Failure Rates Modelling for Heterogenous Populations

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Introduction

One can hardly find homogeneous populations in real life, although most of the studies on the failure rate modelling deal with a homogeneous case. Neglecting existing heterogeneity can lead to substantial errors in stochastic analysis in reliability, survival and risk analysis and other disciplines.

Mixtures of distributions usually present an effective tool for modelling heterogeneity. The origin of mixing in practice can be 'physical' when, for instance, a number of devices of different (heterogeneous) types, performing the same function and not distinguishable in operation, are mixed together. This occurs when we have 'identical' items, but of different manufacturers. A similar situation arises when data from different distributions are pooled to enlarge the sample size.

It is well known that mixtures of decreasing failure rate (DFR) distributions are always DFR (Barlow and Proschan, 1975). On the other hand, mixtures of increasing failure rate distributions (IFR) can decrease at least in some intervals of time, which means that the IFR class of distributions is not closed under the operation of mixing (Lynch, 1999). As IFR distributions usually model lifetimes governed by aging processes, it means that an operation of mixing can change the pattern of aging dramatically, e.g., from positive aging (IFR) to the negative aging (DFR). It should be noted, however, that the change in the aging pattern usually occurs at sufficiently large age of items and therefore the role of asymptotic methods in analysis is evident. These facts and other implications of heterogeneity should be taken into account in applications.

A natural specific approach for this modelling exploits a notion of a non-negative random unobserved parameter (frailty) Z. The term "frailty" was suggested in Vaupel *et al* (1979) for Gamma-distributed Z and the multiplicative failure rate model of the form $Z\lambda(t)$, where $\lambda(t)$ is some baseline failure rate. Since that time multiplicative frailty models were widely used in statistical data analysis (see, e.g., Andersen *et al*, 1993). It is worth noting, however, that the specific case of a Gamma-frailty model was, in fact, first considered by the British actuary R. Beard (1959) (see also his 1971 paper).

A random Z clearly leads to considering a random failure rate $\lambda(t, Z)$ and eventually to a mixture failure rate $\lambda_m(t)$.

The mixture failure rate $\lambda_m(t)$ ***is an observed failure rate in heterogenous populations and the study of its properties is the main goal of this thesis.***

As the failure rate is a conditional characteristic, the 'ordinary' expectation $E[\lambda(t, Z)]$ with respect to Z does not define a mixture (or observed) failure rate $\lambda_m(t)$ and a proper conditioning should be performed (Yashin and Manton, 1997):

$$\lambda_m(t) = E[\lambda(t, Z) \,|\, T > t],$$

where T is a population lifetime random variable, and as usually, when dealing with a failure rate, this notation means considering the risk of failure for survivors at time t. It is worth mentioning that a random failure rate is a specific case of a hazard rate process (see, e.g., Kebir (1991) and Yashin and Manton (1997)) and also, in some sense, of a stochastic intensity, which describes point processes (Aven and Jansen, 2000).

It is already well known for some specific cases, and in Chapter 4 we prove this result analytically for a rather general case, that the mixture failure rate is "bent down" (or decelerated) in the following sense:

$$\lambda_m(t) = E[\lambda(t, Z)|T > t] < E[\lambda(t, Z)], \quad t > 0.$$

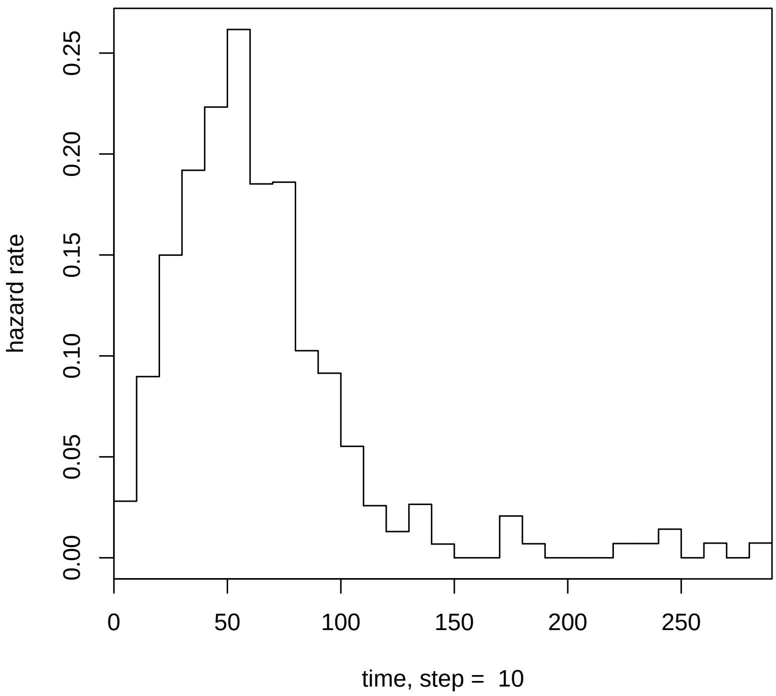
This inequality can be interpreted as follows: if the family of failure rates $\lambda(t, z)$, $z \in [0, \infty)$ is ordered in z (see Chapter 4), then, as the weakest populations are dying out first, the remaining population will have a smaller (better) failure rate, than the population without this dying out effect.

A convincing 'experiment', showing the deceleration (bending down) of the observed (mixture) failure rate is performed by nature. It is well known that the human mortality follows the Gompertz (1825) lifetime distribution with exponentially increasing mortality rate. Assume that heterogeneity can be described by the specific proportional Gamma-frailty model:

$$\lambda(t, Z) = Z\alpha e^{\beta t},$$

where α and β are positive constants. Due to computational simplicity, the Gamma-frailty model is practically the only one used in applications so far. It can be shown (see, e.g., Finkelstein and Esaulova, 2001a) that the mixture failure rate $\lambda_m(t)$ in this case is monotone in $[0, \infty)$ and asymptotically tends to a constant as $t \to \infty$. However, $\lambda_m(t)$ is monotonically increasing for the real values of parameters of this model. This fact explains recently observed deceleration in human mortality for the oldest old (human mortality plateau, as in Thatcher (1999)). A similar result is experimentally obtained for a large cohort of medflies by Carey *et al* (1992).

On the other hand, in engineering applications an operation of mixing can result in increasing in $[0, t_m)$, $t_m > 0$ and decreasing asymptotically to 0 in $[t_m, \infty)$ failure rate, which, e.g., was experimentally observed in Finkelstein (2005b) for the heterogeneous sample of miniature light bulbs (see the following figure).



The graph shows an empirical failure rate for the population of 700 miniature light bulbs under the normal voltage of 5 volts. This shape of the failure rate can be easily explained theoretically via the multiplicative Gamma-frailty model with a baseline failure rate increasing in accordance with the Weibull law as a power function (Gupta and Gupta, 1996; Finkelstein and Esaulova, 2001a). On the other hand, general engineering considerations on stochastic modelling of wearing devices suggest that the accelerated life model (ALM) is more appropriate in the case of light bulbs than a multiplicative one.

It should be noted that there are practically no results for mixture failure rate modelling for the ALM in the literature. In Chapter 2 we consider this model as a specific case of our general lifetime model and prove somehow an unexpected result that irrespectively of the mixing and baseline distributions and under very mild additional assumptions the asymptotic mixture failure rate is proportional to $1/t$. No wonder that it perfectly corresponds to the tail of the mixture failure rate for the multiplicative Gamma-frailty model with a baseline Weibull distribution, as only for this case the accelerated life model can be reparameterized to end up with the multiplicative (proportional hazards) model.

Chapter 1 is devoted to initial considerations, main settings and a brief literature survey. We obtain some results on conditional expectations for the multiplicative and additive hazards models and prove theorems describing the shape of the corresponding mixture failure rate.

The literature on heterogeneity modelling can be roughly divided into three parts. The first one is connected with demographic studies and is mostly based on papers by J. Vaupel with different co-authors. The second one is in the field of bio-medical and bio-statistical applications. The path breaking results in this area are developed in the papers of O. Aalen and P. Hougaard. Finally, general mathematical and asymptotical properties were studied in a number of papers by H. Block and T. Savits and co-authors. In the current study we are closer to the latter mathematical stream of research, although our approach is different.

Chapter 2 contains main mathematical results of this thesis and considers an important topic of asymptotic behavior of mixture failure rates. In Block *et al* (2003a) it was proved that if the failure rate of each subpopulation converges to a constant and this convergence is uniform, then the mixture failure rate converges to the failure rate of the strongest subpopulation: the weakest subpopulations are dying out first. This result generalizes a case of constant failure rates of populations considered by Clarotti and Spizzichino

(1990) and also presents a further development of Block *et al* (1993) (see also Lynn and Singpurwalla, 1997; Gurland and Sethuraman,1995). In Block and Joe (1997) the following asymptotic result, which already addressed the issue of ultimate monotonicity, was obtained: let z_0 be a realization of a frailty Z, which corresponds to the strongest population. If $\lambda(t,z)/\lambda(t,z_0)$ uniformly decreases as $t \to \infty$, then $\lambda_m(t)/\lambda(t,z_0)$ also decreases. If, in addition, $\lim_{t\to\infty} \lambda(t,z_0)$ exists, then this quotient decreases to 1. Note that analytical restrictions in this findings, e.g., uniform convergence, are rather stringent.

In this Chapter 2 we suggest a class of distributions, which generalizes the proportional hazards, the additive hazards and the accelerated life models and prove asymptotic results for the mixture failure rate for this class of lifetime distributions. We develop a new for this kind of applications approach, related to the ideology of generalized convolutions, e.g., Laplace and Fourier transforms and, especially, Mellin convolutions (Bingham *et al*, 1987). For proving our asymptotic results we use a convenient technique, which is used for deriving asymptotics for the Laplace integrals. Similar methods are used for obtaining Abelian, Tauberian and Mercerian theorems, although our theorems are not the direct corollaries of results in this field. It turns out that the asymptotic behavior of mixture failure rates depends only on the behavior of the mixing distribution in the neighborhood of the left end point of its support and not on the whole mixing distribution. As it was mentioned, we have obtained a striking result that asymptotical mixture failure rate for the specific case of the accelerated life model is proportional to $1/t$ for a wide class of mixing and baseline distributions.

Chapter 3 extends asymptotic univariate results of Chapter 2 to the multivariate (bivariate) case. We consider two specific, but important in practical applications cases. The first one deals with the competing risks problem: each of the two components in the series system has its own frailty and these frailties are dependent random variables. Asymptotic formulas for the failure rate of the series system are derived and the conditions for 'asymptotic independence' of the components are discussed. The second case deals with one component with a bivariate frailty. Asymptotic formulas for the mixture failure rate are also derived and the simplest examples are analyzed.

In Chapter 4 we consider heterogeneous populations in different environments. The problem of mixture failure rates ordering for stochastically ordered mixing random variables naturally arises in this setting. This topic was not addressed in the literature before. We show that the natural type of ordering for mixing models under consideration is ordering in a sense of the likelihood ratio (Ross, 1996; Shaked and Shanthikumar, 1993). This also goes in line with general considerations of Block *et al* (1993) with respect to a burn-in in heterogeneous populations. Specifically, when two frailties are ordered in this way, the corresponding mixture failure rates are naturally ordered as functions of time in $[0, \infty)$. Some specific results for the case of frailties with equal means and different variances are also obtained.

We also discuss a 'combination' of a frailty and a proportional hazards (PH) model. A case of a step-stress change-point in the proportional hazards framework is considered and the corresponding bounds for the mixture failure rate are also obtained. Another example deals with a special type of shocks, which perform a burn-in heterogeneous populations.

Some preliminary results, modified and extended in Chapter 1, were first published in:

Finkelstein, M.S., and V. Esaulova, V. (2001a). Modelling a failure rate for the mixture of distribution functions. *Probability in Engineering and Informational Sciences,* **15,** 383-400.

Finkelstein, M.S., and V. Esaulova, V. (2001b). On inverse problem in mixture hazard rates modelling. *Applied Stochastic Models in Business and Industry,* **17,** 221-229.

The following preprints contain most of the results of this thesis:

Finkelstein M. S., Esaulova V. Asymptotic behavior of mixture failure rates. MPIDR (Max Planck Institute for Demographic Research). Working Paper WP-2005-023.

Finkelstein M.S., Esaulova V. (2005). Failure rates in heterogeneous populations. Technical report #353. University of the Free State.

A paper, based on the first preprint, which contains the description of a new method in the mixture failure rate modelling and the main asymptotic results has been accepted for publication in the "Journal of Applied Probability".

Chapter 1

Settings, initial results, literature

In this chapter we define a mixture failure rate and discuss its properties for the simplest models of mixing in terms of the corresponding conditional characteristics.

1.1 Definitions

Let $T \geq 0$ be a lifetime random variable with the cumulative distribution function (Cdf) $F(t)$ and the survival function $\bar{F}(t) \equiv 1 - F(t)$. Assume that $F(t)$ is indexed by a random variable Z in the following sense:

$$P(T \leq t|Z = z) \equiv P(T \leq t|z) = F(t, z)$$

and that the corresponding probability density function (pdf) $f(t, z)$ exists. The failure rate $\lambda(t, z)$ can be defined in a usual way as

$$\lambda(t, z) = \frac{f(t, z)}{\bar{F}(t, z)}.$$

Let Z be interpreted as a non-negative random variable with support in $[a, b]$, $a \geq 0, b \leq \infty$ and probability density function $\pi(z)$. This random variable has also the meaning of an unobserved parameter, which is very helpful in modelling of heterogeneity and is usually called "frailty" (Vaupel *et al*, 1979).

Assume for convenience of notation that $a = 0, b = \infty$. The cases when $a > 0$ are important and will be specified later. A mixture Cdf is defined by

$$F_m(t) = \int_0^\infty F(t,z)\pi(z)dz. \tag{1.1}$$

As the failure rate is a conditional characteristic, the mixture failure rate $\lambda_m(t)$ should be defined in the following way (see, e.g., Finkelstein and Esaulova, 2001a):

$$\lambda_m(t) = \frac{\int_0^\infty f(t,z)\pi(z)dz}{\int_0^\infty \bar{F}(t,z)\pi(z)dz} = \int_0^\infty \lambda(t,z)\pi(z|t)dz, \tag{1.2}$$

where the conditional pdf (on condition that $T > t$) is:

$$\pi(z|t) \equiv \pi(z|T > t) = \pi(z)\frac{\bar{F}(t,z)}{\int_0^\infty \bar{F}(t,z)\pi(z)dz}. \tag{1.3}$$

Therefore, this pdf defines a conditional random variable $[Z|t]$, $[Z|0] \equiv Z$, with the same support, which can be viewed as the frailty among survivors at time t. In a natural Bayesian interpretation (Spizzichino, 1992) $\pi(z)$ is a prior density of Z, whereas by Bayes formula $\pi(z|t)$ is the posterior density after observing the survival the data $T > t$ (Spizzichino, 1992).

On the other hand, consider the following *unconditional characteristic*

$$\lambda_P(t) = \int_0^\infty \lambda(t,z)\pi(z)dz, \tag{1.4}$$

which, in fact, defines an expected value (as a function of t) for a specific stochastic process $\lambda(t,Z)$. It follows from definitions (1.2) - (1.4) that $\lambda_m(0) = \lambda_P(0)$.

The function $\lambda_P(t)$ is a supplementary one, but as a trend function of a stochastic process, it captures the monotonicity pattern of the family $\lambda(t,z)$. Therefore, $\lambda_P(t)$ under certain conditions has a similar to individual $\lambda(t,z)$ shape: if, e.g., $\lambda(t,z)$, $z \in [a,b]$ is increasing in t, then $\lambda_P(t)$ is increasing as well. For some specific cases (see later) it also characterizes the shape of the baseline failure rate. On the contrary, the mixture failure rate $\lambda_m(t)$ may have a different pattern: it can ultimately decrease, for instance, or preserve an increasing in t property, as in Lynch (1999). There is even a possibility of a few oscillations. However, despite all possible patterns, it will be proved in Chapter 4 that the mixture failure rate is majorized by $\lambda_P(t)$:

$$\lambda_m(t) < \lambda_P(t), \; t > 0 \tag{1.5}$$

and under some additional assumptions that

$$(\lambda_P(t) - \lambda_m(t)) \uparrow, \; t \geq 0. \tag{1.6}$$

Definition 1.1 *Relation (1.5) defines a weak bending down property for the mixture failure rate, whereas relation (1.6) is a definition of a strong bending down property.*

We will mostly deal with continuous mixtures in this study, although some discrete case examples will be helpful for interpreting asymptotic results in Chapter 2.

Similar to (1.1), the discrete mixture is defined by

$$F_m(t) = \sum_k F(t, z_k)\pi(z_k), \tag{1.7}$$

where $\pi(z_k)$ is the probability mass of z_k. The failure rate in this case is defined similar to (1.3).

In general, the mixture Cdf is defined as an integral

$$F_m(t) = \int_0^\infty F(t, z)d\Pi(z), \tag{1.8}$$

where $\Pi(z)$ is a cumulative distribution function of Z. The failure rate and other characteristics are defined correspondingly.

1.2 Conditional characteristics and simplest models

Denote by $E[Z|t]$ the expectation of the earlier defined random variable $[Z|t]$:

$$E[Z|t] = \int_0^\infty z\pi(z|t)dz.$$

An important characteristic for further consideration is $E'[Z|t]$, the derivative with respect to t:

$$E'[Z|t] = \int_0^\infty z\pi'(z|t)dz,$$

where

$$\begin{aligned}\pi'(z|t) &= -\frac{f(t,z)\pi(z)}{\int_0^\infty \bar{F}(t,z)\pi(z)dz} + \frac{\bar{F}(t,z)\pi(z)\lambda_m(t)}{\int_0^\infty \bar{F}(t,z)\pi(z)dz} \\ &= \lambda_m(t)\pi(z|t) - \frac{f(t,z)\pi(z)}{\int_0^\infty \bar{F}(t,z)\pi(z)dz}.\end{aligned} \tag{1.9}$$

While deriving (1.9), equation (1.3) was used. Eventually we obtain

Lemma 1.1 *The following equation for $E'[Z|t]$ holds:*

$$E'[Z|t] = \lambda_m(t)E[Z|t] - \frac{\int_0^\infty zf(t,z)\pi(z)dz}{\int_0^\infty \bar{F}(t,z)\pi(z)dz}. \tag{1.10}$$

Now we shall consider two specific cases, where the mixing r.v. Z can be entered directly into the failure rate model.

1.2.1 Additive model

Suppose that

$$\lambda(t,z) = \lambda(t) + z, \tag{1.11}$$

where $\lambda(t)$ is a baseline failure rate: some deterministic continuous increasing function ($\lambda(t) \geq 0$, $t \geq 0$) to be specified later.

Denote by $F(t)$ the corresponding Cdf. Then, noting that

$$f(t,z) = \lambda(t,z)\bar{F}(t,z),$$

and applying definition (1.2) for this concrete model:

$$\lambda_m(t) = \lambda(t) + \frac{\int_0^\infty z\bar{F}(t,z)\pi(z)dz}{\int_0^\infty \bar{F}(t,z)\pi(z)dz} = \lambda(t) + E[Z|t]. \tag{1.12}$$

Remark 1.1 *It is worth noting that in the additive model the pdf* $\pi(z|t)$ *and* $E[Z|t]$ *do not depend on the baseline distribution.*

Indeed, denoting cumulative failure rate $\Lambda(t) = \int_0^t \lambda(u)du$

$$\begin{aligned}\pi(z|t) &= \frac{\pi(z)\bar{F}(t,z)}{\int_0^\infty \bar{F}(t,z)\pi(z)dz} \\ &= \frac{\pi(z)e^{-\Lambda(t)-tz}}{\int_0^\infty e^{-\Lambda(t)-zt}\pi(z)dz} = \frac{e^{-zt}\pi(z)}{\int_0^\infty e^{-zt}\pi(z)dz}.\end{aligned} \tag{1.13}$$

Therefore, $E[Z|t]$ also does not depend on $F(t)$.

Using (1.12) and Lemma 1.1, a specific form of $E'[Z|t]$ can be easily obtained:

$$\begin{aligned}E'[Z|t] = (\lambda(t) + E[Z|t])\, E[Z|t] \\ - \frac{\int_0^\infty \left(z\lambda(t)\bar{F}(t,z) + z^2\bar{F}(t,z)\right)\pi(z)dz}{\int_0^\infty \bar{F}(t,z)\pi(z)dz} \\ = (E[Z|t])^2 - \int_0^\infty z^2\pi(z|t)dz = -Var(Z|t),\end{aligned} \tag{1.14}$$

where $Var(Z|t)$ denotes the variance of Z given $T > t$, which also does not depend on $F(t)$.

This result can be formulated in the form of

Lemma 1.2 *The conditional expectation of* Z *for the additive model is a decreasing function of* $t \in [0, \infty)$*, which follows from*

$$E'[Z|t] = -Var(Z|t) < 0.$$

Differentiating (1.12) and using relation (1.14), we can obtain now the result that was stated (without proof) in Lynn and Singpurwalla (1997):

Theorem 1.1 *Let* $\lambda(t)$ *be an increasing (non-decreasing) convex function in* $[0, \infty)$*. Assume that* $Var(Z|t)$ *is decreasing in* $t \in [0, \infty)$ *and*

$$Var(Z|0) > \lambda'(0).$$

Then $\lambda_m(t)$ *decreases in* $[0, c)$ *and increases in* $[c, \infty)$*, where* c *can be uniquely defined by equation:* $Var(Z|c) = \lambda'(c)$ *(bathtub shape).*

In addition to Lynn and Singpurwalla (1997) we have included an assumption that $Var(Z|t)$ should decrease in t for all values of t. Intuitively it seems that similar to the fact that $E[Z|t]$ is decreasing in $[0, \infty)$ (which follows from (1.14)), $Var(Z|t)$ should also decrease, as the "weak populations are dying out first" while t increases. But this is not true for the general case. The counter-example is presented below, showing that the conditional variance is increasing in the neighborhood of 0. It is shown also that $Var(Z|t)$ decreases in $[0, \infty)$ when Z is exponentially distributed. First, a technical lemma:

Lemma 1.3 *The second derivative of the conditional expectation $E[Z|t]$ is given by:*

$$E''[Z|t] = 2E[Z|t]^3 + E[Z^3|t] - 3E[Z^2|t]E[Z|t].$$

Proof Indeed, by (1.13) in the additive model:

$$\pi(z|t) = \frac{e^{-zt}\pi(z)}{\int_0^\infty e^{-zt}\pi(z)dz}, \qquad E[Z|t] = \int_0^\infty z\pi(z|t)dz = \frac{\int_0^\infty ze^{-zt}\pi(z)dz}{\int_0^\infty e^{-zt}\pi(z)dz}.$$

Then

$$\begin{aligned}\pi'(z|t) &= \frac{-ze^{-zt}\pi(z)}{\int_0^\infty e^{-zt}\pi(z)dz} + \frac{e^{-zt}\pi(z)\int_0^\infty ze^{-zt}\pi(z)dz}{\left(\int_0^\infty e^{-zt}\pi(z)dz\right)^2} \\ &= \pi(z|t)(-z + E[Z|t]).\end{aligned}$$

Therefore, using (1.14)

$$\begin{aligned}\pi''(z|t) &= \pi'(z|t)(-z + E[Z|t]) + \pi(z|t)E'[Z|t] \\ &= \pi(z|t)\left(z^2 - 2zE[Z|t] + E[Z|t]^2 + E'[Z|t]\right) \\ &= \pi(z|t)\left(z^2 - 2zE[Z|t] + 2E[Z|t]^2 - E[Z^2|t]\right).\end{aligned}$$

By definition,

$$E''[Z|t] = \int_0^\infty z\pi''(z|t)dz,$$

and from the last two relations we get

$$E''[Z|t] = 2E[Z|t]^3 + \int_0^\infty z^3\pi(z|t)dz - 3E[Z|t]\int_0^\infty z^2\pi(z|t)dz,$$

which completes the proof. □

Example 1.1 The sign of $E''[Z|t]$ is of interest. As we will show now, it is not necessarily positive.

Let $t = 0$ and thus $\pi(z|0) = \pi(z)$. Using Lemma 1.3:

$$E''[Z|0] = 2\left(\int_0^\infty z\pi(z)dz\right)^3 + \int_0^\infty z^3\pi(z)dz - 3\int_0^\infty z\pi(z)dz \int_0^\infty z^2\pi(z)dz.$$

Let, for instance, $\pi(z) = 2z$ with support in $[0,1]$. Then:

$$2\left(2\int_0^1 z^2dz\right)^3 + 2\int_0^1 z^4dz - 3\cdot 4\int_0^1 z^2dz\int_0^1 z^3dz = \frac{134}{135} - 1 < 0.$$

This means that conditional variance is increasing in the neighborhood of 0.

Consider now the exponential distribution $\pi(z) = e^{-z},\ z \in [0,\infty)$:

$$\pi(z|t) = \frac{e^{-z(t+1)}}{\int_0^\infty e^{-z(t+1)}dz} = (t+1)e^{-z(t+1)}.$$

Taking into account that for $k = 1, 2, ...$

$$\begin{aligned}\int_0^\infty z^k\pi(z|t)dz &= \int_0^\infty z^k(t+1)e^{-z(t+1)}dz \\ &= \frac{1}{(t+1)^k}\int_0^\infty z^k e^{-z}dz \\ &= \frac{k!}{(t+1)^k},\end{aligned}$$

we arrive at

$$\begin{aligned}E''[Z|t] &= 2\cdot\left(\frac{1}{t+1}\right)^3 + \frac{6}{(t+1)^3} - 3\cdot\frac{1}{(t+1)}\cdot\frac{2}{(t+1)^2} \\ &= \frac{2}{(t+1)^3} > 0.\end{aligned}$$

Hence, in this case the conditional variance is decreasing for all values of $t \in [0,\infty)$. ♢

Finally, an example of the uniform in $[0,1]$ distribution of Z can be studied. Similar to the previous examples, it can be shown that the conditional variance in this case is decreasing at least for small values of t.

