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ON IDEALS OF MINORS OF MATRICES WITH INDETERMINATE ENTRIES

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1. INTRODUCTION AND NOTATION

This paper has two aims. The first is to study ideals of minors of matrices whose entries are among the variables of a polynomial ring. Specifically, we describe matrices whose ideals of minors of a given size are prime. The “generic” case, where all the entries are distinct variables has been studied extensively (cf. [1] and [2] for a thorough account.) While some special cases, such as catalecticant matrices and other 1-generic matrices, have been studied by other authors (e.g., [4]), the general case is not well understood. The main result in the first part of this paper is Theorem 2.3 which gives sufficient conditions for the ideal of minors of a matrix to be prime. This theorem is general enough to include interesting examples, such as the ideal of maximal minors of catalecticant matrices and their generalisations discussed in the second part of the paper.

The second aim of this paper is to settle a specific problem raised by David Eisenbud and Frank-Olaf Schreyer (cf. [5]) on the primary decomposition of an ideal of maximal minors. We solve this problem by applying Theorem 2.3 together with some ad-hoc techniques.

Throughout this paper \mathbb{K} shall denote a field. For any matrix M with entries in a ring and any $t \geq 1$, $I_t(M)$ will denote the ideal generated by the $t \times t$ minors of M . The results to be presented here rely on well known properties of determinantal rings which we summarise below:

Theorem 1.1. *Let $X = (x_{ij})$ be the generic $m \times n$ matrix and let $T = \mathbb{K}[x_{11}, \dots, x_{mn}]$.*

- (a) $T/I_t(X)$ is a Cohen-Macaulay domain (cf. Theorems 1.10 and 6.7 in [1]),
- (b) $\dim T/I_t(X) = (m + n - t + 1)(t - 1)$ (cf. Theorem 1.10 in [1]).

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2. SOME PRIME IDEALS OF MINORS

Throughout this section $X = (x_{ij})$ will be a generic $m \times n$ matrix with $m \geq n$ and T will denote the polynomial ring over \mathbb{K} whose variables are the entries of X . We fix a $1 \leq t \leq n$ and write $r = t - 1$. The aim of this section is to describe some prime ideals of minors of the image of X under the identification of some of the variables x_{ij} . We shall prove that these ideals are prime by embedding the appropriate quotient rings into domains. As a first step we realise that determinantal varieties are rational:

Proposition 2.1. *Let W be a polynomial ring with variables $\{y_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq r\}$ and $\{z_{ij} \mid 1 \leq i \leq r, 1 \leq j \leq n - r\}$ over \mathbb{K} . Let Y be a $m \times r$ matrix whose (i, j) entry is y_{ij} , let Z be the following $r \times n$ matrix*

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & z_{1,1} & \dots & z_{1,n-r} \\ 0 & 1 & 0 & \dots & 0 & z_{2,1} & \dots & z_{2,n-r} \\ & & \ddots & & & \vdots & & \\ 0 & 0 & 0 & \dots & 1 & z_{r,1} & \dots & z_{r,n-r} \end{pmatrix}$$

and let S be the sub- \mathbb{K} -algebra of W generated by the entries of the product YZ . The map $T/I_t(X) \rightarrow S$ sending the image of x_{ij} to the (i, j) entry of YZ extends to an isomorphism $f : T/I_t(X) \rightarrow S$ of \mathbb{K} -algebras.

Proof. Notice that YZ has rank $r = t - 1$, hence f is well defined. Since f is clearly surjective, we only need to show that it is injective, and we show this by showing that $\dim S = \dim T/I_t(X) = (m + n - r)r$; we achieve this by showing that W and S have the same fraction field.

Obviously, $y_{ij}/1$ is in the fraction field of S for all $1 \leq i \leq m, 1 \leq j \leq r$; we now show that $z_{ij}/1$ is in the fraction field of S for all $1 \leq i \leq r$ and $1 \leq j \leq n - r$. Let M be the $r \times r$ submatrix of Y consisting of its first r rows and denote the classical adjoint of M with $\text{adj } M$. Now the entries of $(\det \text{adj } M)^{-1}(\text{adj } M)MZ$ are in the fraction field of S but $(\det \text{adj } M)^{-1}(\text{adj } M)MZ$ contains $(z_{i,j})$ as a submatrix, and we are done. \square

Let X_r be the submatrix of X consisting of its first r columns. We let $\mathcal{J} = \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ and $\mathcal{J}_r = \{(i, j) \mid 1 \leq i \leq m, 1 \leq j \leq r\}$.

