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On a vector moment problem appearing in controllability of neutral type systems

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Abstract
We consider solvability of a vector moment problem by means of its correspondence to controllability problem for a certain delayed system of neutral type. In this way we succeeded to determine exactly the minimal interval on which the moment problem is solvable.

Key words: vector moment problem, neutral type system, spectral assignment.

Studying controllability problem for the following neutral type system
\[ \dot{z}(t) = A_{-1} \dot{z}(t-1) + \int_{-1}^{0} A_2(\theta) \dot{z}(t+\theta) \, d\theta + \int_{-1}^{0} A_3(\theta) z(t+\theta) \, d\theta + Bu(t), \] (1)
where \( A_{-1} \) is a constant \( n \times n \)-matrix, \( \det A_{-1} \neq 0 \), \( A_2, A_3 \) are \( n \times n \)-matrices whose elements belong to \( L^2(-1,0) \) and \( B \) is a constant \( n \times r \)-matrix, we reduced it to an equivalent moment problem [3]. We consider the operator model of the neutral type system (1) introduced by Burns and al. in product spaces. The state space is \( M_2(-1,0;\mathbb{C}^n) = \mathbb{C}^n \times L^2(-1,0;\mathbb{C}^n) \), shortly \( M_2 \), and (1) is rewritten as
\[ \frac{d}{dt} \begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} = A \begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} + Bu \] (2)
where the operator \( A \) is given by
\[ A \begin{pmatrix} y(t) \\ z_t(\cdot) \end{pmatrix} = \begin{pmatrix} \int_{-1}^{0} A_2(\theta) \dot{z}_t(\theta) d\theta + \int_{-1}^{0} A_3(\theta) z_t(\theta) d\theta \\ \frac{d}{d\theta} z_t(\theta) / d\theta \end{pmatrix} \]
with the domain
\[ \mathcal{D}(A) = \{(y,z(\cdot)) : z \in H^1(-1,0;\mathbb{C}^n), y = z(0) - A_{-1} z(-1)\} \subset M_2. \]
The operator \( A \) is the infinitesimal generator of a \( C_0 \)-group. The operator \( B \) is defined by \( Bu = (Bu,0) \). The relation between the solutions of the neutral type system (1) and the system (2) is given by the substitutions
\[ y(t) = z(t) - A_{-1} z(t-1), \quad z_t(\theta) = z(t+\theta). \]
Complete description of the spectral properties of operator $A$ is given in [4]. Consider for simplicity the case when all eigenvalues of $A^{-1}$ are simple with different arguments. Denote these eigenvalues by $\mu_m$, $m = 1, \ldots, n$. We will use the notation $\tilde{A}$ for $A$ in the particular case $A_2(\theta) = A_3(\theta) = 0$.

Operator $\tilde{A}$ has a simple eigenvalues

$$\tilde{\lambda}_k^m = \ln |\mu_m| + i(\arg \mu_m + 2\pi k), m = 1, \ldots, n, k \in \mathbb{Z}$$

with corresponding eigenvectors

$$\tilde{\varphi}_k^m = \left( e^{\frac{1}{\sqrt{k^2 + m^2}} \Phi_m} \right) \in M_2,$$

where $A^{-1}\Phi_m = \mu_m \Phi_m$, $m = 1, \ldots, n$. Besides $\tilde{A}$ has the eigenvalue $\tilde{\lambda}^0 = 0$ that corresponds to an $n$-dimensional subspace of generalized eigenvectors: $\tilde{\varphi}_0^0, \ldots, \tilde{\varphi}_n^0$. All eigenvectors $\{\tilde{\varphi}_k^m\} \cup \{\tilde{\varphi}_0^m\}$ constitute a Riesz basis in $M_2$. In general case if $A_2(\theta)$ and $A_3(\theta)$ are chosen in such a way that all eigenvalues of $A$, $\lambda_k^m$, $m = 1, \ldots, n$, $k \in \mathbb{Z}$, are still simple then they satisfy the condition

$$\sum_{k,m} |\lambda_k^m - \tilde{\lambda}_k^m|^2 < \infty$$

and the corresponding eigenvectors $\{\varphi_k^m\}$ satisfy

$$\sum_{k,m} ||\varphi_k^m - \varphi_k^m||^2 < \infty.$$

Eigenvectors $\{\varphi_k^m\}$ together with $\{\tilde{\varphi}_k^0\}$ form a Riesz basis in $M_2$ quadratically close to the spectral basis $\{\tilde{\varphi}\}$ of $\tilde{A}$.

