10-24-2011

Can a Road-Driven Car Outrace a Free-Falling Car?

Diego Castano
Nova Southeastern University, castanod@nova.edu

NSUWorks Citation
Castano, D. (2011). Can a Road-Driven Car Outrace a Free-Falling Car?. European Journal of Physics, 32, (6), 1711 - 1718. https://doi.org/10.1088/0143-0807/32/6/024. Retrieved from https://nsuworks.nova.edu/cnso_chemphys_facarticles/37

Can a road-driven car outrace a free-falling car?

2011 Eur. J. Phys. 32 1711

(http://iopscience.iop.org/0143-0807/32/6/024)

IOP PUBLISHING EUROPEAN JOURNAL OF PHYSICS

Eur. J. Phys. **32** (2011) 1711–1718 doi:10.1088/0143-0807/32/6/024

Can a road-driven car outrace a free-falling car?

Diego J Castaño

Nova Southeastern University, Fort Lauderdale, FL, USA

E-mail: castanod@nova.edu

Received 1 July 2011, in final form 30 August 2011
Published 24 October 2011
Online at stacks.iop.org/EJP/32/1711

Abstract

Motivated by an advertising scenario in which a luxury sports sedan races against a similar car falling under the influence of gravity, a calculation using undergraduate physics and calculus is performed to theoretically predict the outcome.

1. Introduction

Although nowadays there are mythbusting teams ready to empirically confirm or deny advertising claims that may seem too good to be true, it is often economically prohibitive to perform the kinds of experiments that are called for. It is therefore sometimes more sensible and efficacious to perform a thought experiment instead, especially if the physics is relatively elementary. Some years ago there was an advertisement affirming that a certain luxury sports sedan could outrace a falling car (in a horizontal attitude) over the same distance. This is clearly a situation of the type alluded to above: the experiment cannot be performed by a single person, nor without significant funding. However, a single person with some basic knowledge of mechanics (at the level of a university physics course) can construct a reasonable (theoretical) model of the scenario and predict the outcome with some confidence. This exercise is intended as an interesting application of Newton's laws, and as such it can serve as a useful instructional example for advanced undergraduates. It goes beyond the typical end-of-the-chapter problem in an introductory physics text by requiring more than one concept to be brought into consideration, to wit, forces and torques, friction, air resistance and mechanical advantage. It also involves the use of mathematical modelling and computational methods. The mathematics, however, never requires anything beyond the level of integral calculus. The basic question this paper attempts to answer is: can a driven car, starting from rest, outrace an identical free-falling car over the same distance?

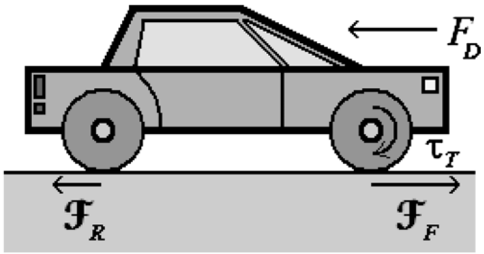

Figure 1. Forces on an accelerating car.

2. Equations of motion

To make the calculation as realistic as possible, all the effective frictional forces should be identified and included. For a car with front wheel drive, the basic collection of horizontal forces during acceleration is depicted in figure 1. The tractive force, $\mathcal{F}_F$, is the reactive frictional force of the ground on the car's front wheels that propels the car forward. Whereas the tractive friction on the front wheels forces the car forward, the resistive frictional force on the rear wheels, $\mathcal{F}_R$, opposes the motion. These frictional forces are static in nature since no slipping is assumed. Rolling friction due to tyre deformation will be included as an effective frictional torque. Assuming air resistance, there is also a drag force, F_D, to contend with. The equation for translational motion to consider would then be

$$\mathcal{F}_F - \mathcal{F}_R - F_D = m\ddot{x}, \tag{1}$$

where

$$F_D = \kappa v^2, \qquad \kappa = \tfrac{1}{2} C_D \rho A, \tag{2}$$

and C_D is the dimensionless drag coefficient, ρ the density of air and A the effective area (orthogonal projection) presented to the air by the car [1, pp 146–8]. As it applies to the front tyres, the assumption of no slipping is optimal to the driven car's performance in the race since $\mathcal{F}_F \leqslant \mu N$, where N is the normal force, and generally $\mu_{\text{static}} > \mu_{\text{kinetic}}$ for two given surfaces of contact [1, p 91]. In other words, the car's translational acceleration is greater if the driver throttles skilfully to avoid any slipping, and this will subsequently be assumed.