Given a sequence $S = ((\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r))$ elements of $\mathcal{J}_r \times \mathcal{J}$ we define a sequence of directed graphs G_0, \dots, G_r whose vertex sets are \mathcal{J} and whose edges given by

$$E(G_j) = \left\{ \overrightarrow{(a, b)(a, \ell)} \mid 1 \leq a \leq m, 1 \leq b \leq r, r + 1 \leq \ell \leq n \right\} \cup \left\{ \overrightarrow{\beta_1 \alpha_1}, \dots, \overrightarrow{\beta_j \alpha_j} \right\}.$$

We call the sequence \mathcal{S} *bad*, if for some $1 \leq j \leq r$ there exists a directed path in G_{j-1} starting at α_j and ending at β_j .

The motivation for this definition is as follows.

Proposition 2.2. *Use the notation of the previous proposition and write $A = YZ$. Let $\mathcal{S} = ((\alpha_1, \beta_1), \dots, (\alpha_s, \beta_s))$ be a set of elements in $\mathcal{J}_r \times \mathcal{J}$ where $\alpha_1, \dots, \alpha_s$ are distinct.*

Define recursively a sequence of matrices $A^{(0)}, \dots, A^{(s)}$ as follows: $A^{(0)} = A$ and, for all $0 \leq i < s$, $A^{(i+1)}$ is obtained from $A^{(i)}$ by replacing each occurrence of $y_{\alpha_{i+1}} = A_{\alpha_{i+1}}^{(i)}$ in $A^{(i)}$ with its β_{i+1} entry.

If for some $\alpha \in \mathcal{J}_r$, $\beta \in \mathcal{J}$ and some $0 \leq j \leq s$ y_α occurs in $A_\beta^{(j)}$ then there exists a directed path in G_j starting at α and ending at β .

Proof. We proceed by induction on j ; when $j = 0$ notice that y_α occurs in A_β for $\alpha \neq \beta$ if and only if $\alpha = (a, b)$ and $\beta = (a, \ell)$ with $1 \leq a \leq m$, $1 \leq b \leq r$ and $r + 1 \leq \ell \leq n$. Assume henceforth that $j > 0$.

Pick a minimal $0 \leq k \leq j$ such that y_α occurs in $A_\beta^{(k)}$. If $k < j$, the induction hypothesis implies that there exists a directed path in G_k starting at α and ending at β , and that path is also a directed path in G_j , and the theorem follows.

We assume now that y_α does not occur in $A_\beta^{(j-1)}$ but it does occur in $A_\beta^{(j)}$. This implies that y_{α_j} occurs in $A_\beta^{(j-1)}$ and that y_α occurs in $A_{\beta_j}^{(j-1)}$, and the induction hypothesis implies that there exist in G_{j-1} a directed path P_1 starting at α and ending at β_j and a directed path P_2 starting at α_j and ending at β . Since both P_1 and P_2 are also directed paths in G_j , and since $\overrightarrow{\beta_j \alpha_j} \in E(G_j)$ we have a directed path in G_j from α to β given by the concatenation of the path P_1 , followed by $\overrightarrow{\beta_j \alpha_j}$ and P_2 . \square

Theorem 2.3. *Let $\mathcal{S} = ((\alpha_1, \beta_1), \dots, (\alpha_s, \beta_s))$ be a sequence of elements in $\mathcal{J}_r \times \mathcal{J}$. Let $L \subseteq T$ be the ideal minimally generated by $x_{\alpha_1} - x_{\beta_1}, \dots, x_{\alpha_s} - x_{\beta_s}$. Assume that*

- (i) $\alpha_1, \dots, \alpha_s$ are distinct,
- (ii) \mathcal{S} is not a bad sequence, and
- (iii) the image of $I_r(X_r)$ in $T/(L + I_t(X))$ has positive height.

Then $T/(L + I_t(X))$ is a Cohen-Macaulay domain of dimension $(m + n - t + 1)(t - 1) - s$.

