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Optimal Weighted Poincaré and Log-Sobolev Inequalities for Cauchy Measures

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Abstract

In this paper, we establish the weighted Poincaré inequalities and Log-Sobolev inequalities for Cauchy distributions with optimal weight functions.

Keywords: Cauchy measure, weighted Poincaré inequality, weighted Log-Sobolev inequality.

AMS Classification Subjects 2000: 60E15 39B62 26Dxx

1 Introduction

A Borel probability $\mu$ on $\mathbb{R}^n$ is said to satisfy a weighted Poincaré inequality with weight function $\omega^2$ (where $\omega$ is a fixed non-negative Borel measurable function), if there exists a constant $C > 0$ such that for every smooth function $f: \mathbb{R}^n \to \mathbb{R}$ with gradient $\nabla f$,

$$\text{Var}_\mu(f) \leq C \int |\nabla f|^2 \omega^2 d\mu,$$

where the variance of $f$ w.r.t. $\mu$ is defined by

$$\text{Var}_\mu(f) = \int \left(f - \int f d\mu\right)^2 d\mu.$$

Similarly, we say that a Borel probability $\mu$ on $\mathbb{R}^n$ satisfies a weighted logarithmic Sobolev inequality with weight function $\omega^2$, if there exists a constant $C > 0$ such that for every smooth function $f: \mathbb{R}^n \to \mathbb{R}$,

$$\text{Ent}_\mu(f^2) \leq C \int |\nabla f|^2 \omega^2 d\mu,$$

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where the entropy is defined by

$$\text{Ent}_\mu(f^2) = \int f^2 \log f^2 d\mu - \left( \int f^2 d\mu \right) \log \left( \int f^2 d\mu \right).$$

The weighted functional inequalities are relatively weak inequalities. Although they can’t deduce the exponential concentration of a probability measure for sure, they can also show some degree of decay for this measure, as can be seen in [5].

Consider the Cauchy measures:

$$d\mu_\beta(x) = \frac{(1 + |x|^2)^{-\beta}}{c(n, \beta)} dx$$

where $\beta > \frac{n}{2}$, and $c(n, \beta)$ is the normalizing constant.

In [3], Bobkov and Ledoux proved that the Cauchy measures $\mu_\beta$ admit a weighted Poincaré inequality with the weight $1 + |x|^2$ and the constant $\left(\sqrt{1 + \frac{2}{\beta}} + \sqrt{\frac{2}{\beta} - 1}\right)^2$, as well as a weighted log-Sobolev inequalities with the weight $(1 + |x|^2)^2$ and the constant $\frac{1}{\beta - 1}$.

Comparing with the results in [3], the better weight functions are found by Hebisch and Zegarlinski (in [13]). However, the authors just prove that the constants exist and are independent of $\beta$.

The aim of this paper is to give the optimal weight functions and the corresponding constants of the inequalities for the one-dimensional Cauchy measures in a different way. But it’s a pity that we can’t reach the similar results for the multi-dimensional case in the same way.

2 Main result

**Theorem 2.1.** (one-dimensional weighted Poincaré inequality) For any $\beta > 1/2$, the probability measure $\mu_\beta$ on $\mathbb{R}$ satisfy the following weighted Poincaré inequality: for any smooth function $f : \mathbb{R} \to \mathbb{R}$,

$$\text{Var}_{\mu_\beta}(f) \leq C_\beta \int_{\mathbb{R}} |f'(x)|^2 (1 + x^2) d\mu_\beta(x),$$

where $C_\beta$ has the same order with $\frac{1}{\beta}$. Moreover, the weight function is optimal in the sense of order.

