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Modal analysis of microcantilever sensors with environmental damping

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The vibration response of microcantilevers to simple harmonic base motion is formulated in terms of base motion plus modal responses relative to the moving base frame. Environmental damping is evaluated. The analysis gives the frequency responses of the first four modes and shows that the phase angle of each mode relative to the moving base frame is constant but that the phase angle of the absolute motion varies along the length of cantilevers. This finding is verified with experimental data. Phase changes along cantilevers depend on frequency ratios and the magnitude of environmental damping. Experimental data are used to establish damping factors, and phase angles are measured experimentally and compared with theoretical predictions. © 2005 American Institute of Physics. DOI: 10.1063/1.1880472

I. INTRODUCTION

Microcantilevers have attracted much attention recently because of their potential as extremely sensitive sensor platforms for chemical and biological detection.1–6 Microcantilever sensors have two operational modes: bending due to changes in surface stress and shifts in resonance frequency due to mass adsorption and viscous damping.7–13 Resonant behavior is important for atomic force microscope (AFM) imaging in liquids because the sensitivity of the AFM depends on the oscillatory response of the cantilever.9

Here, we present an analysis of the vibration response of microcantilevers corresponding to base motion and environmental damping. Cantilever damping is modeled as distributed viscous damping, which links the surface of the cantilever to a ground.

The solution method is based on modal summation, which graphically depicts the behavior of each mode and its contribution to the total response. Other studies have shown the participation of multiple modes in various frequency response data.14 It is assumed that cantilever vibrations are driven by base motion, which can be described by a single harmonic function. The magnitudes of modal responses depend on frequency ratio and damping, which is defined by modal damping factors. The damping factor for the first mode is determined experimentally from frequency response data. Damping factors for the higher modes are ascertained from frequency response data and theory.

The phase angle for each mode relative to the moving base is distinct from the phase angle of absolute motion. The modal phase angle for the relative motion of each mode is typically constant; the modal phase angle for the absolute motion varies along the length of the cantilever.

In this study, the experimental data for the latter case were compared with the predicted data. The study was motivated by the need to find vibration parameters that might be useful for interpreting the data from microcantilever sensors. The obvious parameters are vibration amplitude and phase angle. Vibration models are needed to better understand and interpret experimental data such as vibration amplitude and phase angle.

II. THEORY

A. Natural vibration modes and frequencies of cantilevers

Mode shapes for a uniform cantilever beam can be found in most textbooks on mechanical vibrations.15,16 For example,

\[ X_r(x) = A_r \left[ \cosh \lambda_r x - \cos \lambda_r x - k_r (\sinh \lambda_r x - \sin \lambda_r x) \right], \]

where

\[ k_r = \frac{\cosh \lambda_r L + \cos \lambda_r L}{\sinh \lambda_r L + \sin \lambda_r L} \]

and \( A_r \) is a modal constant. The eigenvalues for the first four modes of vibration are

\[ \lambda_1 L = s_1 = 1.8751, \]
\[ \lambda_2 L = s_2 = 4.6941, \]
\[ \lambda_3 L = s_3 = 7.8548, \]
\[ \lambda_4 L = s_4 = 10.996, \]

where
FIG. 1. Mode shapes of the first four modes of vibration $A_i = 1$.

$\lambda_i^4 = \frac{\rho \omega_i^2}{EI}$

and

$\rho$ = mass per unit length,

$\omega_i$ = circular frequency of $i$th mode,

$E$ = modulus of elasticity,

$I$ = cross-sectional moment of inertia,

$L$ = length of cantilever.

