Ricci-flat Complex Geometry and its Applications

Habilitationsschrift
zur Erlangung des akademischen Grades
doctor rerum naturalium habilitatus (Dr. rer. nat. habil.)

genehmigt
durch die Fakultät für **Mathematik**
der Otto-von-Guericke-Universität Magdeburg

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geboren am 17. Mai 1972 in Hof

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Magdeburg, den 3. Februar 2010

Deutschsprachige Zusammenfassung

Das Interesse des Geometers an Ricci-flacher komplexer Geometrie speist sich aus mehreren Quellen. Gegeben eine komplexe Mannigfaltigkeit X, kann man geometrische Information zu extrahieren, indem man nach "besten" Metriken sucht. Das Beste, was man erwarten kann, ist, dass X lokal dem euklidischen Raum so ähnlich wie möglich sieht. Diese Ähnlichkeit wird durch die Holonomie des Levi-Civita-Zusammenhangs gegeben, d.h. die Automorphismengruppe eines fixierten Tangentialraumes, die durch Paralleltransport eines Tangentialvektors entlang einer geschlossenen Kurve entsteht. Eine komplexe Mannigfaltigkeit hat automatisch eine Untergruppe der $SO(2n)$ als Holonomiegruppe, die Kählerbedingung reduziert bereits auf eine Untergruppe der $U(n)$. Der nächste Schritt wäre $SU(n)$. Im Fall einer kompakten Kählermannigfaltigkeit ist dies äquivalent zur Ricciflachheit der Kählermetrik. Die Existenz einer Ricci-flachen Metrik wiederum ist äquivalent zum Verschwinden der ersten Chernklasse, $c_1(X) = 0$.

Ricci-Flachheit bezeichnet das Verschwinden des komplexen Ricci-Tensors

$$\mathrm{Ric}\omega := i\partial\overline{\partial} \log \det \omega$$

der metrischen Form ω. An dieser Stelle ist Vorsicht geboten: Der komplexe und der reelle (Levi-Civita) Ricci Tensor sind nur identisch, wenn ω eine Kählerform ist, d.h. $d\omega = 0$ gilt.

Die Aussage der von Yau bewiesenen Calabi-Vermutung ist insbesondere, dass eine kompakte komplexe Kählermannigfaltigkeit mit $c_1(X) = 0$ eine Ricci-flache Kählermetrik erlaubt. Jede solche Mannigfaltigkeit hat eine endliche, unverzweigte Überlagerung $\tilde{X}$ mit trivialem kanonischen Bündel $K_X = \Omega_X^n$; diese wiederum haben eine endliche, unverzweigte Überlagerung, die ein Produkt aus einem Torus und einer einfach zusammenhängenden Mannigfaltigkeit mit trivialem kanonischen Bündel ist. Die letzteren heißen Calabi-Yau-Mannigfaltigkeiten. Während das mathematische Interesse aus der Klassifikation komplexer Mannigfaltigkeiten stammt, haben auch Physiker mit dem Aufkommen der Stringtheorie ein Interesse an Calabi-Yau-Mannigfaltigkeiten entwickelt. Die große Familie der Stringtheorien heute beinhaltet auch Modelle mit offenen Strings, die dann nicht-kompakte Calabi-Yau-Mannigfaltigkeiten induzieren; damit sind nicht-kompakte Mannigfaltigkeiten, die eine vollständige Ricci-flache Kählermetrik zulassen gemeint.

In dieser Allgemeinheit kann man nicht hoffen, das Calabi-Problem für nicht-kompakte Mannigfaltigkeiten angehen zu können. Deshalb konzentriert man sich auf den Fall sogenannter offener Mannigfaltigkeiten: Komplemente von Divisoren in kompakten komplexen Mannigfaltigkeiten. Das Analogon zu $c_1(X) = 0$ sollte $K_X + D = 0$ lauten, wenn $X \setminus D$ die untersuchte Mannigfaltigkeit ist. Unter dieser Annahme und, dass D glatt und ample ist, haben Tian und Yau gezeigt, den ersten Schritt der 'offenen Calabi-Vermutung' gezeigt: Es existiert eine vollständige, Ricci-flache Metrik auf

$X \setminus D$. Es ist bis heute aber unbekannt, ob man eine solche Metrik auch in jeder vorgegebenen Kählerklasse finden kann; hierbei ist es angemessener, die Kählerklasse in Bott-Chern-Kohomologie zu verstehen. Ebenso offen ist auch die Existenz noch im Fall singulärer oder nicht-reduzierter Divisoren. Nur der Spezialfall $D = mD'$ wurde in Arbeiten von Tian/Yau und Bando/Kobayashi mit positivem Ergebnis behandelt.

In Kapitel 2 betrachten wir Divisoren und Metriken, die unter großen Symmetriegruppen invariant sind. Gro"s bedeutet hier, dass die Gruppe fast transitiv wirkt. Auf diesem Wege erhält man eine Beispielklasse von singulären Divisoren und vollständigen, Ricci-flachen Metriken auf deren Komplement. Die Kählerbedingung der Metrik wird sich als Kommutativität der Symmetriegruppe erweisen. In diesem singulären Fall ist der Kählerkegel hochdimensional, auch wenn X selbst wenig Topologie hat, wie zum Beispiel $X = \mathbb{P}^n$; es stellt sich also die Frage nach einer vollständigen, Ricci-flachen Kählermetrik in jeder positiven Kohomologieklasse und sie wird positiv beantwortet. Eindeutigkeit in der Kählerklasse ist jedoch nicht notwendig gegeben.

Im Gegensatz dazu zeigt ein Resultat mit Bert Koehler, dass der Kählerkegel trivial ist, wenn D glatt ist und X topologisch einfach genug. In diesem Fall ist jede Kählermetrik von der Form $\omega + i\partial\overline{\partial}u$ für jede beliebige Startmetrik ω. Diese anscheinend komplizierte Formulierung des Sachverhalts trifft genau den Ansatz, den man verfolgen muss, um eine Ricci-flache, vollständige Metrik zu erhalten: Man sucht eine Lösung der komplexen Monge-Ampére-Gleichung

$$\frac{(\omega + i\partial\overline{\partial}u)^n}{\omega^n} = e^f$$

mit $\omega + i\partial\overline{\partial}u > 0$, wobei

$$f = \log \frac{\Omega \wedge \overline{\Omega}}{\omega^n}$$

und Ω eine meromorphe n-Form mit Singularitäten nur entlang D und ohne Nullstellen ist. Tian und Yau haben eine asymptotisch flache Startmetrik konstruiert und dafür eine beschränkte Lösung u gefunden.

Damit kann man ausrechnen, dass das Volumenwachstum geodätischer Bälle mit festem Mittelpunkt mit dem rationalen Exponenten $\frac{2n}{n+1}$ geschieht, während in dem oben angesprochenen symmetrischen Fall für ample D dieser Exponent ganzzahlig ist. Die Asymptotiken bekannter Ricci-flacher Metriken für Komplemente glatte Divisoren unterscheiden sich also qualitativ von denen für Komplemente singulärer Divisoren.

Die hier behandelte Hauptfrage ist, wie nah im amplen, glatten Fall die Startmetrik an der vollständigen, Ricci-flachen Lösungsmetrik liegt. Diese Resultate wurden in Kooperation mit Bert Koehler gefunden. Es wird bewiesen: Sei ω eine bestimmte (im Wesentlichen von Tian und Yau konstruierte) Anfangsmetrik, $\tilde{\omega}$ die konstruierte Ricci-flache, vollständige Metrik, D der Nullstellenort des Schnittes S von $\mathcal{O}(D)$ und $\| \cdot \|$

eine geeignet zu wählende Metrik auf $\mathcal{O}(D)$. Dann gilt: Zu jedem $N > 0$ existiert eine Konstante $C_N > 0$, so dass

$$(1 - C_N(-\log \|S\|^2)^{-N})\omega \leq \tilde{\omega} \leq (1 + C_N(-\log \|S\|^2)^{-N})\omega.$$

Damit ist eine von Tian und Yau gestellte Frage beantwortet. Zur gleichen Zeit und mit anderen Methoden hat Santoro eine ähnliche Aussage bewiesen. Die hier verwendeten Techniken beruhen auf Krümmungsabschätzungen durch Tian und Yau, mit deren Hilfe zunächst eine schwache Abfallrate für die Differenz der Metriken bewiesen werden kann. Die Konstruktion vereinfachter Quasikoordinaten sowie geeigneter "Zylinderkoordinaten" einer Tubenumgebung des Divisors erlauben den hinreichend genauen Vergleich von ω mit einer expliziter berechenbaren Riemannschen Metrik. Spektraltheorie des dazugehörigen linearisierten Problems erlaubt einen Vergleich der Abfallraten für $\Delta\ u$ und $\|\partial\overline{\partial}u\|$; diese liegen nahe genug beieinander, um durch iteratives Einsetzen in die Monge-Ampére-Gleichung beliebig hohe Abfallraten zu erzeugen.

Schließlich erhalten wir auch einen Fortsetzungssatz: Automorphismen auf $X \setminus D$, unter deren Rückzug $\tilde{\omega} = \omega + i\partial\overline{\partial}u$ zu sich äquivalent bleibt, sind als Automorphismen auf X fortsetzbar. Im Rahmen der Untersuchungen zur Eindeutigkeit Ricci-flacher, vollständiger Kählermetriken ist dies interessant.

Kapitel 5 widmet sich krümmungserhaltenden Deformationen holomorpher Vektorbündel. Dies könnte unabhängig von intrinsischem Interesse an Vektorbündeln im Zusammenhang mit der Deformation Ricci-flacher offener Mannigfaltigkeiten interessant werden. Es wird bewiesen, dass die Obstruktion $H^1(X, \mathcal{O}_X)/i_*H^1(X, \mathbb{R})$ ist und diese im Kählerfall verschwindet. In Beispielen von nicht-kählerschen Mannigfaltigkeit werden Bedingungen dafür gegeben, dass alle krümmungserhaltenden Deformationen eines gegebenen Vektorbündels trivial sind.

In Kapitel 6 schließlich wird Ricci-Flachheit kompakter komplexer 3-faltigkeiten genutzt, um Oktiken in $\mathbb{P}^3$ mit vielen Knoten zu konstruieren. Das maximale Beispiel hat 128 Knoten.

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Chapter 1

Introduction

The geometer's interest in Ricci-flat complex geometry arises from different points of view. First of all, given a complex manifold X, a way of retrieving valuable information is to look for "best" metrics on X. The best one can hope for is that X behaves locally in a manner as close to euclidean space as possible. This is encoded in terms of the holonomy of the Levi-Civita connection, i.e. the group of automorphisms of a fixed tangent space obtained by transporting tangent vectors parallel along a closed curve. Whereas a complex manifold has automatically a subgroup of $SO(2n)$ as holonomy group, the Kähler condition already reduces the holonomy to a subgroup of $U(n)$. The next reduction step would be $SU(n)$. In the compact case, these are exactly the Ricci-flat Kähler manifolds. The equivalent condition for the existence of a Ricci-flat Kähler metric on a compact manifold is $c_1(X) = 0$; here c_1 denotes the first Chern class of the tangent bundle.

Ricci-flatness means the vanishing of the complex Ricci tensor

$$Ric\omega = i\partial\overline{\partial}\log\det\omega$$

of the metric form ω. One has to be careful at this stage: The complex and the real (Levi-Civita-) Ricci tensor only coincide, if ω is a Kähler form.

The content of the Calabi conjecture, proved by Yau, is that a compact complex manifold with $c_1(X) = 0$ allows for a Ricci-flat Kähler metric. Moreover, any complex manifolds with $c_1(X) = 0$ allows for an unramified cover $\tilde{X}$ such that the canonical bundle $K_X = \Omega_X^n$ is trivial. Finally, any compact complex Kähler manifold with trivial canonical bundle has an unramified cover splitting into a product of a torus and a simply connected manifold with trivial canonical bundle. The latter are the so called Calabi-Yau-manifolds. First arisen as a kind of classification gap, physicists have evolved some interest in them with the rise of string theory. In turn, physicists' predictions have triggered the wide field of mirror symmetry in complex geometry. The big variety of string theories nowadays also incorporates open string models, leading to a physical interest in non-compact Calabi-Yau manifolds, i.e. non-compact manifolds allowing for a complete, Ricci-flat Kähler metric.

It is illusionary to believe, the analogue of the Calabi conjecture could be settled for arbitrary non-compact manifolds. So it is natural to focus attention first to the case of the so called open manifolds, complements of divisors in compact manifolds. The appropriate analogue for $c_1(X) = 0$ in the usual Calabi conjecture should be $K_X + D = 0$, if $X \setminus D$ is the manifold under consideration. Assuming this setting, together with the condition that D is smooth and ample, Tian and Yau [TY90] proved a first step of the 'open Calabi conjecture': They showed that there exists a complete, Ricci-flat Kähler metric on $X \setminus D$. It remains open up to this day, however, if it is also possible to find one of these in any given Kähler class; here, as outlined later in the book, the Kähler class should be understood in terms of Bott-Chern-cohomology. It also remains open, if it is possible to find such a metric, if singular or non-reduced divisors are admitted.

In chapter 2 we have a look at divisors and metrics invariant under some big symmetry group. This will give us a class of examples of complements of singular divisors with complete, Ricci-flat metrics. The Kähler condition will prove to be an extra condition on the symmetry group, as explained later in chapter 3. In this singular case, the Kähler cone proves to be rather big, so the question of complete, Ricci-flat Kähler metrics in every Kähler class is vivid and we obtain a confirming answer for the highly symmetric case in chapter 3. Contrary to this, a result of the author and Bert Koehler shows triviality of the Kähler cone can be achieved, if D is smooth and X fulfils some conditions of topological simplicity. In this case any complete, Ricci-flat Kähler metric is of the form $\omega + i\partial\overline{\partial}\varphi$ for any initial metric ω. So we look for a solution to the Monge-Ampére equation

$$\frac{(\omega + i\partial\overline{\partial}\varphi)^n}{\omega^n} = e^f$$

and $\omega + i\partial\overline{\partial}\varphi > 0$, if $f = \log \frac{\Omega\wedge\bar{\Omega}}{\omega^n}$ and Ω is a meromorphic n-form with singularities along D and no zeroes. A bounded solution to this problem has been found by Tian and Yau [TY90] for an appropriate initial metric ω. In terms of volume growth of geodesic balls there is a difference between the symmetric and the smooth case. In the symmetric case the polynomial growth rate can jump according to the jump of multiplicities of the divisor, but is always integer whereas in the smooth case it is rational non-integer. So the asymptotics of known complete, Ricci-flat Kähler metrics differ depending on whether D is smooth or not. In chapter 4 we inquire further into the asymptotics of the metric proved to exist by Tian and Yau [TY90]. The results are the joint work of the author with Bert Koehler. We prove that the initial metric and the Ricci-flat solution metric differ only by any given negative power of the radial coordinate when approaching D. This result is expected to have analytic applications also for tackling the singular case. A similar result has been obtained at the same time by Santoro [S08], independently and with a different technique. Further we prove in chapter 4 that the existence result of Tian and Yau implies a posteriori the existence of a sequence of solutions for the disturbed equation

$$\frac{(\omega + i\partial\overline{\partial}\varphi)^n}{\omega^n} = e^{f+\varepsilon\varphi}, \omega + i\partial\overline{\partial}\varphi > 0$$

with $\varepsilon \longrightarrow 0$ converging to a bounded solution of the original Monge-Ampére equation. Finally, we obtain an extendability result for automorphisms using the existence result of Tian and Yau, much in the spirit of Schumacher's extension result in the Ricci-negative case.

Chapter 5 is dedicated to the study of curvature preserving deformations of holomorphic vector bundles. Independently of the intrinsic interest in vector bundles, this could prove to be useful for the study of deformations of Ricci-flat open manifolds. The obstruction for this problem is proved to be $H^1(X, \mathcal{O}_X)/i_*H^1(X, \mathbb{R})$, trivial in the Kähler case. Examples of non-Kähler manifolds are studied and criteria given for the property of a given vector bundle to admit only trivial deformations as curvature preserving deformations-

In the last chapter we apply compact Ricci-flatness in order to construct octic hypersurfaces in $\mathbb{P}^3$ with many nodes. The maximal number of nodes constructed here is 128.

Acknowledgement. Parts of chapter 3 and wide parts of chapter 4 have been obtained in collaboration with Bert Koehler. The papers [K04], [K09], [KK06] and [KK07] establish the basis of this thesis.

I would like to express my gratitude to my mentor Prof. Hans-Christoph Grunau for advice, guidance and his openness to enter new fields; to Prof. Guofang Wang for fruitful discussions and inspiration; to Prof. Klaus Deckelnick for supporting my stay in Magdeburg; to Dr. Bert Koehler and my colleagues Dr. Edoardo Sassone and Dr. Anna Dall'Acqua for fruitful discussions. I would also like to thank my partner in life Silke Splinter for support and encouragement.

Chapter 2

Invariant Ricci-flat metrics

2.1 Introduction

In this chapter and the next we follow the idea that Ricci-flatness should be implied by big continuous symmetry groups of the metric. So we inquire into the structure of such metrics. It turns out that any G-invariant Kähler metric on the complement of a G-invariant, anticanonical divisor, is actually given by a choice of a holomorphic basis of a complex, abelian subalgebra of $\mathfrak{g}$ and the requirement that this basis is pointwise an orthonormal basis of the tangent bundle outside the vanishing set of $\det \mathfrak{g}$, provided G is big enough. A metric constructed in this way is automatically Ricci-flat.

This is motivation enough to inquire into the general structure of metrics given by defining a set of vector fields as pointwise orthonormal. Again the metric is automatically Ricci-flat with respect to the complex connection. This is the same as Ricci-flatness with respect to the Levi-Civita connection, only if the metric is Kähler. We find that completeness of the metric is deeply connected to the complex vector space spanned by the vector fields being a Lie subalgebra. In this case the divisor is invariant under the corresponding complex Lie group action G. The metric itself, however, only shows invariance under G, if G was already an abelian Lie group, and then the metric is Kähler.

Completeness of the metric will imply that the invariant divisor is not smooth as long as $c_2(X) \neq 0$. So we are interested in the asymptotics of the constructed Ricci-flat metric and find that they are quite different from what Tian and Yau [TY90] constructed for the complement of smooth, ample divisors.

2.2 Vocabulary

We consider compact complex manifolds X. Of great importance will be the automorphism group $\mathrm{Aut}(X)$ and its action on X. If $G \subset \mathrm{Aut}(X)$ is a Lie group, we write

G^0 for the connected component of G containing the identity. By $\mathfrak{g}$ we denote the Lie algebra of G. For $D \subset X$ we consider

$$\mathrm{Aut}(X, D) := \{\phi \in \mathrm{Aut}(X)|\ \phi|_D \in \mathrm{Aut}(D)\}.$$

Furthermore, if g is an hermitian metric on $X \setminus D$, we define

$$\mathrm{Aut}(X, D, g) := \{\phi \in \mathrm{Aut}(X, D)|\phi^* g = g\}.$$

If Y is some complex manifold and g a metric on Y, we also denote

$$\mathrm{Aut}(Y, g) := \{\phi \in \mathrm{Aut}(Y)|\phi^* g = g\}.$$

Note that, in general, $\mathrm{Aut}(X, D, g) \neq \mathrm{Aut}(X \setminus D, g)$. For instance, if $X = \mathbb{P}^1 \times \mathbb{P}^1$, $D = [0:1] \times \mathbb{P}^1 \cup \mathbb{P}^1 \times [0:1]$ is a $(1,1)$-divisor consisting of two intersecting fibres and g is the euclidean metric on $\mathbb{C}^2 = \mathbb{P}^1 \times \mathbb{P}^1 \setminus D$, then all automorphisms in $\mathrm{Aut}^0(\mathbb{P}^1 \times \mathbb{P}^1, D, g)$ fixing the point $([0:1],[0:1])$ are $U(1) \times U(1)$, whereas all automorphisms of $\mathrm{Aut}^0(\mathbb{C}^2, g)$ fixing 0 are $U(2)$, so the two groups must differ. In section 4.12 we will prove, however, that both groups coincide, if D is ample, anticanonical and smooth.

In most cases we will further assume that X is *almost homogeneous*. We will often make use of the following equivalences.

Definition+Lemma 2.2.1 *A compact complex manifold X is called* almost homogeneous*, if there is a Lie group $G \subset \mathrm{Aut}(X)$ such that one (and then all) of the following properties are satisfied:*

(i) The action of G has an open orbit,

(ii) the action of G^0 has an open orbit,

(iii) $\mathfrak{g} := T_1 G$ generates T_X at the general point,

(iv) there is a vector space $V \subset \mathfrak{g}$ with $\dim V = \dim X$, which generates T_X at the general point.

If $G \subset \mathrm{Aut}^0(X)$ is a Lie group which has an open orbit, then we say G *acts almost transitively* on X.

Since part of the chapter is written in the language of differential geometry, we use co- and contravariant indexing conventions as well as Einstein's sum convention. In order to be able to do this, we distinguish indices arising from non-differential context by setting them in brackets, if appropriate. The decision, whether such an index is sub- or superscript, is made by considering the beauty of involved formulas. For example, a set of vector fields is denoted by $s^{(i)}$. The same vector fields in local coordinates will be written $s^{ik}\frac{\partial}{\partial z^k}$. Here we omit the brackets, because we want to put the components into a matrix. The only unlucky point of this convention is where powers of coordinates appear. But we believe that also in these cases the meaning will become clear.

2.3 Metrics generated by vector fields

2.3.1 Construction of the metrics

In this section we discuss a method to construct complete, Ricci-flat hermitian metrics. These are neither necessarily invariant nor necessarily Kähler.

The construction of the divisor is widely used in many works about almost homogeneous manifolds. However, in this place we concentrate on the metric which comes with the construction. For this reason we give here a detailed description.

We use the notion of an abelian subspace of $H^0(T_X \otimes \mathcal{M}_X)$. We call $V \subset H^0(T_X \otimes \mathcal{M}_X)$ abelian, if for all $\zeta, \xi \in V$ holds $[\zeta, \xi] = 0$. In general, however, we do not require V to be an algebra.

Construction 2.3.1 *Let X be a projective manifold of dimension n, E an effective divisor and $\mathcal{B} = \{s^{(1)}, ..., s^{(n)}\} \subset H^0(T_X \otimes \mathcal{O}(E))$ meromorphic vector fields generating T_X in the general point and denote $V =< \mathcal{B} >$ the vector space spanned by $\mathcal{B}$. This yields a divisor $D_V \in |-K_X + nE|$ and a Ricci-flat hermitian metric $g_{\mathcal{B}}$ on $X \setminus (D_V \cup E)$.*

Implementation. Since $s^{(1)}, ..., s^{(n)} \in H^0(T_X \otimes \mathcal{O}(E))$ generate T_X in a general point, $\bigwedge_{i=1}^n s^{(i)}$ vanishes exactly on a divisor D_V. Obviously $D_V \in |-K_X + nE|$. Since on $X \setminus (D_V \cup E)$ the $s^{(i)}(x)$ form a basis of $T_{X,x}$, we may construct $s_{(i)} \in \overline{T}^*_{X,x}$ by prescribing $s_{(i)}(\overline{s^{(j)}}) = \delta_{ij}$ on $X \setminus (D_V \cup E)$. Further we can extend this correspondence to a linear map

$$^\dagger : T_X \longrightarrow \overline{T}^*_X$$

and define

$$g_{\mathcal{B}}(s \otimes t) := s^\dagger(t),$$

if $s \in T_{X,x}, t \in \overline{T}_{X,x}$. In a local chart we denote $s^{(i)} = s^{ik} \frac{\partial}{\partial z^k}$. We denote by $(s_{ij}) = \sigma$ the inverse matrix of (s^{ij}). Then

$$g_{\mathcal{B},ij} = g_{\mathcal{B}} \left(\frac{\partial}{\partial z^i} \otimes \frac{\partial}{\partial \overline{z^j}} \right) = s_{ik} \overline{s_{jk}},$$

what yields

$$Ric(g_{\mathcal{B}}) = \frac{i}{2\pi} \partial\overline{\partial} \log \det g_{\mathcal{B}} = \frac{i}{2\pi} \partial\overline{\partial} \log \det \sigma + \frac{i}{2\pi} \partial\overline{\partial} \log \overline{\det \sigma} = 0,$$

since $\det \sigma$ is holomorphic. $\square$

Theorem A and B for projective manifolds imply that the conditions of the construction can be satisfied for any projective X, if we choose E ample enough. Hence there is a Ricci-flat hermitian metric on the complement of an appropriate divisor for any projective X.

Note that by construction $T_{X\setminus(D_V\cup E)}$ is trivial. In [Wi04] the problem is addressed when $T_X(-\log D)$ is trivial and answered in terms of the existence and action of a semi-torus. In the next section we will be able to describe this property in terms of $g_{\mathcal{B}}$ and D_V.

In the remainder we will set $E = 0$. This has two reasons. The exponential map, applied to meromorphic vector fields does *not* yield automorphisms of the whole manifold, hence there is no Lie group we can work with. The other reason is, that we cannot expect completeness of the metric in direction to E. After setting $E = 0$, we can drop the condition of projectiveness of X. On the other hand, the construction then exhibits X as an almost homogeneous manifold.

2.3.2 Completeness of the metrics

Since we are dealing with open manifolds we should address the problem of completeness of the constructed metric. For this purpose we introduce some new notation. First, we define $S := (s^{ik}) = \sigma^{-1}$. Recall that we chose $E = 0$. We interpret $S^t : \mathcal{O}_X^{\oplus n} \longrightarrow T_X$ as a sheaf homomorphism and define

$$\mathcal{L} := \ker S^t.$$

The sheaf $\mathcal{L}$ is supported on D_V and we prove

Lemma 2.3.2 *If D_V is smooth, then $\mathcal{L}$ is a line bundle on D_V.*

Proof. $\mathcal{L}$ is line bundle if and only if $\mathrm{rk} S|D_V = n-1$ everywhere. So assume, that in $x \in D_V$ we have $\mathrm{rk} S(x) < n-1$. Then all $(n-1)\times(n-1)$-minors of S vanish in x, in particular $d\det S(x) = 0$, hence $x \in Sing(D_V)$. □

Now assume that D_V is smooth and consider

$$\omega_{(i)} := \bigwedge_{j\neq i} s^{(j)}|_{D_V} \in H^0(\bigwedge^{n-1} T_X|D_V) = H^0(\Omega^1_X|D_V \otimes N_{D_V|X}).$$

These are related via $\mathcal{L}$ by the equation

$$(-1)^j\lambda_{(j)}\omega_{(i)} + (-1)^i\lambda_{(i)}\omega_{(j)} = 0$$

for all i,j and $x \in D_V, \lambda \in \mathcal{L}_x$. Again, smoothness of D_V implies that in a point $x \in D_V$ not all $\omega_{(i)}$ can vanish. Hence the vector spaces $\mathcal{F}_x :=< (\omega_{(i)})_i >$ are one-dimensional and form a line bundle $\mathcal{F} \subset \Omega^1_X|D_V \otimes N_{D_V|X}$. By looking at the natural local trivialisations of $\mathcal{L}$ and $\mathcal{F}$ it is easy to see that

$$\mathcal{F} \cong \mathcal{L}^\vee.$$

Note that the inclusion $i : D_V \longrightarrow X$ yields via Poincaré Duality a homomorphism $i_* : H^*(D_V, \mathbb{R}) \longrightarrow H^*(X, \mathbb{R})$ of degree 2. With this notation in mind, the very definition of $\mathcal{F}$ implies that

$$i_* c_1(\mathcal{F}) = c_2(X).$$

The tensored dual tangent sequence

$$0 \longrightarrow \mathcal{O}_{D_V} \longrightarrow \Omega^1_X|D_V \otimes N_{D_V|X} \xrightarrow{\pi} \Omega^1_{D_V} \otimes N_{D_V|X} \longrightarrow 0,$$

allows us to formulate the property $\mathcal{F} = \mathcal{O}_{D_V} = \ker \pi$.

Lemma 2.3.3 *If D_V is smooth, then $g_{\mathcal{B}}$ is complete if and only if $\mathcal{F} = \ker \pi$.*

Proof. Let us choose $x \in D_V$ and local coordinates in a small open subset $U \subset X$ such that $D_V = \{z^1 = 0\}$. Furthermore, denote $U' := U \cap D_V$ and $pr : U \longrightarrow U'$ the projection induced by the local coordinates. choose $0 \neq \lambda \in \mathcal{L}(U')$ and an order of $\mathcal{B}$ such that $\lambda_{(1)} \equiv 1$. If we now define B by

$$B_{ij}(z) := \begin{cases} 1 & \text{if } i = j \\ \lambda_{(j)}(pr(z)) & \text{if } i = 1, j \neq 1 \\ 0 & \text{else} \end{cases},$$

then $\tilde{S} := BS$ is just S replaced by a first row vanishing on U'. Now we have a look at $\tilde{\sigma} := (\tilde{S})^{-1}$. Since the first row is identically 0 on U', we conclude that $\tilde{s}_{1i} \in \mathcal{O}(U)$ for $i > 1$. Since

$$g_{11} = \sum_{i>1} |\tilde{s}_{1i}|^2 + 2\mathrm{Re}(\overline{\tilde{s}_{11}} \sum_{i>1} \lambda_{(i)} \tilde{s}_{1i}) + |\tilde{s}_{11}|^2 \sum |\lambda_{(j)}|^2,$$

we see now, that g is complete if and only if $\tilde{s}_{11} \sim \frac{1}{z^1}$ for all such choices of coordinates. The choice of a path to the boundary is reflected by the choice of coordinates. The length of any path to the boundary becomes infinite, if and only if the direct path is infinitely long in any such coordinates.

Indeed, $\tilde{s}_{11} = s_{11}$. Hence, if we denote A_{ij} the (i, j)-entry of the cofactor matrix of S, then the condition is equivalent to $A_{11}(x) \neq 0$. If we now choose other coordinates $z'^1, ..., z'^n$ such that $D_V = \{z'^1 = 0\}$ and denote $J := (\frac{\partial z'^j}{\partial z^i})_{ij}$, $h := \frac{z'^1}{z^1} \neq 0$, then

$$A'_{11} = \det J (h^{-1} A_{11} + \sum_{k>1} J^{-1}_{1k} A_{1k}) \neq 0$$

Since the coordinate transform was arbitrary, we conclude that

$$A_{1k}(x) = 0 \text{ for all } k > 1.$$

This is equivalent to $0 \neq \omega_{(1)} \in H^0(\ker \pi)$. So $\mathcal{F} = \ker \pi$ is equivalent to the completeness of $g_{\mathcal{B}}$. □

Note that completeness of $g_{\mathcal{B}}$ depends only on V.

If D_V is not smooth, but still reduced, then we consider D_V^0, the regular part of D_V and the corresponding objects $\mathcal{F}^0, \mathcal{L}^0, \pi^0$, which are obtained by restriction to D_V^0. We now show the preceding Lemma for the singular case.

Lemma 2.3.4 *Assume that D_V is reduced. Then $g_{\mathcal{B}}$ is complete, if and only if $\mathcal{F}^0 = \ker \pi^0$.*

Proof. '⇒': If $g_{\mathcal{B}}$ is complete, the same arguments as in Lemma 2.3.3 imply that $\mathcal{F}^0 = \ker \pi^0$.

'⇐': Let locally $D_V = \{f = 0\}$ in a small open neighbourhood $U \subset X$. Then we can choose functions $z^2, ...z^n$ which give local coordinates together with f on the set $\tilde{U} := U \setminus \{df \wedge dz^2 \wedge ... \wedge dz^n = 0\}$. Like in the proof above we argue that $g_{\mathcal{B}}$ is complete if $A_{11}(x) \neq 0$ for choices like above and $x \in D_V$. Since by assumption this is true for $x \in D_V \setminus Sing(D_V)$, extension of the holomorphic function A_{11} to U' yields a non-zero function $A_{11} \in \mathcal{O}^*(U')$, if D_V was normal. If D_V is not normal, we choose an embedded normalisation

$$\begin{tikzcd} \tilde{D}_V \arrow[r, "\nu"] \arrow[d, "j"'] & D_V \arrow[d, "i"] \\ \tilde{X} \arrow[r, "\mu"'] & X \end{tikzcd},$$

where i, j denote inclusions and ν the normalisation of D_V. Now we apply the same arguments to the pseudometric $\mu^* g$ and obtain by looking at paths γ such that $\mu|_\gamma$ is a diffeomorphism that g is complete. □

Now we can see a connection to the invariance group.

Lemma 2.3.5 *Assume that D_V is reduced. The metric $g_{\mathcal{B}}$ is complete, if and only if $\exp(V) \subset \mathrm{Aut}^0(X, D_V)$.*

Proof. If we denote by $\mathcal{V}$ the sheaf on D generated by $\mathcal{V}_x :=< \{s^{(i)}(x)\} >$ for $x \in D$, then $\mathcal{F}(\mathcal{V}) \equiv 0$, if we regard $\mathcal{F} \subset Hom(T_X|_{D_V}, N_{D_V|X})$. Since $\mathcal{F}^0 = \ker \pi^0$, figuring out the dualised maps

$$0 \longrightarrow T_{D_V^0} \longrightarrow T_X|D_V^0 \xrightarrow{p^0} \mathcal{F}^{0\vee} \otimes N_{D_V^0|X} \longrightarrow 0$$

yields $\mathcal{V}^0 = \ker p^0 = T_{D_V^0}$. This is equivalent to $V|_{D_V^0} \subset H^0(T_{D_V^0})$, i.e. every $\phi \in \exp(V)$ holds D_V invariant, so $\exp(V) \subset \mathrm{Aut}^0(X, D_V)$ is an equivalent condition. □

Taking into account the results of Section 2.4, which are obtained independently of the considerations about completeness, we even find

Theorem 2.3.6 *If V is a Lie subalgebra, then $g_{\mathcal{B}}$ is complete. If $g_{\mathcal{B}}$ is complete and D_V is reduced, then V is a Lie subalgebra.*

Proof. (i) If V is a Lie subalgebra, Lemma 2.4.1 implies the desired property $\exp(V) \subset \mathrm{Aut}^0(X, D_V)$. Here we are finished, if D_V is reduced. In any case, if we have $D_{V,red} \cap U = \{z^1 = 0\}$, this implies that

$$s^{i1} = z^1 t^{i1}$$

for some $t^{i1} \in \mathcal{O}(U)$ and all i. Now

$$\det S = \sum_i s^{i1} A_{1i} = z^1 \sum_i t^{i1} A_{1i}.$$

So, if $m := \max\{k| \ (z^1)^{-k} A_{1i} \in \mathcal{O}(U) \text{ for all } i\}$, we conclude $(z^1)^{-m-1} \det S \in \mathcal{O}(U)$. Hence

$$g_{11} = \sum |s_{1i}|^2 = (\det S)^{-2} \sum |A_{1i}|^2 \sim |z^1|^{-k}$$

with $k \geq 2$. This procedure generalises easily to the case of $D_{V,red}$ being normal crossings. But this we can achieve by Hironaka's embedded desingularisation. Since $Sing(D_{V,red})$ is also $\exp(V)$-invariant and so is $Sing(Sing(D_{V,red})_{red})$ and so on, we can pull back the vector fields to the normal crossings case and apply the arguments above.

(ii) Since $g_{\mathcal{B}}$ is complete and D_V reduced, by Lemma 2.3.5 we obtain $\exp(V) \subset \mathrm{Aut}^0(X, D_V)$. Hence $\mathrm{Aut}^0(X, D_V)$ acts almost transitively. So Lemma 2.4.2 yields $\dim \exp(V) = \dim X = \dim \mathrm{Aut}^0(X, D_V)$. This means

$$T_1 \mathrm{Aut}^0(X, D_V) = T_1 \exp(V) = V,$$

hence V is a Lie subalgebra. □

Now it is clear from the previous arguments that $T_X(-\log D_V)$ is trivial, if $g_{\mathcal{B}}$ is complete and D_V is a simple normal crossings divisor. In Chapter 3 we will show furthermore that $G := \exp(V)$ is a semi-torus, if $g_{\mathcal{B}}$ is complete and Kähler and D_V is reduced.

The existence of a complete $g_{\mathcal{B}}$ for a smooth D_V restricts the geometry of X significantly:

Corollary 2.3.7 *If D_V is smooth and $g_{\mathcal{B}}$ is complete, then $c_2(X) = 0$.*

Proof. $\mathcal{F} = \ker \pi \cong \mathcal{O}_{D_V}$, hence $c_2(X) = i_* c_1(\mathcal{O}_{D_V}) = 0$. □

Now we have seen that $c_2(X) \neq 0$ implies that the divisor D_V is singular, if V is a Lie subalgebra. We will see later that the Kähler condition allows a more explicit description of the singularities, at least on projective homogeneous manifolds.

Remark 2.3.8 Locally around smooth points of a reduced D_V, in coordinates such that $D \cap U = \{z_1 = 0\}$, calculations very similar to those in the proof of Lemma 2.3.3 show that the metric $g_{\mathcal{B}}$ has the form

$$g_{\mathcal{B}} = a_{1\bar{1}}(z)\frac{dz_1 \otimes d\bar{z}_1}{|z_1|^2} + \sum_{j=2}^{n}(a_{1\bar{j}}(z)\frac{dz_1}{z_1} \otimes d\bar{z}_2 + c.c.) + \sum_{i,j=2}^{n} a_{i\bar{j}}(z)dz_i \otimes d\bar{z}_j$$

for functions $a_{i\bar{j}} \in C^\infty(U)$. So, if $U = B_1(0) \times U'$, under the map

$$\psi : (0, \infty) \times (0, 2\pi) \times U' \longrightarrow U, (R, \varphi, z_2, \dots, z_n) \mapsto (e^{-R+i\varphi}, z_2, \dots, z_n)$$

the metric $g_{\mathcal{B}}$ pulls back to a metric $\psi^* g_{\mathcal{B}}$ equivalent to the euclidean metric. Later we will see that the metric of Tian and Yau constructed on $X \setminus D$ for smooth D shows quite different asymptotics.

