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The algebra of Kleene stars of the plane and polylogarithms

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ABSTRACT

We extend the definition and study the algebraic properties of the polylogarithm \( \text{Li}_T \), where \( T \) is rational series over the alphabet \( X = \{x_0, x_1\} \) belonging to \( (\mathbb{C}(X), +, \text{conv} (\langle x_0 \rangle), +, \text{conv} (\langle x_1 \rangle)) \). We honor this fact with a definition that allows for the abstraction of \( \text{Li}_T \) with credit. To copy otherwise, or republish, on the first page. Copyrights for components of this work owned by others than ACM for profit or commercial advantage and that copies bear this notice and the full citation classroom use is granted without fee provided that copies are not made or distributed

Let us consider also the following morphism

\[
\pi^r: \langle \mathbb{C}(X) \rangle \longrightarrow \langle \mathbb{C}(Y) \rangle,
\]

for \( r \geq 1 \) and, for any \( a \in \mathbb{C}, \pi^r(a) = a \). The extension of \( \pi^r \) over \( \mathbb{C} \) is denoted by \( \pi^r: \langle \mathbb{C}(X) \rangle \longrightarrow \langle \mathbb{C}(Y) \rangle \) satisfying, for any \( p \in \mathbb{C}(X), \pi^r(p) = 0 \). Hence,

\[
\ker(\pi^r) = \langle \mathbb{C}(X) \rangle \quad \text{and} \quad \text{Im}(\pi^r) = \langle \mathbb{C}(Y) \rangle.
\]

We let \( \pi^r \) be the inverse of \( \pi^r \):

\[
\pi^r: \langle \mathbb{C}(Y) \rangle \longrightarrow \langle \mathbb{C}(X) \rangle,
\]

\[
y_0, \ldots, y_n \longmapsto x_0^r x_1 \ldots x_n^r x_1^r \ldots x_1^r
\]

The projectors \( \pi^r \) and \( \pi^r \) are mutual adjoints:

\[
\forall p \in \mathbb{C}(X), \forall q \in \mathbb{C}(Y), \quad (\pi^r(p) | q) = (p | \pi^r(q)).
\]

We extend \( \pi^r \) to \( \mathbb{C}(X) \) and \( \mathbb{C}(Y) \) and require the renormalization of the corresponding divergent

1. Which are all Hopf save the last one due to \( \pi_0 \) which is infiltration-like [2].

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1. Which are all Hopf save the last one due to \( \pi_0 \) which is infiltration-like [2].
polyzetas. It is already done for the corresponding case of polyzetas at positive multi-indices [3, 4, 20] and it is done [8, 11, 22] and completed in [5, 7] for the case of polyzetas at positive multi-indices.

To study the polylogarithms at negative multi-indices, one relies on [5, 7]

1. (the one-to-one) correspondence between the multi-indices \((s_1, \ldots, s_r) \in \mathbb{N}^r\) and the words \(y_k \cdot y_s\) defined over \(Y_0\),

2. indexing these polylogarithms by words \(y_k \cdot y_s\) : 
\[
Li_{y_k \cdots y_s}(z) = \sum_{n_i > \cdots > n_i > 0} n_i^p \ldots n_i^p z^{n_i}.
\]

In the same way, for polylogarithms at positive indices, one relies on [15, 17]

1. (the one-to-one) correspondence between the combinatorial compositions \((s_1, \ldots, s_r)\) and the words \(x_0^{s_1-1} \cdot x_1^{s_2-1} \cdot x_0^{s_3-1} \cdot x_1\) in \(X^r \cdot x_1^r\).

2. the indexing of these polylogarithms by words \(x_0^{s_1-1} \cdot x_1^{s_2-1} \cdot x_0^{s_3-1} \cdot x_1\) :
\[
Li_{x_0^{s_1-1} \cdot x_1^{s_2-1} \cdot x_0^{s_3-1} \cdot x_1}(z) = \sum_{n_i > \cdots > n_i > 0} n_i^p \ldots n_i^p z^{n_i}.
\]

Moreover, one obtained the polylogarithms at positive indices as image by the following isomorphism of theshuffle algebra [15]

\[
Li_u : (\mathbb{C}(X), \cdot, 1_{X^r}) \longrightarrow (\mathbb{C}(Li_u), \cdot, 1_{\Omega}),
\]

\[
x_0^s \mapsto \frac{\log^{(s)}(z)}{n!},
\]

\[
x_0^s \mapsto \frac{\log^{(s)}(1/(1-z))}{n!},
\]

\[
x_0^{s_1-1} \cdot x_1^{s_2-1} \cdot x_0^{s_3-1} \cdot x_1 \mapsto \sum_{n_i > \cdots > n_i > 0} n_i^p \ldots n_i^p z^{n_i}.
\]

Extending over the set of rational power series \[^5\] on non commutative variables, \(\mathbb{C}^{rat}(X)\), for instance, as follows

\[
S = \sum_{n \geq 0 } \langle S \mid x_0^s \rangle x_0^s + \sum_{k \geq 1} \sum_{(s_1, \ldots, s_r) \in \mathbb{N}^r} \langle S \mid x_0^{s_1} x_1^{s_2} \cdots x_0^{s_r} \rangle,
\]

\[
Li_S(z) = \sum_{n \geq 0 } \langle S \mid x_0^s \rangle \frac{\log^{(s)}(z)}{n!} + \sum_{k \geq 1} \sum_{(s_1, \ldots, s_r) \in \mathbb{N}^r} \langle S \mid x_0^{s_1} x_1^{s_2} \cdots x_0^{s_r} \rangle.
\]

The morphism \(Li_u\) is no longer injective over \(\mathbb{C}^{rat}(X)\) but \(\{Li_u\}_{w \in X^r}\) are still linearly independent over \(\mathbb{C}\) [19, 20].

