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Comparisons of elastic and creep deformation linearly dependent upon stress

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The theory of linear elasticity provides a complete description of reversible deformation under small stresses for both isotropic and anisotropic solids. At elevated temperatures, creep deformation sometimes occurs at a rate that is linearly dependent upon stress. When this form of creep arises from vacancy movement, there is possibility of anisotropic behaviour through the orientational dependence of average grain dimensions. This indicates that the elasticity theory may be utilised to provide comparable descriptions of such creep deformation, with creep strain built up of equal increments of strain occurring in equal intervals of time. The extent of this analogy is explored with the conclusion that its usefulness is substantial when grains are small in relation to geometrical features of the component but it is no longer applicable when the grains approach the size of these features and where there is a high gradient of stress.

Keywords: Elasticity, Anisotropy, Creep, Deformation geometry

Introduction

The theory of linear elasticity is well established and continues to provide an invaluable tool for the design of structures and the selection of materials. At elevated temperatures, creep may occur under low stresses at a rate that varies linearly with stress. This suggests that analyses of elastic response to stress might be carried over to predict the effects of this form of creep behaviour under multiaxial stresses. The use of such analogies may be of particular value when the material is anisotropic so that, in the creep case, coefficients may be defined in a way that is analogous to the designation of compliance coefficients in elasticity. A detailed examination of such similarities is undertaken here that identifies the extent to which the theory of elasticity may be employed to predict behaviour in the creep regime both for isotropic and anisotropic materials. Notably, for the creep situation where a linear dependence of creep rate on stress is indicative of a diffusional creep process, linking with elastic behaviour through the concept of an effective relaxation time provides valuable new insights. However, such an analogy breaks down and the concept becomes inapplicable when there is a high stress gradient and when the grain dimensions are of a size approaching the smallest geometrical feature of the component under stress. It is then necessary to evaluate the different patterns of atomic flux in individual grains for which there is no elastic analogue.

Formal representations of elasticity and creep linearly dependent on stress

Linear elastic behaviour is governed by Hooke's law, whereby in the simplest situation, a small reversible elastic strain ε is induced in a solid in the direction of an applied tensile stress σ such that

$$\varepsilon=(1/E)\sigma \tag{1}$$

where E is Young's modulus of the material.

Correspondingly, if an elastic shear strain γ is induced by a shear stress τ, the relationship is written

$$\gamma=(1/G)\tau \tag{2}$$

where G is the shear modulus.

While elasticity is a manifestation of reversible change in atomic spacing, at elevated temperatures atomic movement is facilitated and deformation becomes time dependent and irreversible. This leads to the creep of solids which in the steady state, is often described[1] by the Norton equation $\mathrm{d}\varepsilon/\mathrm{d}t=K\sigma^{\mathrm{n}}$ where n is the stress exponent and K is dependent on the material and its microstructure and on temperature.

At low stresses, for many solids, it is found[2] that $n\approx1$ so that the creep rate is linearly dependent on stress, thus providing a formal analogy with equation (1). This behaviour is often referred to as Newtonian flow, where the strain rate *v.* stress relationship is

$$\mathrm{d}\varepsilon/\mathrm{d}t=K\sigma \tag{3}$$

When linked to flow rate under shear, such behaviour is familiar in formulation of the viscous flow of liquids where the flow rate $\mathrm{d}\gamma/\mathrm{d}t$ under shear stress τ is written

$$\mathrm{d}\gamma/\mathrm{d}t=(1/\eta)\tau \tag{4}$$

where η is the coefficient of dynamical viscosity. Equations (2) and (4) allow a relationship[3] between η and G that can be written in the form

$$\eta=Gt_{\mathrm{r}} \tag{5}$$

by the identification of a relaxation time t_{r} that

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corresponds to the time period over which the increment of non-reversible flow is equal to the elastically induced strain.

Although solid and liquid phases can generally be considered distinct, it is acknowledged[3] that for materials without a sharp melting temperature, it is only possible to ascribe the characteristics of solid or liquid behaviour by arbitrarily selecting a specific value of the viscosity η generally taken to be above or below 10^{14} Pa s. An effective viscosity can similarly be defined for crystalline materials under low stresses, reflecting the wide applicability of these considerations.

Some features of Newtonian flow

A variety of mechanisms,[4–8] that have been extensively debated,[9–18] may lead to the linear strain rate *v.* stress relationship that characterises Newtonian flow in solids. For some materials, there is evidence to suggest that a stress directed diffusional process, first proposed by Nabarro,[4] can be operative whereby vacancies diffuse between and along grain boundaries that act as vacancy sources and sinks depending on their orientation with respect to the stress. This mechanism has a sound theoretical basis that is supported by experimental data, for several but not all materials, through identification of the effects of temperature and of grain size and shape as well as by confirmatory observations[19] of predicted microstructural changes. Results on some other materials including aluminium[6] have shown different relationships suggesting that alternative mechanisms may be operative. These results, however, have proved difficult to reproduce over an extensive range. It has also been pointed out[15] that for significant strains, there might be a parabolic rather than linear strain rate *v.* stress relationship, which would take this area outside the scope of the present considerations.