2.4 Symmetries of the divisor and the metric

In this section we want to relate the construction above (with $E = 0$) to the appearance of symmetries on D and the metric. As it may be not hard to guess, this connection is made by Lie theory.

If G is a complex Lie group we identify $\mathfrak{g} = T_1 G$, and if $G \subset \mathrm{Aut}^0(X)$, then we furthermore identify $\mathfrak{g}$ with the complex subvector space of $H^0(T_X)$ given by the vector fields $s(x) := \frac{\partial}{\partial t} g(t)x|_{t=0}$, where $g(t)$ denotes a holomorphic path in G with $g(0) = 1$ and $\frac{\partial}{\partial t} g(t)|_{t=0} = \xi \in T_1 G$. Furthermore we have an action of G on $T_1 G$ by $h\xi := \frac{\partial}{\partial t} h g(t) h^{-1}|_{t=0}$, if $h \in G$.

Lemma 2.4.1 *Let X be a compact complex manifold of dimension n, $G \subset \mathrm{Aut}^0(X)$ a connected complex Lie group acting almost transitively on X and $\mathfrak{g}$ be the corresponding complex Lie algebra. Then*

(i) $D \in |-K_X|$ is reduced and $G \subset \mathrm{Aut}^0(X, D) \Rightarrow D = D_V$ for all $V \subset \mathfrak{g}$ with $\dim V = n$ and generating T_X in the general point,

(ii) if $\dim G = n$, then $G \subset \mathrm{Aut}^0(X, D_{\mathfrak{g}})$,

(iii) if $\dim G = n$, $\mathcal{B} \subset \mathfrak{g}$ is a basis, then: $G \subset \mathrm{Aut}^0(X, D_{\mathfrak{g}}, g_{\mathcal{B}}) \iff G$ is abelian.

Proof. '(i)': Let $V \subset \mathfrak{g}$ be an n-dimensional vector space generating T_X in the general point and $\mathcal{B} \subset V$ a basis. Since D is G-invariant, for any $s \in \mathfrak{g} \subset H^0(T_X)$ the restriction $s|_{D^\circ}$ gives an element of $H^0(T_D^\circ)$. Since $\dim D = n - 1$, this implies $\bigwedge_{s \in \mathcal{B}} s|_D = 0$, hence $D \subset D_V$. But D and D_V are both elements of $|-K_X|$, hence $D = D_V$.

'(ii)': If $s \in \mathfrak{g} \subset H^0(T_X)$ is given by $\xi \in T_1G$, then for $h \in G$ the pullback h^*s is given by $h^{-1}\xi \in T_1G$, hence $h^*s \in \mathfrak{g}$. Furthermore h^* maps a basis of $\mathfrak{g}$ to a basis again, because h is an automorphism. This proves that $\bigwedge s^{(i)} = 0 \iff \bigwedge h^*s^{(i)} = 0$, if $s^{(1)}, ..., s^{(n)}$ is a basis of $\mathfrak{g}$. Hence $h(D) = D$. This proves that D is G-invariant.

'(iii)': Let ω be the fundamental $(1,1)$-form of $g_{\mathcal{B}}$. The metric $g_{\mathcal{B}}$ is G-invariant if and only if $\mathcal{L}_s\omega = 0$, if $s \in \mathfrak{g}$ and $\mathcal{L}$ denotes the Lie derivative. If C_s denotes the contraction by s, then $\mathcal{L}_s = dC_s + C_sd$ (see e.g. [La, V,5]). Let us choose local coordinates like in the construction. Then

$$0 = \mathcal{L}_{s^{(l)}}\omega = \sum_{i,j,k,m} s^{lm}(s_{ik,m} - s_{mk,i})\overline{s}_{jk} dz^i \wedge d\overline{z}^j.$$

Since S is invertible on $X \setminus D$, we obtain

$$s_{ik,m} - s_{mk,i} = 0$$

for all i, k, m. Converting this condition to the vector field components yields

$$s^{ij}s^{kl}_{,j} = s^{kj}s^{il}_{,j}$$

This is the condition

$$[s^{(i)}, s^{(j)}] = 0$$

for all i, j. Hence $\mathfrak{g}$ is abelian. It is well known that this is equivalent to G to be abelian. $\square$

As a first application, we obtain kind of uniqueness of the vector space V in the construction.

Lemma 2.4.2 *Let X be a compact complex manifold and $D \in |-K_X|$ reduced. If* $\mathrm{Aut}(X, D)$ *acts almost transitively, then* $\dim \mathrm{Aut}(X, D) = \dim X$.

Proof. We abbreviate $G := \mathrm{Aut}(X, D)$ and $n := \dim X$. First of all, $\dim G \geq n$, since otherwise T_1G could not generate T_X in any point. Now choose

$$\mathcal{B} := \{s^{(1)}, ..., s^{(n+1)}\} \subset \mathfrak{g}$$

such that $\mathcal{B} \setminus \{s^{(n+1)}\}$ generates T_X in the general point. Further denote $\eta_{(i)} := \bigwedge_{j \neq i} s^{(j)}$. and by $V_{(i)}$ the vector space generated by $\mathcal{B} \setminus \{s^{(i)}\}$.

Since $\dim X = n$, we can find meromorphic functions $f_{(i)} \in \mathcal{M}_X(X)$ such that $s^{(n+1)} = \sum_{i=1}^n f_{(i)}s^{(i)}$. In order to prove this, we first remark that in the general point the $f_{(i)}$ are uniquely determined and there we compute $f_{(i)} = s^{n+1,k}s_{ki}$, if $s^{(n+1)} = s^{n+1,k}\frac{\partial}{\partial z^k}$. By construction, the s_{ki} are meromorphic, so $f_{(i)}$ are also meromorphic.

By assumption we know $\eta_{(n+1)} \not\equiv 0$.

If $\eta_{(i)} \equiv 0$, then we see by $\eta_{(i)} = f_{(i)}\eta_{(n+1)}$, that $f_{(i)} \equiv 0$.

If $\eta_{(i)} \not\equiv 0$, then $V_{(i)}$ generates T_X in the general point and we may use Lemma 2.4.1 to obtain

$$\eta_{(n+1)} = 0 \iff \eta_{(i)} = 0 \iff f_{(i)}\eta_{(n+1)} = 0.$$

Hence $f_{(i)}$ has no zeroes. By exchanging $s^{(i)}$ and $s^{(n+1)}$ we also see that $f_{(i)}$ has no poles. Hence $f_{(i)}$ is constant.

Now we proved that every f_i is constant, hence $s^{(1)}, ..., s^{(n+1)}$ are linearly dependent and $\dim G = n$. □

Note that the connection between the invariance group and the anticanonical system is essential. For example, the invariance group of a point in $\mathbf{P}^1$ is two-dimensional and acts almost transitively.

Since every $\dim X$-dimensional Lie group which acts almost transitively yields an invariant $D \in |-K_X|$, Lemma 2.4.2 suggests the following definition.

Definition 2.4.3 *If X is compact complex manifold and $G \subset \mathrm{Aut}(X)$ a connected complex Lie group with $\dim G = \dim X$ acting almost transitively on X, we say G is a* divisorial *group. If on the other hand $D \in |-K_X|$ is reduced, we say D has a* divisorial invariance group, *if $\mathrm{Aut}(X, D)$ acts almost transitively. Any object invariant under a divisorial group we call* divisorially invariant.

Lemma 2.4.2 now allows a stronger and more compact formulation of Lemma 2.4.1.

Theorem 2.4.4 *Let X be a compact complex manifold, G a divisorial group and $\mathfrak{g}$ the corresponding Lie algebra. Then for a reduced divisor $D \in |-K_X|$ holds*

$$G = \mathrm{Aut}^0(X, D) \iff D = D_{\mathfrak{g}}.$$

If $\mathcal{B} \subset \mathfrak{g}$ is a basis, then

$$G = \mathrm{Aut}^0(X, D_{\mathfrak{g}}, g_{\mathcal{B}}) \iff G \textit{ is abelian.}$$

Proof. If $D = D_{\mathfrak{g}}$, by Lemma 2.4.1 $G \subset \mathrm{Aut}^0(X, D)$. Hence $\mathrm{Aut}^0(X, D)$ acts almost transitively and by Lemma 2.4.2 we obtain $\dim G = \dim X = \dim \mathrm{Aut}^0(X, D)$, hence $G = \mathrm{Aut}^0(X, D)$. □

In this context it is appropriate to introduce the notion of a *homogeneous pair*.

Definition 2.4.5 *A* homogeneous pair *(X, D) consists of a compact complex manifold X and an effective reduced divisor D such that $\mathrm{Aut}^0(X, D)$ acts transitively on $X \setminus D$. We call a homogeneous pair (X, D) anticanonical, if $D \in |-K_X|$.*

Theorem 2.4.4 provides a close relation between homogeneous pairs and divisorial groups.

Remark 2.4.6 (i) Note that the proof of Lemma 2.4.1 also shows that every analytical G-invariant set S is contained in the G-invariant $D \in |-K_X|$.

(ii) Note that the vector field method is much more general than the invariance approach: There is no need for the vector space $V \subset H^0(T_X)$ to be an algebra, whereas invariant divisors correspond to Lie subalgebras of $H^0(T_X)$.

(iii) However, if g arises by the general vector field method and is Kähler, we have V proved to be abelian, in particular V is a Lie subalgebra. The exponential map $\exp : H^0(T_X) \longrightarrow \mathrm{Aut}^0(X)$ restricted to V maps to an n-dimensional Lie subgroup G leaving D and g invariant.

Now we also see that divisorial invariance is exactly the property we had in mind when we expected that Ricci-flatness should be implied by a high order of symmetry.

Corollary 2.4.7 *Let X be an n-dimensional compact complex manifold, $G \subset \mathrm{Aut}^0(X)$ a divisorial (abelian) Lie group. Then there is a complete Ricci-flat (Kählerian G-invariant) metric on $X \setminus D_{\mathfrak{g}}$.*

In [Wi04] Winkelmann proved that $T_X(-\log D)$ is even holomorphically trivial, if G is a complex semi-torus and acts with only semi-tori as isotropy groups. We will see in later that G being a semi-torus is implied by $D_{\mathfrak{g}}$ being reduced.

Chapter 3

The Kähler cone of open manifolds

3.1 Introduction

On the quest for complete Ricci-flat Kähler metrics the method incorporates finding the flat metric differing from a given (asymptotically flat) one by a Kähler potential $\partial\overline{\partial}\varphi$. In the compact case this ansatz was justified by the $\partial\overline{\partial}$-Lemma, since finding such a potential in the case $c_1(X) = 0$ is equivalent to determining a Ricci-flat representative of each Kähler cohomology class. If X is not compact, however, such a $\partial\overline{\partial}$-Lemma is not valid, in general. Below we give a proof for the $\partial\overline{\partial}$-Lemma for Stein manifolds. But even if we have this tool, the question when we can find a Ricci-flat representative in each Kähler class is widely open. In order to formulate the problem independent of the existence of a $\partial\overline{\partial}$-Lemma it is appropriate to consider the cone of Kähler classes in the framework of Bott-Chern cohomology.

Focussing the attention on open manifolds, i.e. the complement of a divisor of a compact complex manifold, we first prove that topological conditions on X may lead to a triviality of the Kähler cone, if D is smooth. This applies for $X = \mathbb{P}^n, n \geq 3$ and $D \in |-K_X|$ smooth, a case handled in the framework of chapter 4; so any Ricci-flat, complete Kähler metric arises in this case as a solution of the complex Monge-Ampére equation of chapter 4, not necessarily bounded, of course.

On the other hand, for certain reducible D coming from a highly symmetric situation, we prove that the Kähler cone has a rather big dimension, even if X is topologically simple. So smooth and singular divisors D are not only distinguished by the asymptotics of the Ricci-flat metrics constructed as yet, but also by the universality of the ansatz using the complex Monge-Ampére equation.

For the first part of this chapter we need to construct an angular differential on a tubular neighbourhood of D. This construction will also be widely used in chapter 4.

3.2 The angular differential on tubular neighbourhoods

Let η be an arbitrary metric on the compact manifold X and $D \subset X$ a smooth hypersurface. η induces a C^∞-splitting

$$T_X|D \cong T_D \oplus N_{D|X}$$

of holomorphic vector bundles and, moreover, a S^{2n-1}-subbundle $N^1_{D|X}$ of unit length normal vectors. The restricted exponential map (with respect to the Riemannian metric induced by η)

$$\exp : N^1_{D|X} \times [0, \varepsilon) \longrightarrow X, \; (v, x, t) \mapsto \exp_x(tv)$$

may be assumed to be an immersion outside $t = 0$ and then has an open set $U(D)$ as image. Thus

$$U(D) \setminus D \cong N^1_{D|X} \times (0, \varepsilon)$$

as real manifolds. We abbreviate $Y := N^1_{D|X}$ and see that Y is a principal S^1-bundle over D via $v \mapsto \exp(i\alpha)v$. Let

$$\pi : Y \longrightarrow D$$

denote the bundle structure and let $D = \bigcup_{i=1}^{l} U_i$ be a covering of trivialising charts $\pi^{-1}(U_i) \cong S^1 \times U_i$. The transition maps

$$\psi_{jk} : U_{jk} \times S^1 \longrightarrow U_{jk} \times S^1, (x_j^1, \ldots, x_j^{2n-2}, \exp(i\varphi_j)) \mapsto (x_k^1, \ldots, x_k^{2n-2}, \exp(i\varphi_k))$$

have to respect the S^1-action, hence

$$\psi_{jk}(x_j, \exp(i\varphi_j + i\alpha)) = (x_k, \exp(i\varphi_k + i\alpha)).$$

This implies immediately

$$\exp(i\varphi_k) = \beta_{jk}(x_j) \cdot \exp(i\varphi_j)$$

with

$$\beta_{jk} : U_{jk} \longrightarrow S^1.$$

So, $-id \log \beta_{jk}$ are well defined real one-forms on U_{jk} yielding a 1-cocycle of $\mathcal{A}^1_D$. Since for the real sheaves $\mathcal{A}^p_D$ the higher cohomology vanishes, in particular, $H^1(\mathcal{A}^1_D) = 0$, we can find one-forms $\gamma_i \in \mathcal{A}^1(U_i)$ such that

$$\gamma_j - \gamma_k = -id \log \beta_{jk}.$$

On the other hand, also

$$id\varphi_k - id\varphi_j = \pi^* d \log \beta_{jk}$$

and thus

$$\delta := d\varphi_j + \pi^*\gamma_j$$

is independent of j. So it is a global real one-form on Y, called the *angular differential* (although it is not unique but depends on a choice of an element in $\mathcal{A}^1(D)$).

After a choice of δ we have a decomposition

$$\mathcal{A}^1(Y) = \mathcal{A}^0(Y) \cdot \delta \oplus \Gamma(\pi^*\mathcal{A}^1_D),$$

where $\Gamma(\pi^*\mathcal{A}^1_D)$ denotes the $C^\infty(Y)$ sections of the pulled back sheaf. We denote $\mathrm{pr}_1, \mathrm{pr}_2$ the projections onto the corresponding factors.

We refer to [GHV] for the existence of a fibre-wise integration

$$\int : \mathcal{A}^p(Y) \longrightarrow \mathcal{A}^{p-1}(D)$$

commuting with the differentials. We easily see

$$\int \eta \wedge \delta = pr_2\eta.$$

Finally, note that $d\delta = \pi^*\alpha$ for some $\alpha \in \mathcal{A}^2(D)$.

3.3 A $\partial\overline{\partial}$-Lemma for Stein manifolds

In our context we are interested in variations of Kähler metrics via Kähler potentials. So we introduce a modification of the definition of the Kähler cone to suit our needs. If Z is any complex manifold and $G \subset \mathrm{Aut}(Z)$ is a Lie group, we denote by $M_G(Z)$ the set of all Kähler metrics on Z invariant under G.

Definition 3.3.1 *For closed $(1,1)$-forms ω, ω' on a complex manifold Z*

$$\omega \sim \omega' :\Longleftrightarrow \exists \phi \in C^\infty(Z, \mathbb{R}) : \omega - \omega' = i\partial\overline{\partial}\phi$$

is an equivalence relation. The quotient $H^{1,1}_{BC}(Z)$ is an object of Bott-Chern cohomology. If $G \subset \mathrm{Aut}(Z)$, we define

$$K_G^{BC}(Z) := M_G(Z)/\sim \subset H^{1,1}_{BC}(Z)$$

and call it the G-Bott-Chern-Kähler cone *of $X \setminus D$. We also abbreviate $K^{BC}(Z) := K_1^{BC}(Z)$ and call this cone the* Bott-Chern-Kähler cone *of Z.*

We recall that the G-Kähler cone $K_G(Z)$ is the image of $M_G(Z)$ in $H^{1,1}(X)$ and $K(Z) := K_1(Z)$.

If X is a compact complex manifold, $D \subset X$ a divisor, $Z = X \setminus D$ and $G \subset \mathrm{Aut}(X, D)$, we also write $K_G^{BC}(X, D) := K_G^{BC}(Z)$ and $K_G(X, D) := K_G(Z)$.

By definition there is a natural surjective map

$$K_G^{BC}(Z) \longrightarrow K_G(Z).$$

In general the Bott-Chern-Kähler cone will not be the usual Kähler cone as a subcone of $H^{1,1}(Z)$, but we know already that, due to the $\partial\overline{\partial}$-Lemma, for compact manifolds they coincide. We are now going to prove the $\partial\overline{\partial}$-Lemma for Stein manifolds.

Proposition 3.3.2 *A $(1,1)$-form η on a Stein manifold Z is cohomologous to zero if and only if there is a function $F \in C^\infty(Z, \mathbb{C})$ such that $\eta = i\partial\overline{\partial}F$. In particular, $K_G^{BC}(Z) = K_G(Z)$ for any $G \subset \mathrm{Aut}(Z)$.*

Proof. The injective resolution of $\mathbb{C}$

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}_Z \xrightarrow{\partial} \Omega_Z^1 \xrightarrow{\partial} \Omega_Z^2 \xrightarrow{\partial} \ldots$$

yields short exact sequences

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O}_Z \xrightarrow{\partial} \mathcal{H}_Z^1 \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{H}_Z^1 \longrightarrow \Omega_Z^1 \longrightarrow \mathcal{H}_Z^2 \longrightarrow 0.$$

In cohomology we obtain

$$H^1(\mathcal{O}_Z) \longrightarrow H^1(\mathcal{H}_Z^1) \longrightarrow H^2(Z, \mathbb{C}) \longrightarrow H^2(\mathcal{O}_Z).$$

Since Z is Stein we obtain $H^1(\mathcal{O}_Z) = H^2(\mathcal{O}_Z) = 0$, hence

$$H^1(\mathcal{H}_Z^1) = H^2(Z, \mathbb{C}).$$

The second short exact sequence yields

$$H^0(\Omega_Z^1) \longrightarrow H^0(\mathcal{H}_Z^2) \longrightarrow H^1(\mathcal{H}_Z^1) = H^2(Z, \mathbb{C}),$$

where the composition

$$H^0(\mathcal{H}_Z^2) \longrightarrow H^2(Z, \mathbb{C})$$

is simply the projection of the 2-form onto its de Rham class. So for every holomorphic 2-form η on Z with $\partial\eta = 0$ and $[\eta] = 0 \in H^2(X, \mathbb{C})$ there is a holomorphic 1-form φ such that $\eta = \partial\varphi$.

Now let $\omega = d\alpha$ be a $(1,1)$ form. Since Ω_Z^1 is coherent, again Theorem B implies $H^1(\Omega_Z^1) = 0$. Using the Dolbeault interpretation we obtain $\eta \in \mathcal{E}^{1,0}(Z)$ such that $\omega = \overline{\partial}\eta$.

Now look at $\psi := \partial\eta$. Since $0 = \partial\omega = -\overline{\partial}\psi$, we conclude that $\psi \in H^0(\Omega_Z^2)$. The form satisfies $\partial\psi = 0$ and $\psi = d(\eta - \alpha)$. So there is $\varphi \in H^0(\Omega_Z^1)$ such that $\psi = \partial\varphi$. This implies $\partial(\eta - \varphi) = 0$, hence $\overline{\eta} - \overline{\varphi}$ induces a class in $H^{0,1}(Z) = H^1(\mathcal{O}_Z) = 0$. Hence we obtain a function $F : Z \longrightarrow \mathbb{C}$ such that $\overline{\partial}F = \overline{\eta} - \overline{\varphi}$, i.e.

$$\partial\overline{F} = \eta - \varphi.$$

For the $(1,1)$ form this means

$$\omega = \overline{\partial}\eta = \overline{\partial}(\partial\overline{F} + \varphi) = \partial\overline{\partial}(-\overline{F}).$$

If $\omega = \overline{\omega}$ we find

$$\omega = i\partial\overline{\partial}\mathrm{Im}(F)$$

proving the last part. □

3.4 Triviality of the Kähler cone for topologically simple manifolds

In this section we want to demonstrate that the triviality of the Kähler cone of a complement of a *smooth*, ample divisor is implied by a topological simplicity of the compact manifold. This situation changes dramatically once singular divisors are taken into consideration. If X is compact and $D \subset X$ an ample hypersurface, then $X \setminus D$ is Stein. So by Proposition 3.3.2 it remains to find conditions for $b_2(X \setminus D) = 0$.

In a first step we prove that the first Betti number of the complement of a smooth ample divisor vanishes if it did so for the compact manifold. This is the best we can hope for since there are many examples for simply connected compact manifolds with $X \setminus D$ not being simply connected but having infinite fundamental groups.

For the remainder of this section we use $Y, U(D)$ and δ as constructed in Section 3.2.

Lemma 3.4.1 *If X is a projective complex manifold with $b_1(X) = 0$ and $D \subset X$ is a connected, smooth and ample divisor, then*

$$\pi^* : H^1(D, \mathbb{C}) \longrightarrow H^1(Y, \mathbb{C})$$

is an isomorphism. In particular, $b_1(X \setminus D) = 0$.

Proof. Since $U(D) \setminus D$ has Y as a deformation retract, a part of the Mayer-Vietoris sequence reads

$$H^1(X, \mathbb{C}) \longrightarrow H^1(X \setminus D, \mathbb{C}) \oplus H^1(D, \mathbb{C}) \longrightarrow H^1(Y, \mathbb{C}).$$

So $b_1(X\backslash D) = 0$, if $\pi^* : H^1(D, \mathbb{C}) \longrightarrow H^1(Y, \mathbb{C})$ is surjective (and then an isomorphism, indeed).

Let $\zeta \in \mathcal{A}^1(Y)$ be a closed one-form. The closedness of ζ implies that integration along the fibre E_x over $x \in D$

$$c := \int_{E_x} \eta$$

is a constant independent of x.

We prove by contradiction that $c = 0$. Let $c \neq 0$. The local one-forms

$$\beta_i := \eta - \frac{c}{2\pi} d\phi_i$$

satisfy $d\beta_i = 0$ and $\int \beta_i = 0$. So there are functions $g_i \in C^\infty(\pi^{-1}(U_i))$ such that $\beta_i = dg_i$. Averaging the function

$$f_i := \int g_i \delta \in C^\infty(U_i)$$

we obtain

$$df_i = \int d(g_i\delta) = \int dg_i \wedge \delta + g_i d\delta = pr_2(\beta_i) = \beta_i - pr_1(\beta_i).$$

By construction $\gamma := \mathrm{pr}_1(\beta_i)$ is independent of i and so

$$df_i + \frac{c}{2\pi} d\phi_i = \eta - \gamma$$

is independent of i. Now we interpret $f_i : D \longrightarrow \mathbb{R}/c\mathbb{Z} \cong S^1$ as a circle valued function.

The universal coefficient lemma tells us that $H^1(D, S^1) = H_1(D, \mathbb{Z})$ (not canonically), since by the Lefschetz theorem there is no free part of $H_1(D, \mathbb{Z})$. This is a finite group, so there exists some m such that $mg = 0$ for all $g \in H^1(D, S^1)$.

So there are $c_j \in S^1$ such that

$$m(\frac{c}{2\pi}\phi_j + f_j(z)) + c_j$$

represents a global function $\Phi : Y \longrightarrow S^1$. The level sets of Φ induce nowhere vanishing C^∞ sections of $N_{D|X}^{\otimes m}$, so $N_{D|X}^{\otimes m} \cong \mathcal{O}_D$ differentiably. This clearly contradicts the assumption that D is ample. So $c = 0$.

Since $\int_{E_x} : H^1(E_x, \mathbb{R}) \longrightarrow \mathbb{R}$ is an isomorphism the property $c = 0$ implies that for every x there is a function $h_x \in C^\infty(E_x)$ such that $i_{E_x}^* \eta = dh_x$. The dependence on x being able to be chosen C^∞ locally we obtain functions $h_i \in C^\infty(\pi^{-1}(U_i))$ and one-forms $\xi_i \in \Gamma(\pi^* \mathcal{A}^1(\pi^{-1}(U_i)))$ on a cover U_i of D such that

$$\eta|\pi^{-1}(U_i) = dh_i + \xi_i.$$

The decomposition implies $pr_1(dh_i) = \mathrm{pr}_1(dh_j)$ on $\pi^{-1}(U_{ij})$, hence the derivatives of h_i and h_j with respect to ϕ_i coincide. So there are $c_{ij} \in C^\infty(U_{ij})$ with

$$h_i - h_j = \pi^* c_{ij}.$$

Now we apply $H^1(D, \mathcal{A}^0) = 0$ and obtain $c_i \in C^\infty(U_i)$ such that $h := h_i - \pi^* c_i = h_j - \pi^* c_j$ gives a well defined function $h \in C^\infty(Y)$. This again implies

$$\eta = dh + \xi$$

for appropriately chosen $\xi \in \Gamma(\pi^* \mathcal{A}^1_D)$. Using $d\eta = 0$ we see $d\xi = 0$. Thus ξ is ϕ_i-independent, so $\xi = \pi^* \psi$ for some $\psi \in \mathcal{A}^1(D)$ with $d\psi = 0$. In particular, the de Rham classes of η and $\pi^*\psi$ coincide, proving that $\pi^* : H^1(D, \mathbb{C}) \longrightarrow H^1(Y, \mathbb{C})$ is surjective. In fact, it is an isomorphism.

□

Lemma 3.4.2 *If X is a projective manifold with $\dim X \geq 3, b_1(X) = b_3(X) = 0, b_2(X) = 1$ and $D \subset X$ a smooth, ample hypersurface, then $b_2(X \setminus D) = 0$.*

Proof. By the Lefschetz theorem we know that D is connected and $b_1(D) = 0$. So we can apply Lemma 3.4.1 and obtain $b_1(Y) = 0$. Now another part of the Mayer-Vietoris sequence reads

$$0 \longrightarrow H^2(X, \mathbb{C}) \longrightarrow H^2(\tilde{X}, \mathbb{C}) \oplus H^2(D, \mathbb{C}) \longrightarrow H^2(E, \mathbb{C}) \longrightarrow H^3(X, \mathbb{C}).$$

In order to compute the cohomology of E we use the Leray spectral sequence. For a circle bundle, according to [S, 9.5, Thm2] this simplifies in our case to

$$0 \longrightarrow H^0(D, \mathbb{C}) \longrightarrow H^2(D, \mathbb{C}) \longrightarrow H^2(E, \mathbb{C}) \longrightarrow 0.$$

Hence $b_2(E) = b_2(D) - 1$. The assumptions $b_1(X) = b_3(X) = 0, b_2(X) = 1$ imply $b_2(X \setminus D) = 0$ using the Mayer-Vietoris sequence. □

Summing up our arguments we have proved

Corollary 3.4.3 *Let X be a projective manifold satisfying*

$$\dim X \geq 3, b_1(X) = b_3(X) = 0, b_2(X) = 1$$

and $D \subset X$ a smooth ample divisor, then every Kähler metric on $X \setminus D$ allows for a Kähler potential.

$X = \mathbb{P}^n, n \geq 3$ serve as examples for the Corollary. By the Lefschetz theorem, also any smooth complete intersection X of dimension n in a projective space $\mathbb{P}^m$ is subject to the Corollary provided $n \geq 4$.

3.5 Non-compact manifolds with non-trivial Kähler cone

We want to elaborate on complex abelian Lie groups G. Later some of them are shown to be realizable as open manifolds. First note that the exponential map

$$\exp : \mathfrak{g} \longrightarrow G$$

is a group homomorphism in the abelian case and thus exhibits G as a a quotient $\mathbb{C}^n/\Lambda$ for a discrete subgroup Λ. For any subring $R \subset \mathbb{C}$ with 1 we denote the R-span of Λ in $\mathbb{C}^n$ by Λ_R. The discreteness of Λ implies that the kernel of the natural map

$$\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow \Lambda_{\mathbb{R}}$$

onto the $\mathbb{R}$-span of Λ is trivial:

If $\sum_{i=1}^k r_i\lambda_i = 0$, then $\mu_n := \sum_{i=1}^k \{nr_i\}\lambda_i \in \Lambda$ for every $n \in \mathbb{Z}$, if $\{r\}$ denotes the non-integer part of a real number r. If one of the r_i is irrational, then the μ_n form a non-discrete subset of Λ in $\mathbb{C}^n$. Since the abelian subgroup of Λ generated by $\lambda_1, \ldots, \lambda_k$ is a free $\mathbb{Z}$-module and R is torsion-free, rationality of all r_i implies that $\sum_i r_i \otimes \lambda_i = 0$.

So we identify $\Lambda \otimes \mathbb{R}$ and $\Lambda_{\mathbb{R}}$ in the following. Let $\pi := \exp : \mathbb{C}^n \longrightarrow G$ be the projection.

Every $\lambda_{(1)}, \lambda_{(2)} \in \Lambda$ generate an oriented parallelogram, whose image in G is a compact real surface $T_{\lambda_{(1)},\lambda_{(2)}}$. The theory of CW complexes implies that the surfaces $T_{\lambda_{(1)},\lambda_{(2)}}$ generate $H_2(G, \mathbb{Z})$. So by duality and Stokes' Theorem we see that ω is exact if and only if

$$0 = \int_{T_{\lambda_{(1)},\lambda_{(2)}}} \omega$$

for all $\lambda_{(1)}, \lambda_{(2)} \in \Lambda$.

Proposition 3.5.1 *$K(G) = K_G(G)$.*

Proof. It is easy to see that $\tilde{\omega}$ is G-invariant if and only if $\pi^*\tilde{\omega} = i\sum_{i,j} \tilde{\omega}_{i\bar{j}} dz^i \wedge \overline{dz^j}$ with constant hermitian coefficients $\tilde{\omega}_{i\bar{j}}$, i.e.

$$\pi^*\tilde{\omega} \in (\Lambda^2\mathbb{C}^n)^\vee.$$

So, in order to see $K(G) = K_G(G)$ we have to find positive hermitian $\eta \in (\Lambda^2\mathbb{C}^n)^\vee$ such that

$$\int_{T_{\lambda_{(1)},\lambda_{(2)}}} \omega = \eta(\lambda_{(1)} \wedge \lambda_{(2)}).$$

This is done in the following way: The map

$$(\lambda_{(1)}, \lambda_{(2)}) \mapsto \int_{T_{\lambda_{(1)},\lambda_{(2)}}} \omega$$

is an element $\mu_\Lambda \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda^2\Lambda, \mathbb{R})$ inducing naturally an element $\mu \in \mathrm{Hom}_{\mathbb{R}}(\Lambda^2(\Lambda \otimes \mathbb{R}), \mathbb{R}) = \Lambda^2(\Lambda \otimes \mathbb{R})^\vee$. Since $\Lambda \otimes \mathbb{R}$ and $\Lambda_{\mathbb{R}}$ coincide, μ can be extended to an element in $(\Lambda^2\mathbb{C}^n)^\vee$. Taking positivity and the hermitian property of ω into account, we obtain the existence of η like described. $\square$

So now we are left with the problem of computing $K_G(G)$. To this end we compute for $\omega \in (\Lambda^2\mathbb{C}^n)^\vee$

$$\int_{T_{\lambda_{(1)},\lambda_{(2)}}} \omega = \omega_{ij}(\lambda^i_{(1)}\overline{\lambda^j_{(2)}} - \lambda^i_{(2)}\overline{\lambda^j_{(1)}}) = 2\mathrm{Im}(\lambda^t_{(1)}\underline{\omega}\overline{\lambda_{(2)}}),$$

if $\underline{\omega} = (\omega_{ij})_{i,j}$. Note that the ω_{ij} are constant. This property only depends on $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$. It is easy to see that in appropriate complex coordinates every real subspace of $\mathbb{C}^n$ is of the form

$$\Lambda_{\mathbb{R}} = \{z^1 = \ldots = z^{l'} = \mathrm{Im} z^{l'+1} = \ldots = \mathrm{Im} z^{k'} = 0\}.$$

In other words, $\Lambda_{\mathbb{R}}$ is generated by the real standard basis of $\mathbb{C}^k = \mathbb{R}^{2k}$ and the standard basis of $\mathbb{R}^l = \mathrm{Re}(\mathbb{C}^l)$ (for $k = n - k', l = k' - l'$). By this choice the above equations mean that in the standard basis of the decomposition

$$\mathbb{C}^n = \mathbb{C}^k \oplus \mathbb{C}^l \oplus \mathbb{C}^{n-k-l} \tag{3.1}$$

the 2-form ω has a matrix representation of the form

$$\underline{\omega} = \begin{pmatrix} 0 & 0 & * \\ 0 & \mathrm{real} & * \\ * & * & * \end{pmatrix}, \tag{3.2}$$

where every entry stands for the block corresponding to the factors of $\mathbb{C}^n = \mathbb{C}^k \oplus \mathbb{C}^l \oplus \mathbb{C}^{n-k-l}$ and $*$ means, that there is no claim about this entry. So we arrive at

Proposition 3.5.2 *Let $G = \mathbb{C}^n/\Lambda$. We write $\Lambda_{\mathbb{R}} = \mathbb{C}^k \oplus \mathrm{Re}(\mathbb{C}^l)$ (with $k + l \leq n$). Denote $\iota : M(l, \mathbb{C}) \longrightarrow M(k + l, \mathbb{C})$ the embedding which fills up an $l \times l$-matrix with zeroes. Then*

$$K(G) \subset M(k + l, \mathbb{C})/\iota(M(l, \mathbb{R}))$$

is the cone generated by positive definite hermitian matrices. In particular,

$$\dim_{\mathbb{R}} K(G) = (k + l)^2 - \frac{1}{2}l(l + 1).$$

As we already know from Proposition 3.3.2, the Bott-Chern-Kähler cone coincides with the Kähler cone, if G is Stein or compact. These, however, are not the only cases.

Definition 3.5.3 *A complex Lie group G is called* semi-torus*, if there is a discrete subgroup $\Lambda \subset \mathbb{C}^n$ such that the $\mathbb{C}$-span of Λ, denoted $\Lambda_{\mathbb{C}}$, is $\mathbb{C}^n$ and $G \cong \mathbb{C}^n/\Lambda$.*

It is well-known (since a lecture of Remmert in 1959) that any abelian complex Lie group can be uniquely decomposed into a product

$$G \cong T \times (\mathbb{C}^*)^{\tilde{k}} \times \mathbb{C}^{\tilde{l}}$$

with T being a group with only constant holomorphic functions. So T is the obstruction for G being Stein.

Lemma 3.5.4 *$G = T \times (\mathbb{C}^*)^{\tilde{k}} \times \mathbb{C}^{\tilde{l}}$ is a semi-torus if and only if $\tilde{l} = 0$.*

Proof. If G is not a semi-torus, then clearly $\tilde{l} \neq 0$. So let G be a semi-torus. Since also $T = \mathbb{C}^{\tilde{m}}/\tilde{\Lambda}$ for a discrete subgroup $\tilde{\Lambda} \subset \mathbb{C}^{\tilde{m}}$, the decomposition gives us a discrete subgroup $\Lambda' \subset \mathbb{C}^{\tilde{m}+\tilde{k}}$ such that $G = \mathbb{C}^n/\Lambda'$. Standard covering theory yields that the isomorphism $\mathbb{C}^n/\Lambda' \longrightarrow \mathbb{C}^n/\Lambda$ is induced by an isomorphism of vector spaces $\phi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ satisfying $\phi(\Lambda') = \Lambda$. In particular, $\phi(\mathbb{C}^{\tilde{m}+\tilde{k}}) = \phi(\Lambda'_{\mathbb{C}}) = \Lambda_{\mathbb{C}} = \mathbb{C}^n$, so $l = 0$. ☐

For the sake of simplicity, let us denote $\mathbf{k} := \{1, ..., k\}$, $\mathbf{l} := \{k+1, ..., k+l\}$, $\mathbf{m} := \{k+l+1, ..., n\}$, where k, l come from the decomposition (3.1). Note that $\mathrm{Im}(z^i)$ is a Λ-invariant function, if $i \in \mathbf{l}$ and z^j is Λ-invariant, if $j \in \mathbf{m}$. So, if ω is of the form (3.2), we can define $\phi \in C^\infty(G)$ by

$$\begin{aligned} \phi(z) \;:=\; & 2\sum_{i\in\mathbf{l}} \omega_{ii}\mathrm{Im}(z^i)^2 + 4\sum_{i<j\in\mathbf{l}} \omega_{ij}\mathrm{Im}(z^i)\mathrm{Im}(z^j) + \\ & +4\sum_{i\in\mathbf{l},j\in\mathbf{m}} (\omega_{ij}\mathrm{Im}(z^i)\overline{z^j} - \omega_{ji}\mathrm{Im}(z^i)z^j) + \\ & +4\sum_{i\in\mathbf{m}} \omega_{ii}|z^i|^2 + 2\sum_{i<j\in\mathbf{m}} \omega_{ij}z^i\overline{z^j}. \end{aligned}$$

This function satisfies

$$\omega - i\partial\overline{\partial}\phi = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & 0 \\ * & 0 & 0 \end{pmatrix}.$$

If we now assume that G is Stein or a semi-torus, then the first or the third factor does not occur and hence

$$\omega = i\partial\overline{\partial}\phi.$$

Summing up our considerations we obtain with the same notation as in Proposition 3.5.2

Proposition 3.5.5 *(i) If G is Stein, then* $\dim_{\mathbb{R}} K^{BC}(G) = \frac{1}{2}l(l-1)$.

