COLLECTIVE IDENTITY DETERMINATION AS AN AGGREGATION

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MASTER OF SCIENCE IN ECONOMICS

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2006

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BİR AGREGASYON YÖNTEMİ OLARAK
TOPLUMSAL KİMLİK BELİRLEME

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Abstract

A Collective Identity Function (CIF) is a rule which aggregates personal opinions on whether an individual belongs to a certain identity into a social decision. A CIF is qualified as “elementary” whenever it can be expressed in terms of winning coalitions. Elementary CIFs can be characterized with independence axiom. We then investigate the effect of imposing new axioms on the structure of winning coalitions. We further characterize the class of simple CIFs in terms of three axioms, namely independence, monotonicity and self-duality. We also explore the effect of imposing conditions that ensure the equal treatment of individuals as voters or as outcomes. We show that liberalism arises as the unique simple CIF that satisfies axioms which are very natural in the collective identity determination context.

Keywords: Collective identity function, Winnign coalitions, Liberalism.

Özetçe

Toplumsal Kimlik Fonksiyonu (TKF), her bireyin belirli bir kimliğe ait olup olmadığı hakkındaki kişisel görüşlerini toplumsal bir görüşe dönüştüren bir kuraldır. Kazanan koalisyonlar cinsinden ifade edilebilen TKF’ler "temel" olarak nitelendirilmiştir. Temel TKF’ler bağımsızlık (independence) aksiyomu ile karakterize edilebilirler. Daha sonra yeni aksiyomların eklenmesinin kazanan koalisyonların yapıları üzerine etkisi incelenmiştir. Aynı zamanda "sade" TKF’ler grubu bağımsızlık, monotonluk ve kendiliğinden ikilik (self-duality) axiomlarıyla karakterize edilmiştir. Ayrıca, oy verenler veya oy verilenler olarak bireylerin eşit muamele görmelerini temin eden şartların eklenmesinin etkileri incelenmiştir. Liberalizmin sade TKF’ler içerisinde toplumsal kimlik belirleme bağlamında çok doğal olan aksiyonları sağlayan tek kural olduğu gösterilmiştir.

Anahtar Kelimeler: Toplumsal kimlik fonksiyonu, Kazanan koalisyonlar, Liberalizm.

M.M.O.

Acknowledgments

Among so many people I owe to thank, I first start with Remzi Sanver and Göksel Aşan, not only for their support to this thesis, but also their contributions for my academic career. Remzi always supported my thesis with his discussions and comments regardless of time, even midnights. He is a friend and colleague for me. Göksel Aşan has always a special place in my academic progress from an undergraduate student to teaching assistant and a graduate student.

I would also thank to William Zwicker for his helpfull and clever comments and for his interest on my thesis. It is a great pleasure for me to meet William and his wife Catherine.

I presented this work on many conferences and I thank Semih Koray for his critiques, Fuad Aleskerov for his comments and Emre Doğan for his supports and corrections.

I am thankful to Bora Erdamar and Osman Eler; especially to Osman, who is a lawyer and my house mate, for his patience for reading the text and listening the theorems that are meaningless for him. I also want to thank to Turgut Yüksel from graphic design department of Bilgi University for teaching how I can draw the figure in the text.

Contents

1 Introduction

Each individual has an identity in his social life. These identities may vary such as being a member of a club, member of a family, citizen of a country, supporter of a political party, believer of a religion and this list can be expanded. Each person has an opinion (idea, belief) about whether he is a member of an identity or not. It is possible that one's opinion about himself and the social perception (which can be expressed as social opinion or collective identity) about that individual may differ. Hence a question how can we determine an individual's social identity naturally arises and this is the question that we deal with in this work i.e. finding a method of determining identities. Of course when answering this question, the name of identity matters. It can be argued that some identities have some set of strict rules to differentiate whether one carry the identity or not. For example, being a member of a university as a student or as a staff can be an example of such identities. You can look the register of the university and find all who are members of a university.[1] As an other example, consider a club for solidarity of families with children in a specific neighbourhood. One may search a set of rules to separate the members and non-members. The following rule can be applied to determine who are the ones eligible to be a member of the club: the prospective families are welcomed to club if they reside in the given neighbourhood and have at least one child, otherwise they are not allowed to join the club.

However, not all identities fall in this category and the question "who are the members" can not be resolved by applying a strict rule. For example, consider that a group of individuals has to determine a set of representatives who sign a contract or an aggrement that impose a responsibility to all members of the group. In such a case, assuming each individual has an opinion about all individuals (including himself) whether one can be a delegate or not is not so unrealistic. Hence it can be proposed that personal opinions can be aggregated to find a social opinion. Thus one's social decision may depend on all individual's decision about him and there are various ways of aggregating this individual opinions into a social opinion.

Though there may be other suggestions for identity determination problem as well as there may be proponents of the first way stated above, in this study we follow the latter approach and treat the identity determination as an aggregation problem from

[1] Of course we exclude the situations like "As far as, I am legally a student of the University of ..., but I do not feel myself as a student"

individuals opinions to social opinions since aggregation is a commonly discussed and analyzed topic in economics and social choice theory. Moreover, we should say this is not the first attempt known in the literature. The first attempt to analyze the collective identity determination problem through concepts of social choice theory is made by Kasher and Rubinstein (1997) who, based on an exploration of Kasher (1993) about the Jewish identity, propose a method of aggregating personal opinions into social opinion of the identity: Who are the Jews?. Kasher and Rubinstein consider a society and some abstract concept of identity (such as "being a J") to which every member of the society may or may not belong. Each individual has a personal opinion about whom does and whom does not belong to this identity. The collective decision is made by the aggregation of individual opinions - hence the introduction of a *collective identity function* (CIF), which maps individual opinions into a social opinion. The model, while mathematically simple, incorporates a plethora of concepts related to collective identity determination. So, leaving the modesty of its founders aside,[2] it paved the way to a growing literature, the pivots of which will be mentioned in the Section 3 as a start for our analysis. Before summarizing previous results, we present the formal model in Section 2.

Among various aggregation functions that can be defined, the liberal rule appears as a central concept. Under the liberal CIF, an individual is socially conceived as belonging to some identity J if and only if he believes "to carry identity J" -or "to be a J", so to speak. A first axiomatic characterization of liberalism is given by Kasher and Rubinstein (1997).[3] Another strand of the literature views CIFs as a recursive procedure which is also proposed by Kasher (1993). For example, the *procedural CIF* of Kasher (1993) suggests to determine an initial set $J(0)$ of individuals who are unanimously agreed to carry identity J. All individuals who are considered to be a J by at least one member of $J(0)$ are added to $J(0)$, hence expanding the set of Js to $J(1)$. The procedure continues inductively until the set of Js cannot be expanded anymore. A variant of this procedure, where the initial set $J(0)$ consists of individuals who consider themselves as Js, is defined by Dimitrov, Sung, and Xu (2004) who characterize both procedures.[4] More recently, Samet and Schmeidler

[2]Kasher and Rubinstein (1997) present it as a "purely logical exercise"

[3]while Dimitrov and Sung (2003) show that the five axioms used by Kasher and Rubinstein (1997) are logically dependent whereas three of them suffice to establish the desired equivalence. See Section 3.1.

[4]See section 3.2.

(2003) axiomatically characterize a class of CIFs which they call *consent rules*.[5] This class is parametrized by the weights given to individuals in determining their own identity. It contains liberalism at one extreme and majoritarianism[6] at the other. As we discuss in the section 4, the procedural view of CIFs is almost orthogonal to Samet and Schmeidler's conception of consent rules which lie between liberalism and majoritarianism. We devote the section 3 to the results obtained by authors given above.

We propose to approach the collective identity determination problem from a perspective where CIFs can be expressed in terms of *winning coalitions*.[7] In section 4.1, we start by observing under previously used condition of independence which states that one's social decision depends people's opinion only about that individual, we can express the behavior of CIF in terms of winning coalitions. We qualify such CIFs as *elementary*. Under an elementary CIF, the information about the social opinion contained in the set of winning coalitions is the same as that in the corresponding aggregation rule. In other words, elementary CIFs can be examined through their winning coalitions, which brings us a new perspective in the exploration of the collective identity determination problem. Then we investigate the structure of winning coalitions as we introduce new conditions. We add a monotonicity condition stating additional opinions about an individual which are same with social opinion about that individual can not change social opinion of the individual. Then we call independent and monotonic rules as *basic CIFs*. We then introduce blocking coalition, a coalition that can determine an individual's social opinion as non-member by disqualifying him on the contrary of other's qualification. We then investigate the whether a coalition can be winning and/or blocking through three version of self duality axioms. Finally, we characterize *simple* CIFs[8] in terms of independence, monotonicity and self-duality. One can refer Taylor and Zwicker (1999) for details of winning-blocking coalitions and simple games since we follow their terminology in this work. We devote section 4.2 to equal treatment properties for voters and alternatives and offer an alternative characterization for liberal rule in section 4.3. Then we compare our findings with previous

[5] See section 3.3.

[6] Where, as also exemplified by Kasher and Rubinstein (1997), personal opinions about the identity of an individual are aggregated according to the majority rule.

[7] As usual, we say that a coalition K of individuals is winning for individual i if and only if the members of K, on the contrary of the opinions of the rest of the society, are able to determine whether i carries identity J.

[8] A detailed discussion of simple social choice rules in a general social choice setting can be found in Austen-Smith and Banks (1999) in addition to Taylor and Zwicker (1999).

results. We finally give conslusion in section 5.

2 Model

We consider a society represented with $N = \{1, ..., n\}$ which is a finite set of individuals with $n \geq 2$. The society is confronted to the problem of deciding on its members who belong to some "group" or who carry a certain given identity. For each $i \in N$, we write $G_i \subseteq N$ for the set of individuals whom i perceives as a member of the group. We refer to G_i as the *opinion* of i. $j \in G_i$ is interpreted as individual i believes that individual j carry the identity or in other words individual j is qualified by individual i. Thus for individual i, the set G_i represents the set of individuals that i believes they carry the identity. An *opinion profile* is an n-tuple $G = (G_1, ..., G_n) \in \Gamma$ where $\Gamma = (2^N)^n$ is the set of all profiles. It is sometimes referred simply as "profile" in the remaining of this study. A *Collective Identity Function* (CIF) is a mapping $F : \Gamma \to 2^N$ that assigns a subset of individuals to each profile. For any profile, we call $F(G)$ as social opinion which is also a subset of society. $i \in F(G)$ is interpreted as individual i is socially qualified as a member of identity. Let $\mathcal{F}$ represents the set of all CIFs.

For any profile $G \in \Gamma$, we define $\overline{G} \in \Gamma$ as $\overline{G}_i = N \setminus G_i$ for all $i \in N$. In same manner, for any social opinion $F(G)$, we write $\overline{F(G)}$ as the complement of $F(G)$ i.e. $\overline{F(G)} = N \setminus F(G)$. If $G \in \Gamma$ is the collection of personal opinions that reflects members of an identity, $\overline{G} \in \Gamma$ defined as above can be interpreted as the opinions where each individual express non-members as his opinion.

For each personal opinion of j, $G_j|_i$ represents the opinion of j only about i. So we write $G_j|_i = \varnothing$ if $i \notin G_j$ and $G_j|_i = \{i\}$ if $i \in G_j$. Therefore, for any profile $G \in \Gamma$ and any $i \in N$, the opinions restricted to individual i is represented by $G|_i$ which is an element of n-tuple $(\varnothing, \{i\})^n$, that is $G|_i \in (\varnothing, \{i\})^n$. In same manner, $F(G)|_i$ represents the social opinion of individual i and can be either $\varnothing$ or $\{i\}$, that is $F(G)|_i \in \{\varnothing, \{i\}\}$. With the help of restricting opinions to individuals, we can write for any $G, H \in \Gamma$, $G|_i = H|_i$ if and only if $i \in G_j \iff i \in H_j$ for all $j \in N$, that is all people in society has the same personal opinion about i in the profiles G, H while they may possibly differ in opinions about individuals other than i.

Given a bijection $\Pi : N \to N$, we write, by a slight abuse of notation, $\Pi(K) = \{\Pi(j) : j \in K\}$ for any non-empty $K \subseteq N$. By a more considerable abuse of notation, for any $G \in \Gamma$, we mean by $\Pi(G)$ a new profile H such that $H_{\Pi(j)} = \Pi(G_j)$ for each $j \in N$. The bijection can be interpreted as changing the names of individuals.

For example, $\Pi(i)$ can be interpreted as the new name of individual whose old name is i. Applying the permutation Π to a set of individuals give the new names of all in the set whereas applying Π to a profile (G) gives a new profile (H) where each individual (j) expresses his opinions with his new name $(\Pi(j))$ as the set of new names of individuals $(H_{\Pi(j)} = \Pi(G_j))$ he previously qualifies (G_j). In other words, under old names if i qualifies j, then $\Pi(i)$ qualifies $\Pi(j)$ under new names.

Whereas the problem is treated as an aggregation problem, our model differs from other well-known aggregation models such as Arrow's Social Welfare Functions. We will charactare a class of Collective Identity Functions which we call them "simple" including the liberal rule as well as majoritarion rule. Both rules are discussed in the literature in different contexts. For example, May (1952) characterizes simple majority rule where a finite set of individuals confronts two alternatives, usually interpreted as yes/no voting.[9] Later on, Arrow (1951) introduces a social welfare function (SWF) which aggregates individual's preferences into a transitive and complete social preferences. In Arrow's model, there are finite number of alternatives which has at least three cardinality, hence an expansion of May's model and a finite number of individuals express their preferences over alternatives. In his pioneering work, he showed the impossibility of finding a non-dictatorial aggregation rule which satisfies Pareto optimality and independence of irrelevant alternatives under full domain[10] and transitive social outcomes. Sen (1970) extends Arrow's result into impossibility of Paretian Liberal Social Welfare Functions satisfying the liberal principle introduced by Sen. He calls an individual decisive on two alternatives if the function orders these two alternatives in the same way the individual orders regardless of others' preferences over these two alternatives. Sen's minimal liberalism axiom states that there is at least two decisive individuals over two alternatives. He shows, then, that this axiom contradicts Pareto optimality, referring this contradiction as the liberal paradox.

Not only our characterization includes liberal rule, but also we are able to offer alternative characterizations of liberal rule at the end of each section of previous works mentioned in the next section. The possibilty of liberal rule in our model may be because of the differences of our model and previous models in social choice theory some of which are mentioned above. Note that in our model, the liberal is not only possible, rather it satisfies most of mild axioms and can be characterized in many

[9] Note individuals are allowed to be indifferent between alternatives.

[10] Which means that individuals are not prohibited to express any preferences provided that it is complete and transitive.

different ways. We now list main differences of our model from previous models.

- Social alternatives in previous models such as Arrow's has no special meaning. They are any set of abstract alternatives such as bundles of goods, political parties, political/social issues that a society confronts and so on. Because the alternatives does not carrry any given characteristics, one need to exogenously assign some alternatives to an individual to be decisive over. But in our model, the alternatives are the members of society, hence they carry a very certain characteristic and we are able to endegenously allow an individual is decisive over the social decision about himself.

- Social welfare functions in previous models gives a social preference from individual preferences rather than choosing a socially acceptable alternative. Our aggregation function CIF gives a subset of individuals that can be interpreted as the choice of society hence it is closer to social choice functions rather than social welfare functions. Though, it is still possible to think individual opinions as dichotomous preferences like preference in May (1952) but indifferences are not allowed in contrast May's model and the aggregation function as social welfare function where outcome is restricted to strict dichotomous preference.

- In almost all previous model's the set of alternatives and the set of society are different set. Although, there are models at which a set of individuals is faced with the problem of choosing members from another distinct set of individuals. One can see Barbera, Maschler, and Shalev (2001). But in our model, the individuals have to decide over themselves hence there are no two distinct sets; one of which choose from other. As another example, matching problems have two distinct sets of individuals (such as men and women, workers and firms) who have complete and transitive preferences over the members of other set. But in our model, the preferences are restricted to dichotomous preferences and there are no two separate gruops of individuals. This coincidence of sets of alternatives and society will also cause difficulties when one try to define equal-treatment conditions among voters and alternatives. In classical social choice theory, there two well-known axioms; anonimity and neutraliy where first requires equal-treatment among voters and the latter stands for equal-treatment of alternatives. For example, Samet and Schmeidler (2003) offers an axiom, symmetricity which incorporates both anonymity and neutrality. One of main attempts of this study is try to resolve this distinction.

3 Previous Works and Offered Alternative Characterizations

In this section, we summarize some of previous results in the literature. These are Kasher and Rubinstein (1997), Dimitrov, Sung, and Xu (2004) and Samet and Schmeidler (2003). Details of proofs are moved to appendix.

3.1 On the Question of "Who is a J?"

The first formal treatment of identity determination problem in the literature was made by Kasher and Rubinstein (1997) with the title "On the question of "who is a J?"". The title is originated from the previous work of Kasher (1993) who wrote on Jewish identity and propose a procedure to determine Js from individual opinions.[11] Kasher and Rubinstein (1997) formally characterize three different type of collective identity functions: the *liberal* CIF defined as "a J is whoever defines oneself to be J", the *dictatorial* CIF where one's social opinion depends only on the dictator's opinion about that individual and finally *oligarchic* CIF where the power of determining social opinion about any individuals is held by some groups of individuals which is called oligarchy. The details of first characterization will be presented whereas the latter two will be omitted since their characterization based on equivalence relation that is developed in Rubinstein and Fishburn (1986) and expressed as a corollaries of results of Rubinstein and Fishburn (1986).