The first four mode shapes are given in Fig. 1. For a microcantilever having properties as follows:

$L = 120 \ \mu m$ (length),

$b = 30 \ \mu m$ (width),

$h = 2 \ \mu m$ (thickness),

$\rho_o = 1000 \ \text{kg/m}^3$ (material density),

$E = 179 \ \text{GPa}$ (modulus of elasticity),

the natural frequencies of the first four modes are

$f_1 = 300.21 \ \text{kHz},$

$f_2 = 1881.37 \ \text{kHz},$

$f_3 = 267.93 \ \text{kHz},$

$f_4 = 10323.79 \ \text{kHz}.$

B. Base motion

In this analysis, it is assumed that the cantilever is being excited by base motion, as depicted in Fig. 2. Base motion is defined mathematically by

$z(t) = z_0 \cos \omega t.$

The absolute motion of any point on the cantilever is defined by

$y(x,t) = z(t) + w(x,t),$

where $w(x,t)$ is the motion relative to the moving base.

C. Differential equations for microcantilever motion

The differential equation of the motion\textsuperscript{16} for microcantilevers vibrating in air is

$EI \frac{d^4y}{dx^4} + c \frac{dy}{dt} + \rho \frac{d^2y}{dt^2} = 0,$

where the damping depends on the absolute motion $y(t)$ of the cantilever. Damping $c$ is assumed to be grounded as opposed to internal. Substituting Eq. (3) into Eq. (4) gives

$\rho \frac{d^2w}{dt^2} + c \frac{dw}{dt} + EI \frac{d^2w}{dx^2} = -c \ddot{z}(t) - \rho \ddot{z}(t).$

The response of microcantilevers to the base motion can be viewed as the superposition of the responses of each mode. The modal analysis explained below seeks to determine the response of each mode relative to the base motion. The total motion relative to the base is

$w(x,t) = \sum_{i=1}^{4} X_i(x) \eta_i(t),$

where $X_i(x)$ is a function of $x$ and $\eta_i(t)$ is a function of $t$. Only the first four modes will be considered here.

Substituting Eq. (6) into the differential equation of motion [Eq. (5)], multiplying each term by $X_j$, and integrating them gives\textsuperscript{17}

$\sum_{i=1}^{4} \left( \rho \int_{0}^{L} X_i X_j dx \ddot{\eta}_i + c \int_{0}^{L} X_i X_j dx \dot{\eta}_i + EI \int_{0}^{L} X''_i X_j dx \eta_j \right) = -\rho \dddot{z} \int_{0}^{L} X_j dx - c \dddot{z} \int_{0}^{L} X_j dx.$

Because the modes are orthogonal, Eq. (7) becomes

$M_j \ddot{\eta}_j + C_j \dot{\eta}_j + K_j \eta_j = Q_j(t),$

where

$M_j = \rho \int_{0}^{L} X_j^2 dx$ (modal mass),

$C_j = c \int_{0}^{L} X_j^2 dx$ (modal damping coefficient),

$K_j = EI \int_{0}^{L} X''_j X_j dx$ (modal stiffness),
\[ Q_i = (-p\dddot{x} - c\dot{x}) \int_0^L X_i dx \] (modal force).

