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THE HILBERT SERIES OF ALGEBRAS OF VERONESE TYPE

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ABSTRACT. This paper gives an explicit formula for the Hilbert series of algebras of Veronese type.

1. INTRODUCTION

In this paper we describe the Hilbert series of algebras of Veronese type:

Definition 1.1. Fix a positive integer d and a sequence of integers $\mathbf{a} = (a_1, \dots, a_n)$ such that $1 \leq a_1 \leq \dots \leq a_n \leq d$ and $\sum_{i=1}^n a_i > d$. Let $\mathcal{V}(\mathbf{a}; d)$ be the k -subalgebra of $k[x_1, \dots, x_n]$ generated by all monomials $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ with $\sum_{i=1}^n \alpha_i = d$ and $\alpha_i \leq a_i$ for all $1 \leq i \leq n$.

We shall also denote by \mathcal{S} all subsets S of $\{1, \dots, n\}$ with $\sum_{i \in S} a_i < d$; for any $S \subset \{1, \dots, n\}$ we define ΣS to be $\sum_{i \in S} a_i$.

Note that $\mathcal{V}(d, d, \dots, d; d)$ is the classical Veronese algebra while $\mathcal{V}(1, 1, \dots, 1; d)$ is the monomial algebra associated with the d th hypersimplex.

For the purpose of computing Hilbert series and a -invariants we will use a normalized grading on these algebras so that the degree of their generators equals one.

These monomial algebras have recently attracted considerable interest; E. DeNegri, T. Hibi ([3]) have recently classified those which are Gorenstein and B. Sturmfels ([5]) described Gröbner bases arising from presentations of these algebras.

It is known that algebras of Veronese type are normal and in [3] the authors classified all such algebras which are Gorenstein. Also, in [2] the authors described the canonical modules and a -invariants of $\mathcal{V}(1, 1, \dots, 1; d)$. The aim of this section is to extend and complement these results by producing an explicit formula for the h -vectors of all algebras of Veronese type. Additionally, the explicit formulas provide a very efficient way for computing these Hilbert series.

The results in this paper are part of an ongoing project to understand certain monomial algebras. Since these results have been obtained several articles have referred to them in a preliminary form (e.g., [4]); it is hoped that this paper will make the results more widely available.

2. THE HILBERT SERIES OF ALGEBRAS OF VERONESE TYPE

We start by describing the Hilbert function of algebras of Veronese type:

Theorem 2.1. *The Hilbert function of $\mathcal{V}(\mathbf{a}; d)$ is given by*

$$H(i) = \sum_{S \in \mathcal{S}} (-1)^{|S|} \binom{i(d - \Sigma S) - |S| + n - 1}{n - 1}.$$

Proof. By Lemma 2.1 in [3] $H(i)$ is the number of sequences $(\alpha_1, \dots, \alpha_n)$ satisfying $\sum_{j=1}^n \alpha_j = id$ and $1 \leq \alpha_j \leq ia_j$. This number is readily seen to be the coefficient of t^{id} in

$$\prod_{i=1}^n (1 + t + \dots + t^{ia_i}) = \frac{\prod_{i=1}^n (1 - t^{ia_i+1})}{(1 - t)^n} = \left(\sum_{S \subset \{1, \dots, n\}} (-1)^{|S|} t^{i \Sigma S + |S|} \right) \left(\sum_{r=0}^{\infty} \binom{r + n - 1}{n - 1} t^r \right)$$

and the coefficient of t^{id} in this expression is

$$\sum_{S \in \mathcal{S}} (-1)^{|S|} \binom{i(d - \Sigma S) - |S| + n - 1}{n - 1}$$

□

Corollary 2.2. *The Hilbert function $H_{nd}(i)$ of $\mathcal{V}(1, 1, \dots, 1; d)$ is given by*

$$H_{nd}(i) = \sum_{s=0}^{d-1} (-1)^s \binom{n}{s} \binom{i(d - s) - s + n - 1}{n - 1}.$$

Remark 2.3. Since $\mathcal{V}(1, 1, \dots, 1; d)$ are normal, the previous Corollary also gives the Ehrhart polynomial of the d th hypersimplex. This was also proved for $d = 2$ in Chapter 9 of [5] using Gröbner bases techniques.

T. Hibi has also noted that one could obtain the Hilbert functions of algebras of Veronese type using their initial ideals discussed in the last chapter of [5].