Let $\{\tilde{\psi}\}$ and $\{\psi\}$ be biorthogonal bases for $\{\tilde{\varphi}\}$ and $\{\varphi\}$ respectively. It is easily checked that $\{\tilde{\psi}\}$ and $\{\psi\}$ are spectral bases of adjoint operators $\tilde{A}^*$ and $A^*$,

$$\tilde{A}^* \tilde{\psi}_k^m = \tilde{\lambda}_k^m \tilde{\psi}_k^m, \quad A^* \psi_k^m = \lambda_k^m \psi_k^m.$$

Besides these bases are also quadratically close [2]:

$$\sum_{k,m} ||\tilde{\psi}_k^m - \tilde{\psi}_k^m||^2 < \infty.$$  

Finally note that as it is shown in [5] vectors $\tilde{\psi}_k^m$, $k \neq 0$, are of the form

$$\tilde{\psi}_k^m = \left( \frac{1}{k^2 + \mu_m^2} \Psi_m \right) \in M_2 \quad \psi_k^m = \left( \frac{1}{k^2 + \mu_m^2} \Psi_{mk} \right),$$

where $\Psi_m$, $m = 1, \ldots, n$ are eigenvectors of $A^{-1}$,

$$A^{-1}\Psi_m = \mu_m \Psi_m,$$

and

$$\sum_{k,m} ||\Psi_{mk} - \Psi_m||^2 < \infty, \quad m = 1, \ldots, n.$$  

Let now $x \in M_2$. Then

$$x = \sum_{\varphi \in \{\varphi\}} (x, \varphi) \varphi.$$
A state \( x = \begin{pmatrix} y \\ z(\cdot) \end{pmatrix} \in M_2 \) is reachable at time \( T \) by a control \( u(\cdot) \in L_2(0,T; \mathcal{C}^r) \) iff the steering condition

\[
x = \begin{pmatrix} y \\ z(\cdot) \end{pmatrix} = \int_0^T e^{At}Bu(t)dt.
\]

holds. This steering condition may be expanded using the basis \( \{ \varphi \} \). A state \( x \) is reachable iff

\[
\sum_{\varphi \in \{ \varphi \}} \langle x, \psi \rangle \varphi = \sum_{\varphi \in \{ \varphi \}} \int_0^T \langle e^{At}Bu(t), \psi \rangle dt, \quad \psi \in \{ \psi \}.
\]

for some \( u(\cdot) \in L_2(-1,0; \mathbb{R}^r) \). Then the steering condition (6) can be substituted by the following system of equalities

\[
\langle x, \psi \rangle = \int_0^T \langle e^{At}Bu(t), \psi \rangle dt, \quad \psi \in \{ \psi \}.
\]

Let \( \{ b_1, \ldots, b_r \} \) be columns of \( B \) and \( b_i = \begin{pmatrix} b_i \\ 0 \end{pmatrix} \in M_2, \ i = 1, \ldots, r. \) Then the right hand side of (7) takes the form

\[
\int_0^T \langle e^{At}Bu(t), \psi \rangle dt = \sum_{i=1}^r \int_0^T \langle e^{At}b_i, \psi \rangle u_i(t) dt.
\]

Taking into account the fact that \( \{ \psi \} \) is a spectral basis of \( \mathcal{A}^r \) we rewrite the steering conditions as

\[
s_i^m = \left(k + \frac{1}{2}\right) \langle X, \Psi^m \rangle = \int_0^T e^{A^r t} \left(b_{k,m}^i u_1(t) + \ldots + b_{k,m}^r u_r(t)\right) dt, \quad (8)
\]

\( m = 1, \ldots, n, \ k \in \mathbb{Z}, \) where \( b_{k,m}^i = \left(k + \frac{1}{2}\right) \langle b_j, \psi^m \rangle. \) Equations (8) pose a vector moment problem with respect to unknown \( u_j(t), j = 1, \ldots, r. \) Using (4) we observe that

\[
\left(k + \frac{1}{2}\right) \langle b_j, \psi^m \rangle = \left\langle \begin{pmatrix} b_j \\ 0 \end{pmatrix}, \begin{pmatrix} \Psi^m \\ 0 \end{pmatrix} \right\rangle = \langle b_j, \Psi^m \rangle = b_m^j.
\]