Thus, the purpose of this example was to show that the assumption of decreasing conditional variance is quite natural and usually holds for widely used in practice mixing distributions, but it is not valid for arbitrary distributions.

Lynn and Singpurwalla (1997) were primarily interested in the bathtub shape of $\lambda_m(t)$. We, along with other features, are interested in preservation of IFR properties:

Corollary 1.1 *Let all other assumptions of Theorem 1.1 hold, whereas*

$$Var(Z|0) \leq \lambda'(0).$$

Then $\lambda_m(t)$ *increases in* $[0, \infty)$.

It means that the IFR-closure property exists under this assumption. In other words: the family of distributions with increasing failure rates is closed under the operation of mixing. We shall call it the IFR-stability property. In certain situations it can be very important to know that non-parametric properties of distributions (e.g., characteristics of aging) are not changed after the operation of mixing. It is worthwhile noting that the governing factor in this analysis is the conditional variance $Var(Z|t)$: if it is sufficiently small, then the mixture is IFR-stable. This conclusion also follows from some general considerations: the variance is responsible for possible decreasing of $\lambda_m(t)$. When $Var(Z|t) = 0$ (Z is deterministic), $\lambda_m(t)$ is increasing due to the assumption that $\lambda(t)$ is increasing.

1.2.2 Multiplicative model

Suppose that

$$\lambda(t, z) = z\lambda(t), \tag{1.15}$$

where $\lambda(t)$ is some deterministic, increasing (non-decreasing) at least for sufficiently large t, continuous function ($\lambda(t) \geq 0, t \geq 0$). Thus, we are mostly interested in the case of ultimately increasing (non-decreasing) baseline failure rate $\lambda(t)$, especially when $\lambda(t) \to \infty$ as $t \to \infty$. The frailty model (1.15) is often called a proportional hazards (PH) model while the previous one is called an additive hazards (AH) model. See Section 4.2.1 for a brief discussion of 'difference' between frailty and proportional hazards models, which, in fact, is not important for our study. Applying relation (1.2):

$$\lambda_m(t) = \int_0^\infty \lambda(t,z)\pi(z|t)dz = \lambda(t)E[Z|t]. \qquad (1.16)$$

After differentiating

$$\lambda'_m(t) = \lambda'(t)E[Z|t] + \lambda(t)E'[Z|t]. \qquad (1.17)$$

It follows immediately from this equation that when $\lambda(0) = 0$, the failure rate $\lambda_m(t)$ increases in the neighborhood of $t = 0$. Further behavior of this function depends on the other parameters involved.

The unconditional characteristic (1.4) is simplified to

$$\lambda_P(t) = \int_0^\infty \lambda(t,z)\pi(z)dz = \lambda(t)E[Z]$$

and in some sense defines the baseline model.

Example 1.2 Consider the specific type of the Weibull distribution with linear failure rate $\lambda(t,z) = 2zt$ and assume that $\pi(z)$ is the Gamma probability density function:

$$\pi(z) = \frac{\nu^\beta z^{\beta-1} e^{-z\nu}}{\Gamma(\beta)}, \qquad \beta, \nu > 0, z \geq 0.$$

The mixture failure rate can be easily obtained via direct integration:

$$\lambda_m(t) = \frac{2\beta t}{\nu + t^2}.$$

It is equal to zero at $t = 0$ and tends to zero as $t \to \infty$ with a single maximum at $t = \sqrt{\nu}$. Hence, the mixture of IFR distributions has a decreasing (tending to zero!) failure rate for sufficiently large and this is rather surprising. Furthermore, the same result asymptotically holds for arbitrary Weibull distributions with increasing failure rates. The light bulbs example of the Introduction suggests that the underlying (baseline) Cdf in this case is likely to be the Weibull distribution. $\diamondsuit$

It follows from (1.17) that the mixture will be IFR-stable if and only if for all $t \in [0, \infty)$:

$$\frac{\lambda'(t)}{\lambda(t)} \geq -\frac{E'[Z|t]}{E[Z|t]}. \tag{1.18}$$

Substituting $\lambda_m(t)$ and

$$f(t, z) = \lambda(t, z)\bar{F}(t, z) = z\lambda(t)\bar{F}(t, z)$$

in (1.10), similar to (1.14), we obtain the explicit formula for the derivative of the conditional expectation in this case, which shows that $E[Z|t]$ is a decreasing in t function:

Lemma 1.4 *The conditional expectation of Z for the multiplicative model is a decreasing function of $t \in [0, \infty)$, as follows from*

$$E'[Z|t] = -\lambda(t)Var(Z|t) < 0. \tag{1.19}$$

Combining this result with (1.15) and (1.16) for the specific case $\lambda(t) \equiv const$, we arrive at the well known result on the DFR property of the mixture of exponentials. Thus, the foregoing can be considered as a **new proof** of this fact.

With the help of (1.19) inequality (1.18) can be written as

$$\frac{\lambda'(t)}{\lambda^2(t)} \geq \frac{Var(Z|t)}{E[Z|t]}. \tag{1.20}$$

Thus, the first two conditional moments and the function $\lambda(t)$ are responsible for the IFR (DFR) properties of the mixture distribution. A similar result was obtained in a different way by Gurland and Sethuraman (1995). In this paper the right hand side of inequality (1.20) was written in terms of the moment generating function of the mixing Cdf $\Pi(z)$. Considering these problems with the help of conditional characteristics of Z can be often more convenient for the analysis of the corresponding monotonicity properties.

1.3 Laplace transform and inverse problem

The Laplace transform methodology in multiplicative and additive models gives us a convenient way of dealing with the mixture failure rates and conditional expectations, especially if the Laplace transforms of the mixing distribution can be explicitly calculated. It is also useful for solving the inverse problem.

Consider now a rather general class of mixing distributions, called the exponential family, which is given by the following relation

$$\pi(z) = e^{-\theta z}\mu(z)/\eta(\theta), \tag{1.21}$$

where $\mu(z), \eta(\theta)$ are some functions, θ is a parameter, varying in the distribution family. The function $\eta(\theta)$ plays the role of a normalizing constant, chosen so that the integral $\int_0^\infty \pi(z)dz = 1$. It is a very convenient representation of the distribution family to work with. It allows for the Laplace transform to be easily calculated. Many distributions can be represented in this form, such as Gamma, Inverse Gaussian (and Generalized inverse Gaussian), PVF distributions, positive stable distributions.

In this case the Laplace transform depends only on the function $\eta(\theta)$, which is really remarkable:

$$L_\pi(s) = \frac{1}{\eta(\theta)}\int_0^\infty e^{-sz}e^{-\theta z}\mu(z)dz = \frac{\eta(\theta+s)}{\eta(\theta)}. \tag{1.22}$$

It is well known that given only the failure data, a mixing distribution is non-identifiable. (Yashin and Manton, 1997). On the other hand, with the help of the Laplace transform approach the following inverse problem can be easily solved analytically:

Given the mixture failure rate and the mixing distribution, obtain the failure rate of the governing (baseline) distribution.

Now we consider additive and multiplicative models separately.

1.3.1 Additive model

In the additive model

$$\lambda(t,z) = \lambda(t) + z$$

the survival function and the corresponding pdf are

$$\bar{F}(t,z) = e^{-\Lambda(t)-zt}, \qquad f(t,z) = (\lambda(t)+z)e^{-\Lambda(t)-zt},$$

respectively, where

$$\Lambda(t) = \int_0^t \lambda(u)du$$

is a cumulative baseline failure rate. The mixture survival function $\bar{F}_m(t)$ in (1.1) can be writen via the Laplace transform as:

$$\bar{F}_m(t) = e^{-\Lambda(t)} \int_0^\infty e^{-zt}\pi(z)dz = e^{-\Lambda(t)} L_\pi(t), \tag{1.23}$$

where $L_\pi(s) = E[e^{-sZ}]$ is the Laplace transform of the mixing distribution $\pi(z)$. Thus, the mixture failure rate is

$$\lambda_m(t) = \lambda(t) + \frac{\int_0^\infty z e^{-zt}\pi(z)dz}{\int_0^\infty e^{-zt}\pi(z)dz} = \lambda(t) - [\log L_\pi(t)]'. \tag{1.24}$$

In accordance with (1.12), the second term equals to the conditional expectation

$$E[Z|t] = -[\log L_\pi(t)]',$$

which, as we noted before, is free from the initial baseline function and depends only on the mixing distribution.

In this case the solution to the inverse problem is trivial:

$$\lambda(t) = \lambda_m(t) - E[Z|t] = \lambda_m(t) + [\log L_\pi(t)]'.$$

If the Laplace transform of the mixing distribution can be explicitly calculated, we have a simple analytical solution to the inverse problem. We consider some examples of the Laplace transforms for specific distributions in the next section.

1.3.2 Multiplicative model

Consider the multiplicative model:

$$\lambda(t, z) = z\lambda(t),$$

with the corresponding survival function

$$\bar{F}(t, z) = e^{-z\Lambda(t)}.$$

The mixture survival function $\bar{F}_m(t)$ in (1.1) is then defined as

$$\bar{F}_m(t) = \int_0^\infty e^{-z\Lambda(t)}\pi(z)dz = L_\pi\{\Lambda(t)\}, \tag{1.25}$$

where, as previously, $L_\pi(s) = E[e^{-sZ}]$ is the Laplace transform of the mixing pdf $\pi(z)$.

The mixture failure rate is given by

$$\lambda_m(t) = -\frac{\bar{F}'_m(t)}{\bar{F}_m(t)} = -\frac{(L_\pi\{\Lambda(t)\})'}{L_\pi\{\Lambda(t)\}} = -\left(\log L_\pi\{\Lambda(t)\}\right)'. \tag{1.26}$$

Taking into account (1.17), the conditional expectation in this case can be written as

$$E[Z|t] = -\frac{L'_\pi\{\Lambda(t)\}}{L_\pi\{\Lambda(t)\}} = -\left(\log L_\pi\right)'\{\Lambda(t)\}. \tag{1.27}$$

As we see, the use of the Laplace transform in the multiplicative model is also very convenient. For computational reasons, the only condition is that the Lapalce transform of the mixing distribution should be calculated explicitly. This is why the models with Gamma distributed frailties are so popular (see e.g. Vaupel *et al.* (1979), Lancaster (1979), Lancaster and Nickel (1980)). Other distributions: the uniform distribution, the Weibull and the log normal, have been also proposed (Vaupel and Yashin, 1985). Hougaard (1984, 1986, 2000) generalized this approach on a broader class of distributions: exponential families and especially Power Variance Function (PVF) distributions.

Specifically, for the exponential family of mixing densities (1.21) and the multiplicative model, the mixture failure rate is obtained from (1.22) and (1.26):

$$\lambda_m(t) = -\left[\log \eta(\theta + \Lambda(t))\right]' = -\lambda(t)\frac{\eta'(\theta + \Lambda(t))}{\eta(\theta + \Lambda(t))} \tag{1.28}$$

and

$$E[Z|t] = -\frac{\eta'(\theta + \Lambda(t))}{\eta(\theta + \Lambda(t))}.$$

It follows from (1.26) that the general solution to the inverse problem in the multiplicative model is

$$\lambda(t) = \Lambda'(t) = \left(L_\pi^{-1}(e^{-\Lambda_m(t)})\right)', \tag{1.29}$$

Specifically, for the exponential family, using (1.28):

$$\Lambda(t) = \eta^{-1}\left(e^{-\Lambda_m(t)}\eta(\theta)\right) - \theta$$

and

$$\lambda(t) = \left(\eta^{-1}[\eta(\theta)e^{-\Lambda_m(t)}]\right)' = \frac{\eta(\theta)\lambda_m(t)e^{-\Lambda_m(t)}}{\eta'[\eta^{-1}\left(\eta(\theta)e^{-\Lambda_m(t)}\right)]}.$$

We now consider some particular distributions for the multiplicative model.

Example 1.3 Gamma distribution. The mixing density is:

$$\pi(z) = \frac{b}{\Gamma(c)}(zb)^{c-1}e^{-zb}.$$

In accordance with (1.21):

$$\eta(b) = \frac{\Gamma(c)}{b^c}, \qquad L_\pi(t) = \frac{b^c}{(b+t)^c},$$

and

$$\lambda_m(t) = \frac{c\lambda(t)}{b+\Lambda(t)}, \qquad E[Z|t] = \frac{c}{b+\Lambda(t)}.$$

We will see that these formulas coincide with those of Chapters 2 and 4 in this particular case.

The inverse problem simply follows:

$$\lambda(t) = -c\frac{E_t'[Z|t]}{E[Z|t]^2} = \frac{b}{c}\lambda_m(t)e^{\Lambda_m(t)/c},$$

where $\Lambda_m(t) = \int_0^t \lambda_m(u)du$ is the cumulative mixture failure rate. $\diamondsuit$

Example 1.4 Inverse Gaussian distribution. Consider the density

$$\pi(z) = (2\pi)^{-1/2}z^{-3/2}\nu^{1/2}e^{\sqrt{\theta\nu}}e^{-\theta z/2-\nu/2z}.$$

Then, in accordance with (1.21), the corresponding functions $\mu(z)$ and $\eta(\theta)$ in the exponential family are

$$\mu(z) = (2\pi)^{-1/2}e^{-\nu/2z}\nu^{1/2}z^{-3/2}, \qquad \eta(\theta) = e^{-\sqrt{\theta\nu}},$$

thus,

$$\lambda_m(t) = \frac{\sqrt{\nu}\lambda(t)}{2\sqrt{\theta + \Lambda(t)}}, \qquad E[Z|t] = \frac{\sqrt{\nu}}{2\sqrt{\theta + \Lambda(t)}}.$$

and the solution to the inverse problem is given by

$$\lambda(t) = \frac{2}{\nu}\lambda_m(t)(\sqrt{\theta\nu} + \Lambda_m(t)).$$

$\diamondsuit$

Example 1.5 Positive stable distributions. A distribution is strictly stable (See Feller, 1971 p. 169, Bingham *et al.*, p. 343) if the normalized sum of independent random variables from the distribution follows the same distribution:

$$\mathcal{D}(Z_1 + ... + Z_n) = \mathcal{D}(c_n Z_1)$$

for any n.

It turns out that the constant c_n must be of the form $n^{1/\alpha}$ for some $\alpha \in (0, 2]$. The stable distributions with finite variance are the normal, $\alpha = 2$, and the degenerate distributions, $\alpha = 1$. The positive stable distributions have $\alpha \in (0, 1]$ and apart from the scale factors have the Laplace transform

$$L(t) = e^{-\beta t^\alpha/\alpha} \tag{1.30}$$

and the density (see, for example Hougaard, 2000, p. 503)

$$\pi(z) = -\frac{1}{\pi z}\sum_{k=1}^{\infty}\frac{\Gamma(k\alpha + 1)}{k!}(-z^{-\alpha}\beta/\alpha)^k \sin(ak\pi). \tag{1.31}$$

The Laplace transform is already given, therefore:

$$\lambda_m(t) = (\beta\Lambda(t)^\alpha/\alpha)' = \lambda(t)\beta\Lambda(t)^{\alpha-1}, \qquad E[Z|t] = \beta\Lambda(t)^{\alpha-1}$$

and we obtain the solution to the inverse problem

$$\lambda(t) = \frac{\alpha^{1/\alpha-1}}{\beta^{1/\alpha}}\lambda_m(t)\Lambda_m(t)^{1/\alpha-1}.$$

$\diamondsuit$

The next family of distributions has been suggested by Tweedie (1984) and later independently derived by Hougaard (1986) and Bar-Lev and Enis (1986). It is thoroughly discussed in Hougaard (2000). We give its definition and the formulas for the density, and also obtain the mixture failure rate, a conditional expectation and solve the inverse problem for the multiplicative model.

Example 1.6 Power Variance Function. This family of distributions unites all three examples above. It is a distribution, which Laplace transform solves the equation

$$(\log L(t))' = -\beta(\gamma + t)^{\alpha-1} \tag{1.32}$$

The distribution is denoted $PVF(\alpha, \beta, \gamma)$. The parameter set is $\alpha \leq 1$, $\beta > 0$, with $\gamma \geq 0$ for $\alpha > 0$, and $\gamma > 0$ for $\alpha \leq 0$. The distribution is concentrated on positive numbers for $\alpha \geq 0$, and is positive or zero for $\alpha < 0$.

For $\alpha = 0$, the Gamma distribution is obtained, for $\alpha = 1/2$ it turns into the inverse Gaussian, for $\gamma = 0$ it is a positive stable distribution.

Simply from definition and relations (1.26) and (1.27) we obtain

$$\lambda_m(t) = \beta\lambda(t)(\gamma + \Lambda(t))^{\alpha-1}, \qquad E[Z|t] = \beta(\gamma + \Lambda(t))^{\alpha-1}.$$

As previously, the solution to the inverse problem is given by

$$\lambda(t) = \frac{1}{\beta}\lambda_m(t)\left(\frac{\alpha}{\beta}\Lambda_m(t) + \gamma^\alpha\right)^{\frac{1}{\alpha}-1}.$$

We do not need the mixing distribution density function to derive all the results, only its Laplace transform.

It is worth noting that the pdf is given by

$$\pi(z) = -e^{-\gamma z}e^{-\beta\gamma^\alpha/\alpha}\frac{1}{\pi z}\sum_{k=1}^{\infty}\frac{\Gamma(k\alpha+1)}{k!}(-z^{-\alpha}\beta/\alpha)^k\sin(ak\pi).$$

When $\alpha < 0$, the Γ-term in the density might not be defined. In this case we can use the alternative expression

$$\pi(z) = e^{-\gamma z+\beta^\alpha/\alpha}\frac{1}{z}\sum_{k=1}^{\infty}\frac{(-\beta z^{-\alpha}/\alpha)^k}{k!\Gamma(-k\alpha)}.$$

This expression holds for $z > 0$ and all α-values, except 0 and 1, with the convention that when the Γ-function in the denominator is undefined (which happens when $k\alpha$ is a positive integer), the whole term in the sum is 0. For $\alpha < 0$, there is probability $e^{\beta\gamma^\alpha/\alpha}$ of the random variable being 0. For $\alpha \geq 0$, the distribution is unimodal. $\diamondsuit$

1.4 Literature review

There was a brief review of some relevant papers in the Introduction. As the notation and initial results were already discussed in the previous sections of this chapter, we are able to turn to a more detailed analysis of the literature.

1.4.1 Monotonicity of mixture failure rates

Probably the first explanation of the decreasing failure rate for the mixture of exponential distributions was presented by Proschan (1963) (see also Barlow, 1985 and Mi, 1998). The proof of the closure property of the family of DFR distributions is given by Barlow and Proschan (1975) (see also Ross, 1996). Although mathematically simple, it was very important to show engineers, biologists and demographers that this is what really happens when you mix exponential distributions in practice. A more striking example is the multiplicative model with the Weibull mixing distribution (Example 1.2). The mixture failure rate in this case is equal to zero at $t = 0$ and then tends to zero as $t \to \infty$ with a single maximum. Hence, the mixture of IFR distributions has a decreasing (tending to zero!) failure rate for sufficiently large and this is rather surprising. The corresponding formal asymptotic analysis will be given in the next chapter. However, it can be easily seen from equation (1.17) that in the multiplicative model the resulting shape of the mixture failure rate $\lambda_m(t)$ for the IFR distribution can be described by the product of the increasing baseline $\lambda(t)$ and the decreasing conditional expectation $E[Z|t]$. From this perspective the DFR closure property under operation of mixing is trivially explained as both $\lambda(t)$ and $E[Z|t]$ are decreasing.

It was stated in Barlow and Proschan (1985) that the IFR property is not preserved under the operation of mixing. Therefore, additional assumptions which can guarantee this property are of interest in this respect. Lynch (1999) had derived the corresponding conditions:

The mixture failure rate $\lambda_m(t)$ for increasing $\lambda(t,z)$ is increasing, if:

1. *$\bar{F}(t,z)$ is log-concave in (t,z),*
2. *$\bar{F}(t,z)$ is increasing in z for each t,*
3. *The mixing distribution is IFR.*

The log-concavity is a natural, equivalent requirement for the IFR property of univariate distributions (Barlow and Proschan, 1975). Therefore, the first condition also seems natural for the bivariate case. The important stringent condition is, however, the second one. It is clear that for the multiplicative model (1.15) this condition does not hold, as the survival function

$$\bar{F}(t,z) = e^{-z\int_0^t \lambda(u)du}$$

is decreasing in z for each $t \geq 0$ (larger values of z result in larger values of the failure rate). Therefore, the simplest candidate e.g., for the linear failure rate, which meets the above conditions, would be

$$\lambda(t,z) = 2\frac{t}{z}.$$

It can be easily seen that $\int_0^t \lambda(u,z)du$ for this specific case is a convex function of (t,z) (Block et al, 2003), but the practical application of this inverse variation law with respect to z is not so evident as of the proportional law (1.15). The choice of the IFR mixing distribution is not so important and therefore the last assumption is not so restrictive. Note that some general preservation properties for mixtures were studied in Block, Li and Savits (2003b).

1.4.2 Mixture models

There can be various specific mixing models. In this chapter we have discussed only additive and multiplicative models. Another model that speaks for itself and has a clear meaning is the one based on the accelerated life model (ALM) (Cox and Oakes, 1984, Finkelstein, 1999, Bagdonavicius and Nikulin, 2000).

It is clear that relation (1.1) for absolutely continuous baseline $F(t)$ can be written for the multiplicative model as

$$F_m(t) = \int_a^b F^z(t)\pi(z)dz, \tag{1.33}$$

whereas the ALM mixing model is given by

$$F_m(t) = \int_a^b F(zt)\pi(z)dz. \tag{1.34}$$

Therefore, the corresponding mixture model is defined by (compare with (1.15)):

$$\lambda(t,z) = z\lambda(zt) \tag{1.35}$$

The linear scale transformation in the argument of the baseline failure rate in (1.35) makes our life much more difficult for modelling the shape of the corresponding mixture failure rate and other derivations, compared with the multiplicative model (1.15)-(1.16). This is why this model was practically not considered in the literature before. One can probably mention a paper by Anderson and Lois (1995), were the differences between the mixture failure rates defined via the ALM model (1.35), the multiplicative $\lambda_m(t)$ (1.16) and the baseline $\lambda_P(t)$ (1.4) are illustrated graphically. The formal equations for the difference between the mixture failure rates in the multiplicative and ALM models are given in Shaked and Spizzichino (2001). Note that these relations are too general and do not allow the relevant analysis. Shaked (1981) studied bounds on the distance of a mixture distribution function (not of a mixture failure rate!) from its parental distribution for some specific cases. We shall consider asymptotic properties of the mixture failure rate for the ALM in the next chapter.

Lynch (1999) has also noted that the convolution of two IFR distributions is IFR. On the other hand, this convolution can be viewed as specific mixture model, when we observe only T_1+T_2 and do not observe T_1 and T_2, where T_1, T_2 are the corresponding lifetimes. A similar mixture model was considered in Finkelstein (2002). The failure rate of a distribution, which describes a lifetime random variable with unknown initial age, was studied in this paper. In accordance with this mixing model, it turned out that under certain assumptions the operation of mixing distributions with increasing failure rates could result in the mixture failure rate with a bathtub shape: decreasing in the initial interval of time and then increasing, asymptotically approaching the failure rate of the baseline distribution.

As it has already been mentioned in the Introduction, our definition of the mixture failure rate (1.2) can be written in a more general form as

$$\lambda_m(t) = E[\lambda(t,Z)|T > t]. \tag{1.36}$$

It follows from Yashin and Manton (1997), that this equation is true when Z is not just a random variable, but an unobserved or partially observed stochastic process.

It is shown in Block *et al* (1993) that the natural ordering for the conditional random variables $[Z|T > t]$ is the likelihood ratio order, which means that

$$[Z|T > t] \geq_{LR} [Z|T > t'], 0 < t < t'. \tag{1.37}$$

In Chapter 4 we shall use this ordering for comparison of mixture failure rates with different mixing distributions. In fact, our results of this chapter suggest the same conclusion, but in a different way.

1.4.3 Biological and demographic aspects

Our study is devoted to mathematical properties and results in mixture failure rate modelling. Therefore, we shall only briefly discuss heterogeneity analysis in demographic and bio-medical applications. In fact, the input of these studies is hard to overestimate. J. Vaupel (Vaupel *et al.* 1979, Vaupel and Yashin, 1985) was the first to apply the frailty approach to human mortality. He had explained the departure (deceleration) of the oldest-old mortality from the Gompertz curve (see also Carey *et al*, 1992). Vaupel and Yashin (1985) had presented the graphs of the failure rate function for several mixtures. They have shown graphically (see also, Wang *et al*, 1998) that the failure rates of mixtures of some distributions may strictly decrease over some interval in $[0, \infty)$, even in the case, when all of the distributions that are being mixed have strictly increasing failure rates. They also studied graphically the combination of a classical proportional hazards model with a frailty model. In Chapter 4 we prove mathematically some of these results. Hougaard (1984) had generalized the multiplicative gamma-frailty model on some other mixing distributions. As we have seen in Section 1.3, the 'nice' Laplace transform is crucial for that. Hougaard (1984) describes some general classes of distributions (exponential families) and considers the subject in more generality in his 2000 book, defining the general class of distributions which he calls Power Variance Function (PVF) distributions.

Gurland and Sethuraman (1994) considered some further examples of mixtures that have strictly decreasing failure rates, although each of the distributions that are being mixed has a non-decreasing failure rate. They showed, for example, that a truncated extreme value distribution with exponentially increasing failure rate, when it is slightly mixed (5 %) with an exponential distribution, gives rise to a mixed distribution with a strictly decreasing failure rate in $[0, \infty)$.