(ii) If G is a semi-torus, then $\dim_{\mathbb{R}} K_G^{BC}(G) = n^2 - \frac{1}{2}l(l+1)$.

A synopsis of Propositions 3.5.2 and 3.5.5 proves

Corollary 3.5.6 *For an abelian complex connected Lie group G the conditions*

(i) $K(G) = 0$,

(ii) $K^{BC}(G) = 0$ *and*

(iii) $G = \mathbb{C}^n$ *or* $G = \mathbb{C}^* \times \mathbb{C}^{n-1}$

are equivalent.

3.6 Realisations of certain complex abelian Lie groups as open manifolds

Our aim is to construct open manifolds with a group action leaving a Kähler metric g invariant. Given this situation we will show that g is Ricci-flat. So we start with a compact manifold X with an action of a complex connected Lie group G. Our first aim is to construct a natural G-invariant divisor D. There is only hope to be able to do so if G acts almost transitively. The correct property of G is determined by the following definition

Definition 3.6.1 *If X is compact complex manifold and $G \subset \mathrm{Aut}^0(X)$ a connected complex Lie group with $\dim G = \dim X$ acting almost transitively on X, we say G is a* divisorial *group. Any object invariant under a divisorial group we call* divisorially invariant.

Given a divisorial group G on X, the Lie algebra $\mathfrak{g}$ is an n-dimensional subspace of $H^0(T_X)$ generating T_X in the general point. Taking the determinant we obtain an embedding

$$\det \mathfrak{g} \hookrightarrow H^0(-K_X)$$

as a 1-dimensional subspace. We define $D_{\mathfrak{g}}$ as the vanishing locus (with multiplicities) of $\det \mathfrak{g}$. This gives an element $D_{\mathfrak{g}} \subset |-K_X|$.

Lemma 3.6.2 *$X \setminus D_{\mathfrak{g}}$ is the unique (Zariski-)open orbit of G.*

Proof. Let S be an arbitrary non-open orbit of G and S° its smooth part. Any $s \in \mathfrak{g}$ yields by restriction an element $s|S^\circ \in H^0(T_{S^\circ})$. Since $\dim S < n$ we obtain $\det \mathfrak{g}(S) = 0$, so $S \subset D_{\mathfrak{g}}$. On the other hand, any $h \in G$ is an automorphism such that $h^* : \mathfrak{g} \longrightarrow \mathfrak{g}$ is an isomorphism. In particular, $h(D_{\mathfrak{g}}) = D_{\mathfrak{g}}$, so $D_{\mathfrak{g}}$ is G invariant. □

To us even more importantly, any basis $\mathcal{B} = \{s^{(1)}, \ldots, s^{(n)}\}$ of $\mathfrak{g}$ yields an hermitian metric $g_{\mathcal{B}}$ on $X \setminus D_{\mathfrak{g}}$ uniquely determined by the condition

$$g_{\mathcal{B}}(s^{(i)} \otimes \overline{s^{(j)}}) = \delta_{ij}$$

as explained in greater detail in chapter 2. There we also discussed when this metric is G-invariant. Here it is of great importance to determine $g_{\mathcal{B}}$ is Kähler. The next lemma gives us a necessary condition.

Lemma 3.6.3 *Let Y be a complex manifold and g a Kähler metric on Y. Then any connected complex Lie group $G \subset \mathrm{Aut}^0(Y, g)$ is abelian.*

Proof. Let ω denote the Kähler form of g and $\mathfrak{g} \subset H^0(T_Y)$ the Lie algebra of G. Since g is G-invariant, for all $s \in \mathfrak{g}$ we obtain

$$\mathcal{L}_s \omega = 0,$$

where $\mathcal{L}$ denotes the Lie derivative. If furthermore C denotes the contraction by the subscript vector field, $\mathcal{L}_s = dC_s + C_s d$ and hence we conclude $dC_s\omega = 0$ for all $s \in \mathfrak{g}$. Again using an elementary formula (see e.g. [La, V,5]) and $d\omega = 0$ we obtain for $s, t \in \mathfrak{g}$

$$C_{[s,t]}\omega = (\mathcal{L}_s C_t - C_t \mathcal{L}_s)\omega = \mathcal{L}_s C_t \omega = dC_s C_t \omega + C_s dC_t \omega.$$

Since ω is a $(1,1)$-form and s, t are holomorphic, $C_s C_t \omega = 0$. We already saw that $dC_t\omega = 0$, hence both summands of the right hand side vanish, yielding $C_{[s,t]}\omega = 0$. In local coordinates this means

$$g_{\alpha\bar{\beta}}[s, t]^\alpha = 0.$$

Since the matrix $g_{\alpha\bar{\beta}}$ is invertible this implies $[s, t] = 0$. Hence $\mathfrak{g}$ is abelian and therefore also G is abelian. □

So only abelian G are admissible. We will now see that this is exactly the right condition.

Lemma 3.6.4 *Let G be a divisorial group on X and $\mathcal{B} \subset \mathfrak{g}$ an arbitrary basis. The conditions*

(i) G is abelian,

(ii) $g_{\mathcal{B}}$ is Kähler, and

(iii) $G \subset \mathrm{Aut}(X, D_{\mathfrak{g}}, g_{\mathcal{B}})$

are equivalent.

Proof. Let $\mathcal{B} = \{s^{(1)}, \ldots, s^{(n)}\}$. In a local chart we write $s^{(i)} = s^{ik}\frac{\partial}{\partial z^k}$. We denote by $(s_{ij}) = \sigma$ the inverse matrix of (s^{ij}). Now

$$g_{\mathcal{B},ij} = g_{\mathcal{B}}\left(\frac{\partial}{\partial z^i} \otimes \frac{\partial}{\partial \overline{z^j}}\right) = s_{ik}\overline{s_{jk}},$$

$(i) \iff (ii)$: A short calculation shows that $g_{\mathcal{B}}$ is Kähler if and only if

$$s_{ij,l} = s_{lj,i}$$

for all i, j, l. This we identified in the proof of Lemma 2.4.1 as the condition

$$[s^{(i)}, s^{(k)}] = 0$$

for all i, k. So $\mathfrak{g}$ is abelian and thus G is abelian.

$(i) \iff (iii)$: This has already been proved in Lemma 2.4.1.

□

Let $x_0 \in X \setminus D_{\mathfrak{g}}$ and $\alpha : G \longrightarrow X \setminus D_{\mathfrak{g}}$ be the action map $g \mapsto gx_0$. Again it is known (cf. [On1]) that α has constant rank. Since α is surjective, it has to be a covering map. If G is abelian, $yz := \alpha(\alpha^{-1}(y)\alpha^{-1}(z))$ is well-defined and turns $X \setminus D_{\mathfrak{g}}$ itself into an abelian Lie group of dimension n and α into a group homomorphism. Since elements of $\ker \alpha$ induce the identity on $X \setminus D_{\mathfrak{g}}$, the property $G \subset \mathrm{Aut}^0(X)$ implies, that α is an isomorphism. Hence we will identify G and $X \setminus D_{\mathfrak{g}}$ from now on. Our arguments sum up to

Proposition 3.6.5 *Let X be a compact complex manifold and G a divisorial group on X. If G is abelian, then X is an equivariant compactification of G by a divisor. In particular, G is an open manifold.*

In order to classify complex abelian Lie groups being open manifolds, we look at the Albanese map $\alpha : X \longrightarrow \mathrm{Alb}(X)$ and the decomposition

$$G = T \times (\mathbb{C}^*)^k \times \mathbb{C}^l.$$

By the universal property, α is G-equivariant. So, after a suitable choice of the group structure, the restricted map

$$\beta := \alpha|G : G \longrightarrow \mathrm{Alb}(X)$$

is a group homomorphism. Using Stein factorisation and the universality of the Albanese map, we conclude that α (and hence also β) has connected fibres. Since G acts almost transitive on X, the induced action on $\mathrm{Alb}(X)$ is transitive, hence β is surjective. We obtain for the complex Lie algebra $\mathfrak{k}$ of $\ker\beta$ the description

$$\begin{aligned}\mathfrak{k} &= \{s \in \mathfrak{g} |\ \eta(s) = 0\ \forall\ \eta \in H^0(\Omega_X)\} \\ &= \{s \in \mathfrak{g} |\ s \text{ has a zero }\}\end{aligned}$$

by [S74, Lemma I]. This implies that $\ker\beta$ has no torus factor, but not only that: any factor $\mathbb{C}^*$ in G corresponds to a vector field of the form $z\frac{\partial}{\partial z}$ having a zero for $z \longrightarrow 0$ and any factor $\mathbb{C}$ corresponds to a vector field $\frac{\partial}{\partial z}$ having a zero of order two at $z \longrightarrow \infty$. We see this by looking at the local coordinate $y := \frac{1}{z}$ around ∞. So $\{0\} \times (\mathbb{C}^*)^k \times \mathbb{C}^l \subset \ker\beta$, implying that $\beta|T : T \times \{0\} \longrightarrow \mathrm{Alb}(X)$ is surjective.

Proposition 3.6.6 *Let X be an almost homogeneous Kähler manifold such that $|-mK_X|$ is base point free for a certain $m > 0$. Assume further that there is an abelian complex Lie subgroup $G \subset \mathrm{Aut}^0(X)$ acting almost transitive. There is an equivariant decomposition $X = T \times Y$ into the product of a torus T and a projective manifold Y with $b_1(Y) = 0$ being an equivariant compactification of $(\mathbb{C}^*)^k \times \mathbb{C}^l$.*

Proof. We may assume that G has dimension n and use the decomposition $G = T \times (\mathbb{C}^*)^k \times \mathbb{C}^l = T \times H$. First, we want to prove that T is a torus. The assumption that $|-mK_X|$ is base point free enables us to choose for every $x \in D_{\mathfrak{g}}$ a meromorphic function $\tilde{f}$ with poles exactly along $D_{\mathfrak{g}}$ and x is not in the locus of indeterminacy of $\tilde{f}$. If T is not compact, we fix $z \in \mathbb{C}^k \times (\mathbb{C}^*)^l$ and $x \in \overline{T \times \{z\}}$. Since $\tilde{f}|_{X\setminus D_{\mathfrak{g}}}$ is holomorphic, in particular $f := \tilde{f}|_{T\times\{z\}}$ is holomorphic. Hence f is constant and we obtain $\overline{T \times \{z\}} \subset X \setminus D_{\mathfrak{g}}$. This implies that T is compact. Since $T = \mathbb{C}^k/\Lambda$, the lattice Λ is complete and hence T is a torus.

Now we have to prove that the projection onto T is extendable. Since $\ker\beta$ does not contain a torus factor, $\beta|T : T \longrightarrow \mathrm{Alb}(X)$ has to be injective, hence an isomorphism. So the projection $G \longrightarrow T$ can be extended to a holomorphic map $X \longrightarrow T$ and this is the Albanese map.

Now choose $h := (t', 1) \in G = T \times H$ and denote $F_t := \alpha^{-1}(t)$. Of course, $h : X \longrightarrow X$ satisfies $h(F_t) = F_{t+t'}$ and the map $\psi : F_0 \times T \longrightarrow X, (y,t) \mapsto (t,1)y$ is an isomorphism. Since $Y := F_0$ is a fibre of α, it is projective and satisfies $b_1(Y) = 0$. □

The second part of the description concerns the divisor.

Lemma 3.6.7 *If $D_{\mathfrak{g}}$ is reduced, then G is a semi-torus.*

Proof. We assume that G is not a semi-torus, hence by Lemma 3.5.4 we obtain a decomposition $G = \mathbb{C} \times G'$. We choose $s \in H^0(-K_X)$ such that $D_{\mathfrak{g}} = \{s = 0\}$. Since G' is abelian, $G' = \mathbb{C}^{n-1}/\Lambda$ and we may choose local coordinates $\tilde{z}^1, ..., \tilde{z}^{n-1}$ induced by canonical coordinates of $\mathbb{C}^{n-1}$. In those coordinates of $X \setminus D_{\mathfrak{g}}$ we write

$$s = f(x, \tilde{z}^1, ..., \tilde{z}^{n-1}) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial \tilde{z}^1} \wedge ... \wedge \frac{\partial}{\partial \tilde{z}^{n-1}},$$

where x denotes the coordinate of the factor $\mathbb{C}$. If $x \longrightarrow \infty$, we will approximate a point in $D_{\mathfrak{g}}$. In order to approximate other points (and indeed by this procedure all other points of a certain component of $D_{\mathfrak{g}}$), we choose an appropriate holomorphic $\lambda : \mathbb{C} \longrightarrow G'$ and look at the curve $(x, \tilde{z}' + \lambda(x))$ for a fixed point $\tilde{z}' := (\tilde{z}^1, ..., \tilde{z}^{n-1}) \in G'$. Let $p = \lim_{x \longrightarrow \infty}(x, \tilde{z}' + \lambda(x)) \in D_{\mathfrak{g}}$. By $x \mapsto \frac{1}{x} =: y, \tilde{z}^i \mapsto \tilde{z}^i - \lambda^i(x) =: z^i$ we get local coordinates in a neighbourhood $U(p) \setminus D_{\mathfrak{g}}$. Moreover, we obtain a holomorphic map

$$\Psi : V := \{|y| < \delta\} \times V' \longrightarrow U(p)$$

for a neighbourhood $V' \subset \mathbb{C}^{n-1}$. The form $\eta := \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z^1} \wedge \cdots \wedge \frac{\partial}{\partial z^{n-1}}$ is well-defined on V. If Ψ is finite, after shrinking V, if necessary, we obtain a well-defined form $\Psi_* \eta(p)$ in p. If Ψ contracts a complex curve to p, we find a tangent direction τ in every $q \in \Psi^{-1}(p)$ such that $\Psi_* \tau(p) = 0$, in particular $\Psi_* \eta(p)$ is again well defined and $\Psi_* \eta(p) = 0$. Since the form η does not depend on the choice of λ, we may regard η now as a well-defined form on $U(p)$.

In these coordinates,

$$s = -f\left(\frac{1}{y}, z' + \lambda\left(\frac{1}{y}\right)\right) y^2 \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z^1} \wedge ... \wedge \frac{\partial}{\partial z^{n-1}}.$$

Let us denote $h(y) := -f(\frac{1}{y}, z' + \lambda(\frac{1}{y}))y^2$. The group action of $\mathbb{C}$ now is

$$\mu \cdot y = \frac{y}{1 + \mu y}.$$

The invariance of $D_{\mathfrak{g}}$ under G implies for $\mu \in \mathbb{C}$ that $\mu^* s = c(\mu)s$, hence

$$h(\mu \cdot y)(1 + \mu y)^2 = c(\mu)h(y).$$

Since $c(\mu + \kappa)s = (\mu + \kappa)^* s = \mu^* \kappa^* s = c(\mu)c(\kappa)s$, the function $c(\mu) = \exp(\rho\mu)$. This implies

$$h(\mu \cdot y) = \frac{\exp(\rho\mu)}{(1 + \mu y)^2} h(y).$$

Now fixing $y = 1$ yields

$$h(\frac{1}{1 + \mu}) = c\frac{\exp(\rho(1 + \mu))}{(1 + \mu)^2},$$

hence
$$h(y) = cy^2 \exp(\frac{\rho}{y}).$$
Since we have the additional requirement that $s|D_{\mathfrak{g}} = 0$ and $\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z^1} \wedge ... \wedge \frac{\partial}{\partial z^{n-1}}(p)$ has a finite vanishing order, we conclude $\rho = 0$. Now we see that h vanishes of order 2 in 0. As discussed above, it may happen that $\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z^1} \wedge ... \wedge \frac{\partial}{\partial z^{n-1}}(p) = 0$. Hence we only conclude that the vanishing order of s on the limit point p is at least 2. Since we could do this construction for every point of a component containing p, we conclude that this component is multiple. $\square$

This result allows a more detailed description of D in some cases. If, for instance, $b_1(X) = 0, X$ is projective, and $D_{\mathfrak{g}}$ is reduced, then $G \cong (\mathbb{C}^*)^n$ and the action of G on X is algebraic. So X is a toric variety and D is the sum of the invariant toric divisors.

Finally, we want to see that any G-invariant Kähler metric arises as some $g_{\mathcal{B}}$. For this purpose, note that for a Lie group G being abelian implies that $G = \mathbb{C}^n/\Lambda$, where Λ is a discrete subgroup. This is proved by looking at the exponential map
$$\exp : \mathfrak{g} \longrightarrow G,$$
which is easily seen to be a group homomorphism of $(\mathfrak{g}, +)$ into G. Since exp maps some neighbourhood of 0 diffeomorphically to a neighbourhood of 1, say U, and $\bigcup_{k=1}^{\infty} U^k = G$, the map exp is surjective. Hence $G = (\mathfrak{g}, +)/\ker(\exp) = \mathbb{C}^n/\Lambda$, where $\Lambda := \ker(\exp)$ must be discrete, since $n = \dim G = \dim \mathfrak{g}$.

In particular, we see that $X \setminus D_{\mathfrak{g}} = \mathbb{C}^n/\Lambda$, if G is abelian. We will use this in the next proof.

Lemma 3.6.8 *Let X be a compact complex manifold and $G \subset \mathrm{Aut}^0(X)$ a divisorial Lie group. If g is a Kähler metric on $X \setminus D_{\mathfrak{g}}$ such that $\mathrm{Aut}^0(X, D_{\mathfrak{g}}, g) = G$ then g is complete and there is a basis $\mathcal{B} \subset \mathfrak{g}$ such that $g = g_{\mathcal{B}}$.*

Proof. We already know by Lemma 3.6.3 that G is abelian and hence $g_{\mathcal{B}}$ is Kähler and G-invariant, if $\mathcal{B} \subset \mathfrak{g}$ is a basis. Since $G = X \setminus D_{\mathfrak{g}} = \mathbb{C}^n/\Lambda$ we choose the images of the canonical coordinates $z_1, ..., z_n$ of $\mathbb{C}^n$ as local coordinates of $X \setminus D_{\mathfrak{g}}$. For the sake of simplicity we call them also $z_1, ..., z_n$. Of course, $g = g_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta$ is G-invariant, if and only if $g_{\alpha\bar{\beta}}$ is constant for all α, β. Hence g is complete and corresponds one to one to $g(0)$ what we identify with the matrix $\underline{g} = (g_{\alpha\bar{\beta}}(0))$. The corresponding matrix $\underline{g}_{\mathcal{B}}$ is $\underline{g}_{\mathcal{B}} = \sigma\sigma^*$. Note that σ is constant since G is abelian (cf. proof of Theorem 2.4.4). Recall $S = \sigma^{-1}$ and define $H := S\underline{g}S^*$. Since H is hermitian, we can find $A \in Gl(n)$ such that $H = A^*A$. Now $\underline{g} = \sigma H \sigma^* = \sigma A A^* \sigma^*$. Set $B := A^{-1}$. Then g is given by the vector fields $t^{(i)} = \sum_j b_{ij} s^{(j)}$, which form another basis of $\mathfrak{g}$. (Indeed, this shows by Theorem 2.3.6 once more that g is complete.) $\square$

Corollary 3.6.9 *Let X be a compact complex manifold and $S \subset X$ analytic with* codim$S > 1$*. If X allows for a divisorially invariant Kähler metric on $X \setminus S$, then X is a torus and S is empty (if chosen minimal).*

Proof. Assume g is such a metric and G the divisorial abelian Lie group. By Lemma 3.6.8 $g|X \setminus D_{\mathfrak{g}}$ is constructed by a basis of $\mathfrak{g}$. If $D_{\mathfrak{g}}$ is given by $\sigma \in H^0(-K_X)$, then $\det g = |\sigma|^{-2}$, hence is singular on $D_{\mathfrak{g}}$. This implies $D_{\mathfrak{g}} = 0$. In particular, $X = G = \mathbb{C}^n/\Lambda$. Since X is compact, Λ is a complete lattice and X is a torus. □

Note that this proof works also, if codim$S = 1$, but $S \neq D_{\mathfrak{g},red}$.

3.7 Example: $X = \mathbb{P}^2$

If $X = \mathbb{P}^2$, then the tangent bundle may be described by the vector fields homogeneous of degree 1 divided by the vector fields parallel to the orbits of the group action $z \mapsto cz$, i.e. $\mathcal{O}_X \cdot (z^0\frac{\partial}{\partial z^0} + z^1\frac{\partial}{\partial z^1} + z^2\frac{\partial}{\partial z^2})$. Hence the global vector fields are

$$H^0(T_X) \cong (l^0\frac{\partial}{\partial z^0} + l^1\frac{\partial}{\partial z^1} + l^2\frac{\partial}{\partial z^2})/\mathbb{C} \cdot (z^0\frac{\partial}{\partial z^0} + z^1\frac{\partial}{\partial z^1} + z^2\frac{\partial}{\partial z^2}),$$

where l^i are homogeneous linear forms. Now let $V := \mathbb{C}v^{(1)} \oplus \mathbb{C}v^{(2)} \subset H^0(T_X)$ with $v_j = [\sum_i l^{ji}\frac{\partial}{\partial z^i}]$. In order to compute

$$D := \{z | v^{(1)} \wedge v^{(2)} = 0\},$$

we first localise to U_0 and then homogenise the result again. This procedure yields

$$D = \left\{ \det \begin{pmatrix} z^0 & z^1 & z^2 \\ l^{10} & l^{11} & l^{12} \\ l^{20} & l^{21} & l^{22} \end{pmatrix} = 0 \right\}.$$

Now let us assume that $[v^{(1)}, v^{(2)}] = 0$ and $v^{(1)} \wedge v^{(2)} \not\equiv 0$. Denote $G := \exp(V) = \mathrm{Aut}^0(X, D)$. By assumption G is divisorial and abelian, hence the metric $g_{\mathcal{B}}$ is Kähler and $G = \mathrm{Aut}(X, D, g_{\mathcal{B}})$. If D is reduced, Theorem 3.6.6 and Lemma 3.6.7 tell us that D is an equivariant compactification of $G \cong (\mathbb{C}^*)^2$. The action of G being algebraic by construction, D is the union of the invariant toric divisors, i.e. D is the union of three lines in general position. If D is not reduced, $\deg D = 3$ shows that we obtain two lines one of which is double, or a triple line.

So the only position of three lines not occurring in this list is that they are intersecting in one common point. We will now see how this corresponds to a G which acts not almost transitively. After a change of coordinates we may assume that the three lines intersect in $[1 : 0 : 0]$. Let $v^{(1)} := z^1\frac{\partial}{\partial z^0}, v^{(2)} := z^2\frac{\partial}{\partial z^0}$. Of course, $[v^{(1)}, v^{(2)}] = 0$ and

$v^{(1)} \wedge v^{(2)} \equiv 0$, hence G is abelian (indeed, $G \cong \mathbb{C}^2$) and acts not almost transitively. G is given by the matrices

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a, b \in \mathbb{C}.$$

It is not hard to see that G leaves $\{f = 0\}$ invariant for a homogeneous $f \in \mathbb{C}[z^0, z^1, z^2]$ if and only if $f = f(z^1, z^2)$. This factors into linear terms. Hence G leaves all lines through $[1 : 0 : 0]$ invariant. So the not almost transitively case corresponds to the existence of a family of invariant divisors, which are not necessarily anticanonical. If D consists of three lines intersecting in $[1 : 0 : 0]$, the full automorphism group $\mathrm{Aut}(X, D)$ consists of matrices

$$\begin{pmatrix} c & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, a, b, c \in \mathbb{C},$$

hence is not abelian and acts not almost transitively in accordance with Lemma 2.4.2; also $\dim \mathrm{Aut}(X, D) > \dim X$.

We come back to the case of almost transitive G and reduced D. Let us choose coordinates such that $D = \{z^0 z^1 z^2 = 0\}$. Of course, D is invariant under the group G given by $[z^0 : z^1 : z^2] \mapsto [a_0 z^0 : a_1 z^1 : a_2 z^2]$, with $a = [a_0 : a_1 : a_2] \in \mathbb{P}^2 \setminus \{a_0 a_1 a_2 = 0\} \cong \mathbb{C}^* \times \mathbb{C}^*$. The group G is abelian and divisorial. Theorem 2.4.4 and Lemma 2.4.1 tell us that $G = \mathrm{Aut}^0(X, D)$. Lemma 3.6.8 states that every G-invariant Kähler metric on $X \setminus D$ is given by a basis of $\mathfrak{g}$. Carrying out the calculations in the chart $U_0 = \{z_0 \neq 0\}$ yields that all G-invariant Kähler metrics on $X \setminus D \cong \mathbb{C}^* \times \mathbb{C}^*$ are of the form

$$g = g_C = \sum_{i,j=1,2} c_{ij} \frac{dx^i}{x^i} \otimes \frac{\overline{dx^j}}{\overline{x^j}},$$

with $c_{ij} = \overline{c_{ji}}$ and $C = (c_{ij}) > 0$. This description implies that the volume of geodesic balls $B_\rho(x)$ around a fixed point x grows like ρ^2.

According to Theorem 3.5.2 and Proposition 3.5.1

$$K(X, D) \subset M(2, \mathbb{C})/M(2, \mathbb{R})$$

is given by the classes of positive, hermitian matrices. Hence $K_G(X, D)$ has dimension one. Moreover

$$C(r) := \begin{pmatrix} \cosh(r) & i \sinh(r) \\ -i \sinh(r) & \cosh(r) \end{pmatrix}$$

for $r \in \mathbb{R}$ represent every class in $K_G(X, D)$ uniquely.

Similarly, in the case of a double line and a single line, $X = \mathbb{C} \times \mathbb{C}^*$ and the volume of $B_\rho(x)$ grows like ρ^3. If D is a triple line, $X = \mathbb{C}^2$ and the volume growth of $B_\rho(x)$ is like ρ^4. So volume growth happens with integer polynomial order in all these examples. We will see in Chapter 4 that for smooth D volume growth may be of fractional order of ρ.

If $X = \mathbb{P}^3$, then for the corresponding construction $\dim K(\mathbb{P}^3, D) = 3$.

Chapter 4

Asymptotics

4.1 Introduction

For compact complex manifolds the problem of finding Ricci-flat Kähler metrics is reduced to determining $c_1(X) \in H^2(X, \mathbb{Z})$. If $c_1(X) = 0$, then due to the proof of the Calabi conjecture, there is a Ricci-flat Kähler metric in every cohomological class represented by some metric. Of course, if $c_1(X) \neq 0$, there can be no such metric. By the $\partial\overline{\partial}$-Lemma this means that given an initial metric $\tilde{g}$ and provided $c_1(X) = 0$ we can find $u \in C^\infty(X, \mathbb{R})$ such that

$$g := \tilde{g} + i\partial\overline{\partial}u$$

is a Ricci-flat metric.

The situation complicates as soon as we inquire into non-compact manifolds. In this case, we are of course not interested in Ricci-flat metrics of some compactification, but in Ricci-flat, complete Kähler metrics. First of all, we restrict ourselves to *open manifolds* in the following sense: Let X be a compact complex manifold and $D \subset X$ a reduced divisor. We call $\tilde{X} := X \setminus D$ an open manifold.

Tian and Yau proved in 1991 the existence of Ricci-flat, complete Kähler metrics on open with $D \in |-K_X|$ being smooth and ample. To be more precise, they proved that starting with a specifically constructed complete Kähler form ω_{TY} on $\tilde{X}$ there is $u \in C^\infty(\tilde{X}, \mathbb{R})$ such that

$$\tilde{\omega} := \omega_{TY} + i\partial\overline{\partial}u$$

is complete and Ricci-flat. The condition $D \in |-K_X|$ seems to be of great importance, since only then adjunction yields an isomorphism $\Omega^n_{\tilde{X}} \cong \mathcal{O}_{\tilde{X}}$. The existence being settled it remained unclear, however, what the asymptotics of such a Ricci-flat metric looks like and how many metrics of this kind can be found.

In this place it will be proved that the asymptotics of ω_{TY} resemble the asymptotics of ω up to the order of any polynomial decay satisfied by $\log\det\omega_{TY}$; actually, this function decays faster than any negative power of the distance function.

The chapter is organised in the following way: First we explain how an explicit initial metric can be constructed, much in the spirit of Tian and Yau but with a better decay of the volume form; then we construct quasi-coordinates for this metric and use them to estimate higher derivatives of the solution. Another key technique is to find a way to identify a neighbourhood of D with an open part of the holomorphic normal bundle such that $Re(\omega_{TY})$ is strongly asymptotic to an explicitly given metric on the bundle. After this we explain how to tackle the problems using these techniques.

If Ω denotes a holomorphic n-form with simple poles exactly along D and ω a Kähler metric, then the volume form $\Omega \wedge \bar{\Omega}$ is Ricci-flat, of course. So we want to determine the function u such that

$$(\omega + i\partial\overline{\partial}u)^n = \Omega \wedge \bar{\Omega},$$

hence for $f := \log \frac{\Omega\wedge\bar{\Omega}}{\omega^n}$ we obtain

$$\frac{(\omega + i\partial\overline{\partial}u)^n}{\omega^n} = e^f,$$

a complex Monge-Ampère equation. This has been solved by Tian and Yau [TY90].

In [TY90] Tian and Yau proved the existence of a complete Ricci-flat Kähler metric on the complement of a smooth anticanonical divisor D of a Fano manifold. For this purpose they constructed an initial metric ω with volume form deviating only $O((-\log\|S\|^2)^{-N})$ from the volume form of the solution metric; here S denotes the section of $-K_X$ defining $D \in |-K_X|$, $\|.\|$ an appropriate metric on $\mathcal{O}(D)$ and N an arbitrarily big integer. They posed the question how far the initial metric itself is away from the Ricci-flat metric given by the solution of the Monge-Ampère equation.

In this chapter we first construct an initial metric with an exponentially asymptotically flat volume form and give the answer that it is indeed very close, namely $O((-\log\|S\|^2)^{-N})$ for every N. As a first step we prove that there is a slow approach (Section 4.5):

$$\|\partial\overline{\partial}u\|_\omega \in O((-\log\|S\|^2)^{-\frac{1}{6n}}),$$

where u is the bounded solution given by [TY90] of the Monge-Ampère equation

$$\frac{(\omega + i\partial\overline{\partial}u)^n}{\omega^n} = e^f \tag{4.1}$$

for $f \in O((-\log\|S\|^2)^{-N})$.

In order to prove this we use an appropriately disturbed equation that allows for a maximum principle (Section 4.4). This is given by

$$\frac{(\omega + i\partial\overline{\partial}u_\varepsilon)^n}{\omega^n} = e^{\varepsilon u_\varepsilon + f}$$

We prove strong decay for the unique bounded solution constructed in [CY80] and give a description of the behaviour of the involved constants in dependence of ε. Here we make use of a weak maximum principle. By curvature estimates provided by [TY90] and again by a weak maximum principle we are able to prove the theorem.

For this result we use the information from the Monge-Ampère equation (4.1)

$$|\Delta_\omega u| \in O((-\log\|S\|^2)^{-p}) \Rightarrow \|\partial\overline{\partial}u\|_\omega \in O((-\log\|S\|^2)^{-\frac{p}{2}})$$

if $0 < p < N$ (cf. Lemma 4.5.2). One easily observes that the maximal approach rate would apply if the second derivatives would decay with the same order as $\Delta_\omega u$ (see Section 4.10).

Switching the attention from the form of the equation to the form of the initial metric we can prove in Section 4.9 this much better decay property for the linear problem as the main result of this section provided $\Delta_\omega u$ behaves 'almost radially'. In order to explain this we have to introduce

$$\tilde{Y}_R := \{x \in X \setminus D| \ -\log\|S(x)\|^2 = 2R\}$$

equipped with the induced Riemannian metric $\omega_R := i^*_{\tilde{Y}_R}\omega$.

Theorem 4.1.1 *Let ω be the Tian-Yau initial metric, $p > 0$ and $u \in C^\infty(X \setminus D) \cap C^{3n+6,\alpha}_\omega$. If $\Delta_\omega u = f$ with $f \in O((-\log\|S\|^2)^{-p}), \Delta^q_{\omega_R} f|_{\tilde{Y}_R} \in O((-\log\|S\|^2)^{-p-\frac{q}{n}})$ for all $1 \leq q \leq \frac{3}{2}n + 3$, then $\|\partial\overline{\partial}u\|_\omega \in O((-\log\|S\|^2)^{-p})$.*

The extra conditions on $\Delta^q_\omega f$ are satisfied for $f = R^{-p}$. So they are for the function f of (4.1) constructed below. Note that here u is not necessarily a solution of the Monge-Ampère equation (4.1). Our strategy here will be to construct a Riemannian metric strongly asymptotic to ω (meaning $O(\|S\|^q)$-near) on a tubular neighbourhood of D equipped with a principal fibre bundle structure (Section 4.7). This metric can be computed explicitly in terms of polar coordinates of the fibre disks. We will use Fourier analysis on the fibre bundle to convert the Laplace equation to a system of uncoupled ODEs with explicitly computable solutions (Section 4.8). These will give rise to the claim of Theorem 4.1.1.

Once Theorem 4.1.1 is proved it is only a technicality to arrive at

Corollary 4.1.2 *Let ω be the Tian-Yau initial metric and u the bounded solution of the Monge-Ampère equation (4.1). For every $N > 0$ there is a constant $C_N > 0$ such that*

$$(1 - C_N(-\log\|S\|^2)^{-N})\omega \leq \omega + i\partial\overline{\partial}u \leq (1 + C_N(-\log\|S\|^2)^{-N})\omega.$$

The proof is given in Section 4.10.

