THE INFLUENCE OF LIMITED KINEMATIC HARDENING IN SHAKEDOWN ANALYSIS

Domingos E.S. Nery¹, Nestor Zouain² and Reinaldo R. Jospin¹

¹ Instituto de Engenharia Nuclear - IEN/CNEN
Rua Helio de Almeida, 75
Cidade Universitária - Ilha do Fundão
21941-906 Rio de Janeiro, RJ
domingos@ien.gov.br, rj.jospin@ien.gov.br

² Programa de Engenharia Mecânica - COPPE - UFRJ
Cidade Universitária - Ilha do Fundão
Rio de Janeiro, RJ
nestor@ufrj.br

ABSTRACT

The use of the Design by Analysis concept is a trend in modern pressure vessel and piping calculations. DBA flexibility allow us to deal with unexpected configurations detected at in-service inspections. It is also important, in life extension calculations, when deviations of the original standard hypothesis adopted initially in Design by Formula, can happen. To apply the DBA to structures under variable mechanic and thermal loads, it is necessary that, alternate plasticity and incremental collapse (with instantaneous plastic collapse as a particular case), be precluded. These are two basic failure modes considered by ASME or European Standards in DBA. The shakedown theory is the tool available to achieve this goal. In order to apply it, is necessary only the range of the variable loads and the material properties. Precise, robust and efficient algorithms to solve the very large nonlinear optimization problems generated in numerical applications of the shakedown theory is a recent achievement. Zouain and co-workers developed one of these algorithms for elastic ideally-plastic materials. But, it is necessary to consider more realistic material properties in real practical applications. This paper shows an enhancement of this algorithm to dealing with limited kinematic hardening, a typical property of the usual steels. This is done using internal thermodynamic variables. A discrete algorithm is obtained using a plane stress, mixed finite element, with internal variable. An example, a beam encased in an end, under constant axial force and variable moment is presented to show the importance of considering the limited kinematic hardening in a shakedown analysis.

1. INTRODUCTION

When designing mechanical components as pipes and pressure vessels subject to variable thermal and mechanical loads, the use of the Design by Analysis (DBA) route to demonstrate the structural safety is a modern trend and it makes us possible a more flexible approach. For example, when, “in service inspections” detect defects and other deviations from the hypothesis adopted by Design by Formulas (DBF) route, the use of DBA is necessary to proceed a safety assessment. This is important when considering life
extension of components as for example, in nuclear industry. In the implementation of the DBA, it is necessary to assure that alternating plasticity (AP) and incremental collapse (IC or ratcheting) be precluded, as prescribed by the ASME and European Standards. Another plastic failure mode, plastic collapse, is in fact a particular case of incremental collapse. Shakedown analysis is based in the Bleich-Melan static theorem [1] and needs only the knowledge of the material properties and the extremum values of variable loads. Shakedown theory was established for ideal plasticity in the 50th and, ever since, a number of developments has been made. But, the implementation of the theory, in real cases, result in a very large optimization problem with non-linear constraints. Due this fact, only in recent years, the development and extensions in the shakedown algorithms and the development of robust finite elements, efficient optimization methods and the ease access to more powerful computers has become the Design by Analysis route, possible [2],[3],[4]. For elastic-ideally plastic structures, Zouain and co-workers developed an algorithm to proceed the shakedown analysis, described in [5] and [6]. Industrial level applications require to consider realistic material properties, as for example, limited kinematic hardening. By other side, in order to represent the ratcheting phenomenon, it is necessary to consider limited kinematic hardening. The extensions of the basic theory to include nonlinear or limited hardening behaviors came only recently (see e.g. [7],[8] and [9]). Our study is based on the theory of shakedown with thermodynamic internal variables to represent hardening that can be found in [8],[9]and [10]. The constitutive model proposed by E. Stein and coworkers [7],[10]and [11] will be used here. Based in that model, Nery [12] extended the Zouain algorithm to deal with limited kinematic hardening and developed a 2D mixed axisymmetric finite element with internal variable to treat axisymmetric shakedown problems.