Aalen (1992, 2005) uses the compound Poisson distribution as a frailty distribution in the disease modelling It allows some individuals in a population to be non-susceptible to a disease, which can be useful in many settings. A randomization of the Poisson parameter is also performed to end up with a new frailty distribution, which seems to be flexible and convenient especially for the shared frailty models. In Aalen (1988) a very helpful general discussion of the frailty concept in bio-medical applications is presented. These papers had a substantial impact on the field.

Brooks *et al* (1994) demonstrated the decrease in mortality rate with age in an entire population of nematodes even though each of several subpopulations showed continuously increasing mortality.

In Block *et al* (2003a) different shapes of the failure rate of the mixture of two distributions were studied. This is a rather complicated problem. For instance, in Block et al (2003b) the failure rate of a mixture of two distributions with linearly increasing failure rates can exhibit a rather bizarre behavior: it was proved that there can be up to four changes in monotonicity.

1.4.4 Asymptotic behavior

Most of the basic mathematical results of this thesis are presented in Chapter 2, dealing with modelling of tails of mixture failure rates. We develop a new approach and obtain some explicit and implicit results. Asymptotic theory for mixture failure rates had attracted attention of a number of researchers. Among the first to consider the limiting behavior of mixture failure rates were Clarotti and Spizzichino (1993), who stated that for the mixture of exponential subpopulations $\lambda(t, z) = z,\ z \in S \subset (0, \infty)$ the following convergence to the mixture failure rate of the strongest population takes place:

$$\lim_{t \to \infty} \lambda_m(t) = \inf\{z : z \in S\}. \qquad (1.38)$$

In Block *et al* (2003a) the following theorem (Theorem 2.1), generalizing the last equation and improving the results of Block *et al* (1993), was proved:

Consider the mixture failure rate as given by (1.2). Assume that

1. The failure rate $\lambda(t,z)$ converges to a function $a(z)$ uniformly on S, where $0 \leq a(z) \leq \infty$.

Let $I = \{z \in S: \ a(z) = \infty\}$. If $0 < P(I) < 1$, assume that

2. There exist constants L, D, such that $0 \leq L, D < \infty$ and $\lambda(t,z) \leq e^{Lt}$ for all $z \in I$ and $t \geq D$.

Then

$$\lim_{t\to\infty} \lambda_m(t) = \alpha, \tag{1.39}$$

where

$$\alpha = essinf_{z\in S}\, a(z) = \inf\{0 \leq c < \infty: \ P(\{z \in S: \ a(z) \leq c\}) > 0\}.$$

This theorem can be interpreted as saying that the mixture failure rate is asymptotically converging to the failure rate of the strongest population.

The assumptions of this theorem are rather stringent, especially the assumption 1 of the uniform convergence. For instance, it fails for our simplest Example 1.2. Block *et al* (2003a) also show some other important for practical analysis of heterogeneity examples, where one or both of these assumptions fail. Note that our approach of Chapter 2 does not need the stringent assumptions of this kind.

In Block and Joe (1997) the following asymptotic result, which already addressed the issue of ultimate monotonicity, was obtained: let z_0 be a realization of a frailty Z, which corresponds to the strongest population. If $\lambda(t,z)/\lambda(t,z_0)$ uniformly decreases as $t \to \infty$, then $\lambda_m(t)/\lambda(t,z_0)$ also decreases. If, in addition, $\lim_{t\to\infty} \lambda(t,z_0)$ exists, then this quotient decreases to 1. Again, an extremely stringent assumption of the uniform convergence is imposed.

The recent paper of Li (2005) generalizes the results of Block *et al* (2003a), using the similar analytical tools and approaches. Instead of assuming, as in Assumption 1 of the foregoing theorem, that each individual failure rate has a limit, the author assumes that there exists an asymptotic baseline function $\lambda(t)$ such that the ratio of each individual failure rate with the asymptotic baseline function $\lambda(t,z)/\lambda(t)$ has a limit. He shows that under certain conditions the ratio of the mixture failure rate with the asymptotic baseline function has a limit. As in Block *et al* (2003a) it is shown that this limit is the corresponding essential infimum. Again, the stringent condition of the uniform convergence of $\lambda(t,z)/\lambda(t)$ to some $a(z)$ is imposed. Therefore this paper combines the analytical reasoning of Block *et al* (2003a) with the 'ratio approach' of Block and Joe (1997).

The models in the foregoing papers are generalized proportional hazards models. The most general are based on the asymptotic equivalence $\lambda(t,z) \sim \lambda(t)a(z)$ in the sense of the uniform convergence of the ratio to 1.

Our approach of Chapter 2 is totally different. Our main focus is on explicit asymptotic formulas and on proving the results linking asymptotic behavior of mixture failure rates with the behavior of a mixing distribution in the neighborhood of the left end point of its support. Since mixtures are defined via integrals, it is very natural to exploit the corresponding analytical technique for analyzing these integrals (regular variation, generalized convolutions, etc).

1.4.5 Mean remaining lifetime

In this study we are looking at the mixture failure rate and consider general and specific models of mixing. The mean remaining lifetime (MRL) function, uniquely defined by the failure rate, can also constitute a convenient and reasonable in applications model of mixing, although we think that this approach did not receive the proper attention in the literature so far. The MRL function is defined as

$$m(t) = \frac{\int_t^\infty \bar{F}(u)du}{\bar{F}(t)}.$$

Therefore, the mixing model is given by

$$m(t,z) = \frac{\int_t^\infty \bar{F}(u,z)du}{\bar{F}(t,z)}, \tag{1.40}$$

and the mixture MRL function is

$$m_m(t) = \frac{\int_t^\infty \int_0^\infty \bar{F}(u,z)\pi(z)dzdu}{\int_0^\infty \bar{F}(t,z)\pi(z)dz} = \frac{\int_t^\infty \bar{F}_m(u)du}{\bar{F}_m(t)}. \tag{1.41}$$

These models and comparison of $\lambda_m(t)$ with $m_m(t)$ were considered in Finkelstein (2003), Zahedi (1991) and Badia *et al* (2001).

Chapter 2

Asymptotic theory for mixture failure rates

In this chapter we obtain explicit asymptotic results for the mixture failure rate $\lambda_m(t)$. We suggest a new class of distributions and formulate the results on asymptotic behavior. For proving our asymptotic results we use a convenient technique for Laplace integrals, which is similar to the one used for obtaining Abelian, Tauberian, and Mercerian-type theorems, although our theorems are not the direct corollaries of results in this field. Then we discuss possible generalizations.

It should be noted that the developed approach is new and differs from the one described in Block *et al* (2003a, 2003b), Li (2005). On one hand, we obtain explicit asymptotic formulas in a direct way, on the other hand, we are also able to discuss some general asymptotic properties of our models.

But first, we turn to some introductory results that will help us in understanding the nature of the problem and demonstrate some examples in the settings, which are already familiar.

2.1 Initial results. Discrete mixtures in the multiplicative model

Let the frailty Z be a discrete random variable taking values in a set $z_1, ..., z_n$. The discrete case can be very helpful for understanding certain basic issues for a more 'general' continuous one.

The conditional probability $\pi(z_i|t)$ of $Z = z_i$ given $T > t$, $1 \le i \le n$, is defined as

$$\pi(z_i|t) = \frac{\pi(z_i)\bar{F}(t, z_i)}{\sum_{j=1}^{n} \bar{F}(t, z_j)\pi(z_j)}.$$

Then the mixture failure rate can be written as

$$\lambda_m(t) = \sum_{j=1}^{n} \lambda(t, z_i)\pi(z_i|t)dz,$$

whereas for the multiplicative case the model (1.15) turns into

$$\lambda(t, z_i) = z_i\lambda(t), \qquad i = 1, 2 \ldots, n.$$

This relation, as in Chapter 1, leads to

$$\lambda_m(t) = \lambda(t)E[Z|t]$$

and can be used for the direct analysis of the limiting behavior of $\lambda_m(t)$ as $t \to \infty$.

For simplicity, let $n = 2$. Denote $\pi(z_1) = p_1$, $\pi(z_2) = p_2$; $p_1 + p_2 = 1$. Let $z_2 > z_1 > 0$. Then

$$\lambda_m(t) = \lambda(t, z_1)\pi(z_1|t) + \lambda(t, z_2)\pi(z_2|t), \tag{2.1}$$

where

$$\pi(z_i|t) = \frac{p_i\bar{F}(t, z_i)}{p_1\bar{F}(t, z_1) + p_2\bar{F}(t, z_2)}, \quad i = 1, 2. \tag{2.2}$$

Example 2.1 Consider the Weibull distribution of the following form:

$$\bar{F}(t, z_i) = e^{-z_i t^b}, \quad \lambda(t, z_i) = z_i b t^{b-1}, \quad b > 1,\ i = 1, 2.$$

Thus, in accordance with (2.1) and (2.2), for the multiplicative model:

$$\lambda_m(t) = z_1 b t^{b-1} \frac{p_1 e^{-z_1 t^b}}{p_1 e^{-z_1 t^b} + p_2 e^{-z_2 t^b}} + z_2 b t^{b-1} \frac{p_2 e^{-z_2 t^b}}{p_1 e^{-z_1 t^b} + p_2 e^{-z_2 t^b}}.$$

These relations suggest that as $t \to \infty$

$$\begin{aligned} \lambda_m(t) - \lambda(t, z_1) &= (z_2 - z_1) b t^{b-1} \frac{p_2 e^{-z_2 t^b}}{p_1 e^{-z_1 t^b} + p_2 e^{-z_2 t^b}} \\ &= (z_2 - z_1) b t^{b-1} \frac{p_2}{p_1} e^{-(z_2 - z_1)t^b}(1 + o(1)) \to 0 \end{aligned} \tag{2.3}$$

and the mixture failure rate is asymptotically "converging" to the failure rate of the strongest population from above. It is interesting to note that although the Weibull distribution has a power function failure rate, the speed of convergence (in the sense of the difference $\lambda_m(t) - \lambda(t, z_1)$) is exponential. When $b = 1$, the setting is reduced to a well-known exponential case.

In addition to this convergence result, the corresponding piecewise monotonicity properties can be analyzed. For this specific case the sign of $\lambda'_m(t)$ is of interest:

$$\begin{aligned}\lambda'_m(t) &= \left(\frac{bt^{b-1}(z_1p_1e^{-z_1t^b} + z_2p_2e^{-z_2t^b})}{p_1e^{-z_1t^b} + p_2e^{-z_2t^b}}\right)' \\ &= (p_1e^{-z_1t^b} + p_2e^{-z_2t^b})^{-2}\Big[\big\{b(b-1)t^{b-2}(z_1p_1e^{-z_1t^b} + z_2p_2e^{-z_2t^b}) \\ &\qquad - b^2t^{2b-2}(z_1^2p_1e^{-z_1t^b} + z_2^2p_2e^{-z_2t^b})\big\}(p_1e^{-z_1t^b} + p_2e^{-z_2t^b}) \\ &\quad + b^2t^{2b-2}(z_1p_1e^{-z_1t^b} + z_2p_2e^{-z_2t^b})^2\Big].\end{aligned}$$

Thus, the sign of $\lambda'_m(t)$ is the same as the sign of

$$\begin{aligned}(b-1)(z_1p_1e^{-z_1t^b} + z_2p_2e^{-z_2t^b})(p_1e^{-z_1t^b} + p_2e^{-z_2t^b}) \\ - bp_1p_2(z_1 - z_2)^2t^be^{-(z_1+z_2)t^b}.\end{aligned}$$

(see also Theorem 1 in Gurland and Sethuraman (1995)).

If $b \leq 1$, then this quantity is negative and, therefore, $\lambda'_m(t) < 0$ for all $t > 0$.

For the case $b > 1$, it is clear that for $t \in [c, \infty)$, where c is sufficiently large, $\lambda'_m(t) > 0$. Hence, the mixture failure rate is increasing in this interval. On the other hand, it also holds for $t \in [0, d)$, where d is sufficiently small. The behavior in the intermediate interval depends on the parameters involved. For instance, one can always find a sufficiently small $\epsilon > 0$ such that if $z_2 - z_1 < \epsilon$, then the inequality holds in $[0, \infty)$. In this case the mixture is IFR-stable, which means that its failure rate is increasing (non-decreasing). ♢

As usually, throughout this thesis we use the terms "increasing", "decreasing" meaning "non-decreasing", "non-increasing" respectively.

It should be noted that the condition

$$\lim_{t\to\infty} \pi(z_2|t) = 0, \quad \lim_{t\to\infty} \pi(z_1|t) = 1$$

is not sufficient for the convergence result (2.3).

When dealing with the limiting behavior of the mixture failure rate $\lambda_m(t)$ for a general case of the multiplicative model with increasing $\lambda(t)$ $(\lambda(t) \to \infty$ as $t \to \infty)$, we are sometimes interested, as in Example 2.1, in the 'strong' convergence of the mixture failure rate to the failure rate of the strongest population:

$$\lambda_m(t) - \lambda(t, z_1) \to 0 \quad \text{as} \quad t \to \infty. \tag{2.4}$$

On the other hand, a conventional weaker asymptotic equivalence

$$\lambda_m(t) = \lambda(t, z_1)(1 + o(1)) \quad \text{as} \quad t \to \infty, \tag{2.5}$$

denoted as

$$\lambda_m(t) \sim \lambda(t, z_1)$$

will be of prime interest in the rest of this chapter.

The following theorem is a simple consequence of the above considerations. It describes the convergence to the failure rate of the strongest population in the case of two populations.

Theorem 2.1 *Let:*

$$\lambda(t, z_1) = z_1\lambda(t), \quad \lambda(t, z_2) = z_2\lambda(t), \quad z_2 > z_1 > 0,$$

where $\lambda(t) \to \infty$ *as* $t \to \infty$.

Then:

a. Relation (2.5) takes place.

b. Relation (2.4) holds if and only if:

$$\lambda(t)e^{-(z_2 - z_1)\Lambda(t)} \to 0 \quad \textit{as} \quad t \to \infty, \tag{2.6}$$

where $\Lambda(t) = \int_0^t \lambda(u)du$ *is a cumulative baseline failure rate.*

Proof The proof of this result is absolutely straightforward. Indeed, denote for convenience $c \equiv z_2/z_1 > 1$. Using simple transformations, similar to Block and Joe (1997)

$$\frac{\lambda_m(t)}{\lambda(t, z_1)} = 1 + \frac{p_1(c-1)}{p_1 + p_2\left(\bar{F}_1(t, z_1)\right)^{1-c}}.$$

As $\bar{F}(t, z_1) \to 0$ for $t \to \infty$, we immediately arrive at (2.5), while the condition

$$\lambda(t, z_1) \left(\bar{F}_1(t, z_1)\right)^{1-c} \to 0 \quad \text{as} \quad t \to \infty,$$

which is equivalent to (2.6), leads to convergence (2.4). □

Condition (2.6) is a rather weak one. In essence it states that the pdf of a distribution with (ultimately) increasing failure rate $(z_2 - z_1)\lambda(t)$ tends to 0 as $t \to \infty$. All distributions, which are typically used in a lifetime data analysis, meet this condition. But one can consider some "bizarre" distributions, for which relation (2.6) does not hold. Let, for instance

$$\lambda(t) = \beta_{n+1} \quad \text{for} \quad t \in [n, n+1), n = 0, 1, 2...$$

and

$$\beta_1 = 1,\ \beta_{n+1} = e^{\sum_{i=1}^{n} \beta_i};\ n = 1, 2, 3, ...$$

The defined $\lambda(t)$ is piecewise continuous, but it can be made continuous increasing (non-decreasing) function in an obvious way. It is easy to verify in this case that (2.6) does not hold. Therefore, there is no convergence defined by relation (2.4). We are grateful to professor Block for this example (personal communication).

As it was already mentioned, a general convergence result for the mixture of distributions in a generalized multiplicative model was obtained in Block, Mi and Savits (1993) (Theorem 4.1). As we are dealing with the specific model of mixing in a direct way, only one assumption on the growth of $\lambda(t)$ in the form of (2.6) is considered. This assumption is weaker than the corresponding condition (C) in Block, Mi and Savits (1993), which states for our case that there exists non-negative constants L and T such that $\lambda(t) \le e^{Lt}$ for all $t \ge T$. Indeed, let, for instance, $\lambda(t) = e^{t^2}$, then assumption (2.6) holds while condition (C) is not satisfied.

The next sections are devoted to a much more general model, which includes the additive, the multiplicative, and the accelerated life models as specific cases. We deal with the absolutely continuous distributions, but some generalizations may also be discussed. We consider the following as the main mathematical results of this thesis.

2.2 The survival model

We define a class of lifetime distributions $F(t,z)$ and will study asymptotic behavior of the corresponding mixture failure rate $\lambda_m(t)$. It is more convenient at the start to give this definition in terms of the cumulative failure rate $\Lambda(t,z) = \int_0^t \lambda(u,z)du$, rather than in terms of the failure rate $\lambda(t,z)$.

The basic model is defined by the following relation:

$$\Lambda(t,z) = A(z\phi(t)) + \psi(t). \tag{2.7}$$

General assumptions for the model (2.7):

Natural properties of the cumulative failure rate of the absolutely continuous distribution $F(t,z)$ (for all $z \in [0,\infty)$) imply that the functions $A(s), \phi(t)$, and $\psi(t)$ are differentiable, the right hand side of (2.7) is non-decreasing in t and tends to infinity as $t \to \infty$, and that $A(z\phi(0)) + \psi(0) = 0$. Therefore, these properties will be assumed throughout, although some of them will not be needed for formal proofs. We will also relax them to a certain extent in Section 2.7.

An important additional simplifying assumption is that

$$A(s), s \in [0,\infty); \phi(t), t \in [0,\infty)$$

are increasing functions of their arguments, although some generalizations (e.g., for ultimately increasing functions) can be easily performed. Therefore, we will consider $1 - e^{-A(z\phi(t))}$, $z \neq 0$ here as a lifetime Cdf.

It should be noted that model (2.7) can be also easily generalized to the form

$$\Lambda(t,z) = A(g(z)\phi(t)) + \psi(t) + \eta(z)$$

for some properly defined functions $g(z)$ and $\eta(z)$. As it was mentioned, we will consider this and some other generalizations in Section 2.7. However, we cannot go generalizing further (at least, at this stage) and the multiplicative form of arguments in $A(g(z)\phi(t))$ is important for our method of deriving asymptotic relations. It is also clear that the additive term $\psi(t)$, although important in applications, gives only a slight generalization for further analysis of $\lambda_m(t)$, as (2.7) can be interpreted in terms of two components in series (or, equivalently, via two competing risks). However, this term will be essential in next section, while defining the strongest population.

The failure rate, corresponding to the cumulative failure rate $\Lambda(t,z)$, is

$$\lambda(t,z) = z\phi'(t)A'(z\phi(t)) + \psi'(t). \tag{2.8}$$

Now we are able to explain why we start with the cumulative failure rate and not with the failure rate itself, as often in lifetime modelling. The reason is that one can easily suggest intuitive interpretations for (2.7), whereas it is certainly not so simple to interpret the failure rate structure in the form (2.8) without stating that it just follows from the structure of the cumulative failure rate.

Relation (2.7) defines a rather broad class of survival models, which can be used, e.g., for modelling an impact of environment on characteristics of survival. The widely used in reliability, survival analysis, and risk analysis proportional hazards (PH), additive hazards (AH), and accelerated life (ALM) models are obvious specific cases of our relations (2.7) or (2.8):

PH (multiplicative) Model:

Let

$$A(u) \equiv u, \quad \phi(t) = \Lambda(t), \quad \psi(t) = 0.$$

Then

$$\lambda(t,z) = z\lambda(t), \quad \Lambda(t,z) = z\Lambda(t). \tag{2.9}$$

Accelerated Life Model:

Let

$$A(u) \equiv \Lambda(u), \quad \phi(t) = t, \quad \psi(t) = 0.$$

Then

$$\Lambda(t,z) = \int_0^{tz} \lambda(u)du = \Lambda(tz), \quad \lambda(t,z) = z\lambda(tz). \tag{2.10}$$

AH Model:

Let

$$A(u) \equiv u, \quad \phi(t) = t, \qquad \psi(t) \text{ is increasing}, \ \psi(0) = 0.$$

Then

$$\lambda(t,z) = z + \psi'(t), \quad \Lambda(t,z) = zt + \psi(t). \tag{2.11}$$

The functions $\lambda(t)$ and $\phi'(t)$ play the role of baseline failure rates in equations (2.9), (2.10) and (2.11), respectively. Note that in all these models the functions $\phi(t)$ and $A(s)$ are monotonically increasing.

Asymptotic behavior of mixture failure rates for PH and AH models was studied for some specific mixing distributions, e.g., in Gurland and Sethuraman (1995) and Finkelstein and Esaulova (2001a). On the other hand, as far as we know, the mixture failure rate for the ALM was considered at a very descriptive level only in Anderson and Louis (1995).

2.3 General results

In this section we formulate the main asymptotic theorems. The corresponding proofs are deferred to Section 2.4, applications to the multiplicative and the accelerated life models are considered in subsequent sections 2.5 and 2.6.

The next theorem derives an asymptotic formula for the mixture failure rate $\lambda_m(t)$ under rather mild assumptions.

Theorem 2.2 *Let the cumulative failure rate $\Lambda(t,z)$ be given by model (2.7) and the mixing pdf $\pi(z)$ be defined as*

$$\pi(z) = z^{\alpha}\pi_1(z), \tag{2.12}$$

where $\alpha > -1$ and $\pi_1(z)$, $\pi_1(0) \neq 0$, is a bounded in $[0,\infty)$ and continuous at $z = 0$ function.

Assume also that

$$\phi(t) \to \infty \quad as \quad t \to \infty \tag{2.13}$$

and

$$\int_0^{\infty} e^{-A(s)} s^{\alpha} ds < \infty, \tag{2.14}$$

where $A(s)$ is also ultimately increasing.

Then

$$\lambda_m(t) - \psi'(t) \sim (\alpha+1)\frac{\phi'(t)}{\phi(t)}. \tag{2.15}$$

By relation (2.15) we, as usual, mean asymptotic equivalence and write $a(t) \sim b(t)$ as $t \to \infty$, if $\lim_{t\to\infty}[a(t)/b(t)] = 1$.

It is easy to see that assumption (2.12) holds for the main lifetime distributions such as Weibull, Gamma, lognormal etc. Assumption (2.13) states a natural condition for the function $\phi(t)$, which can be often viewed as a scale transformation. Condition (2.14) means that the Cdf $1 - e^{-A(s)}$ should not be 'too heavy tailed' (as, e.g., the Pareto distribution $1 - s^{-\beta}$, for $s \geq 1$, $\beta - \alpha > 1$) and in our assumptions is equivalent to the condition of existence of the moment of order $\alpha + 1$ for this Cdf. Examples of the next section will clearly show that these conditions are not stringent at all and can be easily met in most practical situations.

A ***crucial feature of this result*** *is that the asymptotic behavior of the mixture failure rate depends only (omitting an obvious additive term $\psi(t)$) on the behavior of the mixing distribution in the neighborhood of zero and on the derivative of the logarithm of the scale function $\phi(t)$:*

$$(\log\phi(t))' = \phi'(t)/\phi(t).$$

When $\pi(0) \neq 0$ and $\pi(z)$ is bounded in $[0, \infty)$, the result does not depend on the mixing distribution at all, as $\alpha = 0$!

Theorem 2.2 (as well as later theorems 2.3 and 2.5) can be formally generalized to the case when the mixing random variable Z does not necessarily posses an absolutely continuous Cdf in $[0, \infty)$: it is sufficient that it should be absolutely continuous (from the right) at $z = 0$.

We can formulate a more general result, which states a similar dependence on the behavior of the mixing distribution at zero in terms of asymptotic comparison of two mixture failure rates:

If, under some assumptions, two mixing distributions are equivalent at $z = 0$, then the mixture failure rates are equivalent as $t \to \infty$.

Formally:

Theorem 2.3 *Let $f(t, z)$ and $\pi(z)$ be the lifetime and mixing pdf's in a general mixing model (2.7), respectively. Assume that there exists a positive function $\alpha(t)$, which is ultimately decreasing to 0 as $t \to \infty$ and that*

$$\frac{\int_0^{\alpha(t)} f(t,z)\pi(z)dz}{\int_0^{\infty} f(t,z)\pi(z)dz} \to 1. \tag{2.16}$$

Denote another mixing pdf by $\rho(z)$ and assume that $\rho(z)/\pi(z)$ is bounded in $[0,\infty)$, continuous at 0, and $\lim_{z\to 0}\rho(z)/\pi(z)\neq 0$. Then:

$$\lambda_m^{\pi}(t)\equiv\frac{\int_0^{\infty} f(t,z)\pi(z)dz}{\int_0^{\infty}\bar{F}(t,z)\pi(z)dz}\sim\frac{\int_0^{\infty} f(t,z)\rho(z)dz}{\int_0^{\infty}\bar{F}(t,z)\rho(z)dz}\equiv\lambda_m^{\rho}(t) \tag{2.17}$$

as $t\to\infty$.

It is worth noting that if $\psi\equiv 0$ and all other conditions of Theorem 2.2 hold, condition (2.13) of this theorem guarantees assumption (2.16).

It is important that for applying Theorem 2.3 we do not need a specific form of a survival model. As it will be seen from the proof, $\pi(z)$ and $\rho(z)$ also need not necessarily be probability density functions (local integrability, in fact, is sufficient). The following corollary exploits the latter fact for the case $\pi(z)\equiv 1$:

Corollary 2.1 *Let $f(x,t)$ be a lifetime pdf in a general mixing model (2.7). Assume that there exists a positive function $\alpha(t)$ such that $\alpha(t)$ is ultimately decreasing to zero as $t\to\infty$ and*

$$\frac{\int_0^{\alpha(t)} f(t,z)dz}{\int_0^{\infty} f(t,z)dz}\to 1. \tag{2.18}$$

Let $\rho(z)$ be positive function bounded in $[0,\infty)$, continuous at zero, and $\rho(0)\neq 0$. Then:

$$\frac{\int_0^{\infty} f(t,z)\rho(z)dz}{\int_0^{\infty}\bar{F}(t,z)\rho(z)dz}\sim\frac{\int_0^{\infty} f(t,z)dz}{\int_0^{\infty}\bar{F}(t,z)dz}. \tag{2.19}$$

as $t\to\infty$.

