.$$

We abbreviate $\lambda_i = \lambda_i(\mathcal{D}P)$ for all $i = 1, \dots, n$, and let $v_1, \dots, v_n \in \mathcal{D}P$ be linearly independent such that $\lambda_i v_i = z_i \in \mathbb{Z}^n$ for every $i = 1, \dots, n$. These vectors define the subspaces $L_k = \mathrm{lin}\{v_1, \dots, v_k\}$, $k = 0, \dots, n$.

By assumption, the centroid of $P$ is the origin. Hence, by (1.2), the inclusion $\mathcal{D}P \subseteq (n+1)P$ holds. Therefore

$$\mathcal{D}P \subseteq \left\{ x \in \mathbb{R}^n : |a_j^\intercal x| \le (n+1)b_j,\, j = 1, \dots, m \right\}.$$

By $z_i \in \lambda_i \cdot \mathcal{D}P$ this implies

$$(n+1)b_j \ge \frac{1}{\lambda_i}|a_j^\intercal z_i| \quad \text{for every} \quad i = 1, \dots, n \quad \text{and} \quad j = 1, \dots, m.$$

For every $k = 0, \dots, n$ we define $V_k = \{j : a_j \in L_k^\perp\}$. Then, $V_0 = \{1, \dots, m\}$ and $V_k \subseteq V_{k-1}$ for each $k = 1, \dots, n$. Furthermore, we let $q$ be the smallest index such that $V_q = \emptyset$. The integrality of the $a_j$'s and the $z_i$'s then gives

$$(n+1)b_j \ge \frac{1}{\lambda_k}|a_j^\intercal z_k| \ge \frac{1}{\lambda_k} \quad \text{for every} \quad j \in V_{k-1} \setminus V_k \quad \text{and} \quad k = 1, \dots, q.$$

Recalling $F_j^0 = \operatorname{conv}\{0, F_j\}$, we are prepared for the estimate

$$\begin{aligned}
\mathrm{g}_{n-1}(P) &= \frac{1}{2}\sum_{k=1}^{q} \sum_{j \in V_{k-1}\setminus V_k} \frac{\mathrm{vol}_{n-1}(F_j)}{\|a_j\|} \\
&\le \frac{n(n+1)}{2} \sum_{k=1}^{q} \lambda_k \sum_{j \in V_{k-1}\setminus V_k} \frac{\mathrm{vol}_{n-1}(F_j) b_j}{\|a_j\|\, n} \\
&= \frac{n(n+1)}{2} \sum_{k=1}^{q} \lambda_k \left( \sum_{j \in V_{k-1}} \mathrm{vol}(F_j^0) - \sum_{j \in V_k} \mathrm{vol}(F_j^0) \right) \\
&= \frac{n(n+1)}{2} \left( \lambda_1 \mathrm{vol}(P) + \sum_{k=1}^{q-1} (\lambda_{k+1} - \lambda_k) \sum_{a_j \in L_k^\perp} \mathrm{vol}(F_j^0) \right). \qquad (4.10)
\end{aligned}$$

In the last equality, we used $\sum_{j \in V_0} \mathrm{vol}(F_j^0) = \mathrm{vol}(P)$ and $V_q = \emptyset$.

Finally, by Proposition 4.22 and the monotonicity of the successive minima, we derive

$$\begin{aligned}
\frac{\mathrm{g}_{n-1}(P)}{\mathrm{vol}(P)} &\le \frac{n(n+1)}{2} \left( \lambda_1 + \sum_{k=1}^{q-1} \left( \frac{n+1}{k+1} \right)^k \frac{n-k}{n} (\lambda_{k+1} - \lambda_k) \right) \\
&< e^{\frac{n+1}{e}} \frac{n+1}{2} \left( \sum_{k=1}^{q-1} \lambda_k + (n-q+1)\lambda_q \right) \qquad (4.11) \\
&\le e^{\frac{n+1}{e}} \frac{n+1}{2} \sum_{k=1}^{n} \lambda_k = e^{\frac{n+1}{e}} \frac{n+1}{2} \sum_{k=1}^{n} \frac{\lambda_k(\frac{1}{2}\mathcal{D}P)}{2}.
\end{aligned}$$

The Inequality (4.11) follows because the function $k \mapsto \left(\frac{n+1}{k}\right)^k$ has its maximum at $k = \frac{n+1}{e}$. Note that $e^{\frac{n+1}{e}} \approx 1.4447^{n+1}$. $\square$

**Remark 4.25.** *From the proof above we can extract a little more:*

*i) Let $P \in \mathcal{P}^n$ be a lattice polytope. By $\sum_{a_j \in L_k^\perp} \mathrm{vol}(F_j^0) \leq \mathrm{vol}(P)$ for all $k = 1, \ldots, n$, and Equation (4.10), we have*

$$\frac{\mathrm{g}_{n-1}(P)}{\mathrm{vol}(P)} \leq \frac{n(n+1)}{2} \cdot \frac{\lambda_n(\frac{1}{2}\mathcal{D}P)}{2}.$$

*In particular, $\mathrm{g}_{n-1}(P) \leq \binom{n+1}{2} \mathrm{vol}(P)$.*

*ii) Since Conjecture 4.20 holds for lattice simplices $S \in \mathcal{P}^n$, the application of Proposition 4.22 in (4.11) is superfluous and thus*

$$\frac{\mathrm{g}_{n-1}(S)}{\mathrm{vol}(S)} \leq \frac{n+1}{2} \sum_{i=1}^{n} \frac{\lambda_i(\frac{1}{2}\mathcal{D}S)}{2}.$$

Supported by these observations, Theorem 4.23 and the behavior of the standard simplex $S_n$ (see (4.9)), we conjecture that part ii) holds in general.

**Conjecture 4.26.** *Let $P \in \mathcal{P}^n$ be a lattice polytope. Then*

$$\frac{\mathrm{g}_{n-1}(P)}{\mathrm{vol}(P)} \leq \frac{n+1}{2} \sum_{i=1}^{n} \frac{\lambda_i(\frac{1}{2}\mathcal{D}P)}{2}.$$

In the above considerations, the centroid of the polytope plays a distinguished role. Thus it makes sense to reflect this in an alternative definition of the successive minima that avoids symmetrization and try to derive meaningful inequalities similar to the preceding ones.

For every $i \in \{1, \ldots, n\}$ and every $K \in \mathcal{K}^n$ with centroid $c = \mathrm{cen}(K)$, we define

$$\lambda_i^c(K) = \min\{\lambda > 0 : \dim(\lambda(K - c) \cap \mathbb{Z}^n) \geq i\}$$

to be the *ith successive minimum of $K$ with respect to its centroid*. Since $\mathrm{cen}(K+x) = \mathrm{cen}(K)+x$ for all $x \in \mathbb{R}^n$, and $\mathrm{cen}(tK) = t\,\mathrm{cen}(K)$ for all $t > 0$, the minima $\lambda_i^c(K)$ share the translation invariance and the homogeneity of the usual minima $\lambda_i(K)$.

Let $K \in \mathcal{K}^n$ be with centroid at the origin. Milman and Pajor [MP00, Cor. 3] showed that $\mathrm{vol}(K) \leq 2^n \mathrm{vol}(K \cap (-K))$. Because of $K \cap (-K) \subseteq K$ and the definition of $\lambda_1^c(K)$, we get

$$\mathrm{int}\, \lambda_1^c(K)\,(K \cap (-K)) \cap \mathbb{Z}^n = \mathrm{int}\, \lambda_1^c(K)\, K \cap \mathbb{Z}^n = \{0\}.$$

Hence, we can apply Minkowski's 1st Theorem (see Theorem 4.2) to the centrally symmetric body $K \cap (-K)$ and derive

$$\mathrm{vol}(\lambda_1^c(K)\, K) \leq 2^n \mathrm{vol}(\lambda_1^c(K)\,(K \cap (-K))) \leq 4^n.$$

By the translation invariance of both $\lambda_1^c(K)$ and $\mathrm{vol}(K)$, this implies

$$\lambda_1^c(K)^n \mathrm{vol}(K) \leq 4^n \quad \text{for all} \quad K \in \mathcal{K}^n. \tag{4.12}$$

Recall from (1.2), that for $K \in \mathcal{K}^n$ with centroid $c = \mathrm{cen}(K)$, we have $K - c \subseteq \frac{n}{n+1}\mathcal{D}K \subseteq n(K - c)$. Therefore

$$\frac{2}{n+1}\lambda_i^c(K) \leq \lambda_i(\tfrac{1}{2}\mathcal{D}K) \leq \frac{2n}{n+1}\lambda_i^c(K) \quad \text{for all} \quad i = 1, \ldots, n. \tag{4.13}$$

In view of these relations and Minkowski's 2nd Theorem applied to $\frac{1}{2}\mathcal{D}K$, we obtain

$$\lambda_1^c(K) \cdot \ldots \cdot \lambda_n^c(K)\,\mathrm{vol}(K) \le (n+1)^n \quad \text{for all} \quad K \in \mathcal{K}^n. \tag{4.14}$$

However, we conjecture that the standard simplex $S_n$ is an extremal example in such nonsymmetric versions (4.12) and (4.14) of Minkowski's inequalities. More precisely, since $\lambda_i^c(S_n) = n+1$ for all $i = 1, \ldots, n$, we suggest

**Conjecture 4.27.** *Let $K \in \mathcal{K}^n$. Then*

$$\lambda_1^c(K)^n\,\mathrm{vol}(K) \le \frac{(n+1)^n}{n!} \quad \text{and} \quad \lambda_1^c(K) \cdot \ldots \cdot \lambda_n^c(K)\,\mathrm{vol}(K) \le \frac{(n+1)^n}{n!}.$$

María A. Hernández Cifre (personal communication) proved that this conjecture holds in the planar case $n = 2$. Each of these inequalities would imply Ehrhart's conjecture [Ehr55a], which claims that every $K \in \mathcal{K}^n$ with centroid 0 and $\mathrm{int}\,K \cap \mathbb{Z}^n = \{0\}$ has volume at most $\frac{(n+1)^n}{n!}$.

By virtue of (4.13), we see that the results from Theorem 4.23, Proposition 4.24 and Remark 4.25 also hold with respect to $\lambda_i^c(P)$, where the factor $\frac{n+1}{2}$ has to be replaced by $n$ in each case. Even though this does probably not lead to best possible inequalities.

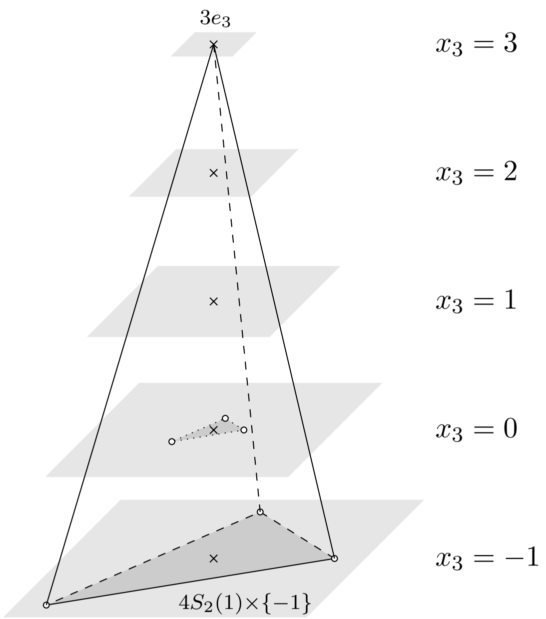



FIGURE 4.2. The pyramid construction in Proposition 4.29 over the reflexive simplex $S_2(1) = \mathrm{conv}\{e_1, e_2, -(e_1+e_2)\}$.

