Cohomologie à coefficients tordus de la réalisation géométrique d'un système de liaison

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THÈSE

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le 17 Juillet 2015

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Directeur de thèse : **Bob OLIVER**

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Pierre Rabhi

Contents

Introduction

The notion of *classifying space of a group* was introduced by Milnor in the 1950s. To each group G (discrete or topological), he associated a topological space BG. When we work with finite (and thus discrete) group, this space is just a CW complex BG with a contractible universal covering space and such that $\pi_1(BG) = G$ (i.e. an Eilenberg MacLane space $K(G,1)$). One canonical way to define it is as the geometric realization of the group category $\mathcal{B}(G)$ (see 2.4 for details). The application which assigns to G the space BG defines a functor $B : G \in \mathrm{Grp} \longrightarrow BG \in \mathrm{Top}$ and creates a link between group theory and homotopy theory. Milnor introduces these spaces to classify certain fiber bundles with a structure determined by the given group. But in the early 1980s, Lannes, Miller and Carlson proved the Sullivan conjecture which implies that BG has homotopy theoretic properties really rigid, and closely connected to the structure of G itself. For example, if G is a finite group, for every $\mathbb{Z}[G]$-module M, there is a natural isomorphism

$$H^*(G, M) \cong H^*(BG, M),$$

where the left term is the usual group cohomology of G and the right one is the cohomology of BG with twisted coefficients by the action of $\pi_1(BG) = G$ on M. Recall that, if a space X has a universal covering space $\widetilde{X}$, the cohomology of X with twisted coefficients in a $\mathbb{Z}[\pi_1(X)]$-module M is the cohomology of the chain complex

$$C^*(X; M) = \mathrm{Hom}_{\mathbb{Z}[\pi_1(X)]}(S_*(\widetilde{X}), M),$$

where $S_*(\widetilde{X})$ is the usual singular chain complex of $\widetilde{X}$. We refer the reader not familiar with algebraic topology to [Hatch], with group cohomology to [AM], [Br2], [Ev] or [Be2] and with general homological algebra to [We].

For p a prime number, we can be interested in the p-completion, $BG^\wedge_p$, of BG and how it is linked to the structure of G. The *p-completion* of a space X, defined by Bousfield and Kan in the 1970s, is a space which allows us to focus on the properties of X "at mod p" (see appendix 2 for more details). It turns out that the homotopy theoretic properties of $BG^\wedge_p$, for G a finite group, are closely linked to the *p-local structure* of G, i.e. the structure of S together with the conjugacy relations between its subgroups. For example, by Cartan-Eilenberg Theorem (Theorem 2.2.4), the cohomology of $BG_p^\wedge$ with coefficients in a trivial abelian p-group is completely determined by the p-local structure of G.

The notion of fusion and p-local structures in finite groups have been of interest for over a century and some results along these lines have been stated in the end of the 18th century by Burnside. Puig was the first, in the early 1970s (in [P1]), to consider the category $\mathcal{F}_S(G)$ to encode the p-local structure of G. The fusion system of a finite group G, denoted by $\mathcal{F}_S(G)$, for S a Sylow p-subgroup of G, is the category with objects the set of all the subgroups of S and the morphisms between two subgroups are given by conjugacy by an element of G. This category seems to be the good object to consider when we want to study the p-local structure of a finite group and many classical results on fusion in a finite group G can be interpreted as results about the fusion system $\mathcal{F}_S(G)$.

Recently, Oliver [O1, O2] proved the Martino-Priddy conjecture, which gives a refine of the Cartan-Eilenberg Theorem.

Theorem (Martino-Priddy-Oliver Theorem)**.** *Let p be a prime number and G_1, G_2 be two finite groups.*

If S_1 and S_2 are, respectively, Sylow p-subgroups of G_1 and G_2, then the following statements are equivalent,

(i) $\mathcal{F}_{S_1}(G_1) = \mathcal{F}_{S_2}(G_2)$,

(ii) $(BG_1)_p^\wedge \simeq (BG_2)_p^\wedge$.

While working on this conjecture, and also trying to understand the group of self homotopy equivalences of $BG_p^\wedge$, Broto, Levi and Oliver [BLO1] were led to investigate the *centric linking system* $\mathcal{L}_S^c(G)$ associated to a finite group G with S as Sylow p-subgroup. They discovered that the p-completed geometric realization $|\mathcal{L}_S^c(G)|_p^\wedge$ of this category has the homotopy type of $BG_p^\wedge$, and also that many of the homotopy properties of $BG_p^\wedge$ can be described in terms of properties of $\mathcal{L}_S^c(G)$.

The notion of fusion system of a finite group G over a Sylow p-subgroup S can be generalized by the notion of saturated fusion system by forgetting the group G and mimicking the Sylow Theorems in terms of morphisms between subgroups of S. This notion was first developed by Puig in the 1990s [P6]. A *saturated fusion system* $\mathcal{F}$ over a p-group S is the category whose objects are the subgroups of S, with the set $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ of morphisms from P to Q consisting of monomorphisms from P to Q, and such that some axioms are satisfied (see Definition 1.1.1 for more details). Of course, the standard example of a saturated fusion system is the category $\mathcal{F}_S(G)$, where G is a finite group and S is a Sylow p-subgroup of G.

Motivated by a work of Ron Solomon, Dave Benson (in [Be3] and unpublished work) predicted that there should be a way of associating classifying spaces to saturated fusion systems, which would generalize the association between $\mathcal{F}_S(G)$ and $BG_p^\wedge$. In that purpose, Broto, Levi and Oliver [BLO2] define the notion, more general, of *centric linking system* associated to a saturated fusion system over a p-group S. As an example, we can mention that, when we work with a finite group G and S a Sylow p-subgroup of G,

$\mathcal{L}_S^c(G)$ is a centric linking system associated to $\mathcal{F}_S(G)$. They also define the notion of *p-local finite group* as a triple $(S, \mathcal{F}, \mathcal{L})$ with S a p-group, $\mathcal{F}$ a saturated fusion system over S and $\mathcal{L}$ an associated centric linking system. Thus, a classifying space of a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ can be defined to be the space $|\mathcal{L}|_p^\wedge$. In particular, if G is a finite group and S a Sylow p-subgroup of G, $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ defines a p-local finite group and $|\mathcal{L}_S^c(G)|_p^\wedge \cong BG_p^\wedge$ is its classifying space. One problem with linking systems was the existence and uniqueness: for a given saturated fusion system, can we always find an associated centric linking system and, if it exists, is there a unique one? Recently, Chermak proved, using his theory of partial groups [Ch], that for any saturated fusion system, there is a unique associated centric linking system. Hence, for a saturated fusion system $\mathcal{F}$, we can define its classifying space $B\mathcal{F}$ as the space $|\mathcal{L}|_p^\wedge$ where $\mathcal{L}$ is the associated centric linking system.

The p-completed classifying space of a finite group have some very remarkable homotopy properties and classifying spaces of saturated fusion systems share many of these properties. Hence, for a p-local finite group $(S, \mathcal{F}, \mathcal{L})$, the homotopy type of $|\mathcal{L}|$ and its homotopy properties are strong invariants of $\mathcal{F}$.

One important result of Broto, Levi and Oliver ([BLO2], Theorem B) and of first interest in this thesis is the following.

Theorem (2.3.4)**.** *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group.*
The inclusion of BS in $|\mathcal{L}|$ induces a natural isomorphism

$$H^*(|\mathcal{L}|, \mathbb{F}_p) \xrightarrow{\cong} H^*(\mathcal{F}, \mathbb{F}_p).$$

Here, $H^*(\mathcal{F}, \mathbb{F}_p) \subseteq H^*(S, \mathbb{F}_p)$ is the submodule of $\mathcal{F}$*-stable* elements (Definition 2.3.3). This theorem is a version of the Cartan-Eilenberg Theorem in the case of a saturated fusion system and with trivial coefficients. In particular, if G is a finite group and if we consider the associated p-local finite group $(S, \mathcal{F}, \mathcal{L})$, it is just another way to state Cartan-Eilenberg Theorem with trivial coefficients.

In this work, for $(S, \mathcal{F}, \mathcal{L})$ a p-local finite group, we are interested in the cohomology of $|\mathcal{L}|$ with twisted coefficients in a $\mathbb{Z}_{(p)}[\pi_\mathcal{L}]$-module. More precisely, we want to generalize Cartan-Eilenberg Theorem and Theorem 2.3.4, trying to express the cohomology of $|\mathcal{L}|$ in terms of "stable" elements in the cohomology of the p-group S. This is a really interesting problem and it corresponds to the 7th open problem ask by Oliver in [AKO], Part III.

The first thing we have to fix is the notion of "stable" elements. Indeed, when we work with untwisted coefficients there is a cohomology functor $\mathcal{F} \longrightarrow \mathbb{Z}_{(p)}\text{-Mod}$ which allows us to define the notion of $\mathcal{F}$-stable element (Definition 2.3.3). But, in general, when we work with coefficients twisted by an action of $\pi_1(|\mathcal{L}|)$, there is not such a functor... However, it is possible to construct a cohomology functor from $\mathcal{L}$ which factor through $\mathcal{F}^c$, the $\mathcal{F}$-centric part of $\mathcal{F}$. Hence we can define the notion of $\mathcal{F}^c$-stable element (Definition 3.2.1) which extends naturally, by Alperin's Fusion Theorem (Theorem 1.1.9), the notion of $\mathcal{F}$-stable elements. Then, for M a $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$-module, if we denote by $H^*(\mathcal{F}^c, M) \subseteq H^*(S, M)$ the submodule of all $\mathcal{F}^c$*-stable elements*, we can ask if the inclusion of BS in

$|\mathcal{L}|$ induces a natural isomorphism,

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M) \ ?$$

Unfortunately several examples (see Chapter 7) show that this is impossible in general. Indeed, sometimes, the cohomology of S is trivial whereas the cohomology of $|\mathcal{L}|$ is not. Nevertheless, under some conditions, we can express the cohomology of $|\mathcal{L}|$ with twisted coefficients in terms of $\mathcal{F}^c$-stable elements in the cohomology of S.

We can first try to look at some restricted classes of fusion systems where we have a good control on the homotopy type of $|\mathcal{L}|$. If we look at constrained fusion systems, almost everything true in the trivial case can be extended, without too much modifications, to the twisted case. In particular we have the following.

Theorem (5.2.5)**.** *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and M a $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$-module.*

If $\mathcal{F}$ is a constrained fusion system, then the inclusion of BS in $|\mathcal{L}|$ induces a natural isomorphism

$$H^*(|\mathcal{L}|, M) \xrightarrow{\cong} H^*(\mathcal{F}^c, M).$$

The second way to attack the problem is by working on restrictions on the action of $\pi_1(|\mathcal{L}|)$. One first idea is to study nilpotent actions and try to use, by induction, the isomorphism given when the action is trivial (Theorem 2.3.4). For this purpose, we have to work on (S,S)-biset. Indeed, an important tool in the proof of Theorem 2.3.4 is the existence of an $\mathcal{F}$*-characteristic* (S,S)-biset which induces an idempotent on $H^*(S, \mathbb{F}_p)$ with image $H^*(\mathcal{F}, \mathbb{F}_p)$. Here we have to be more careful when working with $\mathcal{F}$-characteristic bisets. In Chapter 3, we look at the problem of constructing an idempotent from a $\mathcal{F}$-characteristic biset and assuming an hypothesis (Hypothesis (A)) we construct an idempotent of the δ-functor $\left(H^*(S,-), \delta_{H^*(S,-)}\right)$, with image containing $H^*(\mathcal{F}^c, M)$. From that we can deduce the following theorem.

Theorem (4.2.3)**.** *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and M an abelian p-group with an action of $\pi_1(|\mathcal{L}|)$.*

Assume that Hypothesis (A) is satisfied.

If the action of $\pi_1(|\mathcal{L}|)$ on M is nilpotent, then the inclusion of BS in $|\mathcal{L}|$ induces a natural isomorphism

$$H^*(|\mathcal{L}|, M) \xrightarrow{\cong} H^*(\mathcal{F}^c, M).$$

In particular, if the action of $\pi_1(|\mathcal{L}|)$ on M factors through a p-group, the action is nilpotent and then, we get a natural isomorphism between the cohomology of $|\mathcal{L}|$ and the $\mathcal{F}^c$-stable elements. The action of $\pi_1(|\mathcal{L}|)$ on M factors through a p-group if, and only if, the action factor through $\pi_1(|\mathcal{L}|_p^\wedge) \cong S/\mathfrak{hyp}(\mathcal{F})$, where

$$\mathfrak{hyp}(\mathcal{F}) = \langle g^{-1}\alpha(g) \ \mid \ g \in P \leq S, \alpha \in O^p\left(\mathrm{Aut}_{\mathcal{F}}(P)\right)\rangle \trianglelefteq S$$

is the *hyperfocal subgroup of* $\mathcal{F}$. As a corollary, we can look at the cohomology of $B\mathcal{F} = |\mathcal{L}|_p^\wedge$.

Theorem (4.2.4)**.** *Let $\mathcal{F}$ be a saturated fusion system and M an abelian p-group with an action of $\pi_1(B\mathcal{F})$.*

If Hypothesis (A) is satisfied, then there is a natural isomorphism,

$$H^*(B\mathcal{F}, M) \cong H^*(\mathcal{F}^c, M).$$

Some inductions can also be made using a p-local subgroup which induces a covering space on geometric realizations. For example, by studying p-local subgroups of index prime to p, we get the following theorem.

Theorem (5.1.4)**.** *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and M an abelian p-group with an action of $\pi_1(|\mathcal{L}|)$.*

If the action of $\pi_1(|\mathcal{L}|)$ on M factors through a p'-group, then the inclusion of BS in $|\mathcal{L}|$ induces a natural isomorphism

$$H^*(|\mathcal{L}|, M) \xrightarrow{\cong} H^*(\mathcal{F}^c, M).$$

By Theorem 1.3.5, we also know that p-local subgroups of index a power of p induces covering spaces on the geometric realizations. Unfortunately it is more difficult to work with them. Indeed, as we work on fusion systems over different p-groups, it is more difficult to compare the $\mathcal{F}^c$-stable elements. However, working with realizable p-local finite groups, we get the following

Theorem (5.2.7)**.** *Let G be a finite group and $(S, \mathcal{F}, \mathcal{L})$ be the associated p-local finite group. Let M be a $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$-module.*

If there is a "natural" action of G on M, then there is a natural ismorphism,

$$H^*(|\mathcal{L}|, M) \cong H^*(G, M).$$

We also recall some conditions, due to Grodal [Gr], under which the cohomology of G is isomorphic to the $\mathcal{F}^c$-stable elements. All these Theorems might be generalized and we can conjecture the following.

Conjecture (5.2.9)**.** Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and M a $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$-module. If the action of $\pi_1(|\mathcal{L}|)$ on M is p-solvable, then the inclusion of BS in $|\mathcal{L}|$ induces a natural isomorphism

$$H^*(|\mathcal{L}|, M) \xrightarrow{\cong} H^*(\mathcal{F}^c, M).$$

We finally work, in Chapter 6, on some constructions of p-local finite groups or linking systems with particular forms to give some tools when working on concrete examples. We start with products of p-local finite groups, and using the Kunneth formula we get the following.

Proposition (6.1.5)**.** *Let, for $i \in \{1, 2\}$, $(S_i, \mathcal{F}_i, \mathcal{L}_i)$ be two p-local finite groups and $(S, \mathcal{F}, \mathcal{L})$ the p-local finite group product of $(S_1, \mathcal{F}_1, \mathcal{L}_1)$ and $(S_2, \mathcal{F}_2, \mathcal{L}_2)$.*

Let also M_1 be a $\mathbb{F}_p[\pi_1(|\mathcal{L}_1|)]$-module and M_2 be a $\mathbb{F}_p[\pi_1(|\mathcal{L}_2|)]$-module.

If, for $i \in \{1, 2\}$, δ_{S_i} induces a natural isomorphism $H^*(|\mathcal{L}_i|, M_i) \cong H^*(\mathcal{F}_i^c, M_i)$ then the inclusion of BS in $|\mathcal{L}|$ induces a natural isomorphism

$$H^*(|\mathcal{L}|, M_1 \otimes_{\mathbb{F}_p} M_2) \cong H^*(\mathcal{F}^c, M_1 \otimes_{\mathbb{F}_p} M_2).$$

Secondly, we work on linking systems that we can decompose as the union of two other linking systems which might be more easy to study and we give some criteria (Definition 6.2.1) under which, if we can compute the cohomology of the geometric realizations of the two parts by stable elements, then the initial one can also be compute by stable elements.

Proposition (6.2.5)**.** *Let $(S, \mathcal{F}, \mathcal{L})$ and, for $i = \{1, 2\}$, $(S, \mathcal{F}_i, \mathcal{L}_i)$ be three p-local finite groups on the same p-group S. Let M be a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module.*

Assume that $\{\mathcal{L}_1, \mathcal{L}_2\}$ is an M-cohomological covering of $\mathcal{L}$.

If, for $i \in \{1, 2\}$, we have natural isomorphisms $H^*(|\mathcal{L}_i|, M) \cong H^*(\mathcal{F}_i^c, M)$, then we have a natural isomorphism

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

We apply this machinery to the 2-local finite group associated to $P\Gamma L_3(\mathbb{F}_4)$ and give an example of isomorphism between the cohomology of $|\mathcal{L}|$ and the $\mathcal{F}^c$-stable elements when the action is twisted by a 2-solvable.

We finally study some realizable p-local finite group given by a wreath products. We give informations on the $\mathcal{F}$-essentials subgroups and use it to study the 5-local subgroup associated to $GL_{20}(\mathbb{F}_2) \wr C_5$. We give an example of isomorphism between the cohomology of $|\mathcal{L}|$ and the $\mathcal{F}^c$-stable elements when the action is twisted by a 5-solvable but where there is no natural action of $GL_{20}(\mathbb{F}_2) \wr C_5$ on the coefficients.

Notation

Let G be a finite group.

For $g \in G$, let $c_g : G \longrightarrow G$ be the *conjugation by* g, defined by, for all $x \in G$, $c_g(x) = gxg^{-1}$. For $X \subseteq G$, let $N_G(X) = \{g \in G \mid c_g(X) \subseteq X\}$ be the *normalizer* in G of X, and $C_G(X) = \{g \in G \mid c_g|_X = \mathrm{Id}_X\}$ be the *centralizer* in G of X . For $X, Y \subseteq G$ we denote by $T_G(X, Y) = \{g \in G \mid c_g(X) \subseteq Y\}$ the *transporter* in G from X to Y. We write $\langle X \rangle$ for the subgroup of G generated by X.

For H and G two groups, we write $H \leq G$, $H < G$ or $H \trianglelefteq G$ to indicate that H is a subgroup, proper subgroup or a normal subgroup of G, respectively. Let $\mathrm{Hom}(G, H)$ be the set of all homomorphisms of groups from G to H, $\mathrm{Inj}(G, H) \subseteq \mathrm{Hom}(G, H)$ the set of all injective homomorphisms and $\mathrm{Aut}(G)$ the group of automorphisms of G. Observe that, for $H \leq G$, $c : g \longmapsto c_g$ is a homomorphism from $N_G(H)$ to $\mathrm{Aut}(H)$ with kernel $C_G(H)$; we write $\mathrm{Aut}_G(H)$ its image and call it the *automizer* in G of H and thus, $\mathrm{Aut}_G(H) \cong N_G(H)/C_G(H)$. The *inner automorphism group* of H is $\mathrm{Inn}(H) = \mathrm{Aut}_H(H)$ and the *outer automorphism group* of H is $\mathrm{Out}(H) = \mathrm{Aut}(H)/\mathrm{Inn}(H)$.

For G a finite group, we write $|G|$ for its order. For p a prime number, we say that G is a p-group if $|G|$ is a power of p and a p'-group if $|G|$ is prime to p. We denote by $O_p(G)$ or $O_{p'}(G)$ the largest normal subgroup of G which is a p-group or a p'-group, respectively. We also denote by $O^p(G)$ or $O^{p'}(G)$ the smallest normal subgroup of G of index a power of p or prime to p, respectively. If we consider a $\mathbb{Z}_{(p)}[G]$-module M, we will denote by M^G the submodule of M,

$$M^G = \{x \in M \mid gx = x\}$$

(which is sometimes also denoted by $C_M(G)$ in the literature).

For specific groups, C_n will denote the cyclic group of order n, D_{2n} the dihedral group of order 2n and $A_n \trianglelefteq S_n$ will denote the alternating and symmetric group on n letters.

For p a prime number and q a power of p, we write, $\mathbb{F}_q$ the field of characteristic p with q elements and $\mathbb{Z}_{(p)}$ the ring $\mathbb{Z}$ localized at the prime ideal (p). We will also denote by $GL_n(\mathbb{F}_q)$, $SL_n(\mathbb{F}_q)$, $PGL_n(\mathbb{F}_q)$, $PSL_n(\mathbb{F}_q)$ the general linear, special linear, projective general linear and projective special linear group of $\mathbb{F}_q^n$.

Let $\mathcal{C}$ be a category. We write $\mathrm{Ob}(\mathcal{C})$ for the set of objects, $\mathrm{Mor}(\mathcal{C})$ for the set of all morphisms. If $x, y \in \mathrm{Ob}(\mathcal{C})$, we denote by $\mathrm{Mor}_{\mathcal{C}}(x, y)$ the set of all morphisms from x to y and $\mathrm{Iso}_{\mathcal{C}}(x, y)$ the set of all isomorphisms between x and y. If $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is a functor and $x, y \in \mathrm{Ob}(\mathcal{C}_1)$, we denote by $F_{x,y} : \mathrm{Mor}_{\mathcal{C}_1}(x, y) \longrightarrow \mathrm{Mor}_{\mathcal{C}_2}(F(x), F(y))$ the application induced by F and $F_x = F_{x,x}$.

For X, Y two spaces, we write $X \cong Y$ or $X \simeq Y$ to indicate that X is homeomorphic to Y or has the homotopy type of Y, respectively. We also denote by $[X, Y]$ the set of homotopy classes of maps from X to Y. For a space X, we write $\pi_1(X)$ the fundamental group of X, i.e. the set of homotopy classes of pointed loops with concatenation as composition law.

Finally, for $\mathcal{C}$ a small category, we denote by $\mathcal{N}(\mathcal{C})$ its nerve and $|\mathcal{C}| = |\mathcal{N}(\mathcal{C})|$ its geometric realization (see 2 for more details).

1 Fusion systems and p-local finite groups

In this chapter, we recall the definitions of fusion system and linking system, and some background in homotopy theory of fusion systems. We refer the reader more interested in fusion systems or homotopy theory of fusion systems to [AKO] or [Cr].

1.1 Fusion systems

A fusion system over a p-group S is a way to abstract the action of a finite group $G \geq S$ on the subgroups of S by conjugation.

Definition 1.1.1. Let S be a finite p-group.

A *fusion system* over S is a small category $\mathcal{F}$, where $\mathrm{Ob}(\mathcal{F})$ is the set of all subgroups of S and which satisfies the following two properties for all $P, Q \leq S$:

(a) $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Mor}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$;

(b) each $\varphi \in \mathrm{Mor}_{\mathcal{F}}(P,Q)$ is the composite of an $\mathcal{F}$-isomorphism followed by an inclusion.

The composition in a fusion system is given by composition of homomorphisms. We usually write $\mathrm{Hom}_{\mathcal{F}}(P,Q) = \mathrm{Mor}_{\mathcal{F}}(P,Q)$ to emphasize that the morphims in $\mathcal{F}$ are homomorphisms.

The typical example of fusion system is the fusion system of a finite group G.

Example 1.1.2. Let G be a finite group and S a p-subgroup of G.

The *fusion system of G over S* is the category $\mathcal{F}_S(G)$ where $\mathrm{Ob}(\mathcal{F}_S(G))$ is the set of all subgroups of S and for all $P, Q \leq S$, $\mathrm{Mor}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q)$.

The category $\mathcal{F}_S(G)$ defines a fusion system over S.

In general, it is more convenient to work with fusion system when S is a Sylow p-subgroup of G or more generally, when the fusion system is saturated. For that purpose, we will try to mimic Sylow Theorems in terms of the category $\mathcal{F}_S(G)$. There are several definitions of saturation. Here, we only give one due to Roberts and Shpectorov and the reader interested in other equivalent definitions can find more, for example, in [AKO].

Definition 1.1.3. Let S be a p-group and $\mathcal{F}$ a fusion system over S.

(a) Two subgroups $P, Q \leq S$ are *$\mathcal{F}$-conjugate* if they are isomorphic as objects in $\mathcal{F}$. We denote by $P^{\mathcal{F}}$ the set of all subgroups of S $\mathcal{F}$-conjugate to P.

(b) A subgroup $P \leq S$ is *fully automized* if $\mathrm{Aut}_S(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.

(c) A subgroup $P \leq S$ is *receptive* in $\mathcal{F}$ if it has the following property: for each $Q \leq S$ and $\varphi \in \mathrm{Iso}_{\mathcal{F}}(Q, P)$, if we set

$$N_\varphi = \left\{ g \in N_S(Q) \mid \varphi \circ c_g \circ \varphi^{-1} \in \mathrm{Aut}_S(P) \right\},$$

then there is $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\overline{\varphi}|_Q = \varphi$.

A fusion system $\mathcal{F}$ over a p-group S is *saturated* if each subgroup of S is $\mathcal{F}$-conjugate to a subgroup which is fully automized and receptive.

The case of the fusion system of a finite group over one of its Sylow p-subgroup is a particular case of a saturated fusion system.

Proposition 1.1.4 (Puig). *Let G be a finite group.*
If S is a Sylow p-subgroup of G, then the category $\mathcal{F}_S(G)$ is a saturated fusion system.

A fusion system $\mathcal{F}$ over a p-group S is called *realizable* if there is a finite group G such that S is a Sylow p-subgroup of G and $\mathcal{F} = \mathcal{F}_S(G)$, and *exotic* otherwise. We know examples of exotic fusion systems (see [AKO], Section III.6, for some examples) but we do not know a lot about them and especially how they appear.

Let also distinguish other particular subgroups of S which play an important role.

Definition 1.1.5. Let S be a subgroup and $\mathcal{F}$ a saturated fusion system over S.

(a) A subgroup $P \leq S$ is *fully centralized* in $\mathcal{F}$ if, for all $Q \in P^{\mathcal{F}}$, $|C_S(P)| \geq |C_S(Q)|$.

(b) A subgroup $P \leq S$ is *fully normalized* in $\mathcal{F}$ if, for all $Q \in P^{\mathcal{F}}$, $|N_S(P)| \geq |N_S(Q)|$.

If $\mathcal{F}$ is realizable by a finite group G, then a subgroup $P \leq S$ is fully centralized (resp. fully normalized) if, and only if, $C_S(P)$ (resp. $N_S(P)$) is a Sylow p-subgroup of $C_G(P)$ (resp. $N_G(P)$).

If $\mathcal{F}$ is a fusion system over a p-group S, for each $P \leq S$, we write $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$ and $\mathrm{Out}_S(P) = \mathrm{Aut}_S(P)/\mathrm{Inn}(P)$.

Definition 1.1.6. Let $\mathcal{F}$ be a saturated fusion system over a p-group S.

(a) A subgroup $P \leq S$ is *$\mathcal{F}$-centric* if $C_S(Q) = Z(Q)$ for every $Q \in P^{\mathcal{F}}$.

(b) A subgroup $P \leq S$ is *$\mathcal{F}$-radical* if $O_p(\mathrm{Out}_{\mathcal{F}}(P)) = 1$.

We will denote by $\mathcal{F}^c$, $\mathcal{F}^r$ and $\mathcal{F}^{cr}$ the full subcategory of $\mathcal{F}$ whose objects are the $\mathcal{F}$-centric subgroups, $\mathcal{F}$-radical subgroups and $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroups of S respectively.

If $\mathcal{F}$ is realizable by a finite group G, then a subgroup $P \leq S$ is $\mathcal{F}_S(G)$-centric if, and only if, P is *p-centric*, i.e. $Z(P)$ is a Sylow p-subgroup of $C_G(P)$. The notion of $\mathcal{F}$-radical is not the same as being a radical p-subgroup. A p-subgroup is a *radical p-subgroup of* G if $O_p(N_G(P)/P) = 1$ while, P is $\mathcal{F}_S(G)$-radical if $O_p\left(N_G(P)/PC_G(P)\right) = 1$. The two definitions are closed but there is no implication in between. However, if P is $\mathcal{F}_S(G)$-centric and $\mathcal{F}_S(G)$-radical, then P is a radical p-subgroup of G.

Definition 1.1.7. Let $\mathcal{F}$ be a saturated fusion system over a p-group S.

A subgroup $P \leq S$ is $\mathcal{F}$*-essential* if P is $\mathcal{F}$-centric and fully normalized in $\mathcal{F}$, and if $\mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly p-embedded subgroup.

For a finite group G, a subgroup $H < G$ is *strongly p-embedded*, if $p \mid |H|$ and for each $x \in G \setminus H$, $H \cap xHx^{-1}$ has order prime to p.

The following proposition describes the properties of these $\mathcal{F}$-essential subgroups.

Proposition 1.1.8 ([AKO], Proposition I.3.3)**.** *Let $\mathcal{F}$ be a saturated fusion system over a p-group S.*

(a) Each $\mathcal{F}$-essential subgroup of S is $\mathcal{F}$-centric, $\mathcal{F}$-radical and fully normalized in $\mathcal{F}$.

(b) Let P be a fully normalized proper subgroup of S and let $H_P \leq Aut_{\mathcal{F}}(P)$ be the subgroup generated by those $\alpha \in Aut_{\mathcal{F}}(P)$ which extend to $\mathcal{F}$-isomorphisms between strictly larger subgroups of S. Then, either P is not $\mathcal{F}$-essential and $H_P = Aut_{\mathcal{F}}(P)$; or P is $\mathcal{F}$-essential and $H_P/Inn(P)$ is strongly p-embedded in $Out_{\mathcal{F}}(P)$.

These essential subgroups are important because their automorphisms generate the whole fusion system in a precise sens.

For a p-group S, there is a *universal* fusion system $\mathcal{U}$ over S which contains all other fusion systems over S: for each $P, Q \leq S$, $\mathrm{Hom}_{\mathcal{U}}(P,Q) = \mathrm{Inj}(P,Q)$. Also, the intersection of two fusion systems over S is again a fusion system over S. We can then define, for Ψ a family of monomorphisms between subgroups of S and/or fusion systems over subgroups of S, the fusion system generated by Ψ, denoted $\langle\Psi\rangle_S$ (or $\langle\Psi\rangle$ when there is no confusion on the p-group we consider), as the smallest fusion system over S (not necessarily saturated!) which contains Ψ. Thus, $\langle\Psi\rangle_S$ is the intersection of all the fusion systems over S which contain Ψ and the morphisms in $\langle\Psi\rangle$ are the composites of restrictions of homomorphisms in the set $\Psi \cup \mathrm{Inn}(S)$ and their inverses.

Theorem 1.1.9 (Alperin's Fusion Theorem)**.** *Let $\mathcal{F}$ be a saturated fusion system over a p-group S.*

Then,

$$\mathcal{F} = \langle Aut_{\mathcal{F}}(P) \mid P = S \text{ or } P \text{ is } \mathcal{F}\text{-essential}\rangle_S.$$

1.2 Linking systems

For a finite group G and a subgroup $H \leq G$, we will denote by $\mathcal{T}_H(G)$ the *transporter category* of G over H which is the small category with set of objects the set of all subgroups of H and for all $H_1, H_2 \leq G$, $\mathrm{Mor}_{\mathcal{T}_G(H)}(H_1, H_2) = T_G(H_1, H_2)$. If we just want to consider a family $\mathcal{H}$ of subgroups of H, we will denote by $\mathcal{T}_H^{\mathcal{H}}(G)$ the full subcategory of $\mathcal{T}_H(G)$ with set of objects $\mathcal{H}$. Here, we will consider the case where G is a finite group and $H = S$ is a Sylow p-subgroup of G. We will also sometimes restrict our attention on the full subcategory $\mathcal{T}_S^c(G) = \mathcal{T}_S^{\mathcal{H}}(G)$ with $\mathcal{H} = \mathrm{Ob}(\mathcal{F}^c)$ or $\mathcal{T}_S^{cr}(G) = \mathcal{T}_S^{\mathcal{H}}(G)$ with $\mathcal{H} = \mathrm{Ob}(\mathcal{F}^{cr})$.

These transporter categories are useful in the study of the p-local structure of a finite group G and we can mention that Oliver and Ventura [OV1] extend them to the notion of *transporter systems* to study extension of p-local finite groups by a p-group. Nevertheless, the structure of $\mathcal{T}_S(G)$ is too linked to G (for example, we can show that $|\mathcal{T}_S(G)| = BG$) and even if we restrict our attention on the centric subgroups, two groups G_1 and G_2 with a same Sylow p-subgroup S and the same fusion system over S can have different centric transporter categories (you can for example take G_1 such that $O_{p'}(G_1) \neq 0$ and $G_2 = G_1/O_{p'}(G_1)$). The good object to consider is the centric linking system!

Definition 1.2.1. Let $\mathcal{F}$ be a fusion system over a p-group S.