The rotational equations of motion for the front and rear axles are

$$\tau_T - \tau_{FF} - R\mathcal{F}_F = \frac{I}{R}\ddot{x}, \tag{3}$$

$$-\tau_{RF} + R\mathcal{F}_R = \frac{I}{R}\ddot{x}, \tag{4}$$

where τ_{FF} (τ_{RF}) is the front (rear) effective friction associated with the drivetrain, wheel bearings and rolling, and τ_T is the engine's transmitted torque. It has been assumed that all wheels have the same radius, R, and that the entire axle assemblies (including wheels) of the front and rear have moment of inertia, I. Note that the frictional force $\mathcal{F}_R$ would be tractive instead of resistive if the car had four-wheel drive, and there would be another engine torque τ_T' on the rear axle, resulting in the following changes to (4), in that case

$$\tau_T' - \tau_{RF} - R\mathcal{F}_R = \frac{I}{R}\ddot{x}. \tag{5}$$

Figure 2. Effective engine torque/speed curve for a car.

The equations of motion (1), (3) and (4) can be rewritten as

$$\Delta\mathcal{F} - F_D = m\ddot{x}, \tag{6}$$

$$\tau_T - \tau_F - R\Delta\mathcal{F} = 2\frac{I}{R}\ddot{x}, \tag{7}$$

where $\Delta\mathcal{F} = \mathcal{F}_F - \mathcal{F}_R$ and $\tau_F = \tau_{FF} + \tau_{RF}$. Combining these gives

$$F_{\text{total}} = F_T - F_F - F_D = M\ddot{x}, \tag{8}$$

where $F_T \equiv \tau_T/R$ is the engine's effective force on the ground, $F_F \equiv \tau_F/R$ is the effective frictional force, and $M = m + 2I/R^2$ is the effective inertial mass. Note that when cruising at a constant velocity, the engine's torque matches the resistive forces/torques

$$\tau_T^{(\text{cruising})} = RF_D + \tau_F. \tag{9}$$

The moment of inertia of the axles can be written $I \approx 2\alpha m_{\text{wheel}}R^2$ for some α of order 1 (assuming an axle of negligible mass and/or radius). Since the mass of the car is significantly larger than that of the four wheels, $M \approx m$.

3. Modelling the engine

Automobile engines have varying torque/speed curves, but generally their effective features are displayed in figure 2 [1, pp 126–9]. Through gearing a mechanical advantage is developed [1, pp 129–36], so that in any given gear, with the associated factor γ, the transmitted torque (to the ground) and angular speed are

$$\tau_T = \gamma\tau_E, \tag{10}$$

$$\omega_T = \frac{\omega_E}{\gamma}, \tag{11}$$

where ω_T is the angular speed of the wheels. An engine's transmitted torque and speed through three gears might look something like figure 3. From the general form of the enveloping curve, the engine's effective force (transmitted to the wheels) over all gears is simply assumed to have the general quadratic form

$$F_T(v) = f_0 + f_1 v + f_2 v^2, \qquad f_0 > 0, \qquad f_2 \leqslant 0, \qquad |f_1| + |f_2| \neq 0, \tag{12}$$

such that $F_T(v) \geqslant 0$ for $0 \leqslant v \leqslant v_{\max}$ where $v_{\max}$ is the car's maximum speed. Note that there is therefore a set of four undetermined parameters associated with the engine,

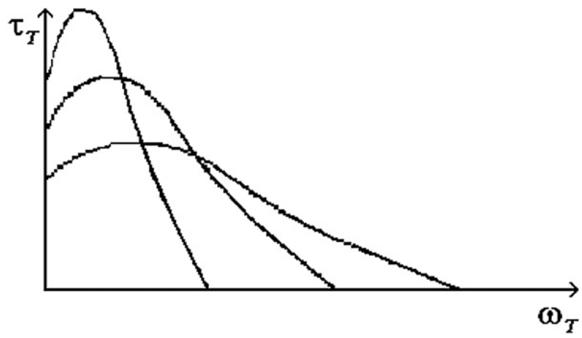

Figure 3. Transmitted torque *versus* transmitted speed.