**Example 1.**

i. \(1_{\Omega} = Li_{1_{X^r}} = Li_{x_0^{s_1-1} \cdot x_1^{s_2-1} \cdot x_0^{s_3-1} \cdot x_1}\);

ii. \(\lambda = Li_{x_0^{s_1-1} \cdot x_1^{s_2-1} \cdot x_0^{s_3-1} \cdot x_1} = \lambda = Li_{1_{X^r}} - 1\);

iii. \(\mathcal{C}(\mathbb{L}_{X^r}) = \mathbb{C}[Li_{1_{X^r}} - Li_{x_0^{s_1-1} \cdot x_1^{s_2-1} \cdot x_0^{s_3-1} \cdot x_1}]\).

Let us consider also the differential and integration operators, acting on \(\{Li_u\}_{w \in X^r}\) [21] :

\[
\partial_z = \frac{d}{dz}, \quad \theta_0 = \frac{d}{dz}, \quad \theta_1 = (1 - z) \frac{d}{dz},
\]

\[
\forall f \in \mathcal{E}, \quad t_0(f) = \int_0^f \partial_z f(s) \omega_0(s) \quad \text{and} \quad t_1(f) = \int_0^f \partial_z f(s) \omega_1(s).\]

Here, the operator \(t_0\) is well-defined (as in definition 1 in section 2.2) then one can check easily [18, 19, 5, 7]

1. The subspace \(\mathcal{E}(\mathbb{L}_w)\) is closed under the action of \(\{\theta_0, \theta_1\}\) and \(\{t_0, t_1\}\).

2. The operators \(\{\theta_0, \theta_1, t_0, t_1\}\) satisfy in particular,

\[
\theta_0 + \theta_1 = [\theta_1, \theta_0] = \partial_z \quad \text{and} \quad \forall k = 0, 1, \theta_k t_0 = 0 \quad \text{and} \quad \theta(t_0 t_0) = (\theta(t_0)) t_0 = \text{Id}.
\]

3. \(\theta(t_1)\) and \(\theta(t_0)\) are scalar operators within \(\mathcal{E}(\mathbb{L}_w)\) respectively with eigenvalues \(\lambda\) and \(1/\lambda\), i.e.

\[
(\theta(t_1)f) = \lambda f \quad \text{and} \quad (\theta(t_0)f) = (1/\lambda)f.
\]

4. Let \(w = y_1 \cdots y_s \in Y^s\) (then \(\mathcal{E}(w) = x_0^{s_1-1} x_1^{s_2-1} \cdots x_1^{s_3-1} x_1\)) and \(u = y_1 \cdots y_s \in Y^s\). The functions \(Li_u, Li_{\bar{w}}\) satisfy

\[
Li_u = (x_0^{s_1-1} \cdot x_1^{s_2-1} \cdot x_0^{s_3-1} \cdot x_1) \Omega, \quad Li_{\bar{w}} = (x_0^{s_1-1} \cdot x_1^{s_2-1} \cdot x_0^{s_3-1} \cdot x_1) \Omega,
\]

\[
t_0 Li_{\bar{w}}(w) = Li_{\bar{w}}(w), \quad t_1 Li_{\bar{w}}(w) = Li_{\bar{w}}(w), \quad \theta Li_{\bar{w}}(w) = Li_{\bar{w}}(w), \quad \theta Li_{\bar{w}}(w) = Li_{\bar{w}}(w).
\]

Here, we explain the whole project of extension of \(Li_u\), study different aspects of it, in particular what is desired of \(i, i = 0, 1\). The interesting problem in here is that what we do expect of \(i, i = 0, 1\). In fact, the answers are

— it is a section of \(\theta_i, i = 0, 1\) (i.e. takes primitives for the corresponding differential operators);

— it extends \(i, i = 0, 1\) (defined on \(C(Li_u)_{w \in X^r}\) and very surprisingly, although not coming directly from Chen calculus, they provide a group-like generating series).

We will use this construction to extend \(Li_u\) to \(\{Li_u\}_{w \in X^r}\) and, after that, we extend it to a much larger rational algebra.