**Corollary 4.28.** *Let $P \in \mathcal{P}^n$ be a lattice polytope.*

*i) We have*

$$\frac{\mathrm{g}_{n-1}(P)}{\mathrm{vol}(P)} < 1.45^{n+1} n \sum_{i=1}^{n} \frac{\lambda_i^c(P)}{2} \quad \text{and} \quad \frac{\mathrm{g}_{n-1}(P)}{\mathrm{vol}(P)} \le n^2 \frac{\lambda_n^c(P)}{2}.$$

*ii) If $n = 2$ or $P$ is a simplex, then*

$$\frac{\mathrm{g}_{n-1}(P)}{\mathrm{vol}(P)} \leq n \sum_{i=1}^{n} \frac{\lambda_i^c(P)}{2}.$$

The worst examples that we found for such an inequality are the following. See also Figure 4.2 for the construction of these polytopes.

**Proposition 4.29.** *Let $P \in \mathcal{P}^{n-1}$ be a reflexive polytope with centroid at the origin and consider the pyramid $P_n = \mathrm{conv}\,\{(n+1)P \times \{-1\}, ne_n\}$. Then*

$$\frac{\mathrm{g}_{n-1}(P_n)}{\mathrm{vol}(P_n)} = \frac{2n}{n+1} \sum_{i=1}^{n} \frac{\lambda_i^c(P_n)}{2}.$$

For the proof we need an auxiliary statement on $\mathrm{g}_{n-1}$ for pyramids over reflexive polytopes.

**Lemma 4.30.** *Let $P \in \mathcal{P}^{n-1}$ be reflexive and let $P' = \mathrm{conv}\,\{P \times \{0\}, e_n\}$. Then, $\mathrm{g}_{n-1}(P') = \mathrm{vol}_{n-1}(P)$.*

Proof. In Proposition 4.7, we computed the Ehrhart polynomial of a bi-pyramid over an $(n-1)$-cube. Analogously, we have in the current situation

$$\begin{aligned}
\mathrm{G}(kP') &= \mathrm{G}(kP) + \sum_{j=0}^{k-1} \mathrm{G}(jP) \\
&= \sum_{j=0}^{n-1} \mathrm{g}_j(P) k^j + \sum_{j=1}^{n} \left( \sum_{i=j-1}^{n-1} \mathrm{g}_i(P) \binom{i+1}{j} \frac{B_{i-j+1}}{i+1} \right) k^j,
\end{aligned}$$

where $B_j$ are again the Bernoulli numbers.

In particular, by $B_0 = 1$ and $B_1 = -\frac{1}{2}$, we have

$$\mathrm{g}_{n-1}(P') = \frac{1}{n-1}\,\mathrm{g}_{n-2}(P) + \frac{1}{2}\,\mathrm{g}_{n-1}(P) = \frac{1}{n-1}\,\mathrm{g}_{n-2}(P) + \frac{1}{2}\,\mathrm{vol}_{n-1}(P).$$

Since $P$ is reflexive, we have $\mathrm{g}_{n-2}(P) = \frac{n-1}{2}\,\mathrm{vol}_{n-1}(P)$ (see [**BHW07**, Lem. 3.1]) and thus $\mathrm{g}_{n-1}(P') = \mathrm{vol}_{n-1}(P)$. □

Proof of Proposition 4.29. The polytope $P_n$ is a dilated and translated pyramid over $P$. More precisely, $P_n = (n+1)\,\mathrm{conv}\,\{P \times \{0\}, e_n\} - e_n$. Thus, writing $P' = \mathrm{conv}\,\{P \times \{0\}, e_n\}$ and using the homogeneity of the Ehrhart coefficients and Lemma 4.30, we get

$$\frac{\mathrm{g}_{n-1}(P_n)}{\mathrm{vol}(P_n)} = \frac{1}{n+1} \frac{\mathrm{g}_{n-1}(P')}{\mathrm{vol}(P')} = \frac{1}{n+1} \frac{\mathrm{vol}_{n-1}(P)}{\frac{1}{n}\,\mathrm{vol}_{n-1}(P)} = \frac{n}{n+1}.$$

Now we compute the successive minima of $P_n$. By the assumption, the centroid of $P$ is the origin. Hence, $\mathrm{cen}(P') = \mathrm{cen}(\mathrm{conv}\,\{P \times \{0\}, e_n\}) = \frac{1}{n+1} e_n$ and thus $\mathrm{cen}(P_n) = (n+1)\,\mathrm{cen}(P') - e_n = 0$. The intersection of $\frac{1}{n} P_n$ with the hyperplane $\{x \in \mathbb{R}^n : x_n = 0\}$ is just $P$. Moreover, the apex of $\frac{1}{n} P_n$ is $e_n$. Since $P$ is reflexive, its only interior lattice point is the origin, which shows that further scaling of $P_n$ excludes any nontrivial lattice point. Therefore, $\lambda_1^c(P_n) = \ldots = \lambda_n^c(P_n) = \frac{1}{n}$, and so

$$\frac{\mathrm{g}_{n-1}(P_n)}{\mathrm{vol}(P_n)} = \frac{2n}{n+1} \sum_{i=1}^{n} \frac{\lambda_i^c(P_n)}{2}. \qquad \square$$

Remark that by choosing $P = S_{n-1}(1) = \mathrm{conv}\{e_1, \ldots, e_{n-1}, -\sum_{i=1}^{n-1} e_i\}$ in Proposition 4.29, we also have simplices among these particular examples.

We conclude our investigations with an inequality on the class of lattice simplices whose factor lies between that in Proposition 4.29 and Corollary 4.28.

**Proposition 4.31.** *Let $S \in \mathcal{P}^n$ be a lattice simplex. Then*

$$\frac{\mathrm{g}_{n-1}(S)}{\mathrm{vol}(S)} \le \frac{n^2+1}{n+1} \sum_{i=1}^{n} \frac{\lambda_i^c(S)}{2}.$$

Proof. As in the proof of Proposition 4.24, it is no restriction to assume that $\mathrm{cen}(S) = 0$. Let $S = \{x \in \mathbb{R}^n : a_j^\intercal x \le b_j,\ j = 0, \ldots, n\}$ and assume that the normal vectors $a_j \in \mathbb{Z}^n$ (corresponding to the facet $F_j$) are primitive. By the assumption, we have $\sum_{j=0}^{n} w_j = 0$, where $w_0, \ldots, w_n$ are the vertices of $S$. Hence, for every $i, j = 0, \ldots, n$,

$$n!\,\mathrm{vol}(F_j^0) = |\det(w_t : t \ne j)| = |\det(w_t : t \ne i)| = n!\,\mathrm{vol}(F_i^0).$$

In particular, $\mathrm{vol}(F_j^0) = \frac{1}{n+1}\mathrm{vol}(S)$ and therefore

$$\frac{\mathrm{g}_{n-1}(S)}{\mathrm{vol}(S)} = \frac{1}{2} \sum_{i=0}^{n} \frac{\mathrm{vol}_{n-1}(F_j)}{\mathrm{vol}(S) \cdot \|a_j\|} = \frac{n}{2(n+1)} \sum_{j=0}^{n} \frac{1}{b_j}. \tag{4.15}$$

Let $\lambda_i^c = \lambda_i^c(S)$ for every $i = 1, \ldots, n$, and let $v_1, \ldots, v_n \in S$ be linearly independent such that $\lambda_i^c v_i = z_i \in \mathbb{Z}^n$ for all $i = 1, \ldots, n$. As in Proposition 4.24, we define the subspaces $L_k = \mathrm{lin}\{v_1, \ldots, v_k\}$, but now refine the definition of the sets $V_k = \{j : a_j \in L_k^\perp\}$ as follows:

$$V_k^+ = \{j : a_j \in L_{k-1}^\perp \text{ and } a_j^\intercal v_k > 0\} \quad \text{and}$$
$$V_k^- = \{j : a_j \in L_{k-1}^\perp \text{ and } a_j^\intercal v_k < 0\}.$$

For every $k = 1, \ldots, n$, we get $V_k \subseteq V_{k-1}$ and $V_k^+ \cup V_k^- = V_{k-1} \setminus V_k$. Since $z_i \in \lambda_i^c \cdot S$ and $S \subseteq (-n)S$ (see [BF87, §7 (34.)] or [LZ91, Thm. 3]), we have

$$b_j \ge \frac{1}{\lambda_i^c} a_j^\intercal z_i \ge -n b_j \quad \text{for all} \quad j = 0, \ldots, n \quad \text{and} \quad i = 1, \ldots, n.$$

The involved scalar product is an integer and so we get

$$b_j \ge \frac{1}{\lambda_k^c} \quad \text{for all} \quad j \in V_k^+, \quad \text{and} \quad n b_j \ge \frac{1}{\lambda_k^c} \quad \text{for all} \quad j \in V_k^-.$$

Using these relations, we continue the lines from Equation (4.15) by

$$\frac{\mathrm{g}_{n-1}(S)}{\mathrm{vol}(S)} = \frac{n}{2(n+1)} \sum_{j=0}^{n} \frac{1}{b_j} = \frac{n}{2(n+1)} \sum_{k=1}^{n} \left( \sum_{j \in V_k^+} \frac{1}{b_j} + \sum_{j \in V_k^-} \frac{1}{b_j} \right)$$

$$\leq \frac{n}{2(n+1)} \sum_{k=1}^{n} \lambda_k^c \left(|V_k^+| + n|V_k^-|\right) \leq \frac{n^2+1}{n+1} \sum_{k=1}^{n} \frac{\lambda_k^c(S)}{2}.$$

For the last inequality we used Lemma 4.32 that is proved below. □

**Lemma 4.32.** *Let $0 < \lambda_1 \leq \ldots \leq \lambda_n$ be real numbers and let $\alpha_k = |V_k^+| + n|V_k^-| = |V_{k-1}| - |V_k| + (n-1)|V_k^-|$ be defined by the conventions from the proof of Proposition 4.31. Then*

$$\sum_{k=1}^{n} \alpha_k \lambda_k \leq \frac{n^2+1}{n} \sum_{k=1}^{n} \lambda_k.$$

Proof. Let $0 = s_0 < s_1 < \ldots < s_p < s_{p+1} = n$ and $n = t_0 > t_1 > \ldots > t_p > t_{p+1} = 0$, where $s_i, t_i, p \in \mathbb{N}$ are such that

$$|V_{s_i+1}| = \ldots = |V_{s_{i+1}}| = t_{i+1} \quad \text{for all} \quad i = 0, \ldots, p.$$

Then, by definition of the sets $V_k$ and $V_k^-$, we have (note that $V_1^+ \neq \emptyset$)

$$|V_{s_i+1}^-| \leq t_i - t_{i+1} \text{ and } |V_{s_i+2}^-| = \ldots = |V_{s_{i+1}}^-| = 0 \quad \text{for all} \quad i = 0, \ldots, p.$$

Therefore,

$$\alpha_1 = |V_0| - |V_1| + (n-1)|V_1^-| \leq n + 1 - t_1 + (n-1)(t_0 - t_1) = n(n - t_1) + 1,$$
$$\alpha_k = |V_{k-1}| - |V_k| + (n-1)|V_k^-| = 0, \forall\, k = s_i + 2, \ldots, s_{i+1}, \forall\, i = 0, \ldots, p,$$

and

$$\alpha_{s_i+1} \leq t_i - t_{i+1} + (n-1)(t_i - t_{i+1}) = n(t_i - t_{i+1}), \forall\, i = 1, \ldots, p.$$

This yields

$$\sum_{k=1}^{n} \alpha_k \lambda_k \leq (n(n-t_1)+1)\lambda_1 + \sum_{i=1}^{p} n(t_i - t_{i+1})\lambda_{s_i+1}$$
$$\leq \lambda_1 + n(\lambda_1 + \ldots + \lambda_n) \leq \frac{n^2+1}{n} \sum_{k=1}^{n} \lambda_k.$$

For the last line of inequalities we need that the $\lambda_i$ form a nondecreasing sequence and that there are at least as many multiples of $n\lambda_j$ 'available' to distribute as there is space between $j$ and $s_i+1$. This means, $n-t_1 \geq s_1$ and $(n-t_1)+\sum_{i=1}^{k}(t_i-t_{i+1})-s_1-\sum_{i=2}^{k}(s_i-s_{i-1}) \geq s_{k+1}-s_k$ for all $k = 1, \ldots, p$. But this is equivalent to $n - t_{k+1} \geq s_{k+1}$ or $t_{k+1} = |V_{s_{k+1}}| \leq n - s_{k+1}$ for all $k = 0, \ldots, p$, which holds because the facet normals $a_j$ of the simplex $S$ are affinely independent. □

## 4.4. On a Conjecture by Wills

We conclude this chapter with a discussion of another proposed generalization of Minkowski's 1st Theorem. To this end, we observe that for a lattice polytope $P \in \mathcal{P}_0^n$ with only the origin as an interior lattice point, we clearly have $\lambda_1(P) \geq 1$. Thus, by Theorem 4.2, Theorem 4.19 and Inequality (4.1) (cf. [Min96]),

$$\mathrm{vol}(P) \leq \mathrm{vol}(C_n) = 2^n,\ \mathrm{g}_{n-1}(P) \leq \mathrm{g}_{n-1}(C_n) = n2^{n-1}$$

and

$$\mathrm{G}(P) \leq \mathrm{G}(C_n) = 3^n.$$

Already in 1981, Wills wondered about a much stronger extremality property of the unit cube and proved it in the three-dimensional case.