Furthermore, $T/(L + I_t(X))$ is rational.

Proof. Let $L' \subseteq T$ be the ideal generated by $x_{\alpha_1} - x_{\beta_1}, \dots, x_{\alpha_{s-1}} - x_{\beta_{s-1}}$. Write

$$U' = \frac{T}{L' + I_t(X)}, \quad U = \frac{T}{L + I_t(X)},$$

$$W' = \mathbb{K}[z_{i,j}][y_\alpha \mid \alpha \in \mathcal{J}_r \setminus \{\alpha_1, \dots, \alpha_{s-1}\}], \quad W = \mathbb{K}[z_{i,j}][y_\alpha \mid \alpha \in \mathcal{J}_r \setminus \{\alpha_1, \dots, \alpha_s\}].$$

Let A be and $A^{(0)}, \dots, A^{(s)}$ be matrices as in Proposition 2.2. Write $B' = (b'_{ij})$ for $A^{(s-1)}$ and $B = (b_{ij})$ for $A^{(s)}$. Proposition 2.2 guarantees that as we transform B' to B , the expression replacing y_{α_s} does not contain y_{α_s} , and so the variable y_{α_s} does not occur in B . Define the \mathbb{K} -algebra map $\rho : W' \rightarrow W$ which sends y_{α_s} to the β_s entry of B' and fixes all other variables.

Let S' be the sub- \mathbb{K} -algebra of W' generated by the entries of B' and let S be the sub- \mathbb{K} -algebra of W generated by the entries of B . We have the following commutative diagram

$$(1) \quad \begin{array}{ccccc} U' & \xrightarrow{f'} & S' & \xrightarrow{i'} & W' \\ \downarrow \phi & & \downarrow \psi & & \downarrow \rho \\ U & \xrightarrow{f} & S & \xrightarrow{i} & W \end{array}$$

where f' is the surjection which maps x_{ij} to the (i, j) entry of B' and f is the surjection which maps x_{ij} to the (i, j) entry of B (these are well defined because the ranks of B and B' are less than t) and where the restriction of ρ to S' induces the map $\psi : S' \rightarrow S$ which replaces any factor y_{α_s} in a generator by the β_s entry of B' . The map $\phi : U' \rightarrow U = U'/(x_{\alpha_s} - x_{\beta_s})$ is the quotient map.

To prove the theorem we show that f is an isomorphism and we do so by induction on s . The case $s = 0$ (i.e., $L = 0$) is a restatement of Proposition 2.1. Assume now that $s > 0$. Notice that the induction hypothesis implies that U' is a Cohen-Macaulay domain, and so $U = U'/(x_{\alpha_s} - x_{\beta_s})$ is also Cohen-Macaulay.

Write

$$D' = \begin{bmatrix} b'_{11} & b'_{12} & \cdots & b'_{1,r} \\ b'_{21} & b'_{22} & \cdots & b'_{2,r} \\ \vdots & \vdots & & \vdots \\ b'_{m1} & b'_{m2} & \cdots & b'_{m,r} \end{bmatrix}, \quad E' = \begin{bmatrix} b'_{1,r+1} & \cdots & b'_{1,n} \\ b'_{2,r+1} & \cdots & b'_{2,n} \\ \vdots & & \vdots \\ b'_{m,r+1} & \cdots & b'_{m,n} \end{bmatrix}.$$

Notice that $D'(z_{i,j}) = E'$ and that, if we localize at any non-zero $\delta' \in I_r(D')$, we can obtain each $z_{i,j}$ as a rational function of the entries of D' and E' . Condition (i) implies that $y_\alpha \in S'$ for all $\alpha \in \mathcal{J}_r \setminus \{\alpha_1, \dots, \alpha_{s-1}\}$ and we deduce that $S'_{\delta'} = W'_{\delta'}$ for every such minor.

Since f is clearly surjective, we conclude the proof by showing that f is injective; write $P = \ker f$.

The image of $I_r(X_r)$ in $T/(L + I_t(X))$ has positive height and since $T/(L + I_t(X))$ is Cohen-Macaulay we can find a d' in the image of $I_r(X_r)$ in $T/(L' + I_t(X))$ such that

$d := \phi(d')$ is not a zero-divisor on $T/(L + I_t(X))$. We now show that the localisation PU_d vanishes and so $P = 0$.