Condition (2.18) is not that unnatural and holds, for instance, for multiplicative model and accelerated life models with the condition $\alpha(t)\Lambda(t)\to\infty$ as $t\to\infty$.

Theorems 2.2 and 2.3 consider the case when the support of a mixing distribution includes 0: $z\in[0,\infty)$. If the support is separated from 0, $z\in[a,\infty)$, $a>0$ the situation changes significantly and we can observe a well-known principle that the mixture failure rate tends to the failure rate of the strongest population (Block and Joe, 1992; Block et al, 2003; Finkelstein and Esaulova, 2001a). Note that when $a=0$, the strongest population cannot be defined.

Theorem 2.4 *Let the class of lifetime distributions be defined by equation (2.7), where* $\phi(t) \to \infty$, $A(s)$ *is twice differentiable,* $\int_0^\infty e^{-A(s)}ds < \infty$. *Assume that as* $s \to \infty$

$$\frac{A''(s)}{(A'(s))^2} \to 0 \tag{2.20}$$

and

$$sA'(s) \to \infty. \tag{2.21}$$

Assume also that for all $b, c > a$, $b < c$ *the quotient* $A'(bs)/A'(cs)$ *is bounded as* $s \to \infty$.

Let the mixing pdf $\pi(z)$ *be defined in* $[a, \infty)$, $a > 0$, *bounded in this interval, continuous at* $z = a$, *and* $\pi(a) \neq 0$.

Then as $t \to \infty$

$$\lambda_m(t) - \psi'(t) \sim a\phi'(t)A'(a\phi(t)). \tag{2.22}$$

It is clear that conditions (2.20) and (2.21) trivially hold for specific multiplicative and additive models of the previous section. We will discuss them within the framework of the accelerated life model later. More generally, these conditions hold if $A(s)$ belongs to a class of functions of smooth variation (Bingham *et al*, 1987).

Assume additionally that the family of failure rates (2.8) is ordered in z, at least, ultimately (this ordering will be very important for our results in Chapter 4):

$$\lambda(t, z_1) < \lambda(t, z_2), \quad \forall z_1, z_2 \in [z_0, \infty),\ z_1 < z_2, z_0 \geq 0, t \geq 0.$$

Then, as it was mentioned, Theorem 2.4 can be interpreted via the principle that the mixture failure rate converges to the failure rate of the strongest population. (Note that the right hand side in (2.22) also can be interpreted in this case as the failure rate of the strongest population for a survival model, defined by a random variable with the Cdf $1 - e^{-A(z\phi(t))}$). An interesting question arises: whether this principle is a 'universal law', or a consequence of sufficient assumptions of Theorem 2.4? Theorem 2.2 gives us an idea for creating counter-examples:

Example 2.2 Assume that all conditions of Theorem 2.2 hold and, additionally, $A'(s)$ is increasing in $[0, \infty)$. Then an ordering of failure rates in the family (2.8) with respect to z (for each fixed $t > 0$) holds resulting **formally** in the strongest population defined as $\lambda(t,0) = \phi'(t)$. Note, however, that $1 - e^{-A(z\phi(t))}$, $z = 0$, cannot be viewed as a Cdf. Therefore, the principle under question implies that

$$\lambda_m(t) \sim \psi'(t).$$

On the other hand, it follows from (2.15) that

$$\lambda_m(t) \sim \psi'(t) + (\alpha + 1)(\log \phi(t))'$$

and if the second term on the right hand side of this relation is increasing faster than $\psi'(t)$ as $t \to \infty$, then this term defines asymptotic behavior of $\lambda_m(t)$. It is clear that it is possible for fast increasing functions (e.g., for $\exp\{t^n\}$, $n \geq 1$). Thus, if

$$\psi'(t) = o((\log \phi(t))'),$$

then

$$\lambda_m(t) \sim (\alpha + 1)(\log \phi(t))',$$

whereas the Principle holds only when $(\log \phi(t))' = o(\psi'(t))$. $\diamondsuit$

Theorem 2.2 gives us only the asymptotics $const \cdot \phi'(t)/\phi(t)$. The next example shows us that it is not the only option and if the mixing distribution $\pi(z)$ behaves differently at $z = 0$, the asymptotics also might be different.

Example 2.3 Consider the multiplicative model

$$\lambda(t,z) = z\lambda(t).$$

The survival function and the pdf are

$$\bar{F}(t,z) = e^{-\Lambda(t)z}, \qquad f(t,z) = z\lambda(t)e^{-\Lambda(t)z},$$

respectively, where $\Lambda(t)$ is a cumulative baseline failure rate.

Consider the mixing distribution density of the form:

$$\pi(z) = \frac{1}{\sqrt{\pi} z \sqrt{z}} e^{-1/z}.$$

Then the mixture failure rate is

$$\lambda_m(t) = \lambda(t)/\sqrt{\Lambda(t)},$$

whereas in terms of the survival model (2.7): $\phi'(t)/\phi(t) = \lambda(t)/\Lambda(t)$.

Indeed, the mixture survival function is

$$\bar{F}_m(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\Lambda(t)z - \frac{1}{z}} \cdot \frac{1}{z\sqrt{z}}\, dz.$$

Changing the variable of integration $z = u/\sqrt{\Lambda(t)}$:

$$\bar{F}_m(t) = \frac{\Lambda(t)^{1/4}}{\sqrt{\pi}} \int_0^\infty e^{-\sqrt{\Lambda(t)}\left(u+\frac{1}{u}\right)} \cdot \frac{1}{u\sqrt{u}}\, du.$$

Similarly, the mixture pdf:

$$\begin{aligned} f_m(t) &= \frac{1}{\sqrt{\pi}} \int_0^\infty z\lambda(t) e^{-\Lambda(t)z - \frac{1}{z}} \cdot \frac{1}{z\sqrt{z}}\, dz \\ &= \frac{\lambda(t)}{\sqrt{\pi}} \int_0^\infty e^{-\Lambda(t)z - \frac{1}{z}} \cdot \frac{1}{\sqrt{z}}\, dz \\ &= \frac{\lambda(t)}{\sqrt{\pi}\Lambda(t)^{1/4}} \int_0^\infty e^{-\sqrt{\Lambda(t)}\left(u+\frac{1}{u}\right)} \cdot \frac{1}{\sqrt{u}}\, du \end{aligned}$$

Changing again the variable of integration, $u = 1/s$, we obtain that

$$\int_0^\infty e^{-\sqrt{\Lambda(t)}\left(u+\frac{1}{u}\right)} \cdot \frac{1}{\sqrt{u}}\, du = \int_0^\infty e^{-\sqrt{\Lambda(t)}\left(\frac{1}{s}+s\right)} \cdot \frac{1}{s\sqrt{s}}\, ds,$$

thus, the integrals in the numerator and denominator cancel out and the mixture failure rate is

$$\begin{aligned} \lambda_m(t) = \frac{f_m(t)}{\bar{F}_m(t)} &= \frac{\lambda(t)}{\sqrt{\pi}\Lambda(t)^{1/4}} \cdot \frac{\sqrt{\pi}}{\Lambda(t)^{1/4}} \\ &= \frac{\lambda(t)}{\sqrt{\Lambda(t)}}. \end{aligned}$$

◇

2.4 Proofs

2.4.1 Proof of Theorem 2.2

First we need a simple lemma for the Dirac sequence of functions:

Lemma 2.1 *Let $g(z), h(z)$ be nonnegative locally integrable functions defined in $[0, \infty)$ and satisfying the following conditions:*

$$\int_0^\infty g(z)dz < \infty,$$

and $h(z)$ is bounded and continuous at $z = 0$.

Then, as $t \to \infty$:

$$t \int_0^\infty g(tz)h(z)dz \to h(0) \int_0^\infty g(z)dz. \tag{2.23}$$

Proof Substituting $u = tz$:

$$t \int_0^\infty g(tz)h(z)dz = \int_0^\infty g(u)h(u/t)du.$$

The function $h(u)$ is bounded and $h(u/t) \to h(0)$ as $t \to \infty$; thus, convergence (2.23) holds by dominated convergence theorem.

□

Now we prove Theorem 2.2. The proof is straightforward, as we use definition (2.7) and Lemma 2.1.

The survival function for the model (2.7) is

$$\bar{F}(t, z) = e^{-A(z\phi(t))-\psi(t)}.$$

Taking into account that $\phi(t) \to \infty$ as $t \to \infty$, and applying Lemma 2.1 to the function $g(u) = e^{-A(u)}u^\alpha$:

$$\begin{aligned} \int_0^\infty \bar{F}(t, z)\pi(z)dz &= \int_0^\infty e^{-A(z\phi(t))-\psi(t)} z^\alpha \pi_1(z)dz \\ &= \frac{e^{-\psi(t)}}{\phi(t)^\alpha} \int_0^\infty e^{-A(z\phi(t))}(z\phi(t))^\alpha \pi_1(z)dz \qquad (2.24) \\ &\sim \frac{e^{-\psi(t)}\pi_1(0)}{\phi(t)^{\alpha+1}} \int_0^\infty e^{-A(s)} s^\alpha ds, \end{aligned}$$

where the integral is finite due to condition (2.14). The corresponding probability density function is:

$$\begin{aligned} f(t,z) &= \left[A'(z\phi(t))z\phi'(t) + \psi'(t)\right] e^{-A(z\phi(t))-\psi(t)} \\ &= A'(z\phi(t))z\phi'(t)e^{-A(z\phi(t))-\psi(t)} + \psi'(t)\bar{F}(t,z). \end{aligned}$$

Similarly, applying Lemma 2.1:

$$\begin{aligned} \int_0^\infty f(t,z)\pi(z)dz - \psi'(t)\int_0^\infty \bar{F}(t,z)\pi(z)dz \\ &= \phi'(t)e^{-\psi(t)}\int_0^\infty A'(z\phi(t))e^{-A(z\phi(t))}z^{\alpha+1}\pi_1(z)dz \qquad (2.25) \\ &\sim \frac{\phi'(t)e^{-\psi(t)}\pi_1(0)}{\phi(t)^{\alpha+2}}\int_0^\infty A'(s)e^{-A(s)}s^{\alpha+1}ds. \end{aligned}$$

Due to condition (2.14) and the fact that $A(s)$ is ultimately increasing,

$$e^{-A(s)}s^{\alpha+1} \to 0 \quad \text{as } s \to \infty. \tag{2.26}$$

Indeed, by the mean value theorem

$$\int_s^{2s} e^{-A(u)}u^\alpha du = se^{-A(s_1)}s_1^\alpha$$

for some $s \le s_1 \le 2s$. The right-hand side tends to 0. For s larger than some s_0 we have $A(s_1) > A(s)$; thus, the left-hand side is smaller than $2^\alpha s^{\alpha+1}e^{-A(s)}$, and this leads to (2.26). Using it while integrating by parts, we get

$$\int_0^\infty A'(s)e^{-A(s)}s^{\alpha+1}ds = (\alpha+1)\int_0^\infty e^{-A(s)}s^\alpha ds. \tag{2.27}$$

Combining (2.24)-(2.27), finally:

$$\frac{\int_0^\infty f(t,z)\pi(z)dz}{\int_0^\infty \bar{F}(t,z)\pi(z)dz} - \psi'(t) \sim (\alpha+1)\frac{\phi'(t)}{\phi(t)}.$$

2.4.2 Proof of Theorem 2.3

Lemma 2.2 *Let* $\{g(t,z),\ z \in [0,\infty)\}$ *be a family of functions and* $h(z)$ *a function, satisfying the following conditions:*

(i) for every $z \in [0,\infty)$ *the function* $g(t,z)$ *is integrable in* t *and for every* $t \in [0,\infty)$ *it is integrable in* z.

(ii) there exists a function $\alpha(t)$, $\alpha(t) \to 0$ *as* $t \to \infty$ *and*

$$\frac{\int_0^{\alpha(t)} g(t,z)dz}{\int_0^{\infty} g(t,z)dz} \to 1 \tag{2.28}$$

as $t \to \infty$.

(iii) a function $h(z)$ *is locally integrable, bounded in* $[0,\infty)$, *and continuous at* $z = 0$.

Then, as $t \to \infty$:

$$\frac{\int_0^{\infty} g(t,z)h(z)dz}{\int_0^{\infty} g(t,z)dz} \to h(0).$$

Proof Let $h(z) \le M$, $z \in [0,\infty)$. Then:

$$\frac{\int_0^{\infty} g(t,z)h(z)dz}{\int_0^{\infty} g(t,z)dz} = \frac{\int_0^{\alpha(t)} g(t,z)h(z)dz}{\int_0^{\infty} g(t,z)dz} + \frac{\int_{\alpha(t)}^{\infty} g(t,z)h(z)dz}{\int_0^{\infty} g(t,z)dz}.$$

The second term is majorized by

$$M\frac{\int_{\alpha(t)}^{\infty} g(t,z)dz}{\int_0^{\infty} g(t,z)dz} = M\left(1 - \frac{\int_0^{\alpha(t)} g(t,z)dz}{\int_0^{\infty} g(t,z)dz}\right),$$

which is due to condition (2.28). The first term converges to $h(0)$ due to the same condition and the fact that $h(z)$ is continuous at $z = 0$. □

For proving Theorem 2.3 we first show in a direct way that for $\bar{F}(t,z)$ there holds a condition similar to (2.16). For every $\varepsilon > 0$ we choose t_ε such that for $u > t_\varepsilon$ the function $\alpha(u)$ already decreases and

$$\int_0^{\alpha(u)} f(u,z)\pi(z)dz > (1-\varepsilon)\int_0^{\infty} f(u,z)\pi(z)dz.$$

Since $\alpha(t)$ decreases

$$\int_0^{\alpha(t)} f(u,z)\pi(z) > \int_0^{\alpha(u)} f(u,z)\pi(z)dz$$

for $u > t > t_\varepsilon$. Thus

$$\begin{aligned}\int_0^{\alpha(t)} \bar{F}(t,z)\pi(z)dz &= \int_0^{\alpha(t)} \int_t^\infty f(u,z)du\,\pi(z)dz \\ &= \int_t^\infty \int_0^{\alpha(t)} f(u,z)\pi(z)dz\,du \\ &> \int_t^\infty \int_0^{\alpha(u)} f(u,z)\pi(z)dz\,du \\ &> (1-\varepsilon)\int_t^\infty \int_0^\infty f(u,z)\pi(z)dz\,du \\ &= (1-\varepsilon)\int_0^\infty \bar{F}(t,z)\pi(z)dz.\end{aligned}$$

Now we apply Lemma 2.2 with $h(z) = \pi_1(z)/\pi(z)$ and $g(t,z) = f(t,z)\pi(z)$, which results in

$$\frac{\int_0^\infty f(t,z)\rho(z)dz}{\int_0^\infty f(t,z)\rho(z)dz} \to h(0).$$

In a similar way $g(t,z) = \bar{F}(t,z)\pi(z)$ with the same $h(z)$ gives

$$\frac{\int_0^\infty \bar{F}(t,z)\rho(z)dz}{\int_0^\infty \bar{F}(t,z)\rho(z)dz} \to h(0),$$

as $t \to \infty$, and relation (2.17) follows immediately.

2.4.3 Proof of Theorem 2.4

This theorem is rather technical and we must first prove three supplementary lemmas, which present consecutive steps on our way to asymptotic relation (2.22).

Lemma 2.3 *Let $h(x)$ be a twice differentiable function with an ultimately positive derivative, and*

$$\int_0^\infty e^{-h(y)}dy < \infty. \tag{2.29}$$

Let also

$$\frac{h''(x)}{(h'(x))^2} \to 0 \tag{2.30}$$

as $x \to \infty$. *Then*

$$\int_x^\infty e^{-h(y)} dy \sim e^{-h(x)} \frac{1}{h'(x)}$$

as $x \to \infty$.

Proof The function $h'(x)$ is ultimately positive. Let x_0 be such that $h'(x) > 0$ for $x > x_0$. Due to (2.29) we have: $h(x) \to \infty$ as $x \to \infty$. Then there exists an inverse function $g(x)$ defined in $[x_0, \infty)$:

$$g(h(x)) \equiv h(g(x)) \equiv 1.$$

The function $g(x)$ is also twice differentiable and $g'(x) = 1/h'(g(x))$. Integrating by parts for $x > x_0$:

$$\begin{aligned} \int_x^\infty e^{-h(y)} dy &= \int_{h(x)}^\infty e^{-u} g'(u) du \\ &= e^{-h(x)} g'(h(x)) + \int_{h(x)}^\infty e^{-u} g''(u) du. \end{aligned} \tag{2.31}$$

Since

$$\frac{g''(u)}{g'(u)} = -\frac{h''(g(u))}{h'(g(u))^2} \to 0$$

as $u \to \infty$, the right-hand side integral vanishes compared with the one on the left-hand side. Therefore, eventually

$$\int_x^\infty e^{-h(y)} dy \sim e^{-h(x)} g'(h(x)) = e^{-h(x)} \frac{1}{h'(x)}\,.$$

□

Lemma 2.4 *Let assumptions of Lemma 2.3 hold. Assume additionally that as* $x \to \infty$

$$x h'(x) \to \infty \tag{2.32}$$

and for any $b, c \geq a > 0$, $b < c$ *the quotient* $h'(bx)/h'(cx)$ *is bounded in* $[0, \infty)$.

Let $\mu(u)$ be a positive, bounded and locally integrable function, defined in $[a,\infty)$, continuous at $u=a$, and $\mu(a)\neq 0$.

Then

$$\int_a^\infty e^{-h(ux)}\mu(u)du \sim \frac{\mu(a)e^{-h(ax)}}{xh'(ax)}$$

as $x\to\infty$.

Proof As the first step, we prove that.

$$I(x)=\int_a^\infty e^{-h(ux)}\mu(u)du \sim \mu(a)\int_a^\infty e^{-h(ux)}du.$$

As $\mu(u)$ is continuous at $u=a$, for $\varepsilon>0$ there is δ such that

$$|\mu(u)-\mu(a)|<\varepsilon, \quad \text{if } |u-a|<\delta.$$

The function $\mu(u)$ is bounded, therefore,

$$\mu(u)<M, \quad \forall u\in[a,\infty)$$

for some positive $M>\mu(a)$. Then

$$I(x)=\int_a^{a+\delta} e^{-h(ux)}\mu(u)du+\int_{a+\delta}^\infty e^{-h(ux)}\mu(u)du$$

and

$$I(x)-\mu(a)\int_a^\infty e^{-h(ux)}du=\int_a^{a+\delta} e^{-h(ux)}(\mu(u)-\mu(a))du + \int_{a+\delta}^\infty e^{-h(ux)}(\mu(u)-\mu(a))du.$$

Therefore,

$$\begin{aligned}|I(x)-\mu(a)\int_a^\infty e^{-h(ux)}du|&\\ &<\varepsilon\int_a^{a+\delta} e^{-h(ux)}du+(M+\mu(a))\int_{a+\delta}^\infty e^{-h(ux)}du\\ &=\varepsilon\int_a^\infty e^{-h(ux)}du+(M+\mu(a)-\varepsilon)\int_{a+\delta}^\infty e^{-h(ux)}du.\end{aligned}$$

Then

$$\left|\frac{I(x)}{\mu(a)\int_a^\infty e^{-h(ux)}du}-1\right|<\frac{\varepsilon}{\mu(a)}+\frac{M+\mu(a)-\varepsilon}{\mu(a)}\cdot\frac{\int_{a+\delta}^\infty e^{-h(ux)}du}{\int_a^\infty e^{-h(ux)}du} \tag{2.33}$$

Using Lemma 2.3:

$$\frac{\int_{a+\delta}^\infty e^{-h(ux)}du}{\int_a^\infty e^{-h(ux)}du}=\frac{\int_{ax+\delta x}^\infty e^{-h(u)}du}{\int_{ax}^\infty e^{-h(u)}du}\sim\frac{h'(ax)}{h'(ax+\delta x)}e^{-(h(ax+\delta x)-h(ax))}.$$

It follows from condition (2.32) and the mean value theorem that

$$h(ax+\delta x)-h(ax)=\delta x h'(s)>sh'(s)\frac{\delta}{a+\delta} \tag{2.34}$$

for some $ax<s<ax+\delta x$. Thus,

$$h(ax+\delta x)-h(ax)\to\infty\quad\text{as}\quad x\to\infty,$$

the quotient $h'(ax)/h'(ax+\delta x)$ is bounded and, therefore, the second summand in (2.33) tends to zero, whereas the first summand can be made arbitrarily small. This yields

$$I(x)\sim\mu(a)\int_a^\infty e^{-h(ux)}du$$

as $x\to\infty$. Applying Lemma 2.3 completes the proof. □

Lemma 2.5 *Under assumptions of Lemma 2.4 the following asymptotic relation holds as* $x\to\infty$

$$\int_a^\infty h'(ux)e^{-h(ux)}u\mu(u)du\sim\frac{a\mu(a)}{x}e^{-h(ax)}.$$

Proof We first show that

$$\int_a^\infty h'(ux)e^{-h(ux)}u\,du\sim\frac{a}{x}\,e^{-h(ax)}. \tag{2.35}$$

Simple calculations give

$$\begin{aligned} x^2 \int_a^\infty h'(ux)e^{-h(ux)}u\,du &= \int_{ax}^\infty h'(u)e^{-h(u)}u\,du \\ &= axe^{-h(ax)} + \int_{ax}^\infty e^{-h(u)}du. \end{aligned}$$

By Lemma 2.4:

$$\int_{ax}^\infty e^{-h(u)}du \sim e^{-h(ax)}\frac{1}{h'(ax)}.$$

We have assumed that

$$axh'(ax) \to \infty \quad \text{as} \quad x \to \infty,$$

thus, $1/h'(ax) = o(ax)$ and

$$x^2 \int_0^\infty h'(ux)e^{-h(ux)}u\,du \sim axe^{-h(ax)},$$

which is the same as (2.35).

The next step is to prove that

$$\int_a^\infty h'(ux)e^{-h(ux)}u\mu(u)du \sim \mu(a)\int_a^\infty h'(ux)e^{-h(ux)}u\,du \tag{2.36}$$

As in Lemma 2.4, we use the same ε, δ, M and the similar reasoning to get

$$\left| \frac{\int_a^\infty h'(ux)e^{-h(ux)}u\mu(u)du}{\mu(a)\int_a^\infty h'(ux)e^{-h(ux)}u\,du} - 1 \right| < \frac{\varepsilon}{\mu(a)} + \frac{\tilde{M}}{\mu(a)} \cdot \frac{\int_{a+\delta}^\infty h'(ux)e^{-h(ux)}u\,du}{\int_a^\infty h'(ux)e^{-h(ux)}u\,du},$$

where $\tilde{M} = M + \mu(a) - \varepsilon$.

Applying (2.35) and using (2.34), we obtain:

$$\frac{\int_{a+\delta}^\infty h'(ux)e^{-h(ux)}u\,du}{\int_a^\infty h'(ux)e^{-h(ux)}u\,du} \sim \frac{a+\delta}{a}e^{-(h(ax+\delta x)-h(ax))} \to 0$$

as $x \to \infty$. Again $\varepsilon/\mu(a)$ can be made arbitrarily small, which gives us (2.36). Combining it with (2.35) completes the proof of the lemma. □

Now we are ready to prove Theorem 2.4 itself. Applying Lemma 2.4 as $t \to \infty$ results in:

$$\begin{aligned}\int_a^\infty \bar{F}(t,z)\pi(z)dz &= \int_a^\infty e^{-A(z\phi(t))-\psi(t)}\pi(z)dz \\ &= e^{-\psi(t)}\int_a^\infty e^{-A(z\phi(t))}\pi(z)dz \\ &\sim \frac{\pi(a)e^{-\psi(t)}}{\phi(t)A'(a\phi(t))}e^{-A(a\phi(t))}.\end{aligned}$$

Similar to the proof of Theorem 2.2:

$$\begin{aligned}&\int_a^\infty f(t,z)\pi(z)dz - \psi'(t)\int_a^\infty \bar{F}(t,z)\pi(z)dz \\ &\qquad = \phi'(t)e^{-\psi(t)}\int_a^\infty A'(z\phi(t))e^{-A(z\phi(t))}z\pi(z)dz.\end{aligned}$$

Using Lemma 2.5:

$$\int_a^\infty A'(z\phi(t))e^{-A(z\phi(t))}z\pi(z)dz \sim \frac{a\pi(a)}{\phi(t)}e^{-A(a\phi(t))}.$$

Combining the last three statements arrive at (2.22)

$$\begin{aligned}\lambda_m(t) - \psi'(t) &= \frac{\int_a^\infty f(t,z)\pi(z)dz}{\int_a^\infty \bar{F}(t,z)\pi(z)dz} - \psi'(t) \\ &\sim \frac{\phi'(t)e^{-\psi(t)}a\pi(a)e^{-A(a\phi(t))}}{\phi(t)} \cdot \frac{A'(a\phi(t))\phi(t)}{\pi(a)e^{-\psi(t)}e^{-A(a\phi(t))}} \\ &= a\phi'(t)A'(a\phi(t)),\end{aligned}$$

which completes the proof.