Before giving details of characterization of the liberal collective identity function, we first give the formal definition.

Definition 3.1 *The Liberal CIF $L \in \mathcal{F}$ is defined for each $G \in \Gamma$ as $L(G) = \{i \in N : i \in G_i\}$*

For each possible profiles, the liberal CIF gives the set of individuals who qualifies themselves. The social qualification of an individual depends on only his opinion about himself. Kasher and Rubinstein (1997) formally characterize liberal CIF L with five axioms: Consensus, symmetricity, monotonicity, independence and the liberal principle. They claimed that these axioms are logically independent but Dimitrov and Sung (2003) showed that the axioms are logically dependent and the liberal CIF can be characterized by only three axioms; namely symmetricity, independence and

[11] Dimitrov, Sung, and Xu (2004) characterize this class of CIFs. See Section 3.2.

the liberal principle. As a result, the definitions of all axioms will be given but the proofs are followed from Dimitrov and Sung (2003) and presented in the appendix.

Axiom 3.1 *A CIF $F \in \mathcal{F}$ satisfies consensus* (C) *if $i \in G_j$ for all $j \in N$, then $i \in F(G)$ and if $i \notin G_j$ for all $j \in N$, then $i \notin F(G)$.*

Consensus axioms states that unanimity on an individual in the personal opinions must result same social opinion with personal opinions. In other words, if a person is qualified by all members of society, then he must be socially qualified and if all members of society believe that the person does not carry the identity, then that person must be socially unqualified.

Axiom 3.2 *A CIF $F \in \mathcal{F}$ satisfies symmetricity* (SYM-KR)[12] *if for any $i, j \in N$ and for any profile $G \in \Gamma$ satisfying the following conditions*

- $G_i \setminus \{i, j\} = G_j \setminus \{i, j\}$
- $i \in G_k \iff j \in G_k$ *for all* $k \in N \setminus \{i, j\}$,
- $j \in G_i \iff i \in G_j$
- $i \in G_i \iff j \in G_j$

we have $i \in F(G) \iff j \in F(G)$.

We say individuals i and j are symmetric in a profile if it satisfies all four conditions above for that two individuals. Symmetricity axiom requires the aggregation rule does not discriminates the individuals who are symmetric in a profile. So for any two individual in a profile, if they agree on all other individuals in their opinions, all other individuals have same opinions about these two individuals, one qualifies other if and only if the other qualifies the one and finally both have same opinions about themselves, then symmetricity requires that the rule must give same social opinions about these two individuals.

Axiom 3.3 *A CIF $F \in \mathcal{F}$ satisfies monotonicity* (MON-KR) *if for any two profiles $G, H \in \Gamma$ such that for all $j \in N \setminus \{k\}$, $G_j = H_j$ and $G_k = H_k \cup \{i\}$, then $i \in F(H)$ implies $i \in F(G)$.*

[12] For further references, some axioms are abbreviated by adding some letters at the end to differentiate them from the axioms defined by other authors with same names. For example, KR stands for Kasher and Rubinstein. But throughout this section, we omit the abbreviations.

Monotonicity states that if an individual i is social qualified in a profile, then a change in some individual's opinion in favour of i being a member of identity can not result disqualification of that individuals.

Axiom 3.4 *A CIF $F \in \mathcal{F}$ satisfies independence* (I-KR) *if for any individual $i \in N$ and for any two profiles $G, H \in \Gamma$ such that $G_j|_i = H_j|_i$ for all $j \in N$ and $F(G) \setminus \{i\} = F(H) \setminus \{i\}$, then $F(G)|_i \iff F(H)|_i$.*

If all individuals (including i) have same opinions about individual i at any two profiles where the social opinion is same except the individuals i, independence states that the social opinion about i must also be same for these two profiles.

Axiom 3.5 *A CIF $F \in \mathcal{F}$ satisfies the liberal principle* (L) *if for any $G \in \Gamma$, there is an individual $i \in N$ with $i \in G_i$ implies $F(G) \neq \varnothing$ and there is an individual $i \in N$ with $i \notin G_i$ implies $F(G) \neq N$.*

The liberal principle states that if there is an individual qualifying himself in a profile, then the outcome can not be empty set and analogously if there is an individual who does not qualify himself, then the social opinion can not be whole society. An equivalent statement of the liberal principle is that, if social opinion is empty set, then each individual believes that he does not carry the identity, and if social opinion is whole society, then each individual qualifies himself.

Kasher and Rubinstein (1997) states that a CIF satisfies *C, SYM, MON, I* and *L* if and only if it is liberal CIF and these five axioms are logically independent. However, as we noted earlier, Dimitrov and Sung (2003) showed that these axioms are logically dependent and proved that *SYM, I* and *L* are enough to characterize the liberal CIF. Kasher and Rubinstein give five examples of CIF for logical independence, each satisfies all but one axioms. Dimitrov and Sung showed that the examples for consensus and monotonicity listed below also fail to satisfy some other axioms and can not be repaired.

Example 3.1 (C) *Let n be odd. The CIF $F \in \mathcal{F}$ defined for all $G \in \Gamma$ as $F(G) = L(G)$ if $\#\{i \in N : i \in G_i\}$ is odd and $F(G) = \{i \in N : i \notin G_i\}$ otherwise.*

This example fails to satisfy not only consensus also the liberal principle. To see this, consider $n = 3$ and the profile $G_i = \varnothing$ for all $i \in \{1, 2, 3\}$. As $\#\{i \in N : i \in G_i\}$ is 0 which is even, the rule gives $F(G) = N$ which contradicts with the liberal principle since there is a profile with an individual who does not qualify himself, but social opinion is whole society.

Example 3.2 (MON) *The CIF $F \in \mathcal{F}$ defined for all $G \in \Gamma$ as $F(G) = \{i \in N : G_i = \{i\}\}$.*

This CIF fails to satisfy *C*, *L* and *I*. To see why F violates C and L, consider the society with three individuals, $N = \{1, 2, 3\}$ and the profile $G_i = \{1, 2\}$ for all $i \in \{1, 2, 3\}$. The rule gives the social opinion $F(G) = \varnothing$, violating consensus as there is a consensus among the members $\{1, 2\}$ and violates L, since there is a profile at which the social opinion is empty set whereas there is an individual who qualify himself, namely individuals 1 or 2. Moreover, to see how the rule violates *I*, consider the profile $H \in \Gamma$ where $H_1 = \{1\}$ and $H_i = \{1, 2\}$ for $i \in \{2, 3\}$. The social opinion for $H \in \Gamma$ is $F(H) = \{1\}$. Note that we have $G_j|_1 = \{1\} = H_j|_1$ for all $j \in N$ and the social opinion about all individuals except 1 is same for two profiles, but $1 \in F(H)$ whereas $1 \notin F(G)$ violates the independence.

In fact C and *MON* is implied from other three axioms. We now stated that *SYM*, *I* and *L* implies *C* after the following lemma which sates if all individuals have a consensus among members of a coalition K as being members and among all other individuals as non-members, then the social outcome must be exactly that coalition, K.

Lemma 3.1 *If a CIF $F \in \mathcal{F}$ satisfies* SYM, I *and* L*, then $F\left(G^K\right) = K$ for all $K \subseteq N$ where $G^K \in \Gamma$ is the profile such that $G_i = K$ for all $i \in N$.*

Theorem 3.1 *If a CIF $F \in \mathcal{F}$ satisfies* SYM, I *and* L*, then it also satisfies* C.

Before showing the liberal CIF is the only CIF that satisfies *SYM, I* and *L*, we need to state partition lemma of Dimitrov and Sung (2003). Let $P = (P_1, P_2, P_3, P_4)$ is any 4-partition of N and let $G^P, H^P \in \Gamma$ are two profiles defined for any 4-partition and for all $j \in N$ as follows:

$$G_j^P = \begin{cases} P_1 \cup P_2 & \text{if } j \in P_1 \cup P_3, \\ P_1 \cup P_2 \cup P_3 & \text{if } j \in P_2 \cup P_4. \end{cases} \text{ and}$$

$$H_j^P = \begin{cases} P_1 & \text{if } j \in P_1 \cup P_3, \\ P_1 \cup P_2 & \text{if } j \in P_2 \cup P_4, \end{cases}.$$

Note that the *liberal* CIF L gives the same social opinion $P_1 \cup P_2$ for both profiles $G^P, H^P \in \Gamma$. The following lemma states any CIF satisfying *SYM, I* and *L* also gives social opinion $P_1 \cup P_2$ as liberal CIF does.

Lemma 3.2 *If a CIF $F \in \mathcal{F}$ satisfies* SYM, I *and* L, *then* $F\left(G^P\right) = F\left(H^P\right) = P_1 \cup P_2$ *for every* 4*-partition* $P = (P_1, P_2, P_3, P_4)$ *of* N.

Note that partition lemma of Dimitrov and Sung (2003) can be viewed as an extension of lemma 3.1 since for particular 4-partition $P = (K, \varnothing, \varnothing, N \setminus K)$ of N, we have $G^K = G^P = H^P$.

Finally we can state the characterization theorem of liberal CIF in terms of the axioms; symmetricity, independence and the liberal principle.[13]

Theorem 3.2 *The Liberal CIF $L \in \mathcal{F}$ is the only CIF that satisfies* SYM, I *and* L.

Note that as a corollary of theorem 3.2, *SYM, I* and *L* implies *MON* since the only CIF which satisfies three axioms is the liberal CIF and it satisfies monotonicity condition proposed by Kasher and Rubinstein (1997). This observation is stated in the following corollary.

Corallary 3.1 *Any CIF $F \in \mathcal{F}$ satisfying* SYM, I *and* L *also satisfies* MON.

However these three axioms are logically independent. To see this, one can check the following three examples each of which fails to satisfies only one axiom. Which axiom the examples fails is demonstrated with the abbreviation of axioms at the beginning of each example.

Example 3.3 (SYM) *The CIF $F \in \mathcal{F}$ defined for each $G \in \Gamma$ as: $F(G) = L(G)$ if $n = 1$ and $F(G) = \{1\}$ otherwise.*

Example 3.4 (I) *The CIF $F \in \mathcal{F}$ defined for each $G \in \Gamma$ as $F(G) = L(G)$ if $L(G) \in \{\varnothing, N\}$ and $F(G) = N \setminus L(G)$ otherwise.*

Example 3.5 (L) *The CIF $F \in \mathcal{F}$ defined as $F(G) = \varnothing$ for all $G \in \Gamma$.*

Among the axioms used in the characterization, symmetricity and independence as well as monotonicity condition has been defined with some differences by other authors who wrote in that literature.

[13] Proofs of all theorems and lemmas stated above can be found in appendix.

3.1.1 An alternative characterization offered for Liberal CIF

Liberal CIF is central at collective identity determination problem. It gives the right of self-determination to each individual. In the previous section, it was shown that the liberal rule satisfies many fairly acceptable axioms. We now offer some alternative characterization for the liberal rule after introducing some new axioms.

Consider any abstract identity such as being G and a society faced with the question of "who are the Gs?". The liberal rule gives the set of individuals who qualifies themselves as a G. Now rename the identity as being non-G. It is natural that each individual express their opinions for the new identity, being non-G as the complement of their previous opinions. In same manner, one may expect the new social outcome is also the complement of previous social opinion. More technically, in aggregation of being non-G, we face with a new profile $\overline{G} \in \Gamma$ such that $\overline{G}_i = N \setminus G_i$ for all $i \in N$, and we have $F\left(\overline{G}\right) = \overline{F\left(G\right)}$. Note that the liberal rule satisfies such a condition. The following axiom which will be introduced again in Section 4, formally define the situation above.

Axiom 3.6 *A CIF $F \in \mathcal{F}$ satisfies self-duality* (SD) *if for any $i \in N$ and any $G \in \Gamma$, we have $i \in F(G) \iff i \notin F(\overline{G})$.*

Self-duality requires that the aggregation rule does not discriminate the name of identity. Aggregating who are the Gs is equivalent to aggregating who are the non-Gs in the sense that aggregation will result social opinions, each one is complement of the other.

The next axiom is the self-exlusion principle. It states that if an individual qualifies all members of society except himself whereas all other individuals has an opinion that he carries the identity and he is not socially qualified, then he has the right of self exclusion, that is, he is not socially qualified whenever he does not qualifies himself.

Axiom 3.7 *A CIF $F \in \mathcal{F}$ satisfies self-exlusion principle* (SE) *if for any $i \in N$, there exist a profile $G \in \Gamma$ such that $G_i = N \setminus \{i\}$ and $i \in G_j$ for all $j \in N \setminus \{i\}$ and $i \notin F\left(G\right)$, then we have $i \notin F\left(H\right)$ for any $H \in \Gamma$ with $i \notin H_i$.*

Note that self-exclusion holds if there is a profile at which the individual i is qualified by all individuals (except him) who are qualified by i but i is not socially qualified. Otherwise self-exclusion does not impose any restriction on a CIF.

The liberal rule can also be characterized by consensus, self-duality, the liberal principle and self-exclusion principle.

Theorem 3.3 *A CIF $F \in \mathcal{F}$ satisfies* (C)*,* (SD)*,* (L) *and* (SE) *if and only if it is the liberal rule.*

Proof. The liberal rule L satisfies all axioms. To see "only if" part, take any CIF F satisfying *(C)*, *(SD)*, *(L)* and *(SE)*. We need to show that $i \in G_i$ implies $i \in F(G)$ and $i \notin G_i$ implies $i \notin F(G)$. But observe that by *SD*, it is enough to show one of them. Thus take some $i \in N$ and consider the profile $G \in \Gamma$ with $G_i = N \setminus \{i\}$ and $G_j = N$ for all $j \in N \setminus \{i\}$. By *C*, we have $N \setminus \{i\} \subseteq F(G)$. Suppose $F(G) = N$. But it contradicts with L since $i \notin G_i$. So $F(G) = N \setminus \{i\}$. But by *SE*, for all $H \in \Gamma$ with $i \notin H_i$, we have $i \notin F(H)$. ■

3.2 Procedural Group Identification

The procedure of Kasher (1993) for identity determination starts an initial set of individuals among whom there is a consensus. Kasher (1993) calls this set as "incontrovertible core" of the collective identity. Then, further individuals are added to this set if and only if they are qualified by some members of initial set. Applying this procedure until the set of members of collective identity does not expand anymore will give the social opinion about an identity. Since the size of society is finite, the process eventually stops. Kasher (1993) express the intuition behind this expansion process as: every socially accepted G as being newly added brings a possibly unique new view of being a G collectively with him, and a collective identity function is supposed to aggregate those views and must pay attention to this new individual's G-concept in order to cover the whole diversity of views in the society about the question "what does it mean to be a G?". Kasher and Rubinstein (1997) refers this procedural way of determining collective identity and discuss it. They point Kasher's procedure satisfies all axioms mentioned by Kasher and Rubinstein (1997) except the liberal principle. Since Kasher (1993) searches a method with only fairness considerations, a condition about self-determination rights may not be considered as derivable from fairness considerations only. Kasher and Rubinstein (1997) criticize the way of determining the initial set and mention another procedure where the initial set is determined by liberal CIF. Quote from Kasher and Rubinstein (1997):

> The axiomatic characterization of Kasher's method remains to be completed. Note that the difficulty in finding a suitable axiomatization is due to the difficulty of justifying why the recursive procedure starts with the set $\{i \in N : i \in G_j \ \forall j \in N\}$ and not with another set, such as $\{i \in N : i \in G_i\}$, for example.

Dimitrov, Sung, and Xu (2004) axiomatically characterize these two procedural collective identity functions namely liberal-start-respecting rule and consensus-start-respecting rule. Each recursive procedure has two parts: An initial set of individuals and how these individuals are determined. Two procedures differ at this step, Kasher's method requires absolute consensus on individuals in this initial set, whereas Kasher and Rubinstein (1997) suggest to apply liberal rule to determine the initial set. The second part is the way of expanding this initial set. For each rule, both authors offer same expansion rule i.e. expanding the initial set by adding all individuals qualified by some members of the initial set. This process continues inductively until the expansion stops.

More formally, take any CIF $F^0 \in \mathcal{F}$ which set up the initial set. For any $G \in \Gamma$ and any non-negative integer k, let $F^{k+1}(G) = F^k(G) \cup \{i \in N : i \in G_j$ for some $j \in F^k(G)\}$. Let $\overline{k}$ be the smallest integer for which $F^{\overline{k}+1}(G) = F^{\overline{k}}(G)$. Define the CIF $F^P \in \mathcal{F}$ as $F^P(G) = F^{\overline{k}}(G)$ for each $G \in \Gamma$. We call F^P the *procedural CIF based on* F^0. Though many procedural CIFs can be generated by changing F^0 as well as changing expansion rule; the liberal-start-respecting procedure, L^P proposed by Dimitrov, Sung, and Xu (2004) and the consensus-start-respecting procedure, C^P proposed by Kasher (1993) are particular procedural CIFs based on the liberal CIF L (that is $F^0 = L$) and the consensus CIF C defined below (that is $F^0 = C$) respectively with same expansion procedure.[14]

Definition 3.2 *The consensus CIF* $C \in \mathcal{F}$ *is defined for all* $G \in \Gamma$ *as* $C(G) = \{i \in N : i \in G_j$ *for all* $j \in N\}$.