To produce the h -vectors of $\mathcal{V}(\mathbf{a}; d)$ we need to understand the generating functions of $\binom{n+id-1}{n-1}$, and these will be described using the following:

Definition 2.4. For any positive integers n and d define the numbers $A_i^{n,d}$ by

$$(1 + T + \dots + T^{d-1})^n = \sum_{i \geq 0} A_i^{n,d} T^i.$$

Theorem 2.5.

$$(1-t)^n \sum_{i=0}^{\infty} \binom{n+id-1}{n-1} t^i = \sum_{j \geq 0} A_{jd}^{n,d} t^j$$

Proof. Let Ξ be the set of (complex) d th roots of 1; we have

$$\begin{aligned} & \frac{1}{d} \sum_{\xi \in \Xi} \frac{1}{(1-\xi t^{1/d})^n} = \\ & \frac{1}{d} \sum_{\xi \in \Xi} \sum_{j=0}^{\infty} \binom{n+j-1}{n-1} (\xi t^{1/d})^j = \\ & \sum_{j=0}^{\infty} \binom{n+jd-1}{n-1} t^j, \end{aligned}$$

the last equality following from the fact that

$$\sum_{\xi \in \Xi} \xi^j = \begin{cases} d & d|j \\ 0 & \text{otherwise.} \end{cases}$$

Multiplying both sides by $(1-t)^n = (1-(\xi t^{1/d})^d)^n$ we obtain

$$\begin{aligned} & (1-t)^n \sum_{j=0}^{\infty} \binom{n+jd-1}{n-1} t^j = \\ & \frac{1}{d} \sum_{\xi \in \Xi} \frac{(1-(\xi t^{1/d})^d)^n}{(1-\xi t^{1/d})^n} = \\ & \frac{1}{d} \sum_{\xi \in \Xi} \left(1 + \xi t^{1/d} + \dots + (\xi t^{1/d})^{d-1}\right)^n. \end{aligned}$$

The coefficient of $(t^{1/d})^s$ in this expression is

$$\frac{1}{d} \sum_{\xi \in \Xi} \xi^s A_s^{n,d} = \begin{cases} A_s^{n,d} & d|s \\ 0 & \text{otherwise,} \end{cases}$$

thus only integer powers of t appear in the sum and the coefficient of t^s is $A_{sd}^{n,d}$. \square

The previous Theorem provides an elementary proof for Theorem 2.4(a) in [3]:

Corollary 2.6. *The h -vector of $\mathcal{V}(d, d, \dots, d; d)$ is symmetric if and only if d divides n , therefore, $\mathcal{V}(d, d, \dots, d; d)$ is Gorenstein if and only if d divides n . The multiplicity is d^{n-1} and the a -invariant equals to $-\left\lceil \frac{n}{d} \right\rceil$.*

Proof. Note that the h -vector is given by the coefficients of

$$(1-t)^n \sum_{i=0}^{\infty} \binom{n+id-1}{n-1} t^i = \sum_{i \geq 0} A_{id}^{n,d} T^i$$

and that $\sum_{i \geq 0} A_{id}^{n,d} T^i$ is a unimodal and reciprocal polynomial of degree $n(d-1)$ (cf. section 3.5 in [1].)

If d divides n then $\sum_{i \geq 0} A_{id}^{n,d} T^i$ is obviously reciprocal. On the other hand, if d does not divide n then d does not divide $n(d-1)$ either and in this case the coefficient of the highest power of T in $\sum_{i \geq 0} A_{id}^{n,d} T^i$ will be greater than $1 = A_0^{n,d}$.

We compute the a -invariant as the degree of the Hilbert series as a rational function: the highest power of t occurring in the numerator is

$$\left\lfloor \frac{n(d-1)}{d} \right\rfloor = n + \left\lfloor \frac{-n}{d} \right\rfloor = n - \left\lceil \frac{n}{d} \right\rceil.$$

The multiplicity will be computed in Corollary 2.11 below. □

Lemma 2.7. *For any non-negative integers n, a and b let*

$$P_{a,b}^n(t) = \sum_{i=0}^{\infty} \binom{ai-b+n-1}{n-1} t^i.$$

For any positive integer b we have $P_{a,b}^n = \sum_{j=0}^b (-1)^j \binom{b}{j} P_{a,0}^{n-j}$.

Proof. Since

$$\binom{ai-b+n-1}{n-1} = \binom{ai-(b-1)+n-1}{n-1} - \binom{ai-(b-1)+n-2}{n-2}$$

we have a recursion relation $P_{a,b}^n = P_{a,b-1}^n - P_{a,b-1}^{n-1}$ and the result follows. □

Combining this with Theorem 2.1 we obtain an explicit expression for the h -vectors of $\mathcal{V}(\mathbf{a}; d)$:

Theorem 2.8.

$$(1-t)^n \sum_{i=0}^{\infty} \sum_{S \in \mathcal{S}} (-1)^{|S|} \binom{i(d-\Sigma S) - |S| + n - 1}{n-1} t^i =$$

$$\sum_{S \in \mathcal{S}} (-1)^{|S|} \sum_{j=0}^{|S|} (-1)^j \binom{|S|}{j} (1-t)^j \sum_{l \geq 0} A_{l(d-\Sigma S)}^{n-j, d-\Sigma S} t^l.$$