2. BASIC ASSUMPTIONS AND NOTATION

Let \( v \in \mathcal{V} \) be a velocity field complying with prescribed boundary conditions. Between the strain rate field \( d \in \mathcal{W} \) and \( v \) there is a relation:

\[
d = \mathcal{D}v
\]

where \( \mathcal{D} \) is the tangent deformation operator, mapping \( \mathcal{V} \) into \( \mathcal{W} \). Small deformations are assumed. \( \sigma \in \mathcal{W}' \) is the stress field and \( L \in \mathcal{F} \), the load systems space. \( \mathcal{W} \) and \( \mathcal{W}' \) are dual spaces. Between \( \sigma \) and \( L \) there is a relation:

\[
\sigma = \mathcal{D}'L
\]

where \( \mathcal{D}' \) is the equilibrium operator. \( \mathcal{D} \) and \( \mathcal{D}' \) are self-adjoint operators.

To derive constitutive relations, the standard generalized material model [13] and isothermal processes (\( \dot{\Theta} = 0 \)) are considered. Aiming to consider kinematic hardening, are adopted the following generalized state variables in the local states method [14]:

\[
\varepsilon = (\varepsilon, 0) \quad \text{generalized strain}
\]

\[
\varepsilon^e = (\varepsilon^e, \omega) \quad \text{generalized reversible strain}
\]

\[
\varepsilon^p = (\varepsilon^p, \beta) \quad \text{generalized irreversible strain}
\]

\[
\sigma = (\sigma, A) \quad \text{generalized stress}
\]

where, \( \varepsilon \) is the observable strain, \( \varepsilon^e \) is the elastic strain, \( \omega \) is the reversible internal hardening variable, \( \varepsilon^p \) is the plastic strain, \( \beta \) is the irreversible internal hardening variable, \( \sigma \) is the Cauchy stress tensor and \( A \) is the back stress.
With additive decomposition of strain we have $\varepsilon = \varepsilon^e + \varepsilon^p$ and then:

$$\varepsilon = \varepsilon^e + \varepsilon^p$$ (3)

$$0 = \omega + \beta$$ (4)

The state laws are obtained from a quadratic free energy potential, in $\varepsilon^e$ and $\beta$. Assuming that the elastic and hardening variables are not coupled, the following relations are obtained:

$$\sigma = E\varepsilon^e$$ (5)

and

$$A = -H\beta$$ (6)

with the tensors $E$ e $H$ constants.

The evolution laws are derived from a dissipation potential defined by Hill’s maximum dissipation principle. As usual we call here $\dot{\varepsilon}^p = d^p$. The flow law is derived from this potential. In the case of Mises criteria and associative flow law, the plastic relations are equivalent to a classical form:

$$\left(d^p, \dot{\beta}\right) = \dot{\lambda}\nabla f(\sigma, A)$$ (7)

Here $\nabla f(\sigma, A)$ denotes the gradient of $f$ ($f$ is the yield surface in stress space) and $\dot{\lambda}$ is a vector field of plastic multipliers. At any body point, the components of $\dot{\lambda}$ are related to each plastic mode in $f$ by the complementarity conditions:

$$\dot{\lambda} f(\sigma, A) = 0 \quad f(\sigma, A) \leq 0 \quad \dot{\lambda} \geq 0$$ (8)

(this inequalities hold componentwise).