4.2 Construction of the initial metric

In this section we use a slightly different approach for the construction of the initial metric used in [TY90]. Let X be smooth projective, $\dim_{\mathbb{C}}(X) = n$, $D \in |-K_X|$ a smooth ample divisor. Let $\|\cdot\|$ be a smooth metric on the line bundle $\mathcal{O}(D)$ such that $\rho := -i\partial\overline{\partial}\log\|\cdot\| > 0$ is positive definite on all X. Let S be a section of $-K_X$ defining D. Since $K_X + D = 0$, there is up to a constant multiple a unique meromorphic n-form Ω on X with poles of first order along D. Adjunction formula provides us with a nowhere vanishing holomorphic $(n-1)$-form $\hat{\Omega}$ on D via

$$\hat{\Omega} := \operatorname{Res}\Omega.$$

After multiplying by a constant $a > 0$ we can assume:

$$\int_D (\rho|_D)^{n-1} = \int_D \hat{\Omega} \wedge \overline{\hat{\Omega}}.$$

So Ω is now uniquely determined. $\rho|_D$ produces a Kähler class on D, so by the Theorem of Yau [Y78] there is a C^∞-function $\varphi_0 : D \longrightarrow \mathbb{R}$ with

$$\left.\begin{array}{rcl} \rho|_D + i\partial\overline{\partial}\varphi_0 & > & 0 \\ \left(\rho|_D + i\partial\overline{\partial}\varphi_0\right)^{n-1} & = & \hat{\Omega} \wedge \overline{\hat{\Omega}} \end{array}\right\} \tag{4.2}$$

Now according to a result of Schumacher [Sch98, Thm 4] the metric $\|\cdot\|_D e^{-\varphi_0}$ for the restricted bundle $\mathcal{O}(D)|_D$ has an extension $\|\cdot\|e^{-\varphi}$, whose curvature is positive on all of X because $\mathcal{O}(D)$ is ample. So we replace the metric $\|\cdot\|$ by the new bundle metric

$$\|\cdot\|_\varphi := \|\cdot\|e^{-\varphi}$$

and set $\rho > 0$ as the curvature form of the new bundle metric. Then we have

$$\rho|_D^{n-1} = \hat{\Omega} \wedge \overline{\hat{\Omega}}.$$

Following [TY90], if $\|S\| < 1$, we define a complete Kähler metric on $X \setminus D$ by

$$\omega := i\frac{n}{n+1}\partial\overline{\partial}(-\log\|S\|^2)^{\frac{n+1}{n}}.$$

First we observe

$$\begin{aligned} \omega &= i\frac{n}{n+1}\partial\overline{\partial}\left(-\log\|S\|^2\right)^{1+\frac{1}{n}} \\ &= \left(-\log\|S\|^2\right)^{\frac{1}{n}}\rho + \frac{i}{n}\left(-\log\|S\|^2\right)^{\frac{1}{n}-1}\partial\log\|S\|^2 \wedge \overline{\partial}\log\|S\|^2 \\ &> 0. \end{aligned}$$

So ω defines a complete Kähler form on $X \setminus D$, if $\|S\| < 1$. But even if this condition is not satisfied everywhere, the form ω^n is still well-defined:

$$\omega^n = \left(-\log \|S\|^2\right) \rho^n + i\rho^{n-1} \wedge \partial \log \|S\|^2 \wedge \overline{\partial} \log \|S\|^2.$$

We are interested in the function

$$F := \frac{\Omega \wedge \overline{\Omega}}{\omega^n}.$$

In a small coordinate chart U such that $S(z) = z_1$ and there is a projection $\mathrm{pr} : U \longrightarrow U \cap D, (z_1, \ldots, z_n) \mapsto (0, z_2, \ldots, z_n)$ we define a holomorphic $(n-1)$-form on U by

$$\tilde{\Omega} := \mathrm{pr}^* \hat{\Omega}.$$

Expanding in this coordinate chart we recognise

$$\begin{aligned} \omega^n \;\; &= \;\; \frac{1}{|z_1|^2} \left(\tilde{\Omega} \wedge \overline{\tilde{\Omega}} \wedge dz_1 \wedge d\overline{z}_1 + (z_1 \tilde{\Omega} \wedge \overline{\tilde{\Omega}} \wedge \partial \log \| \cdot \|^2 \wedge d\overline{z}_1 + c.c.) \right) + \\ & \quad (-\log \|z_1\|^2) \rho^n + \tilde{\Omega} \wedge \overline{\tilde{\Omega}} \wedge \partial \log \| \cdot \|^2 \wedge \overline{\partial} \log \| \cdot \|^2 \end{aligned}$$

and

$$\Omega \wedge \overline{\Omega} = \frac{1}{|z_1|^2} \left(\tilde{\Omega} \wedge \overline{\tilde{\Omega}} \wedge dz_1 \wedge d\overline{z}_1 + z_1 \eta \wedge \overline{\tilde{\Omega}} \wedge d\overline{z}_1 + c.c. \right) + \eta \wedge \overline{\eta}$$

with a holomorphic n-form η on U satisfying $\Omega = \tilde{\Omega} \wedge \frac{dz_1}{z_1} + \eta$. All terms on the left hand side of ω^n and $\Omega \wedge \overline{\Omega}$ are absolutely integrable, except for the coinciding leading terms $\frac{1}{|z_1|^2} \tilde{\Omega} \wedge \overline{\tilde{\Omega}} \wedge dz_1 \wedge d\overline{z}_1$. Thus,

$$\int_X \left| \omega^n - \Omega \wedge \overline{\Omega} \right| < \infty.$$

If we replace φ and φ_0 by $\tilde{\varphi} := \varphi + C, \tilde{\varphi}_0 := \varphi_0 + C$ for a constant C, then $\tilde{\varphi}_0$ is still a solution to (4.2) and $\tilde{\varphi}$ an admissible extension of $\tilde{\varphi}_0$. For the norms we obtain

$$-\log \| \cdot \|^2_{\tilde{\varphi}} = -\log \| \cdot \|^2_{\varphi} + 2C,$$

so the corresponding curvature form $\tilde{\rho}$ coincides with ρ and for the Kähler form $\tilde{\omega}$ corresponding to $\| \cdot \|_{\tilde{\varphi}}$ we obtain

$$\tilde{\omega}^n = \omega^n + 2C\rho^n.$$

Thus, fixing $C := -(2 \int \rho^n)^{-1} \int (\omega^n - \Omega \wedge \overline{\Omega})$ we achieve

$$\int \tilde{\omega}^n - \Omega \wedge \overline{\Omega} = 0. \tag{4.3}$$

This condition is crucial for the existence result of [TY90]. In order to avoid unnecessarily rich notation we replace now ω by $\tilde{\omega}$ and φ by $\tilde{\varphi}$ in the following so as to get rid of the tilde.

But now we have still to ensure that $\|S(z)\| \leq \frac{1}{e}$ on all of X in order to obtain complete Kähler metric ω with volume-form ω^n. This we do by manipulating φ again. This time we replace φ by $\tilde{\varphi} := \varphi + C\|S\|_\varphi^4$ with a positive constant C and thus obtain a new metric $\|\cdot\|_{\tilde{\varphi}} := \|\cdot\|_\varphi e^{-C\|S\|_\varphi^4}$.

Using the formula for any $\alpha = \beta + \gamma$,

$$\begin{aligned}\left(\frac{n}{n+1}\right)^n \left((\partial\overline{\partial}\alpha^{\frac{n+1}{n}})^n \right. & - \left.(\partial\overline{\partial}\beta^{\frac{n+1}{n}})^n\right) = d\left(\sum_{k=1}^n \binom{n}{k}\beta\overline{\partial}\gamma \wedge (\partial\overline{\partial}\gamma)^{k-1} \wedge (\partial\overline{\partial}\beta)^{n-k}\right. \\ & + \binom{n}{k-1}\gamma\overline{\partial}\beta \wedge (\partial\overline{\partial}\gamma)^{k-1} \wedge (\partial\overline{\partial}\beta)^{n-k} \\ & + \binom{n-1}{k}\partial\beta \wedge \overline{\partial}\beta \wedge \overline{\partial}\gamma \wedge (\partial\overline{\partial}\gamma)^{k-1} \wedge (\partial\overline{\partial}\beta)^{n-k-1} \\ & \left. + \binom{n-1}{k-2}\partial\gamma \wedge \overline{\partial}\gamma \wedge \overline{\partial}\beta \wedge (\partial\overline{\partial}\gamma)^{k-2} \wedge (\partial\overline{\partial}\beta)^{n-k}\right),\end{aligned}$$

we obtain for $\alpha := -\log\|S\|_\varphi^2 + C\|S\|_\varphi^4$,

$$(\partial\overline{\partial}\alpha^{\frac{n+1}{n}})^n = (\partial\overline{\partial}(-\log\|S\|_\varphi^2)^{\frac{n+1}{n}})^n + d(\|S\|_\varphi^2\eta)$$

for an $(n-1,n)$-form η with at most logarithmic singularities along D. By Stokes' Theorem, the integral condition (4.3) still holds. Finally, a similar calculation shows that $\tilde{\omega}^n$ and $\Omega \wedge \overline{\Omega}$ are still asymptotically equivalent.

Since

$$i\partial\overline{\partial}\psi^k = k\psi^k i\partial\overline{\partial}\log\psi + k^2\psi^{k-2}i\partial\psi \wedge \overline{\partial}\psi$$

for any function $\psi \in C^\infty(X,\mathbb{R})$, we obtain for the curvature $\tilde{\rho}$ of $\|\cdot\|_{\tilde{\varphi}}$ by this formula with $\psi := \|S\|_\varphi^2$ and $k = 2$

$$\tilde{\rho} = \rho + Ci\partial\overline{\partial}\|S\|_\varphi^4 = (1 - 4C\|S\|_\varphi^4)\rho + 4Ci\partial\|S\|_\varphi^2 \wedge \overline{\partial}\|S\|_\varphi^2.$$

This form is surely positive, if for $A := \max_X \|S\|_\varphi$ and C holds

$$1 - 4CA^4 > 0.$$

So we want to determine $C \in [0, (4A^4)^{-1}]$ such that $\max_X \|S\|_{\tilde{\varphi}}$ is minimal. Surely,

$$\max_X \|S\|_{\tilde{\varphi}} = \max_X \|S\|_\varphi e^{-C\|S\|_\varphi^4} = \max_{z\in[0,A]} ze^{-Cz^4}.$$

For $z \in [0, A]$, the condition on C ensures that the function ze^{-Cz^4} is strictly increasing, so the maximum is attained at $z = A$, hence

$$\max_X \|S\|_{\tilde{\varphi}} = Ae^{-CA^4}.$$

Again this is minimised by the maximal possible C, i.e. $C = (4A^4)^{-1}$ and the corresponding new maximum is $\tilde{A} = Ae^{-\frac{1}{4}}$. Since this value of C may be forbidden by the positivity condition on $\tilde{\rho}$, we have to choose a slightly smaller C and in this way we can still obtain

$$\tilde{A} = Ae^{-\frac{1}{5}}.$$

This means that after a finite number of iterations of this step we can obtain an arbitrarily small maximum of $\|S\|$ while conserving the integral condition (4.3) and the positivity of the curvature of $\|\cdot\|$. The resulting metrics will be denoted $\|\cdot\|$ and ω in the following.

For later use we introduce the Riemannian manifolds

$$\tilde{Y}_R := \{x \in X \setminus D | \ -\log \|S(x)\|^2 = 2R\}$$

equipped with the Riemannian metric $g_R := i^*_{\tilde{Y}_R} \mathrm{Re}(g)$, if g denotes the Hermitian metric on $X \setminus D$ compatible with ω and the complex structure.

4.3 Quasi-Coordinates for ω

We denote for $r \in \mathbb{R}^k_+$

$$P_r := \{(z_1, \dots, z_n) \in \mathbb{C}^k | \ |z_i| < r_i\}.$$

the polycylinder with polradius r.

Let $\psi = (z_1, \dots, z_n) : U \longrightarrow \mathbb{C}^n$ be a coordinate chart intersecting D such that $z_1 = S(z)$ (after the choice of a trivialisation of $\mathcal{O}(D)$ on U) and $\psi(U) \supset P_{(\delta,1,\dots,1)}$. For $x \in P_\varepsilon \setminus \{0\} \subset \mathbb{C}$ for some ε dependent on δ we have quasi-coordinates given by

$$\psi_x : P_{(1,\dots,1)} \longrightarrow X \setminus D, (w_1, \dots, w_n) \mapsto \psi^{-1}(\exp(-\frac{1}{2}(w_1 + \sqrt{-2\log x})^2), w_2, \dots, w_n)$$

with $\psi_x(0, w_2, \dots, w_n) = (x, w_2, \dots, w_n)$. Here, of course, $\sqrt{-2\log x}$ means a choice of a complex number y such that $x = \exp(-\frac{y^2}{2})$ and $\mathrm{Im}(y^2) \in [0, 4\pi)$. We define $U_x := \psi_x(P_{(1,\dots,1)}) \subset U$.

Definition 4.3.1 *Riemannian metrics g_0, g_1 on a real manifold Z are called C^k-equivalent, if there are constants $c, C > 0$, such that on Z for all $h \in C^{k+1}(Z)$ holds*

$$c \sum_{1 \le l \le k+1} \|\nabla^l_{g_0} h\|_{g_0} \le \sum_{1 \le l \le k+1} \|\nabla^l_{g_1} h\|_{g_1} \le C \sum_{1 \le l \le k+1} \|\nabla^l_{g_0} h\|_{g_0}.$$

If Z is a complex manifold and g_0, g_1 are hermitian and Kähler, then g_0 and g_1 are called C^k-equivalent, if $\mathrm{Re}(g_0)$ *and* $\mathrm{Re}(g_1)$ *are.*

Note that the usual equivalence of metrics is the same as their C^0-equivalence.

Lemma 4.3.2 *Let g_0, g_1 be Riemannian metrics on a manifold Z. If g_0 and g_1 are C^0-equivalent and there is a constant $C > 0$ such that*

$$\|\nabla_{g_0}^l g_1\|_{g_0} \leq C$$

for all $1 \leq l \leq k$, then g_0 and g_1 are C^k-equivalent.

Proof. Let $g_t := (1-t)g + th$ and $\nabla_t, \|.\|_t$ denote the covariant derivative resp. the metric on any tensor space associated to g_t. First we note that for any smooth family of metrics g_t and $(0,s)$-tensor V on Z the map

$$V \mapsto \frac{d}{dt}(\nabla_t V)$$

induces a family of $(s, s+1)$-tensors $\gamma = \gamma(t)$ with components

$$\gamma^{i_1 \ldots i_s}_{j_1 \ldots j_{s+1}} = -\dot{\Gamma}^{i_1}_{j_1 j_{s+1}} \delta_{i_2 j_2} \cdots \delta_{i_s j_s} - \ldots - \dot{\Gamma}^{i_s}_{j_s j_{s+1}} \delta_{i_1 j_1} \cdots \delta_{i_{s-1} j_{s-1}},$$

with $\Gamma^i_{jk} = \Gamma^i_{jk}(t)$ being the Christoffel symbols of g_t. We write

$$\gamma \cdot V := \frac{d}{dt}(\nabla_t V)$$

for the application of the map to V. A computation in normal coordinates with respect to g_t confirms the relation

$$2g \cdot \gamma = \nabla_t \dot{g} + \sigma^* \nabla_t \dot{g} - \tau^* \nabla_t \dot{g}$$

with $g = g_t$ being the $(0, 2s)$-tensor induced by the metric g_t and σ, τ being the permutation of tensor components $\sigma = (12)$ resp. $\tau = (13)$. Hence, we can estimate

$$\|\nabla_t^l \gamma\|_0 \leq C \|\nabla_t^{l+1} \dot{g}\|_0.$$

We claim that our assumptions imply that there is a constant C such that

$$\|\nabla_t^l \dot{g}\|_0 \leq C \tag{4.4}$$

for all $0 \leq l \leq k, t \in [0,1]$ and $z \in Z$. Since $\dot{g} = g_1 - g_0$ and g_0 and g_1 are C^0-equivalent the claim is obviously valid for $k = 0$. For the inductive step we compute

$$\begin{aligned}
\frac{d}{dt}\|\nabla_t^{k+1}\dot{g}\| &\leq \|\frac{d}{dt}\nabla_t^{k+1}\dot{g}\| \\
&= \|\sum_{l=0}^{k}\nabla_t^l(\gamma \cdot \nabla_t^{k-l}\dot{g})\|_0 \\
&= \|\sum_{l=0}^{k}\sum_{m=0}^{l}\nabla_t^m\gamma \cdot \nabla_t^{k-m}\dot{g}\|_0 \\
&\leq C(\sum_{l=1}^{k+1}\|\nabla_t^l\dot{g}\|_0)(\sum_{l=0}^{k}\|\nabla_t^l\dot{g}\|_0) \\
&\leq C_1\|\nabla_t^{k+1}\dot{g}\|_0 + C_2
\end{aligned}$$

with constants C_1, C_2 independent of $z \in Z$. Integration of this differential inequality, together with the assumption that $\|\nabla_0^{k+1}\dot{g}\|_0 = \|\nabla_0^{k+1}g_1\|_0$ is uniformly bounded yields the claim.

Note that, in particular,

$$\|\nabla_1^l g_0\|_1 \leq C\|\nabla_1^l g_0\|_0 = C\|\nabla_1^l\dot{g}\|_0 \leq \tilde{C},$$

so the assumptions are symmetric with respect to exchanging g_0 and g_1. This means that we have to prove only

$$\sum_{1\leq l\leq k+1}\|\nabla_1^l h\|_1 \leq C\sum_{1\leq l\leq k+1}\|\nabla_0^l h\|_0$$

in order to prove C^k-equivalence.

So we claim that there exists $C > 0$ such that

$$\sum_{1\leq l\leq k+1}\|\nabla_t^l h\|_t \leq C\sum_{1\leq l\leq k+1}\|\nabla_0^l h\|_0$$

for all $t \in [0,1]$. For $k = 0$, this is surely true, since the covariant derivative of a function does not depend on the underlying metric and all the metrics g_t are C^0-equivalent. In order to do the inductive step we proceed like before and compute for $s \in [0,1]$

$$\begin{aligned} \|\nabla_s^{k+2} h\|_s &\leq C\|\nabla_0^{k+2} h\|_0 + \int_0^s \|\frac{d}{dt}\nabla_t^{k+2} h\|_0 dt \\ &= C\|\nabla_0^{k+2} h\|_0 + \int_0^s \|\sum_{l=0}^{k+1}\sum_{m=0}^{l} \nabla_t^m \gamma \cdot \nabla_t^{k+1-m} h\|_0 dt \\ &\leq C_1\|\nabla_0^{k+2} h\|_0 + C_2 \int_0^s \sum_{l=1}^{k+1} \|\nabla_t^l h\|_0 dt \\ &\leq C\sum_{l=1}^{k+2} \|\nabla_0^l h\|_0 \end{aligned}$$

with a constant C independent of s and $z \in Z$. Here we used the induction hypothesis and inequality 4.4. Clearly, this inequality implies the claim and thus the proof is complete. □

Proposition 4.3.3 *For the pullback metric we obtain*

$$\psi_x^* \omega = (-\log \|S(\psi_x(w))\|^2)^{\frac{1}{n}} \eta_x$$

for a metric η_x on $\prod_{i=1}^n \{|w_i| < 1\}$ uniformly C^k-equivalent (with respect to x) to the euclidean metric for any k.

Proof. Let us denote $h := \log \|\cdot\|^2$ and $\partial = \partial_1 + \partial'$. Note that

$$\frac{\partial h \circ \psi_x}{\partial w_1} = -z_1(w_1 + \sqrt{-2\log x})\frac{\partial h}{\partial z_1}$$

and

$$-c\log\|z_1\|^2 \leq |w_1 + \sqrt{-2\log x}|^2 \leq C_1|\mathrm{Re}(w_1 + \sqrt{-2\log x})^2| \leq -C_2\log\|z_1\|^2 \quad (4.5)$$

with constants c, C_1, C_2 independent of x.

Now we compute

$$\begin{aligned} -i\psi_x^*\omega &= dw_1 \wedge d\overline{w}_1 \left(\frac{1}{n}|z_1\partial_1 h - 1|^2 |w_1 + \sqrt{-2\log x}|^2 (-\log\|z_1\|^2)^{\frac{1}{n}-1} + \right. \\ & \left. \partial_1\overline{\partial}_1 h \cdot |z_1|^2 (-\log\|z_1\|^2)^{\frac{1}{n}} |w_1 + \sqrt{-2\log x}|^2\right) \\ & -\sum_{k=2}^n dw_1 \wedge d\overline{w}_k \left(\overline{\partial}_k h \cdot z_1(w_1 + \sqrt{-2\log x}) \left((-\log\|z_1\|^2)^{\frac{1}{n}} \partial_1\overline{\partial}_1 h \right.\right. \\ & \left.\left. + \ (-\log\|z_1\|^2)^{\frac{1}{n}-1}\partial_1 h\right)\right) + c.c. \\ & + (-\log\|z_1\|^2)^{\frac{1}{n}-1} \partial' h \wedge \overline{\partial}' h + (-\log\|z_1\|^2)^{\frac{1}{n}} \partial'\overline{\partial}' h. \end{aligned}$$

The C^0-equivalence follows now directly from the estimates 4.5: For $x \longrightarrow 0$ the metric η_x converges to the block metric $-i(dw_1 \wedge d\overline{w}_1 + \partial'\overline{\partial}' h)$, hence for δ small enough we obtain uniform equivalence of η_x to the euclidean metric.

In order to obtain C^k-equivalence, we note that it is enough to prove uniform boundedness (with respect to x) of the derivatives of $\psi_x^* \omega_{i\bar{j}}$ up to order k. In order to prove this we define $y := w_1 + \sqrt{-2\log x}$ and the ring

$$R := C^\infty(U, \mathbb{C})[\{(-\log \|z_1\|^2)^\alpha | \alpha \in \mathbb{R}\}, y, y^{-1}, \overline{y}, \overline{y}^{-1}, z_1, z_1^{-1}, \overline{z}_1, \overline{z}_1{}^{-1}].$$

If $P := b \cdot (-\log \|z_1\|^2)^{-d_1} y^{d_2} \overline{y}^{d_3} z_1^{d_4} \overline{z}_1^{d_5} \in R$ is a monomial we denote

$$d(P) := d_1 - \frac{1}{2}(d_2 + d_3) + \infty \cdot (d_4 + d_5)$$

and

$$R_\alpha := \operatorname{span}\{P \in R |\ P \text{ monomial}, d(P) \geq \alpha\}.$$

The elements of R_α decay at least like $(-\log \|z_1\|^2)^{-\alpha}$ for $x \longrightarrow 0$. We prove

Lemma 4.3.4 *$\frac{\partial}{\partial w_1} R_\alpha \subset R_{\alpha+\frac{1}{2}}$, $\frac{\partial}{\partial w_k} R_\alpha \subset R_\alpha$ for all $k = 2, \ldots, n$.*

Proof.

$$\begin{aligned}
\frac{\partial}{\partial w_1}(b \cdot (-\log \|z_1\|^2)^{-d_1} y^{d_2} \overline{y}^{d_3} z_1^{d_4} \overline{z}_1^{d_5}) &= -\partial_1 b \cdot (-\log \|z_1\|^2)^{-d_1} y^{d_2+1} \overline{y}^{d_3} z_1^{d_4+1} \overline{z}_1^{d_5} \\
&\quad -d_1(z_1 \partial_1 h + 1) b \cdot (-\log \|z_1\|^2)^{-(d_1+1)} \\
&\quad \cdot y^{d_2+1} \overline{y}^{d_3} z_1^{d_4} \overline{z}_1^{d_5} \\
&\quad +d_2 b \cdot (-\log \|z_1\|^2)^{-d_1} y^{d_2-1} \overline{y}^{d_3} z_1^{d_4} \overline{z}_1^{d_5} \\
&\quad -d_4 b \cdot (-\log \|z_1\|^2)^{-d_1} y^{d_2+1} \overline{y}^{d_3} z_1^{d_4} \overline{z}_1^{d_5} \\
&\in R_\infty + R_{\alpha+\frac{1}{2}} + R_{\alpha+\frac{1}{2}} + R_\infty = R_{\alpha+\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial}{\partial w_k}(b \cdot (-\log \|z_1\|^2)^{-d_1} y^{d_2} \overline{y}^{d_3} z_1^{d_4} \overline{z}_1^{d_5}) &= -\partial_k b \cdot (-\log \|z_1\|^2)^{-d_1} y^{d_2} \overline{y}^{d_3} z_1^{d_4} \overline{z}_1^{d_5} \\
&\quad +d_1 b \partial_k h \cdot (-\log \|z_1\|^2)^{-(d_1+1)} y^{d_2} \overline{y}^{d_3} z_1^{d_4} \overline{z}_1^{d_5} \\
&\in R_\alpha + R_{\alpha+1} = R_\alpha.
\end{aligned}$$

□

According to our calculations we have $\eta_{x,1\bar{1}} \in R_0, \eta_{x,k\bar{l}} \in R_0$ for $k, l = 2, \ldots, n$ and $\eta_{x,1\bar{k}} \in R_\infty$ for $k > 1$. So Lemma 4.3.4 implies that all derivatives of η_x are in R_0, hence uniformly bounded. □

Proposition 4.3.3 and Lemma 4.3.4 have immediate consequences on the Riemann curvature tensor and its derivatives.

Corollary 4.3.5 *There is a constant $C > 0$ such that*

$$\max\{\|R\|_{\omega(x)}, \|\nabla R\|_{\omega(x)}\}\} \leq C(-\log \|S(x)\|^2)^{-\frac{1}{n}}$$

for all $x \in X \setminus D$.

Finally, the description of $\Omega \wedge \Omega$ and ω^n of section 4.2 in connection with Lemma 4.3.4 allows to claim strong decay of f and its derivatives in quasi-coordinates.

Corollary 4.3.6 *For every $k \geq 0$ and $N > 0$ there is a constant $C > 0$ such that $|\nabla^k(f \circ \psi_x)| \leq C(-\log \|S(x)\|^2)^{-N}$.*

4.4 Strong Asymptotics for the perturbed equation

Let $f := \log F$. Using the bounds of the bisectional and the Ricci curvature proved in [TY90] by Cheng and Yau [CY80] the Monge-Ampère equation (4.1)

$$\frac{(\omega + i\partial\overline{\partial}u_\varepsilon)^n}{\omega^n} = e^{\varepsilon u_\varepsilon + f}$$

has a unique bounded C^∞-solution $u_\varepsilon : X \backslash D \longrightarrow \mathbb{R}$.

First we want to note that the results of [TY90], although not proving it directly, imply a posteriori that the solutions u_ε are uniformly bounded:

Lemma 4.4.1 *There exists a constant $C > 0$ such that for all $\varepsilon > 0$ the estimate*

$$\|u_\varepsilon\|_{C^3_\omega} \leq C$$

holds. In particular, there is a subsequence of u_ε converging in C^2_ω-sense.

Proof. Let u_ε be the unique bounded solution of

$$\frac{\det(\omega + i\partial\overline{\partial}u_\varepsilon)}{\det \omega} = e^{f + \varepsilon u_\varepsilon}, \omega + i\partial\overline{\partial}u_\varepsilon > 0$$

and u a bounded solution of

$$\frac{\det(\omega + i\partial\overline{\partial}u)}{\det \omega} = e^{f}, \omega + i\partial\overline{\partial}u > 0$$

like constructed in [TY90]. Let $\tilde{\omega} := \omega + i\partial\overline{\partial}u$ for the time being. We infer that $v_\varepsilon := u_\varepsilon - u$ satisfies the equation

$$\frac{\det(\tilde{\omega} + i\partial\overline{\partial}v_\varepsilon)}{\det \tilde{\omega}} = e^{\varepsilon u + \varepsilon v_\epsilon}, \tilde{\omega} + i\partial\overline{\partial}v_\varepsilon > 0.$$

By the maximum principle of [CY80] there are sequences $x_m, y_m \in X \setminus D$ such that

$$\begin{aligned} v_\varepsilon(x_m) &\longrightarrow \max v_\varepsilon, \quad \partial v_\varepsilon(x_m) \longrightarrow 0, \overline{\partial} v_\varepsilon(x_m) \longrightarrow 0, \quad \overline{\lim} i\partial\overline{\partial} v_\varepsilon(x_m) \leq 0 \\ v_\varepsilon(y_m) &\longrightarrow \min v_\varepsilon, \quad \partial v_\varepsilon(y_m) \longrightarrow 0, \overline{\partial} v_\varepsilon(y_m) \longrightarrow 0, \quad \overline{\lim} i\partial\overline{\partial} v_\varepsilon(y_m) \geq 0. \end{aligned}$$

The limit is taken with respect to $\tilde{\omega}$. We apply

Lemma 4.4.2 *For hermitian $n \times n$-matrices A, B with $A > 0$ we have:*

$$\begin{aligned} B \geq 0 \quad &\Rightarrow \quad \det(A+B) \geq \det A, \\ B \leq 0, A+B > 0 \quad &\Rightarrow \quad \det(A+B) \leq \det A. \end{aligned}$$

Proof. We have

$$\det(A+B) = \det A \det(\mathrm{Id} + A^{-\frac{1}{2}} B A^{-\frac{1}{2}}).$$

Now $A^{-\frac{1}{2}} B A^{-\frac{1}{2}}$ is still hermitian and so there is a diagonal matrix D and a unitary coordinate change U such that

$$A^{-\frac{1}{2}} B A^{-\frac{1}{2}} = U^* D U.$$

We can continue our calculation

$$\det(\mathrm{Id} + A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) = \det(UU^* + D) = \det(E + D).$$

We have $D \geq 0$ and $E + D > 0$ in the first case, so $\det(E+D) \geq 1$; in the second case, $D \leq 0$ and $E + D > 0$, so $\det(E+D) \leq 1$. □

According to the first claim of Lemma 4.4.2, in the points x_m we obtain

$$e^{\varepsilon(u+v_\varepsilon)} = \frac{\det(\tilde{\omega} + i\partial\overline{\partial} v_\varepsilon)}{\det \tilde{\omega}} \leq 1 + \varepsilon_m,$$

for s sequence $\varepsilon_m \longrightarrow 0$. hence $v_\varepsilon(x_m) \leq -u(x_m) + \varepsilon_m$, so in the limit

$$\max v_\varepsilon \leq \|u\|_{C^0}.$$

In the points y_m we obtain accordingly

$$e^{\varepsilon(u+v_\varepsilon)} = \frac{\det(\tilde{\omega} + i\partial\overline{\partial} v_\varepsilon)}{\det \tilde{\omega}} \geq 1 - \varepsilon_m,$$

hence $-v_\varepsilon(y_m) \leq u(y_m) + \varepsilon_m$, so in the limit

$$-\min v_\varepsilon \leq \|u\|_{C^0}.$$

The two inequalities combine to

$$\|v_\varepsilon\|_{C^0} \leq \|u\|_{C^0},$$

and hence

$$\|u_\varepsilon\|_{C^0} \leq 2\|u\|_{C^0}.$$

Now the estimates of the constant c satisfying

$$c^{-1}\omega \leq \omega + i\partial\overline{\partial}u_\varepsilon \leq c\omega$$

given in [CY80, Prop. 4.2, (4.9), (4.10)] only involve estimates of $\|u_\varepsilon\|_{C^0}, \|f\|_{C^0}$ and $\|\Delta_\omega f\|_{C^0}$. Using interpolation finally proves that $\|du_\varepsilon\|_\omega$ and $\|\partial\overline{\partial}u_\varepsilon\|_\omega$ are bounded uniformly in $0 < \varepsilon < 1$. The bounds for third derivatives are proved using the curvature estimates as it is done in [CY80].

□

Apart from this we can also prove a strong uniform decay property.

Theorem 4.4.3 *Let u_ε be the bounded solution of*

$$\frac{\det(\omega + i\partial\overline{\partial}u)}{\det \omega} = e^{f+\varepsilon u}.$$

Then for all $0 < p$ there exists $C > 0$ independent of ε such that

$$\sum_{i=0}^{3} \|\nabla^i u_\varepsilon\|_\omega \leq C\varepsilon^{-(1+np-\frac{p}{n+1})}(-\log\|S\|^2)^{-p}.$$

Proof. Let us choose $N > p+1+\frac{1}{n}$ and $C > 0$ such that $|f| \leq C(-\log\|S\|^2)^{-N}$. Consider for a $\beta > 0$ the function $v_{\beta\varepsilon}$ defined by

$$u_\varepsilon = v_{\beta\varepsilon}\left(\beta + (-\log\|S\|^2)^{-p}\right).$$

Proposition 4.4.4 *The map*

$$z \longmapsto \left(\beta + (-\log\|S\|^2)^{-p}\right)^{-1}$$

is of bounded geometry.

Proof. Quasicoordinates are given for $C \gg 1$ and $|w_1| \leq \pi$ by

$$w_1 \longmapsto e^{-((w_1+C)^{\frac{2n}{n+1}})} = z_1.$$

So if we denote by $h(w_1)$ the pull back of the above function we obtain

$$h(w_1) \cong \left(\beta + \left(2Re\left((w_1 + C)^{\frac{2n}{n+1}}\right)\right)^{-p}\right)^{-1}.$$

For C large enough we see that

$$\frac{1}{2\beta} \le |h(w_1)| \le \frac{3}{2\beta}$$

is bounded independent of C. Furthermore a calculation shows that we have bounds for all derivatives depending on β but not on $C \gg 1$ and w_1. $\square$

We conclude that $v_{\beta\varepsilon}$ is as a product of two functions of bounded geometry is also within this class and the weak maximum principle is applicable. We know that $|f| \le C(-\log\|S\|^2)^{-N}$ with N arbitrary. As a subsequent discussion shows we can assume that the C^0-bounded function $v_{\beta\varepsilon}$ attains its supremum as a maximum in the interior of $X\backslash D$ in a point $z_{\beta\varepsilon}$ and furthermore that $v_{\beta\varepsilon}(z_{\beta\varepsilon}) \ge 0$. Then we have $(\partial v_{\beta\varepsilon})(z_{\beta\varepsilon}) = (\overline{\partial} v_{\beta\varepsilon})(z_{\beta\varepsilon}) = 0$ and $(\partial\overline{\partial} v_{\beta\varepsilon})(z_{\beta\varepsilon}) \le 0$ negative semidefinite.

We evaluate the Monge-Ampère equation (4.1)

$$\frac{(\omega + i\partial\overline{\partial}u_\varepsilon)^n}{\omega^n} = e^{\varepsilon u_\varepsilon + f}$$

in the point $z_{\beta\varepsilon}$ and Lemma 4.4.2 yields

$$\begin{aligned}
\frac{(\omega + iv_{\beta\varepsilon}\partial\overline{\partial}(-\log\|S\|^2)^{-p})^n}{\omega^n} &\ge \frac{(\omega + i\partial\overline{\partial}u_\varepsilon)^n}{\omega^n} \\
&= e^{\varepsilon(\beta + (-\log\|S\|^2)^{-p})v_{\beta\varepsilon} + f} \\
&\ge 1 + \varepsilon(\beta + (-\log\|S\|^2)^{-p})v_{\beta\varepsilon} - \tilde{f},
\end{aligned}$$

where $|\tilde{f}| \le C(-\log\|S\|^2)^{-N}$. As before we calculate

$$\begin{aligned}
i\partial\overline{\partial}(-\log\|S\|^2)^{-p} &= -p\rho(-\log\|S\|^2)^{-p-1} + \\
&\quad + ip(p+1)\partial\log\|S\|^2 \wedge \overline{\partial}\log\|S\|^2(-\log\|S\|^2)^{-p-2}.
\end{aligned}$$

We realize

$$\Delta_\omega\left((-\log\|S\|^2)^{-p}\right) = (-\log\|S\|^2)^{-p-1-\frac{1}{n}}p(np+1) + O((-\log\|S\|^2)^{-p-2-\frac{1}{n}}$$

and

$$\left|\frac{\omega^{n-k} \wedge \left(\partial\overline{\partial}(-\log\|S\|^2)^{-p}\right)^k}{\omega^n}\right| \le C(-\log\|S\|^2)^{-k(p+1)-\frac{k}{n}}.$$

We have the trivial bound $|u_\varepsilon| \leq \varepsilon^{-1} \max |f| =: C\varepsilon^{-1}$. Then $|v_{\beta\varepsilon}|(-\log \|S\|^2)^{-p} \leq \frac{C}{\varepsilon}$ is also bounded independent of $\beta > 0$. We end up with

$$\frac{(\omega + iv_{\beta\varepsilon}\partial\overline{\partial}(-\log \|S\|^2)^{-p})^n}{\omega^n} = 1 + v_{\beta\varepsilon} p(np+1)(-\log \|S\|^2)^{-p-1-\frac{1}{n}} \left(1 + V_{\beta\varepsilon}(z)\right)$$

where $|V_{\beta\varepsilon}(z)| \leq \frac{C^{n-1}}{\varepsilon^{n-1}}(-\log \|S\|^2)^{-1}$ and this equality holds for all $z \in X \backslash D$. We combine the inequality coming from the Monge-Ampère equation (4.1) and the above equation to get (in the point $z_{\beta\varepsilon}$):

$$\begin{aligned} \tilde{f}(z_{\beta\varepsilon}) &\geq v_{\beta\varepsilon}(z_{\beta\varepsilon})(\varepsilon \left(\beta + (-\log \|S\|^2)^{-p}\right) - \\ &\quad -p(np+1)(-\log \|S\|^2)^{-p-1-\frac{1}{n}}(1 + V_{\beta\varepsilon}(z_{\beta\varepsilon}))). \end{aligned}$$

There are two cases possible:
In the first case we have (in the point $z_{\beta\varepsilon}$):

$$\varepsilon \left(\beta + (-\log \|S\|^2)^{-p}\right) \geq 2p(np+1)(-\log \|S\|^2)^{-p-1-\frac{1}{n}}(1 + \frac{C^{n-1}}{\varepsilon^{n-1}}(-\log \|S\|^2)^{-1}).$$

Then we conclude

$$\begin{aligned} v_{\beta\varepsilon}(z_{\beta\varepsilon}) &\leq \tilde{f}(z_{\beta\varepsilon}) \frac{1}{p(np+1)}(-\log \|S\|^2(z_{\beta\varepsilon}))^{p+1+\frac{1}{n}}(1 + \frac{C^{n-1}}{\varepsilon^{n-1}}(-\log \|S\|^2)^{-1})^{-1} \\ &\leq \tilde{C}(1 + \varepsilon^{n-1}) \end{aligned}$$

because $|\tilde{f}(z_{\beta\varepsilon})| \leq (-\log \|S\|^2(z_{\beta\varepsilon}))^{-N}$ for $0 < p \leq N - 1 - \frac{1}{n}$. So in this case we have an upper bound for $v_{\beta\varepsilon}$ independent of $\beta > 0$ satisfying even a better ε-behaviour than claimed.

In the complementary second case we get a lower bound for $(-\log \|S\|^2(z_{\beta\varepsilon}))^{-1}$:

$$\frac{1}{C^{n-1}}\varepsilon^{n-\frac{1}{n+1}} \leq (-\log \|S\|^2(z_{\beta\varepsilon}))^{-1}.$$

Now we have $u_\varepsilon(z_{\beta\varepsilon}) \leq \frac{C}{\varepsilon}$ and $v_{\beta\varepsilon}(z) \leq v_{\beta\varepsilon}(z_{\beta\varepsilon})$. This implies

$$u_\varepsilon(z) \left(\beta + (-\log \|S\|^2(z))^{-p}\right)^{-1} \leq \frac{C}{\varepsilon} \left(\beta + (-\log \|S\|^2(z_{\beta\varepsilon}))^{-p}\right)^{-1}$$

respectively

$$u_\varepsilon(z) \leq \frac{C}{\varepsilon} \frac{\beta + (-\log \|S\|^2(z))^{-p}}{\beta + (-\log \|S\|^2(z_{\beta\varepsilon}))^{-p}} < \frac{C}{\varepsilon} \frac{\beta + (-\log \|S\|^2(z))^{-p}}{\beta + \left(\frac{1}{C^{n-1}}\varepsilon^{n-\frac{1}{n+1}}\right)^p}.$$

Now we let $\beta \to 0$ tend to 0 and obtain

$$u_\varepsilon(z) \le C^{1+p(n-1)}\varepsilon^{-(1+np-\frac{p}{n+1})}(-\log\|S\|^2(z))^{-p}.$$

The same line of thought applies to the minimum of $v_{\beta\varepsilon}(z)$ so we get also lower bounds. The estimates of the derivatives are provided in the same way as explained in the proof of Lemma 4.4.1.