**Conjecture 4.33** (Wills [**Wil81, GW93**]). *Let $P \in \mathcal{P}_0^n$ be a lattice polytope with* $\operatorname{int} P \cap \mathbb{Z}^n = \{0\}$*. Then*

$$\mathrm{g}_i(P) \leq \mathrm{g}_i(C_n) = 2^i \binom{n}{i} \quad \text{for all} \quad i = 0, \ldots, n.$$

Although this conjecture gets a lot of support by the aforementioned valid cases, it is not true in general. Using `polymake` [**GJ00**], we found that the polytope $P_7 = \operatorname{conv}\{C_6 \times \{0\}, C_6^\star \times \{\pm 1\}\}$ is a counterexample in dimension seven. Its Ehrhart polynomial is given by

$$1 + \frac{1534}{105}k + \frac{3188}{45}k^2 + \frac{7112}{45}k^3 + \frac{1756}{9}k^4 + \frac{7004}{45}k^5 + \frac{4952}{45}k^6 + \frac{15656}{315}k^7.$$

In particular, $\mathrm{g}_1(P_7) = \frac{1534}{105} > 14 = 2^1\binom{7}{1}$. Moreover, the bipyramids $Q^n = \operatorname{conv}\{C_{n-1} \times \{0\}, \pm e_n\}$, that already appeared in Proposition 4.7, show that in higher dimensions also other Ehrhart coefficients are "too big". For instance,

$$\mathrm{g}_1(Q^9) = \frac{494}{15} > 18 = 2^1\binom{9}{1},$$

$$\mathrm{g}_3(Q^{11}) = 1976 > 1320 = 2^3\binom{11}{3} \quad \text{and}$$

$$\mathrm{g}_5(Q^{13}) = \frac{260832}{5} > 41184 = 2^5\binom{13}{5}.$$

The next proposition is folklore and usually used without proof. We provide the argument for the sake of completeness.

**Proposition 4.34.** *Let $P \in \mathcal{P}^p$ and $Q \in \mathcal{P}^q$ be lattice polytopes. Then*

$$\mathrm{g}_j(P \times Q) = \sum_{i=0}^{j} \mathrm{g}_i(P)\, \mathrm{g}_{j-i}(Q) \quad \text{for all} \quad j = 0, \ldots, p+q.$$

Proof. For every $k \in \mathbb{N}$, we have

$$\begin{aligned}
\mathrm{G}(k(P \times Q)) &= \mathrm{G}(kP \times kQ) = \mathrm{G}(kP)\,\mathrm{G}(kQ) \\
&= \left(\sum_{j=0}^{p} \mathrm{g}_j(P)k^j\right)\left(\sum_{i=0}^{q} \mathrm{g}_i(Q)k^i\right) \\
&= \sum_{j=0}^{p+q}\left(\sum_{i=0}^{j} \mathrm{g}_i(P)\,\mathrm{g}_{j-i}(Q)\right)k^j.
\end{aligned}$$

Here we set $\mathrm{g}_i(P) = 0 = \mathrm{g}_j(Q)$ for all $i > p$ and $j > q$. Comparing coefficients gives the claimed identities. ☐

In view of counterexamples to Wills' conjecture, these identities are very useful since they imply that $\mathrm{g}_1(P \times [-1,1]) = \mathrm{g}_1(P) + 2$ for every lattice polytope $P$. That means, every counterexample to Conjecture 4.33 for $\mathrm{g}_1$ can be transferred to a counterexample for $\mathrm{g}_1$ in all higher dimensions. Thus, the polytope $P_7$ from above shows that Wills' Conjecture fails in every dimension $n \geq 7$.

On the positive side, Conjecture 4.33 holds for particular, not necessarily symmetric, reflexive polytopes. Bey, Henk and Wills [**BHW07**, Prop. 1.8] showed that every lattice polytope $P \in \mathcal{P}^n$ whose Ehrhart polynomial has only roots with real part equal to $-\frac{1}{2}$ is unimodularly equivalent to a reflexive polytope and, moreover, its volume is at most $2^n$. We generalize the latter as follows.

**Theorem 4.35.** *Let $P \in \mathcal{P}^n$ be a lattice polytope with the property that all the roots of its Ehrhart polynomial have real part $-\frac{1}{2}$. Then*

$$\mathrm{g}_i(P) \leq 2^i \binom{n}{i} \quad \text{for all} \quad i = 1, \dots, n.$$

*Equality holds for some $i \in \{1, \dots, n\}$ if and only if equality holds for all $i \in \{1, \dots, n\}$, which means that $P$ has the same Ehrhart polynomial as the unit cube $C_n = [-1,1]^n$.*

Proof. Since $\mathrm{g}_n(P) = \mathrm{vol}(P)$, we can write

$$\mathrm{G}(kP) = \mathrm{vol}(P) \prod_{i=1}^{n} (k + \gamma_i(P)).$$

By assumption, the real part of $-\gamma_i(P)$ equals $-\frac{1}{2}$ for all $i = 1, \dots, n$.

We consider the case $n = 2l$ first. There are $b_1, \dots, b_l \in \mathbb{R}$ such that

$$\begin{aligned}
\frac{\mathrm{G}(kP)}{\mathrm{vol}(P)} &= \prod_{j=1}^{l} \left(k + \frac{1}{2} \pm b_j \mathrm{i}\right) = \prod_{j=1}^{l} \left( \left(k + \frac{1}{2}\right)^2 + b_j^2 \right) \\
&= \sum_{j=0}^{l} \sigma_{l-j}\left(b_1^2, \dots, b_l^2\right) \left(k + \frac{1}{2}\right)^{2j} \\
&= \sum_{j=0}^{l} \sigma_{l-j}\left(b_1^2, \dots, b_l^2\right) \sum_{t=0}^{2j} \binom{2j}{t} \left(\frac{1}{2}\right)^{2j-t} k^t \\
&= \sum_{t=0}^{2l} 2^t \left( \sum_{j=\lceil \frac{t}{2} \rceil}^{l} \sigma_{l-j}\left(b_1^2, \dots, b_l^2\right) \binom{2j}{t} \frac{1}{4^j} \right) k^t.
\end{aligned}$$

As usual, $\sigma_j$ denotes the $j$th elementary symmetric polynomial. The volume of $P$ is given by

$$\mathrm{vol}(P) = \prod_{i=1}^{n} \frac{1}{\gamma_i(P)} = \prod_{j=1}^{l} \frac{1}{b_j^2 + \frac{1}{4}} = \frac{4^l}{\sum_{j=0}^{l} \sigma_{l-j}\left(4b_1^2, \dots, 4b_l^2\right)}.$$

Therefore, the Ehrhart coefficients of $P$ are

$$\mathrm{g}_t(P) = \frac{2^t \sum_{j=\lceil \frac{t}{2} \rceil}^{l} \sigma_{l-j}\left(4b_1^2, \dots, 4b_l^2\right) \binom{2j}{t}}{\sum_{j=0}^{l} \sigma_{l-j}\left(4b_1^2, \dots, 4b_l^2\right)} \quad \text{for all} \quad t = 0, \dots, 2l.$$

Hence, we have $\mathrm{g}_t(P) \le 2^t\binom{n}{t}$ if and only if

$$\sum_{j=\lceil \frac{t}{2}\rceil}^{l} \sigma_{l-j}\left(4b_1^2,\dots,4b_l^2\right)\binom{2j}{t} \le \sum_{j=0}^{l}\sigma_{l-j}\left(4b_1^2,\dots,4b_l^2\right)\binom{2l}{t}. \tag{4.16}$$

This holds since all factors are nonnegative and $\binom{2j}{t} \le \binom{2l}{t}$ for $j = 0,\dots,l$. Moreover, equality holds in (4.16) for some $t \in \{1,\dots,n\}$ if and only if $b_1 = \ldots = b_l = 0$. This is equivalent to having equality in (4.16) for all $t \in \{1,\dots,n\}$ and thus $\mathrm{G}(kP) = (2k+1)^n = \mathrm{G}(kC_n)$. It is not clear that this is equivalent to $P \simeq C_n$ (for related work see [**HM08**]).

The case of odd dimensions $n = 2l+1$ is similar. Here we have an additional real zero $-\frac{1}{2}$ and get

$$\begin{aligned}
\frac{\mathrm{G}(kP)}{\mathrm{vol}(P)} &= \left(k+\frac{1}{2}\right)\prod_{j=1}^{l}\left(k+\frac{1}{2}\pm b_j\mathrm{i}\right)\\
&= \left(k+\frac{1}{2}\right)\sum_{t=0}^{2l}2^t\left(\sum_{j=\lceil \frac{t}{2}\rceil}^{l}\sigma_{l-j}\left(b_1^2,\dots,b_l^2\right)\binom{2j}{t}\frac{1}{4^j}\right)k^t\\
&= \sum_{t=1}^{2l+1}2^{t-1}\left(\sum_{j=\lceil \frac{t-1}{2}\rceil}^{l}\sigma_{l-j}\left(b_1^2,\dots,b_l^2\right)\binom{2j}{t-1}\frac{1}{4^j}\right)k^t\\
&\quad+\sum_{t=0}^{2l}2^{t-1}\left(\sum_{j=\lceil \frac{t}{2}\rceil}^{l}\sigma_{l-j}\left(b_1^2,\dots,b_l^2\right)\binom{2j}{t}\frac{1}{4^j}\right)k^t.
\end{aligned}$$

The volume of $P$ is now

$$\mathrm{vol}(P) = \frac{2\cdot 4^l}{\sum_{j=0}^{l}\sigma_{l-j}\left(4b_1^2,\dots,4b_l^2\right)}, \tag{4.17}$$

and thus, for all $t = 1,\dots,n-1 = 2l$, we have

$$\begin{aligned}
\mathrm{g}_t(P) &= \frac{2^t\sum_{j=\lceil \frac{t-1}{2}\rceil}^{l}\sigma_{l-j}\left(4b_1^2,\dots,4b_l^2\right)\binom{2j}{t-1}}{\sum_{j=0}^{l}\sigma_{l-j}\left(4b_1^2,\dots,4b_l^2\right)}\\
&\quad+\frac{2^t\sum_{j=\lceil \frac{t}{2}\rceil}^{l}\sigma_{l-j}\left(4b_1^2,\dots,4b_l^2\right)\binom{2j}{t}}{\sum_{j=0}^{l}\sigma_{l-j}\left(4b_1^2,\dots,4b_l^2\right)}.
\end{aligned}$$