3. SHAKEDOWN ANALYSIS

3.1 Load domain

To proceed the shakedown analysis, one basic data is a prescribed domain of load variation, $\Delta^0$, in the load space, which contains any feasible load history. This domain is assumed to be a convex polyhedron. To unify the approach when dealing with mechanical and thermal loads, we consider another domain, $\Delta^E$, a mapping of $\Delta^0$ in the elastic stress space. $\Delta^E$ is also a convex polyhedron, then, any interior point of polyhedron $\Delta^E$ is a convex combination of its vertex. However, can be necessary to consider a non-linear dependence between the loads, implying in the discretization of the function that defines the load coupling. To avoid this, it is better to consider the total uncoupling of loads, defining for each body point a local uncoupled convex hull $\Delta$ which collect the extremum values of stresses corresponding to the loads, independently of the point in the load cycle. The set of all the local values of elastic stresses associated to any feasible loading is:

$$\forall x \in B, \quad \Delta(x) := \{\sigma^E(x) \mid \forall \sigma^E \in \Delta^E\}$$ (9)

or

$$\Delta := \{\sigma \mid \sigma(x) \in \Delta(x), \forall x \in \Delta^E\}$$ (10)
3.2 Shakedown with limited kinematic hardening

For ideal plasticity, the Bleich-Melan theorem statement is: any load factor $\mu^\ast$ is safe if there exists a time-independent residual (self-equilibrated) stress field $\sigma^r$ such that its superposition with any stress belonging to the amplified load domain $\mu^\ast \Delta$ is plastically admissible. Then, for elastic shakedown, the limit load factor $\mu$ is the supremum of all safe factors. This may be translated as an elastic shakedown static variational principle:

$$\mu := \sup_{(\mu^\ast, \sigma^r) \in \mathbb{R} \times \mathbb{W}} \{ \mu^\ast \geq 0 \mid \mu^\ast \Delta + \sigma^r \subset P, \sigma^r \in S^r \}$$  \hspace{1cm} (11)

$S^r$ is a residual stress space i.e. stress fields in equilibrium with null loads. The limited kinematic hardening was considered in the shakedown theory by Stein et al., [7],[10] and [11] using a 3D overlay-model that approach the behavior of the hardening material by a composite of elastic-ideally plastic micro-elements in a dense spectrum deforming together. Let us to represent the generalized stress deviator $S = (S, A)$, where $S$ is the deviator tensor of macroscopic stress and $A$ is an internal thermodynamic stress like variable, named back stress. The Mises yield function is

$$\Phi(\sigma) := \frac{3}{2} \|S\|^2$$  \hspace{1cm} (12)

The Stein’s work showed that the theorem of Melan can be written for materials with hardening, in terms of a back stress $A$ as: If exist a load factor $m > 1$, a time independent residual stress field, $\sigma^r \in S^r$ and a time-independent back stress field $A$ satisfying

$$\Phi(A) \leq [\sigma_Y - \sigma_{Y0}]^2$$  \hspace{1cm} (13)

such as for all possible loads in the load domain, the condition

$$\Phi(m\sigma^E + \sigma^r - A) \leq [\sigma_{Y}]^2$$  \hspace{1cm} (14)

is fulfilled for all body points beyond a time $t$, where $m > 1$ is a safety factor against non adaptation, then the total plastic energy dissipated within an arbitrary load path contained within the load domain is bounded, i.e. the elastic shakedown occurs. The material parameter $\sigma_{Y0}$ is the initial yield stress and $\sigma_Y$ is the ultimate stress. Is is important to notice that, the Stein’s model does not depend on the hardening curve shape once only $\sigma_{Y0}$ and $\sigma_Y$ appears in equations. Because this fact, we could use a linear model for hardening, to simplify the calculations. The stational principle for hardening materials is:

$$\mu = \sup_{(\mu^\ast, \sigma^r, A)} \{ \mu^\ast \geq 0 \mid \Phi(\mu^\ast \sigma^E + \sigma^r - A) \leq \sigma_{Y0}^2; \Phi(A) \leq (\sigma_Y - \sigma_{Y0})^2; \sigma^r \in S^r \}$$  \hspace{1cm} (15)