**Theorem 2.2.** (one-dimensional weighted log-Sobolev inequality) For any $\beta > 1$, the probability measure $\mu_\beta$ on $\mathbb{R}$ satisfy the following weighted log-Sobolev inequality: for any smooth function $f : \mathbb{R} \to \mathbb{R}$,

$$\text{Ent}_{\mu_\beta}(f^2) \leq C_\beta \int_{\mathbb{R}} |f'(x)|^2 (1 + x^2) \log(1 + x^2) d\mu_\beta(x),$$

where $C_\beta$ has the same order with $\frac{1}{\beta - 1}$. Moreover, the weight function is optimal in the sense of order, that is, for any other function $\omega^2(x)$, if $\lim_{x \to +\infty} \frac{\omega^2(x)}{\log(1 + x^2)} = 0$, then the Cauchy measure $\mu_\beta$ doesn’t satisfy the weighted log-Sobolev inequality with the weight function $(1 + x^2)\omega^2(x)$. 

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Theorem 2.3. (multi-dimensional weighted log-Sobolev inequality) For any \( n \geq 6, \beta > n/2 \), the probability measure \( \mu_\beta \) on \( \mathbb{R}^n \) satisfy the following weighted log-Sobolev inequality: for any smooth function \( f : \mathbb{R}^n \to \mathbb{R} \),

\[
\text{Ent}_{\mu_\beta}(f^2) \leq C \int_{\mathbb{R}^n} |\nabla f(x)|^2 (1 + |x|^2) d\mu_\beta(x),
\]

where \( C = \frac{2}{n - 4 - \frac{4}{\sqrt{n}} - \frac{1}{n}} \) is independent of \( \beta \).

Remark: The weight function is of course optimal. Because it has already been optimal for Poincaré type inequality.

Now we state the previous relative works by Bobkov and Ledoux (13), Hebisch and Żegarliński (13):

Theorem 2.4. (13) For \( \beta \geq n \), and any smooth bounded \( f \) on \( \mathbb{R}^n \),

\[
\text{Var}_{\mu_\beta}(f) \leq \left( \sqrt{1 + \frac{2}{\beta - 2}} + \sqrt{\frac{2}{\beta - 2}} \right)^2 \int |\nabla f(x)|^2 (1 + |x|^2) d\mu_\beta(x)
\]

Theorem 2.5. (13) If \( \beta \geq (n+1)/2, \beta > 1 \), for any smooth bounded \( f \) on \( \mathbb{R}^n \),

\[
\text{Ent}_{\mu_\beta}(f^2) \leq \frac{1}{\beta - 1} \int |\nabla f(x)|^2 (1 + |x|^2)^2 d\mu_\beta(x)
\]

Theorem 2.6. (13) Assume \( \mu_\beta \) is a probability measure on a n-dimensional manifold with metric \( d \), and \( d\mu_\beta = e^{-\beta \log(1+d)} dx \) with \( \beta \geq n, \beta > 1 \). Suppose Ric \( \geq 0 \). Then for any \( q \geq 1 \), there are constants \( M_q, c_q \in (0, +\infty) \), such that

\[
M_q \mu_\beta(|f - \mu_\beta(f)|^q) \leq \mu_\beta((1 + d)^q |\nabla f|^q)
\]

and

\[
\mu_\beta(\frac{|f|^q \log \frac{|f|^q}{\mu_\beta(|f|^q)}}{c_q \mu_\beta((1 + d)^q \log(e + d)|\nabla f|^q)}) \leq c_q \mu_\beta((1 + d)^q \log(e + d)|\nabla f|^q)
\]

Theorem 2.1 does not give a better result than Theorem 2.4 does. we just adapt another different way, and tell that the weight function is optimal.

From the comparison between Theorem 2.2 with Theorem 2.5, it’s clear that our result gives a better weight function. Moreover, the order of the constant isn’t changed.

Contrast to Theorem 2.6, our results give the estimate of order for the constants.