Hence, $\mathrm{g}_t(P) \le 2^t\binom{2l+1}{t}$ if and only if

$$\begin{aligned}
&\sum_{j=\lceil \frac{t-1}{2}\rceil}^{l}\sigma_{l-j}\left(4b_1^2,\dots,4b_l^2\right)\binom{2j}{t-1} + \sum_{j=\lceil \frac{t}{2}\rceil}^{l}\sigma_{l-j}\left(4b_1^2,\dots,4b_l^2\right)\binom{2j}{t}\\
&\le \sum_{j=0}^{l}\sigma_{l-j}\left(4b_1^2,\dots,4b_l^2\right)\binom{2l+1}{t}.
\end{aligned}$$

These inequalities hold since again all factors are nonnegative and $\binom{2j}{t-1} + \binom{2j}{t} = \binom{2j+1}{t} \le \binom{2l+1}{t}$ for all $j = 0,\dots,l$. The case $\mathrm{g}_n(P)$ is immediate by (4.17). Equality is attained for some $t \in \{1, \ldots, n\}$ just as in the even dimensional situation if and only if $b_1 = \ldots = b_l = 0$ and we are done. □

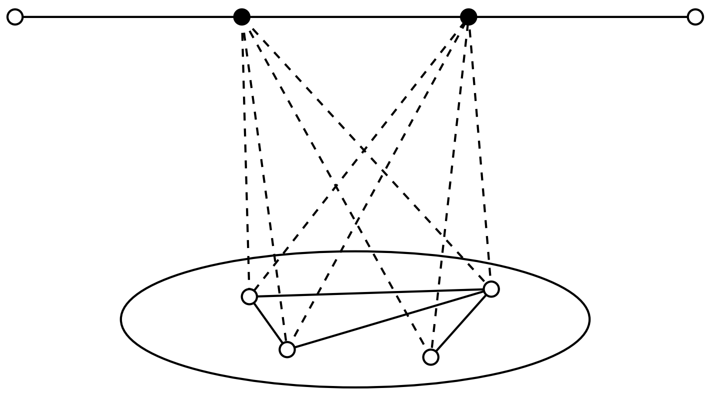
Lattice polytopes whose Ehrhart polynomial has only roots with real part equal to $-\frac{1}{2}$ were also studied in [**BR07**, Ch. 2] and [**HHO11**]. In fact, the question arose whether it is possible to give a characterization of these polytopes. An example that already appeared in [**BHW07**] shows that our Theorem 4.35 is not the final answer in this regard. To this end, consider the triangles $T_1 = \mathrm{conv}\{\binom{-1}{-1}, \binom{2}{-1}, \binom{-1}{2}\}$ and $T_2 = S_2(1) = \mathrm{conv}\{\binom{1}{0}, \binom{0}{1}, \binom{-1}{-1}\}$. Then the cartesian product $T = T_1 \times T_2$ has Ehrhart polynomial

$$\mathrm{G}(kT) = 1 + 6k + \frac{51}{4}k^2 + \frac{27}{2}k^3 + \frac{27}{4}k^4.$$

One finds that the coefficients of $\mathrm{G}(kT)$ fulfill the inequalities in Wills' Conjecture and that $T$ is reflexive. On the other hand, the roots of $\mathrm{G}(kT)$ are given by $-\frac{2}{3}$, $-\frac{1}{3}$ and $-\frac{1}{2} \pm \frac{\sqrt{15}}{6}\mathrm{i}$.

CHAPTER 5

# Face Numbers of Centrally Symmetric Polytopes Produced from Split Graphs



## 5.1. Introduction

The results of this chapter are obtained in joint work with Ragnar Freij, Moritz W. Schmitt and Günter M. Ziegler [FHSZ12]. The problem that we investigate addresses the face structure of certain centrally symmetric polytopes. As mentioned in the introduction, the connection to the previous chapters is the difficulty to exploit the symmetry condition. Even the following question, that can be answered easily in the general case, did not find a solution so far: Which centrally symmetric polytope has the minimal number of faces among all centrally symmetric polytopes of the same dimension?

Let us review some notation and related results from polyhedral combinatorics. For a polytope $P \in \mathcal{P}^n$ and some $i \in \{-1, 0, 1, \dots, n\}$, we let

$$\mathrm{f}_i(P) = \text{number of } i\text{-dimensional faces of } P.$$

We have $\mathrm{f}_{-1}(P) = \mathrm{f}_n(P) = 1$ since the empty set and $P$ itself are the only faces of $P$ of dimension $-1$ and $n$, respectively. These numbers are collected in the f*-vector* $\mathrm{f}(P) = \big(\mathrm{f}_{-1}(P), \mathrm{f}_0(P), \dots, \mathrm{f}_n(P)\big)$ of $P$. The investigation of f-vectors of polytopes has a long history with many brilliant contributions. We refer to Ziegler's book [Zie95] for a detailed exposition of the subject. Prominent achievements are the *upper and lower bound theorems*. Recall that $C_n(m)$ denotes the cyclic polytope with $m$ vertices.

**Theorem 5.1** (McMullen [McM70]). *Let $P \in \mathcal{P}^n$ be a polytope with $m$ vertices. Then*

$$\mathrm{f}_i(P) \leq \mathrm{f}_i(C_n(m)) \quad \textit{for all} \quad i = 0, \dots, n.$$

In order to state the lower bound theorem, we note that a polytope is said to be *simplicial* if all its facets are simplices.

**Theorem 5.2** (Barnette [Bar73]). *Let $P \in \mathcal{P}^n$ be a simplicial polytope with $m$ vertices. Then*

$$\mathrm{f}_i(P) \geq \begin{cases} m\binom{n}{i} - i\binom{n+1}{i+1} & \textit{for all } i = 0, \dots, n-2, \\ m(n-1) - (n+1)(n-2) & \textit{for } i = n-1. \end{cases}$$

Despite a lot of recent work, an upper bound theorem for centrally symmetric polytopes is still not within reach. However, confirming a conjecture of Bárány and Lovász from 1982, Stanley proved a lower bound theorem on that class.

**Theorem 5.3** (Stanley [Sta87]). *Let $P \in \mathcal{P}_0^n$ be a simplicial centrally symmetric polytope with $2m$ vertices. Then*

$$\mathrm{f}_i(P) \geq \begin{cases} 2^{i+1}\binom{n}{i+1} + 2(m-n)\binom{n}{i} & \textit{for all } i = 0, \dots, n-2, \\ 2^n + 2(m-n)(n-1) & \textit{for } i = n-1. \end{cases}$$

Because every centrally symmetric polytope of dimension $n$ has at least $2n$ vertices, a consequence of Stanley's inequalities is that every simplicial polytope $P \in \mathcal{P}_0^n$ has at least $3^n$ nonempty faces. Based on this result, Kalai conjectures that it actually holds for every centrally symmetric polytope. This well-known conjecture is commonly termed the $3^d$*-conjecture*.

**Conjecture 5.4** (Kalai [**Kal89**]). *Every centrally symmetric polytope of dimension $n$ has at least $3^n$ nonempty faces.*

Sanyal, Werner and Ziegler [**SWZ09**] proved the $3^d$-conjecture in dimensions less than or equal to four. In the same paper, they refute two stronger conjectures of Kalai on the number of faces and flags of centrally symmetric polytopes. Their counterexamples are five- and six-dimensional polytopes that are constructed from the stable set structure of certain graphs. Hence, it is desirable to understand these polytopes more thoroughly.

Our main contribution is to give an exact count of the number of faces of centrally symmetric polytopes that are produced from split graphs. In particular, we verify Conjecture 5.4 on this class of polytopes. Before we can introduce the construction of these polytopes, we need to recall some notions from graph theory.

For a graph $G$, we denote its set of nodes by $V(G)$ and its set of edges by $E(G) \subseteq \binom{V(G)}{2}$. All our graphs have finitely many nodes and neither contain loops nor parallel edges. The *graph complement* of $G$ is denoted by $\overline{G}$ and the *complete graph* on $n$ nodes by $K_n$. A *stable set* (or independent set) of $G$ is an edgeless subgraph whereas a *clique* in $G$ is a complete subgraph. For more details on graph theory, we refer to Diestel's textbook [**Die10**].

## 5.2. Hansen Polytopes of Threshold Graphs

A well-known concept to construct a polytope from a graph $G$ is to associate a characteristic vector with each stable set in $G$ and take the convex hull (see Schrijver [**Sch03**, Sec. 64.4]). To this end, we label the nodes of a graph $G$ by $1, \dots, n = \#V(G)$.

**Definition 5.5** (stable set polytope). *The* stable set polytope *of a graph $G$ is defined as*

$$\operatorname{stab}(G) = \operatorname{conv}\Big\{ \sum_{i \in S} e_i : S \subseteq V(G) \text{ a stable set} \Big\}.$$

Since the empty set and each of the nodes are stable sets in every graph $G$, the stable set polytope $\operatorname{stab}(G)$ has a *simple vertex*, that is, a vertex that is contained in exactly $n = \#V(G)$ many edges.

One way to symmetrize the stable set polytope is to apply the *twisted prism* operation. In general, the twisted prism over a polytope $P \in \mathcal{P}^n$ is given by $\operatorname{tp}(P) = \operatorname{conv}\{\{1\} \times P, \{-1\} \times (-P)\}$. This clearly leads to a centrally symmetric polytope in dimension $n+1$. We use the index 0 for the additional dimension.

In 1977, Hansen [**Han77**] investigated twisted prisms over stable set polytopes of graphs. These *Hansen polytopes* are the main object of our studies.

**Definition 5.6** (Hansen polytope). *The* Hansen polytope *of a graph $G$ is defined as* $\mathrm{H}(G) = \operatorname{tp}(\operatorname{stab}(G))$.

The easiest examples of Hansen polytopes are cubes, produced from edgeless graphs, and crosspolytopes, produced from complete graphs.

A *cycle* $C$ of length $k$ is a graph with node set $V(C) = \{1, \dots, k\}$ and edge set $E(C) = \{\{1,2\}, \{2,3\}, \dots, \{k-1,k\}, \{k,1\}\}$. A graph is called *perfect* if it neither contains cycles of odd length of at least five nor their complements as induced subgraphs. This definition of a perfect graph is based on the *Strong Perfect Graph Theorem* proved in [CRST06]. A collection of classes of perfect graphs can be found in [Hou06].

One of the striking results by Hansen is the characterization of perfect graphs $G$ by facet descriptions of $\mathrm{H}(G)$.

**Lemma 5.7** (Hansen [Han77])**.** *Let $G$ be a graph.*

*i) The vertex set of the Hansen polytope of $G$ is*

$$\mathrm{vert}(\mathrm{H}(G)) = \Big\{ \pm \big(e_0 + \sum_{i\in S} e_i\big) : S \subseteq V(G) \text{ a stable set}\Big\}.$$

*ii) $G$ is perfect if and only if*

$$\mathrm{H}(G) = \Big\{x \in \mathbb{R}^{n+1} : -1 \leq -x_0 + 2\sum_{i\in C} x_i \leq 1,\, C \subseteq V(G) \text{ a clique}\Big\}$$

*is an irredundant facet description.*

*iii) If $G$ is perfect, then the polar of the Hansen polytope of $G$ is affinely equivalent to the Hansen polytope of $\overline{G}$, in symbols $\mathrm{H}(G)^\star \cong \mathrm{H}(\overline{G})$.*

Proof. Part i) follows right from the definition of a Hansen polytope. The essential part ii) is Theorem 6 in Hansen's paper [Han77]. Part iii) follows from ii) since stable sets (cliques) in $G$ are cliques (stable sets) in $\overline{G}$. □

We have seen above, that cubes and crosspolytopes are examples of Hansen polytopes. Besides that, they are also instances of *Hanner polytopes*. This important class of polytopes was introduced by Hanner [Han56].