### 2. Background

#### 2.1 Standard topology on \(\mathcal{H}(\Omega)\)

The algebra \(\mathcal{H}(\Omega)\) is that of analytic functions defined over \(\Omega\). It is endowed with the topology of compact convergence whose seminorms are indexed by compact subsets of \(\Omega\), and defined by

\[
p_K(f) = \|f\|_K = \sup_{s \in K} |f(s)|.
\]

Of course,

\[
p_{K_1 \cup K_2} = \sup(p_{K_1}, p_{K_2}),
\]

and therefore the same topology is defined by extending a fundamental subset of seminorms, here it can be chosen denumerable. As \(\mathcal{H}(\Omega)\) is complete with this topology it is a Frechet space and even, as

\[
p_K(fg) \leq p_K(f)p_K(g),
\]

it is a Frechet algebra (even more, as \(p_K(1_{\Omega}) = 1\) a Frechet algebra with unit).

With the standard topology above, an operator \(\phi \in \text{End}(\mathcal{H}(\Omega))\) is continuous iff (with \(K_t\) compacts of \(\Omega\))

\[
(\forall K_t)(\exists M_t > 0)(\forall f \in \mathcal{H}(\Omega))(\|\phi(f)\|_{K_t} \leq M_t \|f\|_{K_t}).
\]

#### 2.2 Study of continuity of the sections \(\theta_i\) and \(\eta_i\)

Then, \(\{Li_u\}_{w \in X^r}\) and \(\mathcal{H}(\Omega)\) being closed under the operators \(\theta_i; i = 0, 1\). We will first build sections of them, then see that
they are continuous and, propose (discontinuous) sections more adapted to renormalisation and the computation of associators.

For \( \alpha \in \Omega \), let us define \( t_\alpha^\Omega \in \text{End}(\mathcal{H}(\Omega)) \) by

\[
t_\alpha^\Omega(f) = \int_{\Omega} f(s) \omega_\alpha(s), \quad t_\alpha^\Omega(f) = \int_{\Omega} f(s) \omega_\alpha(s).
\]

It is easy to check that \( \theta_\gamma t_\alpha^\Omega = I_d \mathcal{H}(\Omega) \) and that they are continuous on \( \mathcal{H}(\Omega) \) for the topology of compact convergence because for all \( K \subset \text{compact} \, \Omega \), we have

\[
|pk(t_\alpha^\Omega(f))| \leq pk(f) \left[ \sup_{s \in K} \int_{\gamma} \omega_\alpha(s) \right],
\]

and, in view or paragraph (2.1), this is sufficient to prove continuity. Hence the operators \( t_\alpha^\Omega \) are also well defined on \( \mathcal{C}(\{ \text{Li}_n \})_{n \in X} \) and it is easy to check that

\[
t_\alpha^\Omega(\mathcal{C}(\{ \text{Li}_n \})_{n \in X}) \subset \mathcal{C}(\{ \text{Li}_n \})_{n \in X}.
\]

Due to the decomposition of \( \mathcal{H}(\Omega) \) into a direct sum of closed subspaces

\[
\mathcal{H}(\Omega) = \mathcal{H}_{\text{ap}}(\Omega) \oplus C(\Omega),
\]

it is not hard to see that the graphs of \( \theta_\gamma \) are closed, thus, the \( \theta_\gamma \) are also continuous.

Much more interesting (and adapted to the explicit computation of associators) we have the operator \( t_0 \) (without superscripts), mentioned in the introduction and (rigorously) defined by means of a \( \mathbb{C} \)-basis of \( \mathcal{C}(\{ \text{Li}_n \})_{n \in X} \).

As \( \mathcal{C}(\{ \text{Li}_n \})_{n \in X} \cong \mathcal{C}(\mathcal{Z}[\alpha]) \), one can partition the alphabet of this polynomial algebra in \( \mathcal{Z}[\alpha] \) and then get the decomposition

\[
\mathcal{C}(\{ \text{Li}_n \})_{n \in X} \cong \mathcal{C}(\{ \text{Li}_n \})_{n \in X} \oplus \mathcal{C}(\{ \text{Li}_n \})_{n \in X}.
\]

In fact, we have an algorithm to convert Li_{a1} as

\[
\text{Li}_{a1}(z) = \sum_{m \geq 0} P_m \log^{m}(z) = \sum_{n} \langle P_m \rangle \text{Li}_n(z) \log^{m}(z).
\]

due to the identity [13]

\[
ux_1 x_0^s = ux_1 x_0^s - \sum_{k=1}^{n} (u = x_0^{s+k}) x_1 x_0^{-k}.
\]

This means that

\[
\mathcal{B} = \left\{ \sum_{k=1}^{n} (1 - z^k) \text{Li}_{a1} \text{Li}_{a2} \right\}_{(k,n,a) \in \mathbb{Z} \times \mathbb{N} \times X},
\]

\[
\mathcal{B} = \left\{ \frac{1}{(1 - z^k)} \text{Li}_{a1} \text{Li}_{a2} \right\}_{(k,n,a) \in \mathbb{Z} \times \mathbb{N} \times X},
\]

\[
\mathcal{B} = \left\{ \frac{1}{(1 - z^k)} \text{Li}_{a1} \text{Li}_{a2} \right\}_{(k,n,a) \in \mathbb{Z} \times \mathbb{N} \times X},
\]

is a \( \mathbb{C} \)-basis of \( \mathcal{C}(\{ \text{Li}_n \})_{n \in X} \).