So this value does not depend on \( k. \) Then from (5) it follows that the coefficients \( b_{k,m}^i \) satisfy the condition

\[
\sum_{k,m} |b_{k,m}^i - b_m^j|^2 = \sum_{k,m} |\langle b_j, \psi^m - \psi^m \rangle|^2 < \infty \quad j = 1, \ldots, r. \quad (9)
\]

In the scalar case \( r = 1, \) under the conditions

\[
b_m^1 = \langle b_1, \psi^m \rangle \neq 0, \quad m = 1, \ldots, n \quad (10)
\]

(controllability of the pair \( (A_{-1}, B) \)) and

\[
\langle b_1, \psi^m \rangle \neq 0, \quad m = 1, \ldots, n; k \in \mathbb{Z} \quad (11)
\]

(ultimate controllability of system (2)) the solvability of (8) can be studied by application of the methods given in [1]. In the vector case the problem is essentially more complicated. We showed [3] that the minimal interval of solvability of (2) is related with the first controllability index \( n_1(A_{-1}, B) \) of the system \( \dot{x} = A_{-1}x + Bu \) (see [6]). This result encouraged
us to consider the inverse problem: to study moment problems of the form (8) (of course, under some special assumptions) by means of their correspondence to the controllability problems for neutral type systems.

We show that the conditions (3), (9) together with some conditions analogous to (10), (11) are not only necessary but also sufficient in order, for the moment problem (8), to be generated by a neutral type system (1).

Theorem 1. Moment problem (2) corresponds to the controllability problem for an exact controllable neutral type system of the form (1) if and only if it satisfies the condition

\[ i) \sum_{k,m} \left| \lambda_k^m - \tilde{\lambda}_k^m \right|^2 < \infty, \text{ where } \lambda_k^m = \ln |\mu_m| + i(\arg \mu_m + 2\pi k), \]
\[ m = 1, \ldots, n, k \in \mathbb{Z}, \mu_1, \ldots, \mu_r \text{ are some nonzero different complex numbers}; \]
\[ ii) \sum_{j,k,m} \left| b_{k,m}^j - b_{m}^j \right|^2 < \infty, \text{ where } b_{m}^j (m = 1, \ldots, n, j = 1, \ldots, r) \text{ are some real numbers such that } \sum_{j} |b_{m}^j| > 0, m = 1, \ldots, n; \]
\[ iii) \sum_{j} |b_{k,m}^j| > 0, m = 1, \ldots, n, k \in \mathbb{Z}. \]

The \( n \times n \) and \( n \times r \)-matrices \( A^{-1} \) and \( B \) being given by

\[ A^{-1} = \text{diag}\{\mu_1, \ldots, \mu_n\}, \quad B = \{b_{m}^j\}_{m=1}^{n} \}_{j=1}^{r}. \]

Our proof of Theorem 1 is based on the following concept. We seek a system (1) in the form

\[ \dot{z}(t) = A^{-1} \dot{z}(t-1) + B \int_{-1}^{0} [F_2(\theta)z(t+\theta) + F_3(\theta)\dot{z}(t+\theta)] d\theta + Bu(t), \]

where matrices \( A^{-1}, B \) are chosen above, \( F_2, F_3 \) are \((r \times n)\)-indetermined matrices with elements from \( L^2(-1,0) \). In infinite-dimensional form this corresponds to the system

\[ \frac{d}{dt} \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} = A \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} + Bu = (\tilde{A} + BP) \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} + Bu, \]

where \( P = \begin{pmatrix} p_1 \\ \vdots \\ p_r \end{pmatrix} \), and \( p_j, j = 1, \ldots, r \) are functionals from space \( X_{-1} \), where

\[ X_{-1} = \left\{ \sum_{\psi \in \{\tilde{\psi}\}} d_{\tilde{\psi}} \psi, \sum_{k,m} (d_{\tilde{\psi}}^m / k)^2 < \infty \right\}, \]

\( \{\tilde{\psi}\} \) is a spectral basis of \( \tilde{A}^* \). One needs to choose an operator \( P \) in such a way that