2.5 Multiplicative model

As previously, denote the baseline failure rate by $\lambda(t)$. Therefore, model (2.9) reads

$$\lambda(t,z) = z\lambda(t), \quad \Lambda(t,z) = z\Lambda(t) = z\int_0^t \lambda(u)du, \tag{2.37}$$

and the mixture failure rate is given by

$$\lambda_m(t) = \frac{\int_0^\infty z\lambda(t)e^{-z\Lambda(t)}\pi(z)dz}{\int_0^\infty e^{-z\Lambda(t)}\pi(z)dz}. \tag{2.38}$$

As $A(u) \equiv u$, $\phi(t) = \Lambda(t)$, $\psi(t) \equiv 0$ in this specific case, theorems 2.2 and 2.4 are simplified to

Corollary 2.2 *Assume that the mixing pdf* $\pi(z)$, $z \in [0, \infty)$ *can be written as*

$$\pi(z) = z^\alpha \pi_1(z), \tag{2.39}$$

where $\alpha > -1$ *and* $\pi_1(z)$ *is bounded in* $[0, \infty)$, *continuous at* $z = 0$, *and* $\pi_1(0) \neq 0$.

Then the mixture failure rate for the multiplicative model (2.37) has the following asymptotic behavior:

$$\lambda_m(t) \sim \frac{(\alpha + 1)\lambda(t)}{\int_0^t \lambda(u)du}. \tag{2.40}$$

Corollary 2.3 *Assume that the mixing pdf* $\pi(z), z \in [a, \infty)$ *(we can define* $\pi(z) = 0, z \in [0, a)$*) is bounded, right semi-continuous at* $z = a$ *and* $\pi(a) \neq 0$.

Then, in accordance with relation (2.22), the mixture failure rate for the model (2.37) has the following asymptotic behavior:

$$\lambda_m(t) \sim a\lambda(t). \tag{2.41}$$

Corollary 2.2 states a **remarkable fact**: asymptotic behavior of the mixture failure rate $\lambda_m(t)$ depends only on the behavior of the mixing pdf in the neighborhood of $z = 0$ and the baseline failure rate $\lambda(t)$.

Corollary 2.3 describes the convergence of a mixture failure rate to the mixture failure rate of the strongest population. In this simple multiplicative case the family of the failure rates is trivially ordered in z and the strongest population has the failure rate $a\lambda(t)$.

The next theorem generalizes the result of Corollary 2.3:

Theorem 2.5 *Assume that the mixing pdf* $\pi(z)$ *in model (2.37) has support in* $[a, b]$, $a > 0$, $b \leq \infty$, *and for* $z \geq a$ *it can be defined as*

$$\pi(z) = (z - a)^{\alpha} \pi_1(z - a), \tag{2.42}$$

where $\alpha > -1$, $\pi_1(z)$ *is bounded in* $[0, b - a]$, *and* $\pi_1(0) \neq 0$.

Then

$$\lambda_m(t) \sim a\lambda(t). \tag{2.43}$$

Proof As in Theorem 2.2, we consider the numerator and the denominator in (2.38) separately. Changing the variables and applying Lemma 2.1:

$$\begin{aligned}\int_0^{\infty} \bar{F}(tz)\pi(z)dz &= \int_a^{\infty} e^{-z\Lambda(t)}(z-a)^{\alpha}\pi_1(z-a)dz \\ &= e^{-a\Lambda(t)} \int_0^{\infty} e^{-z\Lambda(t)} z^{\alpha}\pi_1(z)dz \\ &\sim \frac{e^{-a\Lambda(t)}\pi_1(0)\Gamma(\alpha+1)}{(\Lambda(t))^{\alpha+1}}.\end{aligned} \tag{2.44}$$

Similarly,

$$\begin{aligned}\int_0^{\infty} zf(tz)\pi(z)dz &= \lambda(t) \int_a^{\infty} ze^{-z\Lambda(t)}(z-a)^{\alpha}\pi_1(z-a)dz \\ &= \lambda(t)e^{-a\Lambda(t)} \int_0^{\infty} e^{-z\Lambda(t)} z^{\alpha+1}\pi_1(z)dz \\ &\quad + a\lambda(t)e^{-a\Lambda(t)} \int_0^{\infty} e^{-z\Lambda(t)} z^{\alpha}\pi_1(z)dz.\end{aligned}$$

The first integral on the right hand side is asymptotically equivalent to $\pi_1(0)\Gamma(\alpha + 2)\Lambda(t)^{-\alpha-2}$ and the second to $\pi_1(0)\Gamma(\alpha + 1)\Lambda(t)^{-\alpha-1}$, which decreases slower. Thus,

$$\int_0^{\infty} zf(tz)\pi(z)dz \sim a\pi_1(0)\Gamma(\alpha+1)\lambda(t)\frac{e^{-a\Lambda(t)}}{(\Lambda(t))^{\alpha+1}}\,. \tag{2.45}$$

Finally using (2.44) and (2.45) in (2.38), we arrive at (2.43). □

The asymptotic result in Theorem 2.5 differs from the case $a = 0$ in Corollary 2.2. Relation (2.43) also describes the convergence to the failure rate of the strongest population, which differs dramatically from the convergence described by (2.40). Explanation of this difference is quite obvious and due to the multiplicative nature of the model: the behavior of $z\lambda(t)$ in the neighborhood of $z = 0$ for the pdf (2.39) is different from the behavior of this product in the neighborhood of $z = a$ for the pdf (2.42). Note that the result of Theorem 2.5 does not depend on a mixing distribution even in the case of a singularity at $z = a$.

Block *et al.* (1993) proved that if (under some assumptions) the failure rate $\lambda(t, z)$ converges to a positive function $\nu(z)$, then the mixture failure rate converges to $essinf_z \nu(z)$ with respect to the probability measure generated by the random variable Z. Later Block and Joe (1997) and Li (2005) developed this result further for the ratio $\lambda(t, z)/\lambda(t)$ converging to some positive $\nu(z)$. As we are interested here only in the multiplicative model, the corresponding generalizations will be considered later. In this setting Theorem 2.5 is weaker then the results of Block and Joe (1997) and Li (2005), since it considers the behavior of the mixing distribution described by (2.42), whereas the results of these authors do not specify the mixing distribution.

On the other hand, our theorem is a simple corollary of the technique used. We can also obtain the same "convergence to the strongest population" result. Indeed, directly from (2.38):

$$\begin{aligned}\frac{\lambda_m(t)}{\lambda(t)} &= \frac{\int_a^\infty z e^{-z\Lambda(t)}\pi(z)dz}{\int_a^\infty e^{-z\Lambda(t)}\pi(z)dz} \\ &= \frac{\int_0^\infty (z+a) e^{-(z+a)\Lambda(t)}\pi(z+a)dz}{\int_0^\infty e^{-(z+a)\Lambda(t)}\pi(z+a)dz} \\ &= a + \frac{\int_0^\infty z e^{-\Lambda(t)}\pi(z+a)dz}{\int_0^\infty e^{-z\Lambda(t)}\pi(z+a)dz}\end{aligned}$$

and it is sufficient to prove that the second summand in the last relation converges to 0 as $t \to \infty$. Employing the standard bounds and asymptotic derivations for the Laplace integrals yields the needed convergence, but this result is anyway presented in the foregoing papers and we omit the proof.

To say more, our approach allows for a much stronger result in terms of the rate of convergence to the strongest population. In this case the conditions on the mixing distribution are crucial and cannot be omitted. The following theorem is obtained, using our technique developed previously:

Theorem 2.6 *Under the assumptions of Theorem 2.5 as $t \to \infty$:*

$$\lambda_m(t) - a\lambda(t) \sim (\alpha+1)\frac{\lambda(t)}{\Lambda(t)}\, e^{-a\Lambda(t)}.$$

The proof is rather simple and straightforward, it uses the already known asymptotics (2.41) from Corollary 2.3.

Proof From the form of the mixture failure rate (2.38) we obtain

$$\begin{aligned}\frac{\lambda_m(t)}{\lambda(t)} - a &= \frac{\int_a^\infty z e^{-z\Lambda(t)}(z-a)^\alpha \pi_1(z-a)dz}{\int_a^\infty e^{-z\Lambda(t)}(z-a)^\alpha \pi_1(z-a)dz} - a \\ &= \frac{\int_a^\infty e^{-z\Lambda(t)}(z-a)^{\alpha+1} \pi_1(z-a)dz}{\int_a^\infty e^{-z\Lambda(t)}(z-a)^\alpha \pi_1(z-a)dz} \\ &= e^{-a\Lambda(t)}\frac{\int_0^\infty e^{-z\Lambda(t)} z^{\alpha+1}\pi_1(z)dz}{\int_0^\infty e^{-z\Lambda(t)} z^{\alpha}\pi_1(z)dz}.\end{aligned}$$

Now we are already in the setting of Corollary 2.3 as if we consider the mixing density $z^\alpha \pi_1(z)$, defined on $[0,\infty)$. Thus, the last quotient is equivalent to $(\alpha+1)/\Lambda(t)$ and

$$\frac{\lambda_m(t)}{\lambda(t)} - a \sim \frac{1}{\Lambda(t)}(\alpha+1)e^{-a\Lambda(t)}.$$

□

In Section 1.3 we already discussed the connection between the mixture failure rate in the multiplicative model and the Laplace transform of the mixing distribution. The mixture failure rate is expressed via the Laplace transform by relation (1.26).

We will now give some examples for particular mixing distributions and baseline ones and use Laplace transforms for calculating failure rates. Consider the Gamma mixing distribution, which we already studied in Example 1.3. We are interested in the exact formulas for mixture failure rates and in the asymptotic behavior as well.

Example 2.4 Let the mixing distribution be the Gamma distribution with the pdf

$$\pi(z) = \left(\frac{z}{b}\right)^{c-1} e^{-z/b}\frac{1}{b\Gamma(c)}, \tag{2.46}$$

where $b, c > 0$. The Laplace transform of $\pi(z)$ is

$$\tilde{\pi}(t) = \frac{1}{(tb+1)^c}$$

and, therefore, the mixture failure rate is given by the following expression:

$$\lambda_m(t) = \frac{bc\lambda(t)}{1 + b\int_0^t \lambda(u)du}. \tag{2.47}$$

Since $\int_0^t \lambda(t)(u)du = \Lambda(t) \to \infty$ as $t \to \infty$, the following asymptotic relation holds:

$$\lambda_m(t) \sim \frac{c\lambda(t)}{\Lambda(t)},$$

which coincides with the result obtained from Corollary 2.2.

The expected value of a random variable Z with a pdf (2.46) is bc and the variance is b^2c. Thus, for the fixed expectation $E[Z] = 1$, the variance $\sigma^2 = b$ and equation (2.47) turns into

$$\lambda_m(t) = \frac{\lambda(t)}{1 + \sigma^2\int_0^t \lambda(u)du},$$

which first appeared in Vaupel *et al* (1979) in a demographic context. This form allows to compare different mixtures for the fixed baseline distribution. We can see that when the variance of the mixing distribution increases, the mixture failure rate decreases. More detailed properties of a similar kind will be discussed for a rather general setting in Chapter 4.

Obviously, asymptotic behavior of $\lambda_m(t)$ can be explicitly analyzed. Consider two specific cases, which are important for applications:

1. If the baseline distribution is Weibull with $\lambda(t) = at^\beta$, $\beta > -1$, then the mixture failure rate (2.47) is (see also Gupta and Gupta, 1996):

$$\lambda_m(t) = \frac{(\beta+1)abct^\beta}{(\beta+1) + abt^{\beta+1}},$$

which as $t \to \infty$ converges to 0 and

$$\lambda_m(t) \sim \frac{(\beta+1)c}{t},$$

exactly as prescribed by our formula (2.40) of Corollary 2.2 ($c = \alpha + 1$). Thus, if specifically $\beta > 0$, then the operation of mixing of IFR distributions results in ultimately decreasing to 0 mixture failure rate, which is, in a way, an amazing fact!

2. If the baseline distribution is Gompertz with $\lambda(t) = \mu e^{\beta t}$, $\mu > 0$, $\beta > 0$, then simple transformations result in

$$\lambda_m(t) = \frac{\beta c e^{\beta t}}{e^{\beta t} + \left(\frac{\beta}{\mu b} - 1\right)}.$$

If $b = \beta/\mu$, then $\lambda_m(t) \equiv \beta c$, if $b > \beta/\mu$, then $\lambda_m(t)$ increases to β/μ, and if $b < \beta/\mu$, it decreases to β/μ.

As it was stated in the Introduction, the case $b > \beta/\mu$ describes the deceleration in mortality rates of human populations, which was traditionally, starting with Gompertz (1985), described by $\mu e^{\beta t}$ - the mortality rate of exponentially increasing Gompertz law. $\diamondsuit$

2.6 Accelerated life model

In a conventional notation this model is written as:

$$\lambda(t, z) = z\lambda(tz), \quad \Lambda(t, z) = \Lambda(tz) = \int_0^{tz} \lambda(u)du. \tag{2.48}$$

Although the definition of the ALM is also very simple, the presence of a mixing parameter z in the arguments make analysis of the mixture failure rate more complex than in the multiplicative case. Therefore, as it was already mentioned, this model was practically not studied before. The mixture failure rate in this specific case is

$$\lambda_m(t) = \frac{\int_0^\infty z\lambda(tz)e^{-\Lambda(tz)}\pi(z)dz}{\int_0^\infty e^{-\Lambda(tz)}\pi(z)dz}.$$

Asymptotic behavior of $\lambda_m(t)$ can be described as a specific case of Theorem 2.2 with $A(s) = \Lambda(s)$, $\phi(t) = t$ and $\psi(t) \equiv 0$:

Corollary 2.4 *Assume that the mixing pdf $\pi(z), z \in [0,\infty)$ can be defined as $\pi(z) = z^{\alpha}\pi_1(z)$, where $\alpha > -1$, $\pi_1(z)$ is continuous at $z = 0$ and bounded in $[0,\infty)$, $\pi_1(0) \neq 0$.*

Let the baseline distribution with the cumulative failure rate $\Lambda(t)$ have a moment of order $\alpha + 1$.

Then

$$\lambda_m(t) \sim \frac{\alpha + 1}{t} \tag{2.49}$$

as $t \to \infty$.

The conditions of Corollary 2.4 are not that strong and are relatively natural. The most of the widely used lifetime distributions have all moments. The Pareto distribution will be discussed in the next example.

As it was already stated, the conditions on the mixing distribution hold, e.g., for the Gamma and the Weibull distributions which are commonly used as mixing distributions.

Relation (2.49) is really surprising, as it **does not depend on the baseline distribution**, which seems striking, at least, at the first sight. It is also dramatically different from the multiplicative case (2.40).

It follows from Example 2.4 that both asymptotic results coincide in the case of the Weibull baseline distribution, which is obvious, as only for the Weibull distribution the ALM can be re-parameterized to end up with a PH model and *vise versa*.

The following example formally shows other possibilities for the asymptotic behavior of $\lambda_m(t)$ when one of the conditions of the Corollary 2.4 does not hold.

Example 2.5 Consider the Gamma mixing distribution, written in a more convenient form for this example, than (2.46)

$$\pi(z) = z^{\alpha}e^{-x}/\Gamma(\alpha + 1).$$

Let the baseline distribution be the Pareto-type distribution. For $\beta > 0$ we define the corresponding pdf as

$$f_0(t) = \begin{cases} 0, & 0 \leq t < 1 \\ \beta/t^{\beta+1}, & t \geq 1. \end{cases}$$

For $\beta > \alpha + 1$ the conditions of Corollary 2.4 holds and relation (2.49) takes place. Let $\beta \leq \alpha+1$, which means that the baseline distribution does not have the $(\alpha+1)$th moment. Therefore, one of the conditions of Corollary 2.4 is violated. In this case it can be shown by direct derivations (see the forthcoming proof) that

$$\lambda_m(t) \sim \frac{\beta}{t}$$

as $t \to \infty$, whereas for the general case:

$$\lambda_m(t) \sim \frac{\min(\beta, \alpha+1)}{t}.$$

It can be shown that the same asymptotics holds not only for the Gamma-distribution, but also for any other mixing pdf of the form $\pi(z) = z^{\alpha}\pi_1(z)$. If $\beta > \alpha + 1$, the function $\pi_1(z)$ should be bounded and $\pi_1(0) \neq 0$. $\diamondsuit$

Proof Calculating directly:

$$\begin{aligned} \int_0^{\infty} f_0(tz) z\pi(z)dz &= \int_{1/t}^{\infty} \frac{\beta z}{t^{\beta+1} z^{\beta+1}} \cdot \frac{1}{\Gamma(\alpha+1)} e^{-z} z^{\alpha} dz \\ &= \frac{\beta}{\Gamma(\alpha+1)t^{\beta+1}} \int_{1/t}^{\infty} z^{\alpha-\beta} e^{-z} dz \\ &\sim \frac{\Gamma(\alpha-\beta+1)\beta}{\Gamma(\alpha+1)t^{\beta+1}} \end{aligned}$$

and

$$\int_0^{\infty} \bar{F}_0(tz)\pi(z)dz = \int_0^{1/t} \frac{e^{-z}z^{\alpha}}{\Gamma(\alpha+1)}dz + \int_{1/t}^{\infty} \frac{1}{t^{\beta}z^{\beta}} \cdot \frac{e^{-z}z^{\alpha}}{\Gamma(\alpha+1)}dz,$$

where the fact that $\bar{F}_0(t) = 1$ for $0 \leq t < 1$ was used. As $t \to \infty$, the first integral on the right-hand side is equivalent to

$$\frac{1}{\Gamma(\alpha+1)} \int_0^{1/t} z^{\alpha} dz = \frac{1}{t^{\alpha+1}\Gamma(\alpha+2)}$$

and the second integral is equivalent to

$$\Gamma(\alpha-\beta+1)/\Gamma(\alpha+1)t^{\beta},$$

which in case $\beta < \alpha + 1$ decreases slower; therefore the sum of two integrals is equivalent to $\Gamma(\alpha - \beta + 1)/\Gamma(\alpha + 1)t^{\beta}$.

Eventually

$$\lambda_m(t) \sim \frac{\Gamma(\alpha - \beta + 1)\beta}{\Gamma(\alpha + 1)t^{\beta+1}} \cdot \frac{\Gamma(\alpha + 1)t^{\beta}}{\Gamma(\alpha - \beta + 1)} = \frac{\beta}{t}.$$

If $\beta = \alpha + 1$, then

$$\int_0^{\infty} z f_0(tz)\pi(z)dz = \frac{\alpha + 1}{\Gamma(\alpha + 1)t^{\alpha+2}} \int_{1/t}^{\infty} z^{-1}e^{-z}dz,$$

and since

$$\int_0^{1/t} z^{\alpha}dz = o\left(t^{-\alpha-1}\int_{1/t}^{\infty} z^{-1}e^{-z}dz\right),$$

we obtain:

$$\begin{aligned}\int_0^{\infty} \bar{F}_0(tz)\pi(z)dz &\sim \int_{1/t}^{\infty} \frac{1}{t^{\alpha+1}z} \cdot \frac{e^{-z}}{\Gamma(\alpha + 1)}dz \\ &= \frac{1}{t^{\alpha+1}\Gamma(\alpha + 1)} \int_{1/t}^{\infty} z^{-1}e^{-z}dz.\end{aligned}$$

Therefore,

$$\lambda_m(t) \sim \frac{\alpha + 1}{t} = \frac{\beta}{t}.$$

□

As $A(s) = \Lambda(s)$, $\phi(t) = t$, Theorem 2.4 is simplified in this case to the following result:

Corollary 2.5 *Assume that the mixing pdf $\pi(z)$, $z \in [a, \infty)$ is bounded, continuous at $z = a$ and $\pi(a) \neq 0$. Let*

$$\frac{\lambda'(t)}{(\lambda(t))^2} \to 0, \quad t\lambda(t) \to \infty \tag{2.50}$$

as $t \to \infty$. Assume also that for all $b, c > a$, $b < c$ the quotient $\lambda_0(bx)/\lambda_0(cx)$ Then, in accordance with relation (2.22), the mixture failure rate for the model (2.48) has the following asymptotic behavior:

$$\lambda_m(t) - \psi'(t) \sim a\lambda(at).$$

Conditions (2.50) are rather weak. E.g., the marginal case of the Pareto distribution - the baseline failure rate of the form $\lambda_0(t) = ct^{-1}$, $c > 0, t \geq 1$ does not comply with (2.50), but in mixing we are primarily interested in increasing, at least ultimately, baseline failure rates. This is due to the fact that the family of DFR distributions is closed under the operation of mixing (which means that $\lambda_m(t)$ is always decreasing in this case), whereas the family of IFR distributions is not closed under this operation.

Asymptotic behavior of $\lambda_m(t)$ in the **additive hazards model**, (2.11) due to its simplicity, does not deserve special attention. As $A(s) \equiv s$ and $\phi(t) \equiv t$, conditions (2.13) and (2.14) of Theorem 2.2, for instance, hold and asymptotic result (2.15) is simplified to:

$$\lambda_m(t) - \psi'(t) \sim \frac{\alpha + 1}{t}.$$

2.7 Some generalizations

Here we present some further results including generalizations of the model (2.7). Generalizations to the multivariate case will be considered in the next chapter.

2.7.1 Generalizations of the survival model

As we have noted before, the model (2.7) can be generalized to the following one:

$$\Lambda(t, z) = A(g(z)\phi(t)) + \psi(t) + \eta(z). \tag{2.51}$$

We will now discuss what conditions we should impose on the functions $A(u), \phi(t), \psi(t), \eta(z)$, so that the proofs of theorems 2.2 and 2.4 stay the same. We will turn to discussing the function $g(z)$ afterwards. The natural general assumption is

(C1) The functions $A(u), \phi(t), \psi(t)$ are differentiable.

We do not need to assume anything else specifically for $\psi(t)$, because this additive term cancels out in all the calculations and the asymptotic results deal with $\lambda_m(t) - \psi'(t)$. We do not even need $\psi'(t)$ to be positive.

The next natural condition arises from the fact that $\bar{F}(t,z) = e^{-\Lambda(t,z)}$ is a survival function and $F(0,z) \equiv 0$, i.e.

(C2) $A(g(z)\phi(0)) + \psi(0) + \eta(z) = 0$ *for all* $z \geq 0$.

On the other hand, $\bar{F}(t,z)$ decreases to 0 as $t \to \infty$:

(C3) $A(g(z)\phi(t)) + \psi(t) \to \infty$ *as* $t \to \infty$ *for all* $z \geq 0$,

which imposes the only additional condition on $\psi(t)$. The second half is that $\Lambda(t,z)$ increases with respect to t, i.e.

(C4) $\lambda(t,z) = \Lambda(t,z)'_t = g(z)\phi'(t)A'(g(z)\phi(t)) + \psi'(t) > 0$,

which, in fact, states that the corresponding failure rate should be positive.

These were preliminary conditions for our model. In both theorems 2.2 and 2.4 the following general assumptions were made:

(C5); $\phi(t) \to \infty$ *as* $t \to \infty$,
(C6); $A(s)$ *is ultimately increasing.*

Our goal now is to change the variables and impose some additional assumptions, so that the generalized model (2.51) can be treated similarly to (2.7).

For simplicity, we assume $\psi(t) \equiv 0$, as we have seen before, it does not affect any of the proofs. Assume

(C7) Let $g(z)$ *be positive, differentiable and strictly monotone.*

It is clear that then there exists an inverse function $g^{-1}(z) = k(z)$. If $\pi(z)$ is defined in $[a,b]$, $a \geq 0, b \leq \infty$, then after the change of variable $y = g(z)$, the mixture survival function can be written as

$$\begin{aligned}\bar{F}_m(t) &= \int_a^b e^{-A(g(z)\phi(t))-\eta(z)}\pi(z)dz \\ &= \int_{g(a)}^{g(b)} e^{-A(y\phi(t))}\pi(k(y))e^{-\eta(k(y))}k'(y)dy,\end{aligned}$$

and the mixture density function as:

$$f_m(t) = \int_a^b \phi'(t)g(z)A'(g(z)\phi(t))e^{-A(g(z)\phi(t))-\eta(z)}\pi(z)dz$$
$$= \phi'(t)\int_{g(a)}^{g(b)} yA'(y\phi(t))e^{-A(y\phi(t))}\pi(k(y))e^{-\eta(k(y))}k'(y)dy.$$

Therefore, a new mixing function can be considered:

$$\tilde{\pi}(y) = \pi(k(y))e^{-\eta(k(y))}k'(y). \tag{2.52}$$

Note that in our proofs we did not use the fact that $\pi(z)$ was a proper density function, i.e., $\int_0^\infty \pi(z)dz = 1$, we used only the assumptions stated in the theorems, therefore we can apply our reasoning to $\tilde{\pi}(y)$.

We have several possibilities for the limiting behavior of the mixture failure rate $\lambda_m(t) = f_m(t)/\bar{F}_m(t)$, depending on the domain of the inverse function and the corresponding monotonicity. Also the assumptions on $\tilde{\pi}(y)$ are the same as on $\pi(z)$, depending on the theorem. Which theorem is used for generalization, 2.2 or 2.4, depends on the limits of integration $g(a)$ and $g(b)$.

In all the corollaries we formulate sufficient conditions on $k(y)$, $\pi(z)$ and $\eta(z)$, which guarantee that the conditions of the corresponding theorem hold for $\tilde{\pi}(y)$. A condition on $\eta(z)$ is that $\eta(k(y))$ is continuous at the left end of the interval of integration, i.e., at $\min\{g(a), g(b)\}$, and bounded from above in the whole interval. Keeping in mind that we will state the corresponding condition on $k(y)$ later, we can assume now

(C8) $\eta(z)$ *is bounded from above in* $[a, b]$, *continuous at* $z = k(c) = g^{-1}(c)$, *where* $c = \min\{g(a), g(b)\}$.

Now we are able to formulate generalizations of Theorems 2.2 or 2.4.