A *linking system* associated to $\mathcal{F}$ is a finite category $\mathcal{L}$ together with a pair of functors

$$\mathcal{T}_S^{\mathrm{Ob}(\mathcal{L})}(S) \xrightarrow{\delta} \mathcal{L} \xrightarrow{\pi} \mathcal{F}$$

satisfying the following conditions:

(A1) $\mathrm{Ob}(\mathcal{L})$ is a set of subgroups of S closed under $\mathcal{F}$-conjugacy and overgroups, and contains $\mathrm{Ob}(\mathcal{F}^{rc})$. Each object in $\mathcal{L}$ is isomorphic (in $\mathcal{L}$) to one which is fully centralized.

(A2) δ is the identity on objects, and π is the inclusion on objects. For each $P, Q \in \mathrm{Ob}(\mathcal{L})$ such that P is fully centralized in $\mathcal{F}$, $C_S(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ via $\delta_{P,P}$ and right composition, and

$$\pi_{P,Q} : \mathrm{Mor}_{\mathcal{L}}(P, Q) \longrightarrow \mathrm{Hom}_{\mathcal{F}}(P, Q)$$

is the orbit map for this action.

(B) For each $P, Q \in \mathrm{Ob}(\mathcal{L})$ and each $g \in T_S(P, Q)$, the application $\pi_{P,Q}$ sends $\delta_{P,Q}(g) \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ to $c_g \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$.

(C) For each $P, Q \in \mathrm{Ob}(\mathcal{L})$, all $\psi \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ and all $g \in P$, the diagram

$$\begin{tikzcd} P \arrow[r, "\psi"] \arrow[d, "\delta_P(g)"'] & Q \arrow[d, "\delta_Q(\pi(\psi)(g))"] \\ P \arrow[r, "\psi"] & Q \end{tikzcd}$$

commutes in $\mathcal{L}$.

A *centric linking system* associated to a fusion system $\mathcal{F}$ is a linking system associated to $\mathcal{F}$ such that $\mathrm{Ob}(\mathcal{L}) = \mathrm{Ob}(\mathcal{F}^c)$.

The example of linking system we should have in mind is the centric linking system of a finite group, defined as follows.

Example 1.2.2. Let G be a finite group and S a p-subgroup of G.

The *centric linking system of G over S* is the category $\mathcal{L}_S^c(G)$ where $\mathrm{Ob}(\mathcal{L}_S^c(G))$ is the set of all p-centric subgroups of S and for all $P, Q \leq S$,

$$\mathrm{Mor}_{\mathcal{F}_S(G)}(P,Q) = T_G(P,Q)/O^p(C_G(P))$$

(remark that, as P is p-centric, $O^p(C_G(P))$ has order prime to p).

The category $\mathcal{L}_S^c(G)$, with the obvious functors π and δ, defines a centric linking system associated to $\mathcal{F}_S(G)$.

The following proposition gives some basic properties of linking systems.

Proposition 1.2.3 ([O4], Proposition 4)**.** *Let $\mathcal{F}$ be a saturated fusion system over a p-group S and $\mathcal{L}$ be an associated linking system.*

(a) For each $P, Q \in Ob(\mathcal{L})$, the subgroup

$$E(P) = Ker\left(\pi_P : Aut_{\mathcal{L}}(P) \longrightarrow Aut_{\mathcal{F}}(P) \right)$$

acts freely on $Mor_{\mathcal{L}}(P,Q)$ via right composition and $\pi_{P,Q}$ induces a bijection

$$Mor_{\mathcal{L}}(P,Q)/E(P) \xrightarrow{\cong} Hom_{\mathcal{F}}(P,Q)\ .$$

(b) A morphism $\psi \in Mor(\mathcal{L})$ is an isomorphism if and only if $\pi(\psi)$ is an isomorphism in $\mathcal{F}$.

(c) If $P \in Ob(\mathcal{L})$ is fully normalized in $\mathcal{F}$, then $\delta_P(N_S(P))$ is a Sylow p-subgroup of $Aut_{\mathcal{L}}(P)$.

(d) All morphisms in $\mathcal{L}$ are monomorphisms and epimorphisms in the categorical sense.

When $P \leq Q$ are objects in a linking system $\mathcal{L}$, the morphism $\delta_{P,Q}(1) \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ can be seen as an *inclusion morphism*. This terminology is motivated by axiom (B), because $\pi_{P,Q}(\delta_{P,Q}(1)) = \mathrm{incl}_P^Q \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$.

Definition 1.2.4. Let S be a p-group, $\mathcal{F}$ a saturated fusion system over $\mathcal{F}$ and $\mathcal{L}$ a linking system associated. A *compatible set of inclusions* for $\mathcal{L}$ is a choice of morphisms $\iota_P^Q \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$, one for each pair $P \leq Q$ of objects of $\mathcal{L}$, such that $\iota_S^S = \mathrm{Id}_S$, and the following holds for all $P \leq Q \leq R$,

1. $\pi(\iota_P^Q)$ is the inclusion $P \leq Q$;
2. $\iota_Q^R \circ \iota_P^Q = \iota_P^R$.

We often write $\iota_P = \iota_P^S$. The existence of a compatible set of inclusions for $\mathcal{L}$ is proved in [5a2], Proposition 1.13, but an easy example is given by, for each pair $P \leq Q$ of objects in $\mathcal{L}$,

$$\iota_P^Q = \delta_P^Q(1).$$

In this thesis, we will fix a compatible set of inclusion $\left(\iota_P^Q\right)_{P\leq Q}$.

Once inclusions have been defined, we can consider restrictions and extensions of morphisms in $\mathcal{L}$ and since morphisms in a linking system are monomorphisms and epimorphisms, restrictions and extensions are unique whenever they exist. The following proposition describes the conditions under which they do exist.

Proposition 1.2.5 ([O4], Proposition 4(b,e))**.** *Let $\mathcal{F}$ be a saturated fusion system over a p-group S and $\mathcal{L}$ be an associated linking system.*

(a) For every $P,Q \in Ob(\mathcal{L})$, $\psi \in Mor_{\mathcal{L}}(P,Q)$ and $P_0, Q_0 \in Ob(\mathcal{L})$ such that, $P_0 \leq P$, $Q_0 \leq Q$ and $\pi(\psi)(P_0) \leq Q_0$, there exists a unique morphism, $\psi|_{P_0}^{Q_0}$ such that $\psi \circ \iota_{P_0}^P = \iota_{Q_0}^Q \circ \psi|_{P_0}^{Q_0}$.

(b) Let $P, Q, \overline{P}, \overline{Q} \in Ob(\mathcal{L})$ and $\psi \in Mor_{\mathcal{L}}(P,Q)$ be such that $P \trianglelefteq \overline{P}$, $Q \leq \overline{Q}$, and for each $g \in \overline{P}$, there is an $h \in \overline{Q}$ such that $\iota_Q^{\overline{Q}} \circ \psi \circ \delta_P(g) = \delta_{Q,\overline{Q}}(h) \circ \psi$. Then there is a unique morphism $\overline{\psi} \in Mor_{\mathcal{L}}(\overline{P}, \overline{Q})$ such that $\overline{\psi}|_P^Q = \psi$.

Usually, when we only study one fusion system, the most convenient linking system to work with is the centric linking system. But when we work with two different fusion systems, or with fusion subsystems, it is convenient to work with linking systems with an adapted set of objects. Hence we have to determine what are the possible sets of subgroups of S. For that we have to define the notion of $\mathcal{F}$-quasicentric subgroup.

Definition 1.2.6. Let $\mathcal{F}$ be a saturated fusion system over a p-group S.

A subgroup $P \leq S$ is *$\mathcal{F}$-quasicentric* if for each $Q \leq PC_S(P)$ containing P, and each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Q)$ such that $\alpha|_P = \mathrm{Id}$, α has a p-power order.

We will denote by $\mathcal{F}^q$ the full subcategory of $\mathcal{F}$ with set of objects all the $\mathcal{F}$-quasicentric subgroups of S.

If G is a finite group and S a Sylow p-subgroup of G, a subgroup $P \leq S$ is $\mathcal{F}_S(G)$-quasicentric if, and only if, P is a *p-quasicentric* subgroup of G, i.e. $O^p(C_G(P))$ has order prime to p.

We are now ready to describe the possible sets of subgroups of S we can choose to get a well-defined linking system.

Proposition 1.2.7. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S.*

Let $\mathcal{L}$ be a linking system associated to $\mathcal{F}$.

We have $Ob(\mathcal{F}^{cr}) \subseteq Ob(\mathcal{L}) \subseteq Ob(\mathcal{F}^q)$, and there exists a linking system $\mathcal{L}^q$ associated to $\mathcal{F}$ such that $Ob(\mathcal{L}) = Ob(\mathcal{F}^q)$, and $\mathcal{L}$ is a full subcategory of $\mathcal{L}^q$.

Moreover, for every subset $Ob(\mathcal{F}^{cr}) \subseteq \mathcal{H} \subseteq Ob(\mathcal{F}^q)$ stable by $\mathcal{F}$-conjugacy and overgroups, the full subcategory $\mathcal{L}^{\mathcal{H}}$ of $\mathcal{L}^q$ with set of objects $\mathcal{H}$ is also a linking system associated to $\mathcal{F}$.

Proof. The first point can be found for example in [O4], Proposition 4(g). For the second one, you can find a proof in [AKO], Proposition III.4.8. Finally, the last statement is clear from the axioms of linking system. ✩

A *quasicentric linking system* associated to $\mathcal{F}$ is a linking system $\mathcal{L}$ with $\mathrm{Ob}(\mathcal{L}) = \mathrm{Ob}(\mathcal{F}^q)$.

If G is a finite group and S a Sylow p-subgroup of G, we can define the *quasicentric linking system of* G as the category $\mathcal{L}_S^q(G)$ with set of objects all the p-quasicentric subgroups of S and for all $P, Q \in \mathrm{Ob}\,(\mathcal{L}_S^q(G))$,

$$\mathrm{Mor}_{\mathcal{L}_S^q(G)}(P,Q) = T_G(P,Q)/O^p(C_G(P)).$$

This category, with the obvious functors π and δ, defines a quasicentric linking system associated to $\mathcal{F}_S(G)$ and $\mathcal{L}_S^c(G)$ is a full subcategory of $\mathcal{L}_S^q(G)$.

We finish with the problem of existence and uniqueness of a linking system for a given saturated fusion system over a p-group S and a set $\mathrm{Ob}(\mathcal{F}^{cr}) \subseteq \mathcal{H} \subseteq \mathrm{Ob}(\mathcal{F}^q)$ of subgroups of S stable by $\mathcal{F}$ conjugacy and overgroups. This is a really difficult problem but it have been solved by Andrew Chermak [Ch] using the theory of partial groups (an interpretation by Bob Oliver in terms of obstruction theory is given in [O5]) and the answer is positive! Here we will always explicitly give the linking system and work with a p-local finite group.

Definition 1.2.8. A *p-local finite group* is defined to be a triple $(S, \mathcal{F}, \mathcal{L})$ where S is a p-group, $\mathcal{F}$ a saturated fusion system over S, and $\mathcal{L}$ a linking system associated to $\mathcal{F}$.

If S_0 is a subgroup of S, $\mathcal{F}_0$ a saturated subsystem of $\mathcal{F}$ and $\mathcal{L}_0$ a linking system associated to $\mathcal{F}_0$, the p-local finite group $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ will be called a *p-local subgroup* of $(S, \mathcal{F}, \mathcal{L})$ (**even if** $\mathcal{L}_0 \nsubseteq \mathcal{L}$).

We will always, or it will be stated explicitly, work with a centric linking system.

1.3 Homotopy properties of linking systems

When we work in homotopy theory and with classifying spaces of finite groups, we can be interested in the p-completion of them. If you take a finite group G, its classifying space BG is a space which is strongly linked to the structure of G. This link can be highlighted by p-local structure through the Martino-Priddy conjecture. The proof uses the notion of linking systems of groups and when we work with abstract saturated fusion systems, linking systems allow us to define classifying spaces and make a link between the theory of fusion systems and homotopy theory.

From now on, for a p-local finite group $(S, \mathcal{F}, \mathcal{L})$, π and δ will denote the structural functors

$$\mathcal{T}_S^c(S) \xrightarrow{\ \delta\ } \mathcal{L} \xrightarrow{\ \pi\ } \mathcal{F}.$$

We also write $\pi_{\mathcal{L}} = \pi_1(|\mathcal{L}|, S)$, and $\omega : \mathcal{L} \longrightarrow \mathcal{B}(\pi_{\mathcal{L}})$ denotes the functor which sends each objects of $\mathcal{L}$ to the unique one in $\mathcal{B}(\pi_{\mathcal{L}})$ and, for $P, Q \leq S$, sends $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ to the path $\iota_Q.\varphi.\overline{\iota_P}$ where $\overline{\iota_P}$ is the path ι_P in the other direction (see appendix 2 for more details).

1.3.1 Fundamental groups and covering spaces

Some constructions made with fusion and linking systems can be explained in terms of fundamental group and covering spaces of their geometric realizations.

If $\mathcal{F}$ is a saturated fusion system over a p-group S, an important subgroup of S is the *hyperfocal subgroup* of $\mathcal{F}$:

$$\mathfrak{hyp}(\mathcal{F}) = \langle g^{-1}\alpha(g) \ \mid \ g \in P \leq S, \alpha \in O^p\left(\mathrm{Aut}_{\mathcal{F}}(P)\right)\rangle \trianglelefteq S.$$

The space $|\mathcal{L}|$ is a p-good space and the fundamental group of its p-completion can be computed as follows.

Theorem 1.3.1 ([AKO], Theorem III.4.17)**.** *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group.*
The space $|\mathcal{L}|$ is p-good and the composite

$$\Psi : S \xrightarrow{\delta_S} Aut_{\mathcal{L}}(S) \xrightarrow{\omega_S} \pi_{\mathcal{L}} \xrightarrow{(\lambda_{|\mathcal{L}|})_*} \pi_1(|\mathcal{L}|^\wedge_p)$$

is surjective with $Ker(\Psi) = \mathfrak{hyp}(\mathcal{F})$. Thus Ψ induces an isomorphism

$$\pi_1(|\mathcal{L}|^\wedge_p) \cong S/\mathfrak{hyp}(\mathcal{F}).$$

Hence, the fundamental group of $|\mathcal{L}|^\wedge_p$ only depends on the associated fusion system. We will see later that many of the other homotopy properties of $|\mathcal{L}|^\wedge_p$ depend only on the fusion system.

We can also look at the covering spaces of $|\mathcal{L}|$ and we can wonder when they come from fusion subsystems. For example, we have the following properties (you can see [AKO], Section I.6, for the definition of normal subsystems of a fusion system and a linking system).

Proposition 1.3.2 ([AKO], Proposition III.4.16)**.** *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Let $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ be a p-local subgroup of $(S, \mathcal{F}, \mathcal{L})$ and assume that $\mathcal{F}_0$ is weakly normal in $\mathcal{F}$ and $\mathcal{L}_0$ is normal in $\mathcal{L}$.*

Then the inclusion $\mathcal{L}_0 \subseteq \mathcal{L}$ induces, up to homotopy, a covering space with covering group,

$$\mathcal{L}/\mathcal{L}_0 = Aut_{\mathcal{L}}(S_0)/Aut_{\mathcal{L}_0}(S_0).$$

We can look at some particular p-local subgroups.

Definition 1.3.3. Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ a p-local subgroup of $(S, \mathcal{F}, \mathcal{L})$.

(a) We say that $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ is a *p-local subgroup of index a power of p* if $S_0 \geq \mathfrak{hyp}(\mathcal{F})$ and, for every $P \leq S$, $O^p(\mathrm{Aut}_{\mathcal{F}}(P)) \leq \mathrm{Aut}_{\mathcal{F}_0}(P)$.

(b) We say that $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ is a *p-local subgroup of index prime to p* if $S_0 = S$ and, for every $P \leq S$, $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P)) \leq \mathrm{Aut}_{\mathcal{F}_0}(P)$.

These particular p-local subgroups satisfy the following properties.

Proposition 1.3.4 ([5a2], Proposition 3.8). *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ a p-local subgroup of $(S, \mathcal{F}, \mathcal{L})$.*

(a) If $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ is of index a power of p, then $P \leq S_0$ is $\mathcal{F}_0$-quasicentric if, and only if, P is $\mathcal{F}$-quasicentric.

(b) If $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ is of index prime to p, then $P \leq S$ is $\mathcal{F}_0$-centric if, and only if, P is $\mathcal{F}$-centric.

These p-local subgroups are in one-to-one correspondence with covering spaces of $|\mathcal{L}|$ with index a power of p or prime to p respectively.

Let us first look at p-local finite subgroups of p-power index. According to the previous proposition, we will work with $\mathcal{F}$-quasicentric linking systems.

Theorem 1.3.5 ([5a2], Theorem 4.4). *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group with $\mathcal{L}$ a quasicentric linking system.*

For each T containing $\mathfrak{hyp}(\mathcal{F})$, there is a unique p-local subgroup $(T, \mathcal{F}_T, \mathcal{L}_T)$ of index a power of p and such that $\mathcal{L}_T = \pi^{-1}(\mathcal{F}_T^q)$.

Moreover, the inclusion $\mathcal{L}_T \subseteq \mathcal{L}$ induces, up to homotopy, a covering space of degree $[S : T]$. Hence, $|\mathcal{L}_T|^\wedge_p$ is homotopy equivalent to a covering space of $|\mathcal{L}|^\wedge_p$ with covering group S/T.

Thus, there is a bijective correspondence between the p-local subgroups of $(S, \mathcal{F}, \mathcal{L})$ and the subgroups of $\pi_1(|\mathcal{L}|^\wedge_p)$. This can also be seen as an analogous to the situation for the classifying space of a finite group G: since $\pi_1(BG^\wedge_p) = G/O^p(G)$ there is a bijective correspondence between connected covering spaces of $BG^\wedge_p$ and subgroups of G containing $O^p(G)$.

We can also define, for a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ the *minimal p-local finite group of index a power of p* $(\mathfrak{hyp}(S), O^p(\mathcal{F}), O^p(\mathcal{L}))$ and, by Theorem 1.3.5, we obtain that the space $|O^p(\mathcal{L})|^\wedge_p$ is homotopy equivalent to the normal covering space of $|\mathcal{L}|^\wedge_p$ with covering group $S/\mathfrak{hyp}(\mathcal{F})$.

Let us now look at p-local subgroups of index prime to p. Here, for an infinite group G, we denote by $O^{p'}(G)$ the intersection of all normal subgroups in G of finite index prime to p. For $\mathcal{F}$ a fusion system over a p-group S, let $O^{p'}(\mathcal{F})$ be the fusion system generated by $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ for all $P \leq S$ and define

$$\mathrm{Out}^0_{\mathcal{F}}(P) = \langle \alpha \in \mathrm{Out}_{\mathcal{F}}(S) \ \mid \ \alpha \in \mathrm{Hom}_{O^{p'}(\mathcal{F})}(P, S), \text{ for some } P \leq S \rangle.$$

Since $\mathrm{Aut}_{\mathcal{F}}(S)$ normalizes $O^{p'}(\mathcal{F})$, $\mathrm{Out}^0_{\mathcal{F}}(S) \trianglelefteq \mathrm{Out}_{\mathcal{F}}(S)$.

Proposition 1.3.6 ([5a2], Lemma 3.4 and Proposition 5.2). *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group.*

(a) $\mathcal{F} = \langle Aut_{\mathcal{F}}(S), O^{p'}(\mathcal{F}) \rangle$.

(b) π and the inclusion of $BAut_{\mathcal{F}}(S)$ in $|\mathcal{F}^c|$ induce isomorphisms,

$$\theta : \pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}}) \xrightarrow{\cong} \pi_1(|\mathcal{F}^c|) \xrightarrow{\cong} Out_{\mathcal{F}}(S)/Out^0_{\mathcal{F}}(S).$$

Proof. The point (a) is proved in [5a2], Lemma 3.4, for (b) as $\pi_1(|\mathcal{F}^c|)$ is a p'-group by [AKO] Theorem I.7.7.(a), the second isomorphism is given in [5a2], Proposition 5.2, and the first one in Theorem 5.5 and the comments which follows . ✩

According to 1.3.4, we will work with centric linking systems.

Theorem 1.3.7 ([5a2], Theorem 5.5)**.** *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group with $\mathcal{L}$ a centric linking system.*

For each subgroup $H \leq Out_{\mathcal{F}}(S)$ containing $Out^0_{\mathcal{F}}(S)$, there is a unique p-local finite group $(S, \mathcal{F}_H, \mathcal{L}_H)$ of index prime to p such that $Out_{\mathcal{F}_H}(S) = H$ and $\mathcal{L}_H = \pi^{-1}(\mathcal{F}^c_H)$.

Moreover, $|\mathcal{L}_H|$ is homotopy equivalent, via its inclusion in $|\mathcal{L}|$, to the covering space of $|\mathcal{L}|$ with fundamental group $\widetilde{H} \geq O^{p'}(\pi_{\mathcal{L}})$ such that $\theta(\widetilde{H}/O^{p'}(\pi_{\mathcal{L}})) = H/Out^0_{\mathcal{F}}(S)$ (where θ is the isomorphism of the previous proposition).

Thus, for a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ we can define the *minimal p-local subgroup of index prime to p*, $(S, O^{p'}(\mathcal{F}), O^{p'}(\mathcal{L}))$ corresponding to $(S, \mathcal{F}_H, \mathcal{L}_H)$ with $H = \mathrm{Out}^0_{\mathcal{F}}(S)$ in the previous theorem.

1.3.2 Homotopy properties of classifying spaces

Definition 1.3.8. Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group.

The *classifying space of* $(S, \mathcal{F}, \mathcal{L})$ is the p-completed nerve $|\mathcal{L}|^\wedge_p$.

For example if G is a finite group and $(S, \mathcal{F}, \mathcal{L})$ is the p-local finite group associated, then $|\mathcal{L}|^\wedge_p \simeq BG^\wedge_p$.

Theorem 1.3.9 ([AKO], Theorem III.3.2)**.** *If G is a finite group and S a Sylow p-subgroup of G, then*

$$BG^\wedge_p \simeq |\mathcal{L}^c_S(G)|^\wedge_p.$$

Remark 1.3.10. The homotopy equivalence is given by the following,

$$\mathcal{L}^c_S(G) \xleftarrow{\ \delta\ } \mathcal{T}^c_S(S) \subseteq \mathcal{T}_S(G) \xrightarrow{\ \rho\ } \mathcal{B}(G)$$

where ρ is the functor which send each object on the unique one in the target and for every $P, Q \leq S$ and $g \in G$, $\rho(g) = g$.

We now list some of the other results on the space $|\mathcal{L}|^\wedge_p$ which show that it has many homotopy properties of the p-completed classifying spaces $BG^\wedge_p$. We first verify that the homotopy type of $|\mathcal{L}|$ does not depend on the choice of the object set.

Theorem 1.3.11 ([5a1], Theorem 3.5)**.** *Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Let $\mathcal{L}_0 \subseteq \mathcal{L}$ be two linking systems associated to $\mathcal{F}$ with a different set of objects.*

Then the inclusion induces a homotopy equivalence of space $|\mathcal{L}_0| \simeq |\mathcal{L}|$.

This theorem helps to motivate the use of the homotopy type of $|\mathcal{L}|$ and $|\mathcal{L}|^\wedge_p$, and their homotopy properties, as important invariants of the p-local finite group we consider. It also allows a certain flexibility when working with geometric realization of linking system.

We now look at mapping spaces. If we take a finite group G, and a p-group Q, Mislin has shown that the natural map

$$\mathrm{Rep}(Q,G) = \mathrm{Hom}(Q,G)/\mathrm{Inn}(G) \xrightarrow{\cong} [BQ, BG^\wedge_p]$$

is a bijection. We have an analogue for an abstract fusion system using $|\mathcal{L}|^\wedge_p$ instead of $BG^\wedge_p$. If Q is a finite group and $\mathcal{F}$ is a saturated fusion system over a p-group S, we set,

$$\mathrm{Rep}(Q,\mathcal{F}) = \mathrm{Hom}(Q,S)/\sim,$$

where for $\rho,\sigma \in \mathrm{Hom}(Q,S)$, $\rho \sim \sigma$ if there is $\alpha \in \mathrm{Iso}_{\mathcal{F}}(\rho(Q),\sigma(Q))$ such that $\alpha \circ \rho = \sigma$.

Theorem 1.3.12 ([BLO2], Corollary 4.5)**.** *Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group. If Q is a p-group, then the map*

$$Rep(Q,\mathcal{F}) \xrightarrow{\cong} [BQ, |\mathcal{L}|^\wedge_p],$$

defined by sending the class of $\rho : Q \longrightarrow S$ to the class of the composite

$$BQ \xrightarrow{B\rho} BS \xrightarrow{incl} |\mathcal{L}| \xrightarrow{\lambda_{|\mathcal{L}|}} |\mathcal{L}|^\wedge_p$$

is a bijection.

Another important homotopy property deals with cohomology but we will see it in the next chapter.

Let us finish with a theorem which states that the isomorphism type of a p-local finite group $(S,\mathcal{F},\mathcal{L})$ is completely determined by the homotopy type of the space $|\mathcal{L}|^\wedge_p$.

An isomorphism $(S_1,\mathcal{F}_1,\mathcal{L}_1) \longrightarrow (S_2,\mathcal{F}_2,\mathcal{L}_2)$ of p-local finite groups is a triple

$$S_1 \xrightarrow[\cong]{\alpha} S_2 \qquad \mathcal{F}_1 \xrightarrow[\cong]{\alpha_{\mathcal{F}}} \mathcal{F}_2 \qquad \text{and} \qquad \mathcal{L}_1 \xrightarrow[\cong]{\alpha_{\mathcal{L}}} \mathcal{L}_2$$

of isomorphisms of groups and categories, such that $\alpha_{\mathcal{F}}(P) = \alpha(P)$ for all $P \leq S_1$, $\alpha_{\mathcal{L}}(P) = \alpha(P)$ for all $P \in \mathrm{Ob}(\mathcal{L})$, and such that they commute in the obvious way with the structural functors $\mathcal{L}_i \xrightarrow{\pi_i} \mathcal{F}_i$ and $\mathcal{T}_{\mathrm{Ob}(\mathcal{L})}(S) \xrightarrow{\delta_i} \mathcal{L}_i$.

Theorem 1.3.13 ([AKO], Theorem III.4.25)**.** *If $(S_1,\mathcal{F}_1,\mathcal{L}_1)$ and $(S_2,\mathcal{F}_2,\mathcal{L}_2)$ are p-local finite groups, then any homotopy equivalence*

$$|\mathcal{L}_1|^\wedge_p \xrightarrow{\cong} |\mathcal{L}_2|^\wedge_p$$

induces an isomorphism

$$(S_1,\mathcal{F}_1,\mathcal{L}_1) \xrightarrow{\cong} (S_2,\mathcal{F}_2,\mathcal{L}_2)$$

of p-local finite groups.

2 Cohomology and stable elements

In this chapter, we are interested in the notion of "stable" elements.

After we recall some generalities on group cohomology, we remind the notion of stable elements for a finite group G from Cartan and Eilenberg, their link with the cohomology of G and we translate it in term of projective limit. We also remind the notion of $\mathcal{F}$-stable for a saturated fusion system $\mathcal{F}$ when we work with a $\mathbb{Z}_{(p)}$-module. Then, we define the notion of $\mathcal{F}^c$-stable elements of a p-local finite group with coefficients in a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module. We end this chapter comparing these three notions in the case of a realizable p-local finite group.

2.1 Group cohomology and δ-functors

Let G be a group (not necessarily finite) and M a $\mathbb{Z}_{(p)}[G]$-module. The *group cohomology of G with coefficients in M*, denoted by $H^*(G,M)$, is the cohomology of the chain complex

$$\mathrm{Hom}_{\mathbb{Z}_{(p)}[G]}(P_\bullet, M),$$

where $P_\bullet$ is a projective resolution of $\mathbb{Z}_{(p)}$ as a trivial $\mathbb{Z}_{(p)}[G]$-module. We refer the reader to [CE], [We], [AM] or [Br] for more details about this functor. Here we will use the notion of transfer and the bifunctoriality of group cohomology. We also define the notion of δ-functor and give $H^*(G,-)$ as an example.

Let $\mathcal{D}$ be the category of pairs (G,M) with G a group and M a $\mathbb{Z}_{(p)}[G]$-module. A morphism in $\mathcal{D}$ from (G,M) to (H,N) is a pair, (φ,ρ) where $\varphi : G \longrightarrow H$ is a group homomorphism and $\rho : N \longrightarrow M$ is a linear map such that, for every $n \in N$ and every $g \in G$, $g\rho(n) = \rho(\varphi(g)n)$. Then, we can remark the following.

Proposition 2.1.1.

$$H^*(-,-) : \mathcal{D} \longrightarrow \mathbb{Z}_{(p)}\text{-}Mod$$

defines a contravariant functor.

For a pair (φ, ρ) from (G, M) to (H, N), the morphism $H^*(\varphi, \rho)$ is induced by the chain map,

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}_{(p)}[H]}(P_\bullet^H, N) &\longrightarrow \mathrm{Hom}_{\mathbb{Z}_{(p)}[G]}(P_\bullet^G, M) \\ f &\longmapsto \rho \circ f \circ \varphi_* \end{aligned}$$

where $P_\bullet^H$ and $P_\bullet^G$ are projective resolutions of $\mathbb{Z}_{(p)}$ as trivial $\mathbb{Z}_{(p)}[H]$-module and $\mathbb{Z}_{(p)}[G]$-module respectively and where φ_* is the chain map induced by the identity between the two chain complexes $P_\bullet^G$ and $\varphi_\sharp(P_\bullet^H)$ (Here, for a $\mathbb{Z}_{(p)}[H]$-module M, $\varphi_\sharp M$ is the $\mathbb{Z}_{(p)}[G]$-module M where G acts through φ).

For example, for every group inclusion $H \leq G$ and every $\mathbb{Z}_{(p)}[G]$-module M, the pair $(\mathrm{incl}_H^G, \mathrm{Id})$ induce the restriction map

$$\mathrm{Res}_H^G : H^*(G, M) \longrightarrow H^*(H, M).$$

Another example is given by conjugation. Let G be a group, H be a subgroup of G and $g \in G$. for every $\mathbb{Z}_{(p)}[G]$-module M, the pair (c_g, g^{-1}) induce the conjugation map

$$c_g^* : H^*(gHg^{-1}, M) \longrightarrow H^*(H, M)$$

In that case, if we consider a projective resolution of $\mathbb{Z}_{(p)}$ as a trivial $\mathbb{Z}_{(p)}[G]$-module it also defines a projective resolution for H and gHg^{-1}. Then on the chain level, the map c_g^* is given by,

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}_{(p)}[gHg^{-1}]}(P_\bullet^G, M) &\longrightarrow \mathrm{Hom}_{\mathbb{Z}_{(p)}[H]}(P_\bullet^G, M) \\ f &\longmapsto (u \mapsto g^{-1}f(gu)) \end{aligned}$$

Another important point of view of group cohomology is through the notion of δ-functor.

Definition 2.1.2. Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories.

A (contravariant) *δ-functor* is a functor $F^* : \mathcal{A} \longrightarrow \mathcal{B}$ together with connecting homomorphisms

$$\delta_F^i : F^i(A) \longrightarrow F^{i+1}(C)$$

defined for every short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in $\mathcal{A}$, and such that,

(a) for every short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in $\mathcal{A}$, the long sequence

$$\cdots \longrightarrow F^i(C) \longrightarrow F^i(B) \longrightarrow F^i(A) \xrightarrow{\delta_F^i} F^{i+1}(C) \longrightarrow \cdots$$

is exact.

(b) For every morphism of short exact sequences in $\mathcal{A}$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\varphi_A"] & B \arrow[r] \arrow[d, "\varphi_B"] & C \arrow[r] \arrow[d, "\varphi_C"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0, \end{tikzcd}$$

the following diagram commute for every i,

$$\begin{tikzcd} F^i(A') \arrow[r, "\delta_F^i"] \arrow[d, "F^i(\varphi_A)"'] & F^{i+1}(C') \arrow[d, "F^{i+1}(\varphi_C)"] \\ F^i(A) \arrow[r, "\delta_F^i"'] & F^{i+1}(C). \end{tikzcd}$$

If $(F^*, \delta_F), (G^*, \delta_G) : \mathcal{A} \longrightarrow \mathcal{B}$ are two δ-functors, a *morphism of δ-functors* from (F^*, δ_F) to (G^*, δ_G) is a natural transformation η such that, for every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

We have a commutative diagram,

$$\begin{tikzcd} \cdots \arrow[r] & F^i(C) \arrow[r] \arrow[d, "\eta_C"] & F^i(B) \arrow[r] \arrow[d, "\eta_B"] & F^i(A) \arrow[r, "\delta_F^i"] \arrow[d, "\eta_A"] & F^{i+1}(C) \arrow[r] \arrow[d, "\eta_C"] & \cdots \\ \cdots \arrow[r] & G^i(C) \arrow[r] & G^i(B) \arrow[r] & G^i(A) \arrow[r, "\delta_G^i"'] & G^{i+1}(C) \arrow[r] & \cdots . \end{tikzcd}$$

With the usual composition on natural transformations, we obtain a category. When there is no confusion on the connecting homomorphisms we will just say that F^* is a δ-functor.