In [13], their results still can derive a uniform weighted log-Sobolev inequality with weight \( 1 + |x|^2 \) and constant \( c_q \) independent of \( \beta \). In fact, we can also get the same result by Bakry-Emery criterion, i.e. getting \( \Gamma_2 \geq \rho \Gamma \) for some \( \rho > 0 \).
3 Proofs of main results

3.1 One-dimensional weighted Poincaré inequality

**Theorem 3.1.** ([16]) Let $\mu, \nu$ be Borel measures on $\mathbb{R}$ with $\mu(\mathbb{R}) = 1$ and $d\nu(x) = n(x)dx$. Let $m$ be a median of $\mu$. Let $C_P$ be the optimal constant such that for every smooth function $f : \mathbb{R} \to \mathbb{R}$, one has

$$\text{Var}_\mu(f) \leq C_P \int f'^2 d\nu.$$

Then $\max(b, B) \leq C_P \leq 4 \max(b, B)$, where $b = \sup_{x < m} \mu((-\infty, x]) \int_x^m \frac{1}{n}$, $B = \sup_{x > m} \mu([x, \infty)) \int_m^x \frac{1}{n}$.

**Proof of Theorem 3.1:** By the symmetry of the measure $\mu_\beta$, the median $m$ of $\mu_\beta$ is equal to $0$. Define

$$b(\beta) := \sup_{\alpha \in (-\infty, 0)} \left( \int_x^\infty (1 + y^2)^{-\beta} dy \right) \left( \int_0^x (1 + y^2)^{\beta-1} dy \right),$$

$$B(\beta) := \sup_{\alpha \in (0, +\infty)} \left( \int_x^{+\infty} (1 + y^2)^{-\beta} dy \right) \left( \int_0^x (1 + y^2)^{\beta-1} dy \right).$$

Clearly, by Theorem 3.2 and symmetry, we just need to give an upper bound on $B(\beta)$. Since the point $0$ doesn’t make trouble in our calculation, we can reduce the estimate on $B(\beta)$ to that on $\tilde{B}(\beta)$.

By the following simple estimate,

$$\int_x^{+\infty} (1 + y^2)^{-\beta} dy = 2 \int_x^{+\infty} (1 + t)^{-\beta} t^{-1/2} dt \leq \frac{2 (1 + x^2)^{-\beta + 1}}{x},$$

and

$$\int_0^x (1 + y^2)^{\beta-1} dy = 2 \int_0^{x^2} (1 + t)^{\beta-1} t^{-1/2} dt \leq (1 + x^2)^{\beta-1} x,$$

we get $B(\beta) \leq \frac{2}{\beta - 1}$.

However,

$$B(\beta) \geq \left( \int_{\sqrt{x}}^{+\infty} (1 + y^2)^{-\beta} dy \right) \left( \int_0^{\sqrt{x}} (1 + y^2)^{\beta-1} dy \right) \sim \frac{1}{\beta} \quad (\beta \to +\infty).$$

Thus we get the right order $\frac{1}{\beta}$ of $C_\beta$.

For any other even function $\omega^2(x)$ increasing in $x > 0$ (or increasing in $x \geq M$ for some
M > 0), and \( \lim_{x \to +\infty} \frac{\omega^2(x)}{1 + x^2} = 0 \),

\[
B_{\omega^2}(\beta) := \sup_{\alpha \in (0, +\infty)} \left( \int_x^{+\infty} (1 + y^2)^{-\beta} dy \right) \left( \int_0^x \frac{(1 + y^2)^\beta}{\omega^2(y)} dy \right)
\]

\[
\geq \sup_{x \in (M, +\infty)} \left( \int_x^{+\infty} (1 + y)^{-2\beta} dy \right) \left( \int_0^x \frac{y^{2\beta}}{\omega^2(y)} dy \right)
\]

\[
\geq \sup_{x \in (M, +\infty)} \frac{(1 + x)^{1-2\beta}}{2\beta - 1} \cdot \frac{1}{\omega^2(x)} \cdot \frac{x^{2\beta + 1} - M^{2\beta + 1}}{2\beta + 1}
\]

\[= + \infty.\]

Therefore, the weight function \( 1 + x^2 \) is optimal in the sense of order.