Given that
\[ \int_0^L X_i'^2 dx = A_i^2 L \] (9)
and
\[ \int_0^L X_i'''' dx = X_i^2 \lambda_i^4, \] (10)
it follows that
\[ M_i = \rho A_i^2 L, \]
\[ C_i = c A_i^2 L, \]
\[ K_i = EI A_i^4 L = EI \left( \frac{\lambda_i}{L} \right)^4 A_i^2 L, \]
and
\[ q_i = \int_0^L X_i dx = A_i L p_i, \] (11)
where
\[ p_1 = 0.7930, \]
\[ p_2 = 0.4238, \]
\[ p_3 = 0.2645, \]
\[ p_4 = 0.1716. \]
The time driver for each mode is found in the solution to
\[ A_i^2 L (\rho \ddot{\eta}_i + c \dot{\eta}_i + EI A_i^4 \eta_i) = (-p\dddot{x} - c\dot{x})q_i. \] (12)
Because the \( A_i \)'s are arbitrary and are canceled out when the \( \eta_i \)'s are substituted into Eq. (6), we set them equal to 1 at this point so that
\[ \ddot{\eta}_i + 2\zeta i \omega \dot{\eta}_i + \omega^2 \eta_i = (-\dddot{z} - 2\zeta i \omega \dot{z})p_i. \] (13)
When converted to the complex form,
\[ z(t) = z_0 e^{i\omega t}, \]
\[ \ddot{z}(t) = z_0 i \omega^2 e^{i\omega t}, \]
\[ \dddot{z}(t) = -z_0 \omega^3 e^{i\omega t}. \]
Substitution into the differential equation of motion [Eq. (13)] gives
\[ \ddot{\eta}_i + 2\zeta i \omega \dot{\eta}_i + \omega^2 \eta_i = z_0 e^{i\omega t} (\omega^2 - i2\zeta i \omega \omega)p_i. \] (14)
For the solution,
\[ \eta_i = \bar{\eta}_i e^{i\omega t}. \] (15)
which yields
\[ \bar{H}_i = z_0 \frac{r_i^2}{2 + \frac{r_i^2}{(1 - r_i^2)} + i2\zeta \omega r_i}. \] (16)
Also,
\[ \eta_i = H_i e^{i\phi_i} e^{i\omega t}, \] (17)
where
\[ \bar{H}_i = \frac{[r_i^4 + (2\zeta \omega r_i)^2]^{1/2} p_i}{[(1 - r_i^2)^2 + (2\zeta \omega r_i)^2]^{1/2}} \] (18)
and
\[ \tan \phi_i = -\frac{2\zeta \omega r_i}{r_i^2(1 - r_i^2) - (2\zeta \omega r_i)^2}. \] (19)

The modal response relative to the moving base can now be determined from Eq. (6). Because the reach of each modal response is small compared to the frequency spacing of each resonant frequency and because frequency responses of interest are usually around modal resonance, we can focus on each modal response separately. The advantage here is that a complex modal summation can be reduced to only one mode (i.e., the mode that dominates the total response). To illustrate this point we calculated the contribution of each of the first four modes to the total tip displacement at frequency ratios equal to 1 (see Table I).

The driving frequencies correspond to the natural frequencies of the first four modes. For example, if the driving frequency is 1881.37 kHz, the contribution of the second mode to the amplitude ratio is 738.32, while the contributions of the first, third, and fourth modes are 0.627, 1.077, and 1.012, respectively. Obviously the first, third, and fourth modes are insignificant participants in the total motion and can be ignored.

Recognizing this allows one to focus only on the one mode that is being excited by the frequency of base motion. Focusing on one mode is especially useful in analyzing experimental data, as will be shown later. Based on this observation, Eq. (6) can be reduced to
\[ w(x, t) = X_i(x) \eta_i(t), \] (20)
\[ w(x, t) = H_i X_i(x)\cos(\omega t + \phi_i), \] (21)
where it is understood that the \( i \)th mode dominates the total motion of the cantilever, as illustrated in Table I.

In this case, the motion of each point in the mode has the same phase lag \( \phi_i \) with respect to the base motion, and \( \phi_i \) will be a negative number, as indicated in Eq. (19).

<table>
<thead>
<tr>
<th>Driving frequency (kHz)</th>
<th>First mode</th>
<th>Second mode</th>
<th>Third mode</th>
<th>Fourth mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>300.21</td>
<td>220.28</td>
<td>1.022</td>
<td>1.002</td>
<td>1.0003</td>
</tr>
<tr>
<td>1881.37</td>
<td>0.627</td>
<td>738.32</td>
<td>1.077</td>
<td>1.012</td>
</tr>
<tr>
<td>5267.93</td>
<td>0.591</td>
<td>0.028</td>
<td>1290.24</td>
<td>1.186</td>
</tr>
<tr>
<td>10323.79</td>
<td>0.587</td>
<td>0.123</td>
<td>0.285</td>
<td>1634.29</td>
</tr>
</tbody>
</table>
D. Vibration response in terms of modal summing

In the previous sections, we discussed the modal components of cantilever vibrations relative to the base motion. Let us now consider the absolute vibration, where motion as viewed from a ground or a fixed reference frame is

\[ y(x, t) = z(t) + w(x, t), \]

\[ y(x, t) = z(t) + X_i(x) \eta(t). \]

The summation is omitted because we consider only the dominant mode.