Write $\delta' = f'(d')$ and localise (1) to obtain

$$\begin{array}{ccc} U'_{d'} & \xrightarrow[\cong]{f'_{d'}} & S'_{\delta'} & \xrightarrow[\cong]{i'} & W'_\delta \\ \downarrow \phi_{d'} & & & & \downarrow \rho_{\delta'} \\ U_d & & & & W_\delta \end{array}$$

Now the localisation of i' at δ' is an isomorphism; but $\ker \rho_{\delta'} : W'_{\delta'} \rightarrow W_\delta$ is generated by $y_{\alpha_s} - B'_{\beta_s} = f'_{\delta'}(x_{\alpha_s} - x_{\beta_s})$ so $\ker f'_{d'} \circ i'_{f'(d')} \circ \rho_{\delta'}$ is generated by $x_{\alpha_s} - x_{\beta_s}$, i.e.,

$$\ker f'_{d'} \circ i'_{f'(d')} \circ \rho_{\delta'} = \ker \phi_{d'}.$$

Now $\phi_{d'}^{-1}(PU_d) \subseteq \ker f'_{d'} \circ i'_{f'(d')} \circ \rho_{\delta'}$ so $\phi_{d'}(\phi_{d'}^{-1}(PU_d)) = 0$ and we deduce that $PU_d = 0$. \square

Theorem 2.3 has numerous applications. The following is an instance where conditions (i) and (ii) hold trivially.

Corollary 2.4. *Let R_0 be a polynomial ring. Fix $m \geq n$ and let R be the polynomial ring over R_0 with variables x_{ij} for $1 \leq i \leq m$, $t \leq j \leq n$ where $1 \leq t \leq n$. Let $A = (a_{ij})$ be a $m \times n$ matrix such that a_{ij} are variables in R_0 for $j < t$ while $a_{ij} = x_{ij}$ for $j \geq t$. Write $r = t - 1$ and let A_r be the submatrix of A consisting of its first r columns. If the image of $I_r(A_r)$ in $R/I_t(A)$ has positive height then $R/I_t(A)$ is a Cohen-Macaulay domain.*

Remark 2.5. Let $m = n = t = 3$. Notice that while condition (ii) fails for the matrix

$$\begin{pmatrix} a & X & b \\ X & c & X \\ d & e & f \end{pmatrix},$$

the conditions of the theorem apply to its transpose, and indeed its determinant is irreducible. On the other hand, the matrix $\begin{pmatrix} a & X & b \\ X & c & X \\ d & X & f \end{pmatrix}$ cannot be manipulated to satisfy condition (ii) of the theorem whereas its determinant is irreducible.

Remark 2.6. Among the results describing when ideals of maximal minors of matrices are prime, [4] contains the following one which is quite general.

Theorem: Let M be a $m \times n$ matrix of linear forms where $m \geq n$. If M is the image of a 1-generic matrix modulo $\leq n - 2$ linear forms, then $I_n(M)$ is prime.

One may ask whether the hypothesis of Theorem 2.3 imply those of this theorem in the case of maximal minors– the answer is “no”: consider the matrix

$$M = \begin{pmatrix} A & A & C & x_1 \\ B & A & B & x_2 \\ A & C & B & x_3 \\ A & B & A & x_4 \end{pmatrix}.$$

A computation with using a Macaulay2 ([6]) script written by David Eisenbud shows that M is not the image of a 1-generic matrix modulo ≤ 2 linear forms, while the primality of $I_4(M)$ can be deduced from Theorem 2.3.

3. A PRIMARY DECOMPOSITION

Let $R = \mathbb{K}[a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5]$. We let

$$M_5 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \end{pmatrix}, M_4 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 \\ b_2 & b_3 & b_4 & b_5 \end{pmatrix}, M_3 = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 \\ b_2 & b_3 & b_4 \\ b_3 & b_4 & b_5 \end{pmatrix}.$$

In a lecture in Overwolfach in April 2005, David Eisenbud conjectured that that $I_3(M_4)$ is radical with primary decomposition $I_2(M_5) \cap I_3(M_3)$ (see also section 4 of [5]). In this section we show that this is indeed the case.