### 3.2 One-dimensional weighted Log-Sobolev inequality

Here we’ll make use of the refined characterization from Barthe and Roberto [4]. Of course, we may also use that one from Bobkov, and Götze [6], after all, we can’t give a sharp estimate on the logarithmic Sobolev constants. The refined characterization is stated as follows.

**Theorem 3.2.** (4) Let \( \mu, \nu \) be Borel measures on \( \mathbb{R} \) with \( \mu(\mathbb{R}) = 1 \) and \( d\nu(x) = n(x)dx \). Let \( m \) be a median of \( \mu \). Let \( C \) be the optimal constant such that for every smooth function \( f : \mathbb{R} \to \mathbb{R} \), one has

\[
\text{Ent}_\mu(f^2) \leq C \int f^2 d\nu.
\]

Then \( \max(b_-, b_+) \leq C \leq 4 \max(B_-, B_+) \), where

\[
b_+ = \sup_{x > m} \mu((x, \infty)) \log \left( 1 + \frac{1}{2\mu((x, \infty))} \right) \int_m^x \frac{1}{n} \nonumber
\]

\[
B_+ = \sup_{x > m} \mu((x, \infty)) \log \left( 1 + \frac{e^2}{\mu((x, \infty))} \right) \int_m^x \frac{1}{n} \nonumber
\]

\[
b_- = \sup_{x < m} \mu((\infty, x]) \log \left( 1 + \frac{1}{2\mu((\infty, x])} \right) \int_x^m \frac{1}{n} \nonumber
\]

\[
B_- = \sup_{x < m} \mu((\infty, x]) \log \left( 1 + \frac{e^2}{\mu((\infty, x])} \right) \int_m^x \frac{1}{n} \nonumber
\]

**Proof of Theorem 2.1:** By the symmetry of the measure \( \mu_\beta \), the median \( m \) of \( \mu_\beta \) is equal to 0. Moreover, in order to get our result, by **Theorem 3.2** we just need to give an upper bound on the following quantity,

\[
S(\beta) := \sup_{x \in (0, +\infty)} \left( \int_x^{+\infty} (1 + y^2)^{-\beta} dy \right) \log \left( 1 + \frac{e(1, \beta)}{\int_x^{+\infty} (1 + y^2)^{-\beta} dy} \right)
\]

\[
\cdot \left( \int_0^x \frac{(1 + y^2)^\beta dy}{(1 + y^2) \log(e + y^2)} \right).
\]
Clearly, the superior can’t be obtained on the point 0. If the superior is taken on \( x_\beta \in (0, +\infty) \). Let

\[
I_1 = \int_{x_\beta}^{+\infty} (1 + y^2)^{-\beta} dy \\
I_2 = \int_{0}^{x_\beta} \frac{(1 + y^2)^{\beta-1} dy}{\log(e + y^2)}
\]

Then we’ll discuss \( S(\beta) \) under the following three sorts of situations by \( \lim_{\beta \to +\infty} (1 + (\frac{1}{\sqrt{\beta}})^2)^\beta = e \):

**Convention:** in the following we’ll use the signal ”\( \sim \)” to denote the same order as \( \beta \to +\infty \).

**Case 1:** If \( x_\beta = 0(\frac{1}{\sqrt{\beta}}) \), i.e. \( \lim_{\beta \to +\infty} \frac{x_\beta}{1/\sqrt{\beta}} = 0 \):

\[
I_1 \sim \frac{1}{\sqrt{\beta}} \\
I_2 \sim \int_{0}^{x_\beta} (1 + y^2)^{\beta-1} dy \sim x_\beta
\]

By the monotonicity of \( x \log(1 + \frac{C}{x}) \) in \( x > 0 \), we have immediately

\[
S(\beta) \sim \frac{x_\beta}{\sqrt{\beta}} = 0(\frac{1}{\beta}).
\]