Eq. (23) can be explained in the complex form as follows:

\[ \tilde{y}(x, t) = z_0 e^{i\omega t} + X_i(x) \tilde{H} e^{i\omega t}, \]

\[ \tilde{y} = \tilde{Y}_i e^{i\omega t}, \]

\[ \tilde{Y}_i = z_0 (1 + i0) + \tilde{H}_i X_i(x). \]

See Eq. (16) for \( \tilde{H}_i \). From Eq. (26) we obtain

\[ \frac{\tilde{Y}_i}{z_0} = \frac{[1 - r_i^2] + p_i X_i r_i^2 + i2 \xi_i r_i (1 - p_i X_i)}{(1 - r_i^2) + i2 \xi_i r_i}. \]

The amplitude ratio of any given point on the cantilever is determined from

\[ \frac{Y_i}{z_0} = \frac{\tilde{Y}_i}{z_0}. \]

Frequency responses, as determined from Eq. (28) for the first four modes, are shown in Fig. 3. In each case the point for which the amplitude was calculated was \( x/L = 1 \). The damping factor for each mode is explained in the next section.

For environmental damping, the damping factors are reduced with higher-order modes and can be explained as follows:

\[ \zeta_i = C_i/(C_i)_{ct} = C_i/2M_i \omega_i, \]

\[ \zeta_i = c/2\rho \omega_i, \]

from which

\[ \zeta/\zeta_1 = \omega_1/\omega_i = (s_i/s_1)^2. \]

Therefore, the amplitude ratios for the higher modes are larger than those for the lower modes (Fig. 3).

III. RESULTS AND DISCUSSION

A. Damping factors as determined experimentally

Modal damping factors are needed to determine modal frequency responses and phase angles. These factors are best determined experimentally. They are determined in this study from the frequency response data shown in Fig. 4. Applying Eq. (28) to the data, where \( i \) is taken as “1” (first mode), we determine the damping factor for the first mode to be \( \zeta_1 = 0.0032 \) (air).

Rectangular silicon cantilevers with a length of 350 \( \mu \)m and a four-quadrant AFM head with an integrated laser and a position-sensitive detector were used to measure the frequency responses of cantilevers in air and helium environments. The frequency of a thermally vibrating cantilever was averaged by a spectrum analyzer and was fitted with a Lorenzian curve.

The damping factors for higher modes vary inversely with the natural frequency of the mode \( \omega_i : \omega_1 \) [see Eq. (29)]. For the cantilever specified at the beginning of the paper, and assuming \( \zeta_1 = 0.0036 \) (from Fig. 4),

\[ \zeta_1 = 0.0036, \]

\[ \zeta_2 = 0.1596 \times 0.0036 \approx 0.000574, \]

\[ \zeta_3 = 0.0569 \times 0.0036 \approx 0.000205, \]

\[ \zeta_4 = 0.029 \times 0.0036 \approx 0.000105. \]

These factors were used in generating the frequency responses in Fig. 3. The frequency response of the first mode in Fig. 3 agrees with that in Fig. 4.
FIG. 5. Frequency response of the first mode ($\zeta_1=0.0036$).

The phase angle for the relative motion is constant over the length of the cantilever [see Eq. (21)], but for the absolute motion, the phase angle varies over the cantilever length, as shown below.