**Definition 5.8** (Hanner polytope)**.** *A polytope $P \in \mathcal{P}^n$ is said to be a* Hanner polytope *if it is either a line segment or, for $n \geq 2$, the cartesian product of two Hanner polytopes or the polar of a Hanner polytope.*

Kalai [Kal89] proved that every Hanner polytope of dimension $n$ has exactly $3^n$ nonempty faces and he suggests that these are the only centrally symmetric polytopes with this property. In this context, it is a natural desire to identify those Hansen polytopes that are also Hanner polytopes.

Before we can state our result concerning this question, we need to introduce *threshold graphs*, which form a subclass of perfect graphs. An extensive treatment of this class of graphs is the book of Mahadev and Peled [MP95]. We say that a node in a graph is *dominating* if it is adjacent to all the other nodes, and it is *isolated* if it is not adjacent to any other node.

**Definition 5.9** (threshold graph)**.** *A graph is called a* threshold graph *if it can be constructed from the graph with one node by repeatedly adding either isolated or dominating nodes.*

Clearly, complete graphs and edgeless graphs are threshold graphs. Moreover, this class of graphs is closed under taking complements.

**Theorem 5.10.** *Let $G$ be a graph. Then $\mathrm{H}(G)$ is affinely equivalent to a Hanner polytope if and only if $G$ is a threshold graph.*

Proof. ($\Leftarrow$) We use induction on the number of nodes of $G$. If $V(G) = \emptyset$, then $\mathrm{H}(G)$ is a centrally symmetric line segment and therefore a Hanner polytope. Thus we assume that $G$ has $n+1$ nodes. Since the class of Hanner polytopes is closed under taking polars and, by Lemma 5.7 we have $\mathrm{H}(G)^\star \cong \mathrm{H}(\overline{G})$, we can restrict to the case that $G = T \uplus v$, where $T$ is a threshold graph on $n$ nodes. Here $\uplus$ denotes the disjoint union of graphs and $v$ is a single node not contained in $T$. The stable sets of $G$ are exactly the stable sets of $T$, with and without the node $v$. Therefore, the vertices of $\mathrm{H}(G)$ are of the form $\pm\big(e_0+\sum_{i\in S} e_i\big)$ and $\pm\big(e_0+\sum_{i\in S} e_i+e_{n+1}\big)$, where $S$ is a stable set in $T$ and we assign $v$ the label $n+1$. By the linear transformation defined by $e_0 \mapsto e_0-e_{n+1}$, $e_{n+1} \mapsto 2e_{n+1}$, and $e_i \mapsto e_i$ for all $i = 1,\dots,n$, we get $\mathrm{H}(G) = \mathrm{H}(T \uplus v) \cong \mathrm{H}(T) \times [-1,1]$. Since, by the induction hypothesis $\mathrm{H}(T)$ is a Hanner polytope, this means that $\mathrm{H}(G)$ is a Hanner polytope.

($\Rightarrow$) For the converse we now assume that $\mathrm{H}(G)$ is affinely equivalent to a Hanner polytope. Again it is enough to cover only one case, namely $\mathrm{H}(G) \cong P \times P'$, with $P$ and $P'$ being lower-dimensional Hanner polytopes. The stable set polytope $\mathrm{stab}(G)$ is a facet of $\mathrm{H}(G)$ and therefore can be written as $\mathrm{stab}(G) \cong Q \times Q'$, where $Q$ and $Q'$ are faces of $P$ and $P'$, respectively. Since we have $\dim(Q)+\dim(Q') = \dim(\mathrm{stab}(G)) = \dim(P)+\dim(P')-1$, we can further assume that $Q = P$ and that $Q'$ is a facet of $P'$. Let $q = \dim(Q)$ and $q' = \dim(Q')$.

We construct a threshold graph $H'$ on $q'$ nodes such that $G = \overline{K_q} \uplus H'$. This of course shows that $G$ is a threshold graph as well. Since $\mathrm{stab}(G)$ is a product, we have $\mathrm{vert}(\mathrm{stab}(G)) = \mathrm{vert}(Q) \times \mathrm{vert}(Q')$. Each coordinate of a vertex of $\mathrm{stab}(G)$ corresponds to a node in $G$. Let $V_1 \subseteq V(G)$ be the node set defined by the first $q$ coordinates and $V_2 \subseteq V(G)$ be the set defined by the last $q'$ coordinates. Then

$$\begin{aligned}
\mathrm{vert}(\mathrm{stab}(G)) &= \Big\{ \sum_{i\in S} e_i : S \subseteq V(G) \text{ a stable set in } G \Big\} \\
&= \Big\{ \sum_{i\in S} e_i : S \subseteq V_1 \text{ a stable set in } G[V_1] \text{ and } N(S)\cap V_2 = \emptyset \Big\} \\
&\times \Big\{ \sum_{i\in S} e_i : S \subseteq V_2 \text{ a stable set in } G[V_2] \text{ and } N(S)\cap V_1 = \emptyset \Big\},
\end{aligned}$$

where $N(S)$ is the set of nodes adjacent to some node in $S$ and $G[V_j]$ is the subgraph of $G$ induced by $V_j$, $j = 1,2$. In particular, we have $e_i \in \mathrm{vert}(\mathrm{stab}(G))$ for all $i = 1,\dots,q+q'$. From the equality above, we can therefore deduce that there are no edges between $V_1$ and $V_2$. Setting $H' = G[V_2]$, we get $G = G[V_1] \uplus H'$. So what is left to show is that $G[V_1]$ is an edgeless graph and that $H'$ is a threshold graph.

We first consider $H'$. Since $P \cong \mathrm{stab}(G[V_1])$ is at least one-dimensional, $G[V_1]$ has at least one node, that is, $\#V(H') < \#V(G)$. By [**Han56**, Cor. 3.4 & Thm. 7.4], we know that Hanner polytopes are twisted prisms over any of their facets. By $Q' \cong \mathrm{stab}(G[V_2]) = \mathrm{stab}(H')$, this means that $P' \cong \mathrm{tp}(Q') \cong \mathrm{H}(H')$. Therefore, $H'$ is a threshold graph by the induction hypothesis.

Since $P \cong \mathrm{stab}(G[V_1])$ is a Hanner polytope, it has a center of symmetry. That is, there exists a $c \in \mathbb{R}^q$ such that $\mathrm{stab}(G[V_1]) = -\mathrm{stab}(G[V_1]) + 2c$. The origin and the unit vectors $e_i$, $i = 1,\dots,q$, are vertices of $\mathrm{stab}(G[V_1])$, which implies $c = (\frac{1}{2}, \dots, \frac{1}{2})$. Hence $\operatorname{stab}(G[V_1]) = [0,1]^q$, which in turn yields $G[V_1] \cong \overline{K_q}$. □

Note that not all Hanner polytopes can be represented as Hansen polytopes. For example, the product of two octahedra $P = C_3^\star \times C_3^\star$ is a Hanner polytope but not a Hansen polytope. Indeed, every facet of $P$ is affinely equivalent to $C_3^\star \times S_2$, where $S_2$ is a triangle, and therefore consists of 18 vertices each of which is contained in exactly 6 edges. If $P$ would be a Hansen polytope of a graph $G$, then $\operatorname{stab}(G)$ needs to be a facet of $P$. We have seen, that $\operatorname{stab}(G)$ has a vertex that is contained in exactly 5 edges of $\operatorname{stab}(G)$, a contradiction.

A consequence of the above characterization is the following.

**Corollary 5.11.** *If $G$ is a threshold graph on $n-1$ nodes, then* $\mathrm{H}(G)$ *has exactly $3^n$ nonempty faces.*

Later on, in Corollary 5.19, we see that the converse also holds.

## 5.3. Hansen Polytopes of Split Graphs

In this section, we analyze Hansen polytopes of *split graphs*. These graphs constitute an important subclass of perfect graphs and are defined as follows.

**Definition 5.12** (split graph)**.** *A graph is called a* split graph *if its node set can be partitioned into a clique and a stable set.*

Every threshold graph is an example of a split graph. Indeed, by collecting all the dominating and isolated nodes in the construction of the threshold graph in corresponding subsets, one obtains a partition of the node set.

Our main result in this chapter appears as Theorem 5.17. Its proof is based on a partitioning technique for the faces of Hansen polytopes of split graphs. We begin with describing this partition.

### 5.3.1. Partitioning Faces of Hansen Polytopes of Split Graphs.

In the following, we let $G$ be a split graph on $k+l$ nodes that splits into the clique $C = \{c_1, \dots, c_k\}$ and the stable set $S = \{s_1, \dots, s_l\}$. A stable set of $G$ is either of the form $A$ or $A \cup \{c_i\}$ for some $A \subseteq S$. Similarly, a clique in $G$ either has the form $A$ or $A \cup \{s_j\}$ for some $A \subseteq C$. Thanks to this simple composition of stable sets and cliques, we use Lemma 5.7 to obtain a complete description of the vertices and facets of $\mathrm{H}(G)$. In fact, a vertex of $\mathrm{H}(G)$ is a stable set of $G$ together with a sign that corresponds to $x_0 = \pm 1$. Analogously, a facet of $\mathrm{H}(G)$ is a clique in $G$ together with such a sign. We omit set parentheses of singletons in order to enhance readability.

- The vertices of $\mathrm{H}(G)$ are
  (1) $(\varepsilon, A)$ with $\varepsilon = \pm$ and $A \subseteq S$,
  (2) $(\varepsilon, A \cup c_i)$ with $\varepsilon = \pm$, $A \subseteq S$ and $A \cup c_i$ stable.
- The facets of $\mathrm{H}(G)$ are
  (1) $[\varepsilon, B]$ with $\varepsilon = \pm$ and $B \subseteq C$,
  (2) $[\varepsilon, B \cup s_j]$ with $\varepsilon = \pm$, $B \subseteq C$ and $C \cup s_j$ being a clique.

According to the enumeration above, we refer to the different kinds of vertices and facets as *type-(1)-vertices/-facets* and *type-(2)-vertices/-facets*. By Lemma 5.7, a vertex of $\mathrm{H}(G)$ is contained in a facet if and only if they have the same sign and their defining subsets of $V(G)$ meet in a common node, or if they have different signs and the defining subsets are disjoint. This observation leads us to the following vertex-facet incidences. From now on, we identify a facet of $\mathrm{H}(G)$ with its set of vertices.

- Type-(1)-facets:

$$(\varepsilon, A) \in [\varepsilon', B] \Longleftrightarrow \varepsilon = -\varepsilon'$$
$$(\varepsilon, A \cup c_i) \in [\varepsilon', B] \Longleftrightarrow (c_i \in B \text{ and } \varepsilon = \varepsilon') \text{ or } (c_i \notin B \text{ and } \varepsilon = -\varepsilon')$$

- Type-(2)-facets:

$$(\varepsilon, A) \in [\varepsilon', B \cup s_j] \Longleftrightarrow (s_j \in A \text{ and } \varepsilon = \varepsilon') \text{ or } (s_j \notin A \text{ and } \varepsilon = -\varepsilon')$$
$$\begin{aligned}(\varepsilon, A \cup c_i) \in [\varepsilon', B \cup s_j] \Longleftrightarrow & (\varepsilon' = \varepsilon \text{ and } (c_i \in B \text{ or } s_j \in A)) \text{ or} \\ & (\varepsilon' = -\varepsilon \text{ and } c_i \notin B \text{ and } s_j \notin A)\end{aligned}$$

Observe that, if $A \cup c_i$ is a stable set and $B \cup s_j$ is a clique, then the events $c_i \in B$ and $s_j \in A$ are mutually exclusive. The following lemma is of good use later on.

**Lemma 5.13.** *Let $G$ be a split graph as above. Choose $A, B \subseteq C$ and $U \subseteq S$ such that $A \cup U$ and $B \cup U$ are cliques. Then we have*
*i)* $[\varepsilon, A \cup U] \cap [\varepsilon, B \cup U] = [\varepsilon, (A \cap B) \cup U] \cap [\varepsilon, A \cup B \cup U]$ *and*
*ii)* $[\varepsilon, A \cup U] \cap [-\varepsilon, B \cup U] \subseteq [\varepsilon, A] \cap [-\varepsilon, B]$.