With this basis, we can define the operator \( t_0 \) as follows

**DEFINITION 1.** Define the index map \( \text{ind} : \mathcal{B} \to \mathbb{Z} \) by

\[
\text{ind}(\frac{1}{(1 - z^k)} \text{Li}_{a1} \text{Li}_{a2} \log^{m}(z)) = k + [ux_1].
\]

Now \( t_0 \) is computed by:

1. \( t_0(b) = \int_{X}^{b} \omega_\alpha(s) \) if \( \text{ind}(b) \geq 1 \).
2. \( t_0(b) = \int_{X}^{b} \omega_\alpha(s) \) if \( \text{ind}(b) \leq 0 \).

To show discontinuity of \( t_0 \) with a direct example, the technique consists in exhibiting 2 sequences \( f_n, g_n \in C(\{ \text{Li}_n \})_{n \in X} \) converging to the same limit but such that

\[
\lim_{n \to \infty} f_n \neq \lim_{n \to \infty} g_n.
\]

Here we choose the function \( z \) which can be approached by two limits. For continuity, we should have “equality of the limits of the image-sequences” which is not the case.

\[
z = \sum_{n \geq 0} \frac{\log^{n}(z)}{n!}, \quad z = \sum_{n \geq 1} \frac{(-1)^{n+1}}{m!} \log^{n}(\frac{1}{1-z}).
\]

Let then

\[
f_n = \sum_{0 \leq m \leq n} \frac{\log^{m}(z)}{m!} \quad \text{and} \quad g_n = \sum_{1 \leq m \leq n} \frac{(-1)^{m+1}}{m!} \log^{m}(\frac{1}{1-z}).
\]

We have \( f_n, g_n \in C(\{ \text{Li}_n \})_{n \in X} \) and \( t_0(f_n) = f_{n+1} \). Hence, one has \( \lim_{n \to \infty} \langle f_n \rangle = f_{n+1} \). On the other hand

\[
\lim_{n \to \infty} \langle g_n \rangle = \lim_{n \to \infty} \int_{X}^{g_n} \omega_\alpha(s) \omega_\alpha(s) = \int_{X}^{z} \lim_{n \to \infty} g_n \omega_\alpha(s) = \int_{X}^{z} s \omega_\alpha(s) = z.
\]

The exchange of the integral with the limit above comes from the fact that the operator

\[
\phi \mapsto \int_{X}^{\phi} \omega_\alpha(s) \omega_\alpha(s),
\]

is continuous on the space \( \mathcal{H}_{\text{ap}}(\Omega \cup B(0,1)) \) of analytic functions \( f \in \mathcal{H}(\Omega \cup B(0,1)) \) such that \( f(0) = 0 \) (the open ball of center 0 and radius 1).

**3. Algebraic extension of \( \text{Li}_n \) to \( (\mathbb{C} \cup \{ \infty \}) \).**

We will use several times the following lemma which is characteristic-free.

**LEMMA 1.** Let \( (\mathcal{A}, d) \) be a commutative differential ring without zero divisor, and \( R = \text{ker}(d) \) be its subring of constants. Let \( z \in \mathcal{A} \) such that \( d(z) = 1 \) and \( S = \{ e_\alpha \}_{\alpha \in I} \) be a set of eigenfunctions of \( d \) all different \( (\subset R) \) i.e.

\[ i. \ e_\alpha \neq 0, \]

\[ ii. \ d(e_\alpha) = \alpha e_\alpha ; \alpha \in I. \]

Then the family \( \{ e_\alpha \}_{\alpha \in I} \) is linearly free over \( R[z] \).

**PROOF.** If there is no non-trivial \( R[z] \)-linear relation, we are done. Otherwise let us consider relations

\[
N \sum_{j=1}^{n} P_j(z)e_{\alpha_j} = 0,
\]

with \( P_j \in R[z] \backslash \{ 0 \} \) for all \( j \) (packed linear relations). We choose one minimal w.r.t. the triplet

\[
[N, \deg(P_k), \sum_{j \in N} \deg(P_j)],
\]

6. Here \( R[z] \) is understood as ring adjunction i.e. the smallest subring generated by \( R \cup \{ z \} \).

7. Here \( R[z] \) means the formal univariate polynomial ring (the subscript is here to avoid confusion).
lexicographically ordered from left to right. 8

Remark that d(P(z)) = P'(z) (because d(z) = 1), we apply the operator d - α 0 1d to both sides of (4) and get
\[ N \sum_{j=1}^{N} \left( P_j(z) + (\alpha_j - \alpha_N)P_j(z) \right) e_{\alpha_j} = 0. \] (6)

Minimality of (4) implies that (6) is trivial i.e., formal series) which has no zero divisor.

Now relation (4) implies
\[ \prod_{1 \leq j \leq N-1} \left( \alpha_N - \alpha_j \right) \left( N \sum_{j=1}^{N} P_j(z) e_{\alpha_j} \right) = 0, \] (8)

which, because \( \alpha' \) has no zero divisor, is packed and has the same associated triplet (5) as (4). From (7), we see that for all \( k < N \)
\[ \prod_{1 \leq j \leq N-1} \left( \alpha_N - \alpha_j \right) P_k(z) = \prod_{1 \leq j \leq k} \left( \alpha_N - \alpha_j \right) P_k(z), \]

so, if \( N \geq 2 \), we would get a relation of lower triplet (5). This being impossible, we get \( N = 1 \) and (4) boils down to \( P_k(z)e_{\alpha} = 0 \) which, as \( \alpha' \) has no zero divisor, implies \( P_k(z) = 0 \), contradiction.