Dimitrov, Sung, and Xu (2004) introduce 6 axioms. They have two consensus axioms first of which is same with the consensus axiom defined by Kasher and Rubinstein (1997) and not given below.[15] Second consensus axiom *(C2)* is a weaking

[14] The superscript P reflects that the rule is procedural and differentiate the procedural rules from the liberal CIF L and consensus CIF C.

[15] See Axiom 3.1.

of the standard one. Three axioms are related how insiders' and outsider's views are treated. Finally they offer a stability axiom *(ES).*

Axiom 3.8 *A CIF $F \in \mathcal{F}$ satisfies consensus* 2 (C2) *if for some $i \in N$ we have $i \notin G_j$ for all $j \in N$, then $i \notin F(G)$.*

Axiom 3.9 *A CIF $F \in \mathcal{F}$ satisfies irrelevance of an outsider's view* 1 (IOV1) *if for all $i, j \in N$ and for all $G, H \in \Gamma$ such that $i \notin G_j$, $H_j = G_j \cup \{i\}$, $G_k = H_k$ for all $k \in N \backslash \{j\}$, then $[j \notin F(G)$ and $i \notin H_k$ for some $k \in N]$ implies $F(G)|_i = F(H)|_i$.*

The axiom of irrelevence of outsider's view states that if someone is socially unqualified, then this person's opinion about any qualified individual is irrelevant on deciding that qualified individual. However by existence of some k who disqualifies i, *IOV1* excludes the case where an outsider's view is relevant in one's social decision that is every individual except j qualifies i, hence if i add j to the set of qualified individuals, consensus requires the qualification of individual thus making j's opinion about i relevant. Let us note that *IOV1* is weaker then exclusive self determination axiom introduced by Samet and Schmeidler (2003).[16]

Axiom 3.10 *A CIF $F \in \mathcal{F}$ satisfies equal treatment of insider's view* (ETIV) *if for all $i, j, k \in N$ and for all $G, H \in \Gamma$ such that $i \in G_j$, $H_j = G_j \setminus \{i\}$, $H_k = G_k \cup \{i\}$, $H_l = G_l$ for all $l \in N \setminus \{j, k\}$, then $[j \in F(G)$ and $k \in F(H)]$ implies $F(G)|_i = F(H)|_i$.*

ETIV requires that a CIF must equally treat all socially qualified individuals' opinions. More technically, if an individual i is qualified by a member of identity j in a profile and by a socially qualified individual k in an other profile, then the social opinion about i must be same provided that all individuals except j and k keeps their opinions same in two profiles.

Axiom 3.11 *A CIF $F \in \mathcal{F}$ satisfies irrelevance of an outsider's view* 2 (IOV2) *if for all $i, j \in N$ with $i \neq j$ and for all $G, H \in \Gamma$ such that $H_j = G_j \cup \{i\}$, $G_k = H_k$ for all $k \in N \setminus \{j\}$, then $j \notin F(G)$ implies $F(G)|_i = F(H)|_i$.*

Note that the liberal rule L fails to satisfy *IOV1* since there is no explicit requirement that i is different from j. Hence i's self-qualification immediately determine

[16] See Axiom 3.20

his social opinion which may cause different social opinion contrary to what *IOV1* requires. Other than the distinction above, *IOV2* has same spirit with *IOV1*.

Axiom 3.12 *A CIF $F \in \mathcal{F}$ satisfies external stability* (ES) *if for all $G \in \Gamma$ and for all $i \in N$, $i \notin F(G)$ implies $i \notin G_i$.*

Note that for a CIF, it may be possible for an individual i that $i \in G_i$ but $i \notin F(G)$ or the converse $i \notin G_i$ but $i \in F(G)$. The stability axioms rules out the possibility of the first case. Note that a CIF results a partition $(F(G), N \setminus F(G))$ of the society N and the stability of the CIF F depends on the satisfaction of individuals of each set of partition with the result of F. *ES* deals with the satisfaction of individuals from $N \setminus F(G)$.

Dimitrov et al. (2004) prove the following theorems.

Theorem 3.4 *A CIF F satisfies the axioms* (C2), (ES), (ETIV) *and* (IOV2) *if and only if $F = L^P$.*

Theorem 3.5 *A CIF F satisfies the axioms* (C), (ETIV) *and* (IOV1) *if and only if $F = C^P$.*

3.2.1 An alternative characterization offered for Liberal CIF

In previous section the procedural rules for collective identity determination are characterized, we can still provide some alternative characterization for the liberal rule. We inspire a new axiom, independence of outsiders' view from *IOV2*. It states that socially unqualified members can not reverse social opinion of any qualified member by changing their opinions about members of identity while keeping their opinions about unqualified members same provided that all qualified members keeps their opinions same. In addition, it allows some unqualified members become qualified after the change in opinions, hence weaking the restriction that axiom impose on the result of a CIF. The formal definition is given below.[17]

Axiom 3.13 *A CIF $F \in \mathcal{F}$ satisfies independence of outsiders' views* (IOV) *if for all $G, H \in \Gamma$ such that $H_j = G_j$ for all $j \in F(G)$ and $H_j \cap (N \setminus F(G)) = G_j \cap (N \setminus F(G))$ for all $j \in N \setminus F(G)$, we have $F(G) \subseteq F(H)$.*

[17] Note that *IOV* and *IOV2* are logically independent from each other.

We then introduce an independence axiom which is stronger than the one proposed by Kasher and Rubinstein (1997).

Axiom 3.14 *A CIF $F \in \mathcal{F}$ satisfies independence* (I) *if for all $i \in N$ and for all $G, H \in \Gamma$ such that $G_j|_i = H_j|_i$ for all $j \in N$, then we have $F(G)|_i = F(H)|_i$.*

This stronger independence axiom was introduced also by Samet and Schmeidler (2003) and it will be used in characterization of elementary CIFs in Section 4. We are ready to state an alternative characterization of the liberal rule in terms of consensus, independence and independence of outsiders' views.

Theorem 3.6 *A CIF $F \in \mathcal{F}$ satisfies* C[18], I *and* IOV *if and only if it is the liberal rule. Moreover all three axioms are independent.*

Proof. To see "If" part holds, one can check that the liberal rule satisfies all three axioms. To see "only if" part, take any CIF $F \in \mathcal{F}$ satisfying C, I and IOV and any individual $i \in N$. We first show that for any profiles $G \in \Gamma$ with $G_j \in \{\varnothing, \{i\}\}$ for all $j \in N$, we have

(1) $i \in G_i$ implies $i \in F(G)$ and

(2) $i \notin G_i$ implies $i \notin F(G)$.

To see (1), consider $G \in \Gamma$ where $G_j = \{i\}$ for all $j \in N$. By C, we have $F(G) = \{i\}$. Now take any $H \in \Gamma$ with $H_j \in \{\varnothing, \{i\}\}$ for all $j \in N$ where $i \in H_i$. Since we have $G_i = H_i = \{i\}$ and $H_j \cap (N \setminus \{i\}) = G_j \cap (N \setminus \{i\})$ for all $j \in N \setminus \{i\}$ by choice of H, IOV implies that $\{i\} \subseteq F(H)$. In addition, C requires $F(H) = \{i\}$.

To see (2), consider $G \in \Gamma$ with $G_j \in \{\varnothing, \{i\}\}$ for all $j \in N$ while $G_i = \varnothing$ and assume for the sake of a contradiction that $i \in F(G)$. Note that by C, we have $j \notin F(G)$ for all $j \in N \setminus \{i\}$, hence $F(G) = \{i\}$. Let $H \in \Gamma$ be a profile such that $H_j = \varnothing$ for all $j \in N$. By C, we have $F(H) = \varnothing$. But as $G_i = H_i = \varnothing$ and $H_j \cap (N \setminus \{i\}) = G_j \cap (N \setminus \{i\})$ for all $j \in N \setminus \{i\}$ by choice of H, we must have $\{i\} \subseteq F(H)$ by IOV which is not the case, thus leads us the desired contradiction.

Now we extend our analysis to any $G \in \Gamma$ and show that

(1′) $i \in G_i$ implies $i \in F(G)$ and

(2′) $i \notin G_i$ implies $i \notin F(G)$.

To see (1′), take any $H \in \Gamma$ with $i \in H_i$. Let $M = \{j \in N : i \in H_j\}$ and $G \in \Gamma$ such that $G_j = \{i\}$ for all $j \in M$ and $G_j = \varnothing$ for all $j \in N \setminus M$. By (1), we have

[18] See axiom 3.1.

$F(G) = \{i\}$ and by *I*, we have $F(H)|_i = F(G)|_i = \{i\}$ since $H_j|_i = G_j|_i$ for all $j \in N$. Hence $i \in G_i$ implies $i \in F(G)$.

To see (2′), take any $H \in \Gamma$ with $i \notin H_i$. Let $M = \{j \in N : i \notin H_j\}$ and $G \in \Gamma$ such that $G_j = \varnothing$ for all $j \in M$ and $G_j = \{i\}$ for all $j \in N \setminus M$. By (2), we have $F(G) = \varnothing$ and by *I*, we have $F(H)|_i = F(G)|_i = \varnothing$ since $H_j|_i = G_j|_i$ for all $j \in N$. Hence $i \notin G_i$ implies $i \notin F(G)$ showing the equivalence of F and the liberal CIF L.

To see the logical independence of axioms, one can check the following examples. For an odd n, the simple majority rule M defined as follows: For each $G \in \Gamma$ and each $i \in N$, we have $i \in M(G)$ if and only if $\#\{j \in N : i \in G_j\} > \frac{n}{2}$ satisfies consensus and independence but violates *IOV*. To see this consider $N = \{1, 2, 3\}$ and the profiles $G, H \in \Gamma$ where $G_j = \{1\}$ for all $j \in N$ and $H_1 = \{1\}$ while $H_j = \varnothing$ for $j \in \{2, 3\}$. We have $F(G) = \{1\}$ but $F(H) = \varnothing$ violating *IOV* since $\{1\} \nsubseteq F(H)$. The CIF $F \in \mathcal{F}$ defined as $F(G) = \{1, 2\}$ for all $G \in \Gamma$ where $N = \{1, 2\}$ satisfies *I* and *IOV* but clearly violates *C*. Finally one check the following CIF F defined for any $i, j \in \{1, 2\} = N$ satisfies *IOV* and *C* but violates *I*.

$$F(G) = \begin{cases} \varnothing & \text{if } G_i = \varnothing \text{ and } G_j = \{i\} \\ K & \text{if } G_i = K \text{ for all } i \in \{1,2\} \\ \{i\} & \text{if } G_i = \{i\} \text{ and } G_j = \varnothing \\ N & \text{otherwise} \end{cases} . \quad \blacksquare$$

3.3 Between Liberalism and Democracy

Samet and Schmeidler (2003) recently study a class of CIFs they called consent rules. Consent rules are parameterized by the weights given to individuals in determining their own qualification. For example, in liberalism one's social qualification depends only his opinion. On the other hand, in majoritarianism one needs to consent of majority of society to consent his opinion to society. These are the two extremes of consent rules of Samet and Schmeidler.

Samet and Schmeidler (2003) formally characterize the consent rules which is formally defined below.

Definition 3.3 *A consent rule (with consent quotas s and t such that $s + t \leq n + 2$) is a CIF $F^{st} \in \mathcal{F}$ such that given any $G \in \Gamma$ and any $i \in N$,*

- *if $i \in G_i$, then $i \in F^{st}(G) \Longleftrightarrow \#\{j \in N : i \in G_j\} \geq s$*

- *if* $i \notin G_i$, *then* $i \notin F^{st}(G) \Longleftrightarrow \#\{j \in N : i \notin G_j\} \geq t$

The quotas in the definition reflects the level of social consent that one need to make acceptable his opinion on himself as the social opinion. For any given paramaters s and t[19], if a particular individual qualifies himself, then his qualification of himself is socially adopted if and only if there are $s-1$ other individual in the society who also qualifies that individual and in the case of one's disqualification of oneself, there must be $t-1$ other individuals who disqualify him for social disqualification of that individual. Therefore the larger the quota s, the less the individual power to consent his self-qualification and the greater the value of t, the the greater social power to act against one's self-disqualification. For example, consider the case $s = t = 1$. Then one's social qualification or disqualification only depends on one's opinion about himself. Hence F^{11} is equivalent to the liberal rule.[20] Now consider, $s = 1$ and $t = n+1$. If an individual qualifies himself, then he is socially qualfied by the rule $F^{1,n+1}$. On the other hand, if an individual does not qualifies himself, then he needs to meet quota $n+1$ which is greater than the size of society, hence according to the rule $F^{1,n+1}$, the individual is again socially qualified. Thus $F^{1,n+1}$ is equivalent to the constant rule that each individual is always socially qualified whatever the profile is.[21] Analagously, the rule $F^{n+1,1}$ turns out the constant rule which disqualifies all individuals regardless of the opinion profile.

In addition, Samet and Schmeidler not only mention the values of s and t but also the difference between s and t, $|s-t|$. The smaller difference, the more equally the rule treats one's qualification versus disqualification. The smallest possible difference occurs at $s = t$.[22] Samet and Schmeidler advise such rules where being or not-being a member of identity is socially neutral such as the example they give; being Democrat and being Republican. Note that the liberal rule is among this neutral rules with smallest values of quotas as well as the simple majority rule. For an odd n where both s and t equals to $(n+1)/2$, the consent rule F^{st} becomes the simple majority rule. In a simple majority rule, one's opinion about himself equally treated with every other individuals' opinion about him in contrast to the liberal rule.[23] Though

[19] The condition $s + t \leq n + 2$ is related with monotonicity and reflects a restriction on the power of society. This relation will be introduced in proposition 3.1.

[20] Hence an individual has the full power on his social opinion.

[21] In this case, an individual has maximum power to consent his self-qualification, meanwhile the society also have maximum power to act one's self-disqualification.

[22] We call them as symmetric consent rules in section 4.1 and show that they are the only consent rules that are also members of simple CIFs.

[23] See remark 4.6.

the liberal rule is neutral in the sense that an individual faces with same quota in each possible state of the world (self-qualification and self-disqualification), one's vote has a superior power than another individuals' vote which clearly differentiate the liberal rule from simple majority rule each of which are extremes of neutral rules. Samet and Schmeidler note that the simple majority is the only nontrivial rule in which one's vote concerning one's qualification has no special weight.[24] Samet and Schmeidler also discuss the possiblity $s \neq t$ that reflects the situations that being qualified or not-qualified have different implications. Consider a case that we need to determine who have the rights to do some acts which are related with others' rights. Samet and Schmeidler gives the right to drive in the public domain which can be related with being able to cross the road safely as an example. They suggest F^{st} with $s > t$ for such a situaiton since when one gives up the right to drive, the social consent is expected to be smaller compared to one wishes to exert his right. On the other hand, F^{st} with $s < t$ seems appropriate if the identity in question is imposing an obligation. Think of the example where we wish to determine the one's that works as a volunteer in an organization or foundation. Since one's self-qualification requires less consent in contrast to one's withdrawal requires a wider social consent, F^{st} with $s < t$ may be a suitable way of such a collective identity determination.

Samet and Schmeidler formally characterize consent rules with the following three axioms: Monotonicity, independence and symmetrcity.

Axiom 3.15 *A CIF $F \in \mathcal{F}$ satisfies monotonicity* (MON-SS) *if for all $G, H \in \Gamma$ such that $G_i \subseteq H_i$ for all $i \in N$, we have $F(G) \subseteq F(H)$.*

Monotocity requires that if each individual expand their set of qualified members of society, then all individuals who have previously qualified must remain qualified.

Their next axiom is independence axiom which was defined before.[25] Independence axioms states that the social decision about an individuals can be determined by only knowing each individuals' personal opinion about that individuals. Thus for any two profiles at which all members of society have same opinion about a particular individual, then social opinion of that individual must be same for this two profiles.

Axiom 3.16 *A CIF $F \in \mathcal{F}$ satisfies symmetricity* (SYM-SS) *if given any permutation $\Pi : N \to N$, any $G \in \Gamma$ and any $i \in N$, we have $i \in F(G) \Leftrightarrow \Pi(i) \in F(\Pi(G))$.*

[24] Clearly, all votes are ineffective in the trivial rules $F^{1,n+1}$ and $F^{n+1,1}$.

[25] See Axiom 3.14.

Note that in Section 2, it is stated that the permutation over society can be interpreted as changing names of individuals. Hence for a symmetric rule, the social opinion do not depend on names or alternatively if a previously qualified individual's name changed, then he must be qualified with his new name under same rule provided that each individual updates their opinions with respect to new names.