When a diffusional process is operative, resulting in the stress directed diffusion of vacancies between differently oriented grain boundaries, there is a counter flow of atoms so that creep occurs at a rate that is linearly dependent upon stress. Analysis of this process,[8] when the temperature is sufficiently high for lattice diffusion to be predominant, leads to a formula for creep rate under a tensile stress

$$d\varepsilon/dt = (BD\Omega/kTd^2)\sigma \tag{6}$$

where D is the lattice diffusion coefficient, d is the grain size, Ω is the atomic volume, k is Boltzmann's constant, T is absolute temperature and B is a dimensionless constant. This can readily be put in terms of shear stress and strain rate by noting that the flow of plastically deforming solids occurs at constant volume. In this case, an analogous Poisson's ratio $v=0{\cdot}5$ can be ascribed. The elastic analogy here is that $E=2G(1+v)=3G$ and so, it follows that $(d\varepsilon/dt)/\sigma=(d\gamma/dt)/3\tau$. Hence, introducing the new constant B_1 ($=3B$), equation (6) takes the form

$$d\gamma/dt = (B_1 D\Omega/kTd^2)\tau \tag{7}$$

It follows from equations (4), (5) and (7) that an increment of creep strain reaches the magnitude of the elastic strain in a time t_r where

$$t_r = kTd^2/B_1 GD\Omega \tag{8}$$

The similarity of this form of creep in solids with viscous flow in liquids now becomes apparent. If a liquid is regarded essentially as an amorphous solid with grain dimensions reduced to the size of single atoms with atomic spacing b, then $d\approx b$ and $\Omega\approx b^3$, so that the viscous flow of a liquid becomes comparable with the process of diffusional creep. Following the notations in equations (2) and (7), the relaxation time for the liquid $t_{rL}=kT/B_1G\ D_Lb$, where D_L is now the self-diffusion coefficient of the liquid. From this, we arrive at the equation for viscosity that followed from the work of Einstein and was presented by Frenkel[20] in the form

$$\eta = Gt_{rL} = kT/6\pi D_L b \tag{9}$$

The dimensionless coefficient B in equation (6) is calculated[21] to be ~ 8 so that $B_1=3B\approx 24$ which is close to the value of $6\pi\approx 19$ in equation (9) to confirm the validity of this comparison.

The extent of this analogy between diffusional creep in solids and viscous flow in liquids will not be pursued further but it illustrates the underlying basis of the present approach and the concept of relaxation time that it will be considered later in relation to the limits of its applicability.

Since equation (9) relates to liquid behaviour, there is no indication of factors that may lead to anisotropy. This is a distinguishing feature from the diffusional creep of solids where anisotropy may be introduced into equation (7) through modification of the grain dimensional term d when its value may not be the same in all directions.[21] A viable interpretation of this aspect is important since it opens the opportunity to utilise the theory of anisotropic elasticity to assist in the evaluation of the effects of the time dependent process of diffusional creep.

Analogies between diffusional creep and elasticity in non-isotropic materials

Representation of elastic anisotropy

The well established theory of elasticity[22,23] can incorporate the effects of both multiaxial stresses designated σ_{kl} and anisotropy of the material to induce corresponding strains ε_{ij}. These terms can be linked through compliance coefficients[23] that in the general case, form the components of a fourth rank tensor S_{ijkl} such that

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl}\ (i, j, k, l = 1-3) \tag{10}$$

Writing equation (10) in expanded form leads to nine equations, each with nine terms with the requirement of 81 constants in total. However, ε_{ij} and σ_{kl} are symmetric tensors from which it follows that $S_{ijkl}=S_{ijlk}$. Because of this symmetry, only 36 of the compliance coefficients are independent and distinct terms. These terms can be defined in a 6×6 compliance matrix and adopting a simpler notation, the following can be written to represent the six equations contained within the matrix

$$\varepsilon_i = \sum_{j=1}^{6} S_{ij}\,\sigma_j \quad (i=1-6) \tag{11}$$

Further simplifications can be made by considering a material with aligned grains of orthorhombic symmetry such that there are three mutually perpendicular mirror planes. With the axes of loading normal to these mirror planes, we can make use of the relationships of orthotropic symmetry so that the compliance matrix can be reduced to 12 compliance coefficients and written in the form

$$S_{ij} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & & & \\ S_{21} & S_{22} & S_{23} & & & \\ S_{31} & S_{32} & S_{33} & & & \\ & & & S_{44} & & \\ & & & & S_{55} & \\ & & & & & S_{66} \end{bmatrix} \quad (12)$$

Further reduction ensues since $S_{ij}=S_{ji}$. Thus in the compliance matrix equation (12), only nine of the coefficients are independent.

Perpendicular to the mirror planes, the respective values of Young's moduli E_1, E_2 and E_3 are then represented by $S_{11}=1/E_1$, $S_{22}=1/E_2$ and $S_{33}=1/E_3$. The three shear moduli are correspondingly given by $G_{23}=1/S_{44}$, $G_{31}=1/S_{55}$ and $G_{12}=1/S_{66}$.

Poisson's ratio v_{ij} is defined as the negative of the strain in the j direction divided by the strain resulting from the stress in the (perpendicular) i direction so that $v_{12}=-S_{21}/S_{11}$. Recalling that reciprocal conditions require that $S_{21}=S_{12}$, it follows that $v_{12}/E_1=v_{21}/E_2$.

These considerations, involving both the orientation dependent Young's and shear moduli, now allow the influence of multiaxial stresses and the response of material that is elastically anisotropic to be determined, as described by the following compliance matrix for a material with orthotropic symmetry

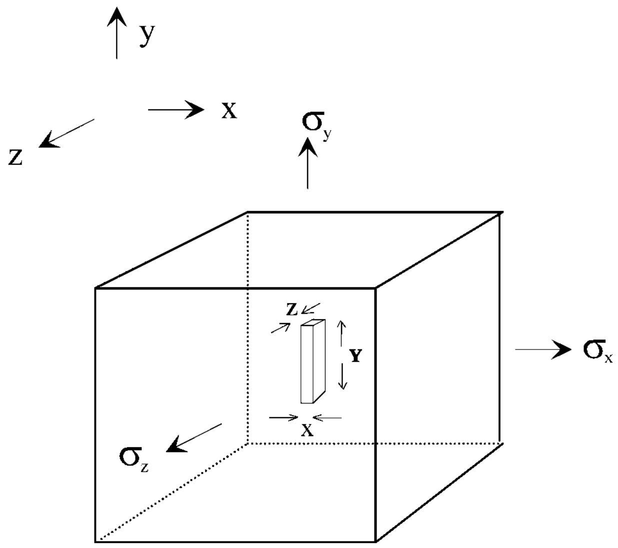
$$S_{ij} = \begin{bmatrix} \frac{1}{E_1} & \frac{-v_{21}}{E_2} & \frac{-v_{31}}{E_3} & & & \\ \frac{-v_{12}}{E_1} & \frac{1}{E_2} & \frac{-v_{32}}{E_3} & & & \\ \frac{-v_{13}}{E_1} & \frac{-v_{23}}{E_2} & \frac{1}{E_3} & & & \\ & & & \frac{1}{G_{23}} & & \\ & & & & \frac{1}{G_{31}} & \\ & & & & & \frac{1}{G_{12}} \end{bmatrix} \quad (13)$$