B. Measurements of phase angle along a microcantilever

The preceding theory was used to predict phase angles along the microcantilever used in the experiments. Since phase angle measurements were taken near the natural frequency of the first mode, only the phase angles near the first mode are considered below. The predicted response of the first mode is defined by

$$\tilde{y}(x,t) = z_0 e^{iat} + X_1(x)\tilde{H}_1 e^{iat},$$

$$\tilde{y} = \tilde{Y}_1 e^{iat},$$

$$\tilde{Y}_1 = z_0 (1 + i0) + \tilde{H}_1 X(x).$$

From Eq. (32) we obtain

$$\tilde{Y}_1 = \frac{[(1 - r_1^2) + p_1 X_1 r_1^2]}{(1 - r_1^2) + i2\zeta_1 r_1}.$$  

(33)

The silicon cantilevers used in these tests were rectangular and had a length of 350 $\mu$m, a width of 35 $\mu$m, and a thickness of 1 $\mu$m. The spring constant 0.03 N/m was given by the supplier. The cantilever was forced to vibrate by a piezoelectric stage, and its optical deflection was measured by a four-quadrant AFM head with an integrated laser and a position-sensitive detector (Digital Instruments, Santa Barbara, CA). The phase difference between the applied voltage and the measured vibration was obtained by a lock-in amplifier (Stanford Research Systems, SR830).

The calculated fundamental frequency was 17.64 kHz. However, the fundamental frequency was measured at 13 kHz. The amplitude ratios for the first mode (Fig. 5) were produced for a range of frequencies near 13 kHz [Eq. (33)]. The damping factor for the first mode in this case was assumed to be $\zeta_1=0.0036$.

FIG. 6. Phase angle vs frequency ratio for the first mode ($s/L=1$).

The corresponding phase angle versus the frequency ratio is shown in Fig. 6, but it does not predict the experimental data well. Since the cantilever is mounted on a quartz cell and held in place with a spring, the damping of the system is the sum of the environmental damping around the cantilever and the friction within the holder [Eq. (33)]. The phase angle distribution along a microcantilever for the first mode is found in

$$\tan \theta_i = \frac{\text{Im}(x)}{\text{Re}(x)},$$

(34)

where

$$\text{Im}(x) = -2\zeta_1 r_1^2 [p_1 X_1(x)],$$

$$\text{Re}(x) = (1 - r_1^2)^2 + 2\zeta_1 r_1 [r_1^2 (1 - r_1^2) - (2\zeta_1 r_1)^2].$$

Of interest is the variation of the phase angle $\theta_i$ with respect to the ratio $x/L$, given that

$$X_1(x) = \cos \lambda_1 L \left(\frac{x}{L}\right) - \sin \lambda_1 L \left(\frac{x}{L}\right)$$

and $\lambda_1 L = 1.8751$. Figure 7 displays how the phase angle $\theta_i$ varies over the cantilever length for the first mode [obtained from Eq. (34)]. The predicted phase angle distribution is sensitive to changes in the frequency ratio.

Experimental measurements of phase angles along the microcantilever are superimposed in Fig. 7 (round dots). Changes in optical deflection caused by cantilever vibrations were measured by a four-quadrant AFM head with an integrated laser and a position-sensitive detector. The phase difference between the applied potential and the measured vibration was obtained by a lock-in amplifier.

The experimental data show that the greatest change in phase angle comes over the first 25% of the cantilever length. This finding is consistent with theoretical predictions (see Fig. 7). Other tests produced the data plotted in Fig. 6 as...
phase angle versus base frequency. These data (solid squares) show a large divergence from the $\zeta_1 = 0.0036$ curve in Fig. 6. The measured phase angle data are more consistent with a damping factor of $\zeta_1 = 0.036$.

IV. CONCLUSIONS

The frequency response of each mode has a narrow bandwidth with little reach. Therefore, only one mode needs to be considered when driving cantilevers with a simple harmonic base motion near resonance. However, for other types of driving mechanisms, all modes may participate simultaneously.

- Total motion can be viewed as the base motion plus the sum of all modes relative to the base reference frame. Each mode maintains its identity relative to the base reference frame.
- The theory shows that for environmental damping, amplitude ratios increase with the order of the modes.
- The phase angle between any vibration mode (relative to the moving base frame) does not vary along a cantilever.

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