From this principle, mixed and kinematic principles can be derived. All that principles can be used to be discretized in a numerical procedure. Introducing the restriction over $\sigma^r$ into the objective function as a penalty we obtain the mixed principle which will be used here:

$$\mu = \sup_{(\mu^\ast, \sigma, A)} \inf_v \{ \mu^\ast + \langle \sigma, Dv \rangle \mid \Phi(\mu^\ast \sigma^E + \sigma - A) \leq \sigma_{Y0}^2; \Phi(A) \leq (\sigma_Y - \sigma_{Y0})^2 \}$$  \hspace{1cm} (16)

The two yield functions of Stein’s statement are:

$$f_1(\sigma, A) = \frac{3}{2} \|S - A^{dev}\|^2 - (\sigma_{Y0})^2$$  \hspace{1cm} (17)
\[ f_2(A) = \frac{3}{2} \| A^{dev} \|^2 - (\sigma_Y - \sigma_{Y_0})^2 \]  

(18)

Associated flow rules for both the plastic strain rate \( d^p \) and the hardening flux \( \dot{\beta} \) completes the model. For the two plastic modes, one have:

\[
d^p = \dot{\lambda}_1 \nabla_{\sigma} f_1 + \dot{\lambda}_2 \nabla_{\sigma} f_2 \quad \dot{\beta} = \dot{\lambda}_1 \nabla_A f_1 + \dot{\lambda}_2 \nabla_A f_2
\]

(19)

where \( \nabla_{\sigma} f_1 \) is the partial gradient of \( f_1(\sigma, A) \) with respect to \( \sigma \), and so on. Deriving Eq.(17) and Eq.(18), the evolution equations are obtained

\[
d^p = 3\dot{\lambda}_1 (S - A) \quad \dot{\beta} = -3\dot{\lambda}_1 (S - A) + 3\dot{\lambda}_2 A
\]

(20)  

(21)

Complementarity constraints (see e.g. [15]), for \( i = 1, 2 \), completes the model.

\[
\dot{\lambda}_i f_i(\sigma, A) = 0 \quad f_i(\sigma, A) \leq 0 \quad \dot{\lambda}_i \geq 0
\]

(22)

3.3 The discrete problem

We discretize the mixed principle presented at Eq.(16) to obtain a numerical solution. Are used here, mixed plane stress six nodes triangular finite elements, with internal variable, interpolation. The velocity field is interpolated quadratically, the stress field is interpolated linearly and the internal variable \( A \) are considered constant over the element. It is important to notice related with \( A \) tensor, which in the equations, only their deviatoric part is present. In the Stein’s model the hydrostatic part remaining undetermined. We can use this freedom to define one, among all the possible \( A \) tensors, adding then a complementary condition to make the calculations simpler (see figure 1). We adopted for \( A \):

i) \( trA = 0 \) in the triaxial cases

ii) \( A_z = 0 \) in the plane stress case (note that in this case the deviatoric part needs to be calculated).

Note also that in the first case, \( A_z \) is not necessarily null and that in the second case, \( trA \) is not necessarily null. In the discrete formulation, we work on the optimality conditions of the mixed principle. We introduce the approximation functions in the principle of virtual power and compute the usual discrete strain-displacement matrix \( B \) such that the kinematic compatibility and self-equilibrium equations read now

\[ d = Bv \quad B^T \sigma^r = 0 \]

(23)

Next, we consider the whole set of constraints in the mixed principle optimality conditions for the \( n_{\text{elem}} \) elements mesh. The plastic admissibility will be enforced only at the triangle \( p \) vertices, for each basic load \( n_\Delta \) of the load domain. As the load domain \( \Delta \) is convex and the stress interpolation is linear, is necessary only to impose the plastic admissibility at the triangle vertices to assure this condition over the whole element. Thus, there are \( pn_{\text{elem}} \) points in the mesh where plastic admissibility is explicitly enforced for each basic load. This, results, for the Stein’s bimodal yield surface in \( m := 2pn_{\text{elem}}n_\Delta \) inequality constraints, that are enumerated using a single index \( k = 1, m \) in correspondence to \( (\ell, i, j) \) with \( \ell = 1, n_\Delta \), \( i = 1, 2 \) and \( j = 1, p n_{\text{elem}} \).
Figure 1: Internal variable $A$ for plane stress.

