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Integrable difference equation and quasilinear partial differential equation

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Integrable Volterra difference equation and non-viscous Burgers equation are investigated from the viewpoint of integrable difference equation and quasilinear partial differential equation, respectively. They have the relation that the former difference equation is discretized of the latter differential equation and inversely the differential equation is recognizes as the original equation. But behaviors of their solutions are quite different each other. A difference method is proposed for the solution of the quasilinear partial differential equation.

Since Newton and Leibnitz found differential methods, dynamical systems have been described in the forms of differential equations. But only few cases have analytical solutions in these differential equations. Many of them could not be exactly solved, but recently it becomes easy that we can solve these equations numerically thanks to development of computer systems. There are many problems, however, when differential equations are treated as difference equations. On the other hand natural and social phenomena in the various fields have been accounted in discrete time-spaces, and informations in our systems are treated often in digital form. So we need new discrete methods, just now the new difference method is proposed as the calculus in the information age by Hirota [1]. But the infinitesimal calculus: the differential and integral equations are very useful and profound with the long history. We must deep
understand the connections between the infinitesimal and the discrete calculus. There are many methods to discretize differential equations and it is the problem which difference equation is a good approximation. Some results of discretization show chaotic behaviors against the smoothing solution of the original differential equation. In this article we consider the integrability and the connections between differential and difference equations; in the cases of the ordinal differential logistic equation and non-viscous Burgers equation of the quasilinear partial differential equation.

As the typical example to show smoothing solution or chaos depending on the discretizing method, logistic dynamical system is well investigated [2, 3]: the equation

\[
\frac{dN}{dt} =aN(1-bN). \tag{1}
\]

has been discretized in several difference equations, for example
\[
N^{n+1} - N^n = aN^n(1-bN^{n+1}), \tag{2}
\]
\[
N^{n+1} - N^n = aN^{n+1}(1-bN^n), \tag{3}
\]
\[
N^{n+1} - N^n = aN^n(1-bN^n). \tag{4}
\]

The first two types show exact solution of the logistic differential equation, but the last one show chaotic for \(a > 3.57\), the former is sometimes called integral difference analogue and the latter is called non-integral [4].

The concept and property of the integrability may not always be unique. In general it is called to be integrable that there are \(n\) conserved quantities in \(n\)-dimensional systems (Liouville theorem) or equations satisfy the zero curvature representation [5], which is equivalent to the former in the sense that the latter have infinitely many integrals of the motion. Now let us consider the latter representation which is generalized the Lax equation. Historically the equation was derived from the inverse scattering method to solve the Korteweg-de Vries (KdV) equation [6]

\[
\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{5}
\]

which is equivalent to the operator equation

\[
\frac{\partial L}{\partial t} = [L, A] = LA - AL, \tag{6}
\]

where
\[ L = -\frac{\partial^2 u}{\partial x^2} + u(x,t), \tag{7} \]
\[ A = 4 \frac{\partial^3 u}{\partial x^3} - 3(u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} u), \tag{8} \]

Equation (6) is known as the Lax equation, and the pair of operators L and A is called a Lax pair. This Lax equation is extended to the following condition. Let us consider two linear systems of the form

\[ \frac{\partial F}{\partial t} = L(x,t,\lambda) F, \tag{9} \]
\[ \frac{\partial F}{\partial x} = M(x,t,\lambda) F, \tag{10} \]

Where \( L(x,t,\lambda) \) and \( M(x,t,\lambda) \) are \( N \times N \) matrix-valued functions of \( t \) and \( x \) with the spectral parameter \( \lambda \). The compatibility condition \( \left( \frac{\partial^2 F}{\partial x \partial t} = \frac{\partial^2 F}{\partial t \partial x} \right) \) for the systems is of the form

\[ \frac{\partial M}{\partial t} - \frac{\partial L}{\partial x} + [M,L] = 0, \tag{11} \]

this is called the zero curvature representation \([5]\). This definition of the integrability need be used to the partial differential equation with plural variables because of the meaning of the compatibility.