Remark 2.1.3. A δ-functor can be seen as a functor from the category $\mathcal{S}_{\mathcal{A}}$ of short exact sequences in $\mathcal{A}$ to $\mathrm{Ch}(\mathcal{B})$ which sends any short exact sequence to an acyclic chain complex.

If $\eta : F \longrightarrow G$ is a natural transformation then, to show that it is a morphism of δ-functors, it is enough to prove that, for every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

the following diagram is commutative for every i.

$$\begin{tikzcd} F^i(A) \arrow[r, "\delta_F^i"'] \arrow[d, "\eta_A"] & F^{i+1}(C) \arrow[d, "\eta_C"] \\ G^i(A) \arrow[r, "\delta_G^i"'] & G^{i+1}(C) \end{tikzcd}$$

One important example of a contravariant δ-functor is the *left derived functor* of an additive and left exact functor. We refer the reader to [We], Chapter 2, where he can find the notion of δ-functor, derived functor and their properties. For a group G, the group cohomology functor $H^*(G,-) : \mathbb{Z}_{(p)}[G]\text{-Mod} \longrightarrow \mathbb{Z}_{(p)}\text{-Mod}$, with the usual connecting homomorphism $\delta_{H^*(G,-)}$, is the left derived functor of the fixed point functor $M \longmapsto M^G$ and then, it defines a universal δ-functor (by [We], Theorem 2.4.7).

Definition 2.1.4. A (contravariant) δ-functor $(F^*, \delta_F) : \mathcal{A} \longrightarrow \mathcal{B}$ is *universal* if, given another δ-functor $(G^*, \delta_G) : \mathcal{A} \longrightarrow \mathcal{B}$ and a natural transformation $\eta^0 : F^0 \longrightarrow G^0$, there exists a unique morphism of δ-functors $\eta^* : (F^*, \delta_F) \longrightarrow (G^*, \delta_G)$ extending η^0.

In particular, if for two groups G and H, we have a natural transformation between the functors $(-)^G$ and $(-)^H$, there is a unique morphism of δ-functors from $\big(H^*(G,-), \delta_{H^*(G,-)}\big)$ to $\big(H^*(H,-), \delta_{H^*(H,-)}\big)$ which extends this natural transformation.

For G a finite group, H a subgroup of G and M a $\mathbb{Z}_{(p)}[G]$-module, restriction and conjugation by an element $g \in G$, defined in degree 0 by

$$\begin{aligned} \big(\mathrm{Res}^G_H\big)^0 : M^G &\longrightarrow M^H \\ x &\longmapsto x \end{aligned}$$

and

$$\begin{aligned} c^0_g : \quad M^{gHg^{-1}} &\longrightarrow M^H \ , \\ x &\longmapsto g^{-1}x \end{aligned}$$

give examples of morphisms of δ-functors. Another example is given by the transfer. If G is a group and H is a subgroup of G of finite index, we can define a *transfer map*

$$\mathrm{tr}^G_H : H^*(H,-) \longrightarrow H^*(G,-)$$

as the morphism of δ-functors induced by the natural transformation $(-)^H \longrightarrow (-)^G$ given by, for M a $\mathbb{Z}_{(p)}[G]$-module,

$$\begin{aligned} M^H &\longrightarrow M^G \\ x &\longmapsto \textstyle\sum_{g \in [G/H]} gx. \end{aligned}$$

2.2 Cartan-Eilenberg Theorem

Let us now talk about an important theorem in group cohomology. This theorem, due to Cartan and Eilenberg, expresses the cohomology of a finite group G with coefficients in a $\mathbb{Z}_{(p)}[G]$-module as the submodule of "stable elements" in the cohomology of a Sylow p-subgroup of G.

Notation 2.2.1. For $H \leq K$ two subgroups of G and M a $\mathbb{Z}_{(p)}[G]$-module, we will denote by $\mathrm{Res}_H^K : H^*(K,M) \longrightarrow H^*(H,M)$ the morphism induced by the inclusion, and by $\mathrm{tr}_H^K : H^*(H,M) \longrightarrow H^*(K,M)$ the transfer homomorphism.

If $H \leq G$ and $g \in G$, we will denote by $c_g^* : H^*(gHg^{-1},M) \longrightarrow H^*(H,M)$ the morphism induced by c_g.

These morphisms satisfy the following properties.

Proposition 2.2.2 ([CE], Chapter XII, §8). *Let G be a finite group and M be a $\mathbb{Z}_{(p)}[G]$-module.*

We have the following properties.

(a) For every $g, h \in G$, $c_h^* \circ c_g^* = c_{gh}^*$.

(b) For every $g \in H$, $c_g^* = Id_{H^*(H,M)}$.

(c) For every $H \leq K \leq G$ and $g \in G$, $c_g^* \circ Res_H^K = Res_{g^{-1}Hg}^{g^{-1}Kg} \circ c_g^*$

(d) For every $H \leq K \leq G$ and $g \in G$, $c_g^* \circ tr_H^K = tr_{g^{-1}Hg}^{g^{-1}Kg} \circ c_g^*$

(e) For every $H \leq G$, $tr_H^G \circ Res_H^G = [G:H] Id_{H^*(G,M)}$.

(f) For every $H \leq G$, $Res_H^G \circ tr_K^G = \sum_{g \in [H \backslash G/K]} tr_{H \cap g^{-1}Kg}^H \circ Res_{H \cap g^{-1}Kg}^{g^{-1}Kg} \circ c_g^*$.

where $[H \backslash G/K]$ denotes the set of all double classes modulo H and K of G.

We define the stable elements as follows.

Definition 2.2.3. Let G be a finite group, S a Sylow p-subgroup of G and M a $\mathbb{Z}_{(p)}[G]$-module.

We say that $x \in H^*(S,M)$ is *stable* if, for every $g \in G$,

$$\mathrm{Res}_{S \cap g^{-1}Sg}^S(x) = c_g^* \circ \mathrm{Res}_{gSg^{-1} \cap S}^S(x).$$

For example, if S is a normal subgroup of G, $x \in H^*(S,M)$ is stable if, and only if, $x \in H^*(S,M)^G$.

Theorem 2.2.4 (Cartan-Eilenberg). *Let G be a finite group, S a Sylow p-subgroup of G and M a $\mathbb{Z}_{(p)}[G]$-module.*

The morphism $Res_S^G : H^*(G,M) \longrightarrow H^*(S,M)$ is injective and its image is the submodule of all the stable elements.

Proof. Let $q = [G:S]$ be the index of S in G. As S is a Sylow p-subgroup of G, q is invertible in $\mathbb{Z}_{(p)}$. Thus, by Proposition 2.2.2 (d), tr_S^G is surjective and Res_S^G is injective.

We then have to show that $\mathrm{Im}\,(\mathrm{Res}_S^G)$ is the set of all the stable elements.

Let $a \in \operatorname{Im}(\operatorname{Res}_S^G)$. There exists $b \in H^*(G, M)$ such that $a = \operatorname{Res}_S^G(b)$. For every $x \in G$, c_x is the identity on $H^*(G, M)$, hence,

$$\begin{aligned} c_x(a) &= c_x \circ \operatorname{Res}_S^G(b) \\ &= \operatorname{Res}^G_{xSx^{-1}} \circ c_x(b) && \text{(by Proposition 2.2.2 } (c)) \\ &= \operatorname{Res}^G_{xSx^{-1}}(b). \end{aligned}$$

and in particular,

$$\begin{aligned} \operatorname{Res}^{xSx^{-1}}_{S\cap xSx^{-1}} \circ c_x(a) &= \operatorname{Res}^{xSx^{-1}}_{S\cap xSx^{-1}} \circ \operatorname{Res}^G_{xSx^{-1}}(b) \\ &= \operatorname{Res}^G_{S\cap xSx^{-1}}(b) \\ &= \operatorname{Res}^S_{S\cap xSx^{-1}} \circ \operatorname{Res}^G_S(b) \\ &= \operatorname{Res}^S_{S\cap xSx^{-1}}(a). \end{aligned}$$

and a is stable.

Conversely, let $a \in H^*(S, M)$ be a stable element. Recall that, by [CE] Proposition XII.9.2, we have $\sum_{x\in[S\backslash G/S]}[S : S \cap xSx^{-1}] = [G : S]$. By Proposition 2.2.2 ($e$) and ($f$), we have

$$\begin{aligned} \operatorname{Res}_S^G \circ \operatorname{tr}_S^G(a) &= \sum_{x\in[S\backslash G/S]} [S : S\cap xSx^{-1}]c_x^*(a) && \text{(by Proposition 2.2.2 } (e) \text{ and } (f)) \\ &= \sum_{x\in[S\backslash G/S]} [S : S\cap xSx^{-1}]a && \text{(because } a \text{ is stable)} \\ &= [G : S]a && \text{(by [CE] Proposition XII.9.2)} \\ &= qa. \end{aligned}$$

Then, if l is the inverse of $q \in \mathbb{Z}_{(p)}$,

$$\operatorname{Res}_S^G \circ \operatorname{tr}_S^G(la) = a.$$

Thus, $a \in \operatorname{Im}(\operatorname{Res}_S^G)$. ✩

This result can be interpreted as a projective limit.

Lemma 2.2.5. *Let $\mathcal{C}$ be small category, k a ring and $F : \mathcal{C} \longrightarrow k\text{-}Mod$ a contravariant functor.*

Assume that there exists $x_0 \in Ob(\mathcal{C})$ such that for all $y \in Ob(\mathcal{C})$, there exists $\varphi_y \in Mor_{\mathcal{C}}(y, x_0)$.

Denote E_{stable} the set of $m \in F(x_0)$ such that for all $y \in Ob(\mathcal{C})$ and for all $\psi, \psi' \in Mor_{\mathcal{C}}(y, x_0)$, we have $F(\psi)(m) = F(\psi')(m)$.

Then,

$$E_{stable} \cong \varprojlim_{\mathcal{C}} F.$$

Proof. Let A be a k-module and $(\gamma_x)_{x\in\mathcal{C}}$ be a family of linear applications such that, for $x \in \mathrm{Ob}(\mathcal{C})$, $\gamma_x \in \mathrm{Hom}_{k\text{-Mod}}(A, F(x))$ and such that for all $y, z \in \mathrm{Ob}(\mathcal{C})$ and $\varphi \in \mathrm{Mor}_{\mathcal{C}}(y, z)$, we have the following commutative diagram.

$$\begin{tikzcd}
A \arrow[r, "\gamma_y"] \arrow[dr, "\gamma_z"'] & F(y) \\
& F(z) \arrow[u, "F(\psi)"']
\end{tikzcd}$$

Then, by definition of E_{stable}, for all $y, z \in \mathrm{Ob}(\mathcal{C})$ and all $\psi \in \mathrm{Mor}_{\mathcal{C}}(y, z)$,

$$\begin{tikzcd}
A \arrow[dr, "\gamma_{x_0}"] \arrow[rrrdd, bend left, "\gamma_y"] \arrow[rrddd, bend right, "\gamma_z"'] & & & \\
& E_{\text{stable}} \arrow[dr, "i"] & & \\
& & F(x_0) \arrow[r, "F(\varphi_y)"] \arrow[d, "F(\varphi_z)"'] & F(y) \\
& & F(z) \arrow[ur, "F(\psi)"'] &
\end{tikzcd}$$

where i denotes the inclusion $E_{\text{stable}} \subseteq F(x_0)$.

Thus, for every $y \in \mathrm{Ob}(\mathcal{C})$, γ_y factors trough E_{stable} along γ_{x_0}. This decomposition is clearly unique. ✩

Remind that $\mathcal{T}_S(G)$ is the category with set of objects all the subgroups of S and for all $P, Q \leq S$, $\mathrm{Mor}_{\mathcal{T}_S(G)}(P, Q) = T_G(P, Q)$.

Corollary 2.2.6. *Let G be a finite group, S a Sylow p-subgroup of G and M a $\mathbb{Z}_{(p)}[G]$-module.*

Then

$$H^*(G, M) \cong \varprojlim_{\mathcal{T}_S(G)} H^*(-, M).$$

Proof. This is just a consequence of Lemma 2.2.5 with

$$\begin{aligned}
&\mathcal{C} = \mathcal{T}_S(G), \\
&x_0 = S \in \mathcal{T}_S(G), \\
&\varphi_P = 1 \in T_G(P, S), \text{ for all } P \in \mathcal{F}_S(G), \\
&F = H^*(-, M) : \mathcal{T}_S(G) \longrightarrow \mathbb{F}_p\text{-Mod}
\end{aligned}$$

and of Theorem 2.2.4. ✩

Remark 2.2.7. In fact, as P acts trivially on $H^*(P, M)$, we can define our functor $H^*(-, M)$ on the orbit category $\mathcal{O}_S(G)$ with set of objects all the subgroups of S and for $P, Q \leq S$, $\mathrm{Mor}_{\mathcal{O}_S(G)}(P, Q) = Q \backslash T_G(P, Q)$ and then $H^*(G, M) \cong \varprojlim_{\mathcal{O}_S(G)} H^*(-, M)$. This category appears in the *subgroup decomposition* of BG (see [DwH] for more details).

2.3 Stable elements with trivial coefficients

Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group.

First, we look at a realizable fusion system.

Proposition 2.3.1. *Let G be a finite group and S a Sylow p-subgroup of G.*

If M is a $\mathbb{Z}_{(p)}$-module, then

$$H^*(G, M) \cong \varprojlim_{\mathcal{F}_S(G)} H^*(-, M).$$

Proof. By Corollary 2.2.6, we have, $H^*(G, M) \cong \varprojlim_{\mathcal{T}_S(G)} H^*(-, M)$. However, as the action of G on M is trivial, for every $P \leq S$ and every $g \in C_G(P)$, c_g^* is induced by the pair $(c_g, g^{-1}) = (\mathrm{Id}_P, \mathrm{Id}_M)$ and then $c_g^* = \mathrm{Id}_{H^*(P,M)}$. Hence, as, for every $P, Q \leq S$, $\mathrm{Hom}_{\mathcal{F}}(P, Q) = T_G(P, Q)/C_G(P)$, the cohomology functor

$$\begin{tikzcd} \mathcal{T}_S(G) \arrow[rr, "H^*(-,M)"] \arrow[dr, "\pi"'] & & \mathbb{Z}_{(p)}\text{-Mod} \\ & \mathcal{F} \arrow[ur, "H^*(-,M)"'] & \end{tikzcd}$$

factors through $\mathcal{F}_S(G)$. ✩

We have then the following corollary.

Corollary 2.3.2. *Let G_1, G_2 be two finite groups and S_1, resp. S_2, a Sylow p-subgroup of G_1, resp. G_2.*

If $\mathcal{F}_{S_1}(G_1) = \mathcal{F}_{S_2}(G_2)$, then, for every $M \in \mathbb{Z}_{(p)}$-Mod,

$$H^*(G_1, M) \cong H^*(G_2, M).$$

In other words, for M a $\mathbb{Z}_{(p)}$-module $H^*(G, M)$ is completely determined by $\mathcal{F}_S(G)$.

If we consider a p-local finite group (or even just a fusion system), we can define the more generally notion of $\mathcal{F}$-stable elements.

Definition 2.3.3. Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and M a $\mathbb{Z}_{(p)}$-module.

An element $x \in H^*(S, M)$ is called $\mathcal{F}$-*stable* if for all $P \in \mathrm{Ob}(\mathcal{F})$ and all $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$,

$$\varphi^*(x) = \mathrm{Res}_P^S(x).$$

We denote by $H^*(\mathcal{F}, M) \leq H^*(S, M)$ the submodule of all $\mathcal{F}$-stable elements.

By lemma 2.2.5, this submodule of $\mathcal{F}$-stable elements corresponds to the inverse limit of $H^*(-,M)$ on the category $\mathcal{F}$

$$H^*(\mathcal{F},M) = \varprojlim_{\mathcal{F}} H^*(-,M).$$

It is easily a homotopy invariant of p-local finite groups and Broto, Levi and Oliver have shown the following results.

Theorem 2.3.4. *Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group and M an abelian p-group.*
The natural homomorphisms

$$H^*(|\mathcal{L}|_p^\wedge, M) \xrightarrow{\cong} H^*(|\mathcal{L}|, M) \xrightarrow{\cong} H^*(\mathcal{F},M)$$

induced by $\lambda_{|\mathcal{L}|}$ and the inclusion of BS in $|\mathcal{L}|$, are isomorphisms.

Proof. The case $M = \mathbb{F}_p$ have been proved in [BLO2], Theorem 8.1, and the general case can be found in [5a2], Lemma 6.12. ☆

Let us finish with a remark on the set of objects we consider which can be useful for computations.

Proposition 2.3.5. *Let $(S,\mathcal{F},\mathcal{L})$ a p-local finite group.*
Let $\mathcal{H},\mathcal{H}'$ be two families of subgroups of S containing S and all the $\mathcal{F}$-essential subgroups.
If k is a ring and $F : \mathcal{F} \longrightarrow k$-Mod a contravariant functor, then

$$\varprojlim_{\mathcal{F}^{\mathcal{H}'}} F = \varprojlim_{\mathcal{F}^{\mathcal{H}}} F.$$

Proof. The modules $\varprojlim_{\mathcal{F}^{\mathcal{H}'}} F$ and $\varprojlim_{\mathcal{F}^{\mathcal{H}}} F$ can be seen as submodules of $F(S)$. As $\mathcal{F}^{\mathcal{H}} \subseteq \mathcal{F}^{\mathcal{H}'}$ we have the inclusion $\varprojlim_{\mathcal{F}^{\mathcal{H}'}} F \subseteq \varprojlim_{\mathcal{F}^{\mathcal{H}}} F$. The opposite inclusion is given by Alperin's Fusion Theorem (Theorem 1.1.9). ☆

In particular, we obtain that, for every $\mathbb{Z}_{(p)}$-module M,

$$H^*(\mathcal{F},M) = \varprojlim_{\mathcal{F}^c} H^*(-,M).$$

This last term will be of interest when we will work with twisted coefficients.

Even better, if we denote by $\mathcal{F}^{\text{ess}}$ the full subcategory of $\mathcal{F}$ with set of objects the set of S and all the $\mathcal{F}$-essential subgroups of S, a direct corollary of Proposition 2.3.5 is that for every family $\mathcal{H}$ containing $\text{Ob}(\mathcal{F}^{\text{ess}})$, then, for every functor F,

$$\varprojlim_{\mathcal{F}^{\mathcal{H}}} F = \varprojlim_{\mathcal{F}^{\text{ess}}} F.$$

2.4 Stable elements with an action of $\pi_{\mathcal{L}}$

When we work with coefficients twisted by an action of $\pi_{\mathcal{L}}$, we cannot define our cohomology functor on all $\mathcal{F}$. In fact, if we take a morphism φ in $\mathcal{F}$ between two subgroups which are not $\mathcal{F}$-centric, by Alperin's Fusion Theorem (Theorem 1.1.9), we can see it as a composite of restrictions of morphisms in $\mathcal{F}^c$. However, this decomposition is not, in general, unique and two different decompositions can lead to different morphisms in twisted cohomology. As an easy example, we can look at the trivial group $\{e\}$: each morphism in $\mathcal{F}^c$ restricts to the identity on $\{e\}$, but, if M is not a trivial $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module, every $\varphi \in \mathrm{Mor}(\mathcal{F}^c)$ does not act trivially on $M = M^{\{e\}} = H^0(\{e\}, M)$.

Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and M a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module. Remark first that M is naturally a $\mathbb{Z}_{(p)}[S]$-module where the action is given by the following composition:

$$\mathcal{B}(S) = \mathcal{B}(\mathrm{Mor}_{\mathcal{T}_S^c(S)}(S,S)) \xrightarrow{\delta_S} \mathcal{L} \xrightarrow{\omega} \mathcal{B}(\pi_{\mathcal{L}}) .$$

Hence, we can consider, for $P \leq S$, the cohomology of P with coefficients in M.

As we work with an action of $\pi_1(|\mathcal{L}|)$, we can easily define a functor on $\mathcal{L}$ using the bifunctoriality of group cohomology (see section 2.1) and the functor $\omega : \mathcal{L} \longrightarrow \mathcal{B}(\pi_{\mathcal{L}})$ which sends each objects of $\mathcal{L}$ to the unique one in $\mathcal{B}(\pi_{\mathcal{L}})$ and, for $P, Q \leq S$, sends $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ to the path $\iota_Q.\varphi.\overline{\iota_P}$ (see appendix 2 for more details).

$$\begin{array}{rrcl} H^*(-,M): & \mathcal{L} & \longrightarrow & \mathbb{Z}_{(p)}\text{-Mod} \\ & P \in \mathrm{Ob}(\mathcal{L}) & \longmapsto & H^*(P,M) \\ & \varphi \in \mathrm{Mor}_{\mathcal{L}}(P,Q) & \longmapsto & H^*(\varphi, M) = \varphi^* \\ & & & := H^*(\pi(\varphi), \omega(\varphi)^{-1}) \end{array}$$

For P, Q two subgroups of S and $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$, $H^*(\varphi, M)$ can be also defined on the chain level as follows:

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbb{Z}_{(p)}[Q]}(R_\bullet, M) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_{(p)}[P]}(R_\bullet, M) \\ f & \longmapsto & (\omega(\varphi)^{-1} f \circ \pi(\varphi)_*) \end{array}$$

where $(R_\bullet)$ is a projective resolution of the trivial $\mathbb{Z}_{(p)}[S]$-module $\mathbb{Z}_{(p)}$. Finally, it can also be defined as the morphism between the two derived functors of $(-)^Q$ and $(-)^P$ induced by

$$x \in M^Q \longmapsto \omega(\varphi)^{-1} x \in M^P.$$

By construction, this functor extend naturally the group cohomology functor defined on $\mathcal{T}_S^c(S)$.

$$\begin{tikzcd} \mathcal{T}_S^c(S) \arrow[rr, "H^*(-,M)"] \arrow[dr, "\delta"'] & & \mathbb{Z}_{(p)}\text{-Mod} \\ & \mathcal{L} \arrow[ur, "H^*(-,M)"'] & \end{tikzcd}$$

In particular, for every $P \leq S$ and $g \in P$, $H^*(\delta_P(g), M) = c_g^*$.

Remark 2.4.1. By construction, for all $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$, $H^*(\varphi,-)$ defines a morphism of δ-functors from $\big(H^*(Q,-),\delta_{H^*(Q,-)}\big)$ to $\big(H^*(P,-),\delta_{H^*(P,-)}\big)$.

Proposition 2.4.2. *Let $\varphi,\beta \in Mor_{\mathcal{L}}(P,Q)$ with $P,Q \in \mathcal{L}$.*
If $\pi(\varphi)=\pi(\beta)$ then $H^*(\varphi,M)=H^*(\beta,M)$.

Proof. If $\pi(\varphi)=\pi(\beta)$, then there exists $u\in Z(P)$ such that $\varphi=\beta\circ\delta_P(u)$ and thus

$$H^*(\varphi,M)=H^*(\delta_P(u),M)\circ H^*(\beta,M).$$

However $H^*(\delta_P(u),M)=H^*(\pi(\delta_P(u)),\omega(u)^{-1})=H^*(c_u,\omega(u)^{-1})$ is the automorphism of $H^*(P,M)$ induced by the conjugation by u, and, as $u\in Z(P)\le P$, from Proposition 2.2.2, this is the identity. ✩

In particular, if $\pi(\varphi)=\mathrm{incl}_P^Q$, then $H^*(\varphi,M)=H^*(\iota_P^Q,M)=H^*(\mathrm{incl}_P^Q,\mathrm{Id}_M)=\mathrm{Res}_P^Q$. For M a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module $P,Q\le S$ two $\mathcal{F}$-centric subgroups and $\varphi\in\mathrm{Hom}_{\mathcal{F}}(P,Q)$ we will also denote by $H^*(\varphi,M)=\varphi^*:=H^*(\psi,M)$ where $\psi\in\mathrm{Mor}_{\mathcal{L}}(P,Q)$ is such that $\pi(\psi)=\varphi$. Hence, we can factor our functor through $\mathcal{F}^c$ and we can define the $\mathcal{F}$*-centric stable elements*.

Definition 2.4.3. An element $x\in H^*(S,M)$ is called $\mathcal{F}$*-centric stable*, or just $\mathcal{F}^c$*-stable*, if for all $P\in\mathrm{Ob}(\mathcal{F}^c)$ and all $\varphi\in\mathrm{Hom}_{\mathcal{F}}(P,S)$,

$$\varphi^*(x)=\mathrm{Res}_P^S(x).$$

We denote by $H^*(\mathcal{F}^c,M)\subseteq H^*(S,M)$ the submodule of all $\mathcal{F}^c$-stable elements.

By lemma 2.2.5, this submodule of $\mathcal{F}^c$-stable elements corresponds to the inverse limit of $H^*(-,M)$ on the category $\mathcal{F}^c$,

$$H^*(\mathcal{F}^c,M)=\varprojlim_{\mathcal{F}^c}H^*(-,M).$$

Let us finish with a computation of the stable elements in degree 0.

Lemma 2.4.4. *Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group and M a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module. We have the following equality,*

$$M^{\pi_{\mathcal{L}}}=\varprojlim_{\mathcal{L}}H^0(-,M)=H^0(\mathcal{F}^c,M).$$

Proof. We identify $\varprojlim_{\mathcal{L}}H^0(-,M)$ with the submodule of $x\in M^S=H^0(S,M)$ such that for all $P\in\mathrm{Ob}(\mathcal{L})$ and all $\varphi,\psi\in\mathrm{Mor}_{\mathcal{L}}(P,S)$, $\varphi^*(x)=\psi^*(x)$.

As for all $P,Q\in\mathrm{Ob}(\mathcal{L})$ and all $\varphi\in\mathrm{Mor}_{\mathcal{L}}(P,Q)$, the induced morphism φ^* is given in degree 0, by, $x\in M^Q\mapsto\omega(\varphi)x\in M^P$, we get the first inclusion $M^{\pi_{\mathcal{L}}}\subseteq\varprojlim_{\mathcal{L}}H^0(-,M)$.

Conversely, as $\pi_{\mathcal{L}}$ is generated by $\{\omega(\gamma)\ ;\ \gamma\in\mathrm{Mor}(\mathcal{L})\}$ (by Proposition 2.5), it is enough to show that for every $\gamma\in\mathcal{L}$ and $x\in\varprojlim_{\mathcal{L}}H^0(-,M)$, $\omega(\gamma)x=x$. Consider then

$P, Q \in \mathrm{Ob}(\mathcal{L})$, $\gamma \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ and the following diagram (which is not commutative in general).

$$\begin{tikzcd} & M^S \arrow[dl, "\delta_Q^S(1)^*"'] \arrow[dr, "\delta_P^S(1)^*"] & \\ M^Q \arrow[rr, "\gamma^*"'] & & M^P \end{tikzcd}$$

As $x \in \varprojlim_{\mathcal{L}} H^0(-, M) \leq M^S$, we obtain $\gamma^* \circ \delta_Q^S(1)^*(x) = \delta_P^S(1)^*(x)$. Thus, as for every $x \in M^S$ we have $\delta_Q^S(1)^*(x) = x = \delta_P^S(1)^*(x)$, we have $\omega(\gamma)x = \gamma^*(x) = x$. ✩

2.5 Realizable fusion systems and stable elements

Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group.

Assume here that there is a finite group G containing S as a Sylow p-subgroup and such that $\mathcal{F} = \mathcal{F}_S(G)$. We can wonder what is the link between the cohomology of G and the $\mathcal{F}^c$-stable elements. For example, by Proposition 2.3.1, we know that, for every $\mathbb{Z}_{(p)}$-module M, the inclusion of S in G induces an isomorphism between $H^*(G, M)$ and $H^*(\mathcal{F}, M)$, and, by Proposition 2.3.5, this last one corresponds also to $H^*(\mathcal{F}^c, M)$. But what happens when we consider a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module ?

First of all, there is not always an obvious link between G and $\pi_{\mathcal{L}}$ and then, we cannot give to M a natural structure of $\mathbb{Z}_{(p)}[G]$-module. But we can consider here that we have the following commutative diagram

$$\begin{tikzcd} & \pi_{\mathcal{L}} \arrow[dr] & \\ \pi_{\mathcal{T}} \arrow[ur, "\delta_*"] \arrow[dr, "\rho_*"'] & & \mathrm{Aut}(M) \\ & G \arrow[ur] & \end{tikzcd}$$

where $\pi_{\mathcal{T}} = \pi_1(|\mathcal{T}_S^c(G)|)$, and $\rho : \mathcal{T}_S^c(G) \longrightarrow \mathcal{B}(G)$ is the functor which sends, for every $P, Q \in \mathrm{Ob}(\mathcal{T}_S^c(G))$, $g \in \mathcal{T}_G(P, Q)$ to $g \in \mathrm{Mor}(o_G)$. In general, $H^*(G, M)$ will be smaller than $H^*(\mathcal{F}^c, M)$. In fact, we can identify

$$H^*(G, M) \cong \varprojlim_{\mathcal{T}_S(G)} H^*(-, M)$$

and,

$$H^*(\mathcal{F}_S^c(G), M) \cong \varprojlim_{\mathcal{T}_S^c(G)} H^*(-, M).$$

Hence, the inclusion of S in G induces an injective map $H^*(G, M) \subseteq H^*(\mathcal{F}^c, M)$. These inverse limits can also be translated in terms of inverse limits other the orbit categories $\mathcal{O}_S(G)$ and $\mathcal{O}(\mathcal{F}^c)$ respectively. A lot of techniques have been developed to compute limits, and higher limits, over orbit categories (see for example [AKO], Section III.3.5). Grodal also studies these questions in [Gr] and gives some answers defining the notion of M-centric-radical subgroups.

Definition 2.5.1. Let G be a finite group, M a $\mathbb{Z}_{(p)}[G]$-module and $K \leq G$ be the kernel of the action of G on M.

A proper p-subgroup P is call *M-centric* if $Z(P) \cap K$ is a Sylow p-subgroup of $C_G(P) \cap K$. If moreover, $O_p\left(N_G(P)/\left(P(C_G(P) \cap K)\right)\right) = 1$, then P is called *M-centric-radical.*

For example, if the action of G on M is trivial (i.e. $K = G$), P is M-centric-radical if, and only if, P is p-centric and a radical p-subgroup. On the other hand, if $K = 1$, P is M-centric-radical if and only if P is a radical p-subgroup of G.

Theorem 2.5.2 ([Gr], Corollary 10.4)**.** *Let G be a finite group, S a Sylow p-subgroup of G and M a $\mathbb{Z}_{(p)}[G]$-module.*

Let $\mathcal{H}$ be a family of subgroup of S containing S and all the subgroups which are radical and M-centric-radical.

Then, for every $\mathbb{Z}_{(p)}[G]$-module M, the inclusion of S in G induce a natural isomorphism,

$$H^*(G, M) \cong \varprojlim_{\mathcal{T}_S^{\mathcal{H}}(G)} H^*(-, M).$$

Hence, we have a condition under which the cohomology of G is isomorphic to the $\mathcal{F}^c$-stable elements.

Theorem 2.5.3. *Let G be a finite group, S be a Sylow p-subgroup of G and $(S, \mathcal{F}, \mathcal{L})$ be the associated p-local finite group. Let also M be a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module.*

Assume we have the following commutative diagram

$$\begin{tikzcd} & \pi_{\mathcal{L}} \arrow[dr] & \\ \pi_{\mathcal{T}} \arrow[ur, "\delta_*"] \arrow[dr, "\rho_*"'] & & Aut(M) \\ & G \arrow[ur] & \end{tikzcd}$$

and that every M-centric-radical subgroup is p-centric.

Then, the inclusion of S in G induces a natural isomorphism,

$$H^*(G, M) \cong \varprojlim_{\mathcal{T}_S^c(G)} H^*(-, M) = H^*(\mathcal{F}^c, M).$$

3 Idempotents problems

An important result in Broto, Levi and Oliver [BLO2], and a crucial tool in the proof of Theorem 2.3.4, is the existence of an $\mathcal{F}$-characteristic (S,S)-biset which leads to an idempotent of $H^*(S,\mathbb{F}_p)$ with image $H^*(\mathcal{F},\mathbb{F}_p)$. Unfortunately things are more complicated when we work with twisted coefficients.

In this chapter, after we recall few things on left-free bisets and how it works with trivial coefficients, we point out what is the problem when we consider twisted coefficients. We then construct, under some hypothesis, an idempotent from an $\mathcal{F}$-characteristic bisets when we work with any coefficients. We also look at the case of constrained fusion systems where everything true in the non twisted case can be extended. We finally show that the image of this idempotent is a δ-functor.

3.1 Left-free (G,H)-bisets

Let G, H be two finite groups.

Transitive (G,H)-bisets (here, G acts on the left and H on the right) are isomorphic to bisets of the form $(G\times H)/K$ for K a subgroup of $G\times H$. We can then use the Goursat Lemma to describe all these subgroups. Here, we are just interested in isomorphic classes of (G,H)-bisets where the action of G is free. In this setting, the classes of transitive left-free (G,H)-bisets are given by pairs (K,φ), where K is a subgroup of G and $\varphi\in\mathrm{Hom}(K,H)$ a group homomorphism.

Notation 3.1.1. For all (K,φ), with K a subgroup of G and $\varphi\in\mathrm{Hom}(K,H)$ a group homomorphism, we write

$$\Delta(K,\varphi)=\{(k,\varphi(k))\ ;\ k\in K\}\leq G\times H.$$

For a (G,H)-pair (K,φ), the set $\{K,\varphi\}:=(G\times H)/\Delta(K,\varphi)$ defines a (G,H)-biset and the isomorphic class of this biset is determined by the conjugacy class of $\Delta(K,\varphi)$ and we will denote by $[K,\varphi]$ this class.