Samet and Schmeidler (2003) prove the following theorem.

Theorem 3.7 *A CIF $F \in \mathcal{F}$ satisfies MON, I and SYM if and only if it is a consent rule. Moreover all three axioms are independent.*

The proof is omitted, but the idea of proof is as follows: Independence implies that the social decision of j can be determined by only knowing $G\,|_j \in \{\varnothing, \{j\}\}^n$. Symmetricity implies that the names of individuals does not matter rather the distribution of other individuals opinions about j and j's opinion about himself are important for social decision. Monotonicty requires that the number votes are important in this distribution and assigns a minimum value which stands for quota. Finally a reapplication of symmetricity ensures this quota is same for all individuals. The condition $s + t \leq n + 2$ is related with monotonicity and the reason is explaned in the following paragraph and in proposition 3.1.

For any $G \in \Gamma$ and for any $j \in N$, we define a profile G^{-j} as follows: $G_i = G_i^{-j}$ for all $i \in N \setminus \{j\}$, $G_j \setminus \{j\} = G_j^{-j} \setminus \{j\}$ and $j \in G_j \iff j \notin G_j^{-j}$. In word, G^{-j} is a profile same with G except individual j changes his opinion about himself. There are 4 possible outcome that a CIF F may give:

1. $F(G)\,|_j = F\left(G^{-j}\right)|_j = \{i\}$
2. $F(G)\,|_j = F\left(G^{-j}\right)|_j = \varnothing$
3. $F(G)\,|_j = G_j\,|_j$ and $F\left(G^{-j}\right)|_j = G_j^{-j}\,|_j$
4. $F(G)\,|_j = G_j^{-j}\,|_j$ and $F\left(G^{-j}\right)|_j = G_j\,|_j$.

In first two case, the social decision about j is insensitive to his personal opinion about himself. In third, the rule respects the personal opinion of j about himself, whereas in the fourth case, social opinion and personal opinion of j are converse. The following axiom rules out the existence of fourth possibility.

Axiom 3.17 *A CIF $F \in \mathcal{F}$ is said to be non-spiteful* (NS) *if there exist no profile $G \in \Gamma$ and $j \in N$ such that $F(G)\,|_j \neq G_j\,|_j$ and $F\left(G^{-j}\right)|_j \neq G_j^{-j}\,|_j$.*

Proposition 3.1 *Let $F \in \mathcal{F}$ is a consent rule with quotas $s, t \leq n+1$ (without restriction $s+t \leq n+2$). Then the following three conditions are equivalent.*

1. *$s+t \leq n+2$.*

2. *F is monotonic.*

3. *F is non-spitefull.*

Samet and Schmeidler then showed that adding self-duality to the axiom set will result that the neutral rules, as discussed at the beginning of this section, are the only consent rules satisifying SD which will be expressed in the following theorem after definition of self-duality.

Axiom 3.18 *A CIF $F \in \mathcal{F}$ satisfies self-duality* (SD-SS) *if given any $G \in \Gamma$, we have $F\left(\overline{G}\right) = N \setminus F(G)$.*

Theorem 3.8 *A CIF $F \in \mathcal{F}$ satisfies MON, I, SYM and SD if and only if it is a consent rule with equal quotas. Moreover all four axioms are independent.*

Finally Samet and Schmeidler discuss the right of self-determination. Quoting from them:

> The political principle of self-determination says that a group of people recognized as a nation has the right to form its own state and choose its own government. One of the main difficulties in applying self-determination is that it grants the right to exercise sovereignty to well-defined national identities; it assumes that the self is well defined. In many cases the very distinct national character of the group is under dispute. Such disputes can be resolved, at least theoretically, by a voting rule. Here we want to examine rules which grant the self the right to determine itself.

Then they introduce three further conditions: first gives sovereignty to citizens and the latter two are related to the right of self-determination. They give two characterizations for liberalism by including the self-determination axioms to some previous axioms.

The first axiom is in the same spirit of citizen sovereignty condition of Arrow (1951). It requires the existence of at least two profiles for each individual in the society at one of which the individual is socially qualified and at the other the individual is not socially qualified.

Axiom 3.19 *A CIF $F \in \mathcal{F}$ satisfies nondegeneracy* (ND) *if for each individual $i \in N$, there are profiles $G, H \in \Gamma$ such that $F(G)|_i = \{i\}$ and $F(H)|_i = \varnothing$.*

Axiom 3.20 *A CIF $F \in \mathcal{F}$ satisfies exclusive self-determination* (ESD) *if for any $G, H \in \Gamma$ such that $[G_i|_j \neq H_i|_j \implies i \notin F(G)$ and $j \in F(G)]$, then we have $F(G) = F(H)$.*

Exclusive self-determination states that applying any rule F to a profile G and then allowing unqalified members to change their opinions about qualified members which forms a new profile H must result same social opinion under the same rule F.

The next axiom, affirmative self-determination says that for any rule F and any profile G, the set of qualified individuals and the set of individuals who qualifies the qualified ones in their personal opinions coincides. Before formal definition, let for each $G \in \Gamma$ define a new profile $G^T \in \Gamma$ such that for all $i, j \in N$, $j \in G_i \iff i \in G_j^T$.

Axiom 3.21 *A CIF $F \in \mathcal{F}$ satisfies affirmative self-determination* (ASD) *if for any $G \in \Gamma$ we have $F(G) = F\left(G^T\right)$.*

Samet and Schmeidler show that each of these axioms with monotonicity, independence and nondegeneracy characterize the liberal CIF L. This two characterization are stated in the following theorems.

Theorem 3.9 *The liberal CIF L is the only CIF that satisfies ESD, MON, I and ND.*

Theorem 3.10 *The liberal CIF L is the only CIF that satisfies ASD, MON, I and ND.*

4 Elementary, Basic and Simple Collective Identity Functions

4.1 Main Characterizations

For each $i \in N$, we define family $\omega(i) \subseteq 2^N$ of subsets of N. We refer to $\omega(i)$ as the set of *winning coalitions* over i. The family of winning coalitions over i contains the sets of individuals who can qualify individual i as a member of identity if they unanimously agree on individual i carries the identity and they are exactly the set of individuals qualifying i. The coalitions that are not in $\omega(i)$ are called losing. We also define another family $\overline{\omega}(i) \subseteq 2^N$ of subsets of N as the set of *blocking coalitions* over i. Contrary to winning coalitions, family of *blocking coalitions* contains the set of individuals who can determine an individual's social opinion as unqualified by not qualifying that individual in their personal opinions while the rest qualifies that individual. Hence, a coalition K is said to be blocking if its complement $\overline{K} = N \setminus K$ with respect to N is losing, that is not winning. Note that we do not impose a requirement whether a coalition $K \subseteq N$ can be winning or blocking or neither. Up to now, it is possible that a coalition may be both winning and blocking or it can be neither winning nor blocking as well as a coalition can be either an element of $\omega(i)$ or $\overline{\omega}(i)$ for an individual $i \in N$.[26] We will discuss this issue later. Before that we define several collections of winning coalitions where proper, strong and self-dual ones impose some particular restrictions over families of winning and blocking coalitions as discussed above.

A collection $\{\omega(i)\}_{i \in N}$ of winning coalitions is said to be

- elementary if there is no restriction over family of winning coalitions of any individual i, that is $\omega(i) \subseteq 2^N$ for all $i \in N$.
- basic if it is elementary and it satisfies the following condition: for all $i \in N$ and for all $K, K' \subseteq N$ with $K \subseteq K'$, $K \in \omega(i)$ implies $K' \in \omega(i)$.
- proper if it is basic and it satisfies the following condition: for all $i \in N$ and for all $K \subseteq N$, $K \in \omega(i)$ implies $\overline{K} \notin \omega(i)$.[27]

[26] A detailed discussion of winning, losing and blocking coalitions can be found in Taylor and Zwicker (1999).

[27] In words, if a coalition K is winning, then K is also blocking.

- strong if it is basic and it satisfies the following condition: for all $i \in N$ and for all $K \subseteq N$, $K \notin \omega(i)$ implies $\overline{K} \in \omega(i)$.[28]

- self-dual if it is both strong and proper, that is, it is basic and it satisfies the following condition: for all $i \in N$ and for all $K \subseteq N$, $K \in \omega(i)$ if and only if $\overline{K} \notin \omega(i)$.

Any type of collection $\{\omega(i)\}_{i\in N}$ of winning coalitions induces a (unique) CIF $F \in \mathcal{F}$ in the following natural way: Given any $G \in \Gamma$ and any $i \in N$, we have $i \in F(G) \iff \{j \in N : i \in G_j\} \in \omega(i)$.[29] We qualify a CIF $F \in \mathcal{F}$ as *elementary* if and only if F is induced by an elementary collection $\{\omega(i)\}_{i\in N}$ of winning coalitions. In addition, we qualify a CIF F with the name of the collection of winning coalitions which induce the CIF with only exception that we qualify a CIF F as *simple* if it is induced by self-dual collection of winning coalitions.

Elementary CIFs can be characterized in terms of the independence axiom which was previously introduced[30] but restated below for the sake of completeness.

Axiom 4.1 *A CIF $F \in \mathcal{F}$ satisfies independence* (I) *if for all $i \in N$ and for all $G, H \in \Gamma$ such that $G_j\,|_i = H_j\,|_i$ for all $j \in N$, then we have $F(G)\,|_i = F(H)\,|_i$.*

If a CIF is elementary, the information about how function behaves on profiles is embedded into the winning coalitions. So by using the collection of winning coalitions, one can construct the same social opinion obtained from an elementary CIF and for an elementary CIF, it is possible to construct a family of winning coalitions for each individual such that the social decision about individuals can be obtained from winning coalitions. Note that there is no restriction over the families of winning coalitions. For example, $\omega(i) = \varnothing$ and $\omega(i) = 2^N$ represent two degenerate elementary CIFs[31] where i is socially unqualified in all profiles by a CIF induced by the former family of winning coalitions (where all coalitions are losing) whereas the CIF induced from latter family of winning coalition (where all coalitions are winning) always qualifies i.

Adding new axioms will impose some particular structures over winning coalitions. So we start by introducing a monotonicity axiom.

[28] In words, if a coalition K is not winning, then K is not blocking as well.

[29] Note that a collection $\{\varpi(i)\}_{i\in N}$ of blocking coalitions induces a (unique) CIF F in the same natural way. The choice does not matter in the sense that one can construct similar result that we obtain by defining CIFs with respect to winning coalitions. So without loss of generality, we choose defining elementary CIFs in terms of winning coalitions.

[30] See Axiom 3.14. This independence axiom is also used by Samet and Schmeidler (2003).

[31] See Axiom 3.19.

Axiom 4.2 *A CIF $F \in \mathcal{F}$ is said to be monotonic* (M) *if given any $i \in N$ and any two profiles $G, H \in \Gamma$ such that*

- $G_j = H_j$ *or* $G_j = H_j \cup \{i\}$ *for all* $j \in N$ *and*
- $\exists k \in N$ *such that* $i \notin H_k$ *but* $G_k = H_k \cup \{i\}$

we have $i \in F(H) \implies i \in F(G)$.

The mononicity condition is quite natural in social choice literature, stating in terms of winning coalitions that if a coalition is winning over an individual, additional members to that coalition can not make the new coalition losing. In general, monotonicity requires that additional opinion about a qualified individual in favour of his qualification can not cause unqualification of that individual in the new opinion profile.[32] For an elementary CIF, monotonicity imposes a particular structure over collection of winning coalitions such that for all $i \in N$ and for all $K, K' \subseteq N$ with $K \subseteq K'$, $K \in \omega(i)$ implies $K' \in \omega(i)$ which is the condition that basic collection of winning coalitions satisfy. Hence monotonic elementary CIFs are basic CIFs. Moreover all basic CIFs satisfy independence (hence elementary) and monotonicity. As a result, basic CIFs[33] which are induced by basic collections of winning coalitions can be characterized with independence and monotonicity.[34]

Remark 4.1 *For a basic CIF F which is induced by the basic collection of winning coalition $\{\omega(i)\}_{i \in N}$, we have either $\omega(i) = \varnothing$ or $N \in \omega(i)$ for all $i \in N$.*

The remark above states the fact that monotonicity does not guarantee an individual's qualification in at least one profile. But it requires that if a coalition is able to qualify an individual, so grand coalition also has power to qualify that individual. We

[32] Let us note that M and *MON-KR* (see Axiom 3.3) are logically equivalent. On the other hand, *MON-SS* (see Axiom 3.15) is logically stronger than our monotonicity. To see why *MON-SS* implies M, observe that the profiles $G, H \in \Gamma$ in the definition of M satisfy $H_j \subseteq G_j$ for all $j \in N$, hence by *MON-SS*, we have $F(H) \subseteq F(G)$. As $i \in F(H)$ is assumed, we have $i \in F(G)$. To see why converse implication may fail, consider the society $N = \{1, 2\}$ and the CIF $F \in \mathcal{F}$ defined for all $i \in N$ and for all $G \in \Gamma$ as $i \in F(G) \iff i \in G_j$ for all $j \in N$ with $j \in G_i$. F satisfies M while violates *MON-SS*. However under independence (See Axiom 4.1), our monotonicity and Samet and Schmeidler's monotonicity turn out to be equivalent.

We also wish to mention that Samet and Schmeidler offer a global version of monotonicity in the sense that both our and Kasher and Rubinstein's monotonicity axioms are defined for a specific individual i (hence local versions), whereas Samet and Schmeidler define monotonicity over sets.

[33] Taylor and Zwicker (1999) call aggregation rules that are induced from a basic collection of winning coalitions as "simple" rather than basic.

[34] One can define a minimal collection of winning coalitions which consists of coalitions all of whose proper subsets are losing for all individuals. Because of monotonicity, basic CIFs can be represented with their minimal collection of winning coalitions.

should also note that monotonicty does not impose a requirement for a coalition to be winning or blocking, it only imposes if a coalition is winning then its all supersets are also winning and it gives possibility to a coalition to be both winning and blocking.

We now introduce three versions of self-duality axioms which are related with the structures of winning and blocking coalitions. To motivate the axioms, we refer the consent rules of Samet and Schmeidler and their suggestions of appropriate rules for the situations: The qualification of individuals having right to drive in public domain and the qualification of individuals who are imposed a duty or obligation if they are qualified. In the former case, Samet and Schmeidler propose a consent rule F^{st} with $s > t$ and in the latter case, they propose F^{st} with $s < t$.[35] Note that consent rules satisfy independence, hence can be represented via collection of winning coalitions.[36] Thus for $N = \{1, 2, 3\}$, consider the rules $F^{2,1}$ and $F^{1,2}$. As an example, we write family of winning coalitions of individual 1. In the former case, we have $\omega(1) = \{\{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$ and $\overline{\omega}(1) = \{\{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. The collection of winning coalitions that induces $F^{2,1}$ is proper but not strong since $\{1\} \notin \omega(1)$ and $\{2, 3\} \notin \omega(1)$. In the latter case, we have $\omega(1) = \{\{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ and $\overline{\omega}(1) = \{\{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}$. The collection of winning coalitions that induces $F^{1,2}$ is strong but not proper since $\{1\} \in \omega(1)$ and $\{2, 3\} \in \omega(1)$.

At that point, we should mention Taylor and Zwicker's observation about strongness and properness. After translating to our model, Taylor and Zwicker states:

> If a collection of winning coalitions is not strong, then it has too few winning coalitions at some individuals' families of winning coalitions in the sense that adding sufficiently many winning coalitions will make collection of winning coalitions strong (and the addition of winning coalitions can never destroy strongness). On the other hand, if a collection of winning coalitions is not proper, then it has too many winning coalitions at some individuals' families of winning coalitions in the sense that deleting sufficiently many winning coalitions will make collection of winning coalitions proper (and the deletion of winning coalitions can never destroy

[35] For details, one can refer Section 3.3 or Samet and Schmeidler (2003).

[36] In addition, consent rules satisfies our monotonicity. Although our monotonicty and Samet and Schmeidler's monotonicity differ, under independence they are equivalent. See footnote 32. Hence, in fact, consent rules can be represented via minimal collection of winning coalitions. See footnote 34.

properness).

At first glance, it may be thought that there are two type of coalitions; winning and losing. The intiution behind is that if a coalition is winning over an individual, it is interpreted as having power to decide that individual's social decision, otherwise it is said to be losing. So this distinction implicitly assumes if a coalition is winning over an individual, this coalition has power to determine the individual's social decision as both qualified and unqualified. But we introduced the families of blocking coalitions which make possible to discriminate a coalition's power on an individual's social decision as having power to qualify and having power to unqualify. Hence a coalition may socially qualify an individual but can not be able to unqualify that individual, that is the coalition is winning but not blocking. The examples above show this discrimination is quite natural. For example, for consent rule $F^{2,1}$, the coalitions $\{1\}$ and $\{2,3\}$ are not winning but blocking whereas for consent rule $F^{1,2}$, the coalitions $\{1\}$ and $\{2,3\}$ are winning but not blocking. Observe that the family of winning coalitions of individual 1 for consent rule $F^{2,1}$ coincides with the family of blocking coalitions of 1 for consent rule $F^{1,2}$ and vice versa. The reason is the symmetricity of quotas of two rules. We now introduce two self duality axioms which lead us to characterization of proper and strong CIFs.