$S_E = \{f_0, f_1, f_2, F_F\}$. The engine's power, $P = vF_T$, is also positive and attains a maximum in the stated range at the speed v_P where

$$v_P = \begin{cases} v_*, & \text{if} \quad 0 \leqslant v_* \leqslant v_{\max} \quad \text{and} \quad \left.\frac{\mathrm{d}P}{\mathrm{d}v}\right|_{v_*} = 0 \\ v_{\max}, & \text{otherwise} \end{cases}. \tag{13}$$

Solving the local maximum condition,

$$\left.\frac{\mathrm{d}P}{\mathrm{d}v}\right|_{v_*} = f_0 + 2f_1 v_* + 3f_2 v_*^2 = 0 \tag{14}$$

for v_* yields

$$v_* = \begin{cases} -\frac{f_0}{2f_1}, & \text{if} \quad f_2 = 0 \quad (\text{with} \quad f_1 < 0) \\ \frac{-f_1 - \sqrt{f_1^2 - 3f_0 f_2}}{3f_2}, & \text{if} \quad f_2 < 0. \end{cases} \tag{15}$$

At the speed v_P, the following equation holds:

$$P(v_P) = P_{\max}, \tag{16}$$

where $P_{\max}$ is the engine's maximum attainable power. This equation represents the first of two constraints on the set S_E of undetermined engine parameters. The second constraint is derived from the fact that the car can no longer accelerate beyond the maximum speed (see figure 4)

$$F_{\text{total}}(v_{\max}) = 0. \tag{17}$$

4. Solving the differential equation

Substituting (2) and (12) into (8) gives the differential equation that must be solved

$$\tilde{f}_0 + f_1 v - \tilde{\kappa} v^2 = M\dot{v}, \tag{18}$$

where $\tilde{f}_0 \equiv f_0 - F_F > 0$ and $\tilde{\kappa} \equiv \kappa - f_2 > 0$ for obvious reasons. The differential equation can be equivalently expressed in the separated form

$$\frac{\mathrm{d}v}{(v - v_+)(v + v_-)} = -\frac{\mathrm{d}t}{\lambda}, \tag{19}$$

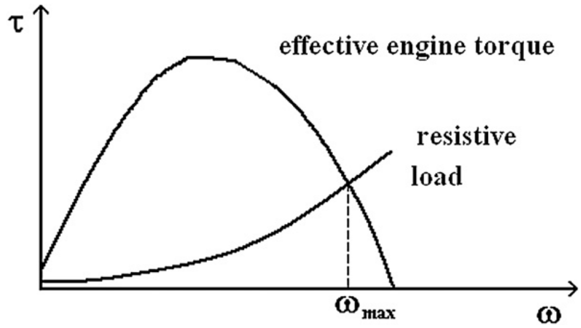

Figure 4. Effective engine torque and resistive load curves.

Table 1.

$0 - 96.5$ km h^{-1} ($v_{60} = 26.8$ m s^{-1}) in $t_{60} = 5.3$ s
$v_{\max} = 228.5$ km h^{-1} (63.5 m s^{-1})
$P_{\max} = 228$ kW (306 Hp)
$m = 1600$ kg
Dimensions: 4.6 m × 1.8 m × 1.4 m

where

$$v_+ = \frac{f_1}{2\tilde{\kappa}} + \sqrt{\left(\frac{f_1}{2\tilde{\kappa}}\right)^2 + \frac{\tilde{f}_0}{\tilde{\kappa}}} \equiv v_{\max}, \tag{20}$$

$$v_- = -\frac{f_1}{2\tilde{\kappa}} + \sqrt{\left(\frac{f_1}{2\tilde{\kappa}}\right)^2 + \frac{\tilde{f}_0}{\tilde{\kappa}}}, \tag{21}$$

$$\lambda = \frac{M}{\tilde{\kappa}}. \tag{22}$$

Integrating the differential equation above gives

$$v(t) = v_+ v_- \frac{1 - e^{-2t/t_0}}{v_- + v_+ e^{-2t/t_0}}, \tag{23}$$

where

$$t_0 = \frac{2\lambda}{v_- + v_+}, \tag{24}$$

and $v \to v_+$ as $t \to \infty$. Integrating for the position further yields

$$x(t) = v_+ t + \lambda \ln\left(\frac{v_- + v_+ e^{-2t/t_0}}{v_- + v_+}\right). \tag{25}$$

5. Data analysis

Table 1 displays some of the available data for the luxury sports sedan used in the advertisement, the Lexus IS350 [2].