We can also define a category $\mathcal{B}$, often called the *Burnside category*, where the objects are the finite groups and, for all finite groups G and H, $\mathcal{B}(G,H)$ is the set of isomorphic classes of (G,H)-bisets. The composition is given by the following construction.

Definition 3.1.2. Let G, H and K be finite groups, Ω a (G,H)-biset and Λ a (H,K)-biset. We define,

$$\Omega \circ \Lambda = \Omega \times_H \Lambda = \Omega \times \Lambda / \sim$$

where, for all $x \in \Omega$, $y \in \Lambda$ and $h \in H$, $(x, h.y) \sim (x.h, y)$.

This construction is compatible with isomorphisms, and then $\mathcal{B}$, endowed with the induced composition law, defines a category.

As we work with left-free bisets, we consider the subcategory $\mathcal{A} \subseteq \mathcal{B}$ where the objects are the same but we restrict the morphisms to isomorphic classes of left-free bisets. This gives us a category and the composition follows from the next lemma.

Lemma 3.1.3. *Let G, H and K be finite groups.*
Let $[K, \varphi] \in \mathcal{A}(G,H)$ and $[L, \psi] \in \mathcal{A}(H,K)$.
Then,

$$[K, \varphi] \circ [L, \psi] = \coprod_{x \in \varphi(K)\backslash H/L} [\varphi^{-1}(\varphi(K) \cap xLx^{-1}), \psi \circ c_{x^{-1}} \circ \varphi].$$

3.2 $\mathcal{F}$-characteristic bisets and trivial coefficients

Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group.

When we work with trivial coefficients, the idea is to consider the category $\mathcal{A}_{\mathcal{F}}$ where the objects are the subgroups of S and for P and Q two subgroups of S, $\mathcal{A}_{\mathcal{F}}(P,Q)$ is the set of isomorphic classes of $\mathcal{F}$-generated left-free (P,Q)-bisets, i.e. the (P,Q)-bisets union of transitive bisets of the form $[R, \varphi]$ with $R \leq P$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R, Q)$.

Then, for M a $\mathbb{Z}_{(p)}$-module, we construct a functor

$$M : \mathcal{A}_{\mathcal{F}} \longrightarrow \mathbb{Z}_{(p)}\text{-Mod}$$

defined on objects by $M(P) = H^*(P, M)$ for every $P \leq S$ and on morphisms as follow. For every $P, Q \leq S$, $R \leq P$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$, $({}_P[R,\varphi]_Q)_* = \mathrm{tr}_R^P \circ \varphi^*$ and, more generally, we define Ω_*, for every $\mathcal{F}$-generated left-free (P,Q)-biset Ω, by sum of its transitive components.

The existence of this functor will help us to construct an idempotent of $H^*(S, M)$ with image $H^*(\mathcal{F}^c, M)$. For that, we also need the notion of $\mathcal{F}$-characteristic (S,S)-biset.

Definition 3.2.1. Let Ω be a left-free (S,S)-biset.

(a) We say that Ω is *$\mathcal{F}$-generated* if it is the union of (S,S)-bisets of the form $[P, \varphi]$ with $P \in \mathrm{Ob}(\mathcal{F})$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$.

(b) We say that Ω is *left-$\mathcal{F}$-stable* if for all $P \in \mathrm{Ob}(\mathcal{F})$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$, we have ${}_\varphi\Omega_S \cong {}_P\Omega_S$, i.e.
$$\left({}_P[P,\varphi]_S\right) \circ [\Omega] = \left({}_P[P, \mathrm{incl}_P^S]_S\right) \circ [\Omega].$$

(c) We say that Ω is *right-$\mathcal{F}$-stable* if for all $P \in \mathrm{Ob}(\mathcal{F})$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$, we have ${}_S\Omega_\varphi \cong {}_S\Omega_P$, i.e.
$$[\Omega] \circ \left({}_S[\varphi(P), \varphi^{-1}]_P\right) = [\Omega] \circ \left({}_S[P, \mathrm{incl}_P^S]_P\right).$$

(d) We say that Ω is *non degenerate* if $|\Omega|/|S| \neq 0$ modulo p.

If Ω satisfies all this four properties, we say that Ω is an $\mathcal{F}$-characteristic (S,S)-biset.

The notion of $\mathcal{F}$-characteristic biset was first motivated by work of Linckelmann and Webb. They are the one who first formulated these conditions and recognized the importance of finding a biset with these properties. Broto, Levi and Oliver proved that such a biset always exists if the fusion system is saturated.

Proposition 3.2.2 ([BLO2], Proposition 5.5)**.** *Let $\mathcal{F}$ be a fusion system over a p-group S.*

If $\mathcal{F}$ is saturated, then there exists an $\mathcal{F}$-characteristic (S,S)-biset.

We can also mention that Ragnarsson and Stancu proved ([RS], Theorem A) that a fusion system $\mathcal{F}$ is saturated if, and only if, there exists an $\mathcal{F}$-characteristic (S,S)-biset.

Now, as in [BLO2], Proposition 5.5, we can show that this biset induces, for M a $\mathbb{Z}_{(p)}$-module, an idempotent of $H^*(S,M)$ with image $H^*(\mathcal{F}^c,M)$.

Proposition 3.2.3. *Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group and M be a $\mathbb{Z}_{(p)}$-module (with a trivial action of $\pi_{\mathcal{L}}$).*

If Ω is an $\mathcal{F}$-characteristic biset, then $\frac{|S|}{|\Omega|}\Omega_* \in End(H^*(S,M))$ defines an idempotent with image $H^*(\mathcal{F}^c,M)$.

Proof. We will show that $\mathrm{Im}\,(\Omega_*) \subseteq H^*(\mathcal{F}^c,M)$ and that $\Omega|_{H^*(\mathcal{F}^c,M)} = \mathrm{Id}_{H^*(\mathcal{F}^c,M)}$.

Let us start with the second point. If $x \in H^*(\mathcal{F}^c,M)$, for all $P \in \mathrm{Ob}(\mathcal{F}^c)$ and every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$,

$$[P,\varphi]_*(x) = \mathrm{tr}_P^S \circ \varphi^*(x) = \mathrm{tr}_P^S \circ \mathrm{Res}_P^S(x) = [S:P]x = \frac{|[P,\varphi]|}{|S|}x.$$

Thus, $\frac{|S|}{|\Omega|}\Omega_*(x) = x$.

The first point uses the $\mathcal{F}$-stability of Ω (Definition 3.2.1). For all $x \in H^*(S,M)$, $P \in \mathcal{F}$ and every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$,

$$\begin{aligned}\varphi^* \circ \Omega_*(x) = ({}_P[P,\varphi]_S)_* \circ ({}_S\Omega_S)_*(x) &= ({}_P[P,\varphi]_S \circ {}_S\Omega_S)_* \\ &= ({}_P[P,\mathrm{incl}_P^S]_S \circ {}_S\Omega_S)_*(x) \\ &= \mathrm{Res}_P^S \circ \Omega_*(x).\end{aligned}$$

Hence the image of $\frac{|S|}{|\Omega|}\Omega_*$ is included in $H^*(\mathcal{F}^c,M)$. ✩

3.3 Bisets and twisted coefficients

Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group

When we work with twisted coefficients, we have to be more careful. We cannot define, as in the trivial case, a functor from $\mathcal{A}_{\mathcal{F}}$ to $\mathbb{Z}_{(p)}$-Mod.

In fact, for M a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module, our cohomological functor $H^*(-,M)$ cannot be defined on $\mathcal{F}$ but only on $\mathcal{F}^c$ and thus, we can only consider $\mathcal{F}^c$-generated bisets, i.e. bisets union of transitive bisets of the form $[R,\varphi]$ with $R \in \mathrm{Ob}(\mathcal{F}^c)$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R,Q)$.

Definition 3.3.1. Let P, Q be two $\mathcal{F}$-centric subgroups of S.

A left-free (P, Q)-biset is $\mathcal{F}^c$-*generated*, if it is an union of transitive bisets of the form $[R, \varphi]$ with $R \in \mathrm{Ob}(\mathcal{F}^c)$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R, Q)$.

Unfortunately, by Lemma 3.1.3, we can see that the set of isomorphic classes of $\mathcal{F}^c$-generated bisets is, in general, not stable by composition. Hence, we can not, by analogy with $\mathcal{A}_{\mathcal{F}}$, define a category $\mathcal{A}_{\mathcal{F}^c}$ where the objects are the $\mathcal{F}$-centric subgroups of S and, for P and Q two $\mathcal{F}$-centric subgroups of S, $\mathcal{A}_{\mathcal{F}^c}(P, Q)$ is the set of isomorphic classes of $\mathcal{F}^c$-generated left-free (P, Q)-bisets.

Nevertheless, we have still a map from the set $A_{\mathcal{F}^c}(P, Q)$ of isomorphic classes of $\mathcal{F}^c$-generated left-free (P, Q)-bisets to $\mathrm{Hom}(H^*(P, M), H^*(Q, M))$ for all $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module M and $P, Q \leq S$.

For $P, Q, R \in \mathcal{F}^c$ with $R \leq Q$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R, P)$, we can associate to the (P, Q)-pair $\{R, \varphi\}$ a morphism

$$\{R, \varphi\}_* = \mathrm{tr}_R^Q \circ \varphi^* : H^*(P, M) \longrightarrow H^*(Q, M).$$

If we take another (P, Q)-biset $\{R', \varphi'\}$ isomorphic to $\{R, \varphi\}$ (this implies that R' is also $\mathcal{F}$-centric), we obtain the same morphism by Proposition 2.2.2. Indeed, let $\{R, \varphi\}$ and $\{R', \varphi'\}$ be two isomorphic (P, Q)-bisets. Then, there exists $g \in P$ and $h \in Q$ such that $(g, h)\Delta(R, \varphi)(g, h)^{-1} = \Delta(R', \varphi')$.

$$\begin{aligned}(g, h)\Delta(R, \varphi)(g, h)^{-1} &= \{(g.kg^{-1}, h.\varphi(k).h^{-1}) \ ; \ k \in R\} \\ &= \{(k, c_h \circ \varphi \circ c_{g^{-1}}(k)) \ ; \ k \in gRg^{-1}\}\end{aligned}$$

Hence, $R' = gRg^{-1}$, $\varphi' = c_h \circ \varphi \circ c_{g^{-1}}$ and,

$$\begin{aligned}\{R', \varphi'\}_* &= \mathrm{tr}_{R'}^Q \circ \varphi'^* \\ &= \mathrm{tr}_{hRh^{-1}}^Q \circ c_h^* \circ \varphi^* \circ c_{g^{-1}}^* \\ &= \mathrm{tr}_{hRh^{-1}}^Q \circ c_h^* \circ \varphi^* && \text{(because } g \in P \text{ and by 2.2.2)} \\ &= c_h^* \circ \mathrm{tr}_R^Q \circ \varphi^* && \text{(by 2.2.2)} \\ &= \mathrm{tr}_R^Q \circ \varphi^* && \text{(because } h \in Q \text{ and by 2.2.2)} \\ &= \{R, \varphi\}_*.\end{aligned}$$

Thus we can set $[R, \varphi]_*$ as the composite $\mathrm{tr}_R^Q \circ \varphi^*$ and it is well-defined. We finally define Ω_*, for all left-free $\mathcal{F}^c$-generated (P, Q)-biset Ω, by the sum of its transitive components.

Remark 3.3.2. By Remark 2.4.1, for $\varphi \in \mathrm{Mor}(\mathcal{F}^c)$, $\varphi^* = H^*(\varphi, -)$ is a morphism of δ-functors. Hence, as Ω_* is a sum of composites of transfers, restrictions and φ^*, for $\varphi \in \mathrm{Mor}(\mathcal{F}^c)$, which are all morphisms of δ-functors, it is a morphism of δ-functors.

3.4 Idempotents and twisted coefficients

In general, an $\mathcal{F}$-characteristic biset is not $\mathcal{F}^c$-generated.

Example 3.4.1. Let $(S, \mathcal{F}, \mathcal{L})$ be the p-local of $S = D_8$ in A_6. The lattice of subgroups of S has the following form.

$$\begin{tikzcd}[column sep=small]
 & & S \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & & \\
 & V \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & R \arrow[d, dash] & V' \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
Q_1 \arrow[drr, dash] & Q_2 \arrow[dr, dash] & Z \arrow[d, dash] & Q_2' \arrow[dl, dash] & Q_1' \arrow[dll, dash] \\
 & & \{e\} & &
\end{tikzcd}$$

The fusion is generated by the inner automorphisms of S, α an automorphism of V of order 3 which permutes Q_1, Q_2 and Z, and the analogue $\beta \in \mathrm{Aut}_{\mathcal{F}}(V')$. The minimal $\mathcal{F}$-characteristic biset is given by the following decomposition,

$$[D_8, \mathrm{Id}] \sqcup [V, \alpha] \sqcup [V', \beta] \sqcup [Q_1, \beta^{-1} \circ \alpha] \sqcup [Q_1', \alpha^{-1} \circ \beta].$$

Here, the subgroups Q_1 and Q_1' are not $\mathcal{F}$-centric and every $\mathcal{F}$-characteristic biset contains this minimal $\mathcal{F}$-characteristic biset.

Hence, when we work with twisted coefficients we cannot use the same biset or, at least, not directly. In fact, we will use it indirectly with the following hypothesis.

Hypothesis (A). Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group.
There exists an $\mathcal{F}$-characteristic (S, S)-biset Ω such that $\Omega = \Omega_1 \circ \Omega_2 \circ \cdots \circ \Omega_n$, where, for all i, Ω_i is an $\mathcal{F}^c$-generate (S, S)-biset.

This is for example the case for the p-local finite group of D_8 in A_6, if Ω is the minimal $\mathcal{F}$-characteristic biset and Ω^c the $\mathcal{F}^c$-generate part of Ω. Then, by Lemma 3.1.3, we get that $(\Omega^c)^4 = \Omega^2$. Hence, as Ω^2 is also $\mathcal{F}$-characteristic, it satisfies Hypothesis (A).

Hypothesis (A) seems reasonable but we do not know yet how to prove it. But we can make the following conjecture.

Conjecture 3.4.2. Hypothesis (A) is always true.

From now, we will use $\mathcal{F}$-characteristic bisets satisfying Hypothesis (A). Let us define the notion of Ω-endomorphism of $H^*(S, M)$ for M a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module. This notion is more artificial than really deep, we introduce it to get a better presentation of our propositions and theorems.

Definition 3.4.3. Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Let Ω be a $\mathcal{F}$-characteristic (S, S)-biset satisfying Hypothesis (A) and write $\Omega = \Omega_1 \circ \Omega_2 \circ \cdots \circ \Omega_n$ a decomposition of Ω in $\mathcal{F}^c$-generated bisets.

For M a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module, $\omega_* = \frac{|S|}{|\Omega|}(\Omega_1)_* \circ \cdots \circ (\Omega_n)_* \in \mathrm{End}(H^*(S, M))$ is called a Ω-*endomorphism*.

Remark 3.4.4. If the action of $\pi_{\mathcal{L}}$ on M is trivial, then $\omega_* = \frac{|S|}{|\Omega|}\Omega_*$ and, by Proposition 3.2.3, is an idempotent with image $H^*(\mathcal{F}^c, M)$.

If we assume Hypothesis (A), we can define for every finite $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module M, an idempotent of $H^*(S, M)$ using an Ω-endomorphism. But first let us look at the behavior of endomorphisms induced by $\mathcal{F}^c$-generated (S, S)-bisets with $\mathcal{F}^c$-stable elements.

Lemma 3.4.5. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and M be a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module.*

If Ω is an $\mathcal{F}^c$-generated (S, S)-biset, then the endomorphism $\Omega_* \in End(H^*(S, M))$ restricted to $H^*(\mathcal{F}^c, M)$ is the multiplication by $\frac{|\Omega|}{|S|}$.

In particular, if Ω is non degenerate, the image of $\Omega_* \in End(H^*(S, M))$ contains $H^*(\mathcal{F}^c, M)$.

Proof. If $x \in H^*(\mathcal{F}^c, M)$, for all $P \in \mathrm{Ob}(\mathcal{F}^c)$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$,

$$[P, \varphi]_*(x) = \mathrm{tr}_P^S \circ \varphi^*(x) = \mathrm{tr}_P^S \circ \mathrm{Res}_P^S(x) = [S : P]x = \frac{|[P, \varphi]|}{|S|}x.$$

Hence, as Ω is $\mathcal{F}^c$-generated, $\Omega_*(x) = \frac{|\Omega|}{|S|}x$. ✩

Remark that, when we work with $\mathcal{F}$-generate (S, S)-bisets, they are left-free and right-free (morphisms in a fusion system are injective homomorphisms). Thus, if Ω is a non degenerate $\mathcal{F}$-generated (S, S)-bisets, and $\Omega = \Omega_1 \circ \Omega_2$ with Ω_1 and Ω_2 other $\mathcal{F}$-generate (S, S)-bisets, then

$$|\Omega| = \frac{|\Omega_1| \times |\Omega_2|}{|S|}$$

and Ω_1 and Ω_2 are also non degenerate. Hence, by Lemma 3.4.5, we have the following corollary.

Corollary 3.4.6. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and M be a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module.*

If Ω is an $\mathcal{F}$-characteristic (S, S)-biset satisfying Hypothesis (A), and ω_* is an Ω-endomorphism of $H^*(S, M)$, then, the image of ω_* contains $H^*(\mathcal{F}^c, M)$.

Remark 3.4.7. Remark also that, by construction and Remark 3.3.2, ω_* is a morphism of δ-functors.

Proposition 3.4.8. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and let M be an abelian p-group with an action of $\pi_{\mathcal{L}}$. Let Ω be an $\mathcal{F}$-characteristic (S, S)-biset satisfying Hypothesis (A) and ω_* an Ω-endomorphism.*

For every $k \geq 0$, there is a natural number $N_{k,M} > 0$ such that, $(\omega_k)^{N_{k,M}}$ defines an idempotent $\overline{\omega}_{k,M}$ of $H^k(S, M)$ and we have

$$H^k(\mathcal{F}^c, M) \subseteq Im\,(\overline{\omega}_{k,M}).$$

Proof. To simplify the notations, we write $\omega = \omega_k$. For any $k \geq 0$, we have the following decreasing family of subgroups of $H^k(S, M)$.

$$H^k(\mathcal{F}^c, M) \subseteq \cdots \subseteq \mathrm{Im}\,(\omega^r) \subseteq \mathrm{Im}\,(\omega^{r-1}) \subseteq \cdots \subseteq \mathrm{Im}\,(\omega^1) \subseteq \mathrm{Im}\,(\omega^0) = H^k(S, M).$$

As $H^k(S,M)$ is a finite abelian p-group, this sequence stabilizes and then, there is an $n_0 \geq 1$ such that for all $n \geq n_0$, $\mathrm{Im}\,(\omega^n) = \mathrm{Im}\,(\omega^{n_0})$. In particular, $\omega^{n_0}|_{\mathrm{Im}\,(\omega^{n_0})}$ is a permutation of the finite set $\mathrm{Im}\,(\omega^{n_0})$ and there is an l such that $(\omega^{n_0}|_{\mathrm{Im}\,(\omega^{n_0})})^l = \mathrm{Id}_{\mathrm{Im}\,(\omega^{n_0})}$. Thus, for $N_{k,M} = l \times n_0$, the endomorphism $\overline{\omega}_{k,M} = \omega^{N_{k,M}} \in \mathrm{End}(H^k(S,M))$ is an idempotent with image $\mathrm{Im}\,(\omega^{n_0}) \supseteq H^k(\mathcal{F}^c, M)$. ✩

Hence, we can define an idempotent of $H^*(S,M)$ as following. For every $k \geq 0$ and every $x \in H^k(S,M)$,

$$\overline{\omega}_{k,M}(x) = (\omega_k)^{\prod_{i=0}^k N_{i,M}}(x)$$

Besides, this definition only depends on the Ω-endomorphism ω_*.

Definition 3.4.9. For Ω an $\mathcal{F}^c$-characteristic (S,S)-biset satisfying Hypothesis (A) and ω an Ω-endomorphism, the idempotent $\overline{\omega}_{*,-}$ of $H^*(S,-)$ obtained by this process is called the $\mathcal{F}^c$-*characteristic idempotent* associated to ω. We will also, for any abelian p-group M with an action of $\pi_{\mathcal{L}}$, denote by $I^*_\omega(M) \subseteq H^*(S,M)$ the image of $\overline{\omega}_{*,M}$.

Remark 3.4.10. Remark that, by Remark 3.4.4, if the action on M is trivial, then $I^*_\omega(M) = H^*(\mathcal{F}^c, M)$.

Proposition 3.4.11. *Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group.*

If Ω is an $\mathcal{F}^c$-characteristic (S,S)-biset satisfying Hypothesis (A) and ω_* is an Ω-endomorphism, then $\overline{\omega}_{*,-}$, the $\mathcal{F}^c$-characteristic idempotent induced by ω_*, defines an endomorphism of the δ-functor $\big(H^*(S,-), \delta_{H^*(S,-)}\big)$.

Proof. For M an abelian p-group with an action of $\pi_{\mathcal{L}}$ and $k \geq 0$, we denote by $N_{k,M}$ a natural number as in Proposition 3.4.8.

We have first to show that $\overline{\omega}_{*,-}$, the $\mathcal{F}^c$-characteristic idempotent associated to ω_*, defines a natural transformation from the functor $H^*(S,-)$ to itself. For every pair of abelian p-groups (M,N) with an action of $\pi_{\mathcal{L}}$ and every $\varphi \in \mathrm{Hom}_{\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]}(M,N)$, let us consider, for $k \geq 0$, the following diagram,

$$\begin{tikzcd}
H^k(S,M) \arrow[r, "\mathrm{Id}"] \arrow[d, "\overline{\omega}_{k,M}"'] & H^k(S,M) \arrow[r, "\varphi_k"] \arrow[d, "\tilde{\omega}_{k,M,N}"'] & H^k(S,N) \arrow[r, "\mathrm{Id}"] \arrow[d, "\tilde{\omega}_{k,M,N}"'] & H^k(S,N) \arrow[d, "\overline{\omega}_{k,M}"] \\
H^k(S,M) \arrow[r, "\mathrm{Id}"'] & H^k(S,M) \arrow[r, "\varphi_k"'] & H^k(S,N) \arrow[r, "\mathrm{Id}"'] & H^k(S,N)
\end{tikzcd}$$

where $\tilde{\omega}_{k,M,N} = (\omega_k)^{\prod_{i=0}^k N_{i,M} \times \prod_{i=0}^k N_{i,N}}$. The middle square commutes as $\tilde{\omega}_{k,M,N}$ is a finite iteration of ω_k and ω_* is an endomorphism of δ-functors by Remark 3.4.7. The first square commutes because, as $\overline{\omega}_{k,M}$ is an idempotent of $H^k(S,M)$, $\tilde{\omega}_{k,M,N} = \overline{\omega}_{k,M}^{\prod_{i=0}^k N_{i,N}} = \overline{\omega}_{k,M}$. Finally, the last one commutes because, as $\overline{\omega}_{k,N}$ is an idempotent of $H^k(S,N)$, $\tilde{\omega}_{k,M,N} = \overline{\omega}_{k,N}^{\prod_{i=0}^k N_{i,M}} = \overline{\omega}_{k,N}$. Hence, the exterior diagram commutes.

Now, to show that it defines a morphism of δ-functors, let consider a short exact sequence of abelian p-groups with an action of $\pi_{\mathcal{L}}$, $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$. By the previous argument we just have to show that, for $k \geq 0$, the following diagram commutes,

$$\begin{tikzcd} H^k(S,M) \arrow[r, "\delta"] \arrow[d, "\overline{\omega}_{k,M}"'] & H^{k+1}(S,L) \arrow[d, "\overline{\omega}_{k+1,L}"] \\ H^k(S,M) \arrow[r, "\delta"'] & H^{k+1}(S,L) \end{tikzcd}$$

where $\delta = \delta_{H^*(S,-)}$ corresponds to the connecting homomorphism. Consider then the following diagram,

$$\begin{tikzcd} H^k(S,M) \arrow[r, "\mathrm{Id}"] \arrow[d, "\overline{\omega}_{k,M}"'] & H^k(S,M) \arrow[r, "\delta"] \arrow[d, "\tilde{\omega}_{k,L,M}"'] & H^{k+1}(S,L) \arrow[r, "\mathrm{Id}"] \arrow[d, "\tilde{\omega}_{k+1,L,M}"'] & H^{k+1}(S,L) \arrow[d, "\overline{\omega}_{k+1,L}"] \\ H^k(S,M) \arrow[r, "\mathrm{Id}"'] & H^k(S,M) \arrow[r, "\delta"'] & H^{k+1}(S,L) \arrow[r, "\mathrm{Id}"'] & H^{k+1}(S,L) \end{tikzcd}$$

where

$$\tilde{\omega}_{k,L,M} = (\overline{\omega}_k)^{\prod_{i=0}^{k+1} N_{i,L} \times \prod_{i=0}^{k+1} N_{i,N}}$$

and

$$\tilde{\omega}_{k+1,L,M} = (\overline{\omega}_{k+1})^{\prod_{i=0}^{k+1} N_{i,L} \times \prod_{i=0}^{k+1} N_{i,N}} .$$

The middle square commutes as $\tilde{\omega}_{k,L,M}$ and $\tilde{\omega}_{k+1,L,M}$ are finite iterations of $\overline{\omega}_k$ and $\overline{\omega}_{k+1}$ and $\overline{\omega}_*$ is an endomorphism of δ-functors by Remark 3.4.7. The first square commutes because, as $\overline{\omega}_{k,M}$ is an idempotent of $H^k(S,M)$, $\tilde{\omega}_{k,L,M} = \overline{\omega}_{k,M}^{N_{k+1,M} \times \prod_{i=0}^{k+1} N_{i,L}} = \overline{\omega}_{k,M}$. The last one commutes because, as $\overline{\omega}_{k,L}$ is an idempotent of $H^k(S,L)$, $\tilde{\omega}_{k+1,L,M} = \overline{\omega}_{k+1,L}^{\prod_{i=0}^{k+1} N_{i,M}} = \overline{\omega}_{k+1,L}$. Thus, the exterior diagram commutes.

3.5 Idempotents and constrained fusion systems

When we work with a constrained fusion system, the (S,S)-characteristic biset is $\mathcal{F}^c$-generated and, working with a suitable category, it induces, for every $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module M, an idempotent of $H^*(S,M)$ with image $H^*(\mathcal{F}^c, M)$. Let us first recall the notion of constrained fusion systems.

Definition 3.5.1. Let $\mathcal{F}$ be a fusion system over a p-group S.
A subgroup $Q \leq S$ is *normal in* $\mathcal{F}$ if $Q \trianglelefteq S$, and for all $P, R \leq S$ and every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,R)$, φ extends to a morphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ, RQ)$ such that $\overline{\varphi}(Q) = Q$.
We write $O_p(\mathcal{F})$ for the maximal subgroup of S which is normal in $\mathcal{F}$.
We say that $\mathcal{F}$ is *constrained* if $O_p(\mathcal{F})$ is $\mathcal{F}$-centric.

Define, for $\mathcal{F}$ a fusion system over a p-group S and P_0 a subgroup of S, $\mathcal{A}_{\mathcal{F} \geq P_0}$ as follow.

$$\mathrm{Ob}(\mathcal{A}_{\mathcal{F} \geq P_0}) = \{P_0 \leq P \leq S\} \text{ is the set of all subgroups of } S \text{ containing } P_0$$

and for all $P, Q \in \mathrm{Ob}(\mathcal{A}_{\mathcal{F}\geq P_0})$,

$$\mathcal{A}_{\mathcal{F}\geq P_0}(P,Q) = \{\mathcal{F}\text{-generated left-free } (P,Q)\text{-bisets union of } [R,\varphi] \text{ with } R \geq P_0\}.$$

$A_{\mathcal{F}\geq P_0}$ is not in general a subcategory of $\mathcal{A}_{\mathcal{F}}$. The problem comes from Proposition 3.1.3: the set

$$\mathrm{Mor}(\mathcal{A}_{\mathcal{F}\geq P_0}) = \bigsqcup_{P,Q\in \mathrm{Ob}(\mathcal{A}_{\mathcal{F}\geq P_0})} \mathcal{A}_{\mathcal{F}\geq P_0}(P,Q)$$

is not stable by composition. But it is stable when the subgroup $P_0 \leq S$ is *strongly closed* in $\mathcal{F}$, i.e. no elements of P_0 is $\mathcal{F}$-conjugate to an elements of $S \setminus P$.

Lemma 3.5.2. *Let $\mathcal{F}$ be a fusion system over a p-group S.*

If $P_0 \trianglelefteq S$ is strongly closed in $\mathcal{F}$, then $\mathcal{A}_{\mathcal{F}\geq P_0}$, with the composition defined in 3.1.2, is a subcategory of $\mathcal{A}_{\mathcal{F}}$.

Proof. As P_0 is strongly closed in $\mathcal{F}$, for every $R, P \geq P_0$, every $s \in S$ and every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(R,S)$,

$$\varphi^{-1}(\varphi(R) \cap sPs^{-1}) \geq \varphi^{-1}(\varphi(P_0) \cap sP_0s^{-1}) = P_0.$$

thus, by Proposition 3.1.3, $\mathrm{Mor}(\mathcal{A}_{\mathcal{F}\geq P_0})$ is stable by composition and $\mathcal{A}_{\mathcal{F}\geq P_0}$ defines a subcategory of $\mathcal{A}_{\mathcal{F}}$. ✩

In particular, for $P_0 = O_p(\mathcal{F})$, which is normal in $\mathcal{F}$ and, thus, strongly closed in $\mathcal{F}$, $\mathcal{A}_{\mathcal{F}\geq O_p(\mathcal{F})}$ is a subcategory of $\mathcal{A}_{\mathcal{F}}$.

When $\mathcal{F}$ is constrained, $O_p(\mathcal{F})$ is $\mathcal{F}$-centric. Thus, every biset $\Omega \in \mathrm{Mor}(\mathcal{A}_{\mathcal{F}\geq O_p(\mathcal{F})})$ is $\mathcal{F}^c$-generated. Hence, if $\mathcal{F}$ is constrained, for every $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module M, we have, as in the trivial case, a functor

$$\begin{aligned} \mathcal{A}_{\mathcal{F}\geq O_p(\mathcal{F})} &\longrightarrow \mathbb{Z}_{(p)}\text{-Mod} \\ P &\longmapsto H^*(P,M) \\ {}_P[R,\varphi]_Q &\longmapsto \mathrm{tr}_R^P \circ \varphi^* \end{aligned}$$

Moreover, if we look at the minimal $\mathcal{F}$-characteristic (S,S)-biset, we have the following.

Proposition 3.5.3. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S.*

If Ω is the minimal $\mathcal{F}$-characteristic biset, then $\Omega \in \mathcal{A}_{\mathcal{F}\geq O_p(\mathcal{F})}$.

Proof. This is a direct corollary of [GRh], Proposition 9.11. Indeed, by [GRh], Proposition 9.11, every $[P,\varphi]$ which appears in the decomposition of Ω satisfy $P \geq O_p(\mathcal{F})$. ✩

Hence, using the same argument as for Proposition 3.2.3, we have the following theorem.

Theorem 3.5.4. *Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group, Ω be the minimal $\mathcal{F}$-characteristic (S,S)-biset and M be a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module.*

If $\mathcal{F}$ is constrained, then $\frac{|S|}{|\Omega|}\Omega_* \in End(H^*(S,M))$ is an idempotent with image the $\mathcal{F}^c$-stable elements $H^*(\mathcal{F}^c,M)$.

Proof. The proof is the same as the proof of Proposition 3.2.3. We will show that $\mathrm{Im}\,(\Omega_*) \subseteq H^*(\mathcal{F}^c, M)$ and that $\Omega|_{H^*(\mathcal{F}^c,M)} = \mathrm{Id}_{H^*(\mathcal{F}^c,M)}$.

Let us start with the second point. If $x \in H^*(\mathcal{F}^c, M)$, for all $P \in \mathrm{Ob}(\mathcal{F}^c)$ and every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$,

$$[P, \varphi]_*(x) = \mathrm{tr}_P^S \circ \varphi^*(x) = \mathrm{tr}_P^S \circ \mathrm{Res}_P^S(x) = [S : P]x = \frac{|[P, \varphi]|}{|S|}x.$$

Thus, $\frac{|S|}{|\Omega|}\Omega_*(x) = x$.