In utilising the above relationships and noting geometrical representation in Cartesian coordinates, the three equations relating the tensile stresses and strains can be illustrated by taking σ_x (to replace σ_1) and ε_x (to replace ε_1), etc. Similarly, the shear stresses and strains can be written respectively with τ_{yz} (replacing σ_4) and γ_{yz} (replacing ε_4), etc.

With orthogonal stresses σ_x, σ_y and σ_z acting along these axes, it follows from equation (13) that the strain in the x direction is given by

$$\begin{aligned} \varepsilon_x &= \sigma_x/E_1 - \sigma_y v_{21}/E_2 - \sigma_z v_{31}/E_3 \\ &= (1/E_1)(\sigma_x - \sigma_y v_{12} - \sigma_z v_{13}) \end{aligned} \quad (14)$$

It is now relevant to enquire about the usefulness of these firmly based features of elasticity theory to the analysis of creep at a rate that is linearly dependent on stress.



1 To illustrate the notation throughout the paper, an idealised embedded grain is shown with dimensions *X*, *Y* and *Z* lying respectively in directions of the *x*, *y* and *z* axes

Formal representation of anisotropy in diffusional creep

It is often found experimentally that the resistance to creep at low stresses is the greatest when the largest principal stress is aligned with the longest grain dimensions.[24]

In considering such dimensions, orthotropic symmetry provides a useful and valid, though oversimplified, description of anisotropic grain shape. Such a description, however, is convenient since for polycrystalline materials under conditions in which a diffusional creep process is expected to operate, it allows a direct appraisal of the possible representation of behaviour through equations that are analogous to those of equations (12) and (13).

For diffusional creep in which the average grain dimensions (small in comparison with the specimen dimensions) in the x, y and z directions are respectively X, Y and Z, aligned with the principal stress directions as illustrated in Fig. 1, it has been shown[25] that at sufficiently high temperatures where lattice diffusion predominates, a corresponding matrix can be established, analogous to that in equation (13)

$$S_{ij} = K_1 \begin{bmatrix} \frac{(Y^2+Z^2)}{\beta} & \frac{-Z^2}{\beta} & \frac{-Y^2}{\beta} & & & \\ \frac{-Z^2}{\beta} & \frac{(Z^2+X^2)}{\beta} & \frac{-X^2}{\beta} & & & \\ \frac{-Y^2}{\beta} & \frac{-X^2}{\beta} & \frac{(X^2+Y^2)}{\beta} & & & \\ & & & \frac{4}{(Y^2+Z^2)} & & \\ & & & & \frac{4}{(Z^2+X^2)} & \\ & & & & & \frac{4}{(X^2+Y^2)} \end{bmatrix} \quad (15)$$

where $K_1=12D\Omega/kT$ and $\beta=X^2Y^2+Y^2Z^2+Z^2X^2$.

If stresses σ_x, σ_y and σ_z act along these axes, it follows from equation (15) that the rate of creep in the x direction is given by

$$(\mathrm{d}\varepsilon/\mathrm{d}t)_x = K_1[\sigma_x(Y^2+Z^2)-\sigma_y Z^2-\sigma_z Y^2]/(X^2Y^2+Y^2Z^2+Z^2X^2) \quad (16)$$

Thus, there is a direct analogue with the elastic situation seen by comparing equations (14) and (16) where Young's modulus E_1 in the x direction corresponds to the term $(X^2Y^2+Y^2Z^2+Z^2X^2)/K_1(Y^2+Z^2)$ in the creep equation.

This can be simplified if we only wish to compare the relative strengths in different directions. In the elastic case, this involves determination of the ratios E_1/E_2, E_2/E_3 and E_3/E_1 and in the creep situation, to the ratios of creep strength in the x and y, y and z, and z and x directions. In the latter case, the creep strength ratios are dependent only on the average grain dimensions in the directions under consideration as indicated in Table 1.

Similarly, the ratios of the shear strengths can be considered in the different directions. These are evaluated from the corresponding terms in equations (13) and (15) and are listed in Table 2.

A diffusional creep analogue to Poisson's ratio in the elastic condition can further be identified that can completely be evaluated numerically in terms of the grain dimensions, without the need for further information. Comparing the elastic and diffusional creep matrices, exemplified by equations (13) and (15) respectively, it is noted that ν_{21}/E_2 corresponds to K_1Z^2/β and E_2 corresponds to $\beta/K_1(Z^2+X^2)$ so that ν_{21} can be equated to $Z^2/(Z^2+X^2)$. Similarly, this is reflected in the relationships that are listed for all the ν values as in Table 3.

On this basis, it is apparent that the extensive theoretical analyses relevant to problems in anisotropic elasticity may be transferred to comparable problems where the creep of materials with anisotropic grain shapes is governed by the stress directed diffusion of vacancies. Before enquiring about any limits to this transferability, the effects of grain boundaries will next be considered.

Influence of grain boundaries

Grain boundaries and free surfaces play a major role in diffusional creep and it is implicit in the theory of this mechanism that they are required to act as perfect vacancy sources and sinks[26] whereby the time for emission or absorption of a vacancy at a grain boundary or free surface is negligible compared with the time for diffusion between the source and the sink. This role has not been clearly established in all materials. While there is substantial evidence of this behaviour in copper, magnesium, nickel and several other materials, there is no definitive evidence of a similar property of grain boundaries in some materials including aluminium. The present analysis is applicable only to those materials in which there is perfect vacancy source and sink action.