PROOF. The relations can be easily derived by the vertex-facet incidences described before. Let us consider the first claim. In the case that $U$ is the empty set, both intersections $[\varepsilon, A] \cap [\varepsilon, B]$ and $[\varepsilon, A \cap B] \cap [\varepsilon, A \cup B]$ are given by the following set of vertices:

$$(-\varepsilon, I) \text{ with } I \subseteq S,$$
$$(\varepsilon, I \cup v) \text{ with } v \in A \cap B \text{ and } I \subseteq S,$$
$$(-\varepsilon, I \cup v) \text{ with } v \notin A \cup B \text{ and } I \subseteq S.$$

In the case that, $U = \{i\} \subseteq S$, the two intersections consist of the vertices:

$$(\varepsilon, I) \text{ with } i \in I \subseteq S,$$
$$(-\varepsilon, I) \text{ with } i \notin I \subseteq S,$$
$$(\varepsilon, I \cup v) \text{ with } v \in A \cap B \text{ or } i \in I \subseteq S,$$
$$(-\varepsilon, I \cup v) \text{ with } v \notin A \cup B \text{ and } i \notin I \subseteq S.$$

Part ii) can be shown in a similar way. □

In particular, part i) of the above lemma shows that every face of $\mathrm{H}(G)$ can be written using at most two type-(1)-facets of each sign. Indeed, for $A_1, \ldots, A_t \subseteq C$, we get inductively $\bigcap_{i=1}^t [\varepsilon, A_i] = [\varepsilon, \bigcap_{i=1}^t A_i] \cap [\varepsilon, \bigcup_{i=1}^t A_i]$. The next definition relies on this fact and partitions the faces of $\mathrm{H}(G)$ into convenient subclasses.

**Definition 5.14.** *For a split graph $G$ we define the following four classes of faces of* $\mathrm{H}(G)$*:*

- Primitive faces*, which are not contained in any type-(1)-facet.*
- Positive faces $[+, A] \cap [+, B] \cap F$*, with $A \subseteq B$ and $F$ a primitive face.*
- Negative faces $[-, A] \cap [-, B] \cap F$*, with $A \subseteq B$ and $F$ a primitive face.*

- Small faces*, which are contained in type-(1)-facets of both signs.*

For the primitive faces there is a nice characterization with respect to the containment of certain vertices.

**Lemma 5.15.** *Let $G$ be a split graph. A face $F$ of* $\mathrm{H}(G)$ *is primitive if and only if it contains type-(1)-vertices of both signs.*

Proof. ($\Rightarrow$) Assume that $F$ is primitive, that is, we can write it as

$$F = \bigcap_{i\in I}[+, A_i \cup s_i] \cap \bigcap_{j\in J}[-, B_j \cup s_j]$$

for some multisets $I$ and $J$. If we had $\{s_i : i \in I\} \cap \{s_j : j \in J\} \neq \emptyset$, then Lemma 5.13 ii) would give a contradiction to the primitivity of $F$. Thus, the two multisets are disjoint and we have

$$\begin{aligned}(+, A) \in F &\iff \{s_i : i \in I\} \subseteq A \subseteq S \setminus \{s_j : j \in J\},\\ (-, A) \in F &\iff \{s_j : j \in J\} \subseteq A \subseteq S \setminus \{s_i : i \in I\}.\end{aligned}$$

This means, we can always find positive and negative type-(1)-vertices in $F$.

($\Leftarrow$) According to the vertex-facet incidences, a vertex $(\varepsilon, A)$ cannot be contained in a facet $[\varepsilon, B]$ for every $B \subseteq C$. So if $F$ contains type-(1)-vertices of both signs, then it cannot be contained in any type-(1)-facet. That is, $F$ is primitive. ☐

**5.3.2. On the Number of Faces of Hansen Polytopes of Split Graphs.** Now we use the partitioning described before in order to obtain an exact count of the number of faces of Hansen polytopes of split graphs. The count is in terms of partitions of the stable set and the clique that the graph splits into. These partitions shall obey certain rules that are defined first.

**Definition 5.16.** *Let $G$ be a split graph that splits into a clique $C$ and a stable set $S$. We denote by $p_G(C, S)$ the number of partitions of $C = C^+ \cup C^- \cup C^0$ and $S = S^+ \cup S^- \cup S^0$ such that $C^+ \cup C^- \neq \emptyset$ or $S^+ \cup S^- \neq \emptyset$, and the following hold:*

*(A) Every element of $C^+ \cup C^-$ has a neighbor in $S^+ \cup S^-$.*
*(B) Every element of $S^+ \cup S^-$ has a nonneighbor in $C^+ \cup C^-$.*

It turns out that $p_G(C, S)$ is the number of faces of $\mathrm{H}(G)$ additionally to the conjectured lower bound by Kalai. We write $\mathrm{s}(P) = \sum_{i=0}^{n} \mathrm{f}_i(P)$ for the number of nonempty faces of a polytope $P \in \mathcal{P}^n$.

**Theorem 5.17.** *Let $G$ be a split graph on $n-1$ nodes with clique $C$ and stable set $S$. Then*

$$\mathrm{s}(\mathrm{H}(G)) = 3^n + p_G(C, S).$$

*In particular, Hansen polytopes of split graphs satisfy Conjecture 5.4.*

Proof. Let $\Pi$ be the set of all partitions of $V(G)$ such that both $C$ and $S$ are partitioned into three parts. Further let $\Pi_A, \Pi_B \subseteq \Pi$ be the subsets for which (A) and (B) in Definition 5.16 hold, respectively. Observe that if (A) fails for a given partition in $\Pi$, then there must be a node in $C^+ \cup C^-$ which is not adjacent to any node in $S^+ \cup S^-$. Thus this partition fulfills

(B). From this we get $\Pi_A^c \subseteq \Pi_B$, where $\Pi_A^c$ is the complement of $\Pi_A$ in $\Pi$. Analogously, we have $\Pi_B^c \subseteq \Pi_A$, and therefore, by inclusion-exclusion, it is

$$\begin{aligned} 3^{n-1} = |\Pi| &= |\Pi_A| + |\Pi_B| + |\Pi_A^c| + |\Pi_B^c| - |\Pi_A \cap \Pi_B| - |\Pi_A \cap \Pi_B^c| \\ &\quad - |\Pi_A^c \cap \Pi_B| - |\Pi_A^c \cap \Pi_B^c| \\ &= |\Pi_A| + |\Pi_B| - |\Pi_A \cap \Pi_B|. \end{aligned}$$

Since $p_G(C, S) = |\Pi_A \cap \Pi_B| - 1$, we thus need to show that

$$\mathrm{s}(\mathrm{H}(G)) = 3^n + |\Pi_A \cap \Pi_B| - 1 = 2 \cdot 3^{n-1} + |\Pi_A| + |\Pi_B| - 1.$$

In order to see this, we use Definition 5.14 to partition the faces of $\mathrm{H}(G)$. We exclude the improper faces $\emptyset$ and $\mathrm{H}(G)$ from our considerations. Let $f_p(G)$, $f_+(G)$ and $f_-(G)$ be the number of primitive, positive and negative faces of $\mathrm{H}(G)$, respectively. If $F$ is a small face of $\mathrm{H}(G)$, then, by definition, it is contained in type-(1)-facets of both signs. Via the usual bijection between faces of a polytope and faces of its polar, type-(1)-facets of $\mathrm{H}(G)$ correspond to type-(1)-vertices of the same sign of $\mathrm{H}(G)^\star$. Lemma 5.15 yields that the polar face of $F$ must be primitive in $\mathrm{H}(G)^\star \cong \mathrm{H}(\overline{G})$. Hence,

$$\mathrm{s}(\mathrm{H}(G)) = f_p(G) + f_+(G) + f_-(G) + f_p(\overline{G}) - 1.$$

From this identity the proof is finished by Lemma 5.18 below. $\square$

**Lemma 5.18.** *In the setting of the above proof, we have*

*i)* $f_+(G) = f_-(G) = 3^{n-1}$,
*ii)* $f_p(G) = |\Pi_A|$ *and* $f_p(\overline{G}) = |\Pi_B|$.

Proof. Recall that $C = \{c_1, \dots, c_k\}$ is the clique and $S = \{s_1, \dots, s_l\}$ is the stable set that $G$ splits into. We need to refine the notion of a primitive face. Given multisets $S^+ = \{s_i : i \in I\}$ and $S^- = \{s_j : j \in J\}$, a primitive face of the form

$$\bigcap_{i \in I} [+, A_i \cup s_i] \cap \bigcap_{j \in J} [-, B_j \cup s_j]$$

is called $(S^+, S^-)$-*primitive*.

i) The bijection $x \mapsto -x$ maps a facet $[\varepsilon, A]$ of $\mathrm{H}(G)$ to $[-\varepsilon, A]$. Therefore, we have $f_+(G) = f_-(G)$, and it is enough to show $f_+(G) = 3^{n-1}$. So let us consider a positive face $P = [+, A'] \cap [+, A] \cap F$, where $A' \subseteq A \subseteq C$ and

$$F = \bigcap_{i \in I} [+, A_i \cup s_i] \cap \bigcap_{j \in J} [-, B_j \cup s_j]$$

is primitive. As noted in the proof of Lemma 5.15, the multisets $\{s_i : i \in I\}$ and $\{s_j : j \in J\}$ are disjoint and $P$ contains a vertex $(-, X)$, $X \subseteq S$, if and only if that vertex is contained in $F$. That is, if and only if $\{s_j : j \in J\} \subseteq X \subseteq S \setminus \{s_i : i \in I\}$. Since there are $3^{|S|}$ many possibilities to choose two disjoint subsets from $S$, it suffices to show that for fixed $\{s_i : i \in I\}$ and $\{s_j : j \in J\}$, we have $3^{|C|}$ many positive faces of the above form.

To this end, let $F$ be a fixed $(\{s_i : i \in I\}, \{s_j : j \in J\})$-primitive face. The type-(1)-vertices of a corresponding positive face $P$ are determined as just explained. Thus it is enough to find out which type-(2)-vertices belong to $P$. We can describe them as

$$(+, X \cup z) \in P \Longleftrightarrow z \in A' \text{ and } z \notin \bigcup_{j \in J} B_j \text{ and } z \in \bigcap_{i \in I, s_i \notin X} A_i$$
$$\text{and } \{s_i : i \in I, z \notin A_i\} \subseteq X \subseteq S \setminus \{s_j : j \in J\},$$

and similarly

$$(-, X \cup z) \in P \Longleftrightarrow z \notin A \text{ and } z \notin \bigcup_{i \in I} A_i \text{ and } z \in \bigcap_{j \in J, s_j \notin X} B_j$$
$$\text{and } \{s_j : j \in J, z \notin B_j\} \subseteq X \subseteq S \setminus \{s_i : i \in I\}.$$

These conditions tell us that, for each $z \in C$, either there is an $X \subseteq S$ such that $(+, X \cup z) \in P$, or there is an $X \subseteq S$ such that $(-, X \cup z) \in P$, or none of these is true. Furthermore, these three cases can be controlled independently by the choices of $A'$ and $A$. This gives the desired $3^{|C|}$ positive faces for fixed $\{s_i : i \in I\}$ and $\{s_j : j \in J\}$.

ii) The second part is a bit more complicated. First, as noted before, each partition of $G$ that satisfies (A), automatically satisfies (B) for $\overline{G}$, and vice versa. It is therefore enough to prove $f_p(G) = |\Pi_A|$. This is achieved by constructing a bijection $\mathcal{P} \to \Pi_A$, where $\mathcal{P}$ is the set of all primitive faces of $\mathrm{H}(G)$. For this purpose, we partition the domain and range as follows.