Now we want to discuss the case that $v_{\beta\varepsilon}(z)$ has no maximum in $X\backslash D$ but takes its supremum on D. So we assume $0 < \sup\{v_{\beta\varepsilon}(z)\} = L < \infty$ and $v_{\beta\varepsilon}(z) < L$ for all $z \in X\backslash D$. According to Cheng and Yau [CY80] we have a sequence of points $z_m \in X\backslash D$, $m \in \mathbb{N}$ with the following properties: For $m \longrightarrow \infty$

(i) $v_{\beta\varepsilon}(z_m) \to L$,

(ii) $\partial v_{\beta\varepsilon}(z_m) \to 0$ and $\overline{\partial} v_{\beta\varepsilon}(z_m) \to 0$,

(iii) $\overline{\lim}\partial\overline{\partial} v_{\beta\varepsilon}(z_m) \le 0$ is negative semidefinite.

We put $\breve{\omega} = \omega + \partial\overline{\partial}u_\varepsilon$ and $\hat{\omega} = \breve{\omega} - v_{\beta\varepsilon}\partial\overline{\partial}(-\log\|S\|^2(z))^{-p}$. By Cheng and Yau [CY80] we have C^2-estimates $\frac{1}{\hat{C}_\varepsilon}\omega \le \breve{\omega} \le \hat{C}_\varepsilon\omega$. Furthermore from $|v_{\beta\varepsilon}|(-\log\|S\|^2(z))^{-p} \le C_\varepsilon$ and from the asymptotics of $\partial\overline{\partial}(-\log\|S\|^2(z))^{-p}$ we conclude an estimate $\frac{1}{\tilde{C}_\varepsilon}\omega \le \hat{\omega} \le \tilde{C}_\varepsilon\omega$. We develop as before

$$\begin{aligned}
\breve{\omega}^n &= \left(\hat{\omega} + iv_{\beta\varepsilon}\partial\overline{\partial}(-\log\|S\|^2(z))^{-p}\right)^n \\
&= \hat{\omega}^n + \sum_{j=1}^{n}\begin{pmatrix} n \\ j \end{pmatrix}\hat{\omega}^{n-j}\wedge v_{\beta\varepsilon}^j\left(i\partial\overline{\partial}(-\log\|S\|^2(z))^{-p}\right)^j.
\end{aligned}$$

Looking at the asymptotics we see

$$\left|\frac{1}{\omega^n}\sum_{j=1}^{n}\begin{pmatrix} n \\ j \end{pmatrix}\hat{\omega}^{n-j}\wedge v_{\beta\varepsilon}^j\left(\partial\overline{\partial}(-\log\|S\|^2(z))^{-p}\right)^j\right| \le \tilde{C}_\varepsilon(-\log\|S\|^2(z))^{-p-1-\frac{1}{n}}.$$

For $m \ge m_0$ large we have $z_m \in U_\varepsilon(D)$ and $v_{\beta\varepsilon}(z_m) > 0$. Then the Monge-Ampère equation (4.1) implies

$$\begin{aligned}
\frac{\hat{\omega}^n}{\omega^n}(z_m) \;\ge\;& 1 - |f(z_m)| + v_{\beta\varepsilon}(z_m)\cdot \\
&\left(\varepsilon(\beta + (-\log\|S\|^2(z_m))^{-p}) - \tilde{C}_\varepsilon(-\log\|S\|^2(z_m))^{-p-1-\frac{1}{n}}\right)
\end{aligned}$$

or, respectively,

$$\begin{aligned}
&\left(\varepsilon(\beta + (-\log\|S\|^2(z_m))^{-p}) - \tilde{C}_\varepsilon(-\log\|S\|^2(z_m))^{-p-1-\frac{1}{n}}\right)^{-1}\left(\frac{\hat{\omega}^n}{\omega^n}(z_m) - 1\right) + \\
&\qquad |f(z_m)|\left(\varepsilon(\beta + (-\log\|S\|^2(z_m))^{-p}) - \tilde{C}_\varepsilon(-\log\|S\|^2(z_m))^{-p-1-\frac{1}{n}}\right)^{-1} \;\ge\; v_{\beta\varepsilon}(z_m).
\end{aligned}$$

Now we hold ε,$\beta > 0$ fix and take the limit $m \to \infty$:

$$\begin{array}{rcl} v_{\beta\varepsilon}(z_m) & \to & L > 0 \\ \varepsilon(\beta + (-\log\|S\|^2(z_m))^{-p}) - \tilde{C}_\varepsilon(-\log\|S\|^2(z_m))^{-p-1-\frac{1}{n}} & \to & \varepsilon\beta \\ |f(z_m)| & \to & 0 \\ \overline{\lim}\left(\frac{\hat{\omega}^n}{\omega^n}(z_m) - 1\right) & \leq & 0. \end{array}$$

So we conclude:

$$0 \geq \overline{\lim}\left(\frac{\hat{\omega}^n}{\omega^n}(z_m) - 1\right) \geq \varepsilon\beta L > 0,$$

which is a contradiction, so there must be a maximum inside of $X \backslash D$. □

4.5 Slow decay of the solution

Now we study the original Monge-Ampère equation (4.1) for our problem

$$\frac{(\omega + i\partial\overline{\partial}u)^n}{\omega^n} = e^f.$$

By [TY90] there exists a unique bounded C^2-solution of the problem such that $\tilde{\omega} := \omega + i\partial\overline{\partial}u > 0$. We want to study the decay of $i\partial\overline{\partial}u$ towards D.

Unfortunately, the estimate of Proposition 4.4.3 is not uniform in ε, so we have to add arguments in order to achieve our result. Our decay rate here is much smaller than the one of u_ε provided by Theorem 4.4.3. We begin with the preparation of elementary tools.

Lemma 4.5.1 *There are constants $C_n > 0$ such that for all positive real numbers $a_1, \ldots, a_n$ and $0 < \varepsilon < 1$ satisfying*

$$\sum_{i=1}^{n} a_i \leq n(1+\varepsilon), \prod_{i=1}^{n} a_i = 1,$$

holds: $(1 + C_n\sqrt{\varepsilon})^{-1} \leq a_i \leq 1 + C_n\sqrt{\varepsilon}$ for every i.

Proof. This is easily seen by the inequality

$$\sum_{i=1}^{n}(\sqrt{a_i} - 1)^2 + 2(\sum_{i=1}^{n}\sqrt{a_i} - n(\prod_{j=1}^{n} a_j)^{\frac{1}{2n}}) \leq n\varepsilon.$$

□

As an immediate consequence we obtain that decay of $\Delta_\omega u$ is enough in order to have decay of $\|\partial\overline{\partial}u\|_\omega$:

Lemma 4.5.2 *Let $C > 0$. If $\Delta_\omega u \leq C(-\log \|S\|^2)^{-\beta}$, then*

$$(1 - C(-\log \|S\|^2)^{-\frac{\beta}{2}})\omega \leq \omega + i\partial\overline{\partial}u \leq (1 + C(-\log \|S\|^2)^{-\frac{\beta}{2}})\omega.$$

Proof. Let us choose a point $x \in X$ and coordinates such that in $\omega(x) = \delta_{ij}, \partial\overline{\partial}u(x) = u_{ij}\delta_{ij}$. Let us denote $a_i := 1 + u_{,i\bar{i}}$. By $\omega + i\partial\overline{\partial}u > 0$ we know $a_i > 0$. Furthermore,

$$\prod a_i = e^f = 1 + O((-\log \|S\|^2)^{-N}), \sum a_i = n + \Delta_\omega u \leq n + C(-\log \|S\|^2)^{-\beta}.$$

So we can change a_i in such a way that $b_i := a_i(1 + O((-\log \|S\|^2)^{-N}))$ satisfies $\prod b_i = 1$. Of course, still $\sum b_i \leq n + C(-\log \|S\|^2)^{-\beta}$. Now we apply Lemma 4.5.1 and obtain

$$u_{,i\bar{i}} = O((-\log \|S\|^2)^{-\frac{\beta}{2}}).$$

This implies the claim. □

Lemma 4.5.3 *If $A \in M(n, \mathbb{C})$ is a hermitian matrix and $\|A\| := \sup_{x \in \mathbb{C}^n} \frac{|(x, Ax)|}{\|x\|^2}$, then the estimate holds $|a_{ij}| \leq n\|A\|$ for any i, j.*

Theorem 4.5.4 $\|u\|_{C^{2,\alpha}_\omega(U_x)} \leq C(-\log \|S(x)\|^2)^{-\frac{1}{6n}})$ *for some $\alpha > 0$, in particular*

$$(1 - C(-\log \|S\|^2)^{-\frac{1}{6n}})\omega \leq \omega + i\partial\overline{\partial}u \leq (1 + C(-\log \|S\|^2)^{-\frac{1}{6n}})\omega.$$

Proof. Recall that u is the bounded solution of

$$\frac{\det(\omega + i\partial\overline{\partial}u)}{\det \omega} = e^f$$

and the curvature tensor is given by

$$R_{i\bar{j}k\bar{l}} = \partial_i\overline{\partial}_j\omega_{kl} - \omega^{pq}\partial_i\omega_{kq}\overline{\partial}_j\omega_{pl}.$$

By a computation of [TY90],

$$\|R_{i\bar{j}}^{k\bar{l}}\|_\omega = \|R_{i\bar{j}k\bar{l}}\|_\omega \leq C(-\log \|S\|^2)^{-\frac{1}{n}}.$$

Here $\|R_{i\bar{j}k\bar{l}}\|_\omega$ is the norm of the bisectional curvature $\sup \frac{|R_{i\bar{j}k\bar{l}}\zeta^i\overline{\zeta}^j\xi^k\overline{\xi}^l|}{g(\zeta,\zeta)g(\xi,\xi)}$. Now we choose coordinates centered around x such that $\omega_{ij}(x) = \delta_{ij}, \omega_{ij,k}(x) = 0$. For $\xi = e_k$ the estimate of the bisectional curvature yields

$$\|(R_{i\bar{j}k\bar{k}})_{ij}\| \leq C(-\log \|S\|^2)^{-\frac{1}{n}}$$

in the sense of Lemma 4.5.3. The matrix $(\sqrt{-1}R_{i\bar{j}k\bar{k}})_{ij}$ is hermitian, indeed, hence

$$|R_{i\bar{j}k\bar{k}}| \le C(-\log\|S\|^2)^{-\frac{1}{n}} \tag{4.6}$$

for any i, j, k.

Further, we obtain in x

$$\begin{aligned}
\Delta_{\tilde{\omega}}\Delta_\omega u &= \tilde{\omega}^{ij}\partial_i\overline{\partial}_j(\omega^{kl}\partial_k\overline{\partial}_l u)\\
&= \tilde{\omega}^{ij}\partial_i\overline{\partial}_j(\omega^{kl}(\tilde{\omega}_{kl}-\omega_{kl}))\\
&= \tilde{\omega}^{ij}(\tilde{\omega}_{kl}-\omega_{kl})\partial_i\overline{\partial}_j\omega^{kl} + \tilde{\omega}^{ij}\omega^{kl}\partial_i\overline{\partial}_j\partial_k\overline{\partial}_l u\\
&= \tilde{\omega}^{ij}R^{k\bar{l}}_{i\bar{j}}\tilde{\omega}_{kl} - \tilde{\omega}^{ij}R_{ij} + \tilde{\omega}^{ij}\omega^{kl}\partial_i\overline{\partial}_j\partial_k\overline{\partial}_l u
\end{aligned}$$

by the choice of the coordinates. Now we have to replace the fourth derivatives of u. Since $\tilde{\omega}$ is Ricci-flat, we may use the equations

$$0 = \tilde{R}_{k\bar{l}} = \partial_k\overline{\partial}_l \log\det(\omega + i\partial\overline{\partial}u) = \partial_k\tilde{\omega}^{i\bar{j}}\overline{\partial}_l\partial_i\overline{\partial}_j u + \tilde{\omega}^{i\bar{j}}(\partial_k\overline{\partial}_l\omega_{i\bar{j}} + \partial_i\overline{\partial}_j\partial_k\overline{\partial}_l u).$$

Now we substitute $\partial_k\tilde{\omega}^{i\bar{j}} = -\tilde{\omega}^{i\bar{m}}\tilde{\omega}^{n\bar{j}}\partial_k\overline{\partial}_m\partial_n u$ and obtain for any k, l

$$\tilde{\omega}^{i\bar{j}}\partial_i\overline{\partial}_j\partial_k\overline{\partial}_l u = \tilde{\omega}^{i\bar{m}}\tilde{\omega}^{n\bar{j}}\overline{\partial}_l\partial_i\overline{\partial}_j u\cdot\partial_k\overline{\partial}_m\partial_n u - \tilde{\omega}^{i\bar{j}}R_{k\bar{l}i\bar{j}}.$$

So,

$$\Delta_{\tilde{\omega}}\Delta_\omega u = \tilde{\omega}^{ij}R^{k\bar{l}}_{i\bar{j}}\tilde{\omega}_{kl} - \omega^{k\bar{l}}\tilde{\omega}^{i\bar{j}}R_{k\bar{l}i\bar{j}} - \tilde{\omega}^{ij}R_{ij} + \omega^{k\bar{l}}\tilde{\omega}^{i\bar{m}}\tilde{\omega}^{n\bar{j}}\overline{\partial}_l\partial_i\overline{\partial}_j u\cdot\partial_k\overline{\partial}_m\partial_n u.$$

From now on we specialise our coordinates further in such a way that $\partial_i\overline{\partial}_j u = \lambda_i\delta_{ij}$ in x; this is possible by a locally constant unitary coordinate transformation. The C^2-estimate of [TY90] yield, that $0 < c \le 1+\lambda_i \le C$ with constants c, C independent of x. The first two terms are now easily dealt with by (4.6). Observe that f is explicitly computable in our case. In particular

$$|R_{ij}| = |\partial_i\overline{\partial}_j f| \le C(-\log\|S\|^2)^{-N}.$$

So the only mysterious term is

$$\begin{aligned}
a &:= \omega^{k\bar{l}}\tilde{\omega}^{i\bar{m}}\tilde{\omega}^{n\bar{j}}\overline{\partial}_l\partial_i\overline{\partial}_j u\cdot\partial_k\overline{\partial}_m\partial_n u\\
&= \sum_{i,j,k}(1+\lambda_i)^{-1}(1+\lambda_j)^{-1}|\overline{\partial}_k\partial_i\overline{\partial}_j u|^2
\end{aligned}$$

Fortunately, $a \ge 0$ and we will use this to our advantage in the computation

$$\begin{aligned}
\Delta_{\tilde{\omega}}((-\log\|S\|^2)^p\Delta_\omega u) &= (-\log\|S\|^2)^p\Delta_{\tilde{\omega}}\Delta_\omega u + \Delta_{\tilde{\omega}}(-\log\|S\|^2)^p\Delta_\omega u +\\
&\quad +2(\nabla_{\tilde{\omega}}(-\log\|S\|^2)^p, \nabla_{\tilde{\omega}}\Delta_\omega u)_{\tilde{\omega}}.
\end{aligned}$$

Like above, we have the estimates

$$|\Delta_{\tilde\omega}(-\log\|S\|^2)^\gamma| \le (-\log\|S\|^2)^{\gamma-1-\frac{1}{n}}, \|\nabla_{\tilde\omega}(-\log\|S\|^2)^\gamma\|^2 \le (-\log\|S\|^2)^{2\gamma-1-\frac{1}{n}}.$$

In order to make use of Cauchy-Schwarz we compute

$$\begin{aligned}
\|\nabla_{\tilde\omega}\Delta_\omega u\|^2_{\tilde\omega} &= \|d\Delta_\omega u\|^2_{\tilde\omega}\\
&= \tilde\omega^{i\bar j}\partial_i\Delta_\omega u\overline{\partial}_j\Delta_\omega u\\
&= \tilde\omega^{i\bar j}\omega^{k\bar l}\omega^{m\bar n}\partial_i\partial_k\overline{\partial}_l u\cdot\overline{\partial}_j\partial_m\overline{\partial}_n u\\
&= \sum_{i,k,m}(1+\lambda_i)^{-1}\partial_i\partial_k\overline{\partial}_k u\cdot\partial_i\partial_m\overline{\partial}_m u\\
&\le \frac{1}{2}\sum_{i,k,m}(1+\lambda_i)^{-1}(|\partial_i\partial_k\overline{\partial}_k u|^2+|\partial_i\partial_m\overline{\partial}_m u|^2)\\
&= \sum_{i,k}(1+\lambda_i)^{-1}|\partial_i\partial_k\overline{\partial}_k u|^2
\end{aligned}$$

The coefficient of $|\partial_i\partial_k\overline{\partial}_k u|^2$ in a is $(1+\lambda_k)^{-1}((1+\lambda_i)^{-1}+(1+\lambda_k)^{-1})$. Hence, for some constant D depending on c and C we obtain

$$\|\nabla_{\tilde\omega}\Delta_\omega u\|_{\tilde\omega} \le D\sqrt{a}.$$

Now this yields together with Cauchy-Schwarz

$$\begin{aligned}
\Delta_{\tilde\omega}((-\log\|S\|^2)^p\Delta_\omega u) \;\ge\;& (-\log\|S\|^2)^p a - (-\log\|S\|^2)^{p-\frac{1}{n}} - (-\log\|S\|^2)^{-N} -\\
&-(-\log\|S\|^2)^{p-1-\frac{1}{n}} - (-\log\|S\|^2)^{p-\frac{1}{2}-\frac{1}{2n}}\sqrt{a}.
\end{aligned}$$

The sum of a-terms has a minimum as a function of a. This is attained at $a = O((-\log\|S\|^2)^{-1-\frac{1}{n}})$ and hence

$$\Delta_{\tilde\omega}((-\log\|S\|^2)^p\Delta_\omega u) \ge -(-\log\|S\|^2)^{p-\frac{1}{n}}.$$

Since $\Delta_{\tilde\omega}u = \sum\frac{\lambda_i}{1+\lambda_i} = n - \sum\frac{1}{1+\lambda_i}$,

$$\Delta_{\tilde\omega}((-\log\|S\|^2)^p\Delta_\omega u - u) \ge -(-\log\|S\|^2)^{p-\frac{1}{n}} - n + \sum\frac{1}{1+\lambda_i}.$$

Now we would like to choose x as a maximum of $(-\log\|S\|^2)^p\Delta_\omega u - u$ but, of course, this would assume what we would like to prove. So we do the same considerations for u_ε instead of u. We denote $\omega_\varepsilon = \omega + i\partial\overline{\partial}u_\varepsilon$. All computations are the same except that $R^{(\varepsilon)}_{i\bar j} \neq 0$ now, but

$$\mathrm{Ric}(\omega_\varepsilon) = \mathrm{Ric}\omega + i\partial\overline{\partial}(f+\varepsilon u_\varepsilon) = i\varepsilon\partial\overline{\partial}u_\varepsilon.$$

So, by Lemma 4.4.1 we get $\|\mathrm{Ric}(\omega_\varepsilon)\|_\omega \leq C\varepsilon$. On the other hand, by Proposition 4.4.3 we know $\|\mathrm{Ric}\omega_\varepsilon\|_\omega \leq C\varepsilon^{-(n\alpha-\frac{\alpha}{n+1})}(-\log\|S\|^2)^{-\alpha}$. We combine both inequalities to obtain

$$\|\mathrm{Ric}(\omega_\varepsilon)\|_\omega \leq C(-\log\|S\|^2)^{-\frac{\alpha}{1+\alpha(n-\frac{1}{n+1})}} \leq C(-\log\|S\|^2)^{-\frac{1}{n}}$$

for $\alpha \geq n+1$ and with a constant C independent of ε. Hence we obtain, in particular,

$$\Delta_{\tilde{\omega}}((-\log\|S\|^2)^p\Delta_\omega u_\varepsilon - u_\varepsilon) \geq -C(-\log\|S\|^2)^{p-\frac{1}{n}} - n + \sum \frac{1}{1+\lambda_i^{(\varepsilon)}}$$

with C independent of ε! By the decay property of u_ε and $\Delta_\omega u_\varepsilon$ we can find a point $x_\varepsilon \in X$ where $(-\log\|S\|^2)^p\Delta_\omega u_\varepsilon - u_\varepsilon$ attains its maximum. In x_ε we obtain

$$C(-\log\|S\|^2)^{p-\frac{1}{n}} + n \geq \sum \frac{1}{1+\lambda_i^{(\varepsilon)}}.$$

We know

$$\prod \frac{1}{1+\lambda_i^{(\varepsilon)}} = e^{-(f+\varepsilon u_\varepsilon)}$$

and again we combine the estimate $|\varepsilon u_\varepsilon| \leq C\varepsilon$ from Lemma 4.4.1 and

$$|\varepsilon u_\varepsilon| \leq C\varepsilon^{-(n\alpha-\frac{\alpha}{n+1})}(-\log\|S\|^2)^{-\alpha}$$

by an adequate multiplication to obtain

$$|\varepsilon u_\varepsilon| \leq C(-\log\|S\|^2)^{-\frac{\alpha}{1+\alpha(n-\frac{1}{n+1})}} \leq C(-\log\|S\|^2)^{-\frac{1}{n}}$$

with C independent of ε. Now by Lemma 4.5.1 we obtain for

$$q := Ce^{\frac{1}{n}(f+\varepsilon u_\varepsilon)}(-\log\|S\|^2)^{p-\frac{1}{n}} + e^{\frac{1}{n}(f+\varepsilon u_\varepsilon)} - 1$$

the inequalities

$$(1+C'\sqrt{q})^{-1} \leq \frac{1}{1+\lambda_i^{(\varepsilon)}} \leq 1 + C'\sqrt{q}.$$

So

$$\lambda_i^{(\varepsilon)} \leq C'\sqrt{q} \leq \tilde{C}(-\log\|S\|^2)^{\frac{1}{2}(p-\frac{1}{n})},$$

still only in the point x_ε. In particular, if we assume $p = -\frac{1}{2}(p-\frac{1}{n})$, i.e. $p = \frac{1}{3n}$ then for any $x \in X$

$$(-\log\|S\|^2)^p\Delta_\omega u_\varepsilon - u_\varepsilon \leq C,$$

since u_ε is uniformly bounded. We obtain also in the limit $\varepsilon \longrightarrow 0$

$$\Delta_\omega u \leq C(-\log\|S\|^2)^{-p}.$$

Now we apply Lemma 4.5.2 and C^3-estimates like given in [CY80] in order to obtain the desired result with decay rate $\frac{1}{2}p = \frac{1}{6n}$. Here the curvature estimate of Corollary 4.3.5 comes in. $\square$

4.6 Estimates of higher derivatives of the solution in Quasi-Coordinates

We make use of the quasi-coordinates ψ_x constructed in section 4.3 and denote $P := P_{(1,\dots,1)}$. First we want to find a comparison between Hölder estimates of derivatives in the euclidean and the ω-metric in quasi-coordinates. Let $U_x := \psi_x(P)$. The $C^{k,\alpha}_\omega$ metric on U_x is defined

$$\|v\|_{k,\alpha;\omega} := \sum_{1\le j\le k} \sup_{z\in U_x} \|\nabla^j_\omega v(z)\|_\omega + \sup_{y,z\in U_x} \frac{\|\nabla^k_\omega v(y) - \tau_{y,z}\nabla^k_\omega v(z)\|_\omega}{\mathrm{dist}_\omega(y,z)^\alpha},$$

where $\tau_{y,z}$ denotes the parallel transport from z to y along a shortest geodesic minimising the expression. Like in the euclidean case, this definition is compatible with the C^k norms, i.e. for all $k, l \in \mathbb{N}_0, \alpha, \beta \in [0,1]$ with $k+\alpha \le l+\beta$ there is a constant C such that

$$\|v\|_{k,\alpha;\omega} \le C\|v\|_{l,\beta;\omega}.$$

for all $v \in C^{l,\beta}_\omega$.

Lemma 4.6.1 *There is a constant $C = C(k,\alpha)$ independent of x such that*

$$\|v\|_{C^{k,\alpha}(P)} \le C(-\log\|S(x)\|)^{\frac{k+\alpha}{2n}} \|v\|_{C^{k,\alpha}_{\psi_x^*\omega}(P)}$$

for all $v \in C^{k,\alpha}(P)$.

Proof. First we see by Lemma 4.3.4 and C^1-equivalence of η to the euclidean metric that the Christoffel symbols of $\mathrm{Re}(\psi^*\omega)$

$$\Gamma^i_{jk} = \frac{1}{2} g^{im}(g_{mj,k} + g_{mk,j} - g_{jk,m})$$

are uniformly bounded.

Let γ denote a geodesic from z to y with $\gamma(0) = z, \gamma(\tau) = y$ and $\|\dot\gamma\|_\omega = 1$ (so $\tau = \mathrm{dist}(y,z)$) and X the $(k,0)$-tensor field along γ obtained by parallel transport of $\nabla^k v(z)$, i.e. X satisfies the differential equation

$$\frac{d}{dt} X_{i_1\dots i_k}(t) = \Gamma^j_{i_1 l} X_{j i_2\dots i_k}\dot\gamma^l + \dots + \Gamma^j_{i_k l} X_{i_1\dots i_{k-1} j}\dot\gamma^l.$$

The condition $\|\dot\gamma\| = 1$ translates to $|\dot\gamma| \sim (-\log\|S(x)\|)^{-\frac{1}{2n}}$ for the euclidean length. If we denote by $A(t)$ the (k,k)-tensor on P such that

$$\frac{d}{dt} X(t) = A(t)X(t),$$

then we see that

$$|A(t)| \sim (-\log \|S(x)\|)^{-\frac{1}{2n}}.$$

Standard integration and application of Gronwall's Lemma implies

$$|X(t) - X(0)| \leq |X(0)| t \sup_{0\leq s\leq t} |A(s)| e^{t \sup_{0\leq s\leq t} |A(s)|}.$$

For the distance we also know $\mathrm{dist}_\omega(y,z) \sim (-\log \|S(x)\|)^{\frac{1}{2n}} |z-y|$, so we obtain at $t = \tau$

$$|\tau_{y,z} \nabla^k_{\psi^*\omega} v(z) - \nabla^k_{\psi^*\omega} v(z)| \leq C |\nabla^k_{\psi^*\omega} v(z)| \mathrm{dist}_{\psi^*\omega}(y,z) \cdot (-\log \|S(x)\|)^{-\frac{1}{2n}}.$$

Furthermore we can write

$$\nabla^k_{\psi^*\omega} v(z) = \nabla^k v(z) + \sum_{i=1}^{k-1} S_i \nabla^i v(z)$$

for (euclidean) bounded $(k+1,i)$-tensors S_i. This implies immediately

$$\|\nabla^k_{\psi^*\omega} v(z)\|_{\psi^*\omega(z)} \geq (-\log \|S(x)\|)^{-\frac{k}{2n}} (C_1 |\nabla^k v(z)| - C_2 \|v\|_{k-1}),$$

hence

$$\|v\|_k \leq C((-\log \|S(x)\|)^{\frac{k}{2n}} \|v\|_{k;\psi^*\omega} + \|v\|_{k-1}).$$

The claim of the Lemma being true for $k = \alpha = 0$, we obtain the case $\alpha = 0$ by induction in this way.

Now we can estimate

$$\begin{aligned}
\|\nabla^k_{\psi^*\omega} v(y) - \tau_{y,z} \nabla^k_{\psi^*\omega} v(z)\|_{\psi^*\omega(y)} \geq\ & \|\nabla^k_{\psi^*\omega} v(y) - \nabla^k_{\psi^*\omega} v(z)\|_{\psi^*\omega(y)} \\
& - \|\nabla^k_{\psi^*\omega} v(z) - \tau_{y,z} \nabla^k_{\psi^*\omega} v(z)\|_{\psi^*\omega(y)} \\
\geq\ & \|\nabla^k v(y) - \nabla^k v(z)\|_{\psi^*\omega(y)} \\
& - \sum_{i=1}^{k-1} \|S_i(y) \nabla^i v(y) - S_i(z) \nabla^i v(z)\|_{\psi^*\omega(y)} \\
& - \|\nabla^k_{\psi^*\omega} v(z) - \tau_{y,z} \nabla^k_{\psi^*\omega} v(z)\|_{\psi^*\omega(y)} \\
\geq\ & C_1 (-\log \|S(x)\|)^{-\frac{k}{2n}} |\nabla^k v(y) - \nabla^k v(z)| - \\
& C_2 (-\log \|S(x)\|)^{-\frac{k}{2n}} |y-z| \|v\|_k \\
& - C_3 (-\log \|S(x)\|)^{-\frac{k}{2n}} |y-z| \cdot |\nabla^k v(z)|.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\frac{\|\nabla^k_{\psi^*\omega} v(y) - \tau_{y,z} \nabla^k_{\psi^*\omega} v(z)\|_{\psi^*\omega(y)}}{\mathrm{dist}_{\psi^*\omega}(y,z)^\alpha} \geq\ & C_4 (-\log \|S(x)\|)^{-\frac{k+\alpha}{2n}} \cdot \\
& \cdot \left(\frac{|\nabla^k v(y) - \nabla^k v(z)|}{|y-z|^\alpha} - C_5 \|v\|_k \right).
\end{aligned}$$

Employing the estimate with $\alpha = 0$ we obtain

$$\|v\|_{k,\alpha} \leq C(-\log\|S(x)\|)^{\frac{k+\alpha}{2n}}\|v\|_{k,\alpha;\psi^*\omega}.$$

$\square$

We apply a similar line of thoughts with the triangle inequality applied in the other direction in order to obtain

$$\|\nabla^k_{\psi^*\omega}v(z)\|_{\psi^*\omega(z)} \leq (-\log\|S(x)\|)^{-\frac{k}{2n}}(C_1|\nabla^k v(z)| + C_2\|v\|_{k-1}).$$

So we also have

Lemma 4.6.2 *There is a constant $C > 0$ such that for all $v \in C^k(P)$*

$$\|\nabla^k_{\psi^*\omega}v\|_{\psi^*\omega} \leq C(-\log\|S(x)\|)^{-\frac{k}{2n}}\|v\|_k.$$

Let u be the bounded solution of the Monge-Ampére equation (4.1). We examine the linear elliptic operator

$$\begin{aligned}
L_x(v) &:= (-\log\|S\circ\psi_x\|)^{\frac{1}{n}}\left(\frac{(\psi_x^*\omega + i\partial\overline{\partial}v)\wedge(\psi_x^*\omega + i\partial\overline{\partial}(u\circ\psi_x))^{n-1}}{\psi_x^*\omega^n} - 1\right)\\
&= (-\log\|S\circ\psi_x\|)^{\frac{1}{n}}\frac{1}{\psi_x^*\omega^n}\left(i\partial\overline{\partial}v\wedge\sum_{j=0}^{n-1}\binom{n}{j}\psi_x^*\omega^{n-1-j}\wedge(i\partial\overline{\partial}(u\circ\psi_x))^j\right) =\\
&= \frac{1}{\eta^n}\left(i\partial\overline{\partial}v\wedge\sum_{j=0}^{n-1}\binom{n}{j}\eta^{n-1-j}\wedge(i(-\log\|S\circ\psi_x\|)^{-\frac{1}{n}}\partial\overline{\partial}(u\circ\psi_x))^j\right).
\end{aligned}$$

We combine Lemma 4.3.4, Lemma 4.6.1 and our slow decay result Theorem 4.5.4 in order to estimate

$$\begin{aligned}
\|(-\log\|S\circ\psi_x\|)^{-\frac{1}{n}}\partial\overline{\partial}(u\circ\psi_x))\|_{0,\alpha} &\leq C(-\log\|S(x)\|)^{-\frac{1}{n}}\|u\circ\psi_x\|_{2,\alpha}\\
&\leq C(-\log\|S(x)\|)^{\frac{\alpha}{2n}}\|u\|_{C^{2,\alpha}_\omega(U_x)}\\
&\leq C(-\log\|S(x)\|)^{\frac{3\alpha-1}{6n}}\\
&\leq C,
\end{aligned}$$

if we choose $\alpha < \frac{1}{3}$.

So the linear elliptic operator L_x has uniformly $C^{0,\alpha}$ coefficients and satisfies

$$L_x(u\circ\psi_x) = (-\log\|S\circ\psi_x\|^2)^{\frac{1}{n}}(e^{f\circ\psi_x} - 1).$$

The right hand side and its derivatives with respect to the quasi-coordinates are uniformly bounded, since it is an element of R_∞. We know already that u is bounded, so we obtain by interior Schauder estimates for $Q := P_{1-\varepsilon,\dots,1-\varepsilon}$

$$\|u \circ \psi_x\|_{C^{2,\alpha}(Q)} \leq C$$

for a constant C independent of x.

In order to obtain estimates for higher derivatives we differentiate

$$\det(\psi_x^*\omega + i\partial\overline{\partial}(u \circ \psi_x)) - \det \psi_x^*\omega = (e^{f\circ\psi_x} - 1) \det \psi_x^*\omega.$$

This yields

$$\begin{aligned} & i\partial\overline{\partial}\frac{\partial}{\partial w_i}(u \circ \psi_x) \wedge \sum_{k=0}^{n-1}((\psi^*\omega + i\partial\overline{\partial}(u \circ \psi_x))^k \wedge \psi^*\omega^{n-k-1})+ \\ +i\partial\overline{\partial}(u \circ \psi_x) \wedge \sum_{k=0}^{n-1}&(c_k(i\partial\overline{\partial}(u \circ \psi_x))^k \wedge \frac{\partial}{\partial w_i}\psi^*\omega \wedge \psi^*\omega^{n-k-2}) \quad = \quad \frac{\partial(f \circ \psi_x)}{\partial w_i}e^{f\circ\psi_x}\omega^n + \\ & \qquad\qquad\qquad\qquad\qquad\qquad +(e^{f\circ\psi_x} - 1)\frac{\partial}{\partial w_i}\omega^n. \end{aligned}$$

As before we define a uniformly elliptic operator

$$\begin{aligned} L_x^{(2)}(v) \quad &:= \quad (-\log\|S \circ \psi_x\|)^{\frac{1}{n}}\frac{1}{\psi_x^*\omega^n}\left(i\partial\overline{\partial}v \wedge \sum_{k=0}^{n-1}((\psi^*\omega + i\partial\overline{\partial}(u \circ \psi_x))^k \wedge \psi^*\omega^{n-k-1})\right) \\ &= \quad \frac{1}{\eta^n}\left(i\partial\overline{\partial}v \wedge \sum_{k=0}^{n-1}((\eta + i(-\log\|S \circ \psi_x\|)^{-\frac{1}{n}}\partial\overline{\partial}(u \circ \psi_x))^k \wedge \eta^{n-k-1})\right) \end{aligned}$$

with uniform $C^{0,\alpha}$ coefficients.

For the corresponding second term of the right hand side

$$H := (-\log\|S\circ\psi_x\|)^{\frac{1}{n}}\frac{1}{\omega^n}\left(i\partial\overline{\partial}(u \circ \psi_x) \wedge \sum_{k=0}^{n-1}(c_k(i\partial\overline{\partial}(u \circ \psi_x))^k \wedge \frac{\partial}{\partial w_i}\psi^*\omega \wedge \psi^*\omega^{n-k-2})\right)$$

we employ again Lemma 4.3.4, the C^k-equivalence stated in Proposition 4.3.3 and the $C^{2,\alpha}$- bound of $u \circ \psi$ in order to show

$$\|H\|_{0,\alpha} \leq C.$$

Such a bound is also available for the right hand side of

$$L_x^{(2)}(\frac{\partial(u \circ \psi_x)}{\partial w_i}) + H = (-\log\|S \circ \psi_x\|)^{\frac{1}{n}}\left(\frac{\partial f \circ \psi_x}{\partial w_i}e^{f\circ\psi_x} + (e^{f\circ\psi_x} - 1)\frac{\frac{\partial}{\partial w_i}\omega^n}{\omega^n}\right),$$

because $f \in R_\infty$.

Since we already proved a C^0 bound for $\frac{\partial(u\circ\psi_x)}{\partial w_i}$, in particular, interior Schauder estimates give us a $C^{3,\alpha}$ bound

$$\|u \circ \psi\|_{C^{3,\alpha}(Q_2)} \leq C$$

on a shrunk $Q_2 = P_{1-\varepsilon_2,\dots,1-\varepsilon_2}, \varepsilon_2 \geq \varepsilon$.

This procedure can be applied inductively and hence we have uniform bounds for the derivatives of arbitrarily high degree of u with respect to quasi-coordinates. The shrinking of Q_k can be controlled in such a way that $Q_k \supset P_{\frac{1}{2},\dots,\frac{1}{2}}$.