**Case 2:** If \( x_\beta = O(\frac{1}{\sqrt{\beta}}) \),

\[
I_1 \sim \frac{1}{\sqrt{\beta}} \\
I_2 \sim \int_{0}^{x_\beta} (1 + y^2)^{\beta-1} dy \sim \frac{1}{\sqrt{\beta}}
\]

\[
S(\beta) \sim \frac{1}{\beta}
\]

**Case 3:** If \( \frac{1}{\sqrt{\beta}} = 0(x_\beta) \),

\[
I_1 \sim \frac{1}{\beta - 1} \cdot \frac{1}{x_\beta (1 + x_\beta^2)^{\beta-1}}
\]

\[
I_2 \sim \begin{cases} 
(1 + x_\beta^2)^\beta \\
\beta x_\beta \\
\frac{1}{\log(e + x_\beta^2)} \\
\beta x_\beta 
\end{cases} 
\quad \text{if } \{x_\beta\} \text{ is bounded}
\]

\[
\begin{align*}
(1 + x_\beta^2)^\beta \\
\frac{1}{\log(e + x_\beta^2)} \\
\beta x_\beta
\end{align*} 
\quad \text{if } \{x_\beta\} \text{ is unbounded}
\]

\[
S(\beta) \sim \frac{1}{\beta}
\]

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Now there is the last case left that the superior is got as $x \to +\infty$. For that we might as well do it by reducing the estimate on $S(\beta)$ to that on $\tilde{S}(\beta)$, defined by

\[ \tilde{S}(\beta) := \sup_{x \in (1, +\infty)} \left( \int_x^{+\infty} y^{-2\beta} dy \right) \log \left( 1 + \frac{1/(2\beta - 1)}{\int_x^{+\infty} y^{-2\beta} dy} \right) \left( \int_1^{+\infty} \frac{y^{2(\beta-1)} dy}{\log(e + y^2)} \right) \]

\[ = \sup_{x \in (1, +\infty)} \frac{x^{1-2\beta}}{2\beta - 1} \log \left( 1 + x^{2\beta-1} \right) \int_1^{+\infty} \frac{y^{2(\beta-1)} dy}{\log(e + y^2)}. \]

By the basic formula of differential and integral, one can get readily

\[ \frac{1}{2\beta - 1} \log(e + x) \geq \frac{1}{2\beta - 1} \left[ \log(e + x) \right]_1^x \]

\[ = \int_1^x \frac{y^{2(\beta-1)}}{\log(e + y)} \left( 1 - \frac{y}{(2\beta - 1)(e + y) \log(e + y)} \right) dy \]

\[ \geq \int_1^x \frac{y^{2(\beta-1)}}{\log(e + y)} \left( 1 - \frac{1}{2\beta - 1 \log(e + y)} \right) dy. \]

When $2\beta - 1 > 1$, i.e. $\beta > 1$, we have

\[ \int_1^x \frac{y^{2(\beta-1)} dy}{\log(e + y^2)} \leq \int_1^x \frac{y^{2(\beta-1)} dy}{\log(e + y)} \]

\[ \leq \frac{1}{2\beta - 1} \log(e + x). \]

Therefore,

\[ \tilde{S}(\beta) \leq \sup_{x \in (1, +\infty)} \frac{1}{(2\beta - 1)(2\beta - 2)} \log(1 + x^{2\beta-1}) \leq \frac{1}{2\beta - 2} \]

From the discussion above, $\frac{1}{\beta}$ is the right order of log-Sobolev constants.