Denote by $\mathcal{P}(S^+, S^-)$ the set of all $(S^+, S^-)$-primitive faces. Then

$$\mathcal{P} = \bigcup \{\mathcal{P}(S^+, S^-) : S^+, S^- \subseteq S \text{ disjoint and } S^+ \cup S^- \neq \emptyset\}$$

is a partition of $\mathcal{P}$.

Let $\Pi_A(S^+, S^-)$ be the set of partitions $(C^+, C^-, C^0, S^+, S^-, S^0)$ of $V(G)$ that satisfy (A) and where $S^+, S^-$ are fixed (so only $C^+, C^-$ vary). Then

$$\Pi_A = \bigcup \{\Pi_A(S^+, S^-) : S^+, S^- \subseteq S \text{ disjoint and } S^+ \cup S^- \neq \emptyset\}$$

is a partition of $\Pi_A$.

For the remainder of the proof we let $S^+, S^-$ be fixed and disjoint subsets of $S$ such that $S^+ \cup S^- \neq \emptyset$. We describe mappings

$$\Psi_{(S^+,S^-)} : \mathcal{P}(S^+, S^-) \to \Pi_A(S^+, S^-)$$

and

$$\Phi_{(S^+,S^-)} : \Pi_A(S^+, S^-) \to \mathcal{P}(S^+, S^-),$$

that will turn out to be inverse to each other. This of course shows that there exists a bijective correspondence between different parts of the partitions of $\mathcal{P}$ and $\Pi_A$, which allows us to conclude the existence of a bijection $\mathcal{P} \to \Pi_A$.

Let $\Psi_{(S^+,S^-)}$ be defined by

$$\Psi_{(S^+,S^-)}(F) = (C^+, C^-, C^0, S^+, S^-, S^0),$$

where, for $\varepsilon = \pm$, we let

$$C^\varepsilon = \{c \in C : (\varepsilon, (S^\varepsilon \setminus N(c)) \cup c) \in F \text{ and } (-\varepsilon, J \cup c) \notin F, J \subseteq S\}. \tag{5.1}$$

Here $N(c)$ again denotes the neighborhood of $c$ in $G$. On the other hand, define $\Phi_{(S^+,S^-)}$ by

$$\Phi_{(S^+,S^-)}(C^+, C^-, C^0, S^+, S^-, S^0)$$

$$= \bigcap_{s\in S^+} [+, A'_s \cup s] \cap [+, A_s \cup s] \cap \bigcap_{s\in S^-} [-, B'_s \cup s] \cap [-, B_s \cup s],$$

where we let

$$A'_s = C^+ \cap N(s), A_s = N(s) \setminus C^-, B'_s = C^- \cap N(s) \text{ and } B_s = N(s) \setminus C^+.$$

We use the abbreviations $\Psi = \Psi_{(S^+,S^-)}$ and $\Phi = \Phi_{(S^+,S^-)}$ from now on.

Let us show $\Psi \circ \Phi = \mathrm{id}_{\Pi_A(S^+,S^-)}$: Given a partition $\pi \in \Pi_A(S^+, S^-)$ it is sufficient to prove

$$\pi = (C^+, C^-, C^0, S^+, S^-, S^0) \subseteq \Psi(\Phi(\pi)) = (D^+, D^-, D^0, S^+, S^-, S^0),$$

where inclusion is to be understood componentwise. Indeed, both $\pi$ and its image are, by construction, partitions of $V(G)$. We begin by explaining $C^+ \subseteq D^+$. Let $c \in C^+$. By definition $c \in D^+$ if the vertex $v = (+, (S^+ \setminus N(c)) \cup c) \in \Phi(\pi)$, and, for all $J \subseteq S$, the vertex $w_J = (-, J \cup c) \notin \Phi(\pi)$. Concerning the first condition, we observe that the stable set $(S^+ \setminus N(c)) \cup c$ does not hit any of the $B_s \cup s$. So $v$ is contained in all of the facets with a negative sign. For the facets with a positive sign the containment is clear if $c \in A'_s$, and in the case $c \notin A'_s$, we have $c \notin N(s)$, i.e., $s \in S^+ \setminus N(c)$. Next consider the second condition on $c$ to be contained in $D^+$. Since $\pi$ fulfills (A), there exists a neighbor $s \in S^+ \cup S^-$ of $c$. If $s \in S^+$, then $c \in C^+ \cap N(s) = A'_s$, and therefore $c \in A'_s \cup s$. This rules out that $(-, J \cup c) \in \Phi(\pi)$. If $s \in S^-$, then $c \notin B'_s$ by construction. So if $w_J \in \Phi(\pi)$, we must have $s \in J$ which contradicts the fact that $J \cup c$ is a stable set. Hence $c \in D^+$ and thus $C^+ \subseteq D^+$. Similarly, we obtain the inclusion $C^- \subseteq D^-$.

It remains to explain $C^0 \subseteq D^0$, so assume $c \in C^0$. If $c \notin N(S^+ \cup S^-)$, then $N(c) \cap S \subseteq S^0$ and, in view of the vertex-facet incidences, we get $(+, S^+ \cup c)$, $(-, S^- \cup c) \in \Phi(\pi)$, hence $c \in D^0$. If $c \in N(S^+ \cup S^-)$, then there is an $s \in S^+$ (the case $s \in S^-$ is analogous) such that $\{c, s\} \in E(G)$. Thus, $c \in C^0 \cap N(s) \subseteq A_s$, meaning that $(-, (S^- \setminus N(c)) \cup c) \notin \Phi(\pi)$. We also must have $c \notin A'_s$, from which we get $(+, (S^+ \setminus N(c)) \cup c) \notin \Phi(\pi)$, since $s \in N(c)$. This shows $c \in D^0$, and concludes $\Psi \circ \Phi = \mathrm{id}_{\Pi_A(S^+,S^-)}$.

Now we show $\Phi \circ \Psi = \mathrm{id}_{\mathcal{P}(S^+,S^-)}$: Given an $(S^+, S^-)$-primitive face

$$F = \bigcap_{s\in S^+} [+, A'_s \cup s] \cap [+, A_s \cup s] \cap \bigcap_{s\in S^-} [-, B'_s \cup s] \cap [-, B_s \cup s],$$

we need to show $\Phi(\Psi(F)) = F$. Both $F$ and its image are $(S^+, S^-)$-primitive faces and hence they contain type-(1)-vertices $(\varepsilon, J)$ if and only if $S^\varepsilon \subseteq J \subseteq S \setminus S^{-\varepsilon}$. Therefore, we only need to show that both $F$ and $\Phi(\Psi(F))$ contain the same type-(2)-vertices as well.

We begin with showing that $(\varepsilon, J \cup c) \in F$ implies $(\varepsilon, J \cup c) \in \Phi(\Psi(F))$. To this end, we let $(\varepsilon, J \cup c) \in F$ and we distinguish two cases.

1) Assume there exists a $K \subseteq S$ such that $(-\varepsilon, K \cup c) \in F$. This means, that $c$ cannot be contained in $A_s$ or $B_s$, for $s \in S^+$ or $s \in S^-$, respectively. Due to $(\varepsilon, J \cup c) \in F$, we must have $S^{-\varepsilon} \subseteq K$ and $S^\varepsilon \subseteq J \subseteq S \setminus S^{-\varepsilon}$. From here we see that $c$ has no neighbor in $S^{-\varepsilon}$. Altogether, this yields $(\varepsilon, J \cup c) \in [-\varepsilon, (C^{-\varepsilon} \cap N(s)) \cup s]$, for all $s \in S^{-\varepsilon}$, and $(\varepsilon, J \cup c) \in [\varepsilon, (C^\varepsilon \cap N(s)) \cup s]$, for all $s \in S^\varepsilon$. Hence, $(\varepsilon, J \cup c) \in \Phi(\Psi(F))$ as desired.

2) The other case is $(-\varepsilon, K \cup c) \notin F$, for all $K \subseteq S$. If $s \in S^\varepsilon$ is not adjacent to $c$, then $s \in J$, i.e., $S^\varepsilon \setminus N(c) \subseteq J$. According to (5.1) and

$(\varepsilon, J \cup c) \in F$, we also have $c \in C^\varepsilon$, independently of $s$. So, for any $s \in S^\varepsilon$, either $s \in J$ or $c \in C^\varepsilon \cap N(s)$. From this we get that $(\varepsilon, J \cup c)$ is contained in every facet of sign $\varepsilon$ that defines $\Phi(\Psi(F))$. Since $J \cap S^{-\varepsilon} = \emptyset$, we conclude that $(\varepsilon, J \cup c)$ is also contained in every facet of sign $-\varepsilon$. Again we obtain $(\varepsilon, J \cup c) \in \Phi(\Psi(F))$.

We finally need to prove also the converse direction, that is, $(\varepsilon, J \cup c) \in \Phi(\Psi(F))$ implies $(\varepsilon, J \cup c) \in F$. By the vertex-facet incidences, we have $J \subseteq S \setminus S^{-\varepsilon}$, for all $(\varepsilon, J \cup c) \in \Phi(\Psi(F))$. Again, we distinguish between two cases.

1) Let $S^\varepsilon \subseteq J$. If $c \notin N(S^{-\varepsilon})$, then, by $S^\varepsilon \subseteq J$, $(\varepsilon, J \cup c)$ is contained in all facets with sign $\varepsilon$ that define $F$. Since the cliques corresponding to the facets with sign $-\varepsilon$ contain only nodes from $N(S^{-\varepsilon})$ or $S^{-\varepsilon}$, and $J \subseteq S \setminus S^{-\varepsilon}$, the vertex $(\varepsilon, J \cup c)$ lies in all facets defining $F$. So, now let $c \in N(S^{-\varepsilon})$. By $(\varepsilon, J \cup c) \in \Phi(\Psi(F))$, we have $(\varepsilon, J \cup c) \in [-\varepsilon, (N(s) \setminus D^\varepsilon) \cup s]$, for all $s \in S^{-\varepsilon}$, where $D^\varepsilon$ is a component of $\Psi(F)$. Therefore, $c \notin N(s) \setminus D^\varepsilon$, for all $s \in S^{-\varepsilon}$, and thus $c \in D^\varepsilon$. This implies that $(\varepsilon, (S^\varepsilon \setminus N(c)) \cup c) \in F$, which gives $(\varepsilon, J \cup c) \in F$ because $S^\varepsilon \subseteq J \subseteq S \setminus S^{-\varepsilon}$.

2) On the other hand, consider $S^\varepsilon \not\subseteq J$. Then, because of $(\varepsilon, J \cup c) \in \Phi(\Psi(F))$, we have $c \in D^\varepsilon \cap N(s)$, for all $s \in S^\varepsilon \setminus J$, where $D^\varepsilon$ is again a component of $\Psi(F)$. In particular, $c \in D^\varepsilon$, which means that the vertex $(\varepsilon, (S^\varepsilon \setminus N(c)) \cup c) \in F$. In view of $J \subseteq S \setminus S^{-\varepsilon}$ and $c \in N(s)$, for any $s \in S^\varepsilon \setminus J$, this implies $(\varepsilon, J \cup c) \in F$.

This finishes the argument for $\Psi \circ \Phi = \mathrm{id}_{\Pi_A(S^+, S^-)}$, and therefore establishes the bijection $\mathcal{P} \to \Pi_A$. □

A consequence of Theorem 5.17 is that the particular partition of the split graph is irrelevant for the number of faces of the corresponding Hansen polytope. Therefore, instead of $p_G(C, S)$ we now write $p_G$. What we know about this function $p_G$ is summarized in the following corollary.