Proposition 4.6.3 *For every k there is a constant C_k such that $\|u \circ \psi_x\|_{C^k(Q)} \leq C_k$ uniformly in x for $Q = P_{\frac{1}{2},\dots,\frac{1}{2}}$.*

Application of this proposition for $k = 2$ and Lemma 4.6.2 improves the asymptotic rate to $\frac{1}{n}$ instead of $\frac{1}{6n}$:

Corollary 4.6.4 *There is a constant $C > 0$ such that*

$$(1 - C(-\log\|S\|^2)^{-\frac{1}{n}})\omega \leq \omega + i\partial\overline{\partial}u \leq (1 + C(-\log\|S\|^2)^{-\frac{1}{n}})\omega.$$

4.7 Construction of the strongly asymptotic metric

Definition 4.7.1 *For Riemannian metrics $g, \tilde{g}$ on $X \setminus D$ we say g is strongly C^k-asymptotic to $\tilde{g}$, if there is a constant $C, q > 0$ such that for $h := \tilde{g} - g$ the estimates*

$$-C\|S\|^q g \leq h \leq C\|S\|^q g$$

and

$$\|\nabla_g^l h\|_g \leq C\|S\|^q$$

for all $0 \leq l \leq k$ hold.

We start the examination of the Tian-Yau initial metric on a tubular neighbourhood $U(D)$ of D. Recall the construction of the principal S^1-bundle Y and the angular differential δ from section 3.2.

Let us have a closer look at the angular differential. It depended on choices of a trivialisation of $N^1_{D|X}$ and an element in $\mathcal{A}^1(D)$. With help of $\|.\|$ we may get rid of this ambiguity. For this purpose we choose local coordinates (z_{ij}) on a cover of $U(D) = \bigcup_{j=1}^k U_j$ such that $S = z_1$ on each U_j. The local coordinates induce a local trivialisation of $N^1_{D|X}$ such that

$$\beta_{jk}(z_2, \dots, z_n) = \frac{\frac{\partial z_{1k}}{\partial z_{1j}}}{|\frac{\partial z_{1k}}{\partial z_{1j}}|}|_{z_1=0}.$$

Let e^h represent $\|.\|^2$ in the local coordinates. The formula for β_{jk} implies that

$$\delta := d\varphi - \frac{i}{2}\pi^*(\partial h|_D - \overline{\partial} h|_D)$$

is a global real one-form on Y independent of the chosen coordinate systems. It satisfies

$$d\delta = \pi^*(\rho|_D),$$

but is not determined by this fact. The product structure $U(D) \setminus D = (0, \varepsilon) \times Y$ allows us to interpret δ also as real one-form on $U(D) \setminus D$.

For the following construction we need that the structure group of C^∞ complex line bundles can be reduced to the unitary group.

Lemma 4.7.2 *Let L be a C^∞ complex line bundle on a manifold Z. There is a unitary complex line bundle $\mathcal{U}$ isomorphic to L.*

Proof. Let L be given by the cocycle $f_{ij} \in C^\infty(U_i \cap U_j)$ for some cover $Z = \bigcup_i U_i$. We want to prove that L is isomorphic to $\mathcal{U}$ given by the cocycle

$$g_{ij} := \frac{f_{ij}}{|f_{ij}|}.$$

First we see that $|f_{ij}|$ is a cocycle for a real line bundle $|L|$ on Z. These are classified by the group $H^1(X, \mathbb{Z}_2)$. In particular, $|L|^2$ is trivial, so there are $f_i^2 \in C^\infty(U_i), f_j^2 \in C^\infty(U_j)$ such that

$$|f_{ij}|^2 = \frac{f_i^2}{f_j^2}$$

on $U_i \cap U_j$. We take the square root and obtain that $|L|$ itself is trivial. Also the complex line bundle $|L| \otimes \mathbb{C}$ given by $|f_{ij}|$ is trivial, so $\mathcal{U} = L \otimes (|L| \otimes \mathbb{C})^{-1} \cong L$. □

Having well-defined the angular differential we look at the radial coordinate. Recall that we have chosen an arbitrary hermitian metric η on X; its conformal class induces a C^∞ splitting

$$T_X|D = T_D \oplus N_{D|X}.$$

To be more precise we denote

$$0 \longrightarrow T_D \longrightarrow T_X \overset{i_\eta}{\underset{\longrightarrow}{\longleftarrow}} N_{D|X} \longrightarrow 0.$$

The map i_η is a C^∞-homomorphism of complex vector bundles. Let us further assume that η is Kähler. We will explain the use of this assumption at the appropriate place.

In order to define the radius appropriately we would like to be able to choose the tubular neighbourhood in such a way that $r = \|S\|$. This we can do only approximately. If $x \in D$, $z = \exp_x(i_\eta(v))$ for $v \in N_{D|X,x}$ we define

$$r(z) := \|dS(v)\|.$$

Note that $dS|_D \in H^0(N^\vee_{D|X} \otimes \mathcal{O}(D)|_D)$.

Now we have to exhibit the tubular neighbourhood foliated by level sets of r. To this end we choose a C^∞-isomorphism

$$\xi : \mathcal{U} \longrightarrow N_{D|X}$$

from a C^∞-unitary $\mathbb{C}$-bundle $\mathcal{U}$ to the normal bundle, i.e. $\xi \in \Gamma(N_{D|X} \otimes \mathcal{U}^\vee)$ is a nowhere vanishing C^∞-section. Further, $dS|_D$ being a nowhere vanishing holomorphic section of $N^\vee_{D|X} \otimes \mathcal{O}(D)|D$,

$$dS(\xi) \in \Gamma(\mathcal{O}(D)|D \otimes \mathcal{U}^\vee)$$

is naturally defined and nowhere zero. Since $\mathcal{U}$ is unitary,

$$\|dS(\xi)\| \in C^\infty(D)$$

is well defined and nowhere vanishing, so we may alter ξ in such a way that

$$\|dS(\xi)\| \equiv 1$$

on D. Note that $\eta(i_\eta(\xi), i_\eta(\xi)) \in C^\infty(D)$ is also well defined, since $\mathcal{U}$ is unitary. By a conformal change of η we could achieve $\eta(i_\eta(\xi), i_\eta(\xi)) \equiv 1$, but later we will need η to be Kähler, so we cannot achieve this.

On the other hand, if $z = \exp(i_\eta(v)) \in X$ for some $v \in N_{D|X,x}$, $x \in D$, there exists some $\lambda \in \mathcal{U}_x$ such that $v = \lambda\xi(x)$. We see

$$r(z) = |\lambda|.$$

Indeed, the neighbourhood defined by r

$$\begin{aligned} U(D) &:= \{z \in X | r(z) < \varepsilon\} \\ &= \{\exp(i_\eta(v)) \in X |\ v \in N_{D|X}, \eta(i_\eta(v), i_\eta(v)) < \varepsilon^2 \eta(i_\eta(\xi), i_\eta(\xi))\} \end{aligned}$$

is tubular in the sense that it is a geodesic disc fibration over D with variable radii. So, assuming $e^{-1} < \varepsilon$, we modify the definition of Y by setting

$$Y := \{\exp(i_\eta(v)) \in X |\ v \in N_{D|X}, \eta(i_\eta(v), i_\eta(v)) = e^{-1}\eta(i_\eta(\xi), i_\eta(\xi))\}.$$

In order to compute the differentials we look at the commutative diagram

$$\begin{tikzcd} V := (\exp \circ i_\eta)^{-1}(U(D)) \arrow[d, "\exp"'] \arrow[dr] & \\ U(D) \arrow[r, "\Phi"] & \mathbb{R} \end{tikzcd}$$

where we consider $\Phi = r^2, \|S\|^2$.
For $v \in N_{D|X,x} \cap V$ we obtain

$$\begin{aligned} d(r^2 \circ \exp \circ i_\eta)(x,v) &= d(\|dS(.)\|^2)(x,v) \\ &= d\|.\|^2(dS(v)) \circ d(dS|_D)(x,v) \end{aligned}$$

and

$$d(\|S\|^2 \circ \exp \circ i_\eta)(x,v) = d\|.\|^2(S(\exp_x(v))) \circ dS(\exp_x(v)) \circ d\exp(x, i_\eta(v)) \circ di_\eta(x,v).$$

In order to verify (and make sense of)

$$d(dS|_D)(x,0) + \begin{pmatrix} 0 & 0 \\ i_\eta & 0 \end{pmatrix} = dS(x) \circ d\exp(x,0) \circ di_\eta(x,0) \tag{4.7}$$

we use the natural splitting on the zero section

$$T\mathcal{E}|0_{\mathcal{E}} = 0_*(\mathcal{E} \oplus T_D),$$

where 0 denotes the zero section as map and $0_{\mathcal{E}}$ its image for any holomorphic vector bundle $\mathcal{E}$. For the left hand side of equation 4.7 we use $\mathcal{E} \in \{N_{D|X}, \mathcal{O}(D)|D\}$ and consider $dS|_D : N_{D|X} \longrightarrow \mathcal{O}(D)|D$ as homomorphism of vector bundles. Its differential can be viewed as map

$$d(dS|_D)|0_{N_{D|X}} : N_{D|X} \oplus T_D \longrightarrow \mathcal{O}(D)|D \oplus T_D.$$

Since $dS|D$ is linear along the fibres and maps $0_{N_{D|X}}$ to $0_{\mathcal{O}(D)|D}$,

$$d(dS|_D)|0_{N_{D|X}} = \begin{pmatrix} dS|_D & 0 \\ 0 & id \end{pmatrix}.$$

To compute the right hand side we note that the composition

$$d(\exp|D) \circ di_\eta|0_{N_{D|X}} : N_{D|X} \oplus T_D \longrightarrow T(T_X|D)|0_{T_X|D} \longrightarrow T_X|D$$

is exactly the splitting given by $i_\eta \oplus id$ (cf. [La, VIII, Prop. 5.1]). Moreover, the full differential of S maps

$$dS : T_X|D \longrightarrow 0_*(T\mathcal{O}(D)|0_{\mathcal{O}(D)})|D = \mathcal{O}(D)|D \oplus T_X|D.$$

This map is simply given by $(dS|_D, id)$. Here $dS|_D$ denotes the section of $N^\vee_{D|X} \otimes \mathcal{O}(D)|D$. So we obtain again by $T_D = \ker dS$ and the fact that i_η is a section of the projection for any $v \in N_{D|X,x}, w \in T_{D,x}$

$$\begin{aligned} dS(x) \circ d\exp(x,0) \circ di_\eta(x,0)(v,w) &= (dS(i_\eta(v)+w), i_\eta(v)+w) \\ &= (dS(v), i_\eta(v)+w) \\ &= d(dS|_D)(x,0)(v,w) + (0, i_\eta(v)). \end{aligned}$$

Since for fixed v and $t > 0$ the limits

$$\lim_{t \longrightarrow 0} d\|.\|(dS(tv)) \text{ and } \lim_{t \longrightarrow 0} d\|.\|(S(exp(tv))$$

exist and

$$\lim_{t \longrightarrow 0} d\|.\|(x, tw)(0,.) = 0$$

for any $w \in \mathcal{O}(D)_x$ we finally see that for all $x \in D$

$$\lim_{z \longrightarrow x} (d\|S\| - dr)(z) = 0.$$

Moreover, $d\|S\|^2 - dr^2$ is a C^∞ one-form vanishing on D of order at least two. In particular,

$$\|S\|^2 = r^2 e^b$$

for some $b \in C^\infty(X)$ vanishing at D.

As angles we have

$$e^{2i\psi} := \frac{S(z)}{\overline{S(z)}} \text{ and } e^{2i\varphi} := \frac{dS(v)}{\overline{dS(v)}}$$

for $z = \exp_x(i_\eta(v)), v \neq 0$ as sections of the respective unitary bundles over $U(D) \setminus D$. The angle φ is the same as constructed earlier in this section.

We define

$$R := -\log r = -\frac{1}{2}\log\|S\|^2(1 + O(\|S\|)).$$

Let $X = (1,\infty) \times Y$ and define for $R \in (1,\infty)$ a metric g_R on Y by:

$$g_R = (2R)^{\frac{1}{n}} \pi^* g_D + (2R)^{\frac{1}{n}-1} \delta \otimes \delta.$$

The corresponding volume form is given by

$$\sqrt{\det g_R} = (2R)^{\frac{1}{2}-\frac{1}{2n}} \sqrt{\det g_D} d\varphi_i \wedge dx_{i1} \wedge \cdots \wedge dx_{i,2n-2}.$$

We can easily calculate the inverse tensor. For this purpose denote by γ_i the vector with components $\gamma_{iq}(x_i)$, $q = 1, \cdots, 2n-2$. Then we have on $\pi^{-1}(U_i)$:

$$g_R = \begin{pmatrix} (2R)^{\frac{1}{n}-1} d\varphi_i \otimes d\varphi_i & (2R)^{\frac{1}{n}-1}\gamma_i^t d\varphi_i \\ (2R)^{\frac{1}{n}-1}\gamma_i d\varphi_i & (2R)^{\frac{1}{n}}\pi^* g_D + (2R)^{\frac{1}{n}-1}\gamma_i \otimes \gamma_i^t \end{pmatrix}$$

and

$$g_R^{-1} = \begin{pmatrix} ((2R)^{1-\frac{1}{n}} + (2R)^{-\frac{1}{n}}\gamma_i^t g_D^{-1}\gamma_i)\frac{\partial}{\partial\varphi_i} \otimes \frac{\partial}{\partial\varphi_i} & -(2R)^{-\frac{1}{n}}\gamma_i^t g_D^{-1}\frac{\partial}{\partial\varphi_i} \\ -(2R)^{-\frac{1}{n}} g_D^{-1}\gamma_i\frac{\partial}{\partial\varphi_i} & (2R)^{-\frac{1}{n}} g_D^{-1} \end{pmatrix}.$$

Theorem 4.7.3 *If* $g_D := \mathrm{Re}(i_D^*\rho)$, *then the metric* $\mathrm{Re}(\omega)$ *is strongly* C^k*-asymptotically equivalent to*

$$g = (2R)^{\frac{1}{n}-1} dR \otimes dR + g_R.$$

Proof. As a first step we show that the metrics ω and

$$i\partial\overline{\partial}R^{\frac{n+1}{n}}$$

are C^k-equivalent for any k. By the preceding arguments we have

$$R = -\frac{1}{2}\log\|S\|^2 + b$$

for a function $b \in C^\infty(X)$ such that $b|D \equiv 0$. So $b = z_1\beta + \overline{z}_1\overline{\beta}$ for a function $\beta \in C^\infty(U, \mathbb{C})$ and local coordinates like in section 4.3. We adopt this section's notation here. Since $b \in R_\infty$ Lemma 4.3.4 shows that all derivatives of b with respect to quasi-coordinates are in R_∞. This sums up to the desired claim that

$$i\partial\overline{\partial}R^{\frac{n+1}{n}} = i\partial\overline{\partial}(-\frac{1}{2}\log\|S\|^2 + b)^{\frac{n+1}{n}}$$

is strongly asymptotic to ω in every C^k-sense.

Next we consider the change of the complex structure when changing from $U(D)$ to $N_{D|X}$. We denote

$$\Phi := \exp\circ i_\eta : N_{D|X} \longrightarrow X$$

and want to compare

$$i\partial\overline{\partial}R^{\frac{n+1}{n}} \text{ and } i(\Phi^{-1})^*\partial\overline{\partial}(R\circ\Phi)^{\frac{n+1}{n}}.$$

A choice of local coordinates $z^1, \ldots, z^n$ in a neighbourhood U intersecting D like above induces local coordinates $v^1 := dz^1, z^2, \ldots, z^n$ of $N_{D|X}$ in a neighbourhood $\tilde{U}$ intersecting $0_{N_{D|X}}$. These will also be denoted by $x^1 := v^1, x^i := z^i$ for $i > 1$. Finally, let us abbreviate $\tilde{a} := R^{\frac{n+1}{n}}, a := \tilde{a} \circ \Phi$ and $\Psi := \Phi^{-1} : U(D) \longrightarrow N_{D|X}$. We obtain

$$\begin{aligned} \partial\overline{\partial}R^{\frac{n+1}{n}} - \Psi^*\partial\overline{\partial}(R\circ\Phi)^{\frac{n+1}{n}} &= dz^i \otimes d\overline{z^j}\left(\frac{\partial^2 a}{\partial x^l \partial x^k}\frac{\partial \Psi^l}{\partial z^i}\frac{\partial \Psi^k}{\partial \overline{z}^j} + \frac{\partial^2 a}{\partial \overline{x}^l \partial \overline{x}^k}\frac{\partial \overline{\Psi}^l}{\partial z^i}\frac{\partial \overline{\Psi}^k}{\partial \overline{z}^j} + \right. \\ & \left. \frac{\partial a}{\partial x^k}\frac{\partial^2 \Psi^k}{\partial z^i \partial \overline{z}^j} + \frac{\partial a}{\partial \overline{x}^k}\frac{\partial^2 \overline{\Psi}^k}{\partial z^i \partial \overline{z}^j}\right) - \\ & -\frac{\partial^2 a}{\partial x^k \partial \overline{x}^l}\left(\frac{\partial \Psi^k}{\partial z^i}\frac{\partial \overline{\Psi}^l}{\partial z^j} dz^i \otimes dz^j + \frac{\partial \Psi^k}{\partial \overline{z}^i}\frac{\partial \overline{\Psi}^l}{\partial \overline{z}^j} d\overline{z}^i \otimes d\overline{z}^j\right) \end{aligned} \tag{4.8}$$

In order to prove strong asymptotic C^0-equivalence of the metrics given by the real parts it is enough to show

$$\frac{(\partial\overline{\partial}R^{\frac{n+1}{n}} - \Psi^*\partial\overline{\partial}(R\circ\Phi)^{\frac{n+1}{n}})(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \overline{z}^j})}{\Psi^*\partial\overline{\partial}(R\circ\Phi)^{\frac{n+1}{n}}(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \overline{z}^j})}|_D = 0 \tag{4.9}$$

and

$$\frac{(\partial\overline{\partial}R^{\frac{n+1}{n}} - \Psi^*\partial\overline{\partial}(R\circ\Phi)^{\frac{n+1}{n}})(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j})}{\Psi^*\partial\overline{\partial}(R\circ\Phi)^{\frac{n+1}{n}}(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \overline{z}^j})}|_D = 0. \tag{4.10}$$

To this end we introduce the notion of an "exponential degree" for functions $p \in C^\infty(\tilde{U} \setminus 0_{N_{D|X}}), q \in C^\infty(U \setminus D)$:

$$\begin{aligned} \mathrm{expdeg}(p) &:= \sup\{r \in \mathbb{R} |\ |x^1|^{-r} p \text{ is bounded around } 0_{N_{D|X}}\}, \\ \mathrm{expdeg}(q) &:= \mathrm{expdeg}(q \circ \Phi). \end{aligned}$$

Now we have to compute the exponential degrees of the derivatives of a and Ψ. The function a can simply be expressed as

$$a(x^1, \dots, x^n) = (-\frac{1}{2}(h(0, x^2, \dots, x^n) + \log|x^1|^2))^{\frac{n+1}{n}},$$

so

$$\mathrm{expdeg}(\frac{\partial a}{\partial x^k}) = \mathrm{expdeg}(\frac{\partial a}{\partial \overline{x}^k}) = -\delta_{k1}$$

and

$$\mathrm{expdeg}(\frac{\partial^2 a}{\partial x^k \partial x^l}) = \mathrm{expdeg}(\frac{\partial^2 a}{\partial x^k \partial \overline{x}^l}) = \mathrm{expdeg}(\frac{\partial^2 a}{\partial \overline{x}^k \partial \overline{x}^l}) = -\delta_{k1} - \delta_{l1}.$$

In order to compute the corresponding values for Ψ we carry out the calculations for Φ first. We denote

$$i_\eta(x^1, \dots x^n) = (x^1\alpha, x^2, \dots, x^n)$$

with $\alpha \in T_X|_D(U \cap D)$ satisfying $\alpha^1 = 1$. We obtain

$$\frac{\partial \Phi^k}{\partial \overline{x}^l}(0, x^2, \ldots, x^n) = 0, \frac{\partial \Phi^k}{\partial x^l}(0, x^2, \ldots, x^n) = \begin{cases} \alpha^k & \text{, if } l = 1 \\ \delta_{kl} & \text{, if } l > 1 \end{cases}$$

The computations for $k, l > 1$ are straightforward, as well as for $k = 1, l > 1$ and $k > 1, l = 1$ as soon as you note that $\frac{\partial \alpha^j}{\partial \overline{x}^1} = 0$.

The inverse of the Jacobian of Φ is

$$\frac{\partial \Psi^k}{\partial \overline{z}^l}(0, z^2, \ldots, z^n) = 0, \frac{\partial \Psi^k}{\partial z^l}(0, z^2, \ldots, z^n) = \begin{cases} -\alpha^k & \text{, if } l = 1, k > 1 \\ \delta_{kl} & \text{else} \end{cases}$$

The results for the first of Ψ imply

$$\text{expdeg}(\frac{\partial \Psi^k}{\partial z^l}) = \begin{cases} 0 & \text{, if } l = 1 \text{ or } l = k \\ 1 & \text{else} \end{cases},$$

$$\text{expdeg}(\frac{\partial \Psi^k}{\partial \overline{z}^l}) = \begin{cases} 2 & \text{, if } k = 1 \text{ and } l > 1 \\ 1 & \text{else} \end{cases},$$

$$\text{expdeg}(\frac{\partial^2 \Psi^j}{\partial z^k \partial \overline{z}^l}) = \begin{cases} 2 & \text{, if } j = 1 \text{ and } k, l > 1 \\ 0 & \text{, if } k = 1 \text{ and } j, l > 1 \\ 1 & \text{, if } k > 1 \end{cases}$$

This is now plugged into (4.8) in order to verify (4.9) and (4.10). So $\partial\overline{\partial} R^{\frac{n+1}{n}}$ and $\text{Re}(\Psi^* \partial\overline{\partial}(R \circ \Phi)^{\frac{n+1}{n}})$ are strongly asymptotically C^0-equivalent. The coefficients of the difference terms being in R_∞ we can proceed like before to prove C^k-equivalence of the two metrics using quasi-coordinates.

In the last step we compute the metric in the new coordinates $R, \varphi, z^2, \ldots z^n, \overline{z}^2, \overline{z}^n$ and with respect to the new complex structure. So we have

$$R = -\frac{1}{2}(h|_D + \log |z^1|^2), z^1 = e^{-R - \frac{1}{2}h|_D + i\varphi}$$

and the metric becomes

$$\begin{aligned} i\frac{n}{n+1}\Psi^* \partial\overline{\partial}(R \circ \Phi)^{\frac{n+1}{n}} &= (2R)^{\frac{1}{n}} \pi^*(\rho|_D) + \frac{1}{n} R^{\frac{1}{n}-1}(d \log z^1 + \partial h) \otimes (d \log \overline{z}^1 + \overline{\partial} h) \\ &= (2R)^{\frac{1}{n}} \pi^*(\rho|_D) + \frac{1}{n} R^{\frac{1}{n}-1}(dR + i\delta) \otimes (dR - i\delta). \end{aligned}$$

Hence the real parts satisfy

$$\text{Re}(i\frac{n}{n+1}\Psi^* \partial\overline{\partial}(R \circ \Phi)^{\frac{n+1}{n}}) = (2R)^{\frac{1}{n}} \text{Re}(\pi^*(\rho|_D)) + \frac{1}{n} R^{\frac{1}{n}-1}(dR \otimes dR + \delta \otimes \delta).$$

$\square$

Note that in this sense also the family g_R is strongly C^k-asymptotic to the family ω_R under the isomorphism between $\tilde{Y}_R$ and Y.

As an immediate application, the coefficients of the Laplacians Δ_ω and Δ_g differ only by terms vanishing exponentially (in R) in D. So, if $u \in C^{2,\alpha}_\omega(X \setminus D)$ and

$$\Delta_\omega u = f \in O(R^{-p}),$$

we can put all difference terms to the right hand side and obtain

$$\Delta_g u = \tilde{f} \in O(R^{-p}).$$

The same holds true for any number of k times iterated Laplacians, if we assume $u \in C^{2k,\alpha}_\omega(X \setminus D)$.

4.8 Spectral analysis of the Laplace equation on the fibre bundle

Let $f : Y \longrightarrow \mathbb{R}$ be a C^∞-function. On $\pi^{-1}(U_i)$ we have a Fourier decomposition

$$f = \sum_{k=-\infty}^{\infty} a_{ik}(x_i) e^{ik\varphi_i}, a_{i,-k} = \overline{a_{ik}}$$

with local C^∞-functions $a_{ik} : U_i \longrightarrow \mathbb{C}$. We employ $e^{ik\varphi_j} = \beta^k_{ij}(x_i)e^{ik\varphi_i}$ and conclude from the uniqueness of the Fourier decomposition

$$a_{jk}(x_j) = \beta^{-k}_{ij}(x_i)a_{ik}(x_i).$$

So we arrive at a projection

$$f_k(x_i, \varphi_i) = a_{ik}(x_i)e^{ik\varphi_i} = a_{jk}(x_j)e^{ik\varphi_j},$$

which is a global smooth function on Y.
For every $k \in \mathbb{Z}$ we define a complex line bundle L_k given by the cocycle $\beta^{-k}_{ij}(x_i)$. The volume form of g is

$$\sqrt{\det g} = \sqrt{\det g_D} dR \wedge d\varphi_i \wedge dx_{i1} \wedge \cdots \wedge dx_{i,2n-2}.$$

Let $u, f : U(D) \backslash D \longrightarrow \mathbb{R}$ be two smooth functions with f decreasing for $R \to \infty$ and $\Delta_g u = f$. We see immediately that we must have $\Delta_g u_k = f_k$ as well. We calculate

$$\Delta_g(u_k) = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial R} \left((2R)^{1-\frac{1}{n}} \sqrt{\det g} \frac{\partial u_k}{\partial R} \right) + \Delta_{g_R}(u_k).$$

Furthermore

$$\begin{aligned}\Delta_{g_R}(u_k) &= (2R)^{1-\frac{1}{n}}\frac{\partial^2}{\partial\varphi_i^2}(u_k) - (2R)^{-\frac{1}{n}}\frac{\partial^2}{\partial\varphi_i^2}(u_k) + (2R)^{-\frac{1}{n}}\Delta_{g_{R=\frac{1}{2}}}(u_k)\\ &= -k^2(2R)^{1-\frac{1}{n}}u_k + k^2(2R)^{-\frac{1}{n}}u_k + (2R)^{-\frac{1}{n}}\Delta_{g_{R=\frac{1}{2}}}(u_k).\end{aligned}$$

The map $u_k \longmapsto \Delta_{g_{R=\frac{1}{2}}}(u_k)$ is a selfadjoint elliptic operator, and we use this to define a selfadjoint elliptic operator $P_k : C^\infty(L_k) \longrightarrow C^\infty(L_k)$ by

$$P_k(u_{ki}(x_i))e^{ik\varphi_i} = \Delta_{g_{R=\frac{1}{2}}}(u_{ki}(x_i)e^{ik\varphi_i}).$$

According to [GT] P_k has an L^2-orthonormal eigenbasis $(\xi_{mki}(x_i))_{m\in\mathbb{N}_0}$, $P_k(\xi_{mki}(x_i)) = \lambda_{mk}\xi_{mki}(x_i)$, $\lim_{m\to\infty}\lambda_{mk} = -\infty$ and $\lambda_{mk} \leq C_k$. Note that any λ_{mk} is, by construction, also an eigenvalue of $\Delta_{g_{R=\frac{1}{2}}}$. Now we can write

$$u(R,\varphi_i,x_i) = \sum_{k=-\infty}^{\infty}\sum_{m=0}^{\infty} u_{mk}(R)e^{ik\varphi_i}\xi_{mki}(x_i)$$

and correspondingly for f. Note that the boundedness of u implies that also

$$u_{mk} = \int_Y ue^{-ik\varphi}\overline{\xi}_{mk}dvol_{g_{R=\frac{1}{2}}}$$

is bounded.

The condition $\Delta_g(u_k) = f_k$ translates to a system of decoupled ODEs

$$\frac{d}{dR}\left((2R)^{1-\frac{1}{n}}\frac{du_{mk}}{dR}\right) - k^2(2R)^{1-\frac{1}{n}}u_{mk} + (k^2+\lambda_{mk})(2R)^{-\frac{1}{n}}u_{mk} = f_{mk}.$$

Proposition 4.8.1 *We actually have $\lambda_{mk} \leq -k^2$.*

Proof. Let $\varepsilon > 0$ be small and consider

$$\begin{aligned}&\left\langle \xi_{mki}(x_i)e^{ik\varphi_i}, \Delta_{g_{R=\frac{1}{2}}}(\xi_{mki}(x_i)e^{ik\varphi_i})\right\rangle_{Y,g_{R=\frac{1}{2}}}\\ -&\left\langle \xi_{mki}(x_i)e^{ik\varphi_i}, \Delta_{g_{R=\frac{1}{2}\varepsilon}}(\xi_{mki}(x_i)e^{ik\varphi_i})\right\rangle_{Y,g_{R=\frac{1}{2}}}\varepsilon^{\frac{1}{n}-\frac{1}{2}+\frac{1}{2n}} =\\ &-k^2\left\langle \xi_{mki}(x_i)e^{ik\varphi_i}, \xi_{mki}(x_i)e^{ik\varphi_i}\right\rangle_{Y,g_{R=\frac{1}{2}}}\\ &+k^2\varepsilon\left\langle \xi_{mki}(x_i)e^{ik\varphi_i}, \xi_{mki}(x_i)e^{ik\varphi_i}\right\rangle_{Y,g_{R=\frac{1}{2}}} = -k^2 + k^2\varepsilon.\end{aligned}$$

This implies

$$\left\langle \xi_{mki}(x_i)e^{ik\varphi_i}, \Delta_{g_{R=\frac{1}{2}}}(\xi_{mki}(x_i)e^{ik\varphi_i}) \right\rangle_{Y,g_{R=\frac{1}{2}}} \leq -k^2 + k^2\varepsilon.$$

Letting $\varepsilon \to 0$ tend to 0 shows the claim. $\square$

Corollary 4.8.2 *There is $C > 0$ such that $\lambda_{mk} \leq -C(k^{2n-1} + m)^{\frac{2}{2n-1}}$.*

Proof. Since any λ_{mk} is also some eigenvalue $\mu_{m'}$ of $\Delta_{g_{R=\frac{1}{2}}}$ with $m' \geq m$, the claim follows from the well-known fact

$$\mu_{m'} \sim Cm'^{\frac{2}{2n-1}}.$$

$\square$

We renormalise $\lambda_{mk} + k^2 \to \lambda_{mk} \leq 0$ and obtain the system of ODEs

$$\frac{d}{dR}\left((2R)^{1-\frac{1}{n}}\frac{du_{mk}}{dR}\right) - k^2(2R)^{1-\frac{1}{n}}u_{mk} + (2R)^{-\frac{1}{n}}\lambda_{mk}u_{mk} = f_{mk}. \tag{4.11}$$

From now on we assume

$$f \in O(R^{-p}).$$

Case $k \neq 0$: First we treat the case where $k \neq 0$, so for symmetry we may assume $k > 0$. We look at the homogeneous equation where $f_{mk} = 0$. We substitute

$$x = -2kR,\ Q(x) = u_{mk}(R)e^{-kR}$$

and end up with a confluent hypergeometric equation:

$$x\frac{d^2Q}{dx^2} + (\gamma - x)\frac{dQ}{dx} - \alpha Q = 0.$$

Here we have set

$$\gamma = 1 - \frac{1}{n} \text{ and } \alpha = \frac{1}{2}\left(1 - \frac{1}{n}\right) + \frac{\lambda_{mk}}{4k}.$$

We have to examine a fundamental system of solutions and their behaviour for $R \to \infty$ so $x \to -\infty$. A first solution is given by

$$Q_1(x) = \int_1^\infty e^{xt}t^{\alpha-1}(t-1)^{\gamma-\alpha-1}dt.$$

Here we notice that $\gamma-\alpha-1 =\geq -\frac{1}{2}-\frac{1}{2n} > -1$ and so $Q_1(x)$ is well defined for $x<0$. A second solution is given by

$$Q_2(x) = \int_0^1 e^{xt}t^{\alpha-1}(1-t)^{\gamma-\alpha-1}dt.$$

This makes sense only for $\alpha > 0$, but if $\alpha \leq 0$, then we regard the integral as a formal expression and perform partial integration, so that both exponents $\alpha-1 > -1$ and $\gamma-\alpha-1 > -1$.

Next we determine the asymptotic of $Q_1(x)$ and $Q_2(x)$ for $x \to -\infty$. We have

$$\begin{aligned} Q_1(x) &= (-x)^{\alpha-\gamma}e^x \int_0^\infty e^{-s}\left(1+\frac{s}{-x}\right)^{\alpha-1} s^{\gamma-\alpha-1}ds \\ &\sim (-x)^{\alpha-\gamma}e^x\left(\Gamma(\gamma-\alpha) - \frac{1-\alpha}{-x}\Gamma(\gamma-\alpha+1)+\cdots\right). \end{aligned}$$

For $Q_2(x)$ we have similarly

$$Q_2(x) = (-x)^{-\alpha}\int_0^{-x} e^{-s}\left(1+\frac{s}{-x}\right)^{\gamma-\alpha-1} s^{\alpha-1}ds$$

and

$$\lim_{x\to-\infty}\int_0^{-x} e^{-s}\left(1+\frac{s}{-x}\right)^{\gamma-\alpha-1} s^{\alpha-1}ds = \int_0^\infty e^{-s}s^{\alpha-1}ds = \Gamma(\alpha).$$

So for $k>0$ we obtain a basis of the homogeneous equation whose asymptotic expansion looks like

$$\begin{aligned} w_1(R) &\sim (2kR)^{\alpha-\gamma}e^{-kR} \\ w_2(R) &\sim (2kR)^{-\alpha}e^{kR}. \end{aligned}$$

A solution of the inhomogeneous equation is found by variation of constants. So put $w = c_1w_1 + c_2w_2$ and set

$$\begin{aligned} \frac{dc_1}{dR}w_1 + \frac{dc_2}{dR}w_2 &= 0 \\ \frac{dc_1}{dR}\frac{dw_1}{dR} + \frac{dc_2}{dR}\frac{dw_2}{dR} &= f_{mk}(R)(2R)^{\frac{1}{n}-1}. \end{aligned}$$

We solve for $\frac{dc_1}{dR}$ and $\frac{dc_2}{dR}$:

$$\begin{aligned} \frac{dc_1}{dR} &= -f_{mk}(R)(2R)^{\frac{1}{n}-1}w_2\left(w_1\frac{dw_2}{dR} - \frac{dw_1}{dR}w_2\right)^{-1} \\ \frac{dc_2}{dR} &= f_{mk}(R)(2R)^{\frac{1}{n}-1}w_1\left(w_1\frac{dw_2}{dR} - \frac{dw_1}{dR}w_2\right)^{-1}. \end{aligned}$$

The Wronskian is asymptotically given by

$$w_1 \frac{dw_2}{dR} - \frac{dw_1}{dR} w_2 \sim 2k(2kR)^{-\gamma},$$

so we get the estimates

$$\begin{aligned}
\left| \frac{dc_1}{dR} \right| &\leq \frac{C}{2k}(2R)^{-p+\frac{1}{n}-1}(2kR)^{\gamma-\alpha}e^{kR} \\
\left| \frac{dc_2}{dR} \right| &\leq \frac{C}{2k}(2R)^{-p+\frac{1}{n}-1}(2kR)^{\alpha}e^{-kR}.
\end{aligned}$$

We end up with

$$\begin{aligned}
|c_1(R)| &\leq |c_1(1)| + C(2k)^{\gamma-\alpha-1} \left| \int_1^R t^{-p+\frac{1}{n}-1+\gamma-\alpha} e^{kt} dt \right| \\
&\leq |c_1(1)| + C(2k)^{\gamma-\alpha-1} R^{-p+\frac{1}{n}-1+\gamma-\alpha} e^{kR} \\
|c_2(\infty) - c_2(R)| &\leq C(2k)^{\alpha-1} \left| \int_R^\infty t^{-p+\frac{1}{n}-1+\alpha} e^{-kt} dt \right| \\
&\leq C(2k)^{\alpha-1} R^{-p+\frac{1}{n}-1+\alpha} e^{-kR}.
\end{aligned}$$

So we see that the general solution looks like

$$\begin{aligned}
w(R) &= c_1 w_1 + c_2 w_2 + \tilde{w} \\
|\tilde{w}| &\leq \frac{C}{2k} R^{-p+\frac{1}{n}-1}
\end{aligned}$$

and constants $c_1, c_2 \in \mathbb{R}$. The same asymptotics hold for $|\frac{d\tilde{w}}{dR}|$.

Case $k = 0$ and $\lambda_m < 0$: This leads to the ODE

$$\frac{d^2w}{dR^2} + \frac{\gamma}{R}\frac{dw}{dR} + \frac{\lambda_m}{2R} w = f_{m0}(R)(2R)^{\frac{1}{n}-1}.$$

As before we first treat the homogeneous equation. Two independent solutions are given by

$$w_1(R) := R^{-\frac{\gamma-1}{2}} I_{\gamma-1}(\sqrt{-2\lambda R}),\ w_2(R) = R^{-\frac{\gamma-1}{2}} K_{\gamma-1}(\sqrt{-2\lambda R}),$$

where I_ν and K_ν denote the respective modified Bessel functions of order ν. The asymptotics of these functions are known to be [AS]

$$I_\nu(x) = \frac{e^x}{\sqrt{2\pi x}}(1 + O(x^{-1})),\ K_\nu(x) = \sqrt{\frac{\pi}{2x}} e^{-x}(1 + O(x^{-1})).$$

Differentiating the Wronskian and using the defining ODE we see that

$$w_2(R)\frac{dw_1}{dR} - w_1(R)\frac{dw_2}{dR} = \frac{1}{2}R^{-\gamma}.$$

So the coefficients c_1, c_2 satisfy

$$\frac{dc_1}{dR} = -2f_{m0}(R)x^{\frac{\gamma-1}{2}+\frac{1}{n}}K_{\gamma-1}(\sqrt{2\lambda R}),\ \frac{dc_2}{dR} = 2f_{m0}(R)x^{\frac{\gamma-1}{2}+\frac{1}{n}}I_{\gamma-1}(\sqrt{2\lambda R}).$$

Now using the asymptotics of the Bessel functions we obtain the general solution

$$w(R) = c_1w_1(R) + c_2w_2(R) + \tilde{w}(R)$$

with

$$|\tilde{w}(R)| \le CR^{\frac{1}{n}-p}.$$

By differentiation of w one easily observes

$$\left|\frac{dw}{dR}\right| \le CR^{-\frac{1}{2}}|w|$$

for the general solution.