Then the \( (e_{\alpha})_{\alpha \in \mathbb{C}} \) are \( \mathbb{R}[z] \)-linearly independent.

Remark 1. If \( \alpha' \) is a \( \mathbb{Q} \)-algebra or only of characteristic zero (i.e., \( n_{\alpha} = 0 \) \( \Rightarrow \alpha = 0 \)), then \( d(z) = 1 \) implies that \( z \) is transcendental over \( R \).

First of all, let us prove

Lemma 2. Let \( k \) be a field of characteristic zero and \( Z \) an algebra. Then \( (k(z), \ldots, z_N) \) is a \( k \)-algebra without zero divisor.

Proof. Let \( B = (b_{ij})_{i,j} \) be an ordered basis of \( \mathcal{L}(\mathbb{Q}(Z)) \) and \( \left( \frac{b_{\alpha}}{\alpha} \right)_{\alpha \in \mathbb{N}[0]} \) the divided corresponding PBW basis. One has
\[ \Delta_{\alpha}(\frac{b_{\alpha}}{\alpha}) = \sum_{\alpha_1 + \alpha_2 = \alpha} \frac{b_{\alpha_1}}{\alpha_1} \otimes \frac{b_{\alpha_2}}{\alpha_2}, \]

Hence, if \( S, T \in (k(z), \ldots, z_N) \), considering
\[ \langle S \circ T | \frac{b_{\alpha}}{\alpha} \rangle = \langle S \otimes T | \frac{\alpha}{\alpha} \rangle = \sum_{\alpha_1 + \alpha_2 = \alpha} \langle S | \frac{b_{\alpha_1}}{\alpha_1} \rangle \langle T | \frac{b_{\alpha_2}}{\alpha_2} \rangle, \]

we see that \( (k(z), \ldots, z_N) \cong (k[l][t], \ldots) \) (commutative algebra of formal series) which has no zero divisor.

Lemma 3. Let \( \alpha' \) be a \( \mathbb{Q} \)-algebra (associative, unital, commutative) and \( z \) an indeterminate, then \( e^z \in \alpha'[z] \) is transcendental over \( \alpha'[z] \).

Proof. It is straightforward consequence of Remark (1). Note that this can be rephrased as \( [z, e^z] \) are algebraically independent over \( \alpha'[z] \).

Proposition 1. Let \( Z = \{ z_{\alpha} \}_{\alpha \in \mathbb{N}[0]} \) be an alphabet, then \( [e^z, e^z] \) is algebraically independent on \( \mathcal{C}[Z] \) within \( \mathcal{C}[Z] \).

Proof. Using lemma 3, one can prove by recurrence that
\[ [e^{z_0}, e^{z_1}, \ldots, e^{z_k}, z_0, z_1, \ldots, z_k], \]

are algebraically independent. This implies that \( Z \cup \{ e^z \}_{z \in Z} \) is an algebraically independent set and, by restriction \( Z \cup \{ e^z, e^z \} \) whence the proposition.

4. Extension to \( \mathcal{C}(X, \ldots, 1_{X}) \) of \( S_{P}(X) \)

Indeed, the set \( \{ a_0 x_0 + a_1 x_1 \}_{a_0, a_1 \in \mathbb{C}} \) is a shuffle monoid as
\[ \{ (a_0 x_0 + a_1 x_1)^* \} = \{ (a_0 x_0 + b_0 x_0 + b_1 x_1)^* \} = \{ (a_0 + b_0) x_0 + (a_1 + b_1) x_1 \} \] as there are many relations between these elements (as a mono id it isomorphic to \( \mathbb{C}^2 \), hence far from being free), we study here the vector space
\[ \text{span}_{\mathbb{C}} \{ (a_0 x_0 + a_1 x_1)^* \}_{a_0, a_1 \in \mathbb{C}}. \]

It is a shuffle sub-algebra of \( \mathcal{C}(X, \ldots, 1_{X}) \) which will be called star of the plane. Note that it is also a shuffle sub-algebra of the \( \mathcal{C}(\mathbb{C}^d)(X, \ldots, 1_{X}) \) of exchangeable series. We can give the