The first point uses the $\mathcal{F}$-stability of Ω (Definition 3.2.1). By [AKO] Proposition I.4.5, every $\mathcal{F}$-centric $\mathcal{F}$-radical subgroup of S contains $O_p(\mathcal{F})$. Then, for all $x \in H^*(S, M)$, $P \in \mathrm{Ob}(\mathcal{F}^{cr})$ and every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$, ${}_P[P, \varphi]_S \in \mathcal{A}_{\mathcal{F} \geq P_0}$ and

$$\begin{aligned} \varphi^* \circ \Omega_*(x) = ({}_P[P, \varphi]_S)_* \circ ({}_S\Omega_S)_*(x) &= ({}_P[P, \varphi]_S \circ {}_S\Omega_S)_* \\ &= ({}_P[P, \mathrm{incl}_P^S]_S \circ {}_S\Omega_S)_*(x) \\ &= \mathrm{Res}_P^S \circ \Omega_*(x). \end{aligned}$$

Hence, by Proposition 2.3.5, the image of $\frac{|S|}{|\Omega|}\Omega_*$ is included in $H^*(\mathcal{F}^c, M)$. ✩

3.6 A δ-functor

Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group, Ω be an $\mathcal{F}$-characteristic (S, S)-biset which satisfy (A), and ω_* an Ω-endomorphism.

For M an abelian p-group with an action of $\pi_{\mathcal{L}}$, let $\overline{\omega}_{*,-} \in \mathrm{End}(H^*(S, M))$ be the associated $\mathcal{F}^c$-characteristic idempotent.

Let us start with the behavior of δ-functors with idempotents. We refer the reader to Definition 2.1.2 for the definition of δ-functors and we recall that, by Remark 2.1.3, a δ-functor can be seen as a functor from the category $\mathcal{S}_{\mathcal{A}}$ of short exact sequences in $\mathcal{A}$ to $\mathrm{Ch}(\mathcal{B})$ which sends any short exact sequence to an acyclic chain complex.

Lemma 3.6.1. *Let* $(M_*, f_*) = \left(\cdots \xrightarrow{f_{l-2}} M_{l-1} \xrightarrow{f_{l-1}} M_l \xrightarrow{f_l} M_{l+1} \xrightarrow{f_{l+1}} \cdots\right)_{l \in \mathbb{Z}}$ *be a long exact sequence in an abelian category* $\mathcal{A}$.

Let $i_* : (M_*, f_*) \to (M_*, f_*)$ *be a morphism of long exact sequences such that, for all* $l \in \mathbb{Z}$, i_l *is an idempotent of* M_l.

Then the sequence

$$\cdots \xrightarrow{f_{l-2}} Im\,(i_{l-1}) \xrightarrow{f_{l-1}} Im\,(i_l) \xrightarrow{f_l} Im\,(i_{l+1}) \xrightarrow{f_{l+1}} \cdots$$

is exact.

Proof. Let $l \in \mathbb{Z}$ and $x \in \mathrm{Im}\,(i_l)$ such that $f_l(x) = 0$. By exactness of (M_*, f_*) in l, there is a $y \in M_{l-1}$ such that $f_{l-1}(y) = x$. Thus $x = i_l(x) = i_l \circ f_{l-1}(y) = f_{l-1} \circ i_{l-1}(y)$ and hence we obtain the exactness of $(\mathrm{Im}\,(i_*), f_*)$ in degree l. ✩

Proposition 3.6.2. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories and let $(F^*, \delta_F) : \mathcal{A} \to \mathcal{B}$ be a δ-functor.*

If $i^* : (F^*, \delta_F) \to (F^*, \delta_F)$ is an idempotent of δ-functors, then $(Im(i^*), \delta_F)$ defines a δ-functor.

Proof. By remark 2.1.3, a δ-functor can be seen as a functor from the category $\mathcal{S}_{\mathcal{A}}$ of short exact sequences in $\mathcal{A}$ to $\mathrm{Ch}(\mathcal{B})$ which sends any short exact sequence to an acyclic chain complex. A morphism of δ-functors is then a natural transformation in that setting. With this point of view, this is just a corollary of Lemma 3.6.1. ✩

Theorem 3.6.3. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group, Ω an $\mathcal{F}$-characteristic (S,S)-biset satisfying Hypothesis (A) and ω_* an Ω-endomorphism. Then, the functor $I^*_\omega(-)$, with the connecting homomorphism $\delta_{H^*(S,-)}$, defines a δ-functor from the category of finite $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-modules to $\mathbb{Z}_{(p)}$-Mod.*

Proof. This is a direct corollary of Proposition 3.4.11 and Proposition 3.6.2. ✩

4 The cohomology of classifying spaces of fusion systems

We apply here Theorem 3.6.3 to the study of the cohomology of the classifying space of a fusion system. We first look at the behavior of twisted cohomology after p-completion.

4.1 Cohomology of p-good spaces

Here, for X a topological space,

$$\lambda_X : X \to X_p^\wedge$$

will denote the natural transformation associated to the p-completion.

Lemma 4.1.1. *Let X be a space and let $0 \to L \to M \to N \to 0$ be a short exact sequence of $\mathbb{Z}_{(p)}[\pi_1(X_p^\wedge)]$-modules.*

If λ_X induces isomorphisms $H^*(X_p^\wedge, L) \cong H^*(X, L)$ and $H^*(X_p^\wedge, N) \cong H^*(X, N)$, then λ_X induces an isomorphism

$$H^*(X, M) \cong H^*(X_p^\wedge, M).$$

Proof. Let consider the exact sequences in cohomology induced by the short exact sequence,

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

and look at the following commutative diagram,

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H^{n-1}(X_p^\wedge, N) & \longrightarrow & H^n(X_p^\wedge, L) & \longrightarrow & H^n(X_p^\wedge, M) & \longrightarrow & H^n(X_p^\wedge, N) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \lambda_X^*} & & \downarrow{\scriptstyle \lambda_X^*} & & \downarrow{\scriptstyle \lambda_X^*} & & \downarrow{\scriptstyle \lambda_X^*} & \\
\cdots \longrightarrow & H^{n-1}(X, N) & \longrightarrow & H^n(X, L) & \longrightarrow & H^n(X, M) & \longrightarrow & H^n(X, N) & \longrightarrow \cdots
\end{array}$$

The two lines are exact and λ_X^* gives the isomorphisms $H^*(X_p^\wedge, L) \cong H^*(X, L)$ and $H^*(X_p^\wedge, N) \cong H^*(X, N)$. We then conclude with the five lemma. ✩

Proposition 4.1.2. *Let X be a space and M be an abelian p-group with an action of $\pi_1(X^\wedge_p)$.*

If X is p-good, then λ_X induces a natural isomorphism

$$H^*(X^\wedge_p, M) \cong H^*(X, M).$$

Proof. As X is p-good, λ_X induces an isomorphism, $H^*(X^\wedge_p, \mathbb{F}_p) \cong H^*(X, \mathbb{F}_p)$. and, by Proposition 0.10, $\pi_1(X^\wedge_p)$ is a p-group quotient of $\pi_1(X)$. In particular, the action of $\pi_1(X^\wedge_p)$ on M is nilpotent: there is a sequence

$$\{0\} = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

such that, for any $1 \leq i \leq n$, $M_i/M_{i-1} \cong \mathbb{F}_p$ is the trivial module. We conclude then by induction on n, by Lemma 4.1.1. ✩

Corollary 4.1.3. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group.*

If M is an abelian p-group with an action of $\pi_1(|\mathcal{L}|^\wedge_p)$, $\lambda_{|\mathcal{L}|}$ induces an isomorphism

$$H^*(|\mathcal{L}|^\wedge_p, M) \cong H^*(|\mathcal{L}|, M).$$

Proof. As $|\mathcal{L}|$ is p-good by Theorem 1.3.1, we can apply Proposition 4.1.2. ✩

4.2 Cohomology of $|\mathcal{L}|$ with nilpotent coefficients

Lemma 4.2.1. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. The natural inclusion δ_S of $\mathcal{B}(S)$ in $\mathcal{L}$ induces, for any $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module M, a natural morphism in cohomology*

$$H^*(|\mathcal{L}|, M) \longrightarrow H^*(\mathcal{F}^c, M) \subseteq H^*(S, M).$$

Proof. For $P \in \mathrm{Ob}(\mathcal{F}^c)$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$, let $\widetilde{\varphi} \in \mathrm{Mor}_{\mathcal{L}}(P, \varphi(P))$ be such that $\pi(\widetilde{\varphi}) = \varphi \in \mathrm{Hom}_{\mathcal{F}}(P, \varphi(P))$. From the definition 1.2.1 of a linking system, for every $g \in P$, we have the following commutative diagram in $\mathcal{L}$.

$$\begin{tikzcd}
P \arrow[r, "\widetilde{\varphi}"] \arrow[d, "\delta_P(g)"'] & \varphi(P) \arrow[d, "\delta_{\varphi(P)}(\varphi(g))"] \\
P \arrow[r, "\widetilde{\varphi}"] & \varphi(P)
\end{tikzcd}$$

Thus, we have a natural transformation between the functors δ_P and $\delta_{\varphi(P)} \circ \mathcal{B}(\varphi)$.

$$\begin{tikzcd}
& \mathcal{L} & \\
\mathcal{B}(P) \arrow[ur, "\delta_P"] \arrow[rr, "\mathcal{B}(\varphi)"'] & & \mathcal{B}(\varphi(P)) \arrow[ul, "\delta_{\varphi(P)}"']
\end{tikzcd}$$

Hence, by Proposition 2.2, the maps $|\delta_P|$ and $|\delta_{\varphi(P)}\circ\mathcal{B}(\varphi)| = |\delta_{\varphi(P)}|\circ|\mathcal{B}(\varphi)|$ are homotopic. In particular, the following diagram commute

$$\begin{tikzcd} & H^*(|\mathcal{L}|, M) \arrow[dl, "\delta_P^*"'] \arrow[dr, "\delta_{\varphi(P)}^*"] & \\ H^*(P, M) & & H^*(\varphi(P), M) \arrow[ll, "\varphi^*"] \end{tikzcd}$$

and the lemma follows. ✩

Lemma 4.2.2. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group, Ω be an $\mathcal{F}$-characteristic (S, S)-biset satisfying Hypothesis (A) and ω_* an Ω-endomorphism. Let also $0 \to L \to M \to N \to 0$ be a short exact sequence of finite $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-modules.*

If δ_S induces the following isomorphisms, $H^*(|\mathcal{L}|, L) \cong I^*_\omega(L)$ and $H^*(|\mathcal{L}|, N) \cong I^*_\omega(N)$, then δ_S induces an isomorphism

$$H^*(|\mathcal{L}|, M) \cong I^*_\omega(M).$$

Proof. Consider the exact sequences in cohomology induced by the short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

and look at the following diagram (where $\overline{\omega}_{*,-}$ denotes the $\mathcal{F}$-characteristic idempotent associated to ω_*).

$$\begin{tikzcd} \cdots \arrow[r] & H^{n-1}(|\mathcal{L}|, N) \arrow[r] \arrow[d, "\overline{\omega}_{n-1,N}\circ\delta_S^*"'] & H^n(|\mathcal{L}|, L) \arrow[r] \arrow[d, "\overline{\omega}_{n,L}\circ\delta_S^*"'] & H^n(|\mathcal{L}|, M) \arrow[r] \arrow[d, "\overline{\omega}_{n,M}\circ\delta_S^*"'] & H^n(|\mathcal{L}|, N) \arrow[r] \arrow[d, "\overline{\omega}_{n,N}\circ\delta_S^*"'] & \cdots \\ \cdots \arrow[r] & I^{n-1}_\omega(N) \arrow[r] & I^n_\omega(L) \arrow[r] & I^n_\omega(M) \arrow[r] & I^n_\omega(N) \arrow[r] & \cdots \end{tikzcd}$$

As $H^*(|\mathcal{L}|, -)$ is a δ-functor and, by Theorem 3.6.3, I^*_ω defines also a δ-functor, the two lines are exact and, as by Proposition 3.4.11 the $\mathcal{F}^c$-characteristic idempotent associated to ω_* defines a morphism of δ-functors, this diagram is commutative. We conclude then with the five lemma. ✩

Theorem 4.2.3. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group.*

Assume Hypothesis (A) is satisfied.

If M is an abelian p-group with a nilpotent action of $\pi_1(|\mathcal{L}|)$,

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

Proof. As the action of $\pi_{\mathcal{L}}$ is nilpotent, there is a sequence

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

such that, for every $1 \leq i \leq n$, the action of $\pi_{\mathcal{L}}$ on M_i/M_{i-1} is trivial. We know, by Theorem 2.3.4 and Remark 3.4.10, that for $1 \leq i \leq n$, δ_S induces an isomorphism

$$H^*(|\mathcal{L}|, M_i/M_{i-1}) \cong H^*(\mathcal{F}^c, M_i/M_{i-1}) = I^*_\omega(M_i/M_{i-1}).$$

By induction on n and, by Lemma 4.2.2, we get that $H^*(|\mathcal{L}|, M) \cong I^*_\omega(M)$. Finally we also have from Lemma 4.2.1 that

$$\delta_S(H^*(|\mathcal{L}|, M)) \subseteq H^*(\mathcal{F}^c, M) \subseteq I^*_\omega(M).$$

Then $H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M) = I^*_\Omega(M)$. ✩

Corollary 4.2.4. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group.*

Assume Hypothesis (A) is satisfied.

If M is an abelian p-group with an action of $\pi_1(|\mathcal{L}|^\wedge_p)$, then $\lambda_{|\mathcal{L}|} \circ \delta^*_S$ induces a natural isomorphism

$$H^*(|\mathcal{L}|^\wedge_p, M) \cong H^*(\mathcal{F}^c, M).$$

Proof. By Theorem 1.3.1, $|\mathcal{L}|$ is p-good. Then $\pi_1(|\mathcal{L}|^\wedge_p)$ is a p-group (by Proposition 0.10) and its action on M is nilpotent. Hence, this is just a corollary of Theorem 4.2.3 and 4.1.3. ✩

5 Cohomology with coefficients twisted by a p-solvable action

In this chapter we are interested in the study of p-solvable actions. We first look at p-local subgroups of index prime to p which allows us to study the case of action factoring through a p'-group. In a second section, we start by studying constrained fusion systems and we finish by realizable p-local finite groups and p-solvable action.

5.1 Extension by a p'-group

In this section, we study the case of p'-actions on the coefficients. The main ingredient here is the use of p-local subgroups of index prime to p.

5.1.1 The minimal p-local subgroup of index prime to p

Recall that, for an infinite group G, we denote by $O^{p'}(G)$ the intersection of all normal subgroups in G of finite index prime to p.

Lemma 5.1.1. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and $(S, O^{p'}(\mathcal{F}), O^{p'}(\mathcal{L}))$ its minimal p-local subgroup of index prime to p.*

If M is a $\mathbb{Z}_{(p)}[\pi_\mathcal{L}]$-module, then the inclusion $O^{p'}(\mathcal{L}) \subseteq \mathcal{L}$ induces the following isomorphism,

$$H^*(|\mathcal{L}|, M) \cong H^*(|O^{p'}(\mathcal{L})|, M)^{\pi_\mathcal{L}/O^{p'}(\pi_\mathcal{L})}.$$

Proof. As $|O^{p'}(\mathcal{L})|$ is, up to homotopy, a covering space of $|\mathcal{L}|$ with fundamental group $O^{p'}(\pi_\mathcal{L}) \trianglelefteq \pi_\mathcal{L}$ (Theorem 1.3.6 and Theorem 1.3.7), this is just a consequence of the transfer. ✩

Lemma 5.1.2. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and $(S, O^{p'}(\mathcal{F}), O^{p'}(\mathcal{L}))$ its minimal p-local subgroup of index prime to p.*

If M is a $\mathbb{Z}_{(p)}[\pi_\mathcal{L}]$-module, then

$$H^*(\mathcal{F}^c, M) \cong H^*(O^{p'}(\mathcal{F})^c, M)^{Aut_\mathcal{F}(S)/Aut_{O^{p'}(\mathcal{F})}(S)}.$$

Proof. By Proposition 1.3.6, we have $\mathcal{F} = \langle O^{p'}(\mathcal{F}), \mathrm{Aut}_{\mathcal{F}}(S)\rangle$ and $\mathrm{Ob}(O^{p'}(\mathcal{F})^c) = \mathrm{Ob}(\mathcal{F}^c)$. The result follows as an easy consequence. ✩

Theorem 5.1.3. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and $(S, O^{p'}(\mathcal{F}), O^{p'}(\mathcal{L}))$ its minimal p-local subgroup of index prime to p.*

If M is a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module and if the inclusion δ_S induces an isomorphism

$$H^*(|O^{p'}(\mathcal{L})|, M) \cong H^*(O^{p'}(\mathcal{F})^c, M),$$

then δ_S induces an isomorphism

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

Proof. We have the following commutative diagram (recall that $\pi_{O^{p'}(\mathcal{L})} = O^{p'}(\pi_{\mathcal{L}})$).

$$\begin{tikzcd} \mathcal{B}S \arrow[r, "\delta_S"] \arrow[dr, "\delta_S"'] & \mathcal{L} \arrow[rr, "\omega"] & & \mathcal{B}\pi_{\mathcal{L}} \arrow[r] & \mathcal{B}(\mathrm{Aut}(M)) \\ & \mathcal{B}\left(O^{p'}(\mathcal{L})\right) \arrow[u] \arrow[rr, "\omega"] & & \mathcal{B}\left(O^{p'}(\pi_{\mathcal{L}})\right) \arrow[u] \arrow[ur] & \end{tikzcd}$$

The projection $\pi : \mathcal{L} \longrightarrow \mathcal{F}$ induces, by Proposition 1.3.6 and Theorem 1.3.7, an isomorphism

$$\pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}}) \cong \pi_1(|\mathcal{F}|)/\pi_1(|O^{p'}(\mathcal{F})|)$$

which is naturally isomorphic to $\mathrm{Aut}_{\mathcal{F}}(S)/\mathrm{Aut}_{O^{p'}(\mathcal{F})}(S)$. Then, by the two previous lemmas, we obtain

$$\begin{aligned} H^*(|\mathcal{L}|, M) &\cong H^*(|O^{p'}(\mathcal{L})|, M)^{\pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}})} \\ &\cong \left(\varprojlim_{O^{p'}(\mathcal{F})^c} H^*(-, M)\right)^{\mathrm{Aut}_{\mathcal{F}}(S)/\mathrm{Aut}_{O^{p'}(\mathcal{F})}(S)} \\ &\cong H^*(\mathcal{F}^c, M). \end{aligned}$$

For the second isomorphism, we have to be careful on the actions of $\pi_{\mathcal{L}}$ on the left and $\mathrm{Aut}_{\mathcal{F}}(S)$ on the right. In fact here, by Definition 3.2.1 of $\mathcal{F}^c$-stable elements, we can see it on the chain level. The map $\delta_S^* : H^*(|O^{p'}(\mathcal{L})|, M) \longrightarrow H^*(S, M)$ is induced by the inclusion $\delta_S : BS \longrightarrow |O^{p'}(\mathcal{L})|$, which gives on the chain level,

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}_{(p)}[S]}\left(C_*\left(|\widetilde{O^{p'}(\mathcal{L})}|\right), M\right) &\longrightarrow \mathrm{Hom}_{\mathbb{Z}_{(p)}[\pi_{O^{p'}(\mathcal{L})}]}(C_*(|\mathcal{E}(S)|), M) \\ f &\longmapsto f|_{C_*(|\mathcal{E}(S)|)} \end{aligned}$$

where $\mathcal{E}(S)$ is defined as in the proof of Proposition 2.4, i.e. the category with set of object S and for each $(s, s') \in S$, $\mathrm{Mor}_{\mathcal{E}(S)}(s, s') = \{\varphi_{s,s'}\}$. Then for $\varphi \in \mathrm{Aut}_S(\mathcal{F})$, if we choose a lift $\widetilde{\varphi} \in \mathrm{Aut}_{\mathcal{L}}(S)$, acts on the left by

$$f \longmapsto \omega(\widetilde{\varphi}^{-1}) f \omega(\widetilde{\varphi}),$$

and on the right by,

$$f \longmapsto \omega(\widetilde{\varphi})^{-1} f \circ \varphi^* \ .$$

Finally, the action of φ on $\mathcal{E}(S)$ corresponds to the action of $\omega(\varphi)$ on $|\mathcal{E}(S)|$ (indeed, a lift of $\omega(\varphi)$ join every vertex $s \in S$ of $|\mathcal{E}(S)|$ to the vertex $\varphi(s)$). Hence, the two actions coincide. ✩

Corollary 5.1.4. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and M an abelian p-group with an action of $\pi_{\mathcal{L}}$.*

If the action of $\pi_{\mathcal{L}}$ on M factors through a p'-group then δ_S induces an isomorphism,

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

Corollary 5.1.5. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and M an abelian p-group with an action of $\pi_{\mathcal{L}}$.*

Assume that hypothesis (A) is satisfied.

If the action of $\pi_{\mathcal{L}}$ on M factors through an extension of a p-group by p'-group then δ_S induces an isomorphism,

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

5.1.2 Remarks on actions factoring through a p'-group

Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group.

According to Corollary 5.1.4, the cohomology of $|\mathcal{L}|$ with coefficients in an abelian p-group with an action of $\pi_{\mathcal{L}}$ factoring through a p'-group can be computed by stable elements. That condition appears if, and only if, the action of S on the coefficients is trivial.

Proposition 5.1.6. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and M a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module.*

The action of $\pi_{\mathcal{L}}$ on M factors through a p'-group if, and only if, the action of S on M is trivial.

Proof. We look at the normal subgroup H of $\pi_{\mathcal{L}}$ generated by $\delta_S(S)$ and we want to show that $H \geq O^{p'}(\pi_{\mathcal{L}})$ (the other inclusion is clear). Consider the homomorphism $\pi_* : \pi_{\mathcal{L}} \to \pi_1(|\mathcal{F}^c|)$ induced by $\pi : \mathcal{L} \longrightarrow \mathcal{F}^c$. We know that π_* is surjective with kernel $O^{p'}(\pi_{\mathcal{L}})$ (Proposition 1.3.6). Hence π_* factors through $p : \pi_{\mathcal{L}}/H \to \pi_1(|\mathcal{F}^c|)$. Let construct the inverse $s : \pi_1(|\mathcal{F}^c|) \to \pi_{\mathcal{L}}/H$ on the generators by the following way: for $\alpha \in \mathrm{Mor}_{\mathcal{F}^c}(P, Q)$ we write $s(\theta(\alpha)) = \omega(\widetilde{\alpha})$ where $\widetilde{\alpha} \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ is such that $\pi(\widetilde{\alpha}) = \alpha$ and $\theta : \mathcal{F} \longrightarrow \mathcal{B}(\pi_{\mathcal{F}})$ is the natural functor (defined in 2.5). The application s is well-defined (because the difference between two elements sent by π on $\alpha \in \mathrm{Mor}_{\mathcal{F}^c}(P, Q)$ is in $C_S(P) \leq S$) and defines a group homomorphism ($\widetilde{\alpha_1} \circ \widetilde{\alpha_2} = \widetilde{\alpha_1 \circ \alpha_2}$) which is the inverse of p. ✩

5.2 Constrained fusion systems and coefficients with a p-solvable action

We studied here the case of p-local finite groups associated to constrained fusion systems. We also give some more refine results about realizable p-local finite groups when the action factor trough a p-solvable group.

5.2.1 Constrained fusion systems

Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Here, we assume that $\mathcal{F}$ is a constrained fusion system (see definition 3.5.1). An important and classical result about constrained fusion system is the existence of a *model* (See [AKO] for more details).

Theorem 5.2.1. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group.*
If $\mathcal{F}$ is constrained, there exists a finite group G such that

(a) S is a Sylow subgroup of G,

(b) $C_G(O_p(G)) \leq O_p(G)$,

(c) $\mathcal{F}_S(G) = \mathcal{F}$.

This group G is called a *model* of $\mathcal{F}$ and is unique in a precise way (see [AKO], Theorem I.4.9, for more details). From now let G be a model of $\mathcal{F}$.

Let us now study the homotopy type of $|\mathcal{L}|$. For that purpose we introduce $\mathcal{H} = \{P \in \mathrm{Ob}(\mathcal{L}) \mid P \geq O_p(G)\}$ and $\mathcal{L}^{\mathcal{H}}$ the full subcategory of $\mathcal{L}$ with set of objects $\mathcal{H}$.

Lemma 5.2.2. *The CW complex $|\mathcal{L}^{\mathcal{H}}|$ is a classifying space of G.*

Proof. The functor

$$\begin{array}{rrcl} F: & \mathcal{L}^{\mathcal{H}} & \longrightarrow & \mathcal{L}^{\{O_p(G)\}} \\ & P \in \mathcal{L}^{\mathcal{H}} & \longmapsto & O_p(G) \\ & g \in T_G(P,Q) & \longmapsto & g \in N_G(O_p(G)) = G \end{array}$$

gives us a retraction by deformation of $|\mathcal{L}^{\mathcal{H}}|$ on the geometric realization of the full subcategory of $\mathcal{L}$ with unique object $O_p(G) \leq S$. As $\mathrm{Aut}_{\mathcal{L}}(O_p(G)) = N_G(O_p(G)) = G$, this last category is $\mathcal{B}(G)$. in particular, its geometric realization is a classifying space of G. ✩

When the fusion system is constrained, we always have a retraction by deformation from $|\mathcal{L}|$ to $|\mathcal{L}^{\mathcal{H}}|$.

Proposition 5.2.3. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group.*
If $\mathcal{F}$ is constrained and G is a model of $\mathcal{F}$, then $|\mathcal{L}|$ is a classifying space of G.

Proof. As $O_p(G) \trianglelefteq G$, the functor

$$\begin{array}{rrcl} F: & \mathcal{L} & \longrightarrow & \mathcal{L}^{\mathcal{H}} \\ & P \in \mathcal{L} & \longmapsto & PO_p(G) \in \mathcal{L}^{\mathcal{H}} \\ & g \in \mathrm{Mor}_{\mathcal{L}}(P,Q) & \longmapsto & \overline{g} \in \mathrm{Mor}_{\mathcal{L}}(PO_p(G), QO_p(G)) \end{array}$$

where $\overline{g}$ is the unique extension of $g \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ in $\mathrm{Mor}_{\mathcal{L}}(PO_p(G), QO_p(G))$ (see [AKO], Proposition III.4.3), is well-defined and induces a retraction by deformation of $|\mathcal{L}|$ on $|\mathcal{L}^{\mathcal{H}}|$. We conclude with Lemma 5.2.2. ✩

The fundamental group of $|\mathcal{L}|$ is isomorphic to G and, in particular, every $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module is naturally a $\mathbb{Z}_{(p)}[G]$-module. The problem is now to compare the cohomology of G, a model of $\mathcal{F}$, and the $\mathcal{F}^c$-stable elements.

Proposition 5.2.4. *Let G be a finite group and S a Sylow p-subgroup of G.*

If $C_G(O_p(G)) \leq O_p(G)$, then, for a $\mathbb{Z}_{(p)}[G]$-module M, the inclusion of S in G induces a natural isomorphism

$$H^*(G,M) \cong H^*(\mathcal{F}_S^c(G), M).$$

Proof. We consider here the p-local finite group $(S,\mathcal{F},\mathcal{L}) = (S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$. By assumption, $\mathcal{F}_S(G)$ is constrained and G is a model of $\mathcal{F}_S(G)$. Hence $H^*(\mathcal{F}^c, M)$ is well-defined. We know that $Res_S^G : H^*(G,M) \longrightarrow H^*(S,M)$ is injective and we can easily see that $\mathrm{Im}\,(Res_S^G) \leq H^*(\mathcal{F}^c, M)$. From Cartan-Eilenberg Theorem (Theorem 2.2.4) we know that $\mathrm{Im}\,(\mathrm{Res}_S^G) = \varprojlim_{\mathcal{T}_S(G)} H^*(-,M)$. Consider then $x \in H^*(\mathcal{F}^c, M) = \varprojlim_{\mathcal{F}_S^c(G)} H^*(-,M) = \varprojlim_{\mathcal{T}_S^c(G)} H^*(-,M)$, $P \leq S$ and $g \in N_G(P)$. In $\mathcal{T}_S(G)$ we have the following commutative diagram

$$\begin{tikzcd} PO_p(G) \arrow[r, "g"] & gPg^{-1}O_p(G) \\ P \arrow[u, "e"] \arrow[r, "g"] & gPg^{-1} \arrow[u, "e"'] \end{tikzcd}$$

where e is the trivial element of G. Hence, as the above part of the diagram is in $T_S^c(G)$ and $x \in \varprojlim_{\mathcal{T}_S^c(G)} H^*(-,M)$,

$$c_g^* \circ \mathrm{Res}^S_{gPg^{-1}}(x) = \mathrm{Res}_P^{PO^p(G)} \circ c_g^* \circ \mathrm{Res}^S_{gPg^{-1}O_p(G)}(x) = \mathrm{Res}_P^{PO^p(G)} \circ \mathrm{Res}^S_{PO_p(G)}(x) = \mathrm{Res}^S_P(x).$$

Thus $x \in \varprojlim_{\mathcal{T}_S^c(G)} H^*(-,M) = \varprojlim_{\mathcal{F}_S^c(G)} H^*(-,M)$. ✩

Corollary 5.2.5. *Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group.*

If $\mathcal{F}$ is constrained and M is a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module, then δ_S induces a natural isomorphism,

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

5.2.2 Realizable fusion systems and p-solvable actions

Consider here a finite group G, S a Sylow p-subgroup of G and let $(S, \mathcal{F}, \mathcal{L})$ be the associated p-local finite group. In that setting, working with $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-modules with a natural action of G, we can compare the cohomology of $|\mathcal{L}|$ and the cohomology of G when the action factor through a p-solvable group. The key is the use of p-local subgroups of index p or prime to p.

We will denote by

$$\rho : \mathcal{T}_S(G) \longrightarrow \mathcal{B}(G)$$

the functor which sends each object in the source to the unique one in the target and sends, for every $P, Q \leq S$, $g \in T_G(P, Q)$ to $g \in \mathrm{Mor}_{\mathcal{B}(G)}(o_G)$.

Finally, let $\mathcal{T} = \mathcal{T}_S^c(G)$ be the centric transporter category of G, $\mathcal{L}^q = \mathcal{L}_S^q(G)$ be the quasicentric linking system associated to G and $\mathcal{T}^q = \mathcal{T}_S^q(G)$ be the associated quasicentric transporter category.

Lemma 5.2.6. *Let G be a finite group and $(S, \mathcal{F}, \mathcal{L})$ be the associated p-local finite group with $\mathcal{L}$ a linking system which is not necessarily centric. Let $\mathcal{T} = \mathcal{T}_S^{Ob(\mathcal{L})}(G)$ be the transporter category associated to G with set of objects $Ob(\mathcal{L})$.*

If M is a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module, then the canonical functor $\delta : \mathcal{T} \to \mathcal{L}$ induces a natural isomorphism $H^*(|\mathcal{T}|, M) \cong H^*(|\mathcal{L}|, M)$.

Proof. This is a consequence of [BLO1], Lemma 1.3, with $\mathcal{C} = \mathcal{T}$, $\mathcal{C}' = \mathcal{L}$ and the functor $T : \mathcal{L}^{\mathrm{op}} \to \mathbb{Z}_{(p)}$-Mod which sends each object on M, and each morphism on its action on M. Then δ induces a natural isomorphism $\varprojlim_{\mathcal{T}}{}^*(M) \cong \varprojlim_{\mathcal{L}}{}^*(M)$. Then

$$H^*(|\mathcal{T}|, M) = \varprojlim_{\mathcal{T}}{}^*(M) \cong \varprojlim_{\mathcal{L}}{}^*(M) = H^*(|\mathcal{L}|, M)$$

Where the first and last equality is just an interpretation in terms of functor cohomology and can be found in [LR], Proposition 3.9. ✩

Theorem 5.2.7. *Let G be a finite group, S a Sylow p-subgroup of G, $\mathcal{L} = \mathcal{L}_S^c(G)$ and $\mathcal{T} = \mathcal{T}_S^c(G)$. Let M be a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module and assume we have the following commutative diagram.*

$$\begin{tikzcd} & \pi_{\mathcal{L}} \arrow[dr, "\varphi"] & \\ \pi_{\mathcal{T}} \arrow[ur, "\delta"] \arrow[dr, "\rho_*"'] & & Aut(M) \\ & G \arrow[ur, "\overline{\varphi}"'] & \end{tikzcd}$$

If ρ_* is surjective and $\Gamma = Im(\varphi) = Im(\overline{\varphi})$ is p-solvable, then δ and ρ induce natural isomorphisms

$$H^*(|\mathcal{L}|, M) \cong H^*(|\mathcal{T}|, M) \cong H^*(G, M).$$

Proof. By Lemma 5.2.6, we just have to show that ρ induces a natural isomorphism $H^*(|\mathcal{T}|, M) \cong H^*(G, M)$. We prove this by induction on the minimal number n of extensions by p-groups or p'-groups we need to obtained Γ.

If $n = 0$, $\Gamma = 1$ and the action of $\pi_{\mathcal{T}}$ on M is trivial then it is just Theorem 2.3.4.

Assume that, if Γ is obtained by n extensions, the result is true and suppose that Γ is obtained with $n+1$ extensions. Consider then the last one

$$0 \to \Gamma_n \to \Gamma \to Q \to 0.$$

Denote $H = \overline{\varphi}_*^{-1}(\Gamma_n)$. Thus $(T, \mathcal{F}_H, \mathcal{L}_H) = (S \cap H, \mathcal{F}_{S\cap H}(H), \mathcal{L}^c_{S\cap H}(H))$ is a p-local subgroup of $(S, \mathcal{F}, \mathcal{L})$ of index a power of p or prime to p.