Table 1 Comparisons of directional Young's moduli and tensile creep strengths

Ratio of directional Young's moduli	Ratio of tensile creep strengths determined by grain shape
E_1/E_2	$(Z^2+X^2)/(Y^2+Z^2)$
E_2/E_3	$(X^2+Y^2)/(Z^2+X^2)$
E_3/E_1	$(Y^2+Z^2)/(X^2+Y^2)$

There is a further important property of the grain boundaries and surfaces for consideration, where below some temperature, they provide diffusion paths for vacancy fluxes[5] that predominate over those provided by lattice diffusion.

While this alteration of diffusion paths influences the compliance coefficients in diffusional creep, it does not change the overall conclusions about the analogy with linear elasticity or the conditions under which there is a deviation from such analogy.

There is need for some modification of the previous analysis, based on lattice diffusion, that is necessary in analysing cases where grain boundary diffusion dominates.

For fluxes confined to the grain boundaries, a matrix similar to that of equation (15) can be compiled[23] to evaluate S_{ij} that incorporates the modified values of the compliance coefficients that are represented as in equation (12). This now takes the form

$$S_{ij} = K_2\begin{bmatrix} \frac{(Y^2+Z^2)}{\phi} & \frac{-Z^2}{\phi} & \frac{-Y^2}{\phi} & & & \\ \frac{-Z^2}{\phi} & \frac{(Z^2+X^2)}{\phi} & \frac{-X^2}{\phi} & & & \\ \frac{-Y^2}{\phi} & \frac{-X^2}{\phi} & \frac{(X^2+Y^2)}{\phi} & & & \\ & & & \frac{4}{YZ(Y+Z)} & & \\ & & & & \frac{4}{ZX(Z+X)} & \\ & & & & & \frac{4}{XY(X+Y)} \end{bmatrix} \quad (17)$$

where $K_2=24D_g w\Omega/kT$ and $\phi=XYZ(XY+YZ+ZX)$.

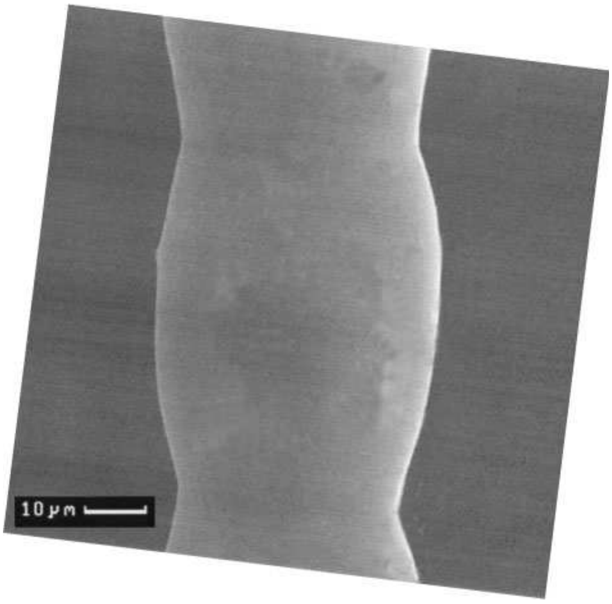
D_g is the grain boundary diffusion coefficient and w the grain boundary width. (It will be shown later that any one of the three coefficients corresponding to S_{44}, S_{55} and S_{66} in the bottom right of the matrix, representative of shear resistance, becomes less accurate if the grain dimension parallel to the shear axis is significantly less than the other grain dimensions. It is then replaced by a modified term. This special case is dealt with in the Appendix.)

Table 2 Comparisons of directional shear moduli with creep strengths in shear when lattice diffusion is predominant

Ratio of directional shear moduli	Ratio of creep strengths in shear determined by grain shape
G_{12}/G_{23}	$(X^2+Y^2)/(Y^2+Z^2)$
G_{23}/G_{31}	$(Y^2+Z^2)/(Z^2+X^2)$
G_{31}/G_{12}	$(Z^2+X^2)/(X^2+Y^2)$

Table 3 The creep analogue of Poission's ratio and influence of grain shape

Creep analogue of Poisson's ratio	Dependence on grain shape
ν_{12}	$Z^2/(Y^2+Z^2)$
ν_{13}	$Y^2/(Y^2+Z^2)$
ν_{23}	$X^2/(Z^2+X^2)$
ν_{21}	$Z^2/(Z^2+X^2)$
ν_{31}	$Y^2/(X^2+Y^2)$
ν_{32}	$X^2/(X^2+Y^2)$



2 Surface profile, after creep, of a copper wire (50 μm in diameter) with bamboo type microstructure formed by grains of length ~65 μm along wire axis. A strain of 5·6% was achieved at 1263 K at a strain rate of $1\cdot1\times10^{-7}$ s^{-1} under an applied stress of 190 kPa. The zero creep stress was 40 kPa. Note the non-uniform contraction of the wire diameter with the minima centred at grain boundaries that are perpendicular to axis

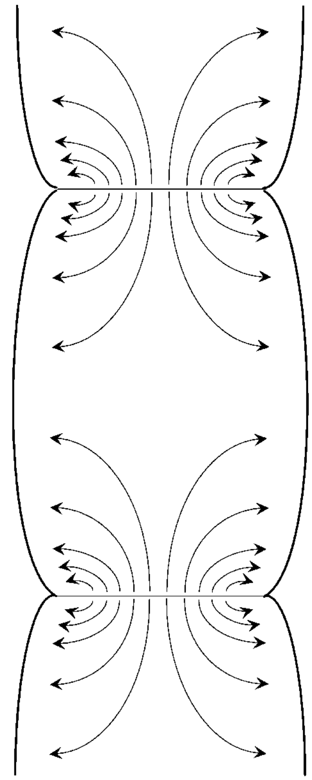
Equation (17) provides an analogous formulation to that in equation (15) with these two equations representing respectively the situations where the fluxes are controlled by lattice and by grain boundary diffusion. There are differences in their terms but they all remain analogues of compliance coefficients familiar in the theory of linear elasticity. It is noted that the numerators of the terms representing the tensile coefficients are the same in both these equations. It follows that Table 1, giving the creep strength ratios in different directions, and Table 3, providing the values of the creep analogue to Poisson's ratio, are independent of whether the vacancy flux is predominantly through the crystal lattice or through the grain boundaries. Hence, these Tables 1 and 3 are valid in both cases. There is some difference, however, for the shear coefficients and so, for the case where grain boundary diffusion predominates, Table 4 below replaces Table 2.