Definition 3. A series is said exchangeable if whenever two words have the same multidegree (here bidegree) then they have the same coefficient within it. Formally for all \( u, v \in X^* \)
\[ \langle \forall x \in X \rangle (|u|_x = |v|_x) \implies \langle S | u \rangle = \langle S | v \rangle. \]
On the other hand, for any $S \in \mathbb{C} \langle \langle X \rangle \rangle$, we can write

$$S = \sum_{n \geq 0} P_n,$$

where $P_n \in \mathbb{C}[X]$ such that $\deg P_n = n$ for every $n \geq 0$. Then $S$ is called exchangeable iff $P_n$ are symmetric by permutations of places for every $n \in \mathbb{N}$. If $S$ is written as above then we can write

$$P_n = \frac{a_n}{n!} x^n 1^{-n}.$$

**Definition 4.** Let $S \in \mathbb{C} \langle \langle X \rangle \rangle$ (resp. $\mathbb{C}(X)$) and let $P \in \mathbb{C}(X)$ (resp. $\mathbb{C}(\langle\langle X \rangle\rangle)$). The left and right residual of $S$ by $P$ are respectively the formal power series $P \circ S$ and $S \circ P$ in $\mathbb{C}(\langle\langle X \rangle\rangle)$ defined by

$$\langle P \circ S \mid w \rangle = \langle S \mid wP \rangle \quad \text{(resp. } \langle S \circ P \mid w \rangle = \langle S \mid Pw \rangle \text{).}$$

For any $S \in \mathbb{C}(\langle\langle X \rangle\rangle)$ (resp. $\mathbb{C}(X)$) and $P, Q \in \mathbb{C}(X)$ (resp. $\mathbb{C}(\langle\langle X \rangle\rangle)$), we straightforwardly get

$$P \circ (Q \circ S) = P \circ Q \circ S, \quad \langle S \circ P \rangle \circ Q = S \circ (P \circ Q), \quad \langle P \circ (Q \circ S) \rangle \circ Q = P \circ \langle Q \circ (P \circ S) \rangle.$$

In case $x, y \in X$ and $w \in X^*$, we get

$$x \circ (w y) = \delta^- w \quad \text{and} \quad x \circ w \circ y = \delta^+ w.$$

**Theorem 1.** Let $\delta \in \mathcal{D}(\mathbb{C}(X), \ldots, 1_X^*)$. Moreover, we suppose that $\delta$ is locally nilpotent. Then the family $(\delta^k)^{k \in \mathbb{N}}$ is summable and its sum, denoted $exp(\delta)$, is a one-parameter group of automorphisms of $(\mathbb{C}(X), \ldots, 1_X^*)$.

**Theorem 2.** Let $L$ be a Lie series. Let $\delta^+_L$ and $\delta^-_L$ be defined respectively by

$$\delta^+_L(P) := P \circ L \quad \text{and} \quad \delta^-_L(P) := L \circ P.$$

Then $\delta^+_L$ and $\delta^-_L$ are locally nilpotent derivations of $(\mathbb{C}(X), \ldots, 1_X^*)$. Therefore, $exp(\delta^+_L)$ and $exp(\delta^-_L)$ are one-parameter groups of $Aut((\mathbb{C}(X), \ldots, 1_X^*))$ and $Aut((\mathbb{C}(X), \ldots, 1_X^*))$.

**4.2 Study of the algebra $\mathbb{C}(X)_{\mathcal{C}} \mathcal{C}^{\text{rat}} \langle \langle x_0 \rangle \rangle \subset \mathbb{C}(\langle\langle X \rangle\rangle)$.**

We will start by the study of the one-letter shuffle algebra, i.e., $(\mathbb{C}(x, \ldots, 1_x^*))$ and use two times Lemma 1 above.

Let us now consider $\delta = \mathcal{C}^{\text{rat}} \langle \langle x_0 \rangle \rangle ; \delta_a = (\alpha x)^* ; \alpha \in C$ endowed with $d = \delta^+_1$ (which is a derivation for the shuffle) and $\delta x = x$. We are in the conditions of Lemma 1 and then $\mathcal{C}(\langle\langle x_0 \rangle \rangle)$ is C-linearly free which amounts to say that

$$B_0 = (x^\circ, x^{\circ \circ}) \in \mathcal{C}(\langle\langle x_0 \rangle \rangle)$$

is C-linearly free in $\mathcal{C}(\langle\langle x_0 \rangle \rangle)$.

To see that it is a basis, it suffices to prove that $B_0$ is (linearly) generating. Consider that

$$\mathcal{C}(\langle\langle x_0 \rangle \rangle) = (P/Q) \mathcal{C}(\langle\langle x_0 \rangle \rangle_{\mathcal{O}(0)}) \text{, and }$$

then, as $\mathbb{C}$ is algebraically closed, we have a basis

$$B_1 \cup B_2 = \{ x^k \}_{k \geq 0} \cup \{ (\alpha x)^* \}_{\alpha \in \mathbb{C}, j \geq 1},$$

and it suffices to see that we can generate $B_2$ by elements of $B_0$, which is a consequence of the two identities

$$x^k \circ (\alpha x)^* = \frac{1}{k!} (x^\circ x^k)^{\circ \circ} (\alpha x)^*, \quad x^k \circ (\alpha x)^* = \frac{1}{k!} (x^\circ x^k)^{\circ \circ} (\alpha x)^*.$$

