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AN EXAMPLE OF AN INFINITE SET OF ASSOCIATED PRIMES OF A LOCAL COHOMOLOGY MODULE

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0. **Introduction**

Let (R, m) be a local Noetherian ring, let $I \subset R$ be any ideal and let M be a finitely generated R-module. It has been long conjectured that the local cohomology modules $H^i_I(M)$ have finitely many associated primes for all i (see Conjecture 5.1 in [H] and [L].)

If R is not required to be local these sets of associated primes may be infinite, as shown by Anurag Singh in [S], where he constructed an example of a local cohomology module of a finitely generated module over a finitely generated $\mathbb{Z}$-algebra with infinitely many associated primes. This local cohomology module has p-torsion for all primes $p \in \mathbb{Z}$.

However, the question of the finiteness of the set of associated primes of local cohomology modules defined over local rings and over k-algebras (where k is a field) has remained open until now. In this paper I settle this question by constructing a local cohomology module of a local finitely generated k-algebra with an infinite set of associated primes, and I do this for any field k.

1. **The example**

Let k be any field, let $R_0 = k[x, y, s, t]$ and let $S = R_0[u, v]$. Define a grading on S by declaring $\deg(x) = \deg(y) = \deg(s) = \deg(t) = 0$ and $\deg(u) = \deg(v) = 1$. Let $f = sx^2v^2 - (t+s)xyuv + ty^2u^2$ and let $R = S/fS$. Notice that f is homogeneous and hence R is graded. Let S_+ be the ideal of S generated by u and v and let R_+ be the ideal of R generated by the images of u and v.

Consider the local cohomology module $H^2_{R_+}(R)$: it is homogeneously isomorphic to $H^2_{S_+}(S/fS)$ and we can use the exact sequence

$$H^2_{S_+}(S)(-2) \xrightarrow{f} H^2_{S_+}(S) \longrightarrow H^2_{S_+}(S/fS) \longrightarrow 0$$

of graded R-modules and homogeneous homomorphisms (induced from the exact sequence

$$0 \longrightarrow S(-2) \xrightarrow{f} S \longrightarrow S/fS \longrightarrow 0)$$

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to study $H^2_{R_+}(R)$. Furthermore, we can realize $H^2_{S_+}(S)$ as the module $R_0[u^-, v^-]$ of inverse polynomials described in [BS, 12.4.1]: this graded S-module vanishes beyond degree -2, and, for each $d \geq 2$, its $(-d)$-th component is a free R_0-module of rank $d-1$ with base $\left(u^{-\alpha}v^{-\beta}\right)_{\alpha,\beta>0,\ \alpha+\beta=-d}$. We will study the graded components of $H^2_{S_+}(S/fS)$ by considering the cokernels of the R_0-homomorphisms

$$f_{-d} : R_0[u^-, v^-]_{-d-2} \longrightarrow R_0[u^-, v^-]_{-d} \quad (d \geq 2)$$

given by multiplication by f. In order to represent these R_0-homomorphisms between free R_0-modules by matrices, we specify an ordering for each of the above-mentioned bases by declaring that

$$u^{\alpha_1}v^{\beta_1} < u^{\alpha_2}v^{\beta_2}$$

(where $\alpha_1, \beta_1, \alpha_2, \beta_2 < 0$ and $\alpha_1 + \beta_1 = \alpha_2 + \beta_2$) precisely when $\alpha_1 > \alpha_2$. If we use this ordering for both the source and target of each f_d, we can see that each f_d $(d \geq 2)$ is given by multiplication on the left by the tridiagonal $d-1$ by $d+1$ matrix

$$A_{d-1} := \begin{pmatrix} sx^2 & -xy(t+s) & ty^2 & 0 & \dots & 0 \\ 0 & sx^2 & -xy(t+s) & ty^2 & 0\dots & 0 \\ 0 & 0 & sx^2 & -xy(t+s) & ty^2\dots & 0 \\ & & & \ddots & & \\ 0 & \dots & & sx^2 & -xy(t+s) & ty^2 \end{pmatrix}.$$

We also define

$$\overline{A}_{d-1} := \begin{pmatrix} s & -(t+s) & t & 0 & \dots & 0 \\ 0 & s & -(t+s) & t & 0\dots & 0 \\ 0 & 0 & s & -(t+s) & t\dots & 0 \\ & & & \ddots & & \\ 0 & \dots & & s & -(t+s) & t \end{pmatrix}$$

obtained by substituting $x = y = 1$ in A_{d-1}.

Let also $\tau_i = (-1)^i(t^i + st^{i-1} + \cdots + s^{i-1}t + s^i)$.