Proof. For any $S \subset \{1, \dots, n\}$ we have

$$\begin{aligned} (1-t)^n (-1)^{|S|} \sum_{i=0}^{\infty} \binom{i(d-\Sigma S) - |S| + n - 1}{n-1} t^i &= \\ (1-t)^n (-1)^{|S|} P_{d-\Sigma S, |S|}^n &= \\ (-1)^{|S|} \sum_{j=0}^{|S|} (-1)^j \binom{|S|}{j} (1-t)^j ((1-t)^{n-j} P_{d-\Sigma S, 0}^{n-j}) & \end{aligned}$$

and by Theorem 2.5 this equals

$$(-1)^{|S|} \sum_{j=0}^{|S|} (-1)^j \binom{|S|}{j} (1-t)^j \sum_{l \geq 0} A_{l(d-\Sigma S)}^{n-j, d-\Sigma S} t^l.$$

□

Corollary 2.9. *The Hilbert series of $\mathcal{V}(1, 1, \dots, 1; d)$ is*

$$(1-t)^{-n} \sum_{s=0}^{d-1} (-1)^s \binom{n}{s} \sum_{j=0}^s (-1)^j \binom{s}{j} (1-t)^j \sum_{l \geq 0} A_{l(d-s)}^{n-j, d-s} t^l.$$

For $d = 2$ this reduces to

$$(1-t)^{-n} \left(\sum_{l \geq 0} \binom{n}{2l} t^l - nt \right).$$

Proof. The first statement follows easily from the previous Theorem. To prove the second statement note that $A_j^{n,2} = \binom{n}{j}$ and that $A_j^{n,1} = 0$ unless $j = 0$, in which case we have $A_0^{n,1} = 1$. □

Even though the expressions for the h -vectors for $\mathcal{V}(\mathbf{a}; d)$ look forbidding, it is possible to extract useful information from them. We shall now proceed to compute the multiplicities and find a (sharp) upper bound for their a -invariants.

Lemma 2.10.

$$\sum_{l \geq 0} A_{ld}^{n,d} = d^{n-1}.$$

Proof. Let Ξ be the set of (complex) d th roots of one. We have

$$\begin{aligned} \sum_{l \geq 0} A_{ld}^{n,d} t^l &= \\ \frac{1}{d} \sum_{\xi \in \Xi} \sum_{l \geq 0} A_l^{n,d} (\xi t^{1/d})^l &= \\ \frac{1}{d} \sum_{\xi \in \Xi} \left(1 + \xi t^{1/d} + (\xi t^{1/d})^2 + \dots + (\xi t^{1/d})^{d-1} \right)^n. & \end{aligned}$$

Setting $t = 1$ the last sum vanishes unless $\xi = 1$ and, therefore, the sum equals d^{n-1} . \square

Corollary 2.11. *The multiplicity of $\mathcal{V}(\mathbf{a}; d)$ is*

$$\sum_{S \in \mathcal{S}} (-1)^{|S|} (d - \Sigma S)^{n-1}.$$

Proof. Substituting $t = 1$ in the numerator of the Hilbert series obtained in Theorem 2.8 we obtain

$$\sum_{S \in \mathcal{S}} (-1)^{|S|} \sum_{l \geq 0} A_{l(d-\Sigma S)}^{n, d-\Sigma S}$$

which by the previous Lemma equals

$$\sum_{S \in \mathcal{S}} (-1)^{|S|} (d - \Sigma S)^{n-1}$$

\square

Corollary 2.12. *If $n \geq d$ the a -invariant of $\mathcal{V}(\mathbf{a}; d)$ is at most $-\left\lceil \frac{n}{d} \right\rceil$.*

Proof. The a -invariant is the degree of the Hilbert series of $\mathcal{V}(\mathbf{a}; d)$ as a rational function. Note that the highest degree of t occurring in a summand of

$$\sum_{S \in \mathcal{S}} (-1)^{|S|} \sum_{j=0}^{|S|} (-1)^j \binom{|S|}{j} (1-t)^j \sum_{l \geq 0} A_{l(d-\Sigma S)}^{n-j, d-\Sigma S} t^l$$

is

$$\begin{aligned} & \max_{S \in \mathcal{S}} \max_{0 \leq j \leq |S|} j + \left\lceil \frac{(n-j)(d-\Sigma S-1)}{d-\Sigma S} \right\rceil = \\ & j + n - j - \min_{S \in \mathcal{S}} \min_{0 \leq j \leq |S|} \left\lceil \frac{n-j}{d-\Sigma S} \right\rceil = \\ & n - \min_{S \in \mathcal{S}} \left\lceil \frac{n-|S|}{d-\Sigma S} \right\rceil \leq \\ & n - \min_{S \in \mathcal{S}} \left\lceil \frac{n-|S|}{d-|S|} \right\rceil = n - \left\lceil \frac{n}{d} \right\rceil \end{aligned}$$

the last equality holding for $n \geq d$. \square

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