1) \( \sigma(\tilde{A} + BP) = \{\lambda_m^0\}, k \in \mathbb{Z}, m = 1, \ldots, n; \)
2) \( \langle b^j, \psi^m \rangle = b_{k,m}^j, \psi^m \) are eigenvalues of \( \tilde{A}^* \).
We reduce this problem to the following one. Denote by $Q(\lambda)$ an $(r \times r)$
matrix-function of the form $Q(\lambda) = PR(\tilde{A},\lambda)B$, $R(\tilde{A},\lambda)$ is resolvent
of $\tilde{A}$. It turns out that all the eigenvalues are the roots of the equation
\[
det(I + Q(\lambda)) = 0
\]
and for and $m = 1, \ldots, n$, $k \in \mathbb{Z}$, vector $w_{mk}^m = (b_{k,m}^1, \ldots, b_{k,m}^m)$ can be
found from the condition:
\[
w_{mk}^m(I + Q(\lambda_{mk}^m)) = 0.
\]
Finally we give a choice of $P$ providing the required values of parameters
\(
\lambda_{mk}^m \in \mathbb{R}_m \), $\{b_{k,m}^j\}$.

Note that Theorem 1 is a direct generalization of pole assignement
Theorem 2.

Let the conditions (i) – (iii) be satisfied. We have:

1. If $T > n_1(A_{-1}, B)$ then $R_T = h_1$.
2. If $T = n_1(A_{-1}, B)$ then $R_T$ is a subspace of $h_1$ of finite co-dimension in $h_1$ (it is possible that $R_T = h_1$).
3. If $T < n_1(A_{-1}, B)$ then $R_T$ is a subspace of $h_1$ of infinite co-dimension.

Example. Consider two vector moment problems:

\[
s_k^1 = \int_0^T e^{((\pi + 2\pi k) + \epsilon k) t}(11u_1(t) + u_2(t))dt,
\]
\[
s_k^2 = \int_0^T e^{((2\pi k + \epsilon) t)}u_1(t) + u_2(t))dt,
\]
\[
s_k^3 = \int_0^T e^{(\log 2 + 2\pi k + \epsilon) t}(u_1(t) - u_2(t))dt,
\]
\[
s_k^4 = \int_0^T e^{(\log 3 + 2\pi k + \epsilon) t}(u_1(t) - u_2(t))dt,
\]
and

\[
s_k^1 = \int_0^T e^{((\pi + 2\pi k) + \epsilon k) t}(6u_1(t) - u_2(t))dt,
\]
\[
s_k^2 = \int_0^T e^{((2\pi k + \epsilon) t)}(4u_1(t) - u_2(t))dt,
\]
\[
s_k^3 = \int_0^T e^{(\log 2 + 2\pi k + \epsilon) t}(3u_1(t) - u_2(t))dt,
\]
\[
s_k^4 = \int_0^T e^{(\log 3 + 2\pi k + \epsilon) t}(2u_1(t) - u_2(t))dt,
\]
k $\in \mathbb{Z}$, $\sum_k (\epsilon_k)^2 \leq \infty$, $j = 1, \ldots, 4$. The both moment problems can be
reduced to the controllability problem for systems of the form (1), where

\[
A_{-1} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
6 & -5 & -5 & 5 \\
\end{pmatrix}. 
\]
\{\mu_1, \mu_2, \mu_3, \mu_4\} = \{-1, 1, 2, 3\}, corresponding eigenvectors of \(A^{-1}\) are

\[
\Psi_1 = \begin{pmatrix} -6 \\ 11 \\ -6 \\ 1 \end{pmatrix}, \Psi_2 = \begin{pmatrix} 6 \\ 1 \\ -4 \\ 1 \end{pmatrix}, \Psi_3 = \begin{pmatrix} 3 \\ -1 \\ -3 \\ 1 \end{pmatrix}, \Psi_4 = \begin{pmatrix} 2 \\ -1 \\ -2 \\ 1 \end{pmatrix},
\]

and in the first case \(B_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}\) and in the second case \(B_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\),

for a certain choice of \(A_2(\theta), A_3(\theta)\). Since

\[n_1(A_{-1}, B_1) = 2, \quad n_1(A_{-1}, B_2) = 3\]

then the moment problem (12) is solvable for all \(\{s_j^k\} \in l_2\) if \(T > 2\) while the problem (13) if \(T > 3\).

References