1. If $g(z)$ is increasing, $g(a) = 0$, then Theorem 2.2 should be applied. The assumptions on $\tilde{\pi}(y)$ are that it must be of the form $\tilde{\pi}(y) = y^{\tilde{\alpha}}\tilde{\pi}_1(y)$, where $\tilde{\alpha} > -1$, $\tilde{\pi}_1(y)$ is bounded in $[0, g(b)]$, continuous at $y = 0$, $\tilde{\pi}_1(0) \neq 0$. Then the following corollary can be formulated:

Corollary 2.6 *Let conditions (C1)-(C8) hold. Let the mixing function* $\pi(z)$ *be defined in* $[a, b]$, $a \geq 0, b \leq \infty$, $g(a) = 0$. *Assume additionally that*

(C9.1) $\pi(z) = \pi_1(z-a)(z-a)^\alpha$, *where* $\pi_1(z)$ *is bounded in* $[0, b-a]$, *continuous at* $z = 0$, $\pi_1(0) \neq 0$, $\alpha > -1$,

(C10.1) $g^{-1}(y) \equiv k(y) = a + y^\beta k_1(y)$, *where again* $k_1(y)$ *is bounded, continuous at* $y = 0$, $k_1(0) \neq 0$, *and its derivative can be written as* $k_1'(y) = y^{\beta-1}k_2(y)$, *where* $k_2(y)$ *is bounded, continuous at* $y = 0$, $k_2(0) \neq 0$, $\beta > 0$.

(C11.1) $\int_0^\infty e^{-A(s)} s^{\beta(\alpha+1)} < \infty$.

Then

$$\lambda_m(t) - \psi'(t) \sim \beta(\alpha+1)\frac{\phi'(t)}{\phi(t)}.$$

Condition (C10.1) follows, for instance, from the assumption that $k'(y)$ is a regularly varying function of the order $\beta - 1$ (see Bingham *et al*, page 21). If the derivative $k_1'(y)$ is slowly varying (i.e. $\beta = 1$), then the asymptotic behavior is the same as in Theorem 2.2.

2. Consider an increasing $g(z)$ and $g(a) > 0$. If $g'(z)$ is continuous at $z = a$, then $k(y)$ and $k'(y)$ continuous at $y = g(a)$, $k'(g(a)) = 1/g'(a) \neq 0$. In this case Theorem 2.4 is generalized to:

Corollary 2.7 *Let conditions (C1)-(C8) hold. Let the mixing function* $\pi(z)$ *be defined in* $[a, b]$, $a \geq 0, b \leq \infty$, $g(a) > 0$. *Assume additionally that*

(C9.2) $\pi(z)$ *is bounded on* $[a, b]$, *continuous at* $z = a$, $\pi(a) \neq 0$.

(C10.2) $g'(z)$ *is continuous at* $z = a$.

(C11.2) $\int_0^\infty e^{-A(s)} ds < \infty$, *and as* $s \to \infty$

$$A''(s)/(A'(s))^2 \to 0, \quad sA'(s) \to \infty,$$

Assume also that for all $c > b > 0$ *the quotient* $A'(bs)/A'(cs)$ *is bounded from above as* $s \to \infty$.

Then

$$\lambda_m(t) - \psi'(t) \sim g(a)\phi'(t)A'(g(a)\phi(t)).$$

Similar corollaries can be formulated for decreasing $g(z)$.

2.7.2 Uniform equivalence of families of distributions

Consider two families of distributions $\mathcal{F}_1(t,z)$ and $\mathcal{F}_2(t,z)$ with failure rates $\lambda_1(t,z)$ and $\lambda_2(t,z)$ respectively. The variable z is, as previously, understood as a realization of a frailty Z, which is the same for both families. We are interested in conditions that guarantee the asymptotic equivalence of the corresponding mixture failure rates $\lambda_{m1}(t)$ and $\lambda_{m2}(t)$. The following result exploits assumptions of the uniform convergence and therefore is close to the reasoning of Block *et al* (2003a) and Li (2005), although our approach is different. We see that the sufficient assumptions are rather strong.

Theorem 2.7 *Let Z be a mixing random variable with a probability measure $\Pi(z)$. Let this measure be concentrated on set $D(z)$.*

Assume that as $t \to \infty$

$$\Lambda_2(t,z) - \Lambda_1(t,z) \to 0, \qquad \lambda_2(t,z)/\lambda_1(t,z) \to 1$$

uniformly in $z \in D(z)$.

Then the corresponding mixture failure rates are also asymptotically equivalent:

$$\lambda_{m1}(t) \sim \lambda_{m2}(t) \qquad as \quad t \to \infty.$$

Proof First we need a simple lemma

Lemma 2.6 *Let $g(t,u)$, $h_1(t,u), h_2(t,u)$ be locally integrable positive functions, $t > 0, u \in D$. Assume that*

$$\int_D g(t,u)h_1(t,u)du < \infty.$$

If $h_2(t,u)/h_1(t,u) \to 1$ as $t \to \infty$ uniformly in $u \in D$, then

$$\int_D g(t,u)h_1(t,u)du \sim \int_D g(t,u)h_2(t,u)du.$$

Proof The condition of uniform convergence of $h_2(t,u)/h(t,u)$ to 1 means that there exists such $c(t) > 0$ that for $u \in D$

$$h_1(t,u)(1 - c(t)) < h_2(t,u) < h_1(t,u)(1 + c(t))$$

and $c(t) \to 0$ as $t \to \infty$. Hence,

$$1 - c(t) < \frac{\int_D g(t,u)h_2(t,u)du}{\int_D g(t,u)h_1(t,u)du} < 1 + c(t),$$

which yields Lemma 2.6. □

The theorem itself immediately follows from the lemma. Indeed, $\Lambda_2(t,z) - \Lambda_1(t,z)$ uniformly converges to 0, then $\bar{F}_2(t,z)/\bar{F}_1(t,z)$ uniformly converges to 1 as $t \to \infty$, and their integrals are equivalent:

$$\bar{F}_{m2}(t) \sim \bar{F}_{m1}(t).$$

The failure rates are also uniformly equivalent, then again the probability density functions $f_i(t,z) = \lambda_i(t,z)\bar{F}_i(t,z)$ are uniformly equivalent, i.e., $f_2(t,z)/f_1(t,z)$ converges to 1 as $t \to \infty$ uniformly in z. Then the mixture densities are equivalent

$$f_{m1}(t) \sim f_{m2}(t)$$

and this completes the proof as

$$\lambda_{mi}(t) = \frac{f_{mi}(t)}{\bar{F}_{mi}(t)}, \quad i = 1, 2.$$

□

Chapter 3

Asymptotic behavior of mixture failure rates for multivariate frailty

In the previous chapter we considered a lifetime random variable T indexed by frailty Z. The next obvious step of generalization is to consider multivariate frailty when Z is a vector. This means that there can be several unobserved parameters (independent or dependent), which is often the case in practice. The simplest model to be considered in Section 3.2 is the bivariate multiplicative model, which is an obvious generalization of the multiplicative model (1.15):

$$\lambda(t, z_1, z_2) = z_1 z_2 \lambda(t). \tag{3.1}$$

For simplicity consider the bivariate case, but our reasoning can be easily generalized to a multivariate one.

Let $Z = (Z_1, Z_2)$ and let Z_1, Z_2 be interpreted as non-negative random variables with supports in $[a_1, b_1], [a_2, b_2]$, respectively, $a_1, a_2 \geq 0, b_1, b_2 \leq \infty$. Similar to Section 1.1,

$$P(T \leq t | Z_1 = z_1, Z_2 = z_2) \equiv P(T \leq t | z_1, z_2) = F(t, z_1, z_2),$$

and

$$\lambda(t, z_1, z_2) = \frac{f(t, z_1, z_2)}{\bar{F}(t, z_1, z_2)}.$$

Assume that Z_1 and Z_2 have a joint probability density function $\pi(z_1, z_2)$. We define the bivariate mixture failure rate in this case as

$$\begin{aligned} \lambda_m(t) &= \frac{\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(t, z_1, z_2)\pi(z_1, z_2)dz_1dz_2}{\int_{a_2}^{b_2}\int_{a_1}^{b_1} \bar{F}(t, z_1, z_2)\pi(z_1, z_2)dz_1dz_2} = \frac{f_m(t)}{\bar{F}_m(t)} \\ &= \int_{a_2}^{b_2}\int_{a_1}^{b_1} \lambda_s(t, z_1, z_2)\pi(z_1, z_2 \,|\, t)dz_1dz_2, \end{aligned} \tag{3.2}$$

where the conditional probability density function, similar to relation (1.3), is defined as

$$\pi(z_1, z_2 \,|\, t) = \pi(z_1, z_2)\frac{\bar{F}(t, z_1, z_2)}{\int_{a_2}^{b_2}\int_{a_1}^{b_1} \bar{F}(t, z_1, z_2)\pi(z_1, z_2)dz_1dz_2}. \tag{3.3}$$

In what follows we consider two specific bivariate models. Our goal is to illustrate the usage of the developed in Chapter 2 methodology in the multivariate setting.

3.1 Competing risks

Consider a system of two statistically independent components in series with lifetimes $T_1 \geq 0, T_2 \geq 0$ (competing risks). The Cdf function of this system is

$$F_s(t) = 1 - \bar{F}_1(t)\bar{F}_2(t),$$

where $F_1(t), F_2(t)$ are the Cdfs of the lifetime random variables T_1, T_2 (and the survival functions $\bar{F}_i(t) \equiv 1 - F_i(t)$), respectively.

As in the univariate case, assume now that $F_i(t),\ i = 1, 2$ are indexed by random variables Z_i in the following sense:

$$P(T_i \leq t \,|\, Z_i = z) \equiv P(T_i \leq t \,|\, z) = F_i(t, z), \quad i = 1, 2$$

and that the pdfs $f_i(t, z)$ exist. Then the corresponding failure rates $\lambda_i(t, z)$ are $f_i(t, z)/\bar{F}_i(t, z)$.

Let $Z_i,\ i = 1, 2$ be interpreted as non-negative random variables with supports in $[a_i, b_i],\ a_1 \geq 0,\ b_i \leq \infty$ and the pdf $\pi_i(z)$. A mixture Cdf for the ith component is defined by

$$F_{m,i}(t) = \int_{a_i}^{b_i} F_i(t, z)\pi_i(z)dz. \tag{3.4}$$

The corresponding mixture failure rate is:

$$\lambda_{m,i}(t) = \frac{\int_{a_i}^{b_i} f_i(t,z)\pi_i(z)dz}{\int_{a_i}^{b_i} \bar{F}_i(t,z)\pi_i(z)dz} = \int_{a_i}^{b_i} \lambda_i(t,z)\pi(z\,|\,t)dz, \tag{3.5}$$

where the conditional pdf (on condition that $T_i > t$) is

$$\pi_i(z\,|\,t) = \pi_i(z)\frac{\bar{F}_i(t,z)}{\int_{a_i}^{b_i} \bar{F}_i(t,z)\pi_i(z)dz}. \tag{3.6}$$

Assume fist that the components of our system are conditionally independent given $Z_1 = z_1, Z_2 = z_2$. Then the Cdf of the system is:

$$F_s(t,z_1,z_2) = 1 - \bar{F}_1(t,z_1)\bar{F}_2(t,z_2) \tag{3.7}$$

and the corresponding probability density function is

$$f_s(t,z_1,z_2) = f_1(t,z_1)\bar{F}_2(t,z_2) + f_2(t,z_2)\bar{F}_1(t,z_1). \tag{3.8}$$

The mixture failure rate in this case is defined similar to (3.2):

$$\begin{aligned}\lambda_{m,s}(t) &= \frac{\int_{a_2}^{b_2}\int_{a_1}^{b_1} f_s(t,z_1,z_2)\pi(z_1,z_2)dz_1dz_2}{\int_{a_2}^{b_2}\int_{a_1}^{b_1} \bar{F}_s(t,z_1,z_2)\pi(z_1,z_2)dz_1dz_2} \\ &= \int_{a_2}^{b_2}\int_{a_1}^{b_1} \lambda_s(t,z_1,z_2)\pi(z_1,z_2\,|\,t)dz_1dz_2,\end{aligned} \tag{3.9}$$

where, similar to (3.3)

$$\pi(z_1,z_2\,|\,t) = \pi(z_1,z_2)\frac{\bar{F}_s(t,z_1,z_2)}{\int_{a_2}^{b_2}\int_{a_1}^{b_1} \bar{F}_s(t,z_1,z_2)\pi(z_1,z_2)dz_1dz_2}, \tag{3.10}$$

and $\pi(z_1, z_2)$ is the bivariate joint probability density function of Z_1 and Z_2. It is clear that for our series system, defined by (3.7):

$$\lambda_s(t,z_1,z_2) = \lambda_1(t,z_1) + \lambda_2(t,z_2). \tag{3.11}$$

It is clear also that if Z_1 and Z_2 are independent, which means

$$\pi(z_1,z_2) = \pi_1(z_1)\pi_2(z_2)$$

for some densities $\pi_1(z_1)$ and $\pi_2(z_2)$; then these random variables are also conditionally independent

$$\pi(z_1, z_2|t) = \pi_1(z_1|t)\pi_2(z_2|t),$$

where, as usual, for $i = 1, 2$

$$\pi_i(z_i|t) = \pi_i(z_i)\frac{\bar{F}_i(t, z_i)}{\int_{a_i}^{b_i} \bar{F}_i(t, z_i)\pi_i(z_i)dz_i}.$$

Indeed, using definitions (3.7) and (3.10), we get

$$\begin{aligned}\pi(z_1, z_2|t) &= \pi_1(z_1)\pi_2(z_2)\frac{\bar{F}_1(t, z_1)\bar{F}_2(t, z_2)}{\int_{a_2}^{b_2}\int_{a_1}^{b_1} \bar{F}_1(t, z_1)\bar{F}_2(t, z_2)\pi_1(z_1)\pi_2(z_2)dz_1dz_2} \\ &= \frac{\pi_1(z_1)\bar{F}_1(t, z_1)\cdot\pi_2(z_2)\bar{F}_2(t, z_2)}{\int_{a_1}^{b_1} \bar{F}_1(t, z_1)\pi_1(z_1)dz_1 \cdot \int_{a_2}^{b_2} \bar{F}_2(t, z_2)\pi_2(z_2)dz_2} \\ &= \pi_1(z_1|t)\pi_2(z_2|t).\end{aligned}$$

Hence, when the components of the system are conditionally independent, the mixture failure rate of the system is the sum of the mixture failure rates of individual components:

$$\lambda_{m,s}(t) = \lambda_{m,1}(t) + \lambda_{m,2}(t).$$

Indeed, using relations (3.9) and (3.11)

$$\begin{aligned}\lambda_{m,s} &= \int_{a_2}^{b_2}\int_{a_1}^{b_1} \lambda_s(t, z_1, z_2)\pi(z_1, z_2 \mid t)dz_1dz_2 \\ &= \int_{a_2}^{b_2}\int_{a_1}^{b_1} \left[\lambda_1(t, z_1) + \lambda_2(t, z_2)\right]\pi_1(z_1|t)\pi_2(z_2|t)dz_1dz_2 \\ &= \int_{a_2}^{b_2} \pi_2(z_2|t)dz_2 \cdot \int_{a_1}^{b_1} \lambda_1(t, z_1)\pi_1(z_1|t)dz_1 \\ &\qquad + \int_{a_2}^{b_2} \lambda_2(t, z_2)\pi_2(z_2|t)dz_2 \cdot \int_{a_1}^{b_1} \pi_1(z_1|t)dz_1 \\ &= \lambda_{m,1}(t) + \lambda_{m,2}(t).\end{aligned}$$

It turns out that asymptotically this property is preserved for a much broader class of joint distributions of Z_1 and Z_2, given that the distribution families are defined by survival model (2.7), where for simplicity of notation we set $\psi_i(t) \equiv 0$. Therefore, let

$$\bar{F}_i(t, z_i) = e^{-A_i(z_i \phi_i(t))}. \tag{3.12}$$

The following generalization of Theorem 2.2 holds:

Theorem 3.1 *Consider the competitive risks model described by (3.7). Let the corresponding survival functions be defined by (3.12).*

Suppose that the mixing variables Z_1 and Z_2 have a joint probability density function $\pi(z_1, z_2)$, which is defined in $[0, b_1] \times [0, b_2]$, $0 < b_1, b_2 \leq \infty$.

Let the following properties hold:

(a) $\pi(z_1, z_2) = z_1^{\alpha_1} z_2^{\alpha_2} \pi_0(z_1, z_2)$, *where* $\alpha_1, \alpha_2 > -1$.

(b) $\pi_0(z_1, z_2)$ *is continuous at* $(0, 0)$, $\pi_0(0, 0) \neq 0$.

(c) assumptions on $A_i(s)$, $i = 1, 2$ in Theorem 2.2: positive ultimately increasing differentiable functions,

$$\int_0^\infty e^{-A_i(s)} s^{\alpha_i} ds < \infty.$$

Assume finally that $\phi_1(t), \phi_2(t) \to \infty$ as $t \to \infty$.

Then

$$\lambda_{m,s}(t) \sim (\alpha_1 + 1) \frac{\phi_1'(t)}{\phi_1(t)} + (\alpha_2 + 1) \frac{\phi_2'(t)}{\phi_2(t)}.$$

The following corollary is proved in exactly the same way as the theorem.

Corollary 3.1 *For a more general model, where not necessarily $\psi_i(t) \equiv 0$, the corresponding asymptotic relation reads:*

$$\lambda_{m,s}(t) - \psi_1'(t) - \psi_2'(t) \sim (\alpha_1 + 1) \frac{\phi_1'(t)}{\phi_1(t)} + (\alpha_2 + 1) \frac{\phi_2'(t)}{\phi_2(t)}.$$

It is clear that under some additional regularity assumptions Theorem 2.2 can be applied to the marginal mixing densities and

$$\lambda_{m,1}(t) \sim (\alpha_1 + 1)\frac{\phi_1'(t)}{\phi_1(t)}, \qquad \lambda_{m,2}(t) \sim (\alpha_2 + 1)\frac{\phi_2'(t)}{\phi_2(t)}.$$

Therefore, Theorem 3.1 states that under some technical assumptions, which are not that stringent

$$\lambda_{m,s}(t) \sim \lambda_{m,1}(t) + \lambda_{m,2}(t).$$

This asymptotic relation can be interpreted in the following way: the mixture failure rate of the described system is (asymptotically) equivalent to the sum of asymptotic failure rates of individual components (given that the other cause of failure does not exist). Therefore conditions of Theorem 3.1 are sufficient for some 'asymptotic independence' of components in the described series system.

Proof We need a supplementary lemma, analogous to Lemma 2.1:

Lemma 3.1 *Let $g(z_1, z_2)$ be a nonnegative integrable function in $[0, \infty)^2$. Let $h(z_1, z_2)$ be a nonnegative locally integrable function defined in $[0, \infty)^2$, such that it is bounded everywhere and continuous at the origin.*

Then as $t_1 \to \infty, t_2 \to \infty$:

$$t_1 t_2 \int_0^\infty \int_0^\infty g(t_1 z_1, t_2 z_2) h(z_1, z_2) dz_1 dz_2 \to h(0,0) \int_0^\infty \int_0^\infty g(z_1, z_2) dz_1 dz_2.$$

Proof The proof is rather straightforward and similar to the univariate case:

$$t_1 t_2 \int_0^\infty \int_0^\infty g(t_1 z_1\,, t_2 z_2) h(z_1, z_2) dz_1 dz_2$$
$$= \int_0^\infty \int_0^\infty g(z_1, z_2) h\left(\frac{z_1}{t_1}, \frac{z_2}{t_2}\right) dz_1 dz_2.$$

Indeed, $h(z_1, z_2)$ is bounded; assume that it is bounded by some M. The function $g(z_1, z_2)$ is integrable, then for any $\epsilon > 0$ there is a finite $b > 0$, such that

$$\iint_{[0,\infty)^2 - [0,b]^2} g(z_1, z_2) dz_1 dz_2 < \epsilon.$$

Then

$$\left|\int_0^\infty \int_0^\infty g(z_1,z_2)\left[h\left(\frac{z_1}{t_1},\frac{z_2}{t_2}\right)-h(0,0)\right]dz_1dz_2\right|$$
$$\leq \int_0^b \int_0^b g(z_1,z_2)\left|h\left(\frac{z_1}{t_1},\frac{z_2}{t_2}\right)-h(0,0)\right|dz_1dz_2$$
$$+2M\iint_{[0,\infty)^2-[0,b]^2} g(z_1,z_2)dz_1dz_2.$$

The first double integral tends to zero since $h(z_1, z_2)$ is continuous at $(0, 0)$, and the second can be made arbitrarily small. □

Now let us turn to the proof of the theorem itself. Substituting (3.7) and (3.8) into (3.9) we get

$$\lambda_{m,s}(t) = \frac{\int_0^{b_1}\int_0^{b_2} f_1(t,z_1)\bar{F}_2(t,z_2)\pi(z_1,z_2)dz_2dz_1}{\int_0^{b_1}\int_0^{b_2} \bar{F}_1(t,z_1)\bar{F}_2(t,z_2)\pi(z_1,z_2)dz_2dz_1} + \frac{\int_0^{b_2}\int_0^{b_1} f_2(t,z_2)\bar{F}_1(t,z_1)\pi(z_1,z_2)dz_1dz_2}{\int_0^{b_2}\int_0^{b_1} \bar{F}_2(t,z_1)\bar{F}_1(t,z_1)\pi(z_1,z_2)}. \tag{3.13}$$

Denote the first term on the right-hand side by $\lambda^1_{m,s}(t)$ and the second one by $\lambda^2_{m,s}(t)$. Then

$$\lambda_{m,s}(t) = \lambda^1_{m,s}(t) + \lambda^2_{m,s}(t).$$

We consider $\lambda^1_{m,s}(t)$ and $\lambda^2_{m,s}(t)$ separately. The probability density function of T_1 is

$$f_1(t,z_1) = A_1'(z_1\phi_1(t))z_1\phi_1'(t)e^{-A_1(z_1\phi_1(t))} \tag{3.14}$$

and

$$\lambda^1_{m,s}(t) = \frac{\int_0^{b_1}\int_0^{b_2} A_1'(z_1\phi_1(t))z_1\phi_1'(t)e^{-A_1(z_1\phi_1(t))-A_2(z_2\phi_2(t))}\pi(z_1,z_2)dz_2dz_1}{\int_0^{b_1}\int_0^{b_2} e^{-A_1(z_1\phi_1(t))-A_2(z_2\phi_2(t))}\pi(z_1,z_2)dz_2dz_1},$$

As in the univariate case, applying Lemma 3.1 to the numerator, we see that it is asymptotically equivalent to

$$\frac{\phi_1'(t)\pi_0(0,0)}{\phi_1(t)^{\alpha_1+2}\phi_2(t)^{\alpha_2+1}}\int_0^\infty A_1'(u)u^{\alpha_1+1}e^{-A_1(u)}du\int_0^\infty s^{\alpha_2}e^{-A_2(s)}ds$$

and the denominator is equivalent to

$$\frac{\pi_0(0,0)}{\phi_1(t)^{\alpha_1+1}\phi_2(t)^{\alpha_2+1}}\int_0^\infty u^{\alpha_1}e^{-A_1(u)}du\int_0^\infty s^{\alpha_2}e^{-A_2(s)}ds.$$

Hence, using (2.27), eventually

$$\begin{aligned}\lambda^1_{m,s}(t) &\sim \frac{\phi_1'(t)}{\phi_1(t)}\cdot\frac{\int_0^\infty A_1'(u)u^{\alpha_1+1}e^{-A_1(u)}du}{\int_0^\infty u^{\alpha_1}e^{-A_1(u)}du}\\ &= (\alpha_1+1)\frac{\phi_1'(t)}{\phi_1(t)}.\end{aligned}$$

Similarly,

$$\lambda^2_{m,s}(t) \sim (\alpha_2+1)\frac{\phi_2'(t)}{\phi_2(t)}.$$

Thus,

$$\begin{aligned}\lambda^1_{m,s}(t) &\sim \lambda_{m,1}(t),\\ \lambda^2_{m,s}(t) &\sim \lambda_{m,2}(t).\end{aligned}$$

□

Another corollary from a univariate case is the next theorem, which is based on Theorem 2.4. Note that distinct from the previous theorem, the case $a_1, a_2 > 0$ is of interest.

Theorem 3.2 *Consider the competing risks model described by (3.7). Let the survival functions be defined by (3.12).*

Suppose that the mixing variables Z_1 *and* Z_2 *have a joint probability density function* $\pi(z_1, z_2)$*, which is defined in* $[a_1, b_1]\times[a_2, b_2]$*,* $a_1, a_2 > 0$*.*

Let the following properties hold:

(a) the function $\pi(z_1, z_2)$ *is bounded in* $[a_1, b_1]\times[a_2, b_2]$*, continuous at* (a_1, a_2)*,* $\pi(a_1, a_2)\neq 0$*.*

(b) conditions from Theorem 2.4 on both $A_1(s)$ *and* $A_2(s)$ *hold.*

Then

$$\lambda_{m,s}(t) \sim a_1\phi_1'(t)A_1'(a_1\phi_1(t)) + a_2\phi_2'(t)A_2'(a_2\phi_2(t)).$$

Note that the meaning of this theorem in terms of a notion of asymptotic independence is the same as discussed for Theorem 3.1.