We shall need the following elementary lemma:

Lemma 3.1. *Let I be an homogeneous ideal in a polynomial ring $P = \mathbb{K}[y_1, \dots, y_n]$, and fix a term ordering in that ring. Let $A \subset I$ be a finite set of homogeneous elements and let $\text{lt } A$ be the set of leading terms of elements in A . $\dim P/I \leq \dim P/\text{lt } A$.*

Proof. Compute dimensions as one plus the degree of Hilbert polynomials, recall that ideals and their initial ideals have identical Hilbert polynomials and notice that $\text{lt } A$ is contained in the initial ideal of I . \square

The following result could be obtained by proving that M_3 is 1-generic and applying Theorem 2.1 in [4]. We give an alternative proof as an example of an application of Theorem 2.3.

Corollary 3.2. *The ideal $I_3(M_3)$ is prime.*

Proof. We apply Theorem 2.3 with $m = 6$, $n = 3$, $t = 3$ the sequence of x_α s taken to be $x_{12}, x_{13}, x_{22}, x_{23}, x_{42}, x_{43}, x_{52}, x_{53}$ and sequence of x_β s taken to be $x_{21}, x_{22}, x_{31}, x_{32}, x_{51}, x_{52}, x_{61}, x_{62}$. Now $R/I_3(M_3) = T/(L + I_3(X))$. It is not hard to verify that conditions (i) and (ii) of Theorem 2.3 hold and it remains to verify that condition (iii) holds.

Define the matrix

$$D = \begin{pmatrix} x & y & & & & \\ y & sx + ty & & & & \\ sx + ty & sy + t(sx + ty) & & & & \\ u & v & & & & \\ v & su + tv & & & & \\ su + tv & sv + t(su + tv) & & & & \end{pmatrix} \begin{pmatrix} 1 & 0 & s \\ 0 & 1 & t \end{pmatrix}$$

$$= \begin{pmatrix} x & y & sx + ty \\ y & sx + ty & sy + t(sx + ty) \\ sx + ty & sy + t(sx + ty) & s(sx + ty) + t(sy + t(sx + ty)) \\ u & v & su + tv \\ v & su + tv & sv + t(su + tv) \\ su + tv & sv + t(su + tv) & s(su + tv) + t(sv + t(su + tv)) \end{pmatrix}$$

and define E to be the sub- \mathbb{K} -algebra of $\mathbb{K}[s, t, x, y, u, v]$ generated by the entries of D . Define $\phi : R \rightarrow E$ to be the surjection of \mathbb{K} -algebras which maps the (i, j) entry of M_3 to the (i, j) entry of D for all $1 \leq i \leq 6, 1 \leq j \leq 3$. Notice that, if $\mathbf{c}_1, \mathbf{c}_2$ and \mathbf{c}_3 are the columns of D , $\mathbf{c}_3 = s\mathbf{c}_1 + t\mathbf{c}_2$, and so $I_3(M) \subseteq \text{Ker } \phi$. We now show that $\dim E = 6$ by showing that the fraction field \mathbb{E} of E is $\mathbb{K}(s, t, x, y, u, v)$, i.e., that s and t are in \mathbb{E} :

$$\frac{[sx + ty]^2 - y[sy + t(sx + ty)]}{x[sx + ty] - y^2} = \frac{s^2x^2 + stxy - sy^2}{sx^2 + txy - y^2} = s$$

and $t = ([sx + ty] - sx)/y$. Now since $I_3(M_3) \subseteq \text{ker } \phi$ and $\dim E = 6$, $\dim R/I_3(M_3) \geq 6$.

If we write

$$N = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \\ a_3 & a_4 \\ b_1 & b_2 \\ b_2 & b_3 \\ b_3 & b_4 \end{bmatrix},$$

since $I_3(M_3) \subseteq I_2(N)$ condition (iii) is equivalent to the statement $\text{ht } I_3(M_3) < \text{ht } I_2(N)$.