<u>If Q is a p'-group:</u>

In that case, $(T, \mathcal{F}_H, \mathcal{L}_H)$ is a p-local finite subgroup of index prime to p (defined in 1.3.4). Then $\mathrm{Ob}(\mathcal{F}^c) = \mathrm{Ob}(\mathcal{F}^c_H)$, $\mathcal{T}_H = \mathcal{T}^c_{S\cap H}(H) \subset \mathcal{T}$ and, by [OV1], Proposition 4.1, this inclusion of category induces, up to homotopy, a covering space with covering group $G/H = Q$. We then have the following commutative diagram with exact row (here, $\twoheadrightarrow$ means onto)

$$\begin{tikzcd}
0 \arrow[rr] & & \pi_{\mathcal{T}_H} \arrow[rr] \arrow[dr] \arrow[dd, two heads] & & \pi_{\mathcal{T}} \arrow[rr] \arrow[dr] \arrow[dd, two heads] & & Q \arrow[rr] \arrow[dr, equal] \arrow[dd, equal] & & 0 & \\
& 0 \arrow[rr] & & \Gamma_n \arrow[rr] & & \Gamma \arrow[rr] & & Q \arrow[rr] & & 0 \\
0 \arrow[rr] & & H \arrow[rr] \arrow[ur, two heads] & & G \arrow[rr] \arrow[ur, two heads] & & Q \arrow[rr] \arrow[ur, equal] & & 0 &
\end{tikzcd}$$

and the following fibration sequences

$$|\mathcal{T}_H| \longrightarrow |\mathcal{T}| \longrightarrow BQ$$

$$BH \longrightarrow BG \longrightarrow BQ\,.$$

Moreover, ρ induces a morphism of fibration sequences between these two.

Hence, we have the following Serre spectral sequences

$$H^{p+q}(|\mathcal{T}|, M) \Leftarrow H^p(Q, H^q(|\mathcal{T}_H|, M)),$$

$$H^{p+q}(G, M) \Leftarrow H^p(Q, H^q(H, M)),$$

and ρ induces a morphism ρ^* of spectral sequences between these two. By induction, ρ^* gives an isomorphism on the E_2 page and then induces an isomorphism of spectral sequences. In particular, ρ induces a natural isomorphism

$$H^*(|\mathcal{T}|, M) \cong H^*(G, M).$$

<u>If Q is a p-group:</u>

In that case, $(T, \mathcal{F}_H, \mathcal{L}_H)$ is a p-local finite subgroup of index a power of p (defined in 1.3.4). We have to be careful and work with all the quasicentric subgroups of S. However, this is not a problem because the inclusion $\mathcal{T} \subset \mathcal{T}^q$ induces a natural isomorphism $H^*(|\mathcal{T}^q|, M) \simeq H^*(|\mathcal{T}|, M)$. Indeed, we have the following commutative diagram.

$$\begin{tikzcd} \mathcal{T} \arrow[r] \arrow[d, "\delta"'] & \mathcal{T}^q \arrow[d, "\delta"] \\ \mathcal{L} \arrow[r] & \mathcal{L}^q \end{tikzcd}$$

The vertical arrows induce isomorphisms in cohomology by Lemma 5.2.6 and the lower horizontal one induces an isomorphism in cohomology because, by Theorem 1.3.11 $|\mathcal{L}| \simeq |\mathcal{L}^q|$. Hence the upper arrow induces an isomorphism $H^*(|\mathcal{T}^q|, M) \simeq H^*(|\mathcal{T}|, M)$. Assume then that, for this part, $\mathcal{T} = \mathcal{T}^q$ and $\mathcal{T}_H = \mathcal{T}_H^q$.

By Proposition 1.3.4, $\mathrm{Ob}(\mathcal{F}_H^q) \subset \mathrm{Ob}(\mathcal{F}^q)$. Then, $\mathcal{T}_H \subset \mathcal{T}$ and, still with Proposition 4.1 in [OV1], this inclusion induces a covering space with covering group $G/H = Q$. We then have the following diagram with exact row

$$\begin{tikzcd} 0 \arrow[rr] & & \pi_{\mathcal{T}_H} \arrow[rr] \arrow[dd, two heads] \arrow[dr] & & \pi_{\mathcal{T}} \arrow[rr] \arrow[dd, two heads] \arrow[dr] & & Q \arrow[rr] \arrow[dd, equal] \arrow[dr, equal] & & 0 & \\ & 0 \arrow[rr] & & \Gamma_n \arrow[rr] & & \Gamma \arrow[rr] & & Q \arrow[rr] & & 0 \\ 0 \arrow[rr] & & H \arrow[rr] \arrow[ur, tail] & & G \arrow[rr] \arrow[ur, tail] & & Q \arrow[rr] \arrow[ur, equal] & & 0 & \end{tikzcd}$$

and the following fibration sequences

$$|\mathcal{T}_H| \longrightarrow |\mathcal{T}| \longrightarrow BQ$$

$$BH \longrightarrow BG \longrightarrow BQ\ .$$

Moreover, ρ induces a morphism of fibration sequences between these two.

Hence, we have the following Serre spectral sequences

$$H^{p+q}(|\mathcal{T}|, M) \Leftarrow H^p(Q, H^q(|\mathcal{T}_H|, M)),$$

$$H^{p+q}(G, M) \Leftarrow H^p(Q, H^q(H, M)),$$

and ρ induces a morphism ρ^* of spectral sequences between these two. By induction, ρ^* gives an isomorphism on the E_2 page and then induces an isomorphism of spectral sequences. In particular, ρ induces a natural isomorphism

$$H^*(|\mathcal{T}|, M) \cong H^*(G, M).$$

Hence, by induction, we get the result. ✩

We can then use Theorem 2.5.3 to get the following.

Corollary 5.2.8. *Let G be a finite group, S a Sylow p-subgroup of G and $(S, \mathcal{F}, \mathcal{L})$ the associated p-local finite group.*

Let M a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module and assume that we have the following commutative diagram,

$$\begin{tikzcd} & \pi_{\mathcal{L}} \arrow[dr, "\varphi"] & \\ \pi_{\mathcal{T}} \arrow[ur, "\delta_*"] \arrow[dr, "\rho_*"'] & & Aut(M) \\ & G \arrow[ur, "\overline{\varphi}"'] & \end{tikzcd}$$

and that $\Gamma := Im\,(\varphi) = Im\,(\overline{\varphi})$.

If Γ is p-solvable and all the M-centric-radical subgroups of S are p-centric, then δ and ρ induce a natural isomorphism,

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

Here we use explicitly that $(S_H, \mathcal{F}_H, \mathcal{L}_H)$ is a p-local subgroup of index p or prime to p of $(S, \mathcal{F}, \mathcal{L})$ because they are the main cases where we know that $\mathcal{T}_H$ is a transporter category associated to a p-local subgroup of $(S, \mathcal{F}, \mathcal{L})$ and $|\mathcal{T}_H|$ a covering space of $|\mathcal{T}|$. But Theorem 5.2.7 might be generalized (at least with technical condition) in view of [OV1], Proposition 4.1.

We can also conjecture that it can be generalize to any abstract p-local finite group and any $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module with a p-solvable action.

Conjecture 5.2.9. Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and M a $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$-module. If the action of $\pi_1(|\mathcal{L}|)$ on M is p-solvable, then the inclusion of BS in $|\mathcal{L}|$ induces a natural isomorphism

$$H^*(|\mathcal{L}|, M) \xrightarrow{\cong} H^*(\mathcal{F}^c, M).$$

From now, we have an example (see Remark 6.3.7) which is not recovered by Corollary 5.2.8 or a previous result. One problem is the difficulty to find good examples which are not too complicated for computation. Indeed, we want a p-solvable action but we also do not want that the fusion system is constrained (in this case we already know the result) or to be under the hypothesis of Corollary 5.2.8. This forces us to consider huge fusion systems...

6

Studies of some constructions

In this chapter, we study some constructions of p-local finite groups and we give some results to study the cohomology of $|\mathcal{L}|$ and to compare it with the $\mathcal{F}^c$-stable elements. This is motivated by the research of examples (or counter examples) in the study of Conjecture 5.2.9.

6.1 Products of fusion systems

Let $(S_1, \mathcal{F}_1, \mathcal{L}_1)$ and $(S_2, \mathcal{F}_2, \mathcal{L}_2)$ be two p-local finite groups. Consider the p-local finite group $(S, \mathcal{F}, \mathcal{L})$ as the *product* of $(S_1, \mathcal{F}_1, \mathcal{L}_1)$ and $(S_2, \mathcal{F}_2, \mathcal{L}_2)$ i.e. $S = S_1 \times S_2$ and $\mathcal{F}$ is generated by the morphisms of the form $(\phi, \psi) \in \mathcal{F}_1 \times \mathcal{F}_2$ (see for example [AKO], Section I.6). Here, we denote by $\mathcal{F}_1 \times \mathcal{F}_2$ the category product of the categories $\mathcal{F}_1$ and $\mathcal{F}_2$ (in particular, this is not a fusion system!) and respectively by $\mathcal{L}_1 \times \mathcal{L}_2$, the category product of the categories $\mathcal{L}_1$ and $\mathcal{L}_2$. We write also, for $i \in \{1, 2\}$, $\mathrm{pr}_i : S \longrightarrow S_i$ the canonical projections.

Proposition 6.1.1. *The inclusion $i : \mathcal{F}_1 \times \mathcal{F}_2 \to \mathcal{F}$ induce an homotopy equivalence on their geometric realizations.*

Proof. Let $P, Q \leq S$ and $\phi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$. As ϕ is a composite of restrictions of morphisms of the form $(\psi, \gamma) \in \mathcal{F}_1 \times \mathcal{F}_2$, ϕ can be factored as follows.

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[d, "\mathrm{pr}_1"'] & Q \arrow[d, "\mathrm{pr}_1"] \\ \mathrm{pr}_1(P) \arrow[r, "\phi_1"] & \mathrm{pr}_1(Q) \end{tikzcd}$$

$$\begin{tikzcd} P \arrow[r, "\phi"] \arrow[d, "\mathrm{pr}_2"'] & Q \arrow[d, "\mathrm{pr}_2"] \\ \mathrm{pr}_2(P) \arrow[r, "\phi_2"] & \mathrm{pr}_2(Q) \end{tikzcd}$$

Consider then $p : \mathcal{F} \to \mathcal{F}_1 \times \mathcal{F}_2$ defined on objects by, for $P \leq S$, $p(P) = \mathrm{pr}_1(P) \times \mathrm{pr}_2(P)$ and on morphisms by, for $\phi \in \mathrm{Mor}(\mathcal{F})$, $p(\phi) = (\phi_1, \phi_2)$ where ϕ_1 and ϕ_2 are defined as above.

It defines a functor and we have the following commutative diagram.

$$\begin{tikzcd} \mathrm{pr}_1(P) \times \mathrm{pr}_2(P) \arrow[r, "(\phi_1,\phi_2)"] & \mathrm{pr}_1(Q) \times \mathrm{pr}_2(Q) \\ P \arrow[u, "\mathrm{pr}_1\times\mathrm{pr}_2"] \arrow[r, "\phi"'] & Q \arrow[u, "\mathrm{pr}_1\times\mathrm{pr}_2"'] \end{tikzcd}$$

Hence, there is a natural transformation from $i \circ p$ to $\mathrm{Id}_{\mathcal{F}}$. As $p \circ i = \mathrm{Id}$, we have, by Proposition 2.3, that $|i|$ and $|p|$ are homotopy equivalence. ✩

To study $\mathcal{L}$, we first have to describe the $\mathcal{F}$-centric subgroups of S.

Lemma 6.1.2. *Let P be a subgroup of S. the following conditions are equivalent,*

(i) P is $\mathcal{F}$-centric,

(ii) $pr_1(P) \times pr_2(P)$ is $\mathcal{F}$-centric,

(iii) $pr_1(P)$ is $\mathcal{F}_1$-centric and $pr_2(P)$ is $\mathcal{F}_2$-centric.

Proof. We can see that $C_S(P) = C_S(\mathrm{pr}_1(P) \times \mathrm{pr}_2(P)) = C_{S_1}(\mathrm{pr}_1(P)) \times C_{S_2}(\mathrm{pr}_2(P))$, $Z(\mathrm{pr}_1(P) \times \mathrm{pr}_2(P)) = Z(\mathrm{pr}_1(P)) \times Z(\mathrm{pr}_2(P))$ and that $Z(P) = P \cap Z(\mathrm{pr}_1(P) \times \mathrm{pr}_2(P))$. Besides, if $P, Q \leq S$ are $\mathcal{F}$-conjugate, then $\mathrm{pr}_1(P) \times \mathrm{pr}_2(P)$ and $\mathrm{pr}_1(Q) \times \mathrm{pr}_2(Q)$ are also $\mathcal{F}$-conjugate (if φ sends P on Q then $p(\varphi)$, with p as in the proof of the last proposition, send $\mathrm{pr}_1(P) \times \mathrm{pr}_2(P)$ on $\mathrm{pr}_1(Q) \times \mathrm{pr}_2(Q)$).

The lemma follows easily. ✩

As on the fusion systems level, we can try to compare $\mathcal{L}$ with the product $\mathcal{L}_1 \times \mathcal{L}_2$. In fact, we can obtain $\mathcal{L}$ as the following pullback (see [CL], Definition 2.15, for more details)

$$\begin{tikzcd} \mathcal{L} \arrow[r] \arrow[d, "\pi"'] & \mathcal{L}_1 \times \mathcal{L}_2 \arrow[d, "\pi_1\times\pi_2"] \\ \mathcal{F} \arrow[r, "p"'] & \mathcal{F}_1 \times \mathcal{F}_2 \end{tikzcd}$$

where p is the retraction defined in the proof of Proposition 6.1.1 and, for $i \in \{1, 2\}$, $\pi_i : \mathcal{L} \longrightarrow \mathcal{F}_i$ is the canonical projection functor. Then we have the following result.

Proposition 6.1.3 ([CL], Proposition 2.17)**.** *Let, for $i \in \{1, 2\}$, $(S_i, \mathcal{F}_i, \mathcal{L}_i)$ be two p-local finite groups and $(S, \mathcal{F}, \mathcal{L})$ the p-local finite group product of $(S_1, \mathcal{F}_1, \mathcal{L}_1)$ and $(S_2, \mathcal{F}_2, \mathcal{L}_2)$. The category $\mathcal{L}_1 \times \mathcal{L}_2$ is a full subcategory of $\mathcal{L}$ and the inclusion induces an homotopy equivalence.*

From this, we can deduce the homotopy type of $|\mathcal{L}|$.

Corollary 6.1.4. *(a)* $|\mathcal{L}| \simeq |\mathcal{L}_1| \times |\mathcal{L}_2|$.

(b) $\pi_{\mathcal{L}} \cong \pi_{\mathcal{L}_1} \times \pi_{\mathcal{L}_2}$

When we look at the cohomology we can use the Kunneth formula.

Proposition 6.1.5. *Let, for $i \in \{1,2\}$, $(S_i, \mathcal{F}_i, \mathcal{L}_i)$ be two p-local finite groups and $(S, \mathcal{F}, \mathcal{L})$ the p-local finite group product of $(S_1, \mathcal{F}_1, \mathcal{L}_1)$ and $(S_2, \mathcal{F}_2, \mathcal{L}_2)$.*

Let also M_1 be a $\mathbb{F}_p[\pi_{\mathcal{L}_1}]$-module and M_2 be a $\mathbb{F}_p[\pi_{\mathcal{L}_2}]$-module.

If, for $i \in \{1,2\}$, δ_{S_i} induces a natural isomorphism $H^*(|\mathcal{L}_i|, M_i) \cong H^*(\mathcal{F}_i^c, M_i)$ then δ_S induces a natural ismorphism

$$H^*(|\mathcal{L}|, M_1 \otimes_{\mathbb{F}_p} M_2) \cong H^*(\mathcal{F}^c, M_1 \otimes_{\mathbb{F}_p} M_2).$$

Here, $\otimes_{\mathbb{F}_p}$ denotes the usual tensor product on $\mathbb{F}_p$-vector spaces but the tensor product $\otimes$ denotes a tensor product of graded $\mathbb{F}_p$-vector spaces, i.e. for two graded $\mathbb{F}_p$-vector spaces V^* and W^*, $V^* \otimes W^*$ is the graded $\mathbb{F}_p$-vector space $(V^* \otimes W^*)^n = \bigoplus_{i=0}^n V^i \otimes_{\mathbb{F}_p} W^{n-i}$.

Proof. As we work with $\mathbb{F}_p$-modules, the Kunneth formula tells us that $H^*(|\mathcal{L}|, M_1 \otimes_{\mathbb{F}_p} M_2) \cong H^*(|\mathcal{L}_1|, M_1) \otimes H^*(|\mathcal{L}_2|, M_2)$ and the following diagram commutes for each $P_1 \leq S_1$, $P_2 \leq S_2$

$$\begin{tikzcd}
H^*(|\mathcal{L}_1|, M_1) \otimes H^*(|\mathcal{L}_2|, M_2) \arrow[r, "\cong"] \arrow[d, "\delta_{P_1} \otimes \delta_{P_2}"'] & H^*(|\mathcal{L}|, M_1 \otimes_{\mathbb{F}_p} M_2) \arrow[d, "\delta_{P_1 \times P_2}"] \\
H^*(P_1, M_1) \otimes H^*(P_2, M_2) \arrow[r, "\cong"] & H^*(P_1 \times P_2, M_1 \otimes_{\mathbb{F}_p} M_2)
\end{tikzcd}$$

Besides, as $\mathcal{F}_1 \times \mathcal{F}_2$ is a retract of $\mathcal{F}$, using the Kunneth formula several times we get

$$\begin{aligned}
H^*(\mathcal{F}^c, M_1 \otimes_{\mathbb{F}_p} M_2) &= \varprojlim_{\mathcal{F}_1^c \times \mathcal{F}_2^c} H^*(-, M_1 \otimes_{\mathbb{F}_p} M_2) \qquad \text{(because } \mathcal{F}^c = \langle \mathcal{F}_1^c \times \mathcal{F}_2^c \rangle\text{)} \\
&\cong \varprojlim_{P_1 \times P_2 \in \mathcal{F}_1^c \times \mathcal{F}_2^c} (H^*(P_1, M_1) \otimes H^*(P_2, M_2)) \\
&\cong \left(\varprojlim_{\mathcal{F}_1^c} H^*(-, M_1) \right) \otimes \left(\varprojlim_{\mathcal{F}_2^c} H^*(-, M_2) \right).
\end{aligned}$$

Hence

$$H^*(|\mathcal{L}|, M_1 \otimes_{\mathbb{F}_p} M_2) \cong H^*(\mathcal{F}^c, M_1 \otimes M_2)$$

and this isomorphism is induced by δ_S. ✩

6.2 Cohomological coverings of p-local finite groups

Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group.

Here we will consider general linking systems. It allows us to work with more flexible sets of objects without changing the homotopy type of the geometric realization. Thus for a p-local finite group $(S, \mathcal{F}, \mathcal{L})$, $\mathrm{Ob}(\mathcal{L})$ will be a set of $\mathcal{F}$-quasicentric subgroups of S which contains all the $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroups. We will write $\mathcal{L}^q$ to indicate that we consider all the $\mathcal{F}$-quasicentric subgroups of S, $\mathcal{L}^c$ if we consider only the $\mathcal{F}$-centric subgroups and $\mathcal{L}^{cr}$ if we consider only the $\mathcal{F}$-centric and $\mathcal{F}$-radical ones.

To explain the idea here, we start by an easy example: $\big(D_8, \mathcal{F}_{D_8}(A_6), \mathcal{L}^{cr}_{D_8}(A_6)\big)$. If one look at the linking system, it has the following form,

$$\begin{tikzcd}
 & D_8 \arrow[loop above, "D_8"] & \\
Q \arrow[ur] \arrow[loop below, "N_{A_6}(Q)"] & & Q' \arrow[ul] \arrow[loop below, "N_{A_6}(Q')"]
\end{tikzcd}$$

where Q and Q' are the two subgroups of D_8 isomorphic to a Klein four group. We can see this category as a union of two other linking systems, $\mathcal{L}_1$ and $\mathcal{L}_2$.

$$\mathcal{L}_1 : \quad \begin{tikzcd}
D_8 \arrow[loop above, "D_8"] \\
Q \arrow[u] \arrow[loop below, "N_{A_6}(Q)"]
\end{tikzcd} \qquad\qquad \mathcal{L}_2 : \quad \begin{tikzcd}
D_8 \arrow[loop above, "D_8"] \\
Q' \arrow[u] \arrow[loop below, "N_{A_6}(Q')"]
\end{tikzcd}$$

These two linking systems correspond to linking systems of p-local finite groups associated to $N_{A_6}(Q)$ and $N_{A_6}(Q')$ respectively, which are both constrained. Thus we can use Van Kampen Theorem to get the fundamental group of $|\mathcal{L}^{cr}_{D_8}(A_6)|$ and the Mayer-Vietoris long exact sequence to study its cohomology.

Here we will try to generalize this idea by looking at some linking systems as union of two others which are easier to work with and hence, using Van Kampen and Mayer-Vietoris, get informations on the homotopy type of the initial one.

Definition 6.2.1. Let $(S, \mathcal{F}, \mathcal{L})$ and, for $i = \{1, 2\}$, $(S, \mathcal{F}_i, \mathcal{L}_i)$ be three p-local finite groups **on the same p-group** S. Let M be a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module.

We will say that $\{\mathcal{L}_1, \mathcal{L}_2\}$ is a *covering of* $\mathcal{L}$ if $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$.

We will say that $\{\mathcal{L}_1, \mathcal{L}_2\}$ is an *M-cohomological covering of* $\mathcal{L}$ if it is a covering of $\mathcal{L}$ and the inclusions of categories induce injective maps $H^*(|\mathcal{L}|, M) \hookrightarrow H^*(|\mathcal{L}_i|, M)$ and $H^*(|\mathcal{L}_i|, M) \hookrightarrow H^*(|\mathcal{L}_1 \cap \mathcal{L}_2|, M)$ for all i.

Remark 6.2.2. If $\{\mathcal{L}_1, \mathcal{L}_2\}$ is a covering of $\mathcal{L}$ then $\mathcal{F}_1$ and $\mathcal{F}_2$ can be seen as subsystems of $\mathcal{F}$ and $\mathcal{F} = \langle \mathcal{F}_1, \mathcal{F}_2 \rangle$.

Lemma 6.2.3. *If $\{\mathcal{L}_1, \mathcal{L}_2\}$ is a covering of $\mathcal{L}$ then we have the following Pushout diagram,*

$$\begin{tikzcd}
|\mathcal{L}_1 \cap \mathcal{L}_2| \arrow[r] \arrow[d] & |\mathcal{L}_2| \arrow[d] \\
|\mathcal{L}_1| \arrow[r] & |\mathcal{L}|
\end{tikzcd}$$

Proof. Let us look at the nerves, $\mathcal{N}(\mathcal{L})$, $\mathcal{N}(\mathcal{L}_1)$ and $\mathcal{N}(\mathcal{L}_2)$ of these categories.

In $\mathcal{L}$, if for example, $\varphi = \psi \circ \theta$ with $\theta \in \mathcal{L}_1(P, Q)$, $\psi \in \mathcal{L}_2(Q, R)$ and $\varphi \in \mathcal{L}_1(P, R)$, then, fixing an inclusion system $\{\iota_P^Q\}_{P \leq Q \leq S}$, we have the following commutative diagram

$$\begin{tikzcd}
 & & R \\
 & Q \arrow[ur, "\psi"] \arrow[r, "\overline{\psi}"'] & Q_R \arrow[u, "\iota_{Q_R}^{P}"'] \\
P \arrow[uurr, bend left, "\varphi"] \arrow[ur, "\theta"] \arrow[r, "\overline{\theta}"'] \arrow[rr, bend right=60, "\overline{\varphi}"'] & P_Q \arrow[u, "\iota_{P_Q}^{Q}"'] \arrow[r, "\overline{\psi}|_{P_Q}^{P_R}"'] & P_R \arrow[u, "\iota_{P_R}^{Q_R}"']
\end{tikzcd}$$

where $P_Q = \mathrm{Im}\,(\theta)$, $Q_R = \mathrm{Im}\,(\psi)$ and $P_R = \mathrm{Im}\,(\psi \circ \theta)$. Thus $\overline{\psi}|_{P_Q}^{P_R} = \overline{\theta}^{-1} \circ \overline{\varphi}$ so $\overline{\psi}|_{P_Q}^{P_R} \in \mathcal{L}_1(P_Q, P_R)$ and, by existence of extensions in $\mathcal{L}_1$ and unicity in $\mathcal{L}$ (Proposition 1.2.5), we have that $\overline{\psi}$ and ψ are in $\mathcal{L}_1$.

We can deduce from this that $\mathcal{N}(\mathcal{L}) = \mathcal{N}(\mathcal{L}_1) \cup \mathcal{N}(\mathcal{L}_2)$. ✩

Proposition 6.2.4. *Let $(S, \mathcal{F}, \mathcal{L})$ and, for $i = \{1, 2\}$, $(S, \mathcal{F}_i, \mathcal{L}_i)$ be three p-local finite groups on the same p-group S. Let M be a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module.*

If $\{\mathcal{L}_1, \mathcal{L}_2\}$ is a covering of $\mathcal{L}$, then $\pi_{\mathcal{L}} = \pi_{\mathcal{L}_1} *_{\pi_1(|\mathcal{L}_1 \cap \mathcal{L}_2|)} \pi_{\mathcal{L}_2}$.

Moreover, if $\{\mathcal{L}_1, \mathcal{L}_2\}$ is an M-cohomological covering of $\mathcal{L}$, then there is a short exact sequence,

$$0 \longrightarrow H^*(|\mathcal{L}|, M) \longrightarrow H^*(|\mathcal{L}_1|, M) \oplus H^*(|\mathcal{L}_2|, M) \longrightarrow H^*(|\mathcal{L}_1 \cap \mathcal{L}_2|, M) \longrightarrow 0.$$

Proof. This is just a consequence of Lemma 6.2.3. We use Van Kampen for the first point and the long exact sequence of Mayer-Vietoris for the second point. ✩

Proposition 6.2.5. *Let $(S, \mathcal{F}, \mathcal{L})$ and, for $i = \{1, 2\}$, $(S, \mathcal{F}_i, \mathcal{L}_i)$ be three p-local finite groups on the same p-group S. Let M be a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module.*

Assume that $\{\mathcal{L}_1, \mathcal{L}_2\}$ is an M-cohomological covering of $\mathcal{L}$ and that, for $i \in \{1, 2\}$, δ_S induces natural isomorphisms $H^*(|\mathcal{L}_i|, M) \cong H^*(\mathcal{F}_i^c, M)$.

Then δ_S induces a natural isomorphism

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

Proof. By the previous proposition, we have the following short exact sequence,

$$0 \longrightarrow H^*(|\mathcal{L}|, M) \longrightarrow H^*(|\mathcal{L}_1|, M) \oplus H^*(|\mathcal{L}_2|, M) \longrightarrow H^*(|\mathcal{L}_1 \cap \mathcal{L}_2|, M) \longrightarrow 0.$$

The maps are given by the inclusions hence, by exactness of the sequence and M-cohomological covering hypothesis, the image of the left arrow is

$$H^*(|\mathcal{L}_1|, M) \cap H^*(|\mathcal{L}_2|, M) \subseteq H^*(|\mathcal{L}_1 \cap \mathcal{L}_2|, M).$$

Besides, we have the following commutative diagram.

$$\begin{tikzcd}
 & & H^*(S,M) & & \\
 & & H^*(|\mathcal{L}_1\cap\mathcal{L}_2|,M) \arrow[u, "\delta_S^*"] & & \\
H^*(|\mathcal{L}_1|,M) \arrow[urr] \arrow[uurr, bend left, "\delta_S^*"] & & & & H^*(|\mathcal{L}_2|,M) \arrow[ull] \arrow[uull, bend right, "\delta_S^*"']
\end{tikzcd}$$

Thus, by hypothesis, δ_S gives the following isomorphism,

$$H^*(|\mathcal{L}_1|,M)\cap H^*(|\mathcal{L}_2|,M)\cong H^*(\mathcal{F}_1^c,M)\cap H^*(\mathcal{F}_2^c,M).$$

Finally, as $\mathcal{F}=\langle\mathcal{F}_1,\mathcal{F}_2\rangle$,we obtain that

$$H^*(\mathcal{F}^c,M)=H^*(\mathcal{F}_1^c,M)\cap H^*(\mathcal{F}_2^c,M)\subseteq H^*(S,M)$$

and that

$$H^*(|\mathcal{L}|,M)\cong H^*(\mathcal{F}^c,M),$$

where the isomorphism is induced by δ_S. ✩

Let us now look at some examples.

One case where we want to use this decomposition is when, for $i\in\{1,2\}$, $\mathcal{L}_1\cap\mathcal{L}_2$ is a linking system, $\mathcal{L}_3$, associated to another p-local finite group over S, $(S,\mathcal{F}_3,\mathcal{L}_3)$. The example of $D_8\leq A_6$ is of this form. Besides, for $i\in\{1,2,3\}$, if we assume that, δ_S induces the isomorphism $H^*(|\mathcal{L}_i|,M)\cong H^*(\mathcal{F}_i^c,M)$, then as $\mathcal{F}_3^{cr}\subseteq\mathcal{F}_i^{cr}$, we have the following inclusion

$$H^*(|\mathcal{L}_i|,M)\cong H^*(\mathcal{F}_i^c,M)\hookrightarrow H^*(\mathcal{F}_3^c,M)\cong H^*(|\mathcal{L}_1\cap\mathcal{L}_2|,M).$$

Then we just have to look at, for $i\in\{1,2\}$, the injectivity of

$$H^*(\mathcal{L},M)\longrightarrow H^*(|\mathcal{L}_i|,M)$$

or, equivalently the injectivity of the direct sum of these two arrows

$$H^*(\mathcal{L},M)\longrightarrow H^*(|\mathcal{L}_1|,M)\oplus H^*(|\mathcal{L}_2|,M)\ .$$

Proposition 6.2.6. *Let $(S,\mathcal{F},\mathcal{L})$ and, for $i=\{1,2,3\}$, $(S,\mathcal{F}_i,\mathcal{L}_i)$ be four p-local finite groups on the same p-group S. Let M a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module.*

Assume that $\{\mathcal{L}_1,\mathcal{L}_2\}$ is a covering of $\mathcal{L}$ and that $\mathcal{L}_1\cap\mathcal{L}_2=\mathcal{L}_3$.

If, for $i\in\{1,2,3\}$, δ_S induces an isomorphism $H^*(|\mathcal{L}_i|,M)\cong H^*(\mathcal{F}_i^c,M)$ then the followings are equivalent,

(i) $\{\mathcal{L}_1,\mathcal{L}_2\}$ is an M-cohomological covering of $\mathcal{L}$,

(ii) $H^*(|\mathcal{L}|,M)\longrightarrow H^*(|\mathcal{L}_1|,M)\oplus H^*(|\mathcal{L}_2|,M)$ is injective,

(iii) $H^*(\mathcal{F}_3^c, M) = H^*(\mathcal{F}_1^c, M) + H^*(\mathcal{F}_2^c, M)$,

(iv) δ_S *induces a natural isomorphism* $H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M)$.

Proof. As remark before, we have, for $i \in \{1, 2\}$ the inclusion

$$H^*(|\mathcal{L}_i|, M) \hookrightarrow H^*(|\mathcal{L}_1 \cap \mathcal{L}_2|, M).$$

Hence (i) is equivalent to (ii).

Besides, by exactness of the Mayer-Vietoris long exact sequence associated to the decomposition $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$, and as we know that, for $i \in \{1, 2, 3\}$, δ_S induces the isomorphism $H^*(|\mathcal{L}_i|, M) \cong H^*(\mathcal{F}_i^c, M)$, (ii) is equivalent to (iii).

Finally, still by exactness, the image of the arrow

$$H^*(|\mathcal{L}|, M) \longrightarrow H^*(|\mathcal{L}_1|, M) \oplus H^*(|\mathcal{L}_2|, M)$$

is $H^*(|\mathcal{L}_1|, M) \cap H^*(|\mathcal{L}_2|, M)$. But, as δ_S induces the isomorphism

$$H^*(|\mathcal{L}_1|, M) \cap H^*(|\mathcal{L}_2|, M) \cong H^*(\mathcal{F}_1^c, M \cap H^*(\mathcal{F}_2^c, M) = H^*(\mathcal{F}^c, M),$$

(ii) is equivalent to (iv). ✩

For example, the hypothesis of this proposition are satisfied if $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ are constrained (as in the example of $D_8 \leq A_6$).

As an example of how we can use this proposition, we can work with realizable fusion systems. We write here $\mathcal{T} = \mathcal{T}_S^{\mathrm{Ob}(\mathcal{L})}(G)$ the transporter category with object set $\mathrm{Ob}(\mathcal{L})$ and $\pi_{\mathcal{T}} = \pi_1(|\mathcal{T}|)$. The following corollary can be for example used with Theorem 5.2.7.