Case $k = 0$ and $\lambda_m = 0$: This is easy. The homogeneous equation reads

$$\frac{d^2w}{dR^2} + \frac{\gamma}{R}\frac{dw}{dR} = 0.$$

The general solution is given by

$$w = c_0R^{-\gamma+1} + c_1.$$

Variation of constants yields $\frac{dc_1}{dR} = f(R)R^{\gamma-1+\frac{1}{n}}$ and $\frac{dc_1}{dR} = -f(R)R^{\frac{1}{n}}$. We end up with

$$\begin{aligned} w &= c_0R^{-\gamma+1} + c_1 + \tilde{w}, \\ |\tilde{w}| &\le CR^{-p+1+\frac{1}{n}} \end{aligned}$$

and

$$\left|\frac{dw}{dR}\right| \le CR^{-1}|w|$$

for the general solution.

For convergence properties we also need information about ξ_{mk} and their derivatives. The derivatives of ξ_{mk} in directions parallel to D can be dealt with by a local maximum principle [GT, Thm 9.20] leading in connection with interior Schauder estimates to the estimates

Lemma 4.8.3 $\|e^{ik\varphi}\xi_{mk}\|_{C^2_{g_{R=\frac{1}{2}}}} \leq C(1+(k^2-\lambda_{mk})^{\frac{n+3}{2}})$.

We will use this to obtain a crucial decay result.

Theorem 4.8.4 *If $u \in C^{3n+6,\alpha}_{\omega}(X\backslash D)$ and $f := \Delta_\omega u \in O(R^{-p})$ satisfies the conditions of Theorem 4.1.1, then $u - \frac{1}{vol_{g_{R=\frac{1}{2}}}(Y)}\int_Y u\, dvol_{g_{R=\frac{1}{2}}}$ and $u_{,R}$ are in $O(R^{-p+\frac{1}{n}})$.*

Proof. First we inquire into the behaviour of the estimating constants C_{mk} such that

$$|f_{mk}| \leq C_{mk}R^{-p}.$$

Due to Lemma 4.8.3 we will need that $\sum_{m,k} C_{mk}(k^2-\lambda_{mk})^{\frac{n+3}{2}}$ converges, so in order to ensure that we compute

$$\begin{aligned} f_{mk} &= \int_Y f e^{-ik\varphi}\overline{\xi_{mk}} dvol_{g_{R=\frac{1}{2}}} \\ &= (\lambda_{mk}-k^2)^{-1}\int_Y \Delta_{g_{R=\frac{1}{2}}} f \cdot e^{-ik\varphi}\overline{\xi_{mk}} dvol_{g_{R=\frac{1}{2}}} \\ &= (\lambda_{mk}-k^2)^{-1}\int_Y ((2R)^{\frac{1}{n}}\Delta_{g_R} f + (1-2R)\frac{\partial^2}{\partial\varphi^2} f)e^{-ik\varphi}\overline{\xi_{mk}} dvol_{g_{R=\frac{1}{2}}} \\ &= (\lambda_{mk}-k^2)^{-1}\left((2R)^{\frac{1}{n}}\int_Y \Delta_{g_R} f\cdot e^{-ik\varphi}\overline{\xi_{mk}} dvol_{g_{R=\frac{1}{2}}} - k^2(1-2R)f_{mk}\right) \end{aligned}$$

and hence

$$\begin{aligned} |(k^2(2R-1)-1)f_{mk}| &= (2R)^{\frac{1}{n}}(k^2-\lambda_{mk})^{-1}|\int_Y \Delta_{g_R} f e^{-ik\varphi}\overline{\xi_{mk}} dvol_{g_{R=\frac{1}{2}}}| \\ &\leq (2R)^{\frac{1}{n}}(k^2-\lambda_{mk})^{-1}\|(\Delta_{g_R} f)\|_{L^2(Y)} \\ &\leq (2R)^{\frac{1}{n}}(k^2-\lambda_{mk})^{-1} vol_{g_{R=\frac{1}{2}}}(Y)^{\frac{1}{2}}\|\Delta_{g_R} f\|_{C^0(Y)} \\ &\leq C_1(k^2-\lambda_{mk})^{-1}R^{-p}. \end{aligned}$$

In the worst case, $k=0$, the right hand side is just an estimate for $|f_{mk}|$.

Iteration and application of the assumption on the iterated Laplacian yields

$$|f_{mk}| \leq C_j(k^2-\lambda_{mk})^{-j}R^{-p}. \tag{4.12}$$

By Corollary 4.8.2 we have $k^2-\lambda_{mk} \geq C(k^{2n-1}+m)^{\frac{2}{2n-1}}$, so choosing $j = \frac{3}{2}n + \frac{5}{2}$ yields the desired property.

We denote by w_{mk} the bounded homogeneous solutions of section 4.8, by $\tilde{w}_{mk}$ the particular inhomogeneous solutions constructed in section 4.8. Then there are constants c_{mk} such that $u_{mk} = c_{mk}w_{mk} + \tilde{w}_{mk}$, hence

$$u = \sum_{m,k}(c_{mk}w_{mk} + \tilde{w}_{mk})e^{ik\varphi}\xi_{mk}.$$

This series converges absolutely. Note that $u_{00} = \frac{1}{vol_{g_{R=\frac{1}{2}}}(Y)} \int_Y u\, dvol_{g_{R=\frac{1}{2}}}$.

Checking the cases $(k, m) \neq (0, 0)$ as discussed in section 4.8 we see that

$$|\tilde{w}|_{mk} \leq C_{mk}R^{-p+\frac{1}{n}},$$

for $(m, k) \neq (0, 0)$, so by Lemma 4.8.3 the series

$$\tilde{u} := \sum_{(m,k)\neq(0,0)} w_{mk}e^{ik\varphi}\xi_{mk}$$

converges absolutely and

$$|\tilde{u}| \leq CR^{-p+\frac{1}{n}}.$$

Hence, also

$$v := u - u_{00} - \tilde{u} = \sum_{(m,k)\neq(0,0)} c_{mk}w_{mk}e^{ik\varphi}\xi_{mk}$$

converges absolutely.

From this and the exponential decay of the w_{mk} we want to deduce an exponential decay of v. We first look at the terms with $k \neq 0$ and estimate

$$\begin{aligned}
|w_{mk}(R)| &= e^{kR}\int_1^\infty e^{-2kRt}t^{\alpha-1}(1-t)^{\gamma-\alpha-1}dt \\
&\leq e^{kR}\int_1^\infty e^{-2kt}e^{-2k(R-1)t}t^{\alpha-1}(1-t)^{\gamma-\alpha-1}dt \\
&\leq e^{-kR+2}|w_{mk}(1)|.
\end{aligned}$$

This gives us

$$\begin{aligned}
\sum_{k\neq 0,m}|c_{mk}w_{mk}(R)\xi_{mk}| &\leq \sum_{k\neq 0,m} e^{-kR+2}|c_{mk}w_{mk}(1)\xi_{mk}| \\
&\leq C\sum_{k\neq 0} e^{-kR} \\
&\leq C\frac{e^{-R}}{1-e^{-R}} \\
&\leq \tilde{C}e^{-R}.
\end{aligned}$$

In case $k = 0, m \neq 0$, we obtain an integral representation of w_{m0}

$$w_{m0}(R) = (2R)^{\frac{1}{n}} \int_1^\infty (t^{\gamma-2} + t^{-\gamma}) e^{-\sqrt{-\frac{1}{2}\lambda_{m0}R}(t+t^{-1})} dt,$$

allowing again for an estimate

$$w_{m0}(R) \leq R^{\frac{1}{n}} e^{-2\sqrt{-\frac{1}{2}\lambda_{m0}R}+2\sqrt{-\frac{1}{2}\lambda_{m0}}} w_{m0}(1).$$

Finally, this yields

$$\begin{aligned}
\sum_{m\neq 0} |c_{m0}w_{m0}(R)\xi_{m0}| &\leq R^{\frac{1}{n}} \sum_{m\neq 0} e^{-2\sqrt{-\frac{1}{2}\lambda_{m0}R}+2\sqrt{-\frac{1}{2}\lambda_{m0}}} |c_{m0}w_{m0}(1)\xi_{m0}| \\
&\leq \tilde{C}R^{\frac{1}{n}} \sum_{m\neq 0} e^{-2\sqrt{-\frac{1}{2}\lambda_{m0}R}+2\sqrt{-\frac{1}{2}\lambda_{m0}}} \\
&\leq \tilde{C}R^{\frac{1}{n}} \sum_{m\neq 0} e^{-C(\sqrt{R}-1)m^{\frac{1}{2n-1}}} \\
&\leq \tilde{C}R^{\frac{1}{n}} \int_1^\infty e^{-C(\sqrt{R}-1)m^{\frac{1}{2n-1}}} dm \\
&= \tilde{C}R^{\frac{1}{n}} \int_1^\infty s^{2n-2} e^{-C(\sqrt{R}-1)s} ds \\
&\leq \tilde{C}e^{-C\sqrt{R}}.
\end{aligned}$$

Putting all results together we obtain

$$|v| \leq \tilde{C}e^{-C\sqrt{R}},$$

and hence $u - u_{00} \in O(R^{-p+\frac{1}{n}})$.

Differentiating the integral representations of the homogeneous solution and applying the results for $\frac{\partial u_{mk}}{\partial R}$ from section 4.8 we obtain with the same techniques the corresponding result for $u_{,R}$. □

4.9 Proof of Theorem 4.1.1

We assume $\Delta_\omega u \in O((-\log\|S\|^2)^{-p})$ and $u \in C^\infty(X \setminus D) \cap C^{3n+6,\alpha}_\omega(X \setminus D)$. Due to Lemma 4.7.3 this implies

$$\Delta_g u = f$$

with

$$|f| \leq C(-\log\|S\|^2)^{-p}$$

and hence

$$(\Delta_g u)_{mk} = f_{mk}$$

with

$$|f_{mk}| \leq C_{mk}(-\log \|S\|^2)^{-p})$$

for all m, k.

In order to apply the results of Section 4.8 we have to compute

$$\|\partial\overline{\partial}u\|^2_\omega = \omega^{i\bar{l}}\omega^{k\bar{j}}u_{,i\bar{j}}u_{,k\bar{l}}.$$

With $z_1 = S$, using the linear theory together with the estimates (4.12), the equalities hold

$$\begin{aligned} u_{,1\bar{1}} &= \frac{\|.\|^2}{4}e^{2R}(u_{,RR} + u_{,\varphi\varphi} + O(e^{-R})) \\ u_{,1\bar{k}} &= -\frac{\|.\|_{,\bar{k}}}{2}e^{R-i\varphi}(u_{,R} + iu_{,\varphi}) - \frac{\|.\|}{2}e^{R-i\varphi}(u_{,R\bar{k}} + iu_{,\varphi\bar{k}}) + O(1) \end{aligned}$$

Abbreviating

$$A(u) := \frac{\|.\|^2}{4}(2R)^{1-\frac{1}{n}}(u_{,RR} + u_{,\varphi\varphi})$$

and

$$B(u) := (2R)^{-\frac{1}{n}}\sum_k(\frac{\|.\|_{,\bar{k}}}{2}(u_{,R} + iu_{,\varphi}) - \frac{\|.\|}{2}(u_{,R\bar{k}} + iu_{,\varphi\bar{k}}))$$

we see

$$\begin{aligned} \|\partial\overline{\partial}u\|^2_\omega \leq\ & C_1|A(u)|^2 + C_2|A(u)||B(u)| + C_3|B(u)|^2 + \\ & + C_{k\bar{l}}(2R)^{-\frac{1}{n}-1}(|A(u)| + |B(u)|)|u_{,k\bar{l}}| + C_{i\bar{j}k\bar{l}}(2R)^{-2-\frac{2}{n}}|u_{,i\bar{j}}u_{,k\bar{l}}|. \end{aligned} \tag{4.13}$$

Here the indices i, j, k, l are running between 2 and n. We note that $u_{00} \in O(R^{-p+1+\frac{1}{n}})$, so the same arguments as in the proof of Theorem 4.8.4 yield with the estimate of Lemma 4.8.3 that $u_{,i\bar{j}} \in O(R^{-p+1+\frac{1}{n}})$, so the last term of equation 4.13 decays with the correct rate $-2p$.

So, in order to prove Theorem 4.1.1 it is enough to show that

$$|A(u)| \in O(R^{-p}) \text{ and } |B(u)| \in O(R^{-p}).$$

For this purpose we translate this condition into the Fourier expansion of Section 4.8. Defining $A_{mk} := e^{-ik\varphi}A(u_{mk}e^{ik\varphi})$ and $B_{mk} := e^{-ik\varphi}B(u_{mk}e^{ik\varphi}\xi_{mk})$ we can plug in the ODEs (4.11) in order to see

$$A_{mk} = f_{mk} - (2R)^{-\frac{1}{n}}u_{mk,R} + (k^2 + \lambda_{mk})(2R)^{-\frac{1}{n}}u_{mk}.$$

Again, checking the cases distinguished in section 4.8 we obtain

$$A_{mk} \in O(R^{-p}),\ B_{mk} \in O(R^{-p})$$

for all $(m,k) \neq 0$. The function u_{00} occurs in A_{00} with coefficient zero, however. So application of the same techniques as in the proof of Theorem 4.8.4 yield $A(u) \in O(R^{-p})$.

For the term $B(u)$ we note that $B_{mk} = B^{(1)}_{mk} + B^{(2)}_{mk}$ with

$$\begin{aligned} B^{(1)}_{mk} &= (2R)^{-\frac{1}{n}} \sum_l \frac{\|.\|_{,\bar{l}}}{2}(u_{mk,R} - ku_{mk})\xi_{mk} \\ B^{(2)}_{mk} &= -(2R)^{-\frac{1}{n}} \sum_l \frac{\|.\|}{2}(u_{mk,R} - ku_{mk})\xi_{mk,\bar{l}}. \end{aligned}$$

For both terms we apply the techniques of the proof of Theorem 4.8.4 in connection with Lemma 4.8.3.

4.10 Proof of Corollary 4.1.2

Let u be the bounded solution of the Monge-Ampère equation (4.1). First we note that f like constructed satisfies the assumptions of Theorem 4.1.1. For $\Delta_\omega u$ we will prove a similar property using the estimates in quasi-coordinates of section 4.6 yielding the same result. Again the spectral decomposition of Section 4.8 can be used to see

$$\begin{aligned} |u_{mk}(R)| &= \left|\int_Y ue^{-ik\varphi}\overline{\xi}_{mk} dvol_{g_{R=\frac{1}{2}}}\right| \\ &\leq (k^2 - \lambda_{mk})^{-1}\left|\int \Delta_{g_{R=\frac{1}{2}}} ue^{-ik\varphi}\overline{\xi}_{mk} dvol_{g_{R=\frac{1}{2}}}\right| \\ &= (k^2 - \lambda_{mk})^{-1}|(\Delta_{g_{R=\frac{1}{2}}} u)_{mk}| \\ &\leq C(k^2 - \lambda_{mk})^{-1} \end{aligned}$$

For the last inequality we use Proposition 4.6.3 after expressing the angular coordinate in quasi coordinates. Iterating this estimate and using Corollary 4.8.2 yields that for every k, m and j there is a constant C_j such that

$$|u_{mk}(R)| \leq C_j(k^{2n-1} + m)^{-j}.$$

The proof of Theorem 4.8.4 told us that independently of the behaviour of higher Laplacians, the exponential decaying part $c_{mk}w_{mk}\xi_{mk}$ of u_{mk} has an exponential decaying bound independent of (m,k),

$$|c_{mk}w_{mk}\xi_{mk}| \leq \tilde{C}e^{-C\sqrt{R}},$$

hence

$$\Delta_\omega u \in O(R^{-p}) \Rightarrow |u_{mk}| \leq CR^{-p+\frac{1}{n}}$$

for a constant C independent of $(m,k) \neq (0,0)$. Again $|u_{00}| \leq CR^{-p+1+\frac{1}{n}}$.

So interpolating both inequalities for every $0 < \alpha < 1$ and l there is a constant $C_{\alpha,l}$ such that

$$|R^{-\frac{1}{n}} u_{mk}| \leq C_{\alpha,l} R^{-\alpha p} (k^{2n-1} + m)^{-l(1-\alpha)}$$

for $(m,k) \neq 0$ and similarly for $(m,k) = (0,0)$. Now we choose l big enough and use Lemma 4.8.3 to conclude

$$|R^{-1-\frac{1}{n}} u_{,i\bar{j}}| \leq \sum_{m,k} |u_{mk} \xi_{mk,i\bar{j}}| \leq C_\alpha R^{-\alpha p}$$

for all $i, j \geq 2$. These two estimates show like in the proof of the Main Theorem (Section 4.9):

Lemma 4.10.1 *Let u be the bounded solution of the Monge-Ampère equation (4.1). For every $p > 0$ and $0 < \alpha < 1$*

$$|\Delta_\omega u| \leq CR^{-p} \Rightarrow \|\partial\overline{\partial} u\|_\omega \leq \tilde{C} R^{-\alpha p}.$$

Now we consider the sets

$$E_1 := \{p \in \mathbb{R}^+ |\ \exists C > 0 : \ |\Delta_\omega u| \leq C(-\log \|S\|^2)^{-p}\}$$

and

$$E_2 := \{p \in \mathbb{R}^+ |\ \exists C > 0 : \ \|\partial\overline{\partial} u\|_\omega \leq C(-\log \|S\|^2)^{-p}\}.$$

By Theorem 4.5.4 E_2 is not empty and $\overline{E_1} = \overline{E_2}$ by Lemma 4.10.1. We may assume that E_1 has a supremum q. So, for any $\varepsilon > 0$ we have a sequence $x_i \in X$ converging to D such that

$$|\Delta_\omega u(x_i)| \geq (-\log \|S(x_i)\|^2)^{-q-\varepsilon}.$$

So,

$$\begin{aligned} e^f - 1 &\geq |\Delta_\omega u| - \left| \sum_{i=2}^n \binom{n}{i} \frac{(\partial\overline{\partial} u)^i \wedge \omega^{n-i}}{\omega^n} \right| \\ &\geq (-\log \|S\|^2)^{-q-\varepsilon} - C(-\log \|S\|^2)^{-2\alpha(q+\varepsilon)} \\ &\geq \frac{1}{2}(-\log \|S\|^2)^{-q-\varepsilon} \end{aligned}$$

on a subsequence of x_i provided $\alpha > \frac{1}{2}$ and $\varepsilon < \frac{2\alpha-1}{2\alpha+1}$. This is impossible.

4.11 Distance function and volume growth

Let $\tilde{\omega}$ be the Ricci-flat Kähler metric constructed above and R the global radial coordinate in a tubular neighbourhood U of D as constructed above.

Proposition 4.11.1 *There are positive constants C_1, C_2, C_3 such that for all $x, y \in U$ with $R(x) \geq R(y)$ holds*

$$\begin{aligned} \mathrm{dist}_{\tilde{\omega}}(x,y) &\geq C_1(R(x)^{\frac{n+1}{2n}} - R(y)^{\frac{n+1}{2n}}) - C_2 R(y)^{\frac{1}{2n}} - C_3 R(y)^{\frac{1-n}{2n}} \\ \mathrm{dist}_{\tilde{\omega}}(x,y) &\leq C_1(R(x)^{\frac{n+1}{2n}} - R(y)^{\frac{n+1}{2n}}) + C_2 R(y)^{\frac{1}{2n}} + C_3 R(y)^{\frac{1-n}{2n}} \end{aligned}$$

Proof. Note that for Riemannian metrics $h, \tilde{h}$ on a manifold Z with $h \leq \tilde{h}$ a look at $\tilde{h}$-geodesics confirms that $\mathrm{dist}_h(x,y) \leq \mathrm{dist}_{\tilde{h}}(x,y)$ for all $x, y \in Z$. Thus, it suffices to prove the claimed inequalities for g instead of $\tilde{\omega}$. Recall the structure of g

$$g = (2R)^{\frac{1}{n}-1} dR \otimes dR + (2R)^{\frac{1}{n}} \pi^* g_D + (2R)^{\frac{1}{n}-1} \delta \otimes \delta.$$

We use the isomorphism $U \cong (1,\infty) \times Y$, write $x = (R(x), z(x)), y = (R(y), z(y))$ and choose a path $\gamma = \gamma_1 \vee \gamma_2 \vee \gamma_3$ from y to x decomposed into paths $\gamma_i : [0,1] \longrightarrow U$ as follows:

(i) $\gamma_3(0) = y, \pi(z(\gamma_3(1))) = \pi(z(x))$, $\pi \circ z \circ \gamma_3$ is a g_D-geodesic on D, $\delta(\gamma_3') \equiv 0$ and $R \circ \gamma_3$ is constant; the block like structure of g implies that γ_3 is a g-geodesic.

(ii) $\gamma_2(0) = \gamma_3(1), z(\gamma_2(1)) = z(x)$, as well $R \circ \gamma_2$ as $\pi \circ z \circ \gamma_2$ are constant and γ_2 is a geodesic on the fibre $\pi^{-1}(\pi(z(x)))$ with respect to the pullback of g to the fibre; again the block like structure implies that γ_2 is a g-geodesic.

(iii) $\gamma_1(t) := (tR(x) + (1-t)R(y), z(x))$; this is again a g-geodesic.

The lengths of the paths γ_i are estimated to be

$$\begin{aligned} l(\gamma_1) &= C_1(R(x)^{\frac{n+1}{2n}} - R(y)^{\frac{n+1}{2n}}) \\ l(\gamma_2) &\leq C_2 R(y)^{\frac{1}{2n}} \\ l(\gamma_3) &\leq C_3 R(y)^{\frac{1-n}{2n}} \end{aligned}$$

This proves the upper bound directly. Using the fact that all γ_i are g-geodesics these estimates also imply the lower bound. □

This will imply less than quadratic volume growth.

Proposition 4.11.2 *Let $x \in X$ be fixed. There are positive constants c, C, r such that for $\tilde{\omega}$-geodesic balls $B_\rho(x)$ of radius $\rho > r$ around x*

$$c\rho^{\frac{2n}{n+1}} \leq \mathrm{vol}_{\tilde{\omega}}(B_\rho(x)) \leq C\rho^{\frac{2n}{n+1}}.$$

Proof. The claim being invariant under the choice of x we may assume $x \in U$ for an appropriate tubular neighbourhood U of D. Since $\tilde{\omega}$ is complete, the compact set $K := X \setminus U$ is bounded, so

$$r_1 := \max_{y \in X \setminus U} \mathrm{dist}_{\tilde{\omega}}(y, x),\ A := \mathrm{vol}_{\tilde{\omega}}(K)$$

are well defined.

We compute

$$\sqrt{\det g} = dR \wedge \delta \wedge \sqrt{\det \pi^* g_D}.$$

Let us denote

$$\tilde{r} := \max(r_1, C_1 R(x)^{\frac{n+1}{2n}} + C_2 R(x)^{\frac{1}{2n}} + C_3 R(x)^{\frac{1-n}{2n}})$$

with C_1, C_2, C_3 being the constants from Proposition 4.11.1. The same proposition implies

$$(R(y) \leq R(x) \Rightarrow \mathrm{dist}_{\tilde{\omega}}(y, x) \leq \tilde{r}).$$

Furthermore there are constants a, b depending on C_1, C_2, C_3 and $R(x)$ such that

$$(R(y) \geq R(x) \Rightarrow \left(\frac{\mathrm{dist}_{\tilde{\omega}}(y, x) - b}{C_1}\right)^{\frac{2n}{n+1}} \leq R(y) \leq \left(\frac{\mathrm{dist}_{\tilde{\omega}}(y, x) + a}{C_1}\right)^{\frac{2n}{n+1}}).$$

So

$$K \cup \left\{ y \in U |\ R(y) \leq \left(\frac{\rho - b}{C_1}\right)^{\frac{2n}{n+1}} \right\} \subset B_\rho(x) \subset K \cup \left\{ y \in U |\ R(y) \leq \left(\frac{\rho + a}{C_1}\right)^{\frac{2n}{n+1}} \right\},$$

whenever $\rho > \tilde{r}$ and hence integrating the volume form of g yields

$$A + 2\pi \mathrm{vol}_{g_D}(D) \left(\left(\frac{\rho - b}{C_1}\right)^{\frac{2n}{n+1}} - 1 \right) \leq \mathrm{vol}_g(B_\rho(x)) \leq A + 2\pi \mathrm{vol}_{g_D}(D) \left(\left(\frac{\rho + a}{C_1}\right)^{\frac{2n}{n+1}} - 1 \right),$$

i.e.

$$C_4 + C_5 \rho^{\frac{2n}{n+1}} \leq \mathrm{vol}_{\tilde{\omega}}(B_\rho(x)) \leq C_6 + C_7 \rho^{\frac{2n}{n+1}}$$

for positive constants C_5, C_7, whenever $\rho > \tilde{r}$. So $r \geq \tilde{r}$, depending on C_4, C_5, C_6, C_7, can be chosen so as to fulfil the claim. $\square$

4.12 Extensions of holomorphic maps

The following Lemma is well-known (cf. [Sch98]) but rarely stated explicitly or proved. It will be applied in this chapter, so here a proof is given.

Lemma 4.12.1 *Let X, Y be complex projective manifolds, $C \subset X$ and $D \subset Y$ smooth, ample divisors. If $f : X \setminus C \longrightarrow Y \setminus D$ is a rational biholomorphic map, then there exists a biholomorphic extension $\tilde{f} : X \longrightarrow Y$.*

Proof. Since X is projective, we may identify $\mathcal{O}_X(C)$ with the germs of rational functions on X having a at most a pole of order one along C. We identify f with the birational map $X \longrightarrow Y$ given by f. The pullback $f^* : K(Y) \longrightarrow K(X)$ is an isomorphism by assumption. Since f is biholomorphic on $X \setminus C$, the poles of functions in $f^*\mathcal{O}_Y(D)$ must lie in C. The maximal pole order is realized by a general element, so we infer by a symmetry argument $f^*\mathcal{O}_Y(D) \subset \mathcal{O}_X(C)$ and again by symmetry $f^*\mathcal{O}_Y(D) = \mathcal{O}_X(C)$. Now consider the projective embeddings $\phi_{|mC|} : X \longrightarrow \mathbb{P}^N$ and $\phi_{|mD|} : Y \longrightarrow \mathbb{P}^N$ (note that N is the same for both embeddings) given by bases of $\mathcal{B} \subset H^0(\mathcal{O}_Y(D))$ and $f^*\mathcal{B} \subset H^0(\mathcal{O}_X(C))$. The triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\phi_{|mC|}"'] & & Y \arrow[dl, "\phi_{|mD|}"] \\ & \mathbb{P}^N & \end{tikzcd}$$

commutes, i.e. f is the restriction of the identity on $\mathbb{P}^N$ to $X \setminus C$. In particular, f is extendable to a biholomorphic map $f : X \longrightarrow Y$. $\square$

Theorem 4.12.2 *Let X be a Fano manifold, $D \in |-K_X|$ smooth and $\phi \in \mathrm{Aut}(X \setminus D)$ such that $\phi^*\tilde{\omega}$ is C^0-equivalent to $\tilde{\omega}$. Then ϕ has an extension $\tilde{\phi} \in \mathrm{Aut}(X, D)$. In particular, $\mathrm{Aut}(X \setminus D, \tilde{\omega}) = \mathrm{Aut}(X, D, \tilde{\omega})$.*

Proof. We fix constants such that

$$a\tilde{\omega} \le \phi^*\tilde{\omega} \le A\tilde{\omega}$$

and

$$c\tilde{\omega} \le \omega \le C\tilde{\omega}$$

in a tubular neighbourhood of D (cf. [TY90, Thm 1.1]). Choose a sequence $(x_n) \longrightarrow x \in D$ in the manifold topology of X with $x_n \notin D$. We apply Proposition 4.11.1 in order to compute

$$\begin{aligned}
\mathrm{dist}_\omega(\phi(x_n),\phi(x_0)) &\leq C^{\frac{1}{2}}\mathrm{dist}_{\tilde\omega}(\phi(x_n),\phi(x_0)) \\
&= C^{\frac{1}{2}}\mathrm{dist}_{\phi^*\tilde\omega}(x_n,x_0) \\
&\leq (AC)^{\frac{1}{2}}\mathrm{dist}_{\tilde\omega}(x_n,x_0) \\
&\leq (AC)^{\frac{1}{2}}c^{-\frac{1}{2}}\mathrm{dist}_\omega(x_n,x_0) \\
&\sim C_1(-\log\|S(x_n)\|^2)^{\frac{n+1}{2n}}.
\end{aligned}$$

So there exists a subsequence, called again x_n, such that $\phi(x_n) \longrightarrow y \in D$ in the manifold topology of X. This implies

$$\mathrm{dist}_\omega(\phi(x_n),\phi(x_0)) \sim (-\log\|S(\phi(x_n))\|^2)^{\frac{n+1}{2n}}.$$

Hence

$$\|S(\phi(x))\| \geq C_2\|S(x)\|^{C_3},$$

first in a neighbourhood of D, but then also on all of X.

By a standard argument, such an inequality implies rationality of the map ϕ. We give an outline of the argument. A basis of $H^0(\mathcal{O}_X(mD))$ is embedding $X \overset{i}{\hookrightarrow} \mathbb{P}^N$ with homogeneous coordinates $[Z_0 : \cdots : Z_N]$; it can be chosen such that $mD = \{Z_0 = 0\}$. Let U be a neighbourhood around a point $z \in D$ and z_i coordinates, such that $S(z) = z_1$. The map

$$\tilde\phi = i \circ \phi : X \setminus D \longrightarrow \mathbb{P}^N$$

has image in $\{Z_0 \neq 0\}$, so we can respresent $\tilde\phi$ by $[1 : \tilde\phi_1 : \cdots : \tilde\phi_N]$ with holomorphic functions $\tilde\phi_i : X \setminus D \longrightarrow \mathbb{C}$. The norm $\|.\|^m$ on $\mathcal{O}_X(mD)$ and the pullback of the standard metric $\|.\|_{FS}$ on $\mathcal{O}_{\mathbb{P}^N}(1)$ under i differ only by a unit $u \in C^\infty(X)$:

$$\|.\|^{2m} = u \cdot i^*\|.\|^2_{FS}.$$

Since for the sections $Z_i \in H^0(\mathcal{O}_{\mathbb{P}^N}(1)$ the norm is $\|Z_i\|^2_{FS} = \frac{|Z_i|^2}{|Z|^2}$ and $i^*Z_0 = S^m$ we compute

$$\|S(\phi(z))\|^{2m} = u(\phi(z))\frac{1}{1+\sum_{i=1}^N|\tilde\phi_i(z)|^2}.$$

On the other hand, $\|S(z)\|^2 = h|z_1|^2$ for a unit $h \in C^\infty(U)$, so the inequality transforms to

$$1 + \sum_{i=1}^N |\tilde\phi_i(z)|^2 \leq C|z_1|^{-2C_3m},$$

for any relatively compact $V \subset U$. As usual, the Cauchy integral formula implies that all $\tilde\phi_i$ have poles along $D \cap V$ of order at most C_3m, so $\tilde\phi$ is meromorphic, in particular. By GAGA for projective manifolds we obtain that ϕ is rational. We apply Lemma 4.12.1 to conclude the proof. $\square$

Chapter 5

Ricci-flat Deformations of Vector Bundles and their application

5.1 Introduction

Let X be a compact complex manifold, U a complex manifold without holomorphic vector bundles (e.g. a contractible Stein manifold), and $\mathcal{E} \longrightarrow X \times U$ a holomorphic vector bundle. (So the deformations we deal here with are neither global nor small but something in between.) Further let g be a hermitian metric on $\mathcal{E}$. The data of the central fibre will be denoted by $\mathcal{E}_0$ and g^0. This setup is called a deformation of the hermitian vector bundle $(\mathcal{E}_0, g^0)$. As usual we denote $\Theta^0 := \partial\overline{\partial} \log \det g^0$ and $\Theta^t := \partial_X \overline{\partial}_X \log \det g|_{\mathcal{E}_t}$ the curvatures.

Definition 5.1.1 *The deformation $(\mathcal{E}, g)$ is called Ricci-flat, if $\Theta^t = \Theta^0$ for all $t \in U$.*

We tackle the question of existence of Ricci-flat deformations. In Theorem 4.1.1 we show that the obstruction space for this problem is

$$H^1(X, \mathcal{O}_X)/i_* H^1(X, \mathbb{R}),$$

if $i : \mathbb{R} \longrightarrow \mathcal{O}_X$ is the natural inclusion. This group is trivial for Kähler manifolds, so on Kähler manifolds we can extend every metric on the central fibre curvature preservingly to the deformation. In the non-Kähler case we can still show that the property of a given vector bundle to admit only Ricci-flat deformations is independent of the vector bundle and hence a property of the manifold, namely the vanishing of the obstruction space.

The answer to the question, whether for a vector bundle any Ricci-flat deformation is trivial, depends on the bundle, however, and will be denoted by Ricci-rigidity. This generalises the notion of rigidity to manifolds with non-vanishing first cohomology. We will see that this property is connected to a minimising property of $H^1(\mathcal{E}nd(\mathcal{E}))$.

On Hopf manifolds we deepen this connection and so construct Ricci-rigid vector bundles. We relate Ricci-rigidity to the non-triviality of the pullback to the universal cover $\mathbb{C}^n \setminus \{0\}$.

It should be noted that the algebraic and analytic category differ widely in this case. Whereas on $\mathbb{C}^2$ and $\mathbb{C}^2\setminus\{0\}$ there are only trivial algebraic vector bundles [S58, H64], on $\mathbb{C}^2\setminus\{0\}$ some non-trivial holomorphic vector bundles have been constructed before. The paper by Bănică and LePotier [BP87] classified filtrable holomorphic vector bundles on non-algebraic surfaces. In particular, for all integers $r \geq 2, c_2 \geq 0$ there exists a holomorphic rank r bundle $\mathcal{E}$ with $c_2(\mathcal{E}) = c_2$ on a Hopf surface. Calculations below and in [M92] show that the pullback to $\mathbb{C}^2 \setminus \{0\}$ of such a bundle is not trivial provided $c_2 > 0$. The constructed bundles are filtrable. Later Ballico [B02] constructed a non-filtrable rank 2 bundle on $\mathbb{C}^2 \setminus \{0\}$. In [S66] a non-trivial line bundle on $\mathbb{C}^2 \setminus \{0\}$ is constructed.

5.2 The local data

By the compactness of X we can employ the trivialisations of $\mathcal{E}$ in order to obtain a finite cover of open sets $U_i \subset X$ and isomorphisms

$$\psi_i : \mathcal{E}|_{U_i\times U} \longrightarrow pr_1^*\mathcal{E}_0|_{U_i\times U},$$

where $pr_1 : X \times U \longrightarrow X$ is the projection. This yields the data

$$\theta_{ij} := \psi_i \circ \psi_j^{-1}, g_i := (\psi_i^{-1})^* g,$$

satisfying $g_j = \theta_{ij}^* g_i$, obviously. Note that $\theta_{ij}|_{U_{ij}\times\{0\}} = Id$. Similarly, we have the extended metric of the central fibre

$$<,>:= pr_1^* g^0$$

as a comparison metric. We obtain by Lax-Milgram $G_i \in End_{C^\infty}(pr_1^*\mathcal{E}_0|U_i \times U)$ with the property

$$g_i(e_1, e_2) =< e_1, G_i e_2 >$$

for any C^∞ sections e_1, e_2 of $pr_1^*\mathcal{E}_0$ over U_i. Note that again $G_i|_{U_i\times\{0\}} = Id$. On $U_{ij} \times U$ we have the formula

$$G_i = \theta_{ji}^* G_j \theta_{ji}. \tag{5.1}$$

Here θ_{ji}^* denotes the adjoint of θ_{ji} with respect to $<,>$. It is easy to see that the deformation is Ricci-flat if and only if

$$\partial_X \overline{\partial}_X \log\det G_i = 0 \text{ for all } i. \tag{5.2}$$

The inclusions $\mathbb{Z} \subset \mathbb{R} \subset \mathcal{O}_X$ give a commuting triangle

$$\begin{tikzcd} H^1(X,\mathbb{Z}) \arrow[r] \arrow[dr] & H^1(X,\mathbb{R}) \arrow[d, "i_*"] \\ & H^1(X,\mathcal{O}_X) \end{tikzcd}$$

of injective maps. We view all unnamed maps as natural inclusions.