We may choose a monomial order (say, reverse lexicographical) so that the set of leading

terms of the 2×2 minors of N contains $J = \{a_2^2, a_3^2, a_4 b_1, b_2^2, b_3^2\}$ and its not hard to see that $\dim R/RJ \leq 3$ hence Lemma 3.1 implies that $\dim R/I_2(N) \leq 3$ and $\text{ht } I_2(N) \geq 8 - 3 = 5$. But $\dim R/I_3(M_3) \geq 6$, so $\text{ht } I_3(M_3) \leq 10 - 6 = 4 < \text{ht } I_2(N)$. \square

Consider the permutations σ, τ of the variables of R given by $\sigma(a_i) = a_{5-i+1}$, $\sigma(b_i) = b_{5-i+1}$ and $\tau(a_i) = b_i$, $\tau(b_i) = a_i$ ($1 \leq i \leq 5$). We shall use the fact that these can be extended to automorphisms of R which fix $I_2(M_5)$, $I_3(M_3)$ and $I_3(M_4)$. We also denote henceforth $a_i b_j - a_j b_i$ with Δ_{ij} .

Proposition 3.3. $\Delta_{12}, \Delta_{23}, \Delta_{13}, \Delta_{45}, \Delta_{34}, \Delta_{35} \in (I_3(M_4) : I_3(M_3))$.

Proof. First notice that the generators of $I_3(M_3)$ not in $I_3(M_4)$ are

$$d_1 := \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}, \quad d_2 := \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ b_3 & b_4 & b_5 \end{vmatrix}, \quad d_3 := \begin{vmatrix} a_1 & a_2 & a_3 \\ a_3 & a_4 & a_5 \\ b_2 & b_3 & b_4 \end{vmatrix}, \quad d_4 := \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 \end{vmatrix} = \sigma(d_2)$$

and $\tau(d_1), \tau(d_2), \tau(d_3), \tau(d_4)$. Indeed,

$$d_5 := \begin{vmatrix} a_1 & a_2 & a_3 \\ a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_4 \\ a_2 & a_3 & a_5 \\ b_1 & b_2 & b_4 \end{vmatrix} - \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \end{vmatrix} \in I_3(M_4),$$

$$d_6 := \begin{vmatrix} a_1 & a_2 & a_3 \\ a_3 & a_4 & a_5 \\ b_3 & b_4 & b_5 \end{vmatrix} = \sigma(d_5) \in I_3(M_4).$$

Consider the relation

$$\begin{bmatrix} b_1 & b_2 & b_1 & b_2 & b_3 \\ a_1 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_1 & a_2 & a_3 \\ a_2 & a_3 & a_2 & a_3 & a_4 \\ a_3 & a_4 & a_3 & a_4 & a_5 \end{bmatrix} \begin{bmatrix} a_2 \\ -a_1 \\ -a_2 \\ a_1 \\ 0 \end{bmatrix} = 0$$

and call the 5×5 matrix above N_1 . We expand $\det N_1$ using the first two columns to obtain

$$0 = \det N_1 = \Delta_{12} d_1 - \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} d_5 + \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

hence $\Delta_{12} d_1 \in I_3(M_4)$.

Consider the relation

$$\begin{bmatrix} b_1 & b_2 & b_1 & b_2 & b_3 \\ a_1 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_1 & a_2 & a_3 \\ a_2 & a_3 & a_2 & a_3 & a_4 \\ b_3 & b_4 & b_3 & b_4 & b_5 \end{bmatrix} \begin{bmatrix} a_2 \\ -a_1 \\ -a_2 \\ a_1 \\ 0 \end{bmatrix} = 0$$

and call the 5×5 matrix above N_2 . We expand $\det N_2$ using the first two columns to obtain

$$0 = \det N_2 = \Delta_{12}d_2 - \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} \tau(d_6) + \begin{vmatrix} a_1 & a_2 \\ b_3 & b_4 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

hence $\Delta_{12}d_2 \in I_3(M_4)$.

Consider the relation

$$\begin{bmatrix} b_1 & b_2 & b_1 & b_2 & b_3 \\ a_1 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_1 & a_2 & a_3 \\ a_3 & a_4 & a_3 & a_4 & a_5 \\ b_2 & b_3 & b_2 & b_3 & b_4 \end{bmatrix} \begin{bmatrix} a_2 \\ -a_1 \\ -a_2 \\ a_1 \\ 0 \end{bmatrix} = 0$$

and call the 5×5 matrix above N_3 . We expand $\det N_3$ using the first two columns to obtain

$$0 = \det N_3 = \Delta_{12}d_3 - \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_2 & b_3 & b_4 \end{vmatrix} + \begin{vmatrix} a_1 & a_2 \\ b_2 & b_3 \end{vmatrix} d_5$$

hence $\Delta_{12}d_3 \in I_3(M_4)$.