Corollary 6.2.7. *Let G be a finite group, S a Sylow p-subgroup of G and $(S, \mathcal{F}, \mathcal{L})$ the associated p-local finite group. Let M be a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module.*

Assume that there are three p-local finite subgroups, $(S, \mathcal{F}_i, \mathcal{L}_i)_{i \in \{1,2,3\}}$, of $(S, \mathcal{F}, \mathcal{L})$ such that,

(a) $\{\mathcal{L}_1, \mathcal{L}_2\}$ *is a covering of* $\mathcal{L}$,

(b) $\mathcal{L}_1 \cap \mathcal{L}_2 = \mathcal{L}_3$,

(c) for $i \in \{1, 2, 3\}$, $\mathcal{F}_i$ *is realized by a subgroup* G_i *of* G *containing* S,

(d) there is a commutative diagram,

$$\begin{tikzcd} & \pi_{\mathcal{L}} \arrow[dr] & \\ \pi_{\mathcal{T}} \arrow[ur, "\delta_*"] \arrow[dr, "\rho_*"'] & & Aut(M) \\ & G \arrow[ur] & \end{tikzcd}$$

(e) for $i \in \{1,2,3\}$, δ_S and $\rho|_{\mathcal{T}_S^{Ob(\mathcal{L})}(S)}$ induce natural isomorphisms

$$H^*(|\mathcal{L}_i|, M) \cong H^*(\mathcal{F}_i^c, M) \cong H^*(G_i, M).$$

If, moreover, δ and ρ induces a natural isomorphism $H^*(|\mathcal{L}|, M) \cong H^*(G, M)$, then δ_S and $\rho|_{\mathcal{T}_S^{Ob(\mathcal{L})}(S)}$ induces a natural isomorphism

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M) \cong H^*(G, M).$$

Proof. We are in the condition of the previous proposition and we just have to show the injectivity of the map $H^*(|\mathcal{L}|, M) \longrightarrow H^*(|\mathcal{L}_1|, M) \oplus H^*(|\mathcal{L}_2|, M)$. However, as G_1 and G_2 are subgroups of G containing S,

$$H^*(|\mathcal{L}|, M) \cong H^*(G, M) \hookrightarrow H^*(G_1, M) \oplus H^*(G_2, M) \cong H^*(|\mathcal{L}_1|, M) \oplus H^*(|\mathcal{L}_2|, M).$$

This gives us the corollary.

We finish this subsection with an example where our last result can be seen in practice. Consider $G = P\Gamma L_3(\mathbb{F}_4)$ the extension of $PGL_3(\mathbb{F}_4)$ with the Galois automorphism of $\mathbb{F}_4$. We will study its 2-local structure $(S, \mathcal{F}, \mathcal{L})$. We also send the reader to the article of Oliver and Ventura [OV2] which gives some computation around these groups and in particular the computation of the essential subgroups.

When we look at the centric-radical linking system (using GAP) which is, in fact, a transporter system, it has the following form

$$\begin{tikzcd}
 & & P_0 = S \cong UT_3(\mathbb{F}_4) \rtimes C_2 \arrow[d, dash] \arrow[ddll, dash] \arrow[ddl, dash] \arrow[ddr, dash] \arrow[ddrr, dash] & & \\
 & & P_1 \cong (C_2 \times Q_8) \rtimes C_2 \arrow[ddl, dash] \arrow[ddr, dash] & & \\
C_2^4 \rtimes C_2 \cong P_2 \arrow[r, dash, squiggly] \arrow[dr, dash] & P_3 \arrow[d, dash] & & P_4 \arrow[r, dash, squiggly] \arrow[d, dash] & P_5 \cong C_2^4 \rtimes C_2 \arrow[dl, dash] \\
 & P_6 \cong C_2^4 & & P_7 \cong C_2^4 &
\end{tikzcd}$$

where straight lines correspond to inclusions, waved lines to $\mathcal{F}$-conjugate subgroups and $UT_3(\mathbb{F}_4)$ denotes the subgroup of $PGL_3(\mathbb{F}_4)$ of upper triangular matrices with one on the diagonal. We also have the following inclusions (from top to bottom) between the normalizers, where for $i \in \{1, 2, \ldots, 7\}$, N_i is the normalizer of P_i.

$$\begin{tikzcd}
 & & N_0 = S \arrow[d, dash] & & \\
 & & N_1 \arrow[ddl, dash] \arrow[ddr, dash] & & \\
N_2 \arrow[dr, dash] & N_3 \arrow[d, dash] & & N_4 \arrow[d, dash] & N_5 \arrow[dl, dash] \\
 & N_6 & & N_7 &
\end{tikzcd}$$

Moreover, the conjugations $P_2 \sim P_3$ and $P_4 \sim P_5$ are controlled by respectively N_6 and N_7, and $N_7 \cap N_6 = N_1$. Then we can see that $\mathcal{L}$ is a covering of two linking systems $\mathcal{L}_6 = \mathcal{L}_S^{\mathcal{H}_6}(N_6)$ and $\mathcal{L}_7 = \mathcal{L}_S^{\mathcal{H}_7}(N_7)$ and the intersection $\mathcal{L}_6 \cap \mathcal{L}_7$ is also a linking system $\mathcal{L}_1 = \mathcal{L}_S^{\mathcal{H}_1}(N_1)$. These three linking systems correspond to constrained fusion systems with models N_6, N_7, and N_1 respectively (P_6, P_7 and P_1 are centric in G). Thus, the first three conditions of Corollary 6.2.7 holds here. By Proposition 6.2.4, $\pi_{\mathcal{L}} = N_6 *_{N_1} N_7$ and a quotient of this amalgamated product is naturally isomorphic to $Q = G/PSL_3,(\mathbb{F}_4) \cong C_3 \rtimes C_2$ which is p-solvable. Hence, for every $\mathbb{Z}_{(p)}[Q]$-module, (c) and (d) are satisfied. Finally, by Theorem 5.2.7, we obtain the last condition and then for every $\mathbb{Z}_{(p)}[Q]$-module M, δ_S induces a natural isomorphism,

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

Remark 6.2.8. In fact, we can also apply Corollary 5.2.8 because, for every $\mathbb{Z}_{(p)}[Q]$-module M, the set of all M-centric-radical subgroups of S (Definition 2.5.1) is exactly the set of all $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroups of S. Hence this does not give a very useful example for Conjecture 5.2.9.

6.3 The p-local structure of wreath products by C_p

Let G_0 be a finite group, S_0 a Sylow p-subgroup of G_0 and $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ be the associated p-local finite group. We are interested in the wreath product $G = G_0 \wr C_p$, $S = S_0 \wr C_p$ and the associated p-local finite group $(S, \mathcal{F}, \mathcal{L})$. By [CL], Theorem 5.2 and Remark 5.3, we have that $|\mathcal{L}| \simeq |\mathcal{L}_0| \wr BC_p = |\mathcal{L}_0|^p \times_{C_p} EC_p$ and, in particular, $\pi_{\mathcal{L}} = \pi_{\mathcal{L}_0} \wr C_p$.

We first give a lemma on strongly p-embedded subgroups. Recall that, for a finite group G, a subgroup $H < G$ is *strongly p-embedded*, if $p \mid |H|$ and for each $x \in G \setminus H$, $H \cap xHx^{-1}$ has order prime to p.

Lemma 6.3.1. *Let G be a finite group, $G_0 \leq G$ a subgroup of index a power of p.*
If G contains a strongly p-embedded subgroup and $p \mid |G_0|$, then G_0 contains a strongly p-embedded subgroup.

Proof. Let H be a strongly p-embedded subgroup of G. By [AKO], Proposition A.7, H contains a Sylow p-subgroup of G so, up to conjugacy, we can choose H such that H contains a Sylow p-subgroup of G_0. Hence $G_0 \cap H$ contains a Sylow p-subgroup of G_0 and $p \mid |G_0 \cap H|$. We will show that $G_0 \cap H$ is a strongly p-embedded subgroup of G_0.

As $[G : H]$ is prime to p and $[G : G_0]$ is a power of p, $G_0 \cap H$ is a proper subgroup of G_0.

It remains to show that, for each $x \in G_0 \setminus G_0 \cap H$, $(G_0 \cap H) \cap x(G_0 \cap H)x^{-1}$ has order prime to p. But $(G_0 \cap H) \cap x(G_0 \cap H)x^{-1} \leq H \cap xHx^{-1}$, thus, as H is a strongly p-embedded subgroup of G, this last subgroup has order prime to p for every $x \in G \setminus H$. In particular, for each $x \in G_0 \setminus G_0 \cap H$, $(G_0 \cap H) \cap x(G_0 \cap H)x^{-1}$ has order prime to p and $G_0 \cap H$ is a strongly p-embedded subgroup of G_0. ✩

We give also a lemma on $\mathcal{F}_1$-essential subgroups for $\mathcal{F}_1 \subseteq \mathcal{F}$ a subsystem of index a power of p. We recall (Definition 1.1.7) that a proper subgroup $P < S$ is $\mathcal{F}$-essential if P

is $\mathcal{F}$-centric and fully normalized in $\mathcal{F}$, and if $\mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly p-embedded subgroup.

Lemma 6.3.2. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and $(S_1, \mathcal{F}_1, \mathcal{L}_1)$ a p-local subgroup of index a power of p.*

If $P < S_1$ is $\mathcal{F}$-essential, then P is $\mathcal{F}_1$-essential.

Proof. Let $P \leq S_1$ be an $\mathcal{F}$-essential subgroup.

P is $\mathcal{F}_1$-centric:

As P is $\mathcal{F}$-centric, $C_S(Q) = Z(Q)$ for all $Q \in P^{\mathcal{F}}$. In particular, for all $Q \in P^{\mathcal{F}_1} \subseteq P^{\mathcal{F}}$, $C_{S_1}(Q) = Z(Q)$ and P is $\mathcal{F}_1$-centric.

P is fully normalized in $\mathcal{F}_1$. By [AKO] Proposition I.2.6, it is enough to show that P is receptive and fully automized in $\mathcal{F}_1$. But, as P is $\mathcal{F}_1$-centric, it is fully centralized in $\mathcal{F}_1$ and so it is receptive by [AKO] Proposition I.2.5. It remains to show that P is fully automized in $\mathcal{F}_1$.

As P is fully normalized in $\mathcal{F}$ (P is $\mathcal{F}$-essential), by [AKO] Proposition I.2.5, P is fully automized in $\mathcal{F}$. Hence $\mathrm{Aut}_S(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$ and then $\mathrm{Aut}_{S_1}(P) = \mathrm{Aut}_S(P) \cap \mathrm{Aut}_{\mathcal{F}_1}(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}_1}(P)$.

$\mathrm{Out}_{\mathcal{F}_1}(P)$ contains a strongly p-embedded subgroup :

As P is $\mathcal{F}$-essential, $\mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly p-embedded subgroup. As $\mathcal{F}_1$ is a subsystem of $\mathcal{F}$ of index a power of p, $\mathrm{Out}_{\mathcal{F}_1}(P)$ is a subgroup of $\mathrm{Out}_{\mathcal{F}}(P)$ of index a power of p. Moreover, as P is a proper subgroup of S_1, $P < N_{S_1}(P)$ and, as P is $\mathcal{F}_1$-centric, every element of $N_{S_1}(P)$ induces a non trivial element in $\mathrm{Out}_{\mathcal{F}_1}(P)$. Hence $p \mid |\mathrm{Out}_{\mathcal{F}_1}(P)|$ and, by Lemma 6.3.1, $\mathrm{Out}_{\mathcal{F}_1}(P)$ contains a strongly p-embedded subgroup. ✩

We can easily describe the essential subgroups of a product of fusion systems.

Lemma 6.3.3. *Let $(S_1, \mathcal{F}_1, \mathcal{L}_1)$ and $(S_2, \mathcal{F}_2, \mathcal{L}_2)$ be p-local finite groups and $(S = S_1 \times S_2, \mathcal{F}, \mathcal{L})$ be the product (see Section 6.1) of the two.*

The $\mathcal{F}$-essential subgroups of S are of the form $Q_1 \times S_2$ with $Q_1 < S_1$ $\mathcal{F}_1$-essential or $S_1 \times Q_2$ with $Q_2 < S_2$ $\mathcal{F}_2$-essential.

Proof. Let $P \leq S$ be a $\mathcal{F}$-essential subgroup. By Proposition 1.1.8, P is $\mathcal{F}$-centric and $\mathcal{F}$-radical. Thus, By [AOV], Lemma 3.1, $P = P_1 \times P_2$ with $P_i \leq S_i$.

Remark also that, if we have two groups G_1 and G_2 such that p divide $|G_1|$ and $|G_2|$ then $G_1 \times G_2$ cannot contain a strongly p-embedded subgroup. Indeed, by [AKO], Proposition A.7, if S_i is a Sylow p-subgroup of G_i, every strongly p-embedded subgroup of $G_1 \times G_2$ must contain $H = \langle x \in G \mid x(S_1 \times S_2)x^{-1} \cap S_1 \times S_2 \neq 1 \rangle$ and it is not difficult to see that H contains $G_1 \times \{0\}$ and $\{0\} \times G_2$ which implies that $H = G$. We have that $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Out}_{\mathcal{F}_1}(P_1) \times \mathrm{Out}_{\mathcal{F}_2}(P_2)$ hence, the only possibility for P to be $\mathcal{F}$-essential is that $P_1 = S$ and P_2 is $\mathcal{F}_2$-essential or the contrary. ✩

The radical subgroups of a wreath product have been listed explicitly by Alperin and Fong in [AF]. They also compute their normalizers, centralizers and outer-automorphisms and we invite the reader to find the details there.

Let G_0 be a finite group, S_0 a Sylow p-subgroup of G_0 and $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ be the associated p-local finite group. We consider the wreath product $G = G_0 \wr C_p$, $S = S_0 \wr C_p$ and the associated p-local finite group $(S, \mathcal{F}, \mathcal{L})$. Here, for the direct computation, we will take the notation of Alperin and Fong [AF]: an element of G will be represented by permutation matrix corresponding to the powers of $(1, 2, \ldots, p)$ with entries in G_0 and the composition will follow the matrix product with the composition in G_0. Denote by $c \in G$ the element

$$e \otimes P_{(1,2,\ldots,p)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & e \\ e & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & 0 & 0 \\ 0 & \cdots & \cdots & e & 0 \end{pmatrix}$$

where e is the trivial element of G_0. Here, we are interested in the $\mathcal{F}$-essential subgroups.

Lemma 6.3.4. *Let $P \leq S$ be an $\mathcal{F}$-essential subgroup.*

(E_1) If $P \leq S_0^p$, then either $P = S_0^p$ and $N_G(P) = N_{G_0}(S_0) \wr C_p$ or P is $\mathcal{F}_0^p$-essential and $N_G(P) = N_{G_0^p}(P)$.

(E_2) If $P \not\leq S_0^p$, then $P \cong_{\mathcal{F}} Q \wr C_p$ where Q is $\mathcal{F}_0$-essential and $N_G(P)/P \cong N_{G_0}(Q)/Q$ through the diagonal map $G_0 \hookrightarrow G_0^p$.

Proof. Let $P \leq S$ be an $\mathcal{F}$-essential subgroup.

Assume first that $P < S_0^p$. By Lemma 6.3.2, we know that P is $\mathcal{F}_0^p$-essential and the computation of $N_G(P)$ and $N_G(S_0^p)$ are direct.

Secondly, assume that $P \not\leq S_0^p$. All choices of a splitting $C_p \to G$ are conjugate in G and hence we can assume that $P = \langle P_0, c\rangle$ where $P_0 = P \cap S_0^p$. If we write $P_0^{(i)}$ the projection of P_0 on its ith factor, as c normalizes P_0, we have that $P_0^{(i)} = P_0^{(j)}$ for all i, j and if we denote $Q = P_0^{(1)}$, then $P_0 \leq Q^p$. A direct computation give that

$$C_G(P) \cong C_{G_0}(Q) \otimes \mathrm{Id} = \left\{ \begin{pmatrix} g & 0 & \cdots & 0 \\ 0 & g & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & g \end{pmatrix} \; ; \; g \in C_{G_0}(Q) \right\}$$

and as P is p-centric, Q is G_0-centric.

Let us now look at the normalizers. We have that $N_G(P) = \langle N_{G_0^p}(P), c\rangle$ so we just have to study $N_{G_0^p}(P)$. If $g = (g_1, \ldots, g_p) \in N_{G_0^p}(P)$, as g normalizes $P \cap G_0^p = P_0$, we have, for all i, $g_i \in N_{G_0}(Q)$. Moreover, if we denote $h = (h_1, \ldots, h_p) = gcg^{-1}c^{-1} \in P_0$, we have, for all i, $g_i h_i = g_{i-1}$ (with $g_0 = g_p$) and then, there is $h' \in Q^p$ such that $g = (g_1, g_1, \ldots, g_1).h \in \langle N_{G_0}(Q) \otimes \mathrm{Id}, Q^p\rangle \leq N_G(Q \wr C_p)$. Hence, every automorphism $c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$ can be extended to an automorphism of $Q \wr C_p$. As P is $\mathcal{F}$ essential, by Proposition 1.1.8.(b), $P = Q \wr C_p$ and $N_G(P)/P = N_{G_0}(Q)/Q \otimes \mathrm{Id} \cong N_{G_0}(Q)/Q$. Then, as $N_G(P)/P = \mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly p-embedded subgroup, $\mathrm{Out}_{\mathcal{F}_0}(Q) = N_{G_0}(Q)/Q$

as well. Finally, up to conjugacy, we can also assume that Q is fully normalized in $\mathcal{F}_0$ and thus Q is $\mathcal{F}_0$-essential. ✩

Let us now look at some cohomological results. Recall that for a group G, a subgroup $H \leq G$, and M a $\mathbb{F}_p[H]$-module, we define the *induced* and *coinduced* $\mathbb{F}_p[G]$-module by,

$$\mathrm{Ind}_H^G(M) = M \otimes_{\mathbb{F}_p[H]} \mathbb{F}_p[G] \qquad \mathrm{coInd}_H^G(M) = \mathrm{Hom}_{\mathbb{F}_p[H]}(\mathbb{F}_p[G], M).$$

Recall also that, when the index of H in G is finite, these two $\mathbb{F}_p[G]$-modules are isomorphic (by [We], Lemma 6.3.4).

Lemma 6.3.5. *Let X be a CW complex and denote by G its fundamental group.*
If X_0 is a covering space of X with fundamental group $G_0 \trianglelefteq G$ of finite index, then, for every $\mathbb{F}_p[G_0]$-module M, we have a natural isomorphism of $\mathbb{F}_p[G/G_0]$-modules,

$$H^*(X_0, Res_{G_0}^G Ind_{G_0}^G(M)) \cong H^*(X_0, M) \otimes_{\mathbb{F}_p} \mathbb{F}_p[G/G_0].$$

This induces the wanted isomorphism in cohomology.

Proof. This can be easily seen on the chain level. Let $\tilde{X}$ be the universal covering space of X. As $\mathbb{F}_p[G/G_0]$-modules, we have the following

$$\begin{aligned}\mathrm{Hom}_{\mathbb{F}_p[G_0]}(C_*(\tilde{X}), \mathrm{Res}_{G_0}^G \mathrm{Ind}_{G_0}^G(M)) &= \bigoplus_{g \in [G/G_0]} \mathrm{Hom}_{\mathbb{F}_p[G_0]}(C_*(\tilde{X}), g.M) \\ &\cong \mathrm{Hom}_{\mathbb{F}_p[G_0]}(C_*(\tilde{X}), M) \otimes_{\mathbb{F}_p} \mathbb{F}_p[G/G_0].\end{aligned}$$

✩

Proposition 6.3.6. *Let G_0 be a finite group and $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ be the associated p-local finite group. Consider $G = G_0 \wr C_p$, $S = S_0 \wr C_p$ a Sylow p-subgroup of G and $(S, \mathcal{F}, \mathcal{L})$ the associated p-local finite group. Denote also by $G_0^\Delta \cong G_0$ and $S_0^\Delta \cong S_0$ the diagonal subgroups of G_0^p and S_0^p and consider the associated p-local finite subgroup $(S_0^\Delta, \mathcal{F}_0^\Delta, \mathcal{L}_0^\Delta) \cong (S_0, \mathcal{F}_0, \mathcal{L}_0)$. If, for M an $\mathbb{F}_p[\pi_{\mathcal{L}_0}^\Delta]$-module, $\delta_{S_0^\Delta}$ and $\delta_{S_0^p}$ induce natural isomorphisms $H^*(|\mathcal{L}_0^\Delta|, M) \cong H^*((\mathcal{F}_0^\Delta)^c, M)$ and $H^*(|\mathcal{L}_0^p|, coInd_{\pi_{\mathcal{L}_0^\Delta}}^{\pi_{\mathcal{L}_0}^p}(M)) \cong H^*((\mathcal{F}_0^p)^c, coInd_{\pi_{\mathcal{L}_0^\Delta}}^{\pi_{\mathcal{L}_0}^p}(M))$ then δ_S induces a natural isomorphism*

$$H^*(|\mathcal{L}|, coInd_{\pi_{\mathcal{L}_0^\Delta}}^{\pi_{\mathcal{L}}}(M)) \cong H^*(\mathcal{F}^c, coInd_{\pi_{\mathcal{L}_0^\Delta}}^{\pi_{\mathcal{L}}}(M)).$$

Proof. Write $N = \mathrm{coInd}_{\pi_{\mathcal{L}_0^\Delta}}^{\pi_{\mathcal{L}}}(M)$ and, for $i \in \{1, 2\}$, denote by $H^*(\mathcal{F}^{E_i}, N)$ the stable elements of $H^*(S, N)$ under the full subcategory of $\mathcal{F}$ with objects S and all the subgroups of S of type (E_i) defined in Lemma 6.3.4.

By Shapiro (see for example [Ev], Proposition 4.1.3), for every $P = Q \wr C_p$ of type (E_1), we have a natural isomorphism $H^*(Q \wr C_p, N) \cong H^*(Q, M)$ and, by the computation of normalizers in Lemma 6.3.4,

$$H^*(Q \wr C_p, N)^{\mathrm{Aut}_{\mathcal{F}}(Q \wr C_p)} \cong H^*(Q^\Delta, M)^{\mathrm{Aut}_{\mathcal{F}_0}(Q)}.$$

Hence, applying this to all the subgroups of type (E_1) and, by naturality of the Shapiro isomorphisms, we have that,

$$H^*(\mathcal{F}^{E_1}, N) \cong H^*((\mathcal{F}_0^{\Delta})^c, M).$$

On the topological side, as $|\mathcal{L}_0|$ is a covering space of $|\mathcal{L}_0^p|$ which is also a covering space of $|\mathcal{L}|$ (by [CL], Theorem 5.2 and Remark 5.3), $|\mathcal{L}_0|$ is a covering space of $|\mathcal{L}|$. Then, if we denote by X the universal covering space of $|\mathcal{L}|$ (which is also the universal covering space of $|\mathcal{L}_0|$), we have the following isomorphism on the chain level (because Res and coInd are adjoint functors)

$$\mathrm{Hom}_{\mathbb{Z}_{(p)}[\pi_{\mathcal{L}_0^{\Delta}}]}(C_*(X), M) \cong \mathrm{Hom}_{\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]}(C_*(X), N)$$

which is analogue to the Shapiro isomorphism (see [Ev], Proposition 4.1.3). It gives us the following isomorphism on cohomology

$$H^*(|\mathcal{L}_0^{\Delta}|, M) \cong H^*(|\mathcal{L}|, N)$$

and give the following commutative diagram

$$\begin{tikzcd} H^*(S_0^{\Delta}, M) \arrow[r, "\cong"] \arrow[d, "\left(\delta_{S_0^{\Delta}}\right)^*"'] & H^*(S, N) \arrow[d, "\delta_S^*"] \\ H^*(|\mathcal{L}_0^{\Delta}|, M) \arrow[r, "\cong"'] & H^*(|\mathcal{L}|, N) \end{tikzcd}$$

Thus by hypothesis, δ_S induces an isomorphism

$$H^*(\mathcal{F}^{E_1}, N) \cong H^*((\mathcal{F}_0^{\Delta})^c, M) \cong H^*(|\mathcal{L}_0^{\Delta}|, M) \cong H^*(|\mathcal{L}|, N).$$

Secondly, by factoring the Shapiro isomorphism (see [Ev], Proposition 4.1.3), the inclusion of S_0^p in S induces an injection $H^*(S, N) \hookrightarrow H^*(S_0^p, N)$. Hence

$$H^*(\mathcal{F}^{E_2}, N) \cong H^*(\mathcal{F}_0^p, N)^{C_p} \leq H^*(S_0^p, N).$$

By assumption, $\delta_{S_0^p}$ induces an isomorphism

$$H^*((\mathcal{F}_0^p)^c, N) = H^*((\mathcal{F}_0^p)^c, \mathrm{coInd}_{\pi_{\mathcal{L}_0^{\Delta}}}^{\pi_{\mathcal{L}}}(M)^p) \cong H^*(|\mathcal{L}_0^p|, \mathrm{coInd}_{\pi_{\mathcal{L}_0^{\Delta}}}^{\pi_{\mathcal{L}}}(M)^p) H^*(|\mathcal{L}_0^p|, N)$$

and, by Lemma 6.3.5, this is isomorphic, as $\mathbb{F}_p[C_p]$-module, to the following

$$H^*(|\mathcal{L}_0^p|, \mathrm{coInd}_{\pi_{\mathcal{L}_0^{\Delta}}}^{\pi_{\mathcal{L}}}(M)) \otimes \mathbb{F}_p[C_p].$$

In particular it is a projective $\mathbb{F}_p[C_p]$-module.

Consider now the Serre spectral sequence associated to the fibration sequence

$$|\mathcal{L}_0|^p \longrightarrow |\mathcal{L}| \longrightarrow BC_p$$

with coefficients in N. The E_2 page is the following,

$$E_2^{p,q} = H^p(C_p, H^q(|\mathcal{L}_O|^p, N))$$

and, by the previous computation, the E_2 page is concentrated in the first row. Hence, we have that, $H^*(|\mathcal{L}_0^p|, N)^{C_p} = E_2^{0,*} \cong H^*(|\mathcal{L}|, N)$.

In conclusion,

$$H^*(\mathcal{F}^c, N) = H^*(\mathcal{F}^{E_1}, N) \cap H^*(\mathcal{F}^{E_2}, N) \cong H^*(|\mathcal{L}|, N)$$

and the theorem follows. ✩

This proposition is a bit technical but we will use it in a specific case. Consider $p = 5$, the group $G_0 = GL_{20}(F_2)$, the wreath product $G = G_0 \wr C_5$ and $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ and $(S, \mathcal{F}, \mathcal{L})$ the associated 5-local finite groups. By [Ru], Theorem 6.3, we know that $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ admits a 5-local subgroup of index 4 which is exotic $(S_e, \mathcal{F}_e, \mathcal{L}_e)$ and that we have a fibration sequence

$$|\mathcal{L}_e| \longrightarrow |\mathcal{L}| \longrightarrow BC_4.$$

In particular, we have $\pi_{\mathcal{L}}/\pi_{\mathcal{L}_0} = C_4 \wr C_5$ and we can be interested in comparing $H^*(|\mathcal{L}|, N)$ and $H^*(\mathcal{F}^c, N)$ for $N = \mathbb{F}_5[C_4 \wr C_5] = \mathrm{Ind}_{\pi_{\mathcal{L}_0^\Delta}}^{\pi_{\mathcal{L}}}(M) \cong \mathrm{coInd}_{\pi_{\mathcal{L}_0^\Delta}}^{\pi_{\mathcal{L}}}(M)$ (the action factors through a finite group) with $M = \mathbb{F}_5[C_4]$.

By Corollary 5.1.4, we have that $\delta_{S_0^\Delta}$ and $\delta_{S_0^p}$ induce natural isomorphisms

$$H^*(|\mathcal{L}_0^\Delta|, M) \cong H^*((\mathcal{F}_0^\Delta)^c, M) \text{ and}$$

$$H^*(|\mathcal{L}_0^p|, \mathrm{coInd}_{\pi_{\mathcal{L}_0^\Delta}}^{\pi_{\mathcal{L}_0}^p}(M)) \cong H^*((\mathcal{F}_0^p)^c, \mathrm{coInd}_{\pi_{\mathcal{L}_0^\Delta}}^{\pi_{\mathcal{L}_0}^p}(M)),$$

$(\mathrm{Ind}_{\pi_{\mathcal{L}_0^\Delta}}^{\pi_{\mathcal{L}_0}^p}(M) = \mathbb{F}_5[C_4^5])$.

Hence, all the hypothesis of Proposition 6.3.6 are satisfied and

$$H^*(|\mathcal{L}|, N) \cong H^*(\mathcal{F}^c, N).$$

Remark 6.3.7. This gives us an example of isomorphism between the cohomology of $|\mathcal{L}|$ and the stable elements when the action factors through a p-solvable group which cannot be recovered by a previous result. It helps us to settle Conjecture 5.2.9.

7 Examples

In this chapter we study some counter-examples. More precisely, we are looking at p-local finite groups $(S, \mathcal{F}, \mathcal{L})$ and $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-modules M such that $H^*(|\mathcal{L}|, M) \ncong H^*(\mathcal{F}^c, M)$.

7.1 A linking system with a non $\mathbb{F}_p$-acyclic universal covering space

We give here a condition on $\mathcal{F}$ such that the free module $\mathbb{F}_p[\pi_{\mathcal{L}}]$ gives a counter-example.

Proposition 7.1.1. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Assume that we have the following*

1. *The universal covering space $\widetilde{|\mathcal{L}|}$ of $|\mathcal{L}|$ is not $\mathbb{F}_p$-acyclic,*
2. *$\pi_{\mathcal{L}}$ is finite.*
3. *δ_S induces an inclusion $S \leq \pi_{\mathcal{L}}$.*

Then we have

$$H^*(|\mathcal{L}|, \mathbb{F}_p[\pi_{\mathcal{L}}]) \cong H^*(\widetilde{|\mathcal{L}|}, \mathbb{F}_p) \ncong H^*(\mathcal{F}^c, \mathbb{F}_p[\pi_{\mathcal{L}}]).$$

Proof. As $\pi_{\mathcal{L}}$ is finite, $\mathbb{F}_p[\pi_{\mathcal{L}}] = \mathrm{Ind}_1^{\pi_{\mathcal{L}}}\mathbb{F}_p \cong \mathrm{coInd}_1^{\pi_{\mathcal{L}}}\mathbb{F}_p$. Then, we have the following isomorphism on the chain level,

$$\mathrm{Hom}_{\mathbb{F}_p[\pi_{\mathcal{L}}]}(C_*(\widetilde{|\mathcal{L}|}), \mathbb{F}_p[\pi_{\mathcal{L}}]) \cong \mathrm{Hom}_{\mathbb{F}_p}(C_*(\widetilde{|\mathcal{L}|}), \mathbb{F}_p)$$

which induces the first isomorphism. As $\widetilde{|\mathcal{L}|}$ is not $\mathbb{F}_p$-acyclic, $H^i(\widetilde{|\mathcal{L}|}, \mathbb{F}_p) \neq 0$ for some i. However, as S is a subgroup of $\pi_{\mathcal{L}}$, $\mathbb{F}_p[\pi_{\mathcal{L}}]$ is also an $\mathbb{F}_p[S]$-projective module. As S is finite, $H^*(S, \mathbb{F}_p[\pi_{\mathcal{L}}])$, and thus $H^*(\mathcal{F}^c, \mathbb{F}_p[\pi_{\mathcal{L}}]) \subseteq H^*(S, \mathbb{F}_p[\pi_{\mathcal{L}}])$, are concentrated in degree 0. ✩

7.2 Projective linear groups

Here we study the p-local structure of the group $PGL_n(F_q)$ for q a power of p.

7.2.1 Tits buildings and radical subgroups

Let $G = PGL_n(\mathbb{F}_q)$ and $V = \mathbb{F}_q^n$ be the natural $\mathbb{F}_q$-vector space on which G acts.

Definition 7.2.1. The *Tits building* of G, denoted by Δ, is the poset of proper nontrivial subspaces of V.

Definition 7.2.2. A *flag* of V is a sequence of proper nontrivial subspaces of V, $D = (W_1 < W_2 < \cdots < W_k)$ with $W_1 \neq 0$ and $W_k < V$.

For two flags, $D = (W_1 < W_2 < \cdots < W_k)$ and $D' = (W'_1 < W'_2 < \cdots < W'_l)$, we will write $D \leq D'$ if $k \leq l$ and if there is an injective increasing map $\sigma : \{1, 2, \ldots, k\} \to \{1, 2, \ldots, l\}$ such that $W_i = W'_{\sigma(i)}$. It gives to the set of flags of V a poset structure.

We will denote by U_D the set of all $g \in G$ satisfying $g(W_i) = W_i$ which induces identity on W_1, W_i/W_{i-1} and V/W_k. It defines a subgroup of G.

We will denote by Rad the poset of these subgroups.

Remark 7.2.3. (a) For a flag D, U_D is a p-group.

(b) Each Sylow p-subgroup of G is equal to U_D for a maximal flag D.

(c) If $D = (W_1 < W_2 < \cdots < W_k)$, for every i and every $g \in U_D$, W_i is stable by g.

(d) If $D = (W_1 < W_2 < \cdots < W_k)$, $W_1 = V^{U_D}$ and W_k is the smallest subspace of V such that every $g \in U_D$ induces the identity on V/W_k (or, by duality, $W_k = V^{U_D}$ where V is identified with its dual $V*$ using the dual of a basis adapted to U_D and $U_D^* = \{g^* \mid g \in U_D\}$).