Limits to analogy between elasticity and creep linearly dependent on stress

Our analysis, so far, indicates that deformation by diffusional creep can reproduce essentially the same

Table 4 Comparisons of directional shear moduli with creep strengths in shear when grain boundary diffusion predominates

Ratio of directional shear moduli	Ratio of creep strengths in shear determined by grain shape
G_{12}/G_{23}	$X(X+Y)/Z(Y+Z)$
G_{23}/G_{31}	$Y(Y+Z)/X(Z+X)$
G_{31}/G_{12}	$Z(Z+X)/Y(X+Y)$



3 Schematic illustration of the pattern of vacancy flux underlying the form of deformation observed in Fig. 2. It is noted that the flux to the surface is greatest in the regions of the grain boundaries to account for the greatest reduction in wire diameter in these regions

macroscopic change of overall shape, in both isotropic and non-isotropic materials as that created (albeit on a strictly limited scale) by elastic deformation.

Some small differences may first be noted. There is the influence of Poisson's ratio v. For isotropic materials undergoing elastic deformation, its value is typically in the range 0·25–0·33. In contrast, during creep, the volume remains almost constant and this implies that for isotropic materials, the corresponding value of v is 0·5. There are comparable differences between the elastic and creep situations in the value of v in the case of anisotropic materials but the situation becomes more complex. Six values of v are then required to describe both the elastic and creep responses, as listed in Table 3. The formal matrix representations however remain similar for both cases although the numerical values will differ. In the elasticity case, the values of v depend upon the relative strengths of the atomic bonds but in the diffusional creep analogue, they are determined by the grain shape anisotropy.

Other, rather more significant, departures from the correspondence become apparent when some specific cases are examined. This can be demonstrated, for example, on wires with bamboo grain structures in experiments along the lines that led to the first experimental support for the predicted operation of the diffusional creep mechanism. It is seen from Fig. 2 that when a bamboo structured wire is under a small tensile stress at elevated temperature, elongation of the wire by diffusional creep is accompanied by preferential thinning in the regions close to grain boundaries. The lack of uniformity in the reduction of diameter, in contrast with the uniformity of elastic deformation, is a consequence of the pattern of vacancy fluxes, illustrated schematically in Fig. 3, since vacancy sinks at the free surfaces are not constrained to accept equal numbers of vacancies per unit area over the entire cylindrical surface. For specimens in which the grain dimensions are substantially less than the wire diameter, this effect is negligible since behaviour is then dominated by the effect of the internal grain boundaries that are constrained by the need for maintaining contact between neighbouring grains. There is then complete correspondence between the elastic and creep deformation geometries except for the numerical differences arising from the effect of Poisson's ratio.

For an internal grain that is small compared with specimen dimensions, vacancy flux patterns within it are subject to boundary conditions imposed by the need to maintain grain contact. For real grain shapes, this aspect gives rise to substantial complexity in modelling but for some simple idealised shapes, analytical solutions can be readily derived. Notably, for a small internally embedded cubic grain in a large specimen under plain strain conditions, lines of vacancy equipotential with corresponding orthogonal lines of vacancy flux have a simply defined geometrical pattern as illustrated in Fig. 4. Here, there is no influence of Poisson's ratio with a consequent identical geometry of deformation for both the elastic and creep situations. It is further noted that the flux lines are two-dimensional and are independent of the grain dimension in the z direction. This flux pattern is linked to the internal stress distribution that exists on the pairs of grain faces where there is a parabolic variation of stress that is respectively tensile and compressive.

It is noted that throughout the above discussions, the external stress has been applied uniformly. It is recalled that the applications of the theory of elasticity are independent of this condition. It is now important to enquire 'does diffusional creep strength have a similar independence?'. This question will next be considered.

Non-uniform loading

The bending of a beam represents a simple situation of non-uniform loading. In the elastic condition, a thin beam with second moment of area I bends to a radius R under a bending moment M where

$$R = EI/M \tag{18}$$

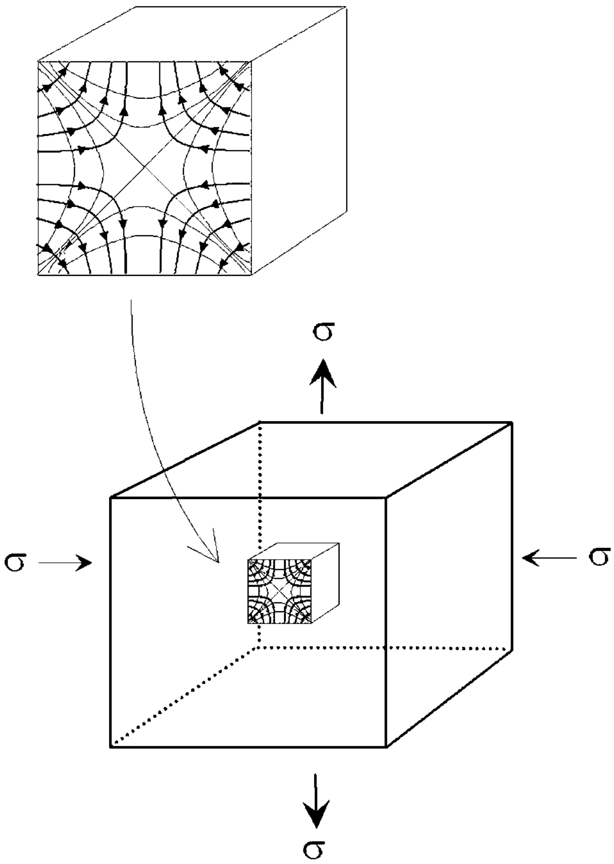
$I = sh^3/12$ where s is the beam dimension parallel to the bending axis and h is the beam thickness, so the curvature $1/R$ is related to the beam dimensions[3] such that