Axiom 4.3 *A CIF $F \in \mathcal{F}$ is said to satisfy self-duality positively* (SD$^+$) *if for any $i \in N$ and for any $G \in \Gamma$ with $i \in F(G)$, we have $i \notin F\left(\overline{G}\right)$.*

Axiom 4.4 *A CIF $F \in \mathcal{F}$ is said to satisfy self-duality negatively* (SD$^-$) *if for any $i \in N$ and for any $G \in \Gamma$ with $i \notin F(G)$, we have $i \in F\left(\overline{G}\right)$.*

For a basic CIF, SD$^+$ impose a particular structure over families of winning coalitions such that for all $i \in N$ and for all $K \subseteq N$, $K \in \omega(i)$ implies $\overline{K} \notin \omega(i)$ which is the condition that proper collection of winning coalitions satisfies and SD$^-$ impose a particular structure such that for all $i \in N$ and for all $K \subseteq N$, $K \notin \omega(i)$ implies $\overline{K} \in \omega(i)$ which is the condition that strong collection of winning coalitions satisfies. In addition all proper CIFs satisfy SD$^+$ and all strong CIFs satisify SD$^-$. Hence, a CIF is proper if and only if it satisfies independence, monotonicity and self-duality positively and a CIF is strong if and only if it satisfies independence, monotonicity and self-duality negatively.

Before combining these two self-duality axioms, we should note that a CIF is proper if and only if the grand coalition N can not be partioned into two disjoing winning coalitions and a CIF is strong if and only if the grand coalition N can not be partitioned into two disjoint losing coalitions.[37] This observation is stated in the following remark.

Remark 4.2 *For a basic CIF F, SD$^+$ impose a particular structure over families of winning coalitions such that for any $i \in N$ and for any $K, K' \in \omega(i)$, we have $K \cap K' \neq \varnothing$ and SD$^-$ impose a particular structure over families of winning coalitions such that for any $i \in N$ and for any $K, K' \notin \omega(i)$, we have $K \cap K' \neq \varnothing$.*

We now combine two self-duality axioms in one definition.

Axiom 4.5 *A CIF $F \in \mathcal{F}$ is said to satisfy self-duality* (SD) *if it satifies self-duality both positively and negatively i.e. for any $i \in N$ and for any $G \in \Gamma$, we have $i \in F(G)$ if and only if $i \notin F\left(\overline{G}\right)$.*

Self-duality axiom is the combination of the former two self-duality axioms.[38] Self-duality axioms neutralize the distinction between collection of winning and blocking coalitions. For an elementary CIF F, self-duality requires the coincidence of families of winning and blocking coalitions for all individuals. More formally, for all $i \in N$ and for all $K \subseteq N$, we have $K \in \omega(i)$ if and only if $K \in \overline{\omega}(i)$. To see this, let K is winning over an individual $i \in N$. Then by self-duality, $N \setminus K$ is losing which requires K is also blocking over individual i by definition of blocking coalitions. Now let K is not winning over an individual $i \in N$. Then by self-duality, $N \setminus K$ is winning which requires K can not be blocking over individual i by definition of blocking coalitions. Thus for a self-dual elementary CIF, if a coalition is winning over an individual, than this coalition has the power to determine that individual's social decision as a member by unanimously qualifying him and as non-member by unanimously unqualifying him.

We now state and prove all characterizations mentioned above.

[37] Because of monotonicity, properness excludes the possibility of disjoint winning coalitions and strongness excludes the possibility of disjoint blocking coalitions (regardless of whether their union is N or not).

[38] Observe that Samet and Schmeidler offer a global version of self-duality (See Axiom 3.18) as in the case of monotonicity. See footnote 32. However in the case of self-duality, our self-duality *(SD)* and Samet and Schmeidler's self-duality *(SD-SS)* axioms are logically equivalent even under absence of independence.

To see why SD implies SD-SS, take any $G \in \Gamma$ and consider $\overline{G}$. Let $F(G) = M$. By SD, we have $i \notin F\left(\overline{G}\right)$ for all $i \in M$. Hence $F\left(\overline{G}\right) \subseteq N \setminus M$. Suppose for a contradiction that $F\left(\overline{G}\right) \subset N \setminus M$, that is, there exist $j \in N \setminus M$ such that $j \notin F\left(\overline{G}\right)$. But by SD, we have $j \in F(G)$, that is, $j \in M$ which leads a contradiction. Thus we have $F\left(\overline{G}\right) = N \setminus M$ showing the desired implication.

To see why SD-SS implies SS, take any $i \in N$ and any $G \in \Gamma$. We need to show that 1) $i \in F(G) \implies i \notin F\left(\overline{G}\right)$ and 2) $i \notin F\left(\overline{G}\right) \implies i \in F(G)$. Consider 1, by SD-SS, we have $F\left(\overline{G}\right) = N \setminus F(G)$, hence if $i \in F(G)$, then $i \notin F\left(\overline{G}\right)$. Now consider 2, if $i \notin F\left(\overline{G}\right)$, then by SD-SS, we have $i \in F(G)$.

Theorem 4.1 *Let $F \in \mathcal{F}$ is a CIF. Then*

1. *F is elementary if and only if it satisfies independence.*
2. *F is basic if and only if it satisfies independence and monotonicity.*
3. *F is proper if and only if it satisfies independence, monotonicity and self-duality positively.*
4. *F is strong if and only if it satisfies independence, monotonicity and self-duality negatively.*
5. *F is simple if and only if it satisfies independence, monotonicity and self-duality.*

Moreover all axioms are independent.

Proof. Before starting the proof, let for some $K \subseteq N$ and for some $i \in N$ define $\Gamma^{K,i} = \{G \in \Gamma : i \in G_k$ for all $k \in K$ and $i \notin G_k$ for all $k \in N \setminus K\}$ as the set of profiles where i is qualified by only the members of $K \subseteq N$.

1. To see "if" part, take any CIF F which satisfies independence and take any individual $i \in N$ and any coalition $K \subseteq N$. As F satisifies I, for all $G \in \Gamma^{K,i}$ we have either $i \in F(G)$ or $i \notin F(G)$. Let $\omega(i) = \left\{K \subseteq N : i \in F(G) \text{ for all } G \in \Gamma^{K,i}\right\}$. By I, $\omega(i)$ is well-defined. Applying this argument for all individuals gives a collection $\{\omega(i)\}_{i \in N}$ of winning coalitions which induces F. To see "only if" part, let take any elementary CIF F and let $\{\omega(i)\}_{i \in N}$ be the family of winning coalitions which induces F. Take any $i \in N$ and any $G, H \in \Gamma$ with $G_j\,|_i = H_j\,|_i$ for all $j \in N$. Hence $\{j \in N : i \in G_j\} = \{j \in N : i \in H_j\}$. Without loss of generality, suppose for a contradiction that $i \in F(G)$ but $i \notin F(H)$. As $i \in F(G)$, we have $\{j \in N : i \in G_j\} \in \omega(i)$, so $\{j \in N : i \in H_j\} \in \omega(i)$. But this implies that $i \in F(H)$ which contradicts with $i \notin F(H)$, establishing the independence of F.

2. To see "if" part, take any CIF F which satisfies independence and monotonicity. By (1), there is a collection of $\{\omega(i)\}_{i \in N}$ that induces F. We will show that $\{\omega(i)\}_{i \in N}$ is a basic collection of winning coalitions. Thus take any individual $i \in N$ and any $K \subseteq K' \subseteq N$ with $K \in \omega(i)$. Suppose for a contradiction $K' \notin \omega(i)$, that is, $i \notin F(G)$ for all $G \in \Gamma^{K',i}$. For any given $G \in \Gamma^{K',i}$, let $H \in \Gamma^{K,i}$ be a profile such that $H_j = G_j$ for all $j \in K \cup (N \setminus K')$ and

$H_j = G_j \setminus \{i\}$ for all $j \in K' \setminus K$. Note that $K = \{j \in N : i \in H_j\} \in \omega(i)$ implies $i \in F(H)$. By monotonicity of F, we have $i \in F(G)$ showing the collection of winning coalitions is basic. To see "only if" part, take any basic CIF. By (1), it satisfies independence. To see F also satisfies monotonicity, take any $i \in N$ and any $G, H \in \Gamma$ such that $G_j = H_j$ or $G_j = H_j \cup \{i\}$ for all $j \in N$ and $\exists k \in N$ such that $i \notin H_k$ but $G_k = H_k \cup \{i\}$. Let assume $i \in F(H)$. Hence $\{j \in N : i \in H_j\} \in \omega(i)$. Moreover $\{j \in N : i \in H_j\} \subseteq \{j \in N : i \in G_j\}$ by construction. As F is basic, then we have $\{j \in N : i \in G_j\} \in \omega(i)$ implying $i \in F(G)$ which establishes monotonicity of F.

3. To see "if" part, take any CIF F which satisfies independence, monotonicity and self-duality positively. By (2), we have F is induced by a basic collection of winning coalitions $\{\omega(i)\}_{i \in N}$. It is enough to show that it satisfies the condition: For all $i \in N$ and for all $K \subseteq N$, $K \in \omega(i)$ implies $\overline{K} \notin \omega(i)$. Thus take any $i \in N$ and any $K \in \omega(i)$ and assume for a contradiction $\overline{K} \in \omega(i)$. As $K \in \omega(i)$, we have $i \in F(G)$ for all $G \in \Gamma^{K,i}$. For each choice of $G \in \Gamma^{K,i}$, we have $i \notin F\left(\overline{G}\right)$ by SD$^+$. But by independence, we have $i \notin F\left(\overline{G}\right)$ for all $\overline{G} \in \Gamma^{\overline{K},i}$ which contradicts the assumption $\overline{K} \in \omega(i)$. To see "only if" part, take any proper CIF F which is induced from a proper collection of winning coalitions $\{\omega(i)\}_{i \in N}$. By (2), F satisfies independence and monotonicity. To see F also satisfies SD$^+$, take any $i \in N$ and any $G \in \Gamma$ with $i \in F(G)$ and let $\overline{G} \in \Gamma$ be the complement of G. Hence $\{j \in N : i \in G_j\} \in \omega(i)$ and as F is proper we have $N \setminus \{j \in N : i \in G_j\} \notin \omega(i)$. But $N \setminus \{j \in N : i \in G_j\} = \left\{j \in N : i \in \overline{G}_j\right\} \notin \omega(i)$ implies $i \notin F\left(\overline{G}\right)$ showing F satisfies self-duality positively.

4. To see "if" part, take any CIF F which satisfies independence, monotonicity and self-duality negatively. By (2), we have F is induced by a basic collection of winning coalitions $\{\omega(i)\}_{i \in N}$. It is enough to show that it satisfies the condition: For all $i \in N$ and for all $K \subseteq N$, $K \notin \omega(i)$ implies $\overline{K} \in \omega(i)$. Thus take any $i \in N$ and any $K \notin \omega(i)$ and assume for a contradiction $\overline{K} \notin \omega(i)$. As $K \notin \omega(i)$, we have $i \notin F(G)$ for all $G \in \Gamma^{K,i}$. For each choice of $G \in \Gamma^{K,i}$, we have $i \in F\left(\overline{G}\right)$ by SD$^-$. But by independence, we have $i \in F\left(\overline{G}\right)$ for all $\overline{G} \in \Gamma^{\overline{K},i}$ which contradicts the assumption $\overline{K} \notin \omega(i)$. To see "only if" part, take any strong CIF F which is induced from a strong collection of winning coalitions $\{\omega(i)\}_{i \in N}$. By (2), F satisfies independence and monotonicity. To see F also

satisfies SD$^-$, take any $i \in N$ and any $G \in \Gamma$ with $i \notin F(G)$ and let $\overline{G} \in \Gamma$ be the complement of G. Hence $\{j \in N : i \in G_j\} \notin \omega(i)$ and as F is strong we have $N \setminus \{j \in N : i \in G_j\} \in \omega(i)$. But $N \setminus \{j \in N : i \in G_j\} = \left\{j \in N : i \in \overline{G}_j\right\} \in \omega(i)$ implies $i \in F\left(\overline{G}\right)$ showing F satisfies self-duality negatively.

5. To see "if" part, take any CIF F which satisfies independence, monotonicity and self-duality. By (2), we have F is induced by a basic collection of winning coalitions $\{\omega(i)\}_{i \in N}$. As SD implies both SD$^+$ and SD$^-$, for all $i \in N$ and for all $K \subseteq N$, we have $K \in \omega(i)$ if and only if $\overline{K} \notin \omega(i)$ by (3) and (4) showing $\{\omega(i)\}_{i \in N}$ is self-dual. To see "only if" part, take any simple CIF F which is induced from a self-dual collection of winning coalitions $\{\omega(i)\}_{i \in N}$. By (2), F satisfies independence and monotonicty. By (3) and (4), F satisfies both SD$^+$ and SD$^-$ respectively, establishes self-duality of F.

To establish the logical independence of independence, monotonicity and self-duality (positive) and self-duality (negative), one can consider consent rules $F^{2,1}$ and $F^{1,2}$ for $N = \{1, 2, 3\}$. As Samet and Schmeidler's and our monotonicities coincide for independent CIFs, both consent rules satisfy independence and monotonicty, but $F^{2,1}$ violates SD$^-$ and $F^{1,2}$ violates SD$^+$. Moreover, both rules violate SD which is the fact that shows independence and monotonicity do not imply self-duality.

In addition, the CIF F which is defined for each $G \in \Gamma$ as $F(G) = N \setminus \{i \in N : i \in G_i\}$ satisfies self-duality and independence but not monotonicity. Finally, to see that the conjunction of self-duality and monotonicity does not imply independence, consider the CIF F defined as follows: At each $G \in \Gamma$ and for all $i \in N$,

- if $i \in G_i$, then $i \in F(G) \iff \#\{j \in N \setminus \{i\} : j \in G_i\} \leq \frac{n-1}{2}$
- if $i \notin G_i$, then $i \notin F(G) \iff \#\{j \in N \setminus \{i\} : j \notin G_i\} \leq \frac{n-1}{2}$

Note that, as SD implies both SD$^+$ and SD$^-$, two examples above can be borrowed for logical independence of independence, monotonicty and SD$^+$ or SD$^-$. ■

To summarize our findings, we first show that all independent CIFs can be expressed in terms of winning coalitions. Adding monotonicity brings a structure on collection of winning coalitions such that if a coalition is winning over an individual, then its all supersets are also winning over that individual. Then we introduce two self-duality axioms, SD$^+$ and SD$^-$ where they make collection of winning coalitions proper and strong respectively. For a CIF induced by a proper collection of winning coalitions, if a coalition is winning over an individual, then it is also losing over that individual and for a CIF induced by a strong collection of winning coalitions, if a coalition is not winning over an individual, then it is not blocking either. However, neither properness nor strongness do not nullify the discrimination between winning and blocking coalitions. Combining two self-duality axioms into one self-duality remove the discrimination above. Thus in the remaining of this section, we concentrate on simple CIFs, whereas we mention elementary, basic, proper and strong CIFs whenever we have specific results for them. Before that we extend our analysis to certain CIFs of the literature:

1. The *liberal CIF* $L \in \mathcal{F}$ introduced by Kasher and Rubinstein (1997) is defined for each $G \in \Gamma$ as $L(G) = \{i \in N : i \in G_i\}$. L satisfies MON, SD and I - hence by Theorem 4.1 is a simple CIF. In fact, its corresponding collection of winning coalitions is defined for each $i \in N$ as $\omega(i) = \{K \in 2^N : i \in K\}$.

2. The *consensus CIF* $C \in \mathcal{F}$ which can be found in Kasher and Rubinstein (1997) is defined for each $G \in \Gamma$ as $C(G) = \{i \in N : i \in G_j \text{ for all } j \in N\}$. Although C satisfies MON, I and SD$^+$, it fails SD - hence by Theorem 4.1 is a proper CIF. In fact, its corresponding collection of winning coalitions is defined for each $i \in N$ as $\omega(i) = N$.

3. The *dictatorial* CIF $F_d \in \mathcal{F}$ (where some $d \in N$ is the dictator) which can also be found in Kasher and Rubinstein (1997) is defined for each $G \in \Gamma$ as $F_d(G) = G_d$. F_d satisfies MON, SD and I - hence by Theorem 4.1 is a simple CIF. Its corresponding collection of winning coalitions is defined for each $i \in N$ as $\omega(i) = \{K \in 2^N : d \in K\}$.

4. Let n be odd. The *majoritarian CIF* $M \in \mathcal{F}$ which can be found in Samet and Schmeidler (2003) is defined as follows: For each $G \in \Gamma$ and each $i \in N$ we have $i \in M(G)$ if and only if $\#\{j \in N : i \in G_j\} > \frac{n}{2}$. Again M satisfies MON, SD and I - hence by Theorem 4.1 is a simple CIF. Its corresponding collection of winning coalitions is defined for each $i \in N$ as $\omega(i) = \{K \in 2^N : \#K > \frac{n}{2}\}$.