Considering $\sum := \sum_{k=1,m}$, the optimality conditions for limited hardening with internal variables can be stated as follows:

\begin{align*}
B^T \sigma^r &= 0 \quad (24) \\
\sum d^k &= Bv \quad (25) \\
\sum \beta^k + \dot{\beta}^A &= 0 \quad (26) \\
\sum \sigma^k \cdot d^k &= 1 \quad (27)
\end{align*}

\begin{align*}
\dot{d}^k &= \dot{\lambda}^k \nabla_{\sigma} f^k \quad k = 1 : m \\
\dot{\beta}^k &= \dot{\lambda}^k \nabla_A f^k \quad k = 1 : m \\
\dot{\beta}^A &= \dot{\lambda}^A \nabla_A f^A \quad (30) \\
\dot{\lambda}^k f^k &= 0 \quad k = 1 : m \quad (31) \\
\dot{\lambda}^A f^A &= 0 \quad (32) \\
f^k := f_1(\mu \sigma^k + \sigma^r, A) &\leq 0 \quad k = 1 : m \quad (33) \\
f^A := f_2(A) &\leq 0 \quad (34) \\
\dot{\lambda}^k &\geq 0 \quad k = 1 : m \quad (35) \\
\dot{\lambda}^A &\geq 0 \quad (36)
\end{align*}

To solve the shakedown problem one needs to find:

$$\{v, \sigma^r, A, \mu, \dot{\lambda}^k, \dot{\lambda}^A\}$$

The internal variable $A$ was considered together with residual stress in a single vector, but not constrained to be residual. The discrete deformation operator $B$ was constructed to have null elements in the positions corresponding to internal variable components. The new obtained vectors were:

$$
\sigma^r = (\sigma^r, A) \quad d^k = (d^k, \dot{d}^k) \quad \sigma^k = (\sigma^k, 0) \quad \lambda^k = (\dot{\lambda}^k, \dot{\lambda}^A)
$$

With this definitions, the optimality conditions are written:

$$
B^T \sigma^r = 0 \quad (39)
$$

$$
\sum \lambda^k \nabla_{\sigma} f^k = Bv \quad (40)
$$

$$
\sum \sigma^k \cdot \lambda^k \nabla_{\sigma} f^k = 1 \quad (41)
$$

$$
\dot{\lambda}^k f^k = 0 \quad k = 1 : m \quad (42)
$$

$$
f^k := f_{S1}(\mu \sigma^k + \sigma^r) \leq 0 \quad k = 1 : m \quad (43)
$$

$$
f^A := f_{S2}(A) \leq 0 \quad (44)
$$

$$
\dot{\lambda}^k \geq 0 \quad k = 1 : m \quad (45)
$$

The above system of nonlinear equations and inequalities is then solved using the algorithm described in Zouain et al. [5] to obtain numerical solutions.

### 4. LIMITED KINEMATIC HARDENING MATERIAL BEAM UNDER CONSTANT AXIAL FORCE AND VARIABLE BENDING MOMENT

We consider now as an application, a shakedown analysis of a beam subjected to a constant axial force and to variable bending moment. This example have analytical solution [16]. Numerical shakedown solution was obtained by Zouain and Silveira [17] for elastic ideally-plastic materials. In this application we will consider a material that presents limited kinematic hardening. This example is a first approach to pressure vessel wall shakedown analysis where we have membrane and bending stresses. In the figure 2 is represented a beam, with retangular cross section with width $t$ and height $h$, encased in one end and subjected to a constant axial load $N_{ext}$ and to a variable bending moment $M_{ext}$ acting alternately. We have:

$$
N_{ext} = t \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma dz, \quad M_{ext} = -t \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma z dz \quad (46)
$$