$$I/R = 12M/Esh^3 \tag{19}$$

The strain parallel to the beam length is tensile at the convex surface and compressive on the concave side and is determined by $\varepsilon = \sigma/E$ where σ represents the tensile stress. Correspondingly, in the creep situation, the rate of change of curvature is given by $\mathrm{d}/\mathrm{d}t(1/R) = M(\mathrm{d}\varepsilon/\mathrm{d}t)/I\sigma$. When the grain size is small compared to the beam thickness, we can substitute for $\mathrm{d}\varepsilon/\mathrm{d}t$ and σ from equation (6) to obtain

$$\mathrm{d}/\mathrm{d}t(1/R) = 12BMD\Omega/kTd^2sh^3 \tag{20}$$

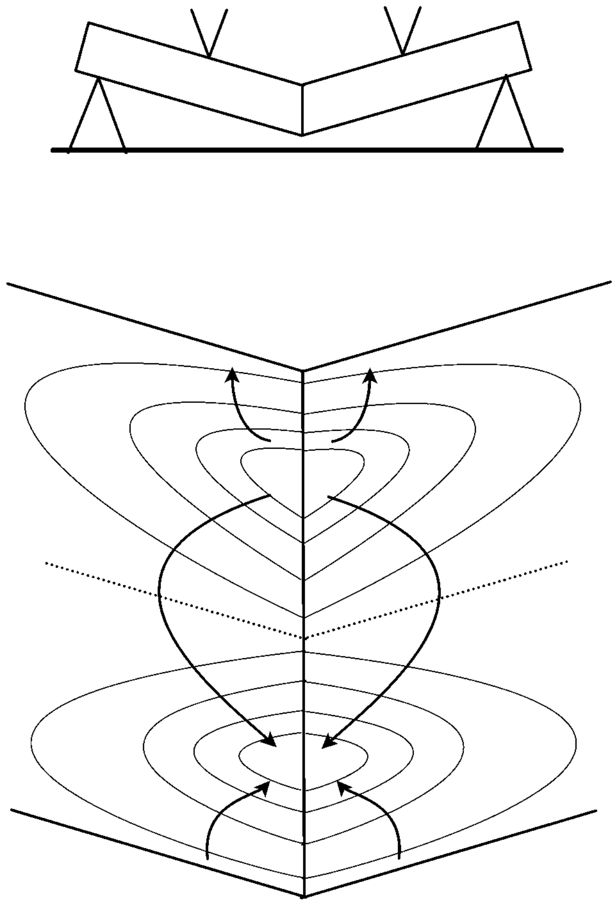
It is noted that the beam dimensions s and h have the



4 Schematic illustration of vacancy flux pattern in an idealised embedded cubic grain with equal tensile and compressive stresses applied perpendicularly to two pairs of faces. Here the geometry of deformation in diffusional creep is identical to that which would be produced elastically with each elastic increment of strain reproduced after a specific relaxation time. The lines of flux are two-dimensional and so, are independent of the grain dimension in the *z* direction. These lines take the form of rectangular hyperbolae that are orthogonal to lines of vacancy equipotential. There is internal stress redistribution with a parabolic stress variation that is tensile on grain boundaries perpendicular to the tensile stress and compressive on boundaries perpendicular to the compressive stress

same effect on the rate of change of curvature by diffusional creep as they have on the curvature that is induced by elastic deformation[27] described respectively by equations (20) and (19). It is clear that the two situations can alternatively relate elasticity and Newtonian creep through the concept of a relaxation time t_r as in equation (8). The above problem can be approached in this alternative way and shown to lead to the same conclusion where the relaxation time has the value $kTd^2/BDE\Omega$.

Further consideration, however, will show that such a relationship does not cover all situations. It is now apparent (as first pointed out for grain boundary diffusion[28]) that in some circumstances, diffusional creep, although at a rate linear to stress, can produce a shape change that is entirely different from the geometry that results from elastic deformation. Such a case arises in extreme form when a bending moment is applied to a thin beam comprising a bicrystal with its grain boundary perpendicular to the beam length (Fig. 5). There is then



5 Upper diagram shows bending of a thin but wide beam comprising a bicrystal with the common grain boundary parallel to the axis of bending. Lower diagram shows the pattern of vacancy flux orthogonal to the vacancy equipotential lines in the grain boundary region. Note that the upper and lower surfaces, as well as the grain boundary, act as vacancy sources and sinks. There is no elastic analogue to this situation

a continuous variation of stress across the boundary creating the maximum tensile strain, normal to the boundary, at the convex surface decreasing to maximum compressive strain at the concave surface. Vacancy movement is associated with an internal stress redistribution across the grain boundary and a vacancy flux, centred on the grain boundary and perpendicular to the lines of vacancy equipotential, is illustrated in the lower diagram in Fig. 5. When the temperature is sufficiently high for lattice diffusion to predominate, this flux induces rotation[29] between the adjacent grains at a rate $d\theta/dt$ given by

$$d\theta/dt = \alpha_L MD\Omega/kTsh^4 \quad (21)$$

where α_L is a dimensionless numerical constant and the value of s and the dimensions of each grain perpendicular to their common boundary are all substantially greater than h.