5. Procedural CIFs: Take any CIF $F^0 \in \mathcal{F}$. For any $G \in \Gamma$ and any non-negative integer k let $F^{k+1}(G) = F^k(G) \cup \{j \in N : j \in G_i \text{ for some } i \in F^k(G)\}$. Let $\overline{k}$ be the smallest integer for which $F^{\overline{k}+1}(G) = F^{\overline{k}}(G)$. Define the CIF $F^P \in \mathcal{F}$ as $F^P(G) = F^{\overline{k}}(G)$ for each $G \in \Gamma$. We call F^P the *procedural CIF based on* F^0.

The consensus-start-respecting procedure proposed by Kasher (1993) and the liberal-start-respecting procedure proposed by Dimitrov, Sung, and Xu (2004) are particular procedural CIFs based on the consensus CIF C and the liberal CIF L respectively. The consensus-start-respecting procedure, which is based on a non-simple CIF, fails independence and self-duality. But this is also the case for the liberal-start-respecting procedure which is based on a simple CIF. In fact, this incompatibility between procedural and simple CIFs is more general, as announced by the following proposition:

Proposition 4.1 *Take any simple CIF $F^0 \in \mathcal{F}$. The procedural CIF F^P based on F^0 fails independence and self-duality.*

Proof. Let F^0 and F^P be as in the statement of the proposition. Let $\{\omega(i)\}_{i\in N}$ be the family of winning coalitions of the simple CIF F^0. We first show that F^P fails independence. Take any $i \in N$ and any $K \in \omega(i)$ which differs from N and $N\backslash\{i\}$. Consider the profile $G \in \Gamma$ where $G_k = N\backslash\{i\}$ for all $k \in K$ and $G_k = N$ for all $k \in N\backslash K$. So $F^0(G) = N\backslash\{i\}$ and, as K differs from $N\backslash\{i\}$ and N, $F^P(G) = N$. Now consider the profile $H \in \Gamma$ where $H_k = \emptyset$ for all $k \in K$ and $H_k = \{i\}$ for all $k \in N\backslash K$. As $N\backslash K \notin \omega(i)$, we have $F^0(H) = \emptyset = F^P(H)$. Remark that $i \in G_k \Leftrightarrow i \in H_k$ for all $k \in N$, while $i \in F^P(G)$ but $i \notin F^P(H)$, showing that F^P fails independence.

To see that F^P fails self-duality, take any $i \in N$ and any $K \in \omega(i)$ which differs from $N\backslash\{i\}$ and N. Consider the profile $G \in \Gamma$ where $G_k = N\backslash\{i\}$ for all $k \in K$ and $G_k = N$ for all $k \in N\backslash K$. So $F^0(G) = N\backslash\{i\}$ and, as K differs from $N\backslash\{i\}$ and N, $F^P(G) = N$. Now consider the profile $H \in \Gamma$ where $H_k = \{i\}$ for all $k \in K$ and $H_k = \emptyset$ for all $k \in N\backslash K$. As $K \in \omega(i)$, we have $F^0(H) = \{i\} = F^P(H)$. Remark that $H_k = N \setminus G_k$ for all $k \in N\backslash K$, while $i \in F^P(G) \cap F^P(H)$, showing that F^P fails self-duality. ■

6. The consent rules of Samet and Schmeidler (2003) are parametrized by two positive integers s and t with $s+t \leq n+2$. *A consent rule (with consent quotas s and t)* is a CIF $F^{st} \in \mathcal{F}$ such that given any $G \in \Gamma$ and any $i \in N$

- if $i \in G_i$, then $i \in F^{st}(G) \Longleftrightarrow \#\{j \in N : i \in G_j\} \geq s$
- if $i \notin G_i$, then $i \notin F^{st}(G) \Longleftrightarrow \#\{j \in N : i \notin G_j\} \geq t$

Taking $s = t$ is a case of particular interest where we call F^{st} a *symmetric consent rule (with quota s)* and denote it F^s. Remark that for symmetric consent rules, the quota varies between $s = 1$ and $s = \lfloor \frac{n}{2} \rfloor + 1$.[39] At one extreme where $s = 1$, F^s coincides with the liberal CIF L. At the other extreme where $s = \lfloor \frac{n}{2} \rfloor + 1$, we go to majoritarianism.[40]

Not every consent rule is simple. In fact, the intersection of the set of consent rules with the set of simple CIFs is the set of symmetric consent rules - a result which we formally state in the following proposition:

Proposition 4.2 *A consent rule $F^{st} \in \mathcal{F}$ is a simple CIF if and only if F^{st} is a symmetric consent rule, i.e., $s = t$.*

Proof. To prove the "only if" part, we refer to Proposition 2 of Samet and Schmeidler (2003) which establishes that a consent rule satisfies self-duality if and only if it is a symmetric consent rule. This result, combined with our Theorem 4.1, implies that a consent rule is a simple CIF only if it is a symmetric consent rule. To show the "if" part, we check that symmetric consent rules satisfy MON, SD and I[41] -hence are simple CIFs by Theorem 4.1. ■

Remark 4.3 *When n is odd while $s \in \{1, ..., \lfloor \frac{n}{2} \rfloor + 1\}$ or n is even while $s \in \{1, ..., \frac{n}{2}\}$, the winning coalitions of the symmetric consent rule F^s are defined for every $i \in N$ as $\omega(i) = \{K \in 2^N : i \in K \text{ and } s \leq \#K \leq n - s\} \cup \{K \in 2^N : \#K > n - s\}$. So $\omega(i)$ consists of coalitions*

- whose cardinality varies from s to $n - s$ while they contain i[42]

- whose cardinality exceeds $n - s$ (independent of whether they contain i or not).

On the other hand, when n is even and $s = \frac{n}{2} + 1$, we have $\omega(i) = \{K \in 2^N : i \notin K$ and $\#K = n/2\} \cup \{K \in 2^N : \#K > \frac{n}{2}\}$ for every $i \in N$.

In addition, the consent rules with different quotas are not simple but they intersect with proper and strong CIFs. In details, all consent rules F^{st} with $s \geq t$ are proper CIFs and all consent rules F^{st} with $s \leq t$ are strong CIFs. This is formally stated in the following proposition.

[39] We write $\lfloor n/2 \rfloor$ for the highest integer less than or equal to $n/2$.

[40] This has two subcases which is worth distinguishing. When n is odd, $F^{\lfloor n/2 \rfloor + 1}$ coincides with the majoritarian CIF M. When n is even, we have two versions of majoritarianism depending on the choice of $s \in \{\frac{n}{2}, \frac{n}{2} + 1\}$. When $s = n/2$, a coalition K of cardinality $n/2$ is winning over an individual i if and only if i is a member of K. On the other hand, when $s = \frac{n}{2} + 1$, a coalition K of cardinality $n/2$ is winning over an individual i if and only if i is not a member of K.

[41] A result which is also established by Theorem 2 of Samet and Schmeidler (2003).

[42] Remark that there is no such coalition when n is odd and $s = \lfloor \frac{n}{2} \rfloor + 1$.

Proposition 4.3 *A consent rule $F^{st} \in \mathcal{F}$ is a proper CIF if and only if $s \geq t$ and a consent rule $F^{st} \in \mathcal{F}$ is a strong CIF if and only if $s \leq t$.*

Proof. To see "if" part of the first statement, we refer to Theorem 1 of Samet and Schmeidler (2003) which establishes all consent rules satisfies independence and monotonicity. Hence all consent rules F^{st} with $s \geq t$ are basic CIFs by our Theorem 4.1 and can be represented with a basic collection of winning coalitions. Thus let $\{\omega(i)\}_{i \in N}$ is the basic collection of winning coalitions that induces F^{st} with $s \geq t$. We need to show that $\{\omega(i)\}_{i \in N}$ be a proper collection of winning coalitions. Note that if $K \in \omega(i)$ for some $i \in N$, we have either $\#K \geq s$ and $i \in K$ or $\#K > n - t$ while $i \notin K$. Consider the case 1, we have $\#\overline{K} \leq n - s$ and $i \notin \overline{K}$. As $s \geq t$, we have $\#\overline{K} \leq n - t$ implying $\overline{K} \notin \omega(i)$. Consider the case 2, we have $\#\overline{K} < t$ and $i \in \overline{K}$. As $s \geq t$, we have $\#\overline{K} < s$ implying $\overline{K} \notin \omega(i)$.

To see "only if" part of the first statement, let F^{st} be a proper consent rule and let $\{\omega(i)\}_{i \in N}$ be the proper collection of winning coalitions that induces F^{st}. Suppose for a contradiction that $s < t$. Let for some $i \in N$, $K \subseteq N$ be a coalition such that $\#K = s$ and $i \in K$. As F^{st} is a consent rule, we have $K \in \omega(i)$. Moreover $\#\overline{K} = n - s$ and $i \notin \overline{K}$. As we assume $s < t$, we have $\#\overline{K} > n - t$ implying $\overline{K} \in \omega(i)$ contradicting $\{\omega(i)\}_{i \in N}$ is a proper collection of winning coalitions.

To see "if" part of the second statement, we refer to Theorem 1 of Samet and Schmeidler (2003) which establishes all consent rules satisfies independence and monotonicity. Hence all consent rules F^{st} with $s \leq t$ are basic CIFs by our Theorem 4.1 and can be represented with a basic collection of winning coalitions. Thus let $\{\omega(i)\}_{i \in N}$ be the basic collection of winning coalitions that induces F^{st} with $s \leq t$. We need to show that $\{\omega(i)\}_{i \in N}$ is a strong collection of winning coalitions. Note that if $K \notin \omega(i)$ for some $i \in N$, we have either $\#K < s$ and $i \in K$ or $\#K \leq n - t$ while $i \notin K$. Consider the case 1, we have $\#\overline{K} > n - s$ and $i \notin \overline{K}$. As $s \leq t$, we have $\#\overline{K} > n - t$ implying $\overline{K} \in \omega(i)$. Consider the case 2, we have $\#\overline{K} \geq t$ and $i \in \overline{K}$. As $s \leq t$, we have $\#\overline{K} \geq s$ implying $\overline{K} \in \omega(i)$.

To see "only if" part of the first statement, let F^{st} be a strong consent rule and let $\{\omega(i)\}_{i \in N}$ be the strong collection of winning coalitions that induces F^{st}. Suppose for a contradiction that $s > t$. Let for some $i \in N$, $K \subseteq N$ be a coalition such that $\#K = s - 1$ and $i \in K$. As F^{st} is a consent rule, we have $K \notin \omega(i)$. Moreover $\#\overline{K} = n - s + 1$ and $i \notin \overline{K}$. As we assume $s > t$, we have $\#\overline{K} < n - t + 1$ implying

$\overline{K} \notin \omega(i)$ contradicting $\{\omega(i)\}_{i \in N}$ is a strong collection of winning coalitions. ■

Note that Proposition 4.2 is a corollary of Proposition 4.3. Since all simple CIFs, by Theorem 4.1, are both proper and strong, by Proposition 4.3, we have $s = t$. Moreover, all symmetric consent rules are proper and strong by Proposition 4.3, hence simple by Theorem 4.1.

4.2 Equal Treatment of Individuals

The literature of social choice theory contains two well-known equal treatment conditions, one for voters (usually called "anonymity") and one for outcomes (usually called "neutrality"). In a framework where voters and alternatives form mutually exclusive sets, the conceptual discrimination between these two conditions is straightforward. On the other hand, the matter is more complicated to handle when voters and outcomes coincide - as is the case in our model.

Samet and Schmeidler (2003) by-pass the problem by introducing a "symmetry" condition which incorporates both kinds of equal treatment conditions.[43] We say that a CIF $F \in \mathcal{F}$ is *Samet-Schmeidler symmetric* if and only if given any permutation $\Pi : N \to N$, any $G \in \Gamma$ and any $i \in N$, we have $i \in F(G) \Leftrightarrow \Pi(i) \in F(\Pi(G))$.

Remark 4.4 *For a simple CIF F, Samet-Schmeidler symmetry imposes a particular structure over the family of winning coalitions $\{\omega(i)\}_{i \in N}$ so that given any $\Pi : N \to N$, any $i \in N$ and any $K \in 2^N$, we have $K \in \omega(i) \Leftrightarrow \Pi(K) \in \omega(\Pi(i))$.*

It is possible to extract from Samet-Schmeidler symmetry, a voters' equal treatment property which requires that while deciding whether some individual $i \in N$ is a J, all individuals, with the possible exception of i him/herself, must be equally treated. For some $i \in N$, let $\Pi^{-i} : N \to N$ stand for some bijection with $\Pi^{-i}(i) = i$. Then the formal definition of anonimity is given below.

Axiom 4.6 *A CIF $F \in \mathcal{F}$ is anonymous if for all $i \in N$ and for all $G \in \Gamma$, we have $i \in F(G) \Leftrightarrow \Pi^{-i}(i) = i \in F(\Pi^{-i}(G))$.*

Clearly, Samet-Schmeidler symmetry implies anonymity.[44]

Remark 4.5 *For a simple CIF F, anonymity imposes a particular structure over the family of winning coalitions $\{\omega(i)\}_{i \in N}$ so that for each $i \in N$ and for all $K \in 2^N$,*

[43] See Section 4.1 of Samet and Schmeidler (2003) where they discuss their symmetry axiom.

[44] While the converse implication does not hold as we show through an example in Footnote 45

$K \in \omega(i) \Leftrightarrow \Pi^{-i}(K) \in \omega(i)$. *In other words, a coalition K, which does not contain individual i, is winning over i if and only if every coalition K' with $\#K' = \#K$ and which does not contain i is winning over i as well. Similarly, a coalition K, which contains i, is winning over i if and only if every coalition K' with $\#K' = \#K$ and which contains i is winning over i as well.*

Anonymous simple CIFs can be characterized in terms of what we call generalized symmetric consent rules. Fix some n-tuple of positive integers $\overline{s} = (s_1, ..., s_n)$ such that $s_i \in \{1, ..., \lfloor \frac{n}{2} \rfloor + 1\}$ for each $i \in N$. *A generalized symmetric consent rule (with quota* $\overline{s}$) is a CIF $F^{\overline{s}} \in \mathcal{F}$ such that given any $G \in \Gamma$ and any $i \in N$

if $i \in G_i$, then $i \in F^{\overline{s}}(G) \Longleftrightarrow \# \{j \in N : i \in G_j\} \geq s_i$

if $i \notin G_i$, then $i \notin F^{\overline{s}}(G) \Longleftrightarrow \# \{j \in N : i \notin G_j\} \geq s_i$

Note that symmetric consent rules of Samet and Schmeidler (2003) are particular cases of $F^{\overline{s}}$ where $\overline{s}$ is such that $s_i = s_j$ for all $i, j \in N$.[45]

Theorem 4.2 *A CIF $F \in \mathcal{F}$ is simple and anonymous if and only if F is a generalized symmetric consent rule $F^{\overline{s}}$ with $s_i \in \{1, ..., \lfloor \frac{n}{2} \rfloor + 1\}$ for each $i \in N$.*

Proof. We leave the "if" part to the reader by asking to check that any generalized symmetric consent rule $F^{\overline{s}}$ satisfies anonymity, MON, SD and I, which, by Theorem 4.1, implies that $F^{\overline{s}}$ is anonymous and simple. To show the "only if" part, take any simple and anonymous CIF $F \in \mathcal{F}$ with a family of winning coalitions $\{\omega(i)\}_{i \in N}$. For each $i \in N$, let $s_i = \min \{\#K : K \in w(i)\}$. First, consider the case where n is odd. As, by the definition of a winning coalition, either $K \in \omega(i)$ or $N \setminus K \in \omega(i)$ holds for each $K \in 2^N$, we have $s_i \in \{1, ..., \lfloor \frac{n}{2} \rfloor + 1\}$. Moreover, there exists $\overline{K} \in \omega(i)$ with $i \in \overline{K}$ and $\#\overline{K} = s_i$. To see this, suppose the contrary. In case $s_i \in \{1, ..., \lfloor \frac{n}{2} \rfloor\}$, there exists, by the anonymity of F, $K, K' \in \omega(i)$ such that $\#K = \#K' = s_i$ while $K \cap K' = \emptyset$, which contradicts the definition of a winning coalition. In case $s_i = \lfloor \frac{n}{2} \rfloor + 1$, there exists, by the definition of a winning coalition, $K \in \omega(i)$ with $i \notin K$ and $\#K = \lfloor \frac{n}{2} \rfloor$, which contradicts the choice of $s_i = \min \{\#K : K \in w(i)\}$. Now, as there exists $\overline{K} \in \omega(i)$ with $i \in \overline{K}$ and $\#\overline{K} = s_i$, by the anonymity of F, we have $K \in \omega(i)$ for any $K \in 2^N$ with $i \in K$ and $\#K = s_i$. Moreover, as F satisfies M, we have $K \in \omega(i)$ for any $K \in 2^N$ with $i \in K$ and $\#K \geq s_i$. So, $\{K \in 2^N : i \in K$ and $\#K \geq s_i\} \subseteq \omega(i)$. As $s_i = \min \{\#K : K \in w(i)\}$, by the definition of a winning

[45] In fact, a generalized symmetric consent rule $F^{\overline{s}}$ where $s_i \neq s_j$ for some $i, j \in N$ is an example of a CIF which is anonymous but not Samet-Schmeidler symmetric. See Footnote 44.

coalition we have $\{K \in 2^N : \#K > n - s_i\} \subseteq \omega(i)$. By the fact that either $K \in \omega(i)$ or $N \setminus K \in \omega(i)$ holds for each $K \in 2^N$, we have $\omega(i) = \{K \in 2^N : i \in K$ and $\#K \geq s_i\} \cup \{K \in 2^N : \#K > n - s_i\}$. So, we conclude, by referring to Remark 4.3, that the decision over i is taken according to a symmetric consent rule with quota s_i.