In the Stein’s hardening model, the plastic admissibility have two constraints:

$$
P_1 : \quad -\sigma_{Y0} \leq \sigma(z) - A(z) \leq \sigma_{Y0} \quad (47)
$$

$$
P_2 : \quad -(\sigma_Y - \sigma_{Y0}) \leq A(z) \leq (\sigma_Y - \sigma_{Y0}) \quad (48)
$$

For pure axial force and bending moment, the plastic collapse loads are:

$$
N_Y = th\sigma_Y, \quad M_Y = th^2\sigma_Y / 4 \quad (49)
$$

and the loads at the first yield, in case of hardening:

$$
N_{Y0} = th\sigma_{Y0}, \quad M_{Y0} = th^2\sigma_{Y0} / 6 \quad (50)
$$
and then,
\[ \frac{M_{Y0}}{M_Y} = \frac{2\sigma_{Y0}}{3\sigma_Y} \quad (51) \]
The external axial force is constant and the bending moment varies between symmetric limits. The variable loads domain is:
\[ \Delta^0 : \quad N_{\text{ext}} = N, \quad -\bar{M} \leq M_{\text{ext}} \leq \bar{M} \quad (52) \]
where $N$ e $M$ are constants. The following non-dimensional parameters are also consider:
\[ \bar{n} := \frac{N}{N_Y}, \quad \bar{m} := \frac{M}{M_Y}, \quad \bar{m}_0 := \frac{M}{M_{Y0}} \quad (53) \]
Denoting:
\[ \sigma_N := \frac{N}{t h} = \bar{n} \sigma_Y, \quad \sigma_M := \frac{6M}{th^2} = \frac{3}{2} \bar{m} \sigma_Y = \bar{m}_0 \sigma_{Y0} \quad (54) \]
as mean axial stress corresponding to $N$ and maximum bending stress, correspondending to $M$, then, the variable elastic stress domain is defined by the inequalities:
\[ \Delta := \sigma_N - \sigma_M \frac{2|z|}{h} \leq \sigma(z) \leq \sigma_N + \sigma_M \frac{2|z|}{h} \quad (55) \]
resulting in the graphical representation of $\Delta$ displayed in the figure 2.

### 4.1 Alternate plasticity in bending

Polizzotto et al. [18] states that, a load factor $\omega^*$ is secure against alternate plasticity if exist a time independent stress field $\sigma^0$, (not necessarily auto-equilibrated) such as when superposed to any stress belonging to the amplified domain of variable stresses $\omega^* \Delta$, the plastic admissibility condition is not violated in any body point. The supreme of these factors $\omega$ is the factor that precludes alternate plasticity.

\[ \omega = \sup_{(\omega^*, \sigma^0, A)} \{ \omega^* \mid \omega^* \Delta + \sigma^0 - A \subset P_1, \quad A \subset P_2 \}, \quad \forall x \in \beta \quad (56) \]
By the way, the shakedown is assured if, additionally, is imposed to the stress field $\sigma^0$, the constraint to be residual. This fact, together with the definition adopted to $\Delta$, to be an uncoupled convex hull, has as consequence:
\[ \mu \leq \omega \quad (57) \]
Therefore, the alternate plasticity factor is an upper bound for the shakedown factor. The factor $\omega$ can be calculated identifying then $z = h/2$ with the critical point and dividing, in this case, the length $2\sigma_Y$ of the plastic admissibility domain by the local stress variation amplitude, $2\sigma_M$.

$$\omega = \frac{2\sigma_Y}{2\sigma_M} = \frac{2\sigma_Y}{3m\sigma_Y}$$

(58)

This results, in the figure 3 diagram, in the horizontal line defined by:

$$\omega m = \frac{2\sigma_Y}{3\sigma_Y}$$

(59)