Moreover, for any other even function $\omega^2(x)$ increasing in $x > M > 0$, and

\[ \lim_{x \to +\infty} \frac{\omega^2(x)}{\log(e + x^2)} = 0, \]

we have

\[ \tilde{S}_{\omega^2}(\beta) := \sup_{x \in (1, +\infty)} \left( \int_x^{+\infty} y^{-2\beta} dy \right) \log \left( 1 + \frac{1/(2\beta - 1)}{\int_x^{+\infty} y^{-2\beta} dy} \right) \left( \int_1^{+\infty} \frac{y^{2(\beta-1)} dy}{\omega^2(y)} \right) \]

\[ \geq \sup_{x \in (1, +\infty)} \frac{x^{1-2\beta}}{2\beta - 1} \log \left( 1 + x^{2\beta-1} \right) \frac{1}{\omega^2(x)} \int_1^{+\infty} \frac{y^{2(\beta-1)} dy}{\omega^2(y)} \]

\[ = +\infty. \]

As a result, the logarithmic-type weight function is optimal in the sense of order.
3.3 Multi-dimensional weighted log-Sobolev inequality

The Cauchy distribution can be represented in the following form:

\[ d\mu_\beta(x) = \frac{e^{-\beta \log(1+|x|^2)}dx}{c(n, \beta)}, \quad x \in \mathbb{R}^n, \beta > n/2. \]

Let

\[ V(x) := \beta \log(1+|x|^2), \quad \omega^2(x) := 1 + |x|^2, \]

we have

\[ \nabla V = \frac{2\beta x}{1 + |x|^2}, \quad \nabla^2 V = \frac{2\beta}{1 + |x|^2} - \frac{4\beta x \otimes x}{(1 + |x|^2)^2} \]

and

\[ \nabla \omega^2 = 2x, \quad \nabla^2 \omega^2 = 2, \Delta \omega = 2n. \]

The generator, associated with the measure \( \mu_\beta \) and the weight \( \omega^2 \), is

\[ L_\omega := \omega^2 \Delta + (\nabla \omega^2 - \omega^2 \nabla V) \nabla. \]  \hspace{1cm} (3.2)

Note that \( \mu_\beta \) is the invariant measure of the diffusion operator \( \mathcal{L}_\omega \). We have the "Carré du champ" operator

\[ \Gamma(f, g) := \frac{1}{2}(\mathcal{L}_\omega(fg) - f\mathcal{L}_\omega g - g\mathcal{L}_\omega f) \]

\[ = \omega^2 \nabla f \nabla g. \]

Define the \( \Gamma_2 \) curvature, see \[2, 1, 14]\]

\[ \Gamma_2(f, f) := \frac{1}{2}(\mathcal{L}_\omega \Gamma(f, f) - 2\Gamma(f, \mathcal{L}_\omega f)). \]

**Proposition 3.3.** Assume that the dimension \( n \geq 6 \), we have the dimension curvature inequality holds, i.e. there exists positive constants \( \rho \), such that

\[ \Gamma_2(f, f) \geq \rho \Gamma(f, f), \]  \hspace{1cm} (3.3)

for all \( f \in C^\infty_0(\mathbb{R}^n) \). \( \rho \) can be chosen to be \( n - 4 - \frac{4}{\sqrt{n}} - \frac{1}{n} \) if necessary.
Proof. By the definition of $\Gamma_2$ curvature,

$$\Gamma_2(f, f) = \omega^4 |\nabla \nabla f|^2 + \frac{\omega^2}{2} |\nabla f|^2 \Delta \omega^2 + 2\omega^2 \nabla f \nabla^2 f \nabla \omega^2 + \frac{1}{2} |\nabla f|^2 |\nabla \omega^2|^2$$

$$- \frac{1}{2} \omega^2 |\nabla f|^2 \nabla \nabla \omega^2 - \omega^2 \Delta f \nabla f \nabla \omega^2 - \omega^2 \nabla f \nabla^2 \omega^2 \nabla f$$

$$+ \omega^2 (\nabla \nabla \nabla f) (\nabla \omega^2 \nabla f) + \omega^4 \nabla f \nabla^2 \nabla f$$

$$= (1 + |x|^2)^2 |\nabla \nabla f|^2 + (2(\beta - 1) + n(1 + |x|^2)) |\nabla f|^2$$