Now consider the case where n is even. As for each $K \in 2^N$, either $K \in \omega(i)$ or $N \setminus K \in \omega(i)$ holds, we have $s_i \in \{1, ..., \frac{n}{2}\}$. In case $s_i \neq \frac{n}{2}$, the arguments we used for the case where n is odd show the existence of $\overline{K} \in \omega(i)$ with $i \in \overline{K}$ and $\#\overline{K} = s_i$ and the anonymity and monotonicity of F similarly establishes that the decision over i is taken according to a symmetric consent rule with quota s_i. In case $s_i = \frac{n}{2}$, we have, by the anonymity of F and the definition of a winning coalition, two possible mutually exclusive cases:

CASE 1: $K \in \omega(i)$ for all $K \in 2^N$ with $i \in K$ and $\#K = \frac{n}{2}$ while $K \notin \omega(i)$ for all $K \in 2^N$ with $i \notin K$ and $\#K = \frac{n}{2}$.

CASE 2: $K \in \omega(i)$ for all $K \in 2^N$ with $i \notin K$ and $\#K = \frac{n}{2}$ while $K \notin \omega(i)$ for all $K \in 2^N$ with $i \in K$ and $\#K = \frac{n}{2}$.

For CASE 1, by the monotonicity of F, we have $\omega(i) = \{K \in 2^N : i \in K$ and $\#K = s_i\} \cup \{K \in 2^N : \#K > n - s_i\}$. So we conclude, by referring to Remark 4.3, that the decision over i is taken according to a symmetric consent rule with quota $s_i = \frac{n}{2}$. For CASE 2, by the monotonicity of F, we have $\omega(i) = \big\{K \in 2^N : i \notin K \text{ and } \#K = n/2\big\} \cup \{K \in 2^N : \#K > \frac{n}{2}\}$. So we conclude, by referring to Remark 4.3, that the decision over i is taken according to a symmetric consent rule with quota $s_i + 1 = \frac{n}{2} + 1$. ■

Remark that while deciding on the identity of some $i \in N$, anonymity does not bring any restriction on the decision power of i compared to the (equal) decision powers of the other individuals. In other words, under anonymity, while all individuals but i are equally treated as voters, the opinion of i about him/herself may be favored, disfavored or equally treated compared to the other individuals' opinions over i. The following three conditions classify the set of simple and anonymous CIFs according to this notion:

SF^+: A simple and anonymous CIF $F \in \mathcal{F}$ with a family of winning coalitions $\{\omega(i)\}_{i \in N}$ is *self-favoring* for $i \in N$ if and only if

- there exists $K \in 2^N$ with $i \notin K$ and $K \notin \omega(i)$ such that given any $j \in K$ we have $(K \setminus \{j\}) \cup \{i\} \in \omega(i)$.

- for all $K \in 2^N$ with $i \notin K$, we have $K \in \omega(i) \Longrightarrow (K \setminus \{j\}) \cup \{i\} \in \omega(i)$ for all $j \in K$

SF^-: A simple CIF $F \in \mathcal{F}$ with a family of winning coalitions $\{\omega(i)\}_{i \in N}$ is *self-disfavoring* for $i \in N$ if and only if

- there exists $K \in 2^N$ with $i \in K$ and $K \notin \omega(i)$ such that given any $j \in N \backslash K$ we have $(K \setminus \{i\}) \cup \{j\} \in \omega(i)$.

- for all $K \in 2^N$ with $i \in K$, we have $K \in \omega(i) \Longrightarrow (K \setminus \{i\}) \cup \{j\} \in \omega(i)$ for all $j \in N \backslash K$

SF^0: A simple CIF $F \in \mathcal{F}$ with a family of winning coalitions $\{\omega(i)\}_{i \in N}$ is *self-disregarding* for $i \in N$ if and only if given any $K \in 2^N$ with $i \in K$ and any $j \in N \backslash K$, we have $K \in \omega(i) \Leftrightarrow (K \setminus \{i\}) \cup \{j\} \in \omega(i)$.

This treatment of self-opinions by generalized symmetric consent rules depends on the quota s_i and the number of individuals n in the society, as we remark below:

Remark 4.6 *A generalized symmetric consent rule $F^{\overline{s}} \in \mathcal{F}$ is*
(i) self-favoring for $i \in N$ if and only if $s_i \in \{1, ..., \lfloor \frac{n}{2} \rfloor\}$
(ii) self-disfavoring for $i \in N$ if and only if n is even and $s_i = \frac{n}{2} + 1$
(iii) self-disregarding for $i \in N$ if and only if n is odd and $s_i = \lfloor \frac{n}{2} \rfloor + 1$

Observe that almost all generalized symmetric consent rules are self-favoring, except two cases: When n is even, it is self-disfavoring to determine the identity of i by a version of majoritarianism where a coalition K of cardinality $n/2$ is winning over an individual i if and only if i is not a member of K.[46] Similarly, when n is odd, determining individual identities by (usual) majoritarianism is self-disregarding.

We now turn to the equal treatment of individuals as outcomes. Recall that Samet-Schmeidler symmetry is pretended to reflect the equal treatment property both for voters and outcomes. Our anonymity condition extracts the former part of this. Hence, we allow ourselves to say that a CIF $F \in \mathcal{F}$ is *Samet-Schmeidler neutral* if and only if given any permutation $\Pi : N \to N$ with $\Pi(i) \neq i$, any $G \in \Gamma$ and any $i \in N$, we have $i \in F(G) \Leftrightarrow \Pi(i) \in F(\Pi(G))$. Samet-Schmeidler neutrality is quite

[46] See footnote 40

a demanding condition. In fact, it is equivalent to Samet-Schmeidler symmetry.[47] Moreover, dictatorial CIFs, which are perfectly consistent with the idea of using the same decision rule for all individuals, fail to satisfy it.[48] Not only this clashes with the standard connotation of neutrality and dictatoriality in social choice theory, but it also seems to impose a structure more than necessary to ensure that "the same rule is used by society to determine the qualification of each individual".[49]

Thus, we look for a less demanding neutrality condition which ensures the equal treatment of individuals as outcomes while it is congruous to our model as well as to the usual connotations of social choice theory. The complication of the matter arises from the fact that voters and outcomes coincide. So we propose to impose the usual neutrality requirement only for cases where voters and outcomes differ in the following axiom.

Axiom 4.7 *A CIF $F \in \mathcal{F}$ is essentially neutral if and only if given any $i, j \in N$, the existence of some $G \in \Gamma$ with $i \notin G_i \cup G_j$ and $i \in F(G)$ implies the existence of some $G' \in \Gamma$ with $j \in G'_k \Leftrightarrow i \in G_k$ for all $k \in N$ such that $j \in F(G')$.*

Remark 4.7 *For simple CIFs, essential neutrality imposes a particular structure over the family of winning coalitions $\{\omega(i)\}_{i \in N}$ so that for any $i, j \in N$ and for any $K \in 2^N$ with $i, j \notin K$, we have $K \in \omega(i) \implies K \in \omega(j)$.*[50]

In words, for simple CIFs, essential neutrality imposes that a coalition K which excludes some $i, j \in N$ is winning over i if and only if K is winning over j - a requirement which incorporates the usual neutrality idea to our model for cases where those who decide and those over which the decision is made form disjoint sets.

Note that even when simple CIFs are essentially neutral, two individuals may have smallest winning coalitions of different cardinalities - a fact which certainly

[47] Samet-Schmeidler symmetry, by definition, implies Samet-Schmeidler neutrality. To see the converse implication, define for each $i, j \in N$, a bijection $\Pi^{ij} : N \to N$ as $\Pi^{ij}(i) = j$, $\Pi^{ij}(j) = i$ and $\Pi^{ij}(k) = k$ for all $k \in N \setminus \{i, j\}$. Note that any bijection over N can be expressed as the composition of some family of bijections Π^{ij} over N. Thus every CIF that is Samet-Schmeidler neutral is also Samet-Schmeidler symmetric.

[48] To see this, consider a society $N = \{1, 2\}$. Let F_1 be the CIF where individual 1 is the dictator. Thus, we have $\omega(1) = \omega(2) = \{\{1\}, \{1, 2\}\}$. On the other hand, given that $\{1\} \in \omega(1)$, Samet-Schmeidler neutrality requires that $\{2\} \in \omega(2)$. This requirement implicitly assumes that $\{1\} \in \omega(1)$ because the opinion of individual 1 about himself is particular - which is not the case when 1 is the dictator. To be sure, under F_1, the opinion of individual 1 about him/herself is fully decisive. However, this is a result of the fact that the CIF in question concentrates all decision power to 1 and not because that the opinions of individuals about themselves are favored.

[49] See Samet and Schmeidler (2003), Section 4.1, p.225.

[50] This can be expressed, in terms of permutations, as follows: Given any $i, j \in N$, any $\Pi^{ij} : N \to N$ (as defined in Footnote 47) and any $K \in 2^N$ with $i, j \notin K$, we have $K \in \omega(i) \implies \Pi^{ij}(K) = K \in \omega(j)$.

contradicts the idea of using the "same" rule for all individuals.[51] So we strengthen essential neutrality by adding a requirement such that all individuals' smallest winning coalitions must have the same cardinality.

Axiom 4.8 *A CIF $F \in \mathcal{F}$ as neutral whenever F is essentially neutral and given any $i, j \in N$, any $G \in \Gamma$ with $i \in F(G)$, there exists $G' \in \Gamma$ with $\#\{k \in N : j \in G'_k\} = \#\{k \in N : i \in G_k\}$ such that $j \in F(G')$.*

This last condition can be translated to the world of simple CIFs as the requirement that all individuals have smallest winning coalitions of the same cardinality.[52] Our next result is a characterization of simple, anonymous and neutral CIFs.

Theorem 4.3 *A CIF $F \in \mathcal{F}$ is simple, anonymous and neutral if and only if F is a symmetric consent rule F^s with $s \in \{1, ..., \lfloor \frac{n}{2} \rfloor + 1\}$.*

Proof. To show the "if" part, take any symmetric consent rule $F^s$with $s \in \{1, ..., \lfloor \frac{n}{2} \rfloor + 1\}$. As every symmetric consent rule is, by definition, a generalized consent rule, we know by Theorem 4.2 that F^s is simple and anonymous. It is straightforward to check that symmetric consent rules are neutral. To prove the "only if" part, take any simple, anonymous and neutral CIF F. As F is simple and anonymous, by Theorem 4.2, it is a generalized symmetric consent rule $F^{\overline{s}}$. Consider first the case where n is odd. As F^s is neutral, hence the smallest winning coalitions of all individuals are of the same cardinality, we have $s_i = s_j$ for all $i, j \in N$, showing that F^s is a symmetric consent rule. Now consider the case where n is even. As F^s is neutral, hence the smallest winning coalitions of all individuals are of the same cardinality, we have $s_i \neq s_j$ for some $i, j \in N$ only if $s_i, s_j \in \{\frac{n}{2}, \frac{n}{2} + 1\}$. But essential neutrality implies that there exists no $i, j \in N$ such that $s_i = \frac{n}{2}$ and $s_j = \frac{n}{2} + 1$, showing that $F^{\overline{s}}$ is a symmetric consent rule. ■

Remark that Theorem 4.3 is related to Theorem 2 of Samet and Schmeidler (2003) which characterizes symmetric consent rules in terms of independence, monotonicity, self-duality and Samet-Schmeidler symmetry. It immediately follows from juxtaposing Theorem 2 of Samet and Schmeidler (2003) with our Theorems 4.1 and 4.3 that for simple CIFs, Samet-Schmeidler symmetry is equivalent to the conjunction

[51] For example, when $N = \{1, 2, 3\}$, the generalized symmetric consent rule $F^{\overline{s}}$ with $s = (1, 1, 2)$ is essentially neutral while individuals 2 and 3 have smallest winning coalitions of different cardinalities.

[52] This requirement does not imply essential neutrality, as one can check through the generalized symmetric consent rule $F^{\overline{s}}$ with $s = (2, 2, 2, 3)$ used in the society $N = \{1, 2, 3, 4\}$.

of anonymity and neutrality.[53] On the other hand, on the general domain of CIFs, Samet-Schmeidler symmetry is stronger than anonymity and neutrality.[54] So over the domain of simple CIFs, the anonymity and neutrality conditions we propose successfully decompose the symmetry condition of Samet and Schmeidler (2003) which is an incorporation of both equal treatment properties.

4.3 Characterizing Liberalism

This section contains various characterizations of the liberal CIF. We start by considering a specific profile G^0 with $G_i^0 = \{i\}$ for all $i \in N$, where each individual only considers him/herself as a J. As we remark below, the behavior of simple CIFs on G^0 determines the set of individuals who have the right of self-determination.

Remark 4.8 *Take any simple CIF $F \in \mathcal{F}$ with a family of winning coalitions $\{\omega(i)\}_{i\in N}$. For any $i \in N$, we have $\omega(i) = \{K \in 2^N : i \in K\}$ if and only if $i \in F(G^0)$.*

Liberalism is the assignment of the right of self-determination to each individual. So the behavior of CIFs over G^0 is critical in characterizing liberalism. We say that a CIF$F \in \mathcal{F}$ satisfies the *weak equal treatment property* (WETP) if and only if $F(G^0) \in \{\emptyset, N\}$. WETP is satisfied *positively* in case $F(G^0) = N$ and negatively when $F(G^0) = \emptyset$. The following theorem characterizes the liberal CIF as the unique simple CIF that satisfies WETP positively.

Theorem 4.4 *A simple CIF $F \in \mathcal{F}$ satisfies the weak equal treatment property positively if and only if F is the liberal CIF.*

Proof. The "if" part immediately follows from the definitions of the liberal CIF and the positive WETP. To see the "only if" part, take any simple CIF F that satisfies WETP positively. By Remark 4.8, we have $\omega(i) = \{K \in 2^N : i \in K\}$ for each $i \in N$, which means that F is the liberal CIF. ■

Theorem 4.4 paves the way to another characterization of the liberal CIF through a liberalism axiom introduced by Kasher and Rubinstein (1997) who say that a CIF

[53]Recall that although the Samet and Schmeidler (2003) monotonicity condition is stronger than ours, the two monotonicities coincide under independence.

[54]Samet-Schmeidler symmetry implies anonymity directly by the definitions of the two concepts. To see that Samet-Schmeidler symmetry implies neutrality, one can use Π^{ij} permutation for any $i, j \in N$ and for any $G \in \Gamma$. Finally, in a society $N = \{1, 2, 3\}$, the (non-simple) CIF $F : \Gamma \to 2^N$ which is defined at each $G \in \Gamma$ as $1 \in F(G) \Leftrightarrow \{i \in N : 1 \in G_i\} \supseteq \{2, 3\}$ and for $k \in \{2, 3\}$ we have $k \in F(G) \Leftrightarrow \#\{i \in N : k \in G_i\} \geq 2$ examplifies a CIF that is anonymous, neutral but not Samet-Schmeidler symmetric.

$F \in \mathcal{F}$ satisfies the *liberal principle* if and only if for each $G \in \Gamma$, $\exists\, i \in N$ with $i \in G_i$ $\Longrightarrow F(G) \neq \varnothing$ and $\exists\, i \in N$ with $i \notin G_i \Longrightarrow F(G) \neq N$.

Theorem 4.5 *A simple CIF $F \in \mathcal{F}$ satisfies the weak equal treatment property and the liberal principle if and only if F is the liberal CIF.*

Proof. The "if" part can be seen by checking that the liberal CIF satisfies WETP and the liberal principle. To see the "only if" part, take any simple CIF F that satisfies WETP and the liberal principle. By WETP, we have $F(G^0) \in \{\emptyset,\, N\}$ while the liberal principle rules $F(G^0) = \emptyset$ out. So $F(G^0) = N$, which means that F satisfies WETP positively and, by Theorem 4.1, F is the liberal CIF. ∎

Finally, we show that as the liberal CIF is the unique symmetric consent rule that satisfies liberal principle.