1.1. **Lemma.**

(i) *Let B_i be the submatrix of $\overline{A}_i$ obtained by deleting its first and last columns. Then* $\det B_i = \tau_i$ *for all $i \geq 1$.*

(ii) *Let $\mathcal{S}$ be an infinite set of positive integers. Suppose that either k has characteristic zero or that k has prime characteristic p and $\mathcal{S}$ contains infinitely many integers of the form $p^m - 2$. The ($k[s,t]$-)irreducible factors of $\{\tau_i\}_{i\in\mathcal{S}}$ form an infinite set.*

Proof. We prove the first statement by induction on i. Since

$$\det B_1 = \det(-t-s) = -t-s \text{ and } \det B_2 = \det \begin{pmatrix} -t-s & t \\ s & -t-s \end{pmatrix} = t^2+st+s^2,$$

the lemma holds for $i=1$ and $i=2$. Assume now that $i \geq 3$. Expanding the determinant of B_i by its first row and applying the induction hypothesis we obtain

$$\begin{aligned}\det B_i &= (-t-s)\det B_{i-1} - st\det B_{i-2} \\ &= (-1)^{i-1}(-t-s)(t^{i-1}+\cdots+s^{i-2}t+s^{i-1}) - (-1)^{i-2}st(t^{i-2}+\cdots+s^{i-3}t+s^{i-2}) \\ &= (-1)^i\left[(t^i+\cdots+s^{i-2}t^2+s^{i-1}t)+(st^{i-1}+\cdots+s^{i-1}t+s^i)-(st^{i-1}+\cdots+s^{i-2}t^2+s^{i-1}t)\right] \\ &= (-1)^i(t^i+st^{i-1}+\cdots+s^{i-1}t+s^i).\end{aligned}$$

We now prove the second statement. Define $\sigma_i = t^i+t^{i-1}+\cdots+t+1$ and notice that it is enough to show that the set of irreducible factors of $\{\sigma_i\}_{i\in\mathcal{S}}$ is infinite. Let $\mathcal{I}$ be the set of irreducible factors of $\{\sigma_i\}_{i\in\mathcal{S}}$. If k has characteristic zero consider $\mathbb{Q}[\mathcal{I}] \supseteq \mathbb{Q}$, the splitting field of this set of irreducible factors. If $\mathcal{I}$ is finite, $\mathbb{Q}[\mathcal{I}] \supseteq \mathbb{Q}$ is finite extension which contains all ith roots of unity for *all* $i \in \mathcal{S}$, which is impossible.

Assume now that k has prime characteristic p. Let $\mathbb{F}$ be the algebraic closure of the prime field of k. For any positive integer m

$$\frac{d}{dt}t(t^{p^m-1}-1) = -1$$

so $\sigma_{p^m-2} = (t^{p^m-1}-1)/(t-1)$ has p^m-2 distinct roots in $\mathbb{F}$ and, therefore, the roots of $\{\sigma_s\}_{s\in\mathcal{S}}$ form an infinite set. □

1.2. **Theorem.** *For every $d \geq 2$ the R_0-module $H^2_{R_+}(R)_{-d}$ has τ_{d-1}-torsion. Hence $H^2_{R_+}(R)$ has infinitely many associated primes.*

Proof. For the purpose of this proof we introduce a bigrading in R_0 by declaring $\deg(x) = (1,0)$, $\deg(y) = (1,1)$ and $\deg(t) = \deg(s) = (0,0)$.

We also introduce a bigrading on the free R_0-modules R_0^n by declaring $\deg(x^\alpha y^\beta s^a t^b \mathbf{e}_j) = (\alpha+\beta, \beta+j)$ for all non-negative integers α, β, a, b and all $1 \leq j \leq n$. Notice that R_0^n is a bigraded R_0-module when R_0 is equipped with the bigrading mentioned above.

Consider the R_0-module $\operatorname{Coker} A_{d-1}$; the columns of A_{d-1} are bihomogeneous of bidegrees

$$(2,1),(2,2),\ldots,(2,d+1).$$

We can now consider $\operatorname{Coker} A_{d-1}$ as a $k[s,t]$ module generated by the natural images of $x^\alpha y^\beta \mathbf{e}_j$ for all non-negative integers α, β and all $1 \leq j \leq d-1$. The $k[s,t]$-module of relations among these generators is generated by $k[x,y]$-linear combinations of the columns of A_{d-1}, and since these columns are bigraded, the $k[s,t]$-module of relations will be bihomogeneous and we can write

$$\operatorname{Coker} A_{d-1} = \bigoplus_{0\le D,\ 1\le j} (\operatorname{Coker} A_{d-1})_{(D,j)} \,.$$

Consider the $k[s,t]$-module $(\operatorname{Coker} A_{d-1})_{(d,d)}$, the bihomogeneous component of $\operatorname{Coker} A_{d-1}$ of bidegree (d,d). It is generated by the images of

$$xy^{d-1}\mathbf{e}_1, x^2y^{d-2}\mathbf{e}_2, \dots, x^{d-2}y^2\mathbf{e}_{d-2}, x^{d-1}y\mathbf{e}_{d-1}$$

and the relations among these generators are given by $k[s,t]$-linear combinations of

$$y^{d-2}\mathbf{c}_2, xy^{d-3}\mathbf{c}_3, \dots, x^{d-3}y\mathbf{c}_{d-1}, x^{d-2}\mathbf{c}_d$$

where $\mathbf{c}_1, \dots, \mathbf{c}_{d+1}$ are the columns of A_{d-1}. So we have

$$(\operatorname{Coker} A_{d-1})_{(d,d)} = \operatorname{Coker} B_{d-1}$$

where B_{d-1} is viewed as a $k[s,t]$-homomorphism $k[s,t]^{d-1} \to k[s,t]^{d-1}$.