Proposition 7.2.4. *Let $D = (W_1 < W_2 < \cdots < W_k)$ be a flag of V. We have the following*

(a) $N_G(U_D) = \{g \in G \mid \forall i, g(W_i) = W_i\}$,

(b) $C_G(U_D) \leq U_D$ (U_D is a centric subgroup of G),

(c) $O_p(N_G(U_D)/U_D)$ is trivial (U_D is a radical p-subgroup of G).

Proof. To show the first point we can remark that, for a subgroup $H \leq G$ and $g \in G$, $V^{gHg^{-1}} = g(V^H)$. In particular $gU_Dg^{-1} = U_{g(D)}$ for every $g \in N_G(U_D)$.

For the point (b), in a basis associated to D, the matrix of an element of U_D is an upper triangular block matrix with identity block on the diagonal and the matrix of an element of $N_G(U_D)$ in the same base is just an upper triangular block matrix. Hence, using matrix arguments, we can show that $C_G(U_D) \leq U_D$. Indeed, if $g \in C_G(U_D)$, it normalizes U_D and thus its block matrix (in a basis adapted to D) has the following form,

$$\begin{pmatrix} A_1 & \star & \cdots & \star \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \star \\ 0 & \cdots & 0 & A_k \end{pmatrix}$$

and commutes for example with matrices of the form

$$\begin{pmatrix} \mathrm{Id} & R & 0 & \cdots & 0 \\ 0 & \mathrm{Id} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \mathrm{Id} \end{pmatrix} \in U_D.$$

In other words, $A_1 R = R A_2$ for every matrix R. Then the endomorphism

$$R \in \mathcal{M}_{n_1,n_2}(\mathbb{F}_q) \mapsto A_1 R - R A_2 \in \mathcal{M}_{n_1,n_2}(\mathbb{F}_q)$$

(where n_1 and n_2 are respectively the dimension of A_1 and A_2) is null. In particular, by looking at elementary matrices, we get that A_1 and A_2 are identity matrices. Finally, with the same arguments (changing the position of R), for every i, $A_i = \mathrm{Id}$.

For the third point, we can first remark that, if we write for $i > 1$, $d_i = dim(W_i) - dim(W_{i-1})$ and $d_1 = dim(W_1)$,

$$N_G(U_D)/U_D \cong PGL_{d_1}(\mathbb{F}_q) \times PGL_{d_2}(\mathbb{F}_q) \times \cdots \times PGL_{d_k}(\mathbb{F}_q).$$

As $O_p(PGL_k(\mathbb{F}_q)) = 1$ for every $k > 1$, U_D is a radical p-subgroup of G. ✩

Proposition 7.2.5. *Let D and D' be two flags.*

The following are equivalent,

(i) $D \leq D'$

(ii) $U_D \leq U_{D'}$.

Proof. $(i) \Rightarrow (ii)$ is clear using the matrix representation in a basis adapted to D'.

Conversely, let $D = (W_1 < \cdots < W_k)$ and $D' = (W'_1 < \cdots < W'_l)$ and assume that $U_D \leq U_{D'}$. By Remark 7.2.3, $W'_1 = V^{U_{D'}} \leq V^{U_D} = W_1$ and $W_k \leq W'_l$. If $l = 1$ then $W'_1 \leq W_1 \leq W_k \leq W'_l = W'_1$ and thus $k = 1$ and $W_1 = W'_1$. Assume now that $l \geq 2$ (i.e. $W'_1 < W'_k$) and assume there is a $i \in \{1, \ldots, k\}$ such that, for every j, $W_i \neq W'_j$. As $W'_1 < W_i < W'_l$, there is $j_0 \in \{1, \ldots, l-1\}$ such that $W'_{j_0} < W_i$ and $W'_{j_0+1} \nleq W_i$. But we have $W'_{j_0+1}/W'_{j_0} = (V/W'_{j_0})^{U_{D'}} \leq (V/W'_{j_0})^{U_D} \leq W_i/W'_{j_0}$ and hence, $W_{j_0+1} \leq W_i$ which contradicts the definition of j_0. ✩

Corollary 7.2.6. *The application $D \mapsto U_D \in Rad$ induces a G-equivariant homeomorphism between the two CW complexes $|Rad|$ and $|\Delta|$.*

Proof. The G-poset of flags of V corresponds to the G-poset of the simplexes of $|\Delta|$. Then the geometric realization of the nerve of Rad is the barycentric subdivision of $|\Delta|$. As the action of G on the flags and Rad coincide, the action of G on $|\mathrm{Rad}|$ and $|\Delta|$ coincide. ✩

These subgroups of G are also important in the study of the radical p-subgroups of G.

Proposition 7.2.7. *Let P be a proper nontrivial p-subgroup of G.*
There is a flag D_P such that,

(a) $P \leq U_{D_P}$

(b) $N_G(P) \leq N_G(U_{D_P})$.

Proof. P acts on V, and, as we are in characteristic p and P is a p-group, $V^P \neq \{0\}$. Moreover, $P \neq \{\mathrm{Id}\}$, thus $V^P \neq V$ and we can define $W_1 = V^P$. By induction, we assume that for $i \leq 0$, the proper nontrivial subspace W_i is defined. If $(V/W_i)^P < (V/W_i)$, we set $W_i < W_{i+1} < V$ such that $W_{i+1}/W_i = (V/W_i)^P$, else we stop.
We obtain by this process a flag $D_P = (W_1 < W_2 < \cdots < W_k)$ such that $P \leq U_{D_P}$ and $N_G(P) \leq N_G(U_{D_P})$ (using the fact that, for a subgroup H of G and $g \in G$, $V^{gHg^{-1}} = g(V^H)$). ✩

In particular, we can show that Rad corresponds to the set of all the radical p-subgroups of G.

Corollary 7.2.8. *The set of all the radical p-subgroups of G is Rad.*

Proof. We already know that, for a flag D, U_D is a radical p-subgroup of G. Consider then H a radical p-subgroup of G and assume that $H < U_{D_H}$ where D_H is the flag from the last proposition. Hence $H < N_{U_{D_H}}(H) \leq U_{D_H}$ (each subgroup of a p-group is strictly contains in its normalizer) and as U_{D_H} is normal in $N_G(H)$, $H < O_p(N_G(H))$ which contradicts the definition of radical p-subgroup. ✩

7.2.2 The homotopy type of the linking system

Let $G = PGL_n(\mathbb{F}_q)$ and $(S, \mathcal{F}, \mathcal{L})$ be the associated p-local finite group.
We are interested in the homotopy type of $|\mathcal{L}|$. As we know, it has the homotopy type of the geometric realization of the subcategory of $|\mathcal{L}|$ with objects the $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroups of S.

Lemma 7.2.9. *Let G be a finite group, S a Sylow p-subgroup of G and $(S, \mathcal{F}, \mathcal{L})$ the associated p-local finite group.*
If a subgroup P of S is $\mathcal{F}$-centric and $\mathcal{F}$-radical, then P is a radical p-subgroup of G.

Proof. Let $P \leq S$ be an $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroup of S. We have, if we write $O^p(C_G(P)) = C'_G(P)$, $O_p(N_G(P)/PC'_G(P)) = 1$. If P is not a radical p-subgroup of G, then $K = O_p(N_G(P)/P) > 1$ and then $1 < K/C'_G(P) \leq N_G(P)/PC'_G(P)$ is a normal p-subgroup of $N_G(P)/PC'_G(P)$ which leads to a contradiction. Hence P is a radical p-subgroup of G. ✩

Corollary 7.2.10. *Let $G = PGL_n(\mathbb{F}_q)$ and S a Sylow p-subgroup of G.*

If we denote by $\mathcal{L}^{cr}$ the subcategory of $\mathcal{L}^c_S(G)$ with set of objects all the $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroups of S and $\mathcal{T}^r$ the subcategory of $\mathcal{T}_S(G)$ with set of objects all the subgroups of S which are radical p-subgroups of G, then

$$\mathcal{L}^{cr} = \mathcal{T}^r.$$

Proof. From the corollary 7.2.8, every radical p-subgroup of G is centric, the set of $\mathcal{F}$-radical and $\mathcal{F}$-centric subgroups of S is the set of subgroups of S which are radical p-subgroups. Thus $\mathrm{Ob}(\mathcal{L}^{cr}) = \mathrm{Ob}(\mathcal{T}^r)$ and, as they are all centric, $\mathcal{L}^{cr} = \mathcal{T}^r$. ✩

Proposition 7.2.11. *$|Rad|$ is a covering space of $|\mathcal{L}^{cr}_S(G)|$ with covering group G.*

Proof. By Corollary 7.2.10, $\mathcal{L}^{cr} = \mathcal{T}^r$ and in particular $|\mathcal{L}^c_S(G)| \cong |\mathcal{L}^{cr}| = |\mathcal{T}^r|$. Also, we can easily see that $|\mathcal{T}^r|$ has the homotopy type of the Borel construction $|\mathcal{E}G \times_G \mathrm{Rad}|$. The result then follows as a consequence. ✩

Let us finish with a result of Tits on the homotopy type of the geometric realization of the Tits building Δ_G.

Proposition 7.2.12 (Tits)**.** *Let $G = PGL_n(\mathbb{F}_q)$.*

The geometric realization of the Tits Building of G has the homotopy type of a wedge of $(n-2)$-spheres. In particular, $H_*(|\Delta|; \mathbb{F}_p) = H_0(|\Delta|; \mathbb{F}_p) \oplus H_{n-2}(|\Delta|; \mathbb{F}_p)$.

Then, the action of G on Δ induces a structure of $\mathbb{F}_p[G]$-module on $H_{n-2}(|\Delta|; \mathbb{F}_p)$, which defines the *Steinberg representation of G* denoted St_G. This representation have been well studied and for example, we have the following.

Proposition 7.2.13. *The $\mathbb{F}_p[G]$-module St_G is projective.*

7.2.3 The case $n \geq 4$

Let $G = PGL_n(\mathbb{F}_q)$.

In that case, by Proposition 7.2.12, $|\Delta|$ is simply connected but not $\mathbb{F}_p$-acyclic and, $\pi_1(|\mathcal{L}^c_S(G)|) = G$. Hence, as a direct corrolary of Proposition 7.2.11, we have the following proposition.

Proposition 7.2.14. *Let $G = PGL_n(\mathbb{F}_q)$ and S a Sylow p-subgroup of G.*

(a) The universal cover of $|\mathcal{L}^c_S(G)|$ has the homotopy type of $|Rad|$.

(b) $\pi_{\mathcal{L}} := \pi_1(|\mathcal{L}_S^c(G)|) \cong G$.

Thus, for S a Sylow p-subgroup of G, the p-local finite group $(S, \mathcal{F}_S(G), \mathcal{L}_S^c(G))$ satisfies the assumption of 7.1.1 and then, the $\mathbb{F}_p[G]$-module $\mathbb{F}_p[G]$ gives a counter-example.

Moreover, the Steinberg module gives also another counter-example.

Proposition 7.2.15. *Let $G = PGL_n(\mathbb{F}_q)$ and S a Sylow p-subgroup of G.*
If St_G is the Steinberg representation of G, then

$$H^{n-2}(|\mathcal{L}_S^c(G)|, St_G) \neq 0 \qquad \textit{but} \qquad H^{n-2}(\mathcal{F}_S^c(G), St_G) = 0.$$

Proof. First, as the Steinberg representation St_G is $\mathbb{F}_p[G]$-projective, it is $\mathbb{F}_p[S]$-projective and then $H^{n-2}(\mathcal{F}_S^c(G), \mathrm{St}_G) \leq H^{n-2}(S, \mathrm{St}_G) = 0$ is trivial. Secondly, still as St_G is $\mathbb{F}_p[G]$-projective and $|\mathrm{Rad}|$ is the universal cover of $\mathcal{L}^{cr} = \mathcal{T}$ by Proposition 7.2.14 (by using the notation of 7.2.10), for every $m \geq 0$,

$$H^m(|\mathcal{L}^{cr}|, \mathrm{St}_G) = H^m(\mathrm{Hom}_{\mathbb{F}_p[G]}(C_*(|\mathrm{Rad}|), \mathrm{St}_G)) \cong \mathrm{Hom}_{\mathbb{F}_p[G]}(H_m(|\mathrm{Rad}|, \mathbb{F}_p), \mathrm{St}_G).$$

Thus, by Corollary 7.2.6,

$$H^{n-2}(|\mathcal{L}^{cr}|, \mathrm{St}_G) \cong \mathrm{Hom}_{\mathbb{F}_p[G]}(H_{n-2}(|\Delta|), \mathrm{St}_G) = \mathrm{Hom}_{\mathbb{F}_p[G]}(\mathrm{St}_G, \mathrm{St}_G) \neq 0.$$

Finally, as $|\mathcal{L}_S^c(G)| = |\mathcal{L}^{cr}|$, we get the result. ✩

7.2.4 The case $n = 3$

Let $G = PGL_3(\mathbb{F}_q)$.

When $n = 3$, $|\Delta|$ is a wedge of circles but the linking system restricted to $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroups of S has the same form as in the case of $D_8 \leq A_6$ (which, by the way, corresponds to the p-local finite group of $G = PGL_3(\mathbb{F}_2)$!).

Let S be the Sylow p-subgroup of G of upper triangular matrices with ones on the diagonal. Then, by Corollary 7.2.8, the proper $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroups of S are the following ones,

$$P = \left\{ \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \; ; \; (a,b) \in \mathbb{F}_q^2 \right\} \quad \text{and} \quad Q = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; ; \; (a,b) \in \mathbb{F}_q^2 \right\}.$$

Moreover, the normalizers are the followings,

$$N_G(S) = S, \; N_G(P)/P \cong PGL_2(\mathbb{F}_q) \text{ and } N_G(Q)/Q \cong PGL_2(\mathbb{F}_q).$$

Hence, $|\mathcal{L}^{cr}| \cong B\left(N_G(P) *_S N_G(Q)\right)$ and, for $M = \mathbb{F}_p[G]$, we have, by Mayer Vietoris, the following exact sequence,

$$0 \longrightarrow H^0(|\mathcal{L}|, M) \longrightarrow H^0(N_G(P), M) \oplus H^0(N_G(Q), M) \longrightarrow H^0(S, M) \longrightarrow H^1(|\mathcal{L}|, M) \longrightarrow 0.$$

Finally, looking at the indexes of S, $N_G(P)$ and $N_G(Q)$ in G, we can verify that,

$$H^1(|\mathcal{L}|, M) \neq \underbrace{0 = H^1(\mathcal{F}_S(G)^c, M)}_{\text{by projectivity of } M}.$$

Appendices

The Geometric realization of a category

1 Simplicial sets and their realizations

Definition 1.1. The *simplicial category* Δ is the category whose objects are the sets $[n] = \{0, 1, \ldots, n\}$ for $n \geq 0$ and whose morphisms are the order preserving maps between objects.

A *simplicial set* is a functor $K : \Delta^{\mathrm{op}} \longrightarrow \mathfrak{Sets}$, and a *morphism of simplicial sets* between two simplicial sets K_1 and K_2 is a natural transformation of functors.

For K a simplicial set, we often write $K_n = K([n])$, which is regarded as the set of "n-simplices" and an n-simplex σ of K is degenerated if $\sigma \in \mathrm{Im}\,(\chi^*)$ for some $\chi \in \mathrm{Mor}(\Delta)$. More precisely, the morphisms in Δ are generated by the *face* and *degeneracy* morphisms. For each n,

(i) there are $n + 1$ face morphisms $d_n^i \in \mathrm{Mor}_\Delta([n-1], [n])$ for $0 \leq i \leq n$ where d_n^i is the (unique) injective morphism whose image does not contains i,

(ii) there are n degeneracy morphisms $s_n^i \in \mathrm{Mor}_\Delta([n], [n-1])$ for $0 \leq i \leq n-1$ where s_n^i is the (unique) surjective morphism such that $s_n^i(i) = s_n^i(i+1) = i$.

Thus, an n-simplex σ of K is degenerate if, and only if, there exists $i \in \{0, \ldots, n-1\}$ such that $\sigma \in \mathrm{Im}\,({s_n^i}^*)$.

Let, for $n \geq 0$, Δ^n be the n-simplex of $\mathbb{R}^n$. This can be seen as a covariant functor $\Delta : \Delta \longrightarrow \mathrm{Top}$. We define then the geometric realization of a simplicial set as follows.

Definition 1.2. Let K be a simplicial set.

The *geometric realization* of K is the space

$$|K| = \left(\coprod_{n \geq 0} K_n \times \Delta^n \right) \Big/ \sim$$

where $(\sigma, \varphi_*(\tau)) \sim (\varphi^*(\sigma), \tau)$ for all $\sigma \in K_m$, $\tau \in \Delta^n$ and all $\varphi \in \mathrm{Mor}_\Delta([n], [m])$.

The geometric realization of a simplicial set K has a natural structure of CW complex with one vertex for each $\sigma \in K_0$, an edge for each non degenerate element of K_1, a 2-cell for each non degenerated element of K_2, etc.

The geometric realization defines a functor from the category $\mathcal{S}$ of simplicial sets to the category of topological spaces. It is more precisely the left adjoint of the singular simplicial set functor

$$S : \mathrm{Top} \longrightarrow \mathcal{S}$$

defined, for a space X, by $S_n(X)$ as the set of all continuous maps $\Delta \longrightarrow X$.

We can also define the product of two simplicial sets K and L by the simplicial set given by $(K \times L)_n = K_n \times L_n$ and respectively on morphisms. The following proposition shows that geometric realization behaves nicely with respect to products.

Proposition 1.3. *Let K, L be two simplicial sets.*

The application

$$|K \times L| \longrightarrow |K| \times |L|$$

induced by the projections of simplicial sets is a continuous bijection.

It is also a homeomorphism under certain conditions (for example if K or L has finitely many non degenerate simplices). See [GZ], Section III.3, for more details about this and a proof.

2 The nerve of a category and its geometric realization

Definition 2.1. Let $\mathcal{C}$ be a small category.

The *nerve* of $\mathcal{C}$ is the simplicial set $\mathcal{N}(\mathcal{C})$ define on objects by

$$\mathcal{N}(\mathcal{C})_0 = Ob(\mathcal{C})$$

$$\text{and for } n > 0, \qquad \mathcal{N}(\mathcal{C})_n = \{c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} c_n \,|\, c_i \in Ob(\mathcal{C}),\, \alpha_i \in Mor(\mathcal{C})\}$$

and on morphisms by, for $n \geq 0$ and $i \in \{0, \dots, n\}$,

$$d_n^{i\,*}(c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} c_n) = \begin{cases} c_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} c_n & \text{if } i = 0 \\ c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_{i-1}} c_{i-1} \xrightarrow{\alpha_{i+1}\circ\alpha_i} c_{i+1} \xrightarrow{\alpha_{i+2}} \cdots \xrightarrow{\alpha_n} c_n & \text{if } 0 < i < n \\ c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} c_{n-1} & \text{if } i = n \end{cases}$$

$$s_n^{i\,*}(c_0 \xrightarrow{\alpha_1} c_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} c_n) = c_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_i} c_i \xrightarrow{\mathrm{Id}_{c_i}} c_i \xrightarrow{\alpha_{i+1}} \cdots \xrightarrow{\alpha_n} c_n$$

The *geometric realization of* $\mathcal{C}$ is then the geometric realization of $\mathcal{N}(\mathcal{C})$

$$|\mathcal{C}| = |\mathcal{N}(\mathcal{C})|$$

The nerve define a functor from the category of small categories to $\mathcal{S}$ and thus, the geometric realization of a category define a functor from the category of small categories to Top. We can now look at natural transformation of functors. We have the following result at first proved by Segal.

Proposition 2.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be two small categories and f, g two functors from $\mathcal{C}$ to $\mathcal{D}$.*

If there is a natural transformation u between f and g, then the induced application $|f|$ and $|g|$ are homotopic.

Proof. Denote $\mathcal{I} = \ 0 \longrightarrow 1\ $ the category with two objects and a unique non identity morphism between the two objects.

By definition of u, for all $c, c' \in \mathcal{C}$ and all $\alpha \in \mathrm{Mor}(\mathcal{C})$, we have the following commutative diagram.

$$\begin{tikzcd} f(c) \arrow[r, "f(\alpha)"] \arrow[d, "u(c)"'] & f(c') \arrow[d, "u(c')"] \\ g(c) \arrow[r, "g(\alpha)"] & g(c') \end{tikzcd}$$

We can then define a functor $\hat{u} : \mathcal{C} \times \mathcal{I} \to \mathcal{D}$ given on objects by, $\forall c \in \mathcal{C}$, $\hat{u}(c, 0) = f(c)$ and $\hat{u}(c, 1) = g(c)$, and on morphisms by, $\forall c, c' \in \mathcal{C}$ and $\forall \alpha \in \mathrm{Mor}_{\mathcal{C}}(c, c')$, $\hat{u}(\alpha, \mathrm{Id}_0) = f(\alpha)$, $\hat{u}(\alpha, \mathrm{Id}_1) = g(\alpha)$ and $\hat{u}(\alpha,\ 0 \longrightarrow 1\ ) = u(c') \circ f(\alpha) = g(\alpha) \circ u(c)$.

As $|\mathcal{I}| = I$, we have, by Proposition 1.3, $|\mathcal{C} \times \mathcal{I}|$ is homeomorphic to $|\mathcal{C}| \times I$. Thus $|\hat{u}|$ define a homotopy between $|f|$ and $|g|$. ✩

Corollary 2.3. *(a) If $\mathcal{C}$ is a category with an initial or a terminal object, then $|\mathcal{C}|$ is contractible.*

(b) Let $f : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathcal{C}$ be functors. If there are natural transformations $u : f \circ g \to Id_{\mathcal{D}}$ and $v : g \circ f \to Id_{\mathcal{C}}$ then $|f| : |\mathcal{C}| \to |\mathcal{D}|$ and $|g| : |\mathcal{D}| \to |\mathcal{C}|$ are homotopy equivalences.

Proof. For (a), denote by F_{c_0} the constant functor which send each object to c_0, the initial (or terminal) object of $\mathcal{C}$, and each morphisms on Id_{c_0}. We have a natural transformation between F_{c_0} and $\mathrm{Id}_{\mathcal{C}}$ which send $c \in \mathcal{C}$ on the unique morphism from c_0 to c (or c to c_0). Thus $|F_{c_0}|$ and $|\mathrm{Id}_{\mathcal{C}}| = \mathrm{Id}_{|\mathcal{C}|}$ are homotopic. Then $|F_{c_0}|$ is a retraction by deformation of $|\mathcal{C}|$ on $|c_0|$ which is a point.

For (b), this is clear by functoriality and Proposition 2.2. ✩

One important example is given by the construction of a classifying space of a finite group G from the category $\mathcal{B}(G)$.

Proposition 2.4. *Let G be a finite group. Let $\mathcal{B}(G)$ be the category with one object o_G and such that $Mor_{\mathcal{B}(G)}(o_G) = G$.*

Then $|\mathcal{B}G|$ is a classifying space of G.

Proof. Consider the category $\mathcal{E}(G)$ with set of objects G and for each $(g, h) \in G \times G$, $\mathrm{Mor}_{\mathcal{E}(G)}(g, h) = \{\varphi_{g,h}\}$. As every object of $\mathcal{E}G$ is initial and terminal, by 2.3, $EG = |\mathcal{E}G|$ is contractible.

Consider then the functor $p_G : \mathcal{E}G \to \mathcal{B}G$ which send each object on o_G and for all $(g, h) \in G \times G$, send $\varphi_{g,h}$ on hg^{-1}.

We have an action of G on $\mathcal{E}G$ by multiplication on objects and morphisms ($g.\varphi_{h,k} = \varphi_{gh,gk}$). This action induces a free G-action on the geometric realization $EG = |\mathcal{E}G|$ and $|p_G|$ is the orbit map for this action. Hence, EG is the universal cover of $|\mathcal{B}G|$ and $\pi_1(|\mathcal{B}G|) \cong G$. Then $|\mathcal{B}G|$ is a classifying space of G. ✩

We finish with the fundamental group and some covering space constructions of the geometric realization of a category with a special form (which contains all the categories we consider here).

Proposition 2.5 ([AKO], Proposition III.2.8)**.** *Let $\mathcal{C}$ be a small category and $c_0 \in Ob(\mathcal{C})$.*

Assume we can choose morphisms $\iota_c \in Mor_{\mathcal{C}}(c, c_0)$, for each $c \in Ob(\mathcal{C})$, where $\iota_{c_0} = Id_{c_0}$. Define the functor

$$\theta : \mathcal{C} \longrightarrow \mathcal{B}(\pi_1(|\mathcal{C}|, c_0)$$

by sending each object $c \in Ob(\mathcal{C})$ to the unique object and, for $c, d \in Ob(\mathcal{C})$, each $\alpha \in Mor_{\mathcal{C}}(c, d)$ to the class of loop $\iota_d.\alpha.\overline{\iota_c}$.

Then θ induces an isomorphism of $\pi_1(|\mathcal{C}|, c_0)$ with the free group on generators $[\alpha]$ for each $\alpha \in Mor(\mathcal{C})$, modulo the relation $[\iota_c] = 1$ for each $c \in Ob(\mathcal{C})$ and $[\beta \circ \alpha] = [\beta][\alpha]$ for each composable pair $\alpha, \beta \in Mor(\mathcal{C})$.

Proposition 2.6 ([AKO], Proposition III.2.9)**.** *Let $\mathcal{C}$ be a small category, $c_0 \in Ob(\mathcal{C})$, G be a finite group and $F : \mathcal{C} \longrightarrow \mathcal{B}(G)$ be a functor.*

Assume , for each $c \in Ob(\mathcal{C})$ and each $g \in G$, there are $d \in Ob(\mathcal{C})$ and $\psi \in Iso_{\mathcal{C}}(c, d)$ such that $F(\psi) = g$.

For each $H \leq G$, let $\mathcal{C}_H \subseteq \mathcal{C}$ be the subcategory with the same objects, where for all $\psi \in Mor(\mathcal{C})$, $\psi \in Mor(\mathcal{C}_H)$ if and only if $F(\psi) \in H$.

Then, For each $H \leq G$, $|\mathcal{C}_H|$ is homotopy equivalent to the covering space of $|\mathcal{C}|$ with fundamental group $|F|_*^{-1}(H)$.

The Bousfield-Kan p-completion

The Bousfield-Kan p-completion functor is a functor from spaces to spaces, denoted $(-)^\wedge_p$, together with a natural transformation $\lambda : \mathrm{Id} \longrightarrow (-)^\wedge_p$. We refer to [BK] for a precise definition of this functor and its properties. We also refer to [AKO] for some of the properties which are more clearly proved.

Definition 0.7. A continuous map $f : X \longrightarrow Y$ is a *p-equivalence* if f induces an isomorphism from $H^*(X, \mathbb{F}_p)$ to $H^*(Y, \mathbb{F}_p)$ (or, equivalently, f induces an isomorphism from $H_*(X, \mathbb{F}_p)$ to $H_*(Y, \mathbb{F}_p)$).

Many of the important properties of p-completions hold only for certain classes of spaces. But the next proposition holds for all spaces.

Proposition 0.8 ([BK], Lemma I.5.5)**.** *A continuous map $f : X \longrightarrow Y$ induces a homotopy equivalence $f^\wedge_p : X^\wedge_p \longrightarrow Y^\wedge_p$ if, and only if, it is a p-equivalence.*

When we work with p-completion, the classes of p-complete spaces and p-good spaces play a central role.

Definition 0.9. Let X be a space.

(i) The space X is *p-complete* if $\lambda_X : X \longrightarrow X^\wedge_p$ is a homotopy equivalence.

(ii) The space X is *p-good* if $\lambda_X : X \longrightarrow X^\wedge_p$ is a p-equivalence. X is called *p-bad* otherwise.

One important family of examples of p-complete spaces are the classifying spaces of p-groups (See [AKO], Proposition III.1.10).

We can show ([BK], Proposition I.5.2) that a space X is p-good if, and only if, $X^\wedge_p$ is p-complete (i.e. $X^\wedge_p \simeq (X^\wedge_p)^\wedge_p$). On the other hand, if a space is p-bad, then all of its iterated p-completions are also p-bad ([BK], Proposition I.5.2).

The most important criterion for checking if a space is p-good, at least for the type of spaces we work with in this thesis, is given by the following proposition. Recall that a finite group G is call p-perfect if G is generated by its commutators and p-th powers.

Proposition 0.10. *Let X be a space.*

If $\pi_1(X)$ contains a p-perfect subgroup of finite index, then X is p-good and if K is the maximal p-perfect group of $\pi_1(X)$, then $\pi_1(X_p^\wedge) = \pi_1(X)/K$.

In particular, if $\pi_1(X)$ is finite, then X is p-good and $\pi_1(X_p^\wedge) = \pi_1(X)/O^p(\pi_1(X))$.

A proof is sketched in [AKO], Proposition III.1.11. Remark that, in particular, this implies that, if X is p-good then $\pi_1(X_p^\wedge)$ is a p-group.

The next proposition states that when X is p-good, $\lambda_X : X \longrightarrow X_p^\wedge$ is universal among all p-equivalence $X \longrightarrow Y$.

Proposition 0.11. *For every p-good space X, and every p-equivalence $f : X \longrightarrow Y$, there is a map $g : Y \longrightarrow X_p^\wedge$ unique up to homotopy, such that $g \circ f \simeq \lambda_X$. Thus $\lambda_X : X \longrightarrow X_p^\wedge$ is a final object among homotopy classes of p-equivalences defined on X.*

This is just a direct consequence of [BK], Lemma I.5.5.

Corollary 0.12. *If X and Y are two spaces, and one of them is p-good, then their p-completions are homotopy equivalent if, and only if, there exists some space Z and maps $X \xrightarrow{f} Z \xleftarrow{g} Y$ such that f and g are p-equivalences.*

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Résumé

Nous présentons une étude de la cohomologie à coefficients tordus de la réalisation géométrique des systèmes de liaison. Plus précisément, si $(S, \mathcal{F}, \mathcal{L})$ est un groupe fini p-local, nous travaillons sur la cohomologie $H^*(|\mathcal{L}|, M)$ de la réalisation géométrique de $\mathcal{L}$, avec un $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$-module M en coefficients, et ses liens avec les éléments $\mathcal{F}^c$-stables $H^*(\mathcal{F}^c, M) \subseteq H^*(S, M)$ à travers l'inclusion de BS dans $|\mathcal{L}|$.

Après avoir donné la définition des éléments $\mathcal{F}^c$-stables, nous étudions l'endomorphisme de $H^*(S, M)$ induit par un (S, S)-bi-ensemble $\mathcal{F}^c$-caractéristique et nous montrons que, sous certaine hypothèse et si l'action est nilpotent, alors on a un isomorphisme naturel $H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M)$. Ensuite, nous regardons les actions p-résolubles à travers la notion de sous-groupe p-local d'index premier à p ou une puissance de p. Nous montrons que si l'action de $\pi_1(|\mathcal{L}|)$ sur M se factorise par un p'-groupe alors on a aussi un isomorphisme naturel. Pour une action p-résoluble plus général, nous obtenons un résultat dans le cas des systèmes réalisables. Ces résultats nous conduisent à la conjecture qu'on a un isomorphisme naturel pour tout groupe fini p-local et toute action p-résoluble.

Nous donnons quelque outils pour étudier cette conjecture. Nous travaillons sur les produits de groupes finis p-locaux avec la formule de Kunneth et les systèmes de liaison que se décomposent bien vis-à-vis de la suite exacte longue de Mayer-Vietoris. Finalement, nous étudions les sous-groupes essentiels d'un produit couronné par C_p.

Nous finissons par des exemples qui soulignent, qu'en général, on ne peut espérer un isomorphisme entre $H^*(|\mathcal{L}|, M)$ et $H^*(\mathcal{F}^c, M)$.

Abstract

The aim of this work is to study the cohomology with twisted coefficients of the geometric realization of linking systems. More precisely, if $(S, \mathcal{F}, \mathcal{L})$ is a p-local finite group, we work on the cohomology $H^*(|\mathcal{L}|, M)$ of the geometric realization of $\mathcal{L}$ with coefficients in a $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$-module M and its links with the $\mathcal{F}^c$-stable elements $H^*(\mathcal{F}^c, M) \subseteq H^*(S, M)$ through the inclusion of BS in $\mathcal{L}$.

After we give the definition of $\mathcal{F}^c$-stable elements, we study the endomorphism of $H^*(S, M)$ induced by an $\mathcal{F}^c$-characteristic (S, S)-biset and we show that, if the action is nilpotent and we assume an hypothesis, we have a natural isomorphism $H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M)$. Secondly, we look at p-solvable actions of $\pi_1(|\mathcal{L}|)$ on M through the notion of p-local subgroups of index a power of p or prime to p. If the action factors through a p'-group, we show that there is also a natural isomorphism. We then work on extending this to any p-solvable action and we get some positive answers when the p-local finite group is realizable. Theses leads to the conjecture that it is true for any p-local finite group and any p-solvable actions.

We also give some tools to study this conjecture on examples. We look at products of p-local finite groups with Kunneth Formula and linking systems which can be decomposed in a way which behaves well with Mayer-Vietoris long exact sequence. Finally, we study essential subgroups of wreath products by C_p.

We finish with some examples which illustrate that, in general, we cannot hope an isomorphism between $H^*(|\mathcal{L}|, M)$ and $H^*(\mathcal{F}^c, M)$.

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