For any deformation the map

$$\eta : U \longrightarrow H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z}) = Pic^0(X)$$

given by

$$t \mapsto [\frac{1}{2\pi i} \log\det\theta_{ij}(.,t)] = \det\mathcal{E}_t \otimes \det\mathcal{E}_0^\vee$$

is a well-defined holomorphic map with $\eta(0) = 0$. If we assume the deformation to be small, we may assume that η maps to $H^1(X,\mathcal{O}_X)$.

5.3 The infinitesimal data

We give the infinitesimal data for the one-dimensional case $U = (\mathbb{C}, 0)$. The generalisation to $U = (\mathbb{C}^n, 0)$ is straightforward.

We take the Taylor series to the first order

$$\theta_{ij} = Id + th_{ij} + h.o.t., G_i = Id + t\gamma_i + \bar{t}\gamma_i^* + h.o.t.$$

with $h_{ij} \in End(\mathcal{E}_0|U_{ij}), \gamma_i \in End_{C^\infty}(\mathcal{E}_0|U_i)$. The h_{ij} satisfy the cocycle condition and hence yield an element $h \in H^1(\mathcal{E}nd(\mathcal{E}_0))$. Indeed, if the deformation is trivial, h_{ij} is a coboundary. By comparison of

$$g_i(e_1,e_2) =< e_1, e_2 > +t < e_1, \gamma_i^* e_2 > +\bar{t} < e_1, \gamma_i e_2 > +h.o.t.$$

and

$$\begin{aligned} \theta_{ji}^* g_j(e_1,e_2) &= < e_1, e_2 > +t(< h_{ji}e_1, e_2 > + < e_1, \gamma_j^* e_2 >) + \\ &\quad +\bar{t} < e_1, (\gamma_j + h_{ji})e_2 > +h.o.t. \end{aligned}$$

we obtain

$$h_{ij} = \gamma_j - \gamma_i \tag{5.3}$$

on U_{ij}. This tells us that the γ_i trivialise the deformation in the C^∞ sense to the first order.

Now let us consider curvature

$$\begin{aligned}\Theta_i &:= \partial_X\overline{\partial}_X \log\det g_i \\ &= \Theta^0 + \partial\overline{\partial}\log\det G_i \\ &= \Theta^0 + \partial\overline{\partial}\log(1 + t\mathrm{tr}\gamma_i + h.o.t) \\ &= \Theta^0 + t\partial\overline{\partial}\mathrm{tr}\gamma_i + h.o.t.\end{aligned}$$

So, any Ricci-flat deformation satisfies

$$\partial\overline{\partial}\mathrm{tr}\gamma_i = 0. \tag{5.4}$$

5.4 Existence

We start with an arbitrary metric $\tilde{g}$ on a deformation $\mathcal{E}$. This yields the data $\tilde{\psi}_i$, $\tilde{g}^0$ and $\tilde{G}_i$. By the Cartan decomposition we can find a unique positive hermitian $A \in End_{C^\infty}(\mathcal{E}_0)$ with respect to $\tilde{g}^0$ such that

$$g^0(e_1, e_2) = \tilde{g}^0(Ae_1, Ae_2)$$

for any local sections e_1, e_2 of $\mathcal{E}_0$. Setting $\psi_i := A^{-1} \circ \tilde{\psi}_i, G_i := A^{-1}\tilde{G}_i A$ (now with respect to $<,>:= pr_1^* g^0$) we obtain a deformation of g^0.

The question of existence of a Ricci-flat deformation is more subtle.

Theorem 5.4.1 *A deformation $\mathcal{E} \longrightarrow X \times U$ of a hermitian holomorphic vector bundle $(\mathcal{E}_0 \longrightarrow X, g^0)$ allows for a curvature preserving metric g extending g^0 if and only if*

$$\eta(t) \in i_* H^1(X, \mathbb{R})/H^1(X, \mathbb{Z})$$

for all $t \in U$ and the inclusion $i : \mathbb{R} \hookrightarrow \mathcal{O}_X$; moreover, this condition being satisfied, in every conformal class of metrics g deforming g^0 there is a Ricci-flat representative.

Proof. Necessity: We shrink U_i appropriately such that they are simply connected. By (5.2) we find for every $t \in U$ holomorphic $h_i(t) \in \mathcal{O}_X(U_i)$ such that

$$\log\det G_i(t) = \mathrm{Re}(h_i(t)).$$

So we have by (5.1) on U_{ij}

$$2\mathrm{Re}(\log\det\theta_{ij}(t)) = \mathrm{Re}(h_i(t) - h_j(t)),$$

hence there are $c_{ij}(t) \in \mathbb{R}$ such that

$$\frac{1}{2\pi i}(2\log\det\theta_{ij}(t) - h_i(t) + h_j(t)) = c_{ij}(t),$$

but this means exactly

$$\eta(t) \in i_* H^1(X, \mathbb{R})/H^1(X, \mathbb{Z}).$$

Sufficiency: Since $\theta_{ij}|U_{ij} \times \{0\} = Id$, by shrinking U we may assume that $\theta_{ij} = \exp(2\pi i k_{ij})$ for some $k_{ij} \in End(pr_1^* \mathcal{E}_0 | U_{ij} \times U)$. Note that (k_{ij}) is no cocycle, unless they commute. Nevertheless, $\det \theta_{ij} = \exp(2\pi i \mathrm{tr} k_{ij})$, so $(\mathrm{tr} k_{ij}) \in \mathcal{O}(U_{ij} \times U)$ is a cocycle, defined uniquely by θ_{ij} up to an integer. In particular, $(\mathrm{tr} k_{ij}|U_{ij} \times \{t\}) \in \mathcal{O}(U_{ij})$ is a cocycle for all $t \in U$. Since we assumed that $\eta(t) = [\mathrm{tr} k_{ij}(t)] \in i_* H^1(X, \mathbb{R})/H^1(X, \mathbb{Z})$, we can choose $K_i(t) \in \mathcal{O}(U_i)$ and $\phi_{ij}(t) \in \mathbb{R}$ such that

$$\mathrm{tr} k_{ij} = K_j - K_i + \phi_{ij}$$

and $K_i(0) = 0, \phi_{ij}(0) = 0$. The dependence on t of K_i and ϕ_{ij} is not holomorphic anymore, but it can be chosen to be C^∞. For $H_i := \exp(2\pi i K_i)$ we obtain

$$\det \theta_{ij} \exp(-2\pi i \phi_{ij}) = \frac{H_j}{H_i}$$

and hence the deformed metric has to satisfy

$$\det G_j = \frac{|H_j|^2}{|H_i|^2} \det G_i. \tag{5.5}$$

Now we can take any metric $\tilde{g}$ deforming g^0 (with data $\tilde{G}_i$) and take a conformal change:

$$G_i := \left(\frac{|H_i|^2}{\det \tilde{G}_i} \right)^{\frac{1}{n}} \tilde{G}_i.$$

We obtain $\det G_i = |H_i|^2$ and hence $\partial\overline{\partial} \log \det G_i = 0$, so we have a Ricci-flat deformation.

Now covering the original U by neighbourhoods like above and using the uniqueness of $\mathrm{tr} k_{ij}$ up to an integer, we extend the criterion to all of U. □

In the infinitesimal information we loose sufficiency of the condition:

Proposition 5.4.2 *A Ricci-flat deformation germ of hermitian vector bundles satisfies*

$$\frac{1}{2\pi i} trh \in V,$$

where V is the maximal complex subspace of $i_* H^1(X, \mathbb{R})$.

Proof. We shrink U_i appropriately such that they are simply connected. By (5.4) we find holomorphic $f_i, g_i \in \mathcal{O}(U_i)$ such that $\mathrm{tr}\gamma_i = f_i + \overline{g}_i$. So (5.3) tells us that

$$\mathrm{tr} h_{ij} + f_i - f_j = \overline{g}_j - \overline{g}_i.$$

Since the right hand side is antiholomorphic and the left hand side is holomorphic, we find constants $c_{ij} \in \mathbb{C}$ such that

$$c_{ij} = \mathrm{tr} h_{ij} + f_i - f_j = \overline{g}_j - \overline{g}_i.$$

So we obtain in $i_* H^1(X, \mathbb{C})$

$$\frac{1}{2\pi i}\mathrm{tr} h = \frac{1}{2\pi i}\mathrm{tr} c = \frac{1}{2\pi i}(\overline{c} + 2i\mathrm{Im} c) = \frac{\mathrm{Im} c}{\pi} \in i_* H^1(X, \mathbb{R}).$$

If we look at a base transformation $\tau : (T, 0) \longrightarrow (T, 0)$ and denote the objects corresponding to the deformation $\tau^*\mathcal{E}$ of $\mathcal{E}_0$ by a superscript τ, it is straightforward that

$$h_{ij}^{\tau} = \tau'(0) h_{ij}.$$

This proves the statement. □

5.5 Stable Curvature and Ricci-Rigidity

We see that Ricci-flatness of a deformation does not depend on the initial metric on the central bundle. There are two natural properties connected to Ricci-flat deformations:

Definition 5.5.1 *The vector bundle $\mathcal{E}$ is said to have* stable curvature*, if all deformations of $\mathcal{E}$ over a base U without holomorphic vector bundles are Ricci-flat. $\mathcal{E}$ is* Ricci-rigid*, if every small Ricci-flat deformation of $\mathcal{E}$ is trivial.*

Our first goal is to realise that stable curvature is not a property of a vector bundle, but of the underlying manifold:

Proposition 5.5.2 *Let X be a compact complex manifold. Then the properties*

(i) There exists a vector bundle $\mathcal{E}$ with stable curvature,

(ii) All vector bundles have stable curvature,

(iii) $i_* : H^1(X, \mathbb{R}) \longrightarrow H^1(X, \mathcal{O}_X)$ is an $\mathbb{R}$-isomorphism

are equivalent.

Proof. Since $(ii) \Rightarrow (i)$ is obvious and $(iii) \Rightarrow (ii)$ is Theorem 5.4.1, it remains to show $(i) \Rightarrow (iii)$. So we assume that i_* is not surjective. In particular, $\dim Pic^0(X) > 0$, hence $\mathcal{O}_X$ can be deformed non-trivially. So, for any deformation $\mathcal{L}_t$ of $\mathcal{L}_0 = \mathcal{O}_X$ with $\mathcal{L}_t \notin i_* H^1(X, \mathbb{R})/H^1(X, \mathbb{Z})$ for all real t and arbitrary vector bundle $\mathcal{E}$, the deformation

$\mathcal{E} \otimes \mathcal{L}_t$ satisfies $\eta(t) = \mathcal{L}_t^n \notin i_* H^1(X, \mathbb{R})/H^1(X, \mathbb{Z})$ for all real t. Hence we have shown that no vector bundle has stable curvature. □

Examples of such "stable curvature" manifolds are all compact Kähler manifolds. Counterexamples are all compact manifolds with odd $b_1(X)$, e.g. Hopf surfaces.

Also Ricci-rigidity has geometric implications.

Proposition 5.5.3 *If X allows for a Ricci-rigid vector bundle, then all germs of holomorphic maps $f : (\mathbb{C}, 0) \longrightarrow Pic^0(X)$ with image in*

$$i_* H^1(X, \mathbb{R})/H^1(X, \mathbb{Z})$$

are constant.

Proof. Let f be such a map with $f(0) = 0$ and $\mathcal{E}_0$ a Ricci-rigid vector bundle. Then, of course $\mathcal{E}_0 \otimes f(t)$ is a Ricci-flat deformation, hence $\mathcal{E}_0 \otimes f(t) \cong \mathcal{E}_0$. Taking determinants shows $f(t)^n = \mathcal{O}_X$, but this means that $f(t)$ is constant, so $f \equiv 0$. □

The classical correspondence that a vector bundle $\mathcal{E}$ over X with $H^2(\mathcal{E}nd(\mathcal{E})) = H^2(\mathcal{O}_X)$ is rigid if and only if $H^1(X, \mathcal{E}nd(\mathcal{E})) = 0$ has a nice analogue here. Note that the maps

$$\mathcal{O}_X \xrightarrow{\cdot \mathrm{Id}} \mathcal{E}nd(\mathcal{E}) \xrightarrow{\mathrm{tr}} \mathcal{O}_X$$

compose to multiplication by $\mathrm{rk}\mathcal{E}$. So they do on every level of cohomology, hence $h^1(\mathcal{E}nd(\mathcal{E})) \geq h^1(\mathcal{O}_X)$. Our notion of Ricci-rigidity measures in the infinitesimal setup that at $\mathcal{E}$ the minimal value is attained.

Proposition 5.5.4 *If a vector bundle $\mathcal{E}$ satisfying $H^2(\mathcal{E}nd(\mathcal{E})) = H^2(\mathcal{O}_X)$ is Ricci-rigid, then $H^1(\mathcal{E}nd(\mathcal{E})) \cong H^1(\mathcal{O}_X)$.*

Proof. Let $\mathcal{E}_0$ be Ricci-rigid and $\mathcal{E}_t$ be any small deformation of $\mathcal{E} = \mathcal{E}_0$ and define $\tilde{\mathcal{L}}_t := \det \mathcal{E}_t^\vee \otimes \det \mathcal{E}_0$. Note that $\tilde{\mathcal{L}}_t \in Pic^0(X)$. If we shrink the base of the deformation appropriately we may assume the existence of a holomorphic family $\mathcal{L}_t$ of line bundles with $\mathcal{L}_t^n = \tilde{\mathcal{L}}_t$. Now we have

$$\det(\mathcal{E}_t \otimes \mathcal{L}_t) = \det(\mathcal{E}_0),$$

in particular, $\mathcal{E}_t \otimes \mathcal{L}_t$ is a Ricci-flat deformation. Hence we obtain

$$\mathcal{E}_t \cong \mathcal{E}_0 \otimes \mathcal{L}_t^\vee.$$

By imposing $H^2(\mathcal{E}nd(\mathcal{E})) = H^2(\mathcal{O}_X)$ we ensure that every $\zeta \in H^1(\mathcal{E}nd(\mathcal{E}))$ can be integrated to a small deformation (see [B95, Cor. 5.7]). So

$$\zeta = \frac{d}{dt}|_{t=0} \mathcal{L}_t{}^\vee \in H^1(\mathcal{O}_X)$$

for some deformation $\mathcal{L}_t \in Pic^0(X)$ of $\mathcal{L}_0 = \mathcal{O}_X$. □

5.6 Examples

5.6.1 Kähler manifolds

Corollary 5.6.1 *Every compact Kähler manifold is a "stable curvature" manifold.*

Proof. Let X be a compact Kähler manifold, $i : \mathbb{R} \longrightarrow \mathcal{O}_X$ be the inclusion and $j_p : \mathcal{A}_X^p \longrightarrow \mathcal{A}_X^{0,p}$ be the map mapping a real-valued p-form ω to its complex-valued $(0,p)$-part $\omega^{(0,p)}$ for $p \geq 1$ and the natural inclusion for $p = 0$. Since the maps

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d, "i"] & \mathcal{A}_X^0 \arrow[r, "d"] \arrow[d, "j_0"] & \mathcal{A}_X^1 \arrow[r, "d"] \arrow[d, "j_1"] & \mathcal{A}_X^2 \arrow[r, "d"] \arrow[d, "j_2"] & \cdots \\ 0 \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{A}_X^{0,0} \arrow[r, "\overline{\partial}"] & \mathcal{A}_X^{0,1} \arrow[r, "\overline{\partial}"] & \mathcal{A}_X^{0,2} \arrow[r, "\overline{\partial}"] & \cdots \end{tikzcd}$$

give a cochain map between these two acyclic resolutions, general theory (e.g. [HS, 1,IV,4.4]) yields, that $j_* = i_* : H^p(X, \mathbb{R}) \longrightarrow H^p(X, \mathcal{O}_X)$ on all levels p.

Any harmonic $(0,1)$-form η yields by conjugation a harmonic $(1,0)$-form $\overline{\eta}$ such that $\omega := \eta + \overline{\eta}$ is a real one-form with $i_*[\omega] = [\eta]$, so in the Kähler case i_* is surjective. □

5.6.2 Non-Kähler examples

Proposition 5.6.2 *Let X be a compact manifold with $b_1(X) = h^{0,1}(X) = 1$ and $\mathcal{E}$ a holomorphic vector bundle on X. If*

$$H^1(\mathcal{E}nd(\mathcal{E})) \cong H^1(\mathcal{O}_X),$$

then $\mathcal{E}$ is Ricci-rigid.

Proof. If $H^1(\mathcal{E}nd(\mathcal{E}_0)) = H^1(\mathcal{O}_X)$, and $\mathcal{E}_t$ is a Ricci-flat deformation, then by Grauert's semi-continuity theorem we obtain also $H^1(\mathcal{E}nd(\mathcal{E}_t)) = H^1(\mathcal{O}_X)$ for small t. Shifting the centre of the deformation to t, we obtain a family $h(t) \in H^1(\mathcal{E}nd(\mathcal{E}_t))$. We know now that $\mathrm{tr} : H^1(\mathcal{E}nd(\mathcal{E}_t)) \longrightarrow H^1(\mathcal{O}_X)$ is an isomorphism. By the Ricci-flatness of the deformation we have due to Proposition 5.4.2 that $\mathrm{tr} h(t) = 0$. Hence $h(t) = 0$ for all t and so the deformation is trivial. □

Example 5.6.3 *Let X be the Hopf manifold defined by the automorphism $\phi(z) = 2z$ on $\mathbb{C}^n \setminus \{0\}$. Then there is a natural smooth elliptic fibration $\pi : X \longrightarrow \mathbb{P}^{n-1}$. For any bundle we have $R^1\pi_*\pi^*\mathcal{E} = \mathcal{E}^\vee$. Let $\mathcal{E}$ be a a simple bundle on $\mathbb{P}^{n-1}$ with $H^1(\mathcal{E}nd(\mathcal{E})) = H^2(\mathcal{E}nd(\mathcal{E})) = 0$. Then the Leray spectral sequence implies that $H^1(\mathcal{E}nd(\pi^*\mathcal{E})) = \mathbb{C}$ and hence $\pi^*\mathcal{E}$ is Ricci-rigid. For instance, $T_{\mathbb{P}^{n-1}}$ satisfies these conditions for $n \geq 2$. (For $n = 2$ exactly the line bundles satisfy the requirements.)*

Moreover, it is known that $p^*T_{\mathbb{P}^n}$ is not trivial for $n \geq 2$, if $p : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n$ denotes the natural projection. The following results will recover this and give a connection between Ricci-rigid bundles on some Hopf manifolds and non-trivial vector bundles on $\mathbb{C}^n \setminus \{0\}$.

Proposition 5.6.4 *Let X be the Hopf manifold given by the quotient of $\mathbb{C}^n \setminus \{0\}$ by the automorphism group generated by $\varphi(z_1, \ldots, , z_n) = (\alpha_1 z_1, \ldots, \alpha_n z_n)$, $|\alpha_i| > 1$ and $u : \mathbb{C}^n \setminus \{0\} \longrightarrow X$ the projection. If $\mathcal{E}$ is a Ricci-rigid vector bundle on X with rk$\mathcal{E} > 1$, then $u^*\mathcal{E}$ is not trivial.*

Proof. There is a multiplicative degree $\deg_\varphi : \mathbb{C}[z_1, \ldots, z_n] \longrightarrow \mathbb{C}$ via

$$\deg_\varphi(z_i) := \alpha_i.$$

If $u^*\mathcal{E}$ is trivial, usual techniques allow us to identify $\mathcal{E}$ of rank r with an equivalence class of holomorphic maps $L : \mathbb{C}^n \setminus \{0\} \longrightarrow Gl(r, \mathbb{C})$ where

$$L \cong \tilde{L} :\Longleftrightarrow \quad \exists T \in \mathcal{O}(\mathbb{C}^n \setminus \{0\}, Gl(r, \mathbb{C})) \text{ such that } \tilde{L} = T \circ \varphi \cdot L \cdot T^{-1}.$$

Choosing T carefully we can achieve a normal form of L consisting of blocks L_ν in upper triangle form (cf. [M92] for a very similar normal form) with the property

$$(L_\nu)_{kk} = \prod_{j=1}^{n} \alpha_j^{i_{jk}} \nu$$

for $i_{jk} \geq 0, i_{j1} = 0$ and

$$(L_\nu)_{kl} \in \mathbb{C}[z_1, \ldots, z_n]$$

homogeneous with

$$\deg_\varphi(L_\nu)_{kl} = \prod_{j=1}^{n} \alpha_j^{i_{jl} - i_{jk}}.$$

Now $L(t)_{kl} := L_{kl}$ for $(k, l) \notin \{(1, 1), (2, 2)\}$ and

$$L(t)_{11} := \exp(t)L_{11}, L(t)_{22} := \exp(-t)L_{22}$$

defines a non-trivial Ricci-flat small deformation of $\mathcal{E}$. □

Combining (5.6.4) and (5.6.3) we obtain immediately

Corollary 5.6.5 *If $n > 1, p : \mathbb{C}^{n+1} \setminus \{0\} \longrightarrow \mathbb{P}^n$ is the natural projection and $\mathcal{E}$ a simple vector bundle on $\mathbb{P}^n$ satisfying rk$\mathcal{E} > 1$ and $H^1(\mathcal{E}nd(\mathcal{E})) = H^2(\mathcal{E}nd(\mathcal{E})) = 0$, then $p^*\mathcal{E}$ is not trivial.*

Chapter 6

A Method for the Construction of Octics with Many Nodes

Introduction

Given a nodal octic S in $\mathbb{P}^3$, one can construct a Calabi-Yau-threefold, which is a desingularisation of a double cover of $\mathbb{P}^3$ ramified over S (cf. [Cl83]). So the usual way is first to prove existence of octics with a certain number of nodes and then to construct from this Calabi-Yau-threefolds with certain Euler numbers. This can also be done for octic arrangements with isolated singularities (cf. [Cy99, CS99]).

It has been proven that for every given number not larger than 108 there are octic hypersurfaces with this number of nodes ([We87, Bo90]). Furthermore there are examples for many numbers between 108 and 168. For the probably most complete list of constructed numbers we refer to [Labs]. In the references the interested reader may find the most important sources for constructions of octics with many nodes. The author likes to thank Oliver Labs for pointing out several references. Miyaoka proved in [Mi84] an upper bound of 174 possible nodes.

In this chapter the way is reversed: Calabi-Yau-threefolds that are desingularisations of double covers of $\mathbb{P}^3$, are used to construct octics with a certain number of nodes. To construct the Calabi-Yau-threefolds we look at hypersurfaces of projectivised rank-2-bundles over $\mathbb{P}^3$. For this class it turns out that, indeed, the octics can only have nodes as singularities. Finally, we construct an example with 128 nodes as our maximal case up to now.

6.1 Construction of the octic hypersurface

Let $\mathcal{E} \xrightarrow{p} \mathbf{P}^3$ be a rank-2-bundle and $s \in H^0(-K_{\mathbb{P}(\mathcal{E})})$ a section such that $X = \{s = 0\}$ is smooth. We denote $\gamma(\mathcal{E}) := \deg(c_1^2(\mathcal{E}) - 4c_2(\mathcal{E}))$. This is invariant under tensoring $\mathcal{E}$ with line bundles, and moreover

$$c_3(X) = -8\gamma - 168,$$

which can be computed by standard methods. For details of this and other computations we refer to [K01]. Hence γ is a topological invariant of X.

The restriction to X of the projection p we call $\pi : X \longrightarrow \mathbb{P}^3$. The map π is a generic double cover. If we take the Stein factorization

$$X \xrightarrow{c} X' \xrightarrow{\phi} \mathbb{P}^3, \qquad \pi = \phi \circ c,$$

i.e. the unique factorization such that $c_*\mathcal{O}_X = \mathcal{O}_{X'}$ and ϕ is finite (see e.g. [BHPV, p. 32]), the map ϕ is a double cover, whose ramification locus we denote by $B \subset \mathbb{P}^3$. We call B also the ramification divisor of π and denote $\tilde{B} := \pi^{-1}(B)$. Another description of B is obtained by looking at the discriminant map.

Construction 6.1.1 Let $X = \{s = 0\}$, with $s \in H^0(-K_{\mathbf{P}(\mathcal{E})})$ and

$$B := \{p \in \mathbf{P}^3 | \pi \text{ is locally in } p \text{ not an étale covering}\}.$$

We define the discriminant

$$\Delta_{\mathcal{E}} : S^2\mathcal{E} \otimes \mathcal{F} \longrightarrow (\det(\mathcal{E}) \otimes \mathcal{F})^{\otimes 2}$$

by

$$\Delta_{\mathcal{E}}(\sum_{1 \le i < j \le 2} c_{ij} s_i s_j \otimes f) := (c_{12}^2 - 4c_{11}c_{22})(s_1 \wedge s_2 \otimes f)^{\otimes 2},$$

where s_1, s_2 is a $\mathcal{O}(U)$-basis of $\mathcal{E}(U)$, $\mathcal{F} \longrightarrow \mathbf{P}^3$ a line bundle and $f \in \mathcal{F}(U)$ a generator of $\mathcal{F}(U)$ for a small open set $U \subset \mathbf{P}^3$. It is an easy computation that this definition is independent of the chosen bases.

Now we specify

$$\mathcal{F} = \det \mathcal{E}^\vee \otimes \mathcal{O}(4).$$

Then the discriminant is a map

$$\Delta_{\mathcal{E}} : p_*(-K_{\mathbf{P}(\mathcal{E})}) \longrightarrow \mathcal{O}(8),$$

with

$$\{\Delta_{\mathcal{E}}(p_*s) = 0\} = B$$

set theoretically: in local coordinates

$$s = \sum s_{ij} x_i x_j,$$

where $[x_0 : x_1]$ denotes the coordinates of the fiber and B is the locus where the zeroes of

$$\sum s_{ij}(z) x_i x_j = 0$$

are not two distinct points. By definition this is the discriminant locus of the qudratic equation in x_0, x_1, given by

$$s_{01}^2 - 4 s_{00} s_{11} = 0.$$

This coincides with the discriminant locus of $p_* s$.

Since on a trivialising neighbourhood $U \subset \mathbb{P}^3$ the map $\Delta_{\mathcal{E}}$ is given by

$$\Delta_{\mathcal{E}}(t)|U = t_{12}^2 - 4 t_{11} t_{22},$$

if $t \in H^0(p_*(-K_{\mathbb{P}(\mathcal{E})}))$ and $t|U = (t_{11}, t_{12}, t_{22})$, we see, that, in particular, $H^0(\Delta_{\mathcal{E}})$ is a holomorphic map.

Moreover,

$$H^0(\Delta_{\mathcal{E}})(rt) = r^2 H^0(\Delta_{\mathcal{E}})(t)$$

for $r \in \mathbf{C}, t \in H^0(-K_{\mathbf{P}(\mathcal{E})})$. Hence we can projectivize. However, we cannot exclude that $H^0(\Delta_{\mathcal{E}})(s') = 0$ for some $s' \neq 0$. Therefore we get a rational map

$$\delta_{\mathcal{E}} : \mathbf{P}(H^0(-K_{\mathbf{P}(\mathcal{E})})) \cdots\to \mathbf{P}(H^0(\mathcal{O}(8))) \cong \mathbf{P}^{164}.$$

Let for the moment $B' := \{z \in \mathbf{P}^3 | H^0(\Delta_{\mathcal{E}})(s)(z) = 0\}$ in the sense of ideals. If we denote

$$P := \{z \in \mathbf{P}^3 | \dim \pi^{-1}(z) = 1\},$$

then we see, that

$$P = \{z \in \mathbf{P}^3 | \sum s_{ij} x_i x_j = 0 \text{ for all } [x_o : x_1]\}$$

and hence

$$P = \{z \in \mathbf{P}^3 | s_{00}(z) = s_{01}(z) = s_{11}(z) = 0\} \subset Sing(B').$$

Moreover, this shows

$$P = \{z \in \mathbf{P}^3 | \pi^{-1}(z) \cong \mathbf{P}^1\}.$$

Now let $z \in Sing(B')$. If $s_{00}(z) = s_{01}(z) = s_{11}(z) = 0$, then $z \in P$. So let us assume $s_{00}(z) \neq 0$ or $s_{01}(z) \neq 0$. Let us define

$$x := [s_{01}(z) : -2 s_{00}(z)] \in p^{-1}(z).$$

Since $z \in B$, we get that $\Delta_{\mathcal{E}}(s)(z) = s_{01}(z)^2 - 4s_{00}(z)s_{11}(z) = 0$. Therefore

$$s(x) = s_{00}(z)s_{01}(z)^2 - 2s_{00}(z)s_{01}(z)^2 + 4s_{11}(z)s_{00}(z)^2 = -s_{00}(z)\Delta_{\mathcal{E}}(s)(z) = 0,$$

hence $x \in X$.

We want to show that $x \in X$ is singular. For this we have to compute in the point x

$$\frac{\partial s}{\partial x_0} = 2s_{00}x_0 + s_{01}x_1 = 0 \tag{6.1}$$

$$\frac{\partial s}{\partial x_1} = s_{01}x_0 + 2s_{11}x_1 = 0 \tag{6.2}$$

$$\frac{\partial s}{\partial z_i} = \tfrac{\partial s_{00}}{\partial z_i}x_0^2 + \tfrac{\partial s_{01}}{\partial z_i}x_0x_1 + \tfrac{\partial s_{11}}{\partial z_i}x_1^2 = 0 \tag{6.3}$$

and we know moreover, since $z \in Sing(B')$, that at the point z

$$s_{01}^2 - 4s_{00}s_{11} = 0 \tag{6.4}$$

$$2s_{01}\frac{\partial s_{01}}{\partial z_i} - 4s_{11}\frac{\partial s_{00}}{\partial z_i} - 4s_{00}\frac{\partial s_{11}}{\partial z_i} = 0. \tag{6.5}$$

Using the expression for x in (6.1), (6.2) and (6.3) we compute

$$\begin{aligned}
\frac{\partial s}{\partial x_0} &= 2s_{00}s_{01} - 2s_{00}s_{01} &= 0\\
\frac{\partial s}{\partial x_1} &= s_{01}^2 - 4s_{00}s_{11} &= 0\\
\frac{\partial s}{\partial z_i} &= \tfrac{\partial s_{00}}{\partial z_i}s_{01}^2 - 2\tfrac{\partial s_{01}}{\partial z_i}s_{00}s_{01} + 4\tfrac{\partial s_{11}}{\partial z_i}s_{00}^2 &=\\
&= 4\tfrac{\partial s_{00}}{\partial z_i}s_{00}s_{11} - 2\tfrac{\partial s_{01}}{\partial z_i}s_{00}s_{01} + 4\tfrac{\partial s_{11}}{\partial z_i}s_{00}^2 &=\\
&= -s_{00}(2s_{01}\tfrac{\partial s_{01}}{\partial z_i} - 4s_{11}\tfrac{\partial s_{00}}{\partial z_i} - 4s_{00}\tfrac{\partial s_{11}}{\partial z_i}) &= 0,
\end{aligned}$$

with the last equation using (6.4) as well as (6.5).

Thus we have proved that $x \in X$ is singular. But we assumed X to be smooth. Hence it is proven that $P = Sing(B')$. In particular, B' is reduced and therefore $B' = B$ in the sense of ideals. △

Now we know, if $X = \{s = 0\}$ for some $s \in H^0(-K_{\mathbf{P}(\mathcal{E})})$ then

Lemma 6.1.2 *$B = \delta_{\mathcal{E}}(X) \in |\mathcal{O}(8)|$ and $P = \{p_*s = 0\} = Sing(B)$.*

Note that p_*s gives the three local equations of P. For example, if $\mathcal{E}$ splits, then we can conclude that P is the complete intersection of three hypersurfaces of degrees $4 - \sqrt{\gamma}, 4$ and $4 + \sqrt{\gamma}$. (Indeed, if $\mathcal{E}$ splits, then γ is a square.)

Let us now specify the type of the singularities.

Lemma 6.1.3 *B has only double points of type A_1 as singularities.*

Proof. First, singularities of $\tilde{B}$ can only occur over singularities of B, hence by Lemma 6.1.2

$$s_{00}(z) = s_{01}(z) = s_{11}(z) = 0,$$

if $y \in \tilde{B}$ is singular and $z = \pi(y)$. From this we conclude again by the local descriptions that y is singular in X. Hence $\tilde{B}$ is non-singular. In particular, B has only isolated singularities. Now we look at the rational curves $F = F_p := \pi^{-1}(p)$ for $p \in P$. The adjunction formula yields

$$\mathcal{O}(-2) = K_F = K_{\tilde{B}}|F \otimes N_{F|\tilde{B}}.$$

Since again by adjunction formula $\deg K_{\tilde{B}}|F = \tilde{B}.F = p^{-1}(B).F = 0$ we conclude

$$N_{F|\tilde{B}} = \mathcal{O}(-2)$$

and hence p is a double point of type A_1. □

Lemma 6.1.4 $|P| = 64 - 4\gamma$.

Proof. Since by Lemma 6.1.2 and Lemma 6.1.3 we know that $\dim P \leq 0$ and $P \cap U = \{z \in \mathbb{P}^3 | s_{00}(z) = s_{01}(z) = s_{11}(z) = 0\}$ in a trivialising neighbourhood U, we conclude

$$[P] = c_3(p_*(-K_{\mathbb{P}(\mathcal{E})})).$$

Again by standard methods (cf. [Ha, p. 423]) we compute $c_3(p_*(-K_{\mathbb{P}(\mathcal{E})})) = 64 - 4\gamma$. □

This method has only limitated applications, since by Lemma 6.1.4 we see, that the number of nodes must be divisible by 4. But there is an additional restriction:

Lemma 6.1.5 $\gamma(\mathcal{E}) \mod 8 \in \{0, 1, 4\}$.

Proof. By definition $\gamma(\mathcal{E}) \mod 4$ is a square. The case $\gamma(\mathcal{E}) \mod 8 = 5$ can be excluded by the Schwarzenberger condition $c_1(\mathcal{E}).c_2(\mathcal{E}) \equiv 0(2)$. □

Since Miyaoka proved an upper bound of 174 nodes, by Lemma 6.1.5 the theoretical maximum of our method lies at $\gamma = -24$, i.e. at most 160 nodes.

6.2 Construction of some bundles

By looking at the splitting bundles allowing for smooth X we obtain immediately examples of octics with $28, 48, 60$ and 64 nodes. Moreover, it is not hard to see the existence of elliptic curves in $\mathbf{P}^3$ of degrees $d \leq 7$ which are cut out by quartics (cf. [vB95, Hu]). The Serre construction then yields examples of octics with $80, 96$ and 112 nodes. It is a little bit more work to see the existence of elliptic curves of degree 8 cut out by quartics.

We use the Serre construction of rank-2-bundles to get the desired example (cf. [OSS]). Let $Y \subset \mathbb{P}^3$ be an elliptic curve of degree 8. Then there is a rank-2-bundle with an exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Y(4) \longrightarrow 0,$$

since by adjunction formula $\det N_{Y|\mathbb{P}^3} = \mathcal{O}(4)|Y$. If we choose concretely Y as the image of $C := \{y^2z - x^3 + xz^2 = 0\} \subset \mathbb{P}^2$ via

$$\begin{array}{rcl} i : \mathbb{P}^2 & \cdots\rightarrow & \mathbb{P}^3 \\ [x : y : z] & \mapsto & [x^2y + xyz + z^3 : xy^2 + yz^2 + zx^2 : x^2y + xyz + xz^2 : xy^2 + y^2z + z^3], \end{array}$$

we can compute with MACAULAY [GS], that

$$I_Y := \bigoplus_{n \in \mathbb{N}} H^0(\mathcal{I}_Y(n))$$

is generated by three quartics q_1, q_2, q_3 and four quintics. But also with MACAULAY we can verify that the projective schemes Y and $\{z \in \mathbb{P}^3 | q_1(z) = q_2(z) = q_3(z) = 0\}$ are identical. Hence $\mathcal{I}_Y(4)$ is generated by global sections and we conclude that $\mathcal{E}$ is generated by global sections. Therefore $-K_{\mathbb{P}(\mathcal{E})} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2)$ is globally generated and we can choose $s \in H^0(-K_{\mathbb{P}(\mathcal{E})})$ such that X is smooth.

By construction $c_1(\mathcal{E}) = 4h$ and $c_2(\mathcal{E}) = 8h^2$, hence $\gamma(\mathcal{E}) = -16$ and $|P| = 128$.

Remark 6.2.1 *(i) Note that this case is extremal in some sense: Any elliptic curve in $\mathbf{P}^3$ of degree $d \geq 9$ cannot be cut out by quartics. This can be seen like follows: If the contrary would be the case, the Serre construction would yield a globally generated vector bundle $\mathcal{E}$ and a sequence*

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_Y(4) \longrightarrow 0.$$

Since $-K_{\mathbb{P}(\mathcal{E})} = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(2)$ would then be globally generated as well, we conclude $(-K_{\mathbb{P}(\mathcal{E})})^4 \geq 0$. On the other hand, $\gamma(\mathcal{E}) = 16 - 4d \leq -20$ and hence $(-K_{\mathbb{P}(\mathcal{E})})^4 = 32\gamma + 512 \leq -128$.

(ii) This example is extremal in some other sense, too: A general member of $|-K_{\mathbb{P}(\mathcal{E})}|$ *is an elliptic fibre space over a quadric, which is the restriction of an elliptic fibre space* $\mathbf{P}(\mathcal{E}) \longrightarrow \mathbf{P}^3$ *(see [K01]).*

The cases where γ is odd are more complicated to deal with. In these cases we cannot use elliptic curves. Instead of genus 1 we have to choose negative genera, hence Y is not irreducible and the extendability condition of the normal bundle is harder to check. So the Serre construction does not appear to be useful.

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