Using Lemma 1.1(i) we deduce that for all $d \ge 2$ the direct summand $(\operatorname{Coker} A_{d-1})_{(d,d)}$ of $\operatorname{Coker} A_{d-1}$ has τ_{d-1} torsion, and so does $\operatorname{Coker} A_{d-1}$ itself.

Lemma 1.1(ii) applied with $\mathcal{S} = \mathbb{N}$ now shows that there exist infinitely many irreducible homogeneous polynomials $\{p_i \in k[s,t] : i \ge 1\}$ each one of them contained in some associated prime of the R_0-module $\oplus_{d\ge2} \operatorname{Coker} A_{d-1}$. Clearly, if $i \ne j$ then any prime ideal $P \subset R_0$ which contains both p_i and p_j must contain both s and t.

Since the localisation of $(\operatorname{Coker} A_{d-1})_{(d,d)}$ at s does not vanish, there exist $P_i, P_j \in \operatorname{Ass}_{R_0} \operatorname{Coker} A_{d-1}$ which do not contain s and such that $p_i \subset P_i$, $p_j \subset P_j$, and the previous paragraph shows that $P_j \ne P_j$.

The second statement now follows from the fact that $H^2_{R_+}(R)$ is R_0-isomorphic to $\oplus_{d\ge2} \operatorname{Coker} A_{d-1}$. □

1.3**. Corollary.** *Let T be the localisation of R at the irrelevant maximal ideal $\mathfrak{m} = \langle s,t,x,y,u,v\rangle$. Then $H^2_{(u,v)T}(T)$ has infinitely many associated primes.*

Proof. Since $\tau_i \in \mathfrak{m}$ for all $i \ge 1$, $H^2_{(u,v)T}(T) \cong (H^2_{(u,v)R}(R))_{\mathfrak{m}}$ has τ_i-torsion for all $i \ge 1$. □

2. **A connection with associated primes of Frobenius powers**

In this section we apply a technique similar to the one used in section 1 to give a proof of a slightly more general statement of Theorem 12 in [K]. The new proof is simpler, open to generalisations and it gives a connection between associated primes of Frobenius powers of ideals and of local cohomology modules, at least on a purely formal level.

Let k be any field, let $S = k[x, y, s, t]$, let $F = xy(x - y)(sx - ty) = sx^3y - (t + s)x^2y^2 + txy^3$ and let $R = S/FS$.

2.1. **Theorem.** *Let $\mathcal{S}$ be an infinite set positive integers and suppose that either k has characteristic zero or that k has characteristic p and that $\mathcal{S}$ contains infinitely many powers of p. The set*

$$\bigcup_{n \in \mathcal{S}} \operatorname{Ass}_R \left(\frac{R}{\langle x^n, y^n \rangle} \right)$$

is infinite.

Proof. We introduce a grading in S by setting $\deg(x) = \deg(y) = 1$ and $\deg(s) = \deg(t) = 0$. Since F is homogeneous, R is also graded.

Fix some $n > 0$ and consider the graded R-module $T = R/\langle x^n, y^n \rangle$. For each $d > 4$ consider T_d, the degree d homogeneous component of T, as a $k[s, t]$-module. If $d < n$, T_d is generated by the images of $y^d, xy^{d-1}, \ldots, x^{d-1}y, x^d$ and the relations among these generators are obtained from $y^{d-4}F, xy^{d-5}F, \ldots, x^{d-5}yF, x^{d-4}F$. Using these generators and relations, in the given order, we write $T_d = \operatorname{Coker} M_d$ where

$$M_d = \begin{pmatrix} 0 & 0 & \dots & & 0 \\ t & & & & \\ -t-s & t & & & \\ s & -t-s & & & \\ & s & \ddots & & \\ & & & t & \\ & & & -t-s & t \\ & & & s & -t-s \\ & & & & s \\ 0 & 0 & \dots & & 0 \end{pmatrix}.$$

When $d = n$, T_d is isomorphic to the cokernel of the submatrix of M_d obtained by deleting the first and last rows which correspond to the generators y^n, x^n of T_n.

When $d = n + 1$, T_d is isomorphic to the cokernel of the submatrix of M_d obtained by deleting the first two rows and and last two rows which correspond to the generators $y^{n+1}, xy^n, x^n y, x^{n+1}$ of T_{n+1}, and the resulting submatrix is B_{n-2} defined in Lemma 1.1; the result now follows from that lemma. □

This technique for finding associated primes of non-finitely generated graded modules and of sequences of graded modules has been applied in [BKS] and [KS] to yield further new and surprising properties of top local cohomology modules.

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