Table 2.

$f_0 = 10,749.3$ N
$f_1 = -126.5$ kg s^{-1}
$F_F = 697.6$ N
$\lambda = \frac{M}{\kappa} = 3200$ m
$v_+ = 63.5$ m s^{-1}
$v_- = 316.5$ m s^{-1}
$t_0 = 16.8$ s

Using data from [2] and [1, pp 148–50], the drag constant for this car when racing is

$$\kappa = \tfrac{1}{2} C_D \rho_{\text{air}} A = 0.5\,(0.3)\,(1.29)\,(1.4 \times 1.8) = 0.5. \tag{26}$$

The information collected in table 1 represents only three constraints (the first three rows of the table) on the engine parameter set, S_E. Therefore, one of the engine parameters must be eliminated or fixed. To this end, let $f_2 = 0$ ($f_1 < 0$) in (12), thereby assuming a simple linear fit to the engine's effective transmitted force. The maximum power then occurs at

$$v_P = \begin{cases} v_* \equiv -\dfrac{f_0}{2 f_1}, & \text{if} \quad v_* < v_{\max} \\ v_{\max}, & \text{otherwise.} \end{cases} \tag{27}$$

The simplest way to solve for $\{f_0, f_1, F_F\}$ in terms of $\{v_{\max}, P_{\max}, v_{60}(t_{60})\}$ is to iterate. First guess an initial value for $\tilde{f}_0$, then,

(1) calculate f_1 from (17)

$$f_1 = \kappa v_{\max} - \frac{\tilde{f}_0}{v_{\max}}, \tag{28}$$

(2) calculate v_- using $v_+ v_- = \tilde{f}_0/\kappa$ and $v_+ = v_{\max}$

$$v_- = \frac{\tilde{f}_0}{\kappa v_+}, \tag{29}$$

(3) calculate t_0

$$t_0 = \frac{M}{\kappa\,(v_+ + v_-)}, \tag{30}$$

(4) calculate $\tilde{f}_0$ using (23)

$$\tilde{f}_0 = \frac{\kappa\,(v_- + v_+\,\mathrm{e}^{-2t_{60}/t_0})\,v_{60}}{1 - \mathrm{e}^{-2t_{60}/t_0}}. \tag{31}$$

Using the data in table 1, this iterative procedure converges to $\tilde{f}_0 = 10,051.7$ N and $f_1 = -126.5$ kg s^{-1}. Then, using (16) and (17) gives f_0 and F_F. The values of all the parameters can be found in table 2.

Figure 5 shows the force and power profiles for the simplified engine model used above.

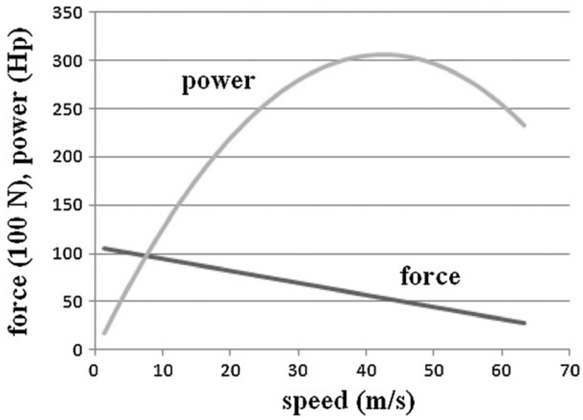

Figure 5. Engine power and force profiles.