Proof As in the previous theorem, we denote

$$\lambda_{m,s}(t) = \lambda^1_{m,s}(t) + \lambda^2_{m,s}(t).$$

where

$$\lambda^1_{m,s}(t) = \frac{\int_{a_1}^{b_1}\int_{a_2}^{b_2} A_1'(z_1\phi_1(t))z_1\phi_1'(t)e^{-A_1(z_1\phi_1(t))-A_2(z_2\phi_2(t))}\pi(z_1,z_2)dz_2dz_1}{\int_{a_1}^{b_1}\int_{a_2}^{b_2} e^{-A_1(z_1\phi_1(t))-A_2(z_2\phi_2(t))}\pi(z_1,z_2)dz_2dz_1}, \tag{3.15}$$

The following lemmas are proved similarly to Lemma 3.1 and the corresponding lemmas from Chapter 2:

Lemma 3.2 *Let the assumptions of Lemma 2.4 on $h_1(x_1)$ and $h_2(x_2)$ hold. Let $\mu(u_1, u_2)$ be a positive bounded and locally integrable function, continuous at (a_1, a_2), $\mu(a_1, a_2) \neq 0$.*

Then, as $x_1 \to \infty$ and $x_2 \to \infty$:

$$\int_{a_1}^{\infty}\int_{a_2}^{\infty} e^{-h_1(x_1u_1)-h(x_2u_2)}\mu(u_1,u_2)du_1du_2 \sim \frac{\mu(a_1,a_2)e^{-h_1(a_1x_1)-h_2(a_2x_2)}}{x_1x_2h_1'(a_1x_1)h_2'(a_2x_2)}.$$

Lemma 3.3 *Under the assumptions of Lemma 3.2 the following asymptotic relation holds as $x_1 \to \infty$, $x_2 \to \infty$:*

$$\int_{a_1}^{\infty}\int_{a_2}^{\infty} h_1'(u_1x_1)e^{-h_1(x_1u_1)-h(x_2u_2)}\mu(u_1,u_2)du_1du_2 \sim \frac{a_1a_2\mu(a_1,a_2)}{x_1x_2}e^{-h_1(a_1x_1)-h_2(a_2x_2)}.$$

From these lemmas, using the proof of Theorem 2.4 we derive

$$\lambda^1_{m,s}(t) \sim a_1\phi_1'(t)A_1'(a_1\phi_1(t)).$$

The similar relation holds for $\lambda^2_{m,s}(t)$ and this completes the proof. □

As it can be seen from the proofs of Theorems 3.1 and 3.2, the generalization from the bivariate to arbitrarily multivariate case is rather straightforward, which means that the competing risks problem for a system of m, $m > 2$ components in series is considered similar to the case $m = 2$.

3.2 Bivariate frailty

In this section we will discuss another (and maybe more natural) bivariate model: when there is only one lifetime random variable T, but two unobserved possibly dependent frailties Z_1, Z_2, as defined by relations (3.2) and (3.3). Let Z_1, Z_2 take values in $[0, \infty)$.

Consider the following survival model, which generalizes the simplest bivariate multiplicative model (3.1):

$$\Lambda(t, z_1, z_2) = \Lambda(t)A(z_1, z_2), \tag{3.16}$$

where $A(z_1, z_2)$ is some positive function. The failure rate is

$$\lambda(t, z_1, z_2) = \lambda(t)A(z_1, z_2).$$

The survival and the probability density functions are

$$\bar{F}(t, z_1, z_2) = e^{-\Lambda(t)A(z_1,z_2)}, \quad f(t, z_1, z_2) = \lambda(t)A(z_1, z_2)e^{-\Lambda(t)A(z_1,z_2)},$$

respectively.

Suppose that $A(z_1, z_2)$ is invertible with respect to z_1, and B is an inverse function, i.e.,

$$B(A(z_1, z_2), z_2) \equiv z_1 \quad \text{and} \quad A(B(z_1, z_2), z_2) \equiv z_1.$$

Substituting $s = A(z_1, z_2)$

$$\begin{aligned}
\bar{F}_m(t, z_1, z_2) &= \int_0^\infty \int_0^\infty e^{-A(z_1,z_2)\Lambda(t)} \pi(z_1, z_2) dz_1 dz_2 \\
&= \int_0^\infty e^{-\Lambda(t)s} \int_0^\infty B_1'(s, z_2)\pi(B(s, z_2), z_2) dz_2 ds \\
&= \int_0^\infty e^{-\Lambda(t)s} g(s) ds,
\end{aligned}$$

where B_1' is the partial derivative of B with respect to the first variable and

$$\begin{aligned} g(s) &= \int_0^\infty B_1'(s, z_2)\pi(B(s, z_2), z_2)dz_2 \\ &= \int_0^\infty \frac{1}{A_1'(B(s, z_2), z_2)}\pi(B(s, z_2), z_2)dz_2. \end{aligned} \tag{3.17}$$

Similarly,

$$\begin{aligned} f_m(t, z_1, z_2) &= \int_0^\infty \int_0^\infty \lambda(t)A(z_1, z_2)e^{-A(z_1,z_2)\Lambda(t)}\pi(z_1, z_2)dz_1dz_2 \\ &= \lambda(t)\int_0^\infty se^{-\Lambda(t)s}\int_0^\infty B_1'(s, z_2)\pi(B(s, z_2), z_2)dz_2ds \\ &= \lambda(t)\int_0^\infty e^{-\Lambda(t)s}sg(s)ds. \end{aligned}$$

If $g(s)$ satisfies the conditions of the main univariate theorems (2.2 and 2.4), then the following corollaries can be easily obtained in terms of this function:

Corollary 3.2 *If the function $g(s)$ can be represented as $g(s) = s^\alpha g_1(s)$, where $\alpha > -1$, $g_1(s)$ is bounded in $[0, \infty)$, continuous at $s = 0$ and $g_1(0) \neq 0$, then*

$$\lambda_m(t) \sim \frac{(\alpha + 1)\lambda(t)}{\Lambda(t)}.$$

Note that this result is formulated via the properties of the function $g(s)$.

The second corollary deals with the case $A(z_1, z_2) \geq a$ for z_1, z_2 such that $\pi(z_1, z_2) > 0$ and some positive a. Then under similar assumptions we again can observe a principle similar to the one in the univariate case: the mixture failure rate tends to the failure rate of the strongest population as $t \to \infty$.

Corollary 3.3 *If the function $g(z)$ is defined in $[a, b]$, $a > 0, b \leq \infty$, and for $s \geq a$ it can be defined as*

$$g(s) = (s - a)^\alpha g_1(s - a),$$

where $\alpha > -1$, $g_1(z)$ is bounded in $[0, b - a]$ and $g_1(0) \neq 0$.

Then

$$\lambda_m(t) - a\lambda(t) \sim (\alpha+1)\frac{\lambda(t)}{\Lambda(t)}\, e^{-a\Lambda(t)}.$$

Conditions on the function $g_1(s)$ need further discussion. So far we managed to formulate the results stating the conditions on the distribution family and on the mixing distribution separately. Note that $g(s)$ can be interpreted as a pdf:

$$\int_0^\infty g(s)ds = \int_0^\infty \int_0^\infty \pi(z_1,z_2)dz_1dz_2 = 1$$

It is also worth noting that the assumption that $g(s)$ is bounded does not seem restrictive for the model. The following remark states rather strong sufficient general conditions in terms of $A(z_1,z_2)$ and $\pi(z_1,z_2)$ for two conditions to hold:

Remark 3.1 *If on the support of* $\pi(z_1,z_2)$*:* $A(z_1,z_2)$ *is continuously differentiable in* $[0,\infty)^2$*,* $A_1'(z_1,z_2) > 0$*,* $\pi(z_1,z_2)$ *is strictly positive and uniformly continuous, then the conditions of the two corollaries hold and* $\alpha = 0$*.*

Consider two specific cases, which will clarify the developed approach.

Example 3.1 Let

$$A(z_1,z_2) = z_1z_2.$$

If $\pi(z_1,z_2)$ has a compact support and is continuous in it, then it is uniformly continuous there. The condition $A_1'(z_1,z_2) > 0$ does not hold since $A_1'(z_1,0) = 0$, whereas the asymptotic behavior stays the same.

Obviously,

$$B(s,z_2) = \frac{s}{z_2}, \quad A_1'(z_1,z_2) = z_2, \quad B_1'(s,z_2) = \frac{1}{z_2}.$$

Consider now the mixing distribution to be a uniform one in $[0,b]^2$ for some $b > 0$

$$\pi(z_1,z_2) = \begin{cases} 1/b^2, & 0 \le z_1, z_2 \le b \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$
\begin{aligned}
\bar{F}_m(t) &= \frac{1}{b^2}\int_0^b\int_0^b e^{-\Lambda(t)xy}dxdy \\
&= \frac{1}{b^2}\int_0^b \frac{1}{\Lambda(t)y}\left(1-e^{-\Lambda(t)by}\right)dy \\
&= \frac{1}{\Lambda(t)b^2}\int_0^{\Lambda(t)b^2}\frac{1}{u}\left(1-e^{-u}\right)du.
\end{aligned}
$$

It is easy to show that as $v \to \infty$

$$
\int_0^v \frac{1}{u}\left(1-e^{-u}\right)du \sim \log v,
$$

thus,

$$
\bar{F}_m(t) \sim \frac{\log\Lambda(t)}{\Lambda(t)b^2} \tag{3.18}
$$

as $t \to \infty$. Similarly,

$$
\begin{aligned}
f_m(t) &= \frac{\lambda(t)}{b^2}\int_0^b\int_0^b xy\,e^{-\Lambda(t)xy}dxdy \\
&= \frac{\lambda(t)}{\Lambda(t)^2b^2}\int_0^b\frac{1}{y}\int_0^{\Lambda(t)by} ue^{-u}dudy \\
&= \frac{\lambda(t)}{\Lambda(t)^2b^2}\int_0^b\frac{1}{y}\left(e^{-\Lambda(t)by}\Lambda(t)by+1-e^{-\Lambda(t)by}\right)dy \\
&= \frac{\lambda(t)}{\Lambda(t)^2b^2}\left[\int_0^{\Lambda(t)b^2}\frac{1}{u}\left(1-e^{-u}\right)du+1-e^{-\Lambda(t)b^2}\right]
\end{aligned}
$$

and

$$
f_m(t) \sim \frac{\lambda(t)\log\Lambda(t)}{\Lambda(t)^2b^2}, \tag{3.19}
$$

hence,

$$
\lambda_m(t) \sim \frac{\lambda(t)}{\Lambda(t)},
$$

which is a remarkably simple asymptotic, similar to the univariate case. $\diamondsuit$

Example 3.2 Consider the following specific case of the model (3.16) with additive $A(z_1, z_2)$:

$$A(z_1, z_2) = z_1 + z_2,$$

Then

$$B(s, z_2) = s - z_2, \quad A'_1(z_1, z_2) \equiv B'_1(s, z_2) \equiv 1,$$

and finally

$$\lambda_{m,s}(t) \sim (\alpha_1 + \alpha_2 + 2)\frac{\lambda(t)}{\Lambda(t)}. \tag{3.20}$$

On the other hand, this model can be also interpreted in terms of the series system of two identical components. Then $\phi_1(t) = \phi_2(t) = \Lambda(t)$ and it follows from Theorem 3.1 that (3.20) also holds.

Indeed, if $\pi(z_1, z_2) = z_1^{\alpha_1} z_2^{\alpha_2} \pi_0(z_1, z_2)$, $\pi_0(z_1, z_2)$ is continuous at the origin and $\pi_0(0, 0) \neq 0$, then as $s \to 0$

$$g(s) = \int_0^s \pi(s - z_2, z_2) dz_2 \sim s^{\alpha_1 + \alpha_2 + 1} \cdot \pi(0, 0).$$

Thus, if $g(s)$ is also bounded in $[0, \infty)$, then the conditions of the both corollaries hold and (3.20) also holds.

$\diamondsuit$

Chapter 4

Stochastic ordering and bounds on mixture failure rate

In this chapter we will look at several comparison problems for the mixture failure rates. First the bending down property will be proved, stating that

$$\lambda_m(t) < \lambda_P(t) \equiv \int_a^b \lambda(t,z)\pi(z)dz, \ t > 0, \quad \lambda_m(0) = \lambda_P(0)$$

(see Definition 1.1). Then we will consider the ordering of mixture failure rates for different mixing distributions. Finally, some bounds for $\lambda_m(t)$ in the proportional hazards framework will be obtained.

4.1 Ordering of failure rates

4.1.1 Comparison with $\lambda_P(t)$

The main additional assumption that will be needed for the following result is that the family of failure rates $\lambda(t,z)$, $z \in [a,b]$ should be ordered in z.

Theorem 4.1 *Let the failure rate $\lambda(t,z)$ in the mixing model (1.2) be differentiable with respect to both arguments and be ordered as*

$$\lambda(t,z_1) < \lambda(t,z_2), \quad z_1 < z_2, \quad \forall z_1, z_2 \in [a,b], \, t \geq 0. \tag{4.1}$$

Assume that conditional and unconditional expectations in relations (1.2) and (1.4), respectively, are finite for all $t \in [0,\infty)$.

Then:

(a) The mixture failure rate $\lambda_m(t)$ bends down with time at least in a weak sense.

(b) If, additionally, $\partial\lambda(t,z)/\partial z$ is increasing in t, then $\lambda_m(t)$ bends down in a strong sense.

Proof It is clear that ordering (4.1) is equivalent to the condition that $\lambda(t,z)$ is increasing in z for each $t \geq 0$. In accordance with equations (1.2) and (1.4) and integrating by parts (Finkelstein, 2004) we obtain

$$\begin{aligned} \triangle\lambda(t) &\equiv \lambda_P(t) - \lambda_m(t) \\ &= \int_a^b \lambda(t,z)[\pi(z) - \pi(z|t)]dz \\ &= \lambda(t,z)[\Pi(z) - \Pi(z|t)]|_a^b - \int_a^b \lambda'_z(t,z)[\Pi(z) - \Pi(z|t)]dz \\ &= \int_a^b -\lambda'_z(t,z)[\Pi(z) - \Pi(z|t)]dz > 0, \quad t > 0, \end{aligned} \tag{4.2}$$

where

$$\Pi(z) = P(Z \leq z); \qquad \Pi(z|t) = P(Z \leq z|T > t),$$

and the term

$$\lambda(t,z)[\Pi(z) - \Pi(z|t)]|_a^b$$

vanishes for $b = \infty$ as well.

Inequality (4.2) and, therefore, the first part of the theorem follows from the fact that $\lambda(t,z)$ increases in z, i.e., $\lambda'_z(t,z) > 0$, and the following inequality:

$$\Pi(z) - \Pi(z|t) < 0, \quad \forall t > 0,\, z \in [a,b]. \tag{4.3}$$

This inequality can be interpreted as *"the weakest populations are dying out first"*. This interpretation is widely used in various specific cases, especially in the demographic literature (e.g., Vaupel, 2003). For obtaining (4.3), it is sufficient to prove that

$$\Pi(z|t) = \frac{\int_a^z \bar{F}(t,u)\pi(u)du}{\int_a^b \bar{F}(t,u)\pi(u)du}$$

increases in t, which can be easily done by considering the corresponding derivative.

The derivative $\Pi_t'(z|t) > 0$, if

$$\frac{\int_a^z \bar{F}_t'(t,u)\pi(u)du}{\int_a^z \bar{F}(t,u)\pi(u)du} > \frac{\int_a^b \bar{F}_t'(t,u)\pi(u)du}{\int_a^b \bar{F}(t,u)\pi(u)du}.$$

As $\bar{F}_t'(t,z) = -\lambda(t,z)\bar{F}(t,z)$, it is sufficient to show that

$$B(t,z) \equiv \frac{\int_a^z \lambda(t,u)\bar{F}(t,u)\pi(u)du}{\int_a^z \bar{F}(t,u)\pi(u)du}$$

increases in z. Inequality $B_z'(t,z) > 0$ is equivalent to:

$$\lambda(t,z)\int_a^z \bar{F}(t,u)\pi(u)du > \int_0^z \lambda(t,u)\bar{F}(t,u)\pi(u)du,$$

which follows from ordering (4.1).

Thus, due to additional assumption in (b), the integrand in the end part of (4.2) is increasing and therefore $\triangle\lambda(t)$ as well, which immediately leads to the strong bending down property (1.6). □

The light bulbs example of the Introduction shows the strong bending property of the mixture failure rate in practice. The results were really convincing: the failure rate is initially increasing (a tentative fit showed the Weibull law) and then decreasing to a very low level. The pattern of the observed failure rate is exactly the same as predicted in Finkelstein and Esaulova (2001a) for the Weibull baseline Cdf. Some biological experiments on frailties and worms suggest the same conclusion (Carey *et al*, 1992).

4.1.2 Likelihood ordering of mixing distributions

We will show now that a natural ordering for our mixing model (1.2) is the likelihood ratio one. A somewhat similar reasoning can be found in Block *et al* (1993) and Shaked and Spizzichino (2001).

Let Z_1 and Z_2 be continuous nonnegative random variables with the same support and densities $\pi_1(z)$ and $\pi_2(z)$, respectively. Recall (Ross, 1996;

Shaked and Shanthikumar, 1993) that Z_2 is smaller than Z_1 in the sense of the likelihood ratio (Kaas *et al*, 1994):

$$Z_1 \geq_{LR} Z_2, \tag{4.4}$$

if $\pi_2(z)/\pi_1(z)$ is a decreasing function.

Definition 4.1 *Let $Z(t)$, $t \in [0, \infty)$ be a family of random variables indexed by parameter t (time) with probability density functions $p(z,t)$. We say that $Z(t)$ is decreasing in t in the sense of the likelihood ratio, if*

$$L(z, t_1, t_2) = \frac{p(z, t_2)}{p(z, t_1)}$$

decreases in z for all $t_2 > t_1$.

The following simple result states that our family of conditional mixing random variables $[Z|t]$, $t \in [0, \infty)$ is decreasing in this sense:

Theorem 4.2 *Let the family of failure rates $\lambda(t,z)$ in the mixing model (1.2) be ordered as in relation (4.1).*

Then the family of random variables $[Z|t] \equiv [Z|T > t]$ is decreasing in $t \in [0, \infty)$ in the sense of the likelihood ratio.

Proof In accordance with definition (1.3) of the density $\pi(z|t)$, we have

$$L(z, t_1, t_2) = \frac{\pi(z|\, t_1)}{\pi(z|\, t_2)} = \frac{\bar{F}(t_2, z) \int_a^b \bar{F}(t_1, u)\pi(u)du}{\bar{F}(t_1, z) \int_a^b \bar{F}(t_2, u)\pi(u)du}. \tag{4.5}$$

Therefore, monotonicity in z of $L(z, t_1, t_2)$ is defined by

$$\frac{\bar{F}(t_2, z)}{\bar{F}(t_1, z)} = e^{-\int_{t_1}^{t_2} \lambda(s,z)ds},$$

which, due to ordering (4.1), is decreasing in z for all $t_2 > t_1$. □

Consider now two different mixing random variables Z_1 and Z_2 with probability density functions $\pi_1(z), \pi_2(z)$, and cumulative distribution functions

$\Pi_1(z)$, $\Pi_2(z)$, respectively. Assuming some type of stochastic ordering for Z_1 and Z_2, we intend to arrive at a simple ordering of the corresponding mixture failure rates. It can be seen using simple examples that the 'usual' stochastic ordering (stochastic dominance) is too weak for this purpose. It was shown in the previous section that the likelihood ratio ordering is a natural one for the family of random variables $[Z|t]$ in our mixing model. Therefore, it seems reasonable to order Z_1 and Z_2 in this sense too. First, consider a supplementary lemma.

Lemma 4.1 *Let*

$$\pi_2(z) = \frac{g(z)\pi_1(z)}{\int_a^b g(z)\pi_1(z)dz}, \tag{4.6}$$

where $g(z)$ is a decreasing function.

Then Z_1 is stochastically larger than Z_2:

$$Z_1 \geq_{st} Z_2 \qquad (\,\Pi_1(z) \leq \Pi_2(z),\; z \in [a,b]\,) \tag{4.7}$$

Proof Indeed,

$$\begin{aligned}\Pi_2(z) &= \frac{\int_a^z g(u)\pi_1(u)du}{\int_a^b g(u)\pi_1(u)du} \\ &= \frac{\int_a^z g(u)\pi_1(u)du}{\int_a^z g(u)\pi_1(u)du + \int_z^b g(u)\pi_1(u)du} \\ &= \frac{g^*(a,z)\int_a^z \pi_1(u)du}{g^*(a,z)\int_a^z \pi_1(u)du + g^*(z,b)\int_z^b \pi_1(u)du} \\ &\geq \int_a^z \pi_1(u)du = \Pi_1(z),\end{aligned} \tag{4.8}$$

where $g^*(a,z)$ and $g^*(z,b)$ are the mean values of the function $g(z)$ in the corresponding integrals. As this function decreases: $g^*(z,b) \leq g^*(a,z)$. □

Equation (4.6) for decreasing $g(z)$ means that $Z_1 \geq_{LR} Z_2$, and it is well known (see, e.g., Ross, 1996) that the likelihood ratio ordering implies the corresponding stochastic ordering. But we need the foregoing reasoning for deriving the following result:

Theorem 4.3 *Let relation (4.6), where $g(z)$ is a decreasing function hold, which means that Z_1 is larger than Z_2 in the sense of the likelihood ratio ordering.*

Assume that ordering (4.1) holds.

Then for all $t \in [0, \infty)$ the corresponding mixture failure rates are ordered as:

$$\lambda_{m1}(t) \equiv \frac{\int_a^b f(t,z)\pi_1(z)dz}{\int_a^b \bar{F}(t,z)\pi_1(z)dz} \geq \frac{\int_a^b f(t,z)\pi_2(z)dz}{\int_a^b \bar{F}(t,z)\pi_2(z)dz} \equiv \lambda_{m2}(t) \qquad (4.9)$$

Proof Inequality (4.9) means that the mixture failure rate, which is obtained for the stochastically larger (in the likelihood ratio ordering sense) mixing distribution, is larger for all $t \in [0, \infty)$ than the one obtained for the stochastically smaller mixing distribution.

We shall prove first that

$$\Pi_1(z|t) = \frac{\int_a^z \bar{F}(t,u)\pi_1(u)du}{\int_a^b \bar{F}(t,u)\pi_1(u)du} \leq \frac{\int_a^z \bar{F}(t,u)\pi_2(u)du}{\int_a^b \bar{F}(t,u)\pi_2(u)du} = \Pi_2(z|t). \qquad (4.10)$$

Indeed, using representation (4.6), we get

$$\begin{aligned}
\frac{\int_a^z \bar{F}(t,u)\pi_2(u)du}{\int_a^b \bar{F}(t,u)\pi_2(u)du} &= \frac{\int_a^z \bar{F}(t,u)g(u)\pi_1(u)(\int_a^b g(s)\pi_1(s)ds)^{-1}du}{\int_a^b \bar{F}(t,u)g(u)\pi_1(u)(\int_a^b g(s)\pi_1(s)ds)^{-1}du} \\
&= \frac{\int_a^z g(u)\bar{F}(t,u)\pi_1(u)du}{\int_a^b g(u)\bar{F}(t,u)\pi_1(u)du} \\
&\geq \frac{\int_a^z \bar{F}(t,u)\pi_1(u)du}{\int_a^b \bar{F}(t,u)\pi_1(u)du},
\end{aligned}$$

where the last inequality follows using exactly the same argument, as in inequality (4.8) of Lemma 4.1. Similar to (4.2), and taking into account relation (4.10), we obtain

$$\lambda_{m1}(t) - \lambda_{m2}(t) = \int_a^b \lambda(t,z)[\pi_1(z|t) - \pi_2(z|t)]dz$$
$$= \lambda(t,z)[\Pi_1(z|t) - \Pi_2(z|t)]|_a^b - \int_a^b \lambda'_z(t,z)[\Pi_1(z|t) - \Pi_2(z|t)]dz$$
$$= \int_a^b -\lambda'_z(t,z)[\Pi_1(z|t) - \Pi_2(z|t)]dz \ \geq 0.$$

A starting point of this theorem was equation (4.6) with a crucial assumption of a decreasing function $g(z)$. It should be noted, however, that this assumption can be rather formally justified directly by considering the difference $\lambda_{m1}(t) - \lambda_{m2}(t)$ and using definitions (1.2)-(1.3). The corresponding numerator (the denominator is positive) is transformed into a double integral in the following way:

$$\int_a^b \lambda(t,z)\bar{F}(t,z)\pi_1(z)dz \int_a^b \bar{F}(t,z)\pi_2(z)dz$$
$$- \int_a^b \lambda(t,z)\bar{F}(t,z)\pi_2(z)dz \int_a^b \bar{F}(t,z)\pi_1(z)dz$$
$$= \int_a^b \int_a^b \bar{F}(t,u)\bar{F}(t,s)[\lambda(t,u)\pi_1(u)\pi_2(s) - \lambda(t,s)\pi_1(u)\pi_2(s)]duds$$
$$= \iint\limits_{b>u>s>a} \bar{F}(t,u)\bar{F}(t,s)[\pi_1(u)\pi_2(s)(\lambda(t,u) - \lambda(t,s))$$
$$+ \pi_1(s)\pi_2(u)(\lambda(t,s) - \lambda(t,u)]duds$$
$$= \iint\limits_{b>u>s>a} \bar{F}(t,u)\bar{F}(t,s)(\lambda(t,u) - \lambda(t,s))[\pi_1(u)\pi_2(s) - \pi_1(s)\pi_2(u)]duds. \tag{4.11}$$

Therefore, the final double integral is positive if ordering (4.1) holds and $\pi_2(z)/\pi_1(z)$ is decreasing. □

4.1.3 Another useful ordering

Consider the multiplicative model with two frailties Z_1 and Z_2. The use of the Laplace transforms technique (see Section 1.3.2) allows us to formulate another elegant result for the ordering of mixture failure rate.

Suppose that the second frailty Z_2 equals in distribution to a sum:

$$Z_2 =_D Z_1 + Y, \tag{4.12}$$

where Y is some random variable, Y and Z_1 are independent.

Theorem 4.4 *The mixture failure rates are ordered:*

$$\lambda_{m2}(t) > \lambda_{m1}(t), \qquad \forall\, t \geq 0$$

if and only if Y is a positive (nonnegative) random variable.

Although this result seems to be intuitively evident (at least, for the case when Y is positive), intuition in mixture failure rate modelling can be sometimes deceiving. The counter-reasoning is based on the fact that the larger the failure rate of a subpopulation $\lambda(t, z)$, the more intensive is the process of dying out.

Note that this theorem states both necessary and sufficient conditions for the corresponding ordering of mixture failure rates, given the representation (4.12).