**Corollary 5.19.** *Let $G$ be a split graph on $n-1$ nodes. Then*

$$\mathrm{s}(\mathrm{H}(G)) = 3^n + 16 \cdot l \quad \textit{for some} \quad l \in \mathbb{N}_0.$$

*Moreover, $l = 0$ if and only if $G$ is a threshold graph.*

Proof. Let us first see that $p_G = 16 \cdot l$. To this end, let $G$ be split into a clique $C$ and a stable set $S$ and consider a partition $C = C^+ \cup C^- \cup C^0$ and $S = S^+ \cup S^- \cup S^0$. If $C^+ \cup C^- = \emptyset$, then condition (B) in Definition 5.16 is only satisfied if $S^+ \cup S^- = \emptyset$. Similarly, if $S^+ \cup S^- = \emptyset$, we have $C^+ \cup C^- = \emptyset$ due to condition (A). In both cases, we consider the trivial partition $C^0 = C$, $S^0 = S$ which is not counted by $p_G$. If $C^+ \cup C^- = \{c\}$, then, by (A), there exists a neighbor of $c$ in $S^+ \cup S^-$. By (B) again, this neighbor must have a nonneighbor in $C^+ \cup C^-$, which clearly cannot be. By similar reasoning we can disregard the case $S^+ \cup S^- = \{s\}$. In conclusion, if our partition should contribute to $p_G$, we must have $|C^+ \cup C^-| \geq 2$ and $|S^+ \cup S^-| \geq 2$. Since we can assign the elements to $C^+$, $C^-$, $S^+$ and $S^-$ arbitrarily, we get $p_G = 16 \cdot l$, for some $l \in \mathbb{N}_0$.

Now, $l = 0$ if and only if $p_G = 0$. Assume that $G$ contains a path $P_4$ on four nodes as an induced subgraph. Then the partition, where $C^+$ consists of the two middle nodes of $P_4$, $S^+$ consists of the two endpoints and $C^- = S^- = \emptyset$, satisfies the conditions (A) and (B). Thus, if $l = 0$, then $G$ is a split graph without induced paths on four nodes. By [MP95, Thm. 1.2.4], this property characterizes threshold graphs. On the other hand, if $G$ is a threshold graph, then $l = 0$ by Theorem 5.10. □

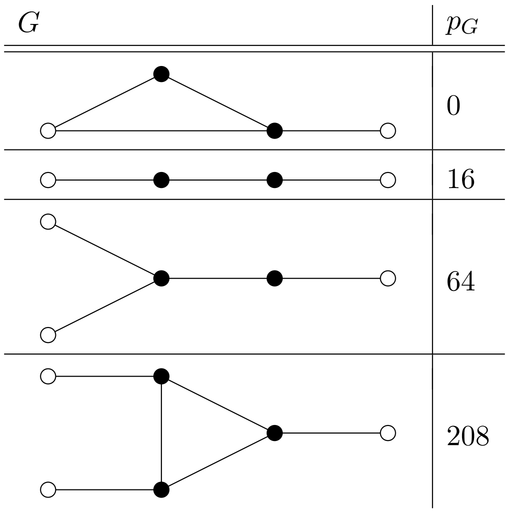



FIGURE 5.1. Some split graphs and their value of $p_G$

Figure 5.1 illustrates some examples of particular split graphs $G$ and their corresponding value of $p_G$. Therein the black nodes constitute a clique and the white nodes a stable set that $G$ splits into.

**5.3.3. High-dimensional Hansen Polytopes with Few Faces.** We finish our considerations of Hansen polytopes of split graphs with a construction of high-dimensional Hansen polytopes with few faces. To this end, consider a threshold graph $T$ on $m$ nodes and a split graph $G$ on $n$ nodes that is split, as usual, into a clique $C$ and a stable set $S$. We construct a new graph $G \ltimes T$ by taking the union of $G$ and $T$ and adding edges between every node of $C$ and every node of $T$. Figure 5.2 illustrates the construction for the case where $G$ is the path on four nodes.

Note that the resulting graph $G \ltimes T$ is again a split graph.

**Proposition 5.20.** *Let $G$ be a split graph on $n$ nodes and let $T$ be a threshold graph on $m$ nodes. Then*

$$\mathrm{s}(\mathrm{H}(G \ltimes T)) = 3^{m+n+1} + p_G.$$

*This means $p_{G \ltimes T} = p_G$, so $p_{G \ltimes T}$ is independent of $T$.*

PROOF. By definition, the threshold graph $T$ can be built by successive adding of isolated and dominating nodes. This induces an ordering on the nodes $v_1, \dots, v_m$ of $T$. Let $C_T = \{v_i : v_i \text{ dominating at step } i\}$ and $S_T = \{v_i : v_i \text{ isolated at step } i\}$. This splits $T$ into the clique $C_T$ and the stable set $S_T$, which in turn splits $G \ltimes T$ into $C \cup C_T$ and $S \cup S_T$, where $C$ is a clique and $S$ is a stable set that $G$ splits into. By construction, any node in $C_T$ and $S_T$ is connected to any node in $C$ and no node in $S$. Now consider

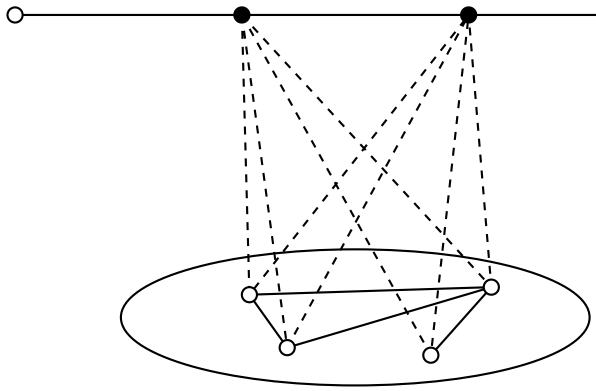


FIGURE 5.2. Appending a threshold graph to a split graph

a partition $(C^+ \cup C^- \cup C^0, S^+ \cup S^- \cup S^0)$ of $V(G \ltimes T)$ which is counted by $p_{G \ltimes T}(C \cup C_T, S \cup S_T)$. In view of condition (A), there exists a neighbor $y \in (S^+ \cup S^-) \cap S_T$ for any $x \in (C^+ \cup C^-) \cap C_T$. This means, that in $T$ the once isolated node $y$ was inserted before the once dominating node $x$. On the other hand, by condition (B), any given node $y \in (S^+ \cup S^-) \cap S_T$ has to have a nonneighbor $z \in (C^+ \cup C^-) \cap C_T$. Such a node $z$ was used before $y$ in the construction of $T$. Clearly, these two events can only hold in the case $(C^+ \cup C^-) \cap C_T = \emptyset = (S^+ \cup S^-) \cap S_T$. Therefore, for this partition, we have $C_T \subseteq C^0$ and $S_T \subseteq S^0$, which implies $p_{G \ltimes T}(C \cup C_T, S \cup S_T) = p_G$. □

This finally yields a series of high-dimensional Hansen polytopes with very few faces.

**Corollary 5.21.** *Let $P_4$ be the path on four nodes and let $T$ be a threshold graph on $m$ nodes. Then*

$$\mathrm{s}(\mathrm{H}(P_4 \ltimes T)) = 3^{m+5} + 16.$$

PROOF. Any partition $(C^+, C^-, C^0, S^+, S^-, S^0)$ of $V(P_4 \ltimes T)$ that contributes to the count of $p_{P_4}$ fulfills $|C^+ \cup C^-| \geq 2$ and $|S^+ \cup S^-| \geq 2$, by the proof of Corollary 5.19. By virtue of the considerations in the proof of Proposition 5.20, we actually have that $C^+ \cup C^-, S^+ \cup S^- \subseteq V(P_4)$. The path $P_4$ restricts the partition to $|C^+ \cup C^-| = |S^+ \cup S^-| = 2$, hence $p_{P_4} = 16$. Proposition 5.20 then gives the claim. □

# List of Symbols

| | |
|---|---|
| $\mathbb{R}^n$ | $n$-dimensional real vector space |
| $\|\cdot\|$ | Euclidean norm |
| $\|\cdot\|_\infty$ | maximum norm |
| $e_i$ | $i$th coordinate unit vector |
| $L^\perp$ | orthogonal complement of a linear space $L$ |
| $K\|L$ | orthogonal projection of $K$ onto $L$ |
| $\mathcal{K}^n$ | set of convex bodies in $\mathbb{R}^n$ |
| $\mathcal{P}^n$ | set of convex polytopes in $\mathbb{R}^n$ |
| $\mathcal{K}_0^n$ | set of centrally symmetric convex bodies in $\mathbb{R}^n$ |
| $\mathcal{P}_0^n$ | set of centrally symmetric convex polytopes in $\mathbb{R}^n$ |
| $\mathcal{L}^n$ | set of $n$-dimensional lattices |
| $\det(\Lambda)$ | determinant of a lattice $\Lambda$ |
| $S^\star$ | polar set of $S$ |
| $\Lambda^\star$ | polar lattice of a lattice $\Lambda$ |
| $\mathrm{aff}(S)$ | affine hull of $S$ |
| $\mathrm{conv}(S)$ | convex hull of $S$ |
| $\mathrm{lin}(S)$ | linear hull of $S$ |
| $\dim(S)$ | affine dimension of $S$ |
| $\mathrm{int}\, S$ | interior of $S$ |
| $\mathrm{relint}\, S$ | relative interior of $S$ |
| $\partial S$ | boundary of $S$ |
| $\mathrm{vol}(S)$ | volume (Lebesgue measure) of $S$ |
| $\mathrm{V}(\cdot)$ | mixed volume |
| $\mathcal{D}S$ | difference set $S - S$ of $S$ |
| $\mathrm{cen}(S)$ | centroid (center of gravity) of $S$ |
| $\mathrm{vert}\, P$ | vertex set of a polytope $P$ |
| $\mathrm{tp}(P)$ | twisted prism over a polytope $P$ |
| $\mathrm{f}_i(P)$ | number of $i$-dimensional faces of a polytope $P$ |
| $\mathrm{s}(P)$ | number of nonempty faces of a polytope $P$ |
| $\mathrm{G}(S, \Lambda)$ | the lattice point enumerator $\#(S \cap \Lambda)$ |
| $\mathrm{g}_i(P)$ | $i$th Ehrhart coefficient of a lattice polytope $P$ |
| $\mathrm{a}_i(P)$ | $i$th a-vector entry of a lattice polytope $P$ |
| $\lambda_i(K, \Lambda)$ | $i$th successive minimum of $K$ with respect to $\Lambda$ |
| $\lambda_i^c(K, \Lambda)$ | $i$th successive minimum of $K$ with respect to $\Lambda$ and $\mathrm{cen}(K)$ |
| $[n]$ | first $n$ natural numbers |
| $\binom{S}{i}$ | set of $i$-element subsets of a finite set $S$ |
| $\gcd(a)$ | greatest common divisor of the entries of a lattice vector $a$ |
| $\lfloor\cdot\rfloor$ | floor function |
| $\sigma_i$ | $i$th elementary symmetric polynomial |

| | |
|---|---|
| $\pi$ | projection that forgets the last coordinate |
| $\pi^{(i)}$ | projection that forgets the last $i$ coordinates |
| $\cong$ | affine equivalence |
| $\simeq$ | unimodular equivalence |
| $V(G)$ | set of nodes of a graph $G$ |
| $E(G)$ | set of edges of a graph $G$ |
| $\overline{G}$ | graph complement of a graph $G$ |
| $K_n$ | complete graph on $n$ nodes |
| $\mathrm{stab}(G)$ | stable set polytope of a graph $G$ |
| $\mathrm{H}(G)$ | Hansen polytope of a graph $G$ |
| $B_n$ | $n$-dimensional Euclidean unit ball |
| $C_n$ | unit $n$-cube $[-1,1]^n$ |
| $C_n^\star$ | standard crosspolytope $\mathrm{conv}\{\pm e_1, \ldots, \pm e_n\}$ |
| $C_n(m)$ | $n$-dimensional cyclic polytope with $m$ vertices |
| $S_n$ | standard simplex $\mathrm{conv}\{0, e_1, \ldots, e_n\}$ |
| $\kappa_n$ | volume of $B_n$ |

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