4.2 Simple mechanism of incremental collapse

Zouain e Silveira [17] and Silveira [19], developed for this example considering elastic ideally-plastic materials, the analytical solution for the simple mechanism of incremental collapse. They used a generalization of the limit analysis, and proposed a variational principle for calculate safety factor $\rho$ related with incremental plasticity. The static formulation of the limit analysis consist in to find for the load $L$, an amplifying load factor $\alpha$ such as:

$$\alpha = \sup_{\alpha^* \in \mathbb{R}} \left\{ \alpha^* | \sigma^c \in P, \sigma^c \in S(\alpha^* L) \right\}$$

(60)

where $S(L)$ is a stress field set equilibrating the load $L$ and $\sigma^c$ is the stress field corresponding to the instantaneous plastic collapse load. The elastic stresses $\sigma^E$ equivalent to each applied load, are obtained solving an elastic boundary value problem. Let $\tilde{\rho}(\sigma^E)$ be the instantaneous collapse factor associated to the load $D'\sigma^E$ that it is in equilibrium with the stress field $\sigma^E$. The limit analysis static formulation is written:

$$\tilde{\rho}(\sigma^E) = \sup_{\rho^* \in \mathbb{R}} \left\{ \rho^* | \sigma^c \in P, \sigma^c \in S(\rho^* D'\sigma^E) \right\}$$

(61)

The second constraint can be transformed:

$$\langle \sigma^c, Dv \rangle = \langle \rho^* D'\sigma^E, v \rangle \quad \forall v \in V$$

(62)

or

$$\langle \sigma^c - \rho^* \sigma^E, Dv \rangle = 0 \quad \forall v \in V$$

(63)

therefore, the stress field $\sigma^c - \rho^* \sigma^E$ is self-equilibrated because $\sigma^c - \rho^* \sigma^E \in S^r$. Thus, the amplifying factor corresponding to instantaneous plastic collapse is

$$\tilde{\rho}(\sigma^E) = \sup_{\rho^* \in \mathbb{R}} \left\{ \rho^* | \rho^* \sigma^E + \sigma^r \in P, \sigma^r \in S^r \right\}$$

(64)

In the case of load domain, $\Delta^E$, one defines the amplifier $\rho$ such that the body don’t suffer instantaneous plastic collapse for any $\sigma^E \in \Delta^E$, as the minimum collapse amplifier $\tilde{\rho}(\sigma^E)$ value, that is:

$$\rho = \inf_{\sigma^E \in \mathbb{R}} \{ \tilde{\rho}(\sigma^E) | \sigma^E \in \Delta^E \}$$

(65)

where $\tilde{\rho}(\sigma^E)$ is the instantaneous collapse factor for an elastic stress field associated to one loading. As we saw, the stress domain $\Delta^E$ is obtained from $\Delta^0$ trough an elastic mapping. Defining an uncoupled point to point hull $\Delta$, the principle is written:

$$\rho = \inf_{\sigma \in \mathbb{R}} \{ \tilde{\rho}(\sigma) | \sigma \in \Delta \}$$

(66)
In this case, $\sigma$ is no more an elastic stress field but only an admissible stress distribution in each body point (see figure 2). In two distinct body points, $x_1$ and $x_2$ one can have, for example, elastic stresses $\sigma^E(x_1)$ due to load $L_1$ and $\sigma^E(x_2)$ due to another load, $L_3$. Thus, the amplifying factor $\rho$ relative to instantaneous collapse for any load $\sigma \in \Delta$, it is truly the incremental plasticity safety factor for the domain $\Delta$ as a whole, that is:

$$\rho = \inf_{\sigma \in W'} \sup_{\rho^* \in \mathbb{R}} \{ \rho^* | \rho^* \Delta + \sigma^r \in P, \sigma^r \in S^r \}$$

(67)

The instantaneous collapse amplifying factor, $\hat{\rho}$, even for a hardening material, only depends to the final yield stress, the same occurring relating to the simple incremental collapse amplifying factor, $\rho$. Thus, the simple incremental collapse solution for the beam, developed in [17] for elastic ideally-plastic materials, can be used in the case of hardening, because it only depends on $\overline{n}$ and $\overline{m}$ in which definition one only have the final yield stress $\sigma_Y$. Thus, taking as reference the Zouain and Silveira work [17], one have two cases:

i) For $8\overline{n} \geq 3\overline{m}$, the solution is:

$$\rho = \frac{4}{4\overline{n} + 3\overline{m}}$$

(68)

resulting in the figure 3 diagram, in the straight line BCC’D:

$$4(\rho \overline{n}) + 3(\rho \overline{m}) = 4$$

(69)