$$+ 4(1 + |x|^2) \nabla f \cdot \nabla \nabla f \cdot x - 2(1 + |x|^2) \Delta f \cdot x \cdot \nabla f$$

$$\geq (1 + |x|^2)^2 |\nabla \nabla f|^2 - 2 \left(2 + \frac{1}{\sqrt{n}}\right) (1 + |x|^2) |\nabla \nabla f||\nabla f|$$

$$+ (2(\beta - 1) + n(1 + |x|^2)) |\nabla f|^2$$

$$= \left((1 + |x|^2)|\nabla \nabla f| - \left(2 + \frac{1}{\sqrt{n}}\right) |x||\nabla f|\right)^2$$

$$+ \left(\left(n - 4 - \frac{4}{\sqrt{n}} - \frac{1}{n}\right) |x|^2 + (2\beta - 2 + n)\right) |\nabla f|^2$$

$$\geq \min\left\{n - 4 - \frac{4}{\sqrt{n}} - \frac{1}{n}, 2\beta - 2 + n\right\} (1 + |x|^2)|\nabla f|^2,$$ where $|\nabla \nabla f|$ is the Hilbert-Schmidt norm of $\nabla \nabla f$ and $(i)$ follows from

$$\Delta f)^2 \leq \frac{|\nabla \nabla f|^2}{n}.$$

If $n \geq 6$, there exists a positive constant $\rho := \rho(n) = n - 4 - \frac{4}{\sqrt{n}} - \frac{1}{n} > 0$ (independent on $\beta$), such that the dimension curvature inequality $CD(\rho, \infty)$, i.e. $(3.3)$ holds. \qed

By the above proposition, Theorem 2.3 is well-known, see [2], also [17, 1, 14].

4 Appendix

Lemma 4.1. The normalized constant $c(1, \beta)$ has the same order with $\frac{1}{\sqrt{\beta}}$ as $\beta$ goes to infinity.

Proof.

$$c(1, \beta) = \int_{\mathbb{R}} (1 + x^2)^{-\beta} \, dx$$

$$= 2 \int_0^{+\infty} (1 + x^2)^{-\beta} \, dx$$

$$= \int_0^{+\infty} (1 + \tau)^{-\beta} \tau^{-1/2} \, d\tau$$

$$= \int_0^{\frac{1}{n}} (1 + \tau)^{-\beta} \tau^{-1/2} \, d\tau + \int_{\frac{1}{n}}^{+\infty} (1 + \tau)^{-\beta} \tau^{-1/2} \, d\tau \sim \frac{1}{\sqrt{\beta}}.$$
This is because of the following facts:

\[ \int_{0}^{\frac{1}{\sqrt{\beta}}} (1 + \tau)^{-\beta} \tau^{-1/2} d\tau \sim \int_{0}^{\frac{1}{\sqrt{\beta}}} \tau^{-1/2} d\tau = \frac{2}{\sqrt{\beta}}. \]

and

\[ \int_{\frac{1}{\sqrt{\beta}}}^{+\infty} (1 + \tau)^{-\beta} \tau^{-1/2} d\tau \leq \sqrt{\beta} \int_{\frac{1}{\sqrt{\beta}}}^{+\infty} (1 + \tau)^{-\beta} d\tau \sim \frac{1}{\sqrt{\beta}}. \]

**Remark:** Let \( C(n, \beta) := \int_{0}^{+\infty} (1+r^2)^{-\beta} r^n dr \), then \( c(n, \beta) \) has the same order with \( C(n, \beta) \) as \( \beta \to +\infty \). Next we give an estimate for \( C(n, \beta) \).

\[ C(n, \beta) = \frac{1}{2} \int_{0}^{+\infty} (1 + \tau)^{-\beta} \tau^{n/2} d\tau \]

By the similar method, we can get

\[ C(n, \beta) \sim \frac{1}{\beta^2}. \]

That is \( c(n, \beta) \) has the order of \( \frac{1}{\beta^2} \) as \( \beta \) large enough.

**References**