The connection coefficients \( L(x,t,\lambda), M(x,t,\lambda) \) are affected by the transformation

\[ F(x,t,\lambda) \longrightarrow G(x,t,\lambda) F(x,t,\lambda), \tag{12} \]

their transformations are

\[ L \longrightarrow \frac{\partial G}{\partial t} G^{-1} + G L G^{-1}, \tag{13} \]
\[ M \longrightarrow \frac{\partial G}{\partial x} G^{-1} + G M G^{-1}. \tag{14} \]

Such transformations are called gauge transformations, so that the zero curvature representation is invariant under gauge transformations \([5]\).

Let us see many constants of motion deduced from the Lax form of the Toda lattice model \([7]\). The equations of motion of the Toda lattice are
\[
\frac{d^2 q_n}{dt^2} = \exp(q_{n-1} - q_n) - \exp(q_n - q_{n+1}), \quad (15)
\]

where \( q_n \) and \( p_n \) are coordinate and momentum of \( n \)-th particle in the \( N \) particles. The Hamiltonian on the periodic boundary conditions is

\[
H = \sum_{n=1}^{N} \frac{1}{2} p_n^2 + \sum_{n=1}^{N-1} \exp(q_n - q_{n+1}) \quad (16)
\]

and the Poisson structure are

\[
\{ p_n, q_m \} = \delta_{nm}, \quad \{ p_n, q_m \} = \{ p_n, q_m \} = 0. \quad (17)
\]

where \( \{ , \} \) mean the Poisson bracket. New variables \( a_n \) and \( b_n \) are given by

\[
a_n = \frac{1}{2} \exp\left(\frac{q_n - q_{n+1}}{2}\right), \quad b_n = \frac{1}{2} p_n. \quad (18)
\]

The equation of motion is transformed into

\[
\frac{\partial L}{\partial t} = [L, A] = LA - AL, \quad (19)
\]

where

\[
L = \begin{pmatrix}
b_1 & a_1 & \cdots & a_N \\
a_1 & b_2 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
a_N & a_{N-1} & \cdots & b_N
\end{pmatrix}, \quad A = \begin{pmatrix}
0 & a_1 & \cdots & -a_N \\
-a_1 & 0 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
a_N & a_{N-1} & \cdots & 0
\end{pmatrix}. \quad (20)
\]

It is clearly seen in this form that time derivative of \( L^k \) is

\[
\frac{d}{dt} L^k = [L^k, A] \quad (k=1,2,\ldots,N). \quad (21)
\]

Taking trace of the both sides of this equation

\[
\frac{d}{dt} \text{tr} L^k = \text{tr} [L^k, A] = 0, \quad (22)
\]

then

\[
\text{tr} L^k = \text{const} \quad (k=1,2,\ldots,N). \quad (23)
\]

Therefore the Toda lattice has as many constants of motion as number of particles,
and it is called completely integrable [8].

Many integrable models are related to this Toda model, the Volterra differential-difference equation

\[
\frac{df_n}{dt} = f_n(f_{n+1} - f_{n-1})
\]

(24)

is one of these integrable models, which is satisfied with the semi-discrete version of zero curvature representation

\[
\frac{dT_n}{dt} + T_n L_n - L_{n+1} T_n = 0
\]

(25)

where

\[
T_n(\lambda) = \begin{pmatrix}
\lambda & f_n \\
-1 & 0
\end{pmatrix},
\]

(26)

\[
L_n(\lambda) = \begin{pmatrix}
f_n & \lambda f_n \\
-\lambda & -\lambda^2 + f_{n-1}
\end{pmatrix}
\]

(27)

We have investigated this equation with some different discretizations in the previous papers [12].

\[
f_n^{t+1} - f_n^t = \delta (f_n^t f_{n+1}^{t+1} - f_n^{t+1} f_{n+1}^t)
\]

(28)

\[
f_n^{t+1} - f_n^t = \delta (f_n^t f_{n+1}^{t+1} - f_n^{t+1} f_{n+1}^t)
\]

(29)

If the difference of functions for spatial variable n in the right hand side of eq. (24) is regarded as the central difference of differential equation, whose interval is 1, the original partial differential equation is

\[
\frac{\partial f}{\partial t} = f \frac{\partial f}{\partial x}.
\]

(30)

This is the non-viscous Burgers equation [10] or is regarded as the dispersionless KdV equation and is sometimes called Hopf equation [11]. If this partial differential equation is regarded as the original differential one of the integrable Volterra model, what difference is good approximation? But behaviors of their solutions are quite different each other. The eq.(30) has a shock wave solution, it is a problem whether such solution can be made by the difference equation.