Considering the situation described by equation (21), a beam containing several such grain boundaries that are parallel and at an average distance x (much greater than h) apart with the bending moment similarly applied, then there is rotation between each of the grains, with an effective hinge at their common grain boundaries separating the straight segments. This leads[27] to an effective overall rate of change of curvature $d/dt(1/R)$ of the beam given by

$$d/dt(1/R) = (d\theta/dt)/x = \alpha_L MD\Omega/kTsh^4 \quad (22)$$

It follows from equation (22) that under a given bending moment, this rate of change of curvature is determined by beam dimensions and is inversely related to h^4, in contrast with the elastic condition in equation (21) where the rate is inversely related to h^3. This demonstrates the impossibility of analysing such a creep situation through the concept of a relaxation time and illustrates that there is no way in which the accumulation with time of increments of elastic deformation can replicate the geometrical dependence that can arise through diffusional creep.

This difference in the elastic and creep relationships is magnified at lower temperatures when grain boundary diffusion predominates, since it can be shown that

$$d/dt(1/R) = \alpha_g MD_g w\Omega/kTsh^5 \quad (23)$$

where α_g is a dimensionless numerical constant.

It is instructive to examine these differences by comparing equations (22) and (23) with equation (20). The origin of their differences lies in the variation of the patterns of diffusional fluxes when grain dimensions approach the thickness of the beam. When many grains are contained within the thickness of the beam, then diffusional creep produces a mode of deformation similar to that which can be progressively built up by an accumulation of elastic strain increments. This is not the case when only few grains lie across the thickness of the beam.

Analyses of this kind can employ numerical methods to cover other beam geometries. A notable example is the case where the beam is of circular cross-section with a bamboo grain structure. Here, the internal stress across each grain boundary, at a temperature where grain boundary diffusion predominates, has been calculated.[30] Since a wire with such microstructure and geometry can be prepared for experimental study, this theoretical analysis now allows determination of grain boundary self-diffusion coefficients on materials for which a measurement of this property may otherwise be difficult.

In general, where one or only a few grains lie across the beam thickness, the stress gradient across individual boundaries creates diffusional fluxes along the grain boundary regions that exceed the fluxes that are routed between boundaries of differing orientation. There is no elastic analogue to this situation.

In contrast, when the grain size is sufficiently small for many grains to be encompassed within the beam thickness, the variation in stress across any individual grain boundary is much less than the variation that is created between boundaries of different orientation. It is then appropriate to consider the vacancy source and sink action of differently oriented grain boundaries under the influence of the tensile and compressive stresses acting respectively in the upper and lower parts of the beam. This is the mechanism envisaged in the derivation of equation (6) that underlies the derivation of equation (21). The concept of a relaxation time linking elastic and creep deformation can be employed in dealing with this situation.

Comparison of equations (22) and (23) with equation (21) highlights the change in formulation that is required by difference in grain size in relation to beam dimensions. In equation (21), it is noted that the effect of beam dimensions on creep rate is the same as its effect on the extent of elastic deflection. This is in contrast with the result of equation (22) where the beam thickness has a larger effect in diffusional creep than in elastic deflection. The effect is further enhanced, as in equation (23) when grain boundary diffusion predominates. It is apparent that when the grain dimensions approach the size of the minimum dimension of the deforming material, the elastic analogy breaks down. Then, there is no way in which the characteristics of deformation can be considered in terms of a relaxation time.

The above considerations have significant consequences, for they assist in answering the question 'when can the extensive analyses available to deal with problems in elasticity be confidently applied to the creep condition?'. This question becomes of further importance when matters of anisotropy are considered. Since the theory of anisotropic elasticity has been extensively and rigorously developed over many years, it is pertinent to ask 'to what extent can this development be applied to evaluate the effect of grain shape anisotropy on diffusional creep?'. It is apparent that the analogies of elastic and creep linearly dependent on stress can be usefully applied but only when the largest grain dimensions are much less than geometrical features of the component.

Conclusions

The well established theory of linear elasticity enables the prediction of the reversible, time independent deformation that occurs under stress below some specific value. This can be of particular importance when it is required to evaluate the effects of a complex, multiaxial stress system. It is applicable to materials of all shapes and also, to those whose elastic properties are anisotropic.

Under low stresses and at elevated temperatures, permanent deformation by creep can occur at a constant rate that can be linearly dependent on stress. Thus, there is an immediate analogy with elastic behaviour but with creep continuously producing additive permanent strains. Sometimes, it is convenient to consider the time interval over which an increment of creep strain is induced that is of similar magnitude to the elastic strain.

There is substantial understanding of such creep when it is governed by the stress directed flow of vacancies and this has been extended to include the influence of multiaxial stresses and to materials with anisotropic grain shapes. This allows coefficients in creep to be determined that are analogous to the compliance coefficients familiar in the theory of anisotropic elasticity theory. They can similarly be incorporated into matrices to assist in their practical application.

From exploration of the analogy between this form of creep and linear elastic deformation, the detailed characteristics of both these processes can be compared.

Such comparison identifies the situations where the theories of linear elastic deformation and of creep rate linearly dependent on stress can be mutually helpful, but it also leads to a clear recognition of cases in which there is a marked divergence, where it would be entirely wrong to invoke the assistance of elasticity theory to evaluate the form of deformation induced by diffusional creep.

The conclusions may be summarised as follows.

1. Where the largest grain dimension is substantially less than the smallest dimension of the component in question, there is almost complete correspondence between the geometrical form of deformation that is induced by elastic stressing and at elevated temperatures, by diffusional creep. They can both be analysed in terms of effective compliance coefficients arranged in a convenient matrix form. There is the ability in creep to reproduce continually, as permanent deformation over equal increments of time, additive increments of strain that are each equal to the elastic strain. The concept of a relaxation time can be useful in this respect. Only a small geometrical difference between the effects of these two different forms of deformation occurs from the numerical influence of an effective Poisson's ratio. This ratio is a material constant related to atomic bonding in elasticity theory but arises in creep from the requirement to preserve constant volume and is influenced by the average grain dimensions and their orientation dependence.

2. When grain dimensions approach the size of the smallest geometrical feature of the component in which they are a constituent and particularly, where substantial stress gradients are imposed, the analogies between elasticity and diffusional creep break down. The shape changes produced by the two forms of deformation are fundamentally different. It is then no longer possible to employ the concept of a relaxation time or to utilise parameters in creep in the form of compliance coefficients to analyse the form of deformation that diffusional creep produces. There is a need to determine the individual flux patterns in the constituent grains.