Figure 3: Bree diagram for the beam under constant axial force and variable bending moment, with $\sigma_{Y0}/\sigma_Y = 0.8$. S indicate the shakedown domain and E the elastic domain. PC indicate plastic collapse, AP indicate alternate plasticity and SMIC indicate simple mechanism of incremental collapse.
ii) for $8n \leq 3m$, the solution is:

$$\rho = \sqrt{4n^2 + m^2 - m^2}$$

resulting in the figure 3 diagram, in the curve:

$$(\rho \overline{m}) + (\rho \overline{m})^2 = 1$$

which coincides with the segment ED of the parabol BFF’DE that represent the instantaneous collapse.

In the figure 3 are plotted the plastic collapse, the alternate plasticity and the simple incremental collapse curves, with and without hardening. The curve EDFF’B represents the plastic collapse. The curves ED and the straight line DCC’B represents the incremental collapse. The straight line ACFG represents the alternate plasticity for elastic-ideally plastic materials and the straight line A’C’F’G’ represents the alternate plasticity for materials with kinematic hardening. In figure 3 can be noted the decrease of the shakedown domain $S$ due to the limited kinematic hardening, emphasizing the importance of this consideration. We notice here that, this occurs due we consider that, using an elastic ideally-plastic material model, the yield stress to be used is $\sigma_Y$ and not $\sigma_{Y0}$. This is a conventional issue and other interpretations are possible and eventually they are done, mainly in comparisons with other works. However, this assumption is justified by the fact that the structure have a unique plastic collapse load. In the figure 3 one can see that the plastic collapse load is unique and does not depends on the hardening existence.

### 4.3 Numerical solutions for the beam

For the beam subjected to constant axial force and variable bending moment, we also obtained a numerical solution for the shakedown problem. To model the beam encased in an extremity, were used eight plane stress finite elements layers displayed in the figure 4. In this model, the plane section condition is imposed.

![Eight layers finite element model for the one end encased beam, under constant axial force and variable bending moment.](image)

**Figure 4:** Eight layers finite element model for the one end encased beam, under constant axial force and variable bending moment.
The numerical shakedown solution were plotted together with the analytical ones in the figure 5 for elastic ideally-plastic materials and for materials presenting limited kinematic hardening with a relation among initial and final yield stress $\sigma_{Y_0}/\sigma_Y = 0.8$. One can observe the excellent agreement between numerical and analytical results and can be also noticed, the reduction in the shakedown domain caused by the hardening.

![Figure 5: Numerical results (Bree diagram) for the beam under constant axial force and variable bending moment. The numerical values are indicated by black squares for the boundary of the shakedown domain and by empty circles for the boundary of the elastic domain. The continuous lines are the analytical solutions.](image)

5. CONCLUSIONS

In this paper, using internal variables, the shakedown algorithm developed by Zouain and co-workers was extended to deal with limited kinematic hardening materials. Analytical results was obtained for a beam encased in an extremity under constant axial load and variable bending moment. The numerical values obtained through the enhanced shakedown algorithm, presented good matching with the analytical ones. In the developments we shown that, in plane stress problems when using the Stein’s model, the trace of the internal tensor $A$ will be not necessarily null but such as resulting in the nullity of the hardening internal variable component $A_z$. The limited kinematic hardening importance in reducing the secure shakedown domain was emphasized, under the assumptions that the structure presents only one plastic collapse load and we explain here why this hypothesis was done.
ACKNOWLEDGMENTS

This paper was sponsored by CNPq proc. 561533/2008-3

REFERENCES