6. The free-falling car

The analysis of the falling car is very similar since its equation of motion is

$$mg - \kappa' v^2 = m\ddot{y} \tag{32}$$

which admits the solutions above with the following identifications:

$$\tilde{f}_0 = mg, \tag{33}$$

$$f_1 = 0, \tag{34}$$

$$\tilde{\kappa} = \kappa'; \tag{35}$$

then,

$$v'_+ = v'_- \equiv v'_T = \sqrt{\frac{mg}{\kappa'}}. \tag{36}$$

For the falling car (in horizontal attitude) [1, p 142]

$$\kappa' = \tfrac{1}{2} C'_D \rho_{\text{air}} A' = 0.5(1)(1.29)(4.6 \times 1.8) = 5.3, \tag{37}$$

so $v'_T = 54.4\ \text{m}\,\text{s}^{-1}$. This implies that

$$v'(t) = v'_T\ \frac{1 - \mathrm{e}^{-2t/t'_0}}{1 + \mathrm{e}^{-2t/t'_0}}, \tag{38}$$

$$y'(t) = v'_T t + \lambda' \ln\left(\frac{1 + \mathrm{e}^{-2t/t'_0}}{2}\right), \tag{39}$$

where

$$\lambda' = \frac{m}{\kappa'} = 302\ \text{m}, \tag{40}$$

$$t'_0 = \frac{\lambda'}{v'_T} = 5.6\ \text{s}. \tag{41}$$

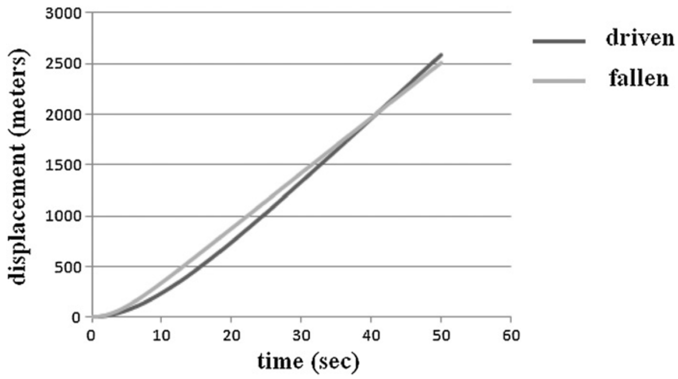

Figure 6. Plot displaying the motion of the two cars.

These numbers imply that the racing car will only overtake the falling one after 40.8 s and at a distance of 2011 m (see figure 6). The television commercial realizing this race claimed that the racing distance was 1220 m. Based on the present analysis, the racing car would need a head start of a few seconds to overtake the falling car over such a distance, which is presumably what was done in the commercial since in the commercial the driven car wins [3].

7. Conclusions

Using relatively basic mechanical principles, the outcome of a race between a free-falling car and a road-driven car can be predicted. Since air resistance is expected to play a significant role in the matter, it is included in the theoretical model, as well as other frictional effects associated with the driven car's inner workings. Also the engine's performance is modelled by a quadratic torque curve. The resulting differential equation of motion turns out to be separable and straightforward to solve with elementary calculus techniques. As in the advertisement, the falling car was assumed to fall in a horizontal attitude which significantly affects its aerodynamics. When representative data for a luxury sports sedan is fitted to the model, the results indicate that the race can go either way depending on the racing distance. For shorter distances, the falling car wins, but for longer distances the road-driven car triumphs. The threshold distance turned out to be about 2000 m. It is interesting to note that in the television commercial realizing this race, the distance chosen was only about 1200 m, perhaps because of the constraints involved in air lifting a car. The falling car should have won, but a head start was given to the driven car to ensure victory.

From the form of the solutions, equations (25) and (39), it follows immediately that the asymptotic behaviour favours the car with the larger terminal velocity, which depends crucially on the drag coefficient. Had the falling car assumed a vertical attitude, i.e. fallen head-on, the victory would belong to it irrespective of the distance.

References

[1] Parker B 2003 *The Isaac Newton School of Driving* (Baltimore, MD: John Hopkins University Press)
[2] Lexus *'IS' Detailed Specifications* http://www.lexus.com/models/IS/detailed_specifications.html (accessed 29 August 2011)
[3] http://www.youtube.com/watch?v=IwrCHvmGT78&feaure=related (accessed 29 August 2011)