Theorem 4.6 *A simple, neutral and anonymous CIF $F \in \mathcal{F}$ satisfies the liberal principle if and only if F is the liberal CIF.*

Proof. The "if" part can be seen by checking that the liberal CIF satisfies the liberal principle. To see the "only if" part, take any simple, neutral and anonymous CIF F which, by Theorem 4.3, is a symmetric consent rule. Let F satisfy the liberal principle. Note that F satisfies WETP by the definition of a symmetric consent rule. So by Theorem 4.5, F is the liberal CIF. ∎

5 Conclusion

We consider a model of aggregating vectors of sets into a set. This mathematical structure is sufficiently rich to allow various interpretations such as the qualification problem where a set of objects is to be partitioned as "goods" and "bads" depending on individuals' opinions[55]; electing committees by approval balloting where voters may approve any set of candidates[56]; deciding over accepting or rejecting a set of issues[57] or more generally the aggregation of individual choices into a social choice.[58] The interpretation we explore is the collective identity determination problem, proposed by Kasher and Rubinstein (1997), where individuals' opinions about "Who is a J" are aggregated into a social decision.[59] We introduce a family of elementary CIFs which are aggregation rules that can be expressed in terms of winning coalitions. We then look into the effects of adding new axioms on the structure of collection of winning coalitions. However our main focus is on simple CIFs which we characterize in terms of three axioms, namely independence, self-duality and monotonicity. Many interesting CIFs of the literature, including (symmetric) consent rules introduced by Samet and Schmeidler (2003), are simple. The class of simple CIFs exhibits the following properties:

- The family of anonymous and neutral CIFs coincides with the family of symmetric consent rules ranging from liberalism to majoritarianism.
- The family of anonymous CIFs coincides with the family of generalized symmetric consent rules.
- All neutral CIFs, but dictatoriality, satisfy WETP.
- An anonymous CIF F fails WETP if and only if F is a generalized symmetric consent rule $F^{\overline{s}}$with $s_i = 1$ for some $i \in N$ and $s_j > 1$ for some $j \in N$.
- Liberalism is the unique CIF that satisfies WETP positively.

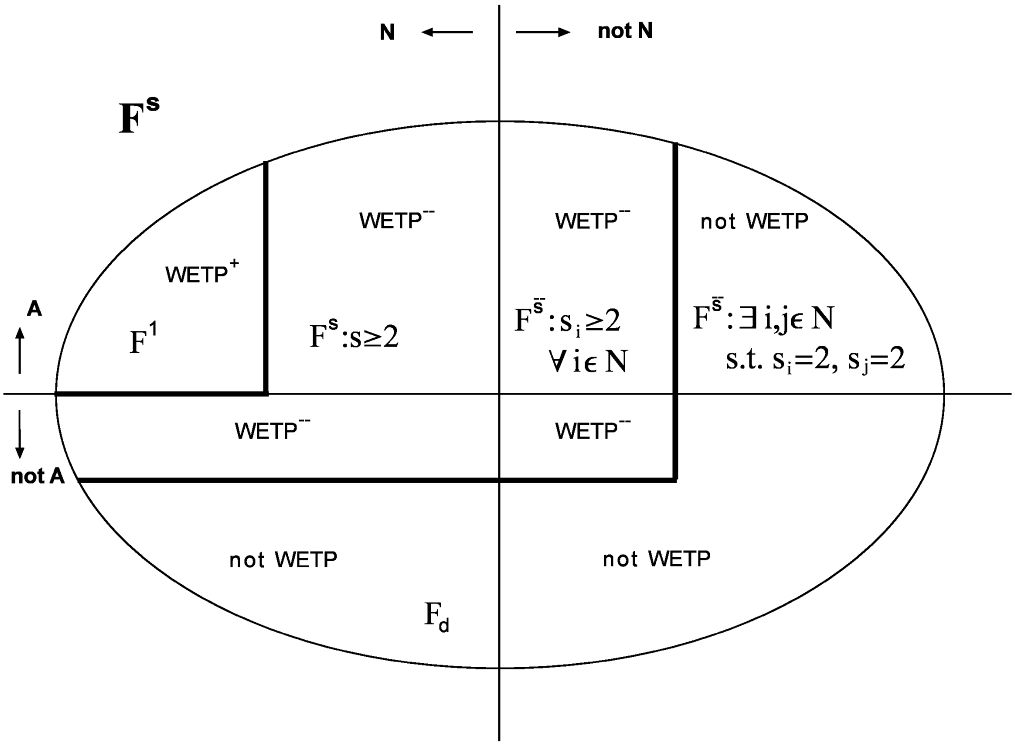
[55] such as the analysis made by Dimitrov et al. (2004) and Ju (2005a)

[56] examples of which can be found in Brams et al. (2005b) and Brams et al. (2005a)

[57] see Ju (2005b)

[58] Aleskerov (1999) and Aizerman and Aleskerov (1995) give an excellent treatment of the choice aggregation problem. We wish to say that the plethora of results they establish in an abstract framework can certainly bring further insights to particular applications of aggregating choices, such as the one we consider in this paper. See also Lahiri (2001) and Quesada (2003).

[59] Under this final interpretation, voters and outcomes coincide, which is not case in the qualification or the committee election problem. This leads to subtleties such as special treatments of "self-qualification" and finer distinctions between the standard anonymity and neutrality conditions of social choice theory - matters which we adress and handle in this paper.



- Among non-anonymous and non-neutral CIFs there are those which do and those which do not satisfy WETP.

The following figure summarizes our findings regarding simple CIFs:

Their ability of expressing the aggregation rule through families of winning coalitions makes simple CIFs of particular interest. Moreover, independence, self-duality and monotonicity are conditions which are very suitable for the identity aggregation context.[60] It is also to emphasize that among simple CIFs, the liberal one arises as the unique CIF that satisfies positive WETP, which is an intuitive and fairly mild axiom. This supports the idea of endowing individuals with the right of self-determination - hence embracing liberalism as a natural solution to the collective identity determination problem.

For further researches in that topic, one can offer a full characterization of neutral simple CIFs. Just to give an idea, writing an acceptable self-dual family of winning coalition for some individual and copying it to all individuals induces a neutral simple

[60] Though they would not be that appropriate for other interpretations of our model, such as the committee election problem.

CIF. However, dictatorial consent rules F_d^{st} defined for some fixed individual $d \in N$ and for any $i \in N$ and any $G \in \Gamma$ as follows:

- if $i \in G_d$, then $i \in F(G) \iff \#\{j \in N : i \in G_j\} \geq s$
- if $i \notin G_d$, then $i \notin F(G) \iff \#\{j \in N : i \notin G_j\} \geq t$

are simple and neutral and families of winning coalitions for each individuals differ.[61] Moreover, in our model, each individual has same weight over social opinion of an individual. It can be thought that there is a matrix $W = [w_{ij}]$ of weights where w_{ij} represents individuals i's power on deciding social opinion of j. In addition to neutral simple CIFs or weighted rules, it is possible to construct different models. For example, one can treat the collective identity determination problem as searching set of strict rules that differentiate socially acceptable members and non-members. Hence, there may be a finite set of criterias $C = (1, ..., c)$ that are related with the identity in question and the degree that each individual i satisfies the criterias k can be represented with a value α_i^k from $[0,1]$ interval. Then $\alpha_i = \left(\alpha_i^k\right)_{k \in C}$ is the vector showing the satisfaction of criteria degrees of individual i and $\alpha = \{\alpha_i\}_{i \in N}$ forms a criteria profile. A CIF may aggregate each possible criteria profiles into a final unique value for each individual. As a last suggestion for further researches, in our model individuals have to express their personal opinions about an individual as either a member or non-member. This construction is appropriate when the size of society is small. But if the size expands, some individuals may not have enough information about some members of society to qualify them as member or non-member. This situation can be modelled with three partition $\left(N^+, N^0, N^-\right)$ of society, N as personal opinions[62] and a CIF may aggregate this personel opinions into a social opinion which may be again a three partition of society or restricted to two partition as in our model. Observe that in the model suggested above, the individuals preferences can be interpreted as trichotomous preferences where each N^+, N^0, N^- stands for an equivalence classes.

[61] In dictatorial consent rule, there is a dictator but he needs to meet some quotas (s and t for each state of world) to consent his opinion about an individual i to society.

[62] Note that, in our model, $N^0 = \varnothing$ for each individual.

Appendix

A Proofs of Previous Theorems

Proofs of Theorems of Section 3.1

Proof of Lemma 3.1. We will prove the statement by induction of $\#K$ of K by showing for all $K \subseteq N$, we have $F\left(G^K\right) = K$ and $F\left(G^{N\setminus K}\right) = N \setminus K$ which is equivalent to show that for all $K \subseteq N$, we have $F\left(G^K\right) = K$. Let take any CIF $F \in \mathcal{F}$ satisfying SYM-KR, I-KR and L and any $K \subseteq N$. Note that any $i, j \in K$ is symmetric in the profile G^K as well as any $k, m \in N \setminus K$, hence by SYM-KR, we have $F\left(G^K\right) \in \{\varnothing, K, N \setminus K, N\}$. If $\#K = 0$, then we have $F\left(G^{\varnothing}\right) \in \{\varnothing, N\}$. As L excludes the possibility of $F\left(G^{\varnothing}\right) = N$, we have $F\left(G^{\varnothing}\right) = \varnothing$. Analogously, we have $F\left(G^N\right) = N$. Moreover note that for any K different from $\varnothing$ and N, we have $F\left(G^K\right)$ and $F\left(G^{N\setminus K}\right)$ is neither $\varnothing$ nor N. Now assume that for some $K \subseteq N$ with $\#K = k$, we have $F\left(G^K\right) = K$ and $F\left(G^{N\setminus K}\right) = N \setminus K$. We will show that for some $K' \subseteq N$ with $\#K' = k+1$, we have $F\left(G^{K'}\right) = K'$ and $F\left(G^{N\setminus K'}\right) = N \setminus K'$. Let $K' = K \cup \{i\}$ for some $i \notin K$. As for all $K \subseteq N$ with $K \neq \varnothing$ and $K \neq N$, we have $F\left(G^K\right) \in \{K, N \setminus K\}$, suppose for the sake of contradiction that $F\left(G^{K'}\right) = N \setminus K'$. Consider the profile $G^{N\setminus K}$. We have $F\left(G^{N\setminus K}\right) \setminus \{i\} = F\left(G^{K'}\right) \setminus \{i\}$ and $i \in G_j^{K'} \iff i \in G_j^{N\setminus K}$ for all $j \in N$ but we have $i \in F\left(G^{N\setminus K}\right)$ and $i \notin F\left(G^{K'}\right)$ which establishes the desired contradiction with I-KR. A similar argument shows that $F\left(G^{N\setminus K'}\right) = N \setminus K'$. ■

Proof of Theorem 3.1. Suppose for a contradiction, there is a profile $G \in \Gamma$ such that for some $i \in N$, we have $i \in G_j$ for all $j \in N$ but $i \notin F(G)$ or $i \notin G_j$ for all $j \in N$ but $i \in F(G)$. Assume first case. Let $K = F(G) \cup \{i\}$, by lemma 3.1, we have $F\left(G^K\right) = K = F(G) \cup \{i\}$. Note that $F\left(G^K\right) \setminus \{i\} = F(G) \setminus \{i\}$ and $i \in G_j \iff i \in G_j^K$ for all $j \in N$ but $i \in F\left(G^K\right)$ and $i \notin F(G)$ violating independence. A similar argument shows non-existence of a profile for any $i \in N$ such that $i \notin G_j$ for all $j \in N$ but $i \in F(G)$. ■

Proof of Lemma 3.2. Let take any CIF $F \in \mathcal{F}$ satisfying SYM-KR, I-KR and L and any $G^P, H^P \in \Gamma$ defined for some 4-partition $P = (P_1, P_2, P_3, P_4)$ of N. By theorem 3.1, we have $P_1 \cup P_2 \subseteq F\left(G^P\right) \subseteq P_1 \cup P_2 \cup P_3$ and $P_1 \subseteq F\left(H^P\right) \subseteq P_1 \cup P_2$.

Moreover, by *SYM*, we have either $F\left(G^P\right) = P_1 \cup P_2$ or $F\left(G^P\right) = P_1 \cup P_2 \cup P_3$ and either $F\left(H^P\right) = P_1$ or $F\left(H^P\right) = P_1 \cup P_2$. Now suppose for a contradiction that $F\left(G^P\right) = P_1 \cup P_2 \cup P_3$ where $P_3 \neq \varnothing$. Let $i \in P_3$.Consider the profile $G' \in \Gamma$ with $G'_k = P_1 \cup P_2 \cup P_3 \setminus \{i\}$ for all $k \in P_1 \cup P_3 \setminus \{i\}$, $G'_k = N$ for all $k \in P_2$ and $G'_k = N \setminus \{k\}$ for all $k \in P_4 \cup \{i\}$. From theorem 3.1, we have $P_1 \cup P_2 \cup P_3 \setminus \{i\} \subseteq F(G')$ and for any $j \in P_4$, we have i and j are symmetric in G', hence by *SYM*, we have either $F(G') = P_1 \cup P_2 \cup P_3 \setminus \{i\}$ or $F(G') = N$. But $F(G') = N$ violates L as $i \notin G_i$. Hence $F(G') = P_1 \cup P_2 \cup P_3 \setminus \{i\}$. But note that $F(G') \setminus \{i\} = F\left(G^P\right) \setminus \{i\}$ and for all $j \in N$, we have $i \in G'_j \iff i \in G^P_j$ hence by I we must have $i \in F\left(G^P\right) \iff i \in F(G')$ which is not the case.

Now suppose for a contradiction that $F\left(H^P\right) = P_1$ where $P_2 \neq \varnothing$. Let $i \in P_2$ and consider the profile $H' \in \Gamma$ such that $H'_k = \{k\}$ for all $k \in P_1 \cup \{i\}$, $H'_k = P_1 \cup \{i\}$ for all $k \in P_2 \cup P_4 \setminus \{i\}$ and $H'_k = \varnothing$ for all $k \in P_3$. From theorem 3.1, we have $(P_2 \cup P_3 \cup P_4 \setminus \{i\}) \cap F(H') = \varnothing$ and we have either $P_1 \cup \{i\} \subseteq F(H')$ or $(P_1 \cup \{i\}) \cap F(H') = \varnothing$ from *SYM* since for any $j \in P_1$ and i are symmetric in the profile H'. Thus we have either $F(H') = \varnothing$ or $F(H') = P_1 \cup \{i\}$. $F(H') = \varnothing$ violates L since $i \in G_i$. But $F(H') = P_1 \cup \{i\}$ violates I since $F\left(H^P\right) \setminus \{i\} = F(H') \setminus \{i\}$ and $i \in H'_j \iff i \in H^P_j$ for all $j \in N$, we have $i \in F(H')$ but $i \notin F\left(H^P\right)$. ■

Proof of Theorem 3.2. Clearly the Liberal CIF satisfies all three axioms. To see the converse, take any CIF $F \in \mathcal{F}$ which satisfies *SYM, I* and *L*. Suppose for a contradiction that there exists a profile $G \in \Gamma$ and an individuals $i \in N$ such that

$i \notin G_i$ but $i \in F(G)$ or

$i \in G_i$ but $i \notin F(G)$.

Consider the first case. Let (M_0, M_1, N_0, N_1) be a 4-partition of $N \setminus \{i\}$ such that

$M_0 = \{j \in F(G) \setminus \{i\} : i \notin G_j\}$

$M_1 = \{j \in F(G) \setminus \{i\} : i \in G_j\}$

$N_0 = \{j \notin F(G) \cup \{i\} : i \notin G_j\}$

$N_0 = \{j \notin F(G) \cup \{i\} : i \in G_j\}$.

Consider the profile $G' \in \Gamma$ defined for each individual $j \in N$ as follows:

$$G'_j = \begin{cases} M_0 \cup M_1 & \text{if } k \in M_0, \\ M_0 \cup M_1 \cup N_0 \cup \{i\} & \text{if } k \in M_1, \\ M_0 \cup M_1 & \text{if } k \in N_0 \cup \{i\}, \\ M_0 \cup M_1 \cup N_0 \cup \{i\} & \text{if } k \in N_1. \end{cases}.$$

Note that $G' = G^P$ where G^P is the profile defined for 4-partition $P = (M_0, M_1, N_0 \cup \{i\}, N_1)$ of N as in the lemma 3.2, hence we have $F(G') = M_0 \cup M_1 = F(G) \setminus \{i\}$. But I is violated since $F(G') \setminus \{i\} = F(G) \setminus \{i\}$ and $i \in G'_j \iff i \in G_j$ for all $j \in N$ but we have $i \in F(G)$ whereas $i \notin F(G')$.

Now consider the second case where $i \in G_i$ but $i \notin F(G)$. Define $H' \in \Gamma$ defined for each individual $j \in N$ as follows:

$$H'_j = \begin{cases} M_0 & \text{if } k \in M_0, \\ M_0 \cup M_1 \cup \{i\} & \text{if } k \in M_1 \cup \{i\}, \\ M_0 \cup M_1 & \text{if } k \in N_0, \\ M_0 \cup M_1 \cup \{i\} & \text{if } k \in N_1. \end{cases}.$$

Note that $H' = H^P$ where H^P is the profile defined for 4-partition $P = (M_0, M_1 \cup \{i\}, N_0, N_1)$ of N as in the lemma 3.2, hence we have $F(H') = M_0 \cup M_1 \cup \{i\} = F(G) \cup \{i\}$. But I is violated since $F(H') \setminus \{i\} = F(G) \setminus \{i\}$ and $i \in H'_j \iff i \in G_j$ for all $j \in N$ but we have $i \in F(H')$ whereas $i \notin F(G)$. ∎

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