Proof As the mixture failure rate for the multiplicative model is given by (1.26), i.e., the failure rates ordering is equivalent to the following inequality for all $t \geq 0$ (and therefore for all $\Lambda(t)$)

$$(\log L_{\pi_2}(t))' < (\log L_{\pi_1}(t))', \tag{4.13}$$

where $L_{\pi_1}(t), L_{\pi_2}(t)$ are the corresponding Laplace transforms:

$$L_{\pi_i}(t) = Ee^{-tZ_i}, \quad i = 1, 2.$$

On the other hand, $Z_2 = Z_1 + Y$, where random variables Y and Z_1 are independent, therefore

$$L_{\pi_2}(t) = Ee^{-tZ_2} = Ee^{-tZ_1 - tY} = Ee^{-tZ_1} Ee^{-tY} = L_{\pi_1}(t) L_{\pi_Y}(t),$$

where $L_{\pi_Y}(t)$ is the Laplace transform of Y. Hence, (4.13) turns into

$$(\log L_{\pi_1}(t))' + (\log L_{\pi_Y}(t))' < (\log L_{\pi_1}(t))',$$

or

$$(\log L_{\pi_Y}(t))' < 0,$$

which is equivalent to the decreasing of the Laplace transform $L_{\pi_Y}(t)$. This is obviously the case if Y is a nonnegative random variable.

Consider the case $P(Y < 0) > 0$. The two-sided Laplace transform of Y exists because $Z_2 =_D Z_1 + Y$, where Z_1, Z_2 are both positive r.v., then

$$L_{\pi_Y}(t) = L_{\pi_2}(t)/L_{\pi_1}(t),$$

then

$$L_{\pi_Y}(t) = \int_{-\infty}^{0} e^{-ts}\pi_Y(s)ds + \int_{0}^{\infty} e^{-ts}\pi_Y(s)ds;$$

where the second integral decreases and the first integral increases to infinity as $t \to \infty$, therefore

$$L_{\pi_Y}(t) \to \infty \qquad \text{as} \quad t \to \infty,$$

which completes the proof. □

This simple theorem results in a non-trivial interpretation: let the possible Y be 'contaminated' (or pertubed) to end up with some random variable $\tilde{Y}$ such that $P(\tilde{Y} < 0) > \epsilon$. Theorem 4.4 says that for arbitrarily small ϵ

$$\lambda_{m2}(t) < \lambda_{m1}(t) \quad \text{as } t \to \infty,$$

which, in fact, again comes from the principle "the weakest populations are dying out first".

4.1.4 Ordering of variances of mixing distributions

If Z_1 and Z_2 are ordered in the sense of the likelihood ordering (4.7), then automatically

$$E[Z_1] \geq E[Z_2].$$

Assume now that distributions $\Pi_1(z)$ and $\Pi_2(z)$ have equal means and let us 'play' with the corresponding variances. It follows from equation (1.16) that for the multiplicative model, which will be considered in this section:

$$\lambda_{m1}(0) = \lambda_{m2}(0).$$

Intuitive considerations and reasoning based on the principle: "the weakest populations are dying out first" suggest that, unlike (4.9), the mixture failure rates will be ordered as

$$\lambda_{m1}(t) < \lambda_{m2}(t)$$

for all $t > 0$ if, e.g., the variance of Z_1 is larger than the variance of Z_2. We will show that this is true for a specific case, whereas for a general multiplicative model this ordering holds only for the sufficiently small time t. Therefore, a stronger condition on ordering 'variabilities' of Z_1 and Z_2 should be imposed.

Example 4.1 For a meaningful specific example, consider again the frailty model (1.15), where Z has a Gamma distribution:

$$\pi(z) = \frac{\beta^\alpha}{\Gamma(\alpha)} z^{\alpha-1} e^{-\beta z}; \quad \alpha, \beta > 0.$$

We have already obtained in Examples 1.3 and 2.4 that

$$\lambda_m(t) = \frac{\lambda(t) \int_0^\infty e^{-z\Lambda(t)} z\pi(z)dz}{\int_0^\infty e^{-z\Lambda(t)} \pi(z)dz} = \frac{\alpha\lambda(t)}{\beta + \Lambda(t)}, \tag{4.14}$$

where $\Lambda(t) = \int_0^t \lambda(u)du$ is a cumulative baseline failure rate. It can be shown that equation (4.14) can be written now in terms of $E[Z]$ and $Var(Z)$:

$$\lambda_m(t) = \lambda(t) \frac{E^2[Z]}{E[Z] + Var(Z)\Lambda(t)}, \tag{4.15}$$

which for the specific case $E[Z] = 1$ gives the widely used in demography result of Vaupel *et al* (1979):

$$\lambda_m(t) = \frac{\lambda(t)}{1 + Var(Z)\Lambda(t)}.$$

Using equation (4.15), we can easily compare and analyse mixture failure rates for two populations with different Z_1 and Z_2 on condition that $E[Z_2] = E[Z_1]$. If, for instance,

$$Var(Z_1) \geq Var(Z_2),$$

the mixture failure rates are ordered in the reversed way:

$$\lambda_{m1}(t) \leq \lambda_{m2}(t).$$

♢

Intuitively it can be expected that this result could be valid for arbitrary mixing distributions in the multiplicative model. However, the mixture failure rate dynamics can be much more complicated even for this specific case and this topic needs further attention in the future research. A somehow similar situation was observed in Finkelstein and Esaulova (2001a): although the conditional variance $Var(Z|t)$ was decreasing in t for the multiplicative Gamma-frailty model, a counter example was constructed for the case of the uniform mixing distribution in $[0, 1]$.

The following theorem shows that ordering of variances is a sufficient and necessary condition for ordering of mixture failure rates, only for the initial time interval.

Theorem 4.5 *Let Z_1 and Z_2 ($E[Z_2] = E[Z_1]$) be two mixing distributions in the multiplicative model (1.15)-(1.16) with the same baseline failure rate $\lambda(t)$.*

Then ordering of variances

$$Var(Z_1) > Var(Z_2) \tag{4.16}$$

is a sufficient and necessary condition for ordering of mixture failure rates in the neighborhood of $t = 0$:

$$\lambda_{m1}(t) < \lambda_{m2}(t); \quad t \in (0, \epsilon), \tag{4.17}$$

where $\epsilon > 0$ is sufficiently small.

Proof *Sufficient condition:*

From results of the previous sections:

$$\triangle\lambda(t) = \lambda_{m1}(t) - \lambda_{m2}(t) = \lambda(t)(E[Z_1|t] - E[Z_2|t]), \tag{4.18}$$

$$E'_t[Z_i|t] = -\lambda(t)Var(Z_i|t) < 0, \quad i = 1, 2,\ t \geq 0, \tag{4.19}$$

where

$$E[Z_i|0] \equiv E[Z_i], \quad Var(Z_i|t) \equiv Var(Z_i). \tag{4.20}$$

As the means of mixing variables are equal, relation (4.18) for $t = 0$ reads: $\triangle\lambda(0) = 0$ and therefore the time interval in (4.17) is open. Thus, if ordering in variances holds, ordering (4.17) follows immediately after considering the derivative of

$$\frac{\lambda_{m1}(t)}{\lambda_{m2}(t)} = \frac{E[Z_1|t]}{E[Z_2|t]}$$

at $t = 0$ and taking into account relations (4.19) and notation (4.20).

Necessary condition:

The proof of the first part of the theorem was, in fact, trivial. The second one is a bit more technical. Similar to (4.11), the numerator of the difference $\triangle\lambda(t)$ is

$$\lambda(t) \int_a^b \int_a^b e^{-\Lambda(t)(s+u)}(u - s)\pi_1(u)\pi_2(s)duds.$$

After changing variables to $x = (u + s)/2$, $y = (u - s)/2$, the double integral is transformed to the iterated integral and denoted by $G(t)$:

$$G(t) \equiv \int_a^b e^{-2\Lambda(t)x} \int_{-x}^{x} y\,\pi_1(x + y)\pi_2(x - y)dydx.$$

Denote the internal integral by $g(x)$. Then:

$$G(t) = \int_a^b e^{-2\Lambda(t)x} g(x)dx.$$

On the other hand, coming back to initial variables of integration and taking into account that $\Lambda(0) = 0$:

$$G(0) = \int_a^b g(x)dx = \int_a^b \int_a^b (u-s)\pi_1(u)\pi_2(s)duds$$
$$= \int_a^b u\pi_1(u)du - \int_a^b u\pi_2(u)du$$
$$= E[Z_1] - E[Z_2] = 0.$$

Assume first that $\lambda(0) \neq 0$. As $G(0) = 0$, the function $G(t)$ is negative in the neighborhood of 0 if $G'(0) < 0$:

$$G'(t) = -2\lambda(t) \int_a^b e^{-2\Lambda(t)x} xg(x)dx,$$

and

$$G'(0) < 0 \implies \int_a^b xg(x)dx > 0.$$

If $\triangle\lambda(t) < 0,\ t \in (0, \epsilon)$ (condition (4.17), then $G(t) < 0,\ t \in (0, \epsilon)$, and taking into account that

$$\int_a^b xg(x)dx = \int_a^b \int_a^b \frac{u+s}{2}(u-s)\pi_1(s)\pi_2(s)duds$$
$$= \frac{1}{2}\int_a^b \int_a^b (u^2 - s^2)\pi_1(s)\pi_2(s)duds$$
$$= \frac{1}{2}(Var(Z_1) - Var(Z_2)),$$

we arrive at ordering (4.16).

Similar considerations are valid for $\lambda(0) = 0$. The function $G(t)$ is negative in this case in the neighborhood of 0, if $G''(0) < 0$. As

$$G''(0) = -2\lambda'(0) \int_a^b xg(x)ds$$

and $\lambda'(0) > 0$ (as $\lambda(t) > 0$, $t > 0$ and $\lambda(0)$), the foregoing reasoning, which was used for the case $\lambda(0) \neq 0$, also takes place. □

A trivial but important consequence of this theorem is:

Corollary 4.1 *Let mixtures failure rate ordering (4.17) holds for* $t \in (0, \infty)$*. Then inequality (4.16) holds.*

4.2 Modelling Impact of Environment

4.2.1 Bounds in the proportional hazards model

Consider a specific multiplicative frailty model (1.15)-(1.16). Combine formally this model with a proportional hazards (PH) model in a following way:

$$\lambda(t, z, k) = zk\lambda(t) \equiv \lambda(t), \tag{4.21}$$

where z, as previously, comes from a realization of an unobserved random frailty Z, and k is a proportional factor from the 'conventional' PH model. As we are not performing data analysis, this factor is written in the 'aggregated' form k (and not as $e^{B^T X}$ for regression analysis). PH model is often used for modelling an impact of environment (covariates), therefore the combined model can be aslo used for this purpose (Wienke, 2003).

In accordance with (4.21), the baseline $F(t)$ can be viewed as being indexed by the random variable $Z_k = kZ$ with the pdf $\pi_k(z) = \pi(z/k)$, whereas the corresponding conditional pdf $\pi_k(z|t)$ is given by the right hand side of equation (1.3), where $\pi(z)$ is substituted by $\pi_k(z)$. Equivalently, (4.21) can be interpreted as a frailty model with a mixing random variable Z and the baseline failure rate $k\lambda(t)$. These two simple equivalent interpretations will help us in what follows. Without loosing generality assume that $a = 0$ and $b = \infty$. Thus, similar to the previous situations, the mixture failure rate in this case is:

$$\lambda_{mk}(t) = k\lambda(t) \int_0^\infty z\pi_k(z|t)dz \equiv \lambda(t)E[Z_k|t], \tag{4.22}$$

As $Z_k = kZ$, its density function is

$$\pi_k(z) = \frac{1}{k}\pi\left(\frac{z}{k}\right).$$

Theorem 4.6 *Let the mixture failure rates for the multiplicative models (1.15) and (4.21) be given by relations (1.16) and (4.22), respectively, where $k > 1$.*

Assume that the following quotient increases in z:

$$\frac{\pi_k(z)}{\pi(z)} = \frac{\pi(z/k)}{k\pi(z)} \quad \uparrow \tag{4.23}$$

Then the following ordering holds:

$$\lambda_{mk}(t) > \lambda_m(t); \quad \forall t \in [0, \infty). \tag{4.24}$$

Proof Although inequality (4.24) seems intuitively trivial at first sight, it is valid only for some specific cases of mixing (e.g., the multiplicative model)! It is clear that (4.24) is always true for sufficiently small t, whereas for larger values of time the ordering can be different for general mixing models. Denote:

$$\triangle\lambda_m(t) = \lambda_{mk}(t) - \lambda_m(t).$$

Using definitions (1.2)-(1.3), it can be seen similar to relation (4.11) that the sign of this difference is defined by the sign of:

$$\begin{aligned}
&\int_0^\infty z\bar{F}(t,z)\pi_k(z)dz \int_0^\infty \bar{F}\pi(z)dz - \int_0^\infty \bar{F}(t,z)\pi_k(z)dz \int_0^\infty z\bar{F}\pi(z)dz \\
&\quad = \int_0^\infty \int_0^\infty \bar{F}(t,u)\bar{F}(t,s)[a\pi_k(u)\pi(s) - s\pi_k(u)\pi(s)]duds \\
&\quad = \iint_{0<s<u<\infty} \bar{F}(t,u)\bar{F}(t,s)[\pi_k(u)\pi(s)(u-s) + \pi_k(s)\pi(u)(s-u)]duds \\
&\quad = \iint_{0<s<u<\infty} \bar{F}(t,u)\bar{F}(t,s)(u-s)[\pi_k(u)\pi(s) - \pi_k(s)\pi(u)]duds.
\end{aligned} \tag{4.25}$$

Therefore, the sufficient condition for inequality (4.24) is condition (4.23), which is, in fact, rather crude. It is easy to verify that this condition is satisfied e.g., for the Gamma and the Weibull densities, which are often used for mixing. □

Example 4.2 Consider the same setting as in Example 4.1. Condition (4.23) is satisfied for the Gamma distribution. The mixture failure rate $\lambda_m(t)$ in this case is given by relation (4.15). A similar equation obviously exists for $\lambda_{mk}(t)$, and the corresponding comparison can be performed explicitly:

$$\begin{aligned}\lambda_{mk}(t) &= \lambda(t)\frac{E^2[Z_k]}{E[Z_k]+Var(Z_k)\Lambda(t)} \\ &= \lambda(t)\frac{k^2E^2[Z]}{kE[Z]+k^2Var(Z)\Lambda(t)} > \lambda_m(t).\end{aligned} \tag{4.26}$$

$\diamondsuit$

Now we will obtain an upper bound for $\lambda_m k(t)$.

Theorem 4.7 *Let the mixture failure rates for the multiplicative models (1.15) and (4.21) be given by relations (1.16) and (4.22), respectively, where* $k > 1$. *Then:*

$$\lambda_{mk} < k\lambda_m(t); \quad \forall t \in (0, \infty). \tag{4.27}$$

Proof Consider the difference $\lambda_{mk}(t) - k\lambda_m(t)$ similar to (4.25), but in a slightly different way: $\lambda_{mk}(t)$ will be equivalently defined by the baseline failure rate $k\lambda(t)$ and the mixing variable Z (in (4.25) it was defined by the baseline $\lambda(t)$ and the mixing variable kZ). This means:

$$\lambda_{mk}(t) - k\lambda_m(t) = k\lambda(t)(\hat{E}[Z|t] - E[Z|t]), \tag{4.28}$$

where conditioning in $\hat{E}[Z|t]$ is different from the one in $E[Z|t]$ in the described sense. Denote:

$$\bar{F}_k(t,z) = e^{-zk\Lambda(t)}.$$

'Symmetrically' to (4.25), $sign[\lambda_{mk}(t) - k\lambda_m(t)]$ is defined by

$$sign \iint_{\infty>u>s>0} \pi(u)\pi(s)(u-s)[\bar{F}_k(t,u)\bar{F}(t,s) - \bar{F}(t,u)\bar{F}_k(t,s)]duds,$$

which is negative for all $t > 0$, as

$$\frac{\bar{F}_k(t,z)}{\bar{F}(t,z)} = e^{-(k-1)z\Lambda(t)}$$

is decreasing in z for $k > 1$. □

It is worth noting that we do not need additional condition for this bound as in the case of Theorem 4.6. Also it is clear that $\lambda_{mk}(0) = k\lambda_m(0)$. As it was already mentioned, model (4.21) defines a combination of a PH and a frailty model. When $Z = 1$, it is an 'ordinary' PH model. In the presence of a random Z, as follows from (4.27), the observed failure rate $\lambda_{mk}(t)$ cannot be obtained as $k\lambda_m(t)$ due to the nature of mixing. Therefore:

The PH model in each realization does not result in the PH model for the corresponding mixture failure rates.

Example 4.2 can be continued to illustrate inequality (4.27):

$$\lambda_{mk}(t) = \lambda(t)\frac{k^2E^2[Z]}{kE[Z] + k^2Var(Z)\lambda(t)}$$
$$< \lambda(t)\frac{kE^2[Z]}{E[Z] + Var(Z)\Lambda(t)} = k\lambda_m(t).$$

4.2.2 Change point in environment

Assume that there are two possible environments (stresses): $\epsilon(t)$ and $\epsilon_s(t)$ - the baseline and a more severe one, respectively. The baseline environment for our heterogeneous population corresponds to the observed failure rate $\lambda_m(t)$ and a more severe one to $\lambda_{mk}(t)$, $k > 1$. As previously, assume also that the PH model for each subpopulation (for each fixed z) holds.

Consider a piece-wise constant step stress with a single change point at t_1:

$$\epsilon(t_1) = \begin{cases} \epsilon, & 0 \le t < t_1 \\ \epsilon_k & t \ge t_1 \end{cases} \tag{4.29}$$

where the stresses ϵ and ϵ_k correspond to the failure rates $z\lambda(t)$ and $zk\lambda(t)$, respectively ($k > 1$, $z \ge 0$), and z, as previously, is a realization of the frailty Z. In accordance with a 'memory-less property' of the PH model, the stress (4.29) results in the following failure rate:

$$\lambda(t, t_1, z, k) = \begin{cases} z\lambda(t), & 0 \le t < t_1 \\ kz\lambda(t) & t \ge t_1 \end{cases} \tag{4.30}$$

for each subpopulation.

Denote the resulting mixture failure rate in this case as:

$$\lambda(t, t_1) = \begin{cases} \lambda_m(t), & 0 \le t < t_1 \\ \tilde{\lambda}_{mk}(t) & t \ge t_1 \end{cases} \tag{4.31}$$

where, similar to the previous section:

$$\tilde{\lambda}_{mk}(t_1) = k\lambda_m(t_1). \tag{4.32}$$

It is worth noting that relation (4.32) means that this model with a step stress is proportional for the mixture failure rates *only at the switching point* t_1.

We want to prove the following inequality:

$$\lambda_{mk}(t) < \tilde{\lambda}_{mk}(t); \quad \forall t \in [t_1, \infty). \tag{4.33}$$

In accordance with (4.31), consider two initial (for the interval $[t_1, \infty)$) mixing random variables: $Z_1 = [Z|T_1 > t_1]$, where T_1 is defined by the baseline failure rate $k\lambda(t)$ and $\tilde{Z}_1 = [Z|\tilde{T}_1 > t_1]$, where $\tilde{T}_1$ is defined by the baseline failure rate $\lambda(t)$. As follows from definition (1.3), the corresponding ratio

$$\frac{\tilde{\pi}(z, t_1)}{\pi(z, t_1)} = e^{(k-1)z\Lambda(t_1)}$$

increases in z for $k > 1$. Then inequality (4.33) follows immediately after taking into account the proof of Theorem 4.1 with obvious alterations caused by the change in the left end point of an interval from 0 to t_1.

Inequality (4.33) was graphically illustrated in Vaupel and Yashin (1985) (fig. 10) for a specific case of a discrete mixture of two subpopulations and the Gompertz baseline failure rate. The demographic meaning was the following: suppose we decrease mortality rates of subpopulations at early life ($[0, t_1)$). Then the observed mortality rate in $[t_1, \infty)$ is *larger* than the observed mortality rate for the initial mixture without changes. In other words: "early successes results in further failures" (Vaupel and Yashin (1985)).

4.2.3 Shocks in heterogeneous populations

In this section we consider another type of environmental impact - shocks. The shock models form an interesting and elaborated area in reliability mathematics. We shall look only at one specific model, but hopefully the approach can be generalized to a wider class of settings.

Consider now a general mixing model (1.2)-(1.3) and assume that at time $t = t_1$ an instantaneous shock had occurred, which affects the whole population: with the corresponding complementary probabilities it either kills an individual, or 'leaves him unchanged'. Without loosing generality let $t_1 = 0$, otherwise, as in previous sections, a new initial mixing variable $[Z|t_1]$ should be defined and the corresponding procedure can be easily adjusted to this case. It is natural to suppose that the more frail individuals or populations (with larger failure rate) are more susceptible to killing.

This setting can be defined probabilistically in a following way: let $\pi_1(z)$ denote a frailty pdf of a random variable Z_1 after a shock and let $\lambda_{ms}(t)$ be the corresponding observed (mixture) failure rate after it. Assume:

$$\pi_1(z) = \frac{g(z)\pi(z)}{\int_a^b g(z)\pi(z)dz}, \tag{4.34}$$

where $\pi(z)$ is a frailty pdf before a shock and $g(z)$ is a decreasing function and, therefore, $\pi_1(z)/\pi(z)$ is decreasing. It means that a shock performs a kind of a burn-in operation (Block et al, 1993) and random variables Z and Z_1 are ordered in the sense of the likelihood ratio (Ross, 1996; Shaked and Shanthikumar,1993):

$$Z \geq_{LR} Z_1. \tag{4.35}$$

Now we able to formulate the following result:

Theorem 4.8 *Let relation (4.34), defining a mixing density after a shock at $t = 0$, where $g(z)$ is a decreasing function, hold.*

Assume also that ordering (4.1) holds. Then:

$$\lambda_{ms}(t) < \lambda_m(t); \quad \forall t \in [0, \infty). \tag{4.36}$$

Proof Inequality (4.1) is a natural ordering in the family of failure rates $\lambda(t, z),\ z \in [0, \infty)$ and trivially holds for the specific multiplicative model

(1.15). Conducting all steps as when obtaining relation (4.25) we obtain that the sign of $\lambda_{ms}(t) - \lambda_m(t)$ is the same as the sign of

$$\iint_{\infty>u>s} \bar{F}(t,u)\bar{F}(t,s)(\lambda(t,u) - \lambda(t,s))(\pi_1(u)\pi(s) - \pi_1(s)\pi(u))duds,$$

which is negative due to definition (4.34) and assumptions of this theorem. $\square$

At $t = 0$, for instance:

$$\lambda_m(0) - \lambda_{ms}(0) = \int_0^\infty \lambda(0,z)(\pi(z) - \pi_1(z))dz.$$

In accordance with inequality (4.36), the curve $\lambda_{ms}(t)$ lies beneath the curve $\lambda_m(t)$ for $t \geq 0$, which means that the weakest populations are 'burned-out' by the shock. This fact seems intuitively evident, but, in fact, it is valid only due to rather stringent conditions of this theorem. It can be shown, for instance, that the replacement of condition (4.35) by a weaker one of stochastic dominance: $Z \geq_{st} Z_1$ will not guarantee ordering (4.36) for all t.

Conclusions

Populations are often heterogeneous in real life and the assumption of homogeneity usually simplifies the corresponding statistical analysis. The failure rate is a crucial characteristic of lifetime random variables, as it probabilistically defines the instantaneous hazard (risk) of failure (death) for survivors. A shape of the failure rate is an important feature, which among other things, is responsible for aging properties of lifetime distributions. If, for instance, the failure rate increases, then the corresponding distribution belongs to the IFR class and is usually appropriate for modelling lifetimes of wearing objects.

This thesis is devoted to a mixture failure rate modelling. Mixtures of distributions usually present an effective tool for modelling heterogeneity. It turns out that the shape of the mixture failure rate differs from the shape of the baseline failure rate and even the pattern can be surprisingly different. Under certain assumptions, for instance, the baseline IFR distribution can change to the DFR one after the operation of mixing.

The main emphasis of the study is on asymptotic properties of the mixture failure rate $\lambda_m(t)$ as $t \to \infty$. These properties are studied in chapters 2 and 3. We develop a new asymptotic approach, which allows for explicit asymptotic formulas for $\lambda_m(t)$. The suggested class of survival models is rather broad and includes the conventional proportional hazards, accelerated life and additive models. It is shown that asymptotic behavior of mixture failure rates under reasonable assumptions depends only on the behavior of the mixing distribution in the neighborhood of the left end point of its support and not on the whole mixing distribution. The approach is generalized to a multivariate (bivariate) case.

The behavior of $\lambda_m(t)$ in $[0, \infty)$ is also studied. Specifically, we develop a methodology for the mixture failure rates ordering for stochastically ordered mixing random variables. We show that the natural type of ordering for mixing models is ordering in the sense of likelihood ratio. It is proved, specifically, that when two frailties are ordered in this way, the corresponding mixture failure rates are naturally ordered as functions of time in $[0, \infty)$. A 'combination' of a frailty and a proportional hazards model is studied and the bounds for the mixture failure rate in this case are also obtained.

The obtained mathematical results are new and they are not extensions or generalizations of the previous results in the literature. Asymptotic analysis is based on the original approach, based on considering the corresponding Laplace integrals. Stochastic ordering of the mixture failure rates was not considered in the literature before.

We see a lot of engineering and biological applications of our results. Human and animal populations are heterogeneous and understanding the shape of the mortality rate is very important. For instance, we can explain the deceleration in human mortality for the oldest old under rather general assumptions without assuming the oversimplified Gamma-frailty model, as it was done before. Another example is the accelerated life model, which is widely used for modelling lifetimes of engineering objects. The mixture failure rate for this model was not studied before and our methodology can help, e.g., in the proper analysis of reliability and maintenance strategies in the presence of random factors.

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