Now, we use again Lemma 1 with

$$\alpha = \mathcal{C}^{\text{rat}} \langle \langle x_0 \rangle \rangle \neq \mathcal{C}^{\text{rat}} \langle \langle x_1 \rangle \rangle \subset \mathcal{C}(\langle\langle x_0, x_1 \rangle \rangle),$$

hence without zero divisor (see Lemma 2), endowed with $d = \delta^+_1$ then $(x^\circ, x^{\circ \circ}, (\alpha x)^*)_{\alpha \in \mathbb{C}, k \leq 0}$ is linearly free over $R = \mathcal{C}(\langle\langle x_0 \rangle \rangle)$. It is easily seen, using a decomposition like

$$S = \sum_{p, q \geq 0} \langle S \mid x^p y^q \rangle x^p y^q,$$

that $\mathcal{C}(\langle\langle x_0 \rangle \rangle) = \ker(d)$ and one obtains then that the arrow

$$\mathcal{C}(\langle\langle x_0 \rangle \rangle) \subset \mathcal{C}(\langle\langle x_1 \rangle \rangle) \rightarrow \mathcal{C}(\langle\langle x_0 \rangle \rangle) \subset \mathcal{C}(\langle\langle x_0, x_1 \rangle \rangle)$$

is an isomorphism. Hence, $(x_{i_0}^\circ, x_{i_0}^{\circ \circ}, (\alpha x_{i_0})^*)_{\alpha \in \mathbb{C}, i_0 \leq 0}$ is a C-basis of $\mathcal{S} = \mathcal{C}(\langle\langle x_0 \rangle \rangle) \subset \mathcal{C}(\langle\langle x_1 \rangle \rangle)$. In order to extend $\mathcal{L}_a$ to $\mathcal{S}$ we send

$$T(x_0, k_0, a_0, a_1) = x_{i_0}^\circ x_{i_0}^{\circ \circ} (\alpha x_{i_0})^* x_{i_0}^{\circ \circ} (\alpha x_{i_0})^*.$$
Using, once more, Lemma 1, one gets

**PROPOSITION 2.** The family \( \{(a_{0}x_0)^{\ast} \cup (a_1x_1)^{\ast}\}_{a_0,a_1 \in \mathbb{C}} \) is a (\( C(X) \times \mathbb{C}^{\ast} \)) basis of \( C(X) \cup C^{\ast}(\langle x_0 \rangle) \cup C^{\ast}(\langle x_1 \rangle) \), then we have a \( C \)-basis \( \{w_{\ast}(a_{0}x_0)^{\ast} \cup (a_1x_1)^{\ast}\}_{a_0,a_1 \in \mathbb{C}} \) of

\[
C(X) \cup C^{\ast}(\langle x_0 \rangle) \cup C^{\ast}(\langle x_1 \rangle) = \mathbb{C}(X) \cup C^{\ast}(\langle x_0 \rangle) \cup C^{\ast}(\langle x_1 \rangle) = C(X) \cup SP_C(X).
\]

**Proof.** We will use a multi-parameter consequence of Lemma 1.

**LEMMA 4.** Let \( Z \) be an alphabet, and \( k \) a field of characteristic zero. Then, the family \( \{a_{0}x_0 \}_{0 \leq k} \subset k[Z] \) is linearly independent over \( k[Z] \).

This proves that, in the shuffle algebra the elements

\[
\{(a_{0}x_0)^{\ast} \cup (a_1x_1)^{\ast}\}_{a_0,a_1 \in \mathbb{C}}
\]

are linearly independent over \( C(X) \simeq C[Z\text{ym}(X)] \) within \( C(\langle x \rangle) \cup \ldots \cup C(\langle x \rangle) \).

Now \( L^{(2)} \) is well-defined and this morphism is not into from \( C(X) \cup C^{\ast}(\langle x \rangle) \) admits \( \{(a_{0}x_0)^{\ast} \cup (a_1x_1)^{\ast}\}_{a_0,a_1 \in \mathbb{C}} \) as a basis for its structure of \( C(X) \)-module, it suffices to remark

\[
L^{(1)}_{a_0,a_1}(z) = z^a \times \frac{1}{(1-z)^b}
\]

is a generating system of \( \mathcal{C} \).

First of all, we recall the following lemma

**LEMMA 5.** Let \( M_1 \) and \( M_2 \) be \( K \)-modules (\( K \) is a unitary ring). Suppose \( \phi : M_1 \rightarrow M_2 \) is a linear mapping. Let \( N \subset ker(\phi) \) be a submodule. If there is a system of generators in \( M_1 \), namely \( \{g_i\}_{i \in I} \), such that

1. For any \( i \in I \) \( g_i \equiv \sum_{j \in J} c_{ij} g_j \mod N \), \( c_{ij} \in K \forall j \in J \);
2. \( \{g_i\}_{i \in I} \) is \( K \)-free in \( M_2 : \)

then \( N = ker(\phi) \).

**PROOF.** Suppose \( P \in ker(\phi) \). Then \( P \equiv \sum_{j \in J} p_j g_j \mod N \) with \( \{p_j\}_{j \in J} \subset K \). Then \( 0 = \phi(P) = \sum_{j \in J} p_j \phi(g_j) \). From the fact that \( \phi(g_j) \) is \( \{g_i\}_{i \in I} \) is \( K \)-free on \( M_2 \), we obtain \( p_j = 0 \) for any \( j \in J \). This means that \( P \in N \). Thus \( ker(\phi) \subset N \). This implies that \( N = ker(\phi) \).