For the ordinary differential equations it is expected to have the integrable differ-
ence equation as a good approximation, and the logistic equation is playing an important role in the problem: depending on discretizations, solutions of the difference equation show the exact solution of the differential equation or chaotic behavior. For the partial differential equation the situations are quite different from the ordinary one, they are more complex and difficult. But we want to have examples expected to play a similar role as logistic equation. The nonlinear differential equations are too complicated and their discretizations are too various, but the equation (30) is a simple quasi-linear equation and is a common part of the above-mentioned non-linear partial differential equations. So the non-viscous Burgers equation is a good candidate, and as the above case integrability of differential and difference equations are not simple.

Now we take a brief summary about the partial differential equations and the discretization of differential operator. The partial differential equations are classified 2 or 3 types: linear, quasilinear and nonlinear equations [9]. The former two types are distinguished from the third one in the sense that the highest order of the differential operator is linear. As against this classification the latter two types are often unified to nonlinear, but the quasilinear and the nonlinear equations need be distinguished because of different way to solve. It is noted that the above-referred inverse scattering method is the usefull way to solve nonlinear differential equations.

The initial value problem of partial differential equations in the normal form has unique solution in the neighbourhood of the origin, guaranteed by the Cauchy-Kovalevskaya existence theorem. The one order quasilinear partial differential equation, which is sometimes called Lagrange equation, is solved like the linear and has the integral hypersurface with the characteristic curve. In the case of eq. (30) the characteristic curve on the point \( x_0 \) is

\[
x - x_0 = f_0(x_0)t,
\]

where \( f_0 \) is the initial condition. The solution \( f(x,t) \) in the neighbourhood of the initial time is formally given by

\[
f(x_0 + f_0(x_0)t,t) = f_0(x).
\]

The informations of the function about the initial values are translated along the characteristic curve [13]. Because all of the curves are not always pararel, as known
in eq. (31), they are crossed at any time-points where the solution is not unique, but a shock wave solution for example.

Many of the differential equations can not be solved analytically, but can be solved numerically. For the numerical calculation the difference method is used. There are many types to discretize differential operator, the typical 3 types and q-difference as following:

advanced or forward difference operator

$$\Delta_{+}\varepsilon f(x) = \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

(33)

backward or retarded difference operator

$$\Delta_{-}\varepsilon f(x) = \frac{f(x) - f(x-\varepsilon)}{\varepsilon}$$

(34)

central difference operator

$$\Delta_0\varepsilon f(x) = \frac{f(x+\varepsilon/2) - f(x-\varepsilon/2)}{\varepsilon}$$

(35)

q-difference

$$\delta_n\varepsilon f(x) = \frac{f(x) - f(qx)}{x - qx}$$

(36)

where $\varepsilon$ is a interval of difference and $q$ is a constant. The others are combinations of these types and non-equidistant difference (Difference quotient), and q-difference method is notices. The interesting new method with new symbols is proposed by Hirota recently, in which Difference quotient is introduced and it is emphasized that non-equidistant difference of space is usefulness [1]. Let us go back to the problem whether shock wave solutions can be made by the difference equation. To answer the question affirmative we propose the difference method that non-equidistant difference of time is added to Difference quotient. The intervals of the difference method change not only in the space but also in the time, in a sense discrete version of the local gauge transformation. And we propose the difference method along the characteristic curve to solve quasilinear differential equation. They are not so strange, there are examples that define the covariant difference on the hypersurface [14].
References