Appendix

The analysis, for the grain boundary diffusion case, leading to equation (17) becomes inaccurate when the shear stress acts on a material with grain dimensions perpendicular to the shear plane smaller than the other two orthogonal grain dimensions.

A more complete analysis[23] for the shear terms, where grain boundary diffusion is predominant and Z is the grain dimension perpendicular to the shear plane, takes the form

$$2(2)^{1/2}(Y^2+2Z^2)(X^2+2Z^2)/ \\ XYZ(XY+YZ+ZX)[(Y^2+2Z^2)^2+(X^2+2Z^2)^2]^{1/2} \tag{24}$$

When Z is larger than X and Y, this expression reduces to $4/XY(X+Y)$ which is the term listed in the matrix in equation (17). However, if Z is significantly less than X and Y, the full expression reduces to the form $2(2)^{1/2}/Z(X^4+Y^4)^{1/2}$ and it is then more appropriate to use this latter formulation. The physical significance of this situation is clear because when there is a large number (inversely proportional to Z) of grain boundary paths in the shear plane, diffusional flux has more easy pathways on which to flow.

It is instructive to consider the limiting situation for the hypothetical case where the Z dimension decreases to the width w of the grain boundary, since the equations for lattice and grain boundary diffusion should then converge. Taking a shear term in equation (15) for lattice diffusion

$$S_{44}=(12D\Omega/kT)[4/(X^2+Y^2)] \tag{25}$$

This can now be compared with the grain boundary case by putting $Z\approx w$ in the shear term, to obtain

$$S_{44}=(24D_g\Omega/kT)[2(2)^{1/2}/(X^4+Y^4)^{1/2}] \tag{26}$$

The derivation of equation (26) involves the contribution of two grain boundaries perpendicular to Z but there is only one boundary, in the limit effectively comprising a single grain, contributing to the flux in equation (25). Noting also that $D\to D_g$, the expressions $24(2)^{1/2}/(X^4+Y^4)^{1/2}$ and $48/(X^2+Y^2)$ are now compared. It is seen that their numerical values are equal when $X=Y$, thus demonstrating compatibility between the analyses of roles of lattice and of grain boundary by their convergence when the grain dimension perpendicular to the axis of shear is reduced to atomic dimensions.

References

1. R. W. Evans and B. Wilshire: 'Introduction to creep', 22; 1993, London, Institute of Materials.
2. H. J. Frost and M. F. Ashby: 'Deformation mechanism maps'; 1982, New York, Pergamon Press.
3. A. H. Cottrell: 'Mechanical properties of matter', 196; 1964, New York, Wiley.
4. F. R. N. Nabarro: 'Report on the conference on the strength of solids', 75; 1948, London, The Physical Society.
5. R. L. Coble: *J. Appl. Phys.*, 1963, **34**, 1679.
6. J. Harper and J. E. Dorn: *Acta Metall.*, 1957, **5**, 654.
7. J. R. Springarn and W. D. Nix: *Acta Metall.*, 1979, **27**, 171.
8. B. Burton: 'Diffusional creep in polycrystalline materials'; 1977, Aedermannsdorf, Trans. Tech. Publications.
9. O. A. Ruano, J. Wadsworth, J. Wolfenstine and O. D. Sherby: *Mater. Sci. Eng. A*, 1993, **A165**, 133.
10. W. Blum and W. Maier: *Phys. Stat. Sol. (a)*, 1999, **61**, 182.
11. F. R. N. Nabarro: *Metall. Mater. Trans. A*, 2002, **33A**, 213.
12. T. J. Ginter and F. A. Mohammed: *Mater. Sci. Eng. A*, 2002, **A322**, 148.
13. K. R. McNee, H. Jones and G. W. Greenwood: in 'Creep and fracture of engineering materials and structures', (ed. J. D. Parker), 185; 2001, London, Institute of Materials.
14. T. G. Langdon: *Mater. Sci. Eng. A*, 2000, **A283**, 266.
15. P. Kumar, M. E. Kassner and T. G. Langdon: *J. Mater. Sci.*, 2007, **42**, 409.
16. B. Wilshire and C. J. Palmer: *Scr. Mater.*, 2002, **436**, 483.
17. F. A. Mohamed: *Mater. Sci. Eng. A*, 2007, **A463**, 177.
18. B. Burton and G. L. Reynolds: *Mater. Sci. Eng. A*, 1995, **A191**, 135.
19. K. R. McNee, G. W. Greenwood and H. Jones: *Philos. Mag.*, 2002, **82**, 2773.
20. J. Frenkel: 'Kinetic theory of liquids', 193; 1955, New York, Dover Publications Inc.
21. G. W. Greenwood: *Philos. Mag.*, 1985, **51**, 537.
22. T. C. T. Ting: 'Anisotropic elasticity'; 1996, Oxford, Oxford University Press.
23. R. F. S. Hearmon: 'An introduction to applied anisotropic elasticity'; 1961, Oxford, Oxford University Press.
24. B. A. Wilcox and A. H. Clauer: *Acta Metall.*, 1972, **20**, 743.
25. G. W. Greenwood: *Proc. R. Soc. Lond. A*, 1992, **436A**, 187.
26. J. B. Bilde-Sorensen and P. A. Thorsen: 'Creep behaviour of advanced materials for the 21st century', (ed. R. S. Mishra *et al.*); 1999, Warrendale, PA, TMS Publication.
27. V. Srivastava, H. Jones and G. W. Greenwood: *Proc. R. Soc. Lond. A*, 2006, **462A**, 2863.
28. B. Burton: *Philos. Mag.*, 2002, **82**, 51.
29. B. Burton: *Philos. Mag.*, 2003, **83**, 2715.
30. B. Burton: *Mater. Sci. Technol.*, 2004, **20**, 1215.