Let now \( \mathcal{F} \) be the ideal generated by \( x_0^a \cup x_1^b - x_0^a + 1_{X} \). It is easily checked, from the following formulas, (for \( l \geq 1 \))

\[
[w_{\ast}(x_0^a) \cup x_1^b]_{l} = [w_{\ast}(x_1^b) \cup (x_0^a)^{l-1}]_{l-1} [\mathcal{F}],\]

in Figure 1, \( (w,l,k) \) codes the element \( w_{\ast}(x_0^a) \cup x_1^b \cup (x_0^a)^{l-1} \).
THEOREM 3 (DERIVATIONS AND AUTOMORPHISMS).

Let \( P, Q \in \mathbb{C}(X) \) (resp. \( C[x_0^*, (x_0^*)^*] \subset \mathbb{C}(X) \), \( T \in \mathcal{L}_{\mathfrak{ic}}(X) \) (resp. \( \mathcal{L}_{\mathfrak{ic}}(X) \)). Then \( \mathcal{T}(T) \) is a derivation in \( (\mathbb{C}[\text{Li}_u]_{w \in X}, \times, 1) \) (resp. \( (\mathbb{C}[\text{Li}_u]_{w \in X}, \times, 1) \)) and \( \exp(\theta(T)) \) is then a one-parameter group of automorphisms of \n
\[
(\mathbb{C}[\text{Li}_u]_{w \in X}, \times, 1) \quad (\text{resp.} \quad (\mathbb{C}[\text{Li}_u]_{w \in X}, \times, 1)).
\]

PROOF. Because \( \text{Li}_{p \cdot u} \cdot Q = \text{Li}_p \cdot \text{Li}_u \cdot Q \) and \( \theta(T) \cdot (\text{Li}_u \cdot Q) = (\theta(T) \cdot \text{Li}_u) \cdot Q \)
and \( \text{Li}_{p \cdot u} \cdot Q = \text{Li}_p \cdot \text{Li}_u \cdot Q \) and \( \theta(T) \cdot (\text{Li}_u \cdot Q) = (\theta(T) \cdot \text{Li}_u) \cdot Q \) for any \( p \in \mathbb{P} \) and again (see Example 1) \n
\[
\text{Li}_{\theta_1 \cdot \gamma_2} = \text{Li}_F = \mathcal{T}(F) \cdot 1_{\Omega},
\]

where \( F \) is the following rational series on \( x_1 \)

\[
F = \sum_{k_i=0}^{x_0} \sum_{k_2=0}^{x_0} \ldots \sum_{k_r=0}^{x_0} \left( s_1 \right)_{k_1} \left( s_1 + s_2 - k_1 \right)_{k_2} \ldots
\]

\[
\left( s_1 + \ldots + s_r - k_1 - \ldots - k_{r-1} \right) \sum_{i=1}^{k_r} \frac{1}{z_i - 1\times i},
\]

if \( k_i = 0 \),

\[
\left( s_1 + \ldots + s_r - k_1 - \ldots - k_{r-1} \right) \sum_{i=1}^{k_r} \frac{1}{z_i},
\]

if \( k_i > 0 \).

Since \( \mathcal{T}(\gamma_1) \cdot 1_{\Omega} = 1/(1 - z) \) then this proves once again that [5, 7]

\[
\text{Li}_{\theta_1 \cdot \gamma_2} = \text{Li}_F = \mathcal{T}(F) \cdot 1_{\Omega},
\]

One can deduce finally that

COROLLARY 3.

\[
\{\text{Li}_u\}_{u \in X^*} \supseteq \mathbb{C}[1/(1 - z)](\{\text{Li}_u\}_{u \in X^*}) = \text{span}_{\mathbb{C}} \left\{ \prod_{n_i>0} \sum_{n_0=0}^{n_0} \frac{1}{z_i}, \frac{1}{z_i} \right\}
\]

6. Conclusion

We have studied the structure of the algebra \( C(X) = C^\mathfrak{lat}(\langle x_0 \rangle) \subset C^\mathfrak{lat}(\langle x_1 \rangle) \),

where \( X = \{x_0, x_1\} \) is an alphabet. We have also considered the ways for denoting the polylogarithms. By the results on the algebra \( C(X) = C^\mathfrak{lat}(\langle x_0 \rangle) \subset C^\mathfrak{lat}(\langle x_1 \rangle) \), we have given an extension of the polylogarithms and have obtained polylogarithmic transseries

\[
C\{x^\mathfrak{lat}(1 - z)\} \subset \mathbb{C}[x_0^*, (x_0^*)^*] \subset \mathbb{C}(X).
\]

With this extension, we have constructed several shuffle bases of the algebra of polylogarithms. In the special case of the “Laurent subalgebra”

\[
C(X, 1, X^\cdot \gamma_1, (x_0^* - x_0^*), x_1^*) \subset \mathbb{C}(X) = C^\mathfrak{lat}(\langle x_0 \rangle) \subset C^\mathfrak{lat}(\langle x_1 \rangle),
\]

we have completely characterized the kernel of the polylogarithmic map \( \text{Li}_u \), providing a rewriting process which terminates to a normal form.

7. References