Consider the relation

$$\begin{bmatrix} b_1 & b_2 & b_1 & b_2 & b_3 \\ a_1 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_1 & a_2 & a_3 \\ b_2 & b_3 & b_2 & b_3 & b_4 \\ b_3 & b_4 & b_3 & b_4 & b_5 \end{bmatrix} \begin{bmatrix} a_2 \\ -a_1 \\ -a_2 \\ a_1 \\ 0 \end{bmatrix} = 0$$

and call the 5×5 matrix above N_4 . We expand $\det N_4$ using the first two columns to obtain

$$0 = \det N_4 = \Delta_{12}\tau(d_4) - \begin{vmatrix} a_1 & a_2 \\ b_2 & b_3 \end{vmatrix} \tau(d_5) + \begin{vmatrix} a_1 & a_2 \\ b_3 & b_4 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_2 & b_3 & b_4 \end{vmatrix}$$

hence $\Delta_{12}\tau(d_4) \in I_3(M_4)$ and $\Delta_{12}d_4 \in I_3(M_4)$.

A similar argument employing the matrices

$$\begin{bmatrix} b_2 & b_3 & b_2 & b_3 & b_1 \\ a_2 & a_3 & 0 & 0 & 0 \\ 0 & 0 & a_2 & a_3 & a_1 \\ a_3 & a_4 & a_3 & a_4 & a_2 \\ a_4 & a_5 & a_4 & a_5 & a_3 \end{bmatrix}, \begin{bmatrix} b_2 & b_3 & b_2 & b_3 & b_1 \\ a_2 & a_3 & 0 & 0 & 0 \\ 0 & 0 & a_2 & a_3 & a_1 \\ a_3 & a_4 & a_3 & a_4 & a_2 \\ b_4 & b_5 & b_4 & b_5 & b_3 \end{bmatrix}, \begin{bmatrix} b_2 & b_3 & b_2 & b_3 & b_1 \\ a_2 & a_3 & 0 & 0 & 0 \\ 0 & 0 & a_2 & a_3 & a_1 \\ a_3 & a_4 & a_3 & a_4 & a_2 \\ b_3 & b_4 & b_3 & b_4 & b_2 \end{bmatrix}, \begin{bmatrix} b_2 & b_3 & b_2 & b_3 & b_1 \\ a_2 & a_3 & 0 & 0 & 0 \\ 0 & 0 & a_2 & a_3 & a_1 \\ a_3 & a_4 & a_3 & a_4 & a_2 \\ b_4 & b_5 & b_4 & b_5 & b_3 \end{bmatrix}$$

shows that $\Delta_{23}I_3(M_3) \subseteq I_3(M_4)$ and a similar argument employing the matrices

$$\begin{bmatrix} b_1 & b_3 & b_1 & b_3 & b_2 \\ a_1 & a_3 & 0 & 0 & 0 \\ 0 & 0 & a_1 & a_3 & a_2 \\ a_2 & a_4 & a_2 & a_4 & a_3 \\ a_3 & a_5 & a_3 & a_5 & a_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_3 & b_1 & b_3 & b_2 \\ a_1 & a_3 & 0 & 0 & 0 \\ 0 & 0 & a_1 & a_3 & a_2 \\ a_2 & a_4 & a_2 & a_4 & a_3 \\ b_3 & b_5 & b_3 & b_5 & b_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_3 & b_1 & b_3 & b_2 \\ a_1 & a_3 & 0 & 0 & 0 \\ 0 & 0 & a_1 & a_3 & a_2 \\ a_2 & a_4 & a_2 & a_4 & a_3 \\ b_2 & b_4 & b_2 & b_4 & b_3 \end{bmatrix}, \begin{bmatrix} b_1 & b_3 & b_1 & b_3 & b_2 \\ a_1 & a_3 & 0 & 0 & 0 \\ 0 & 0 & a_1 & a_3 & a_2 \\ a_2 & a_4 & a_2 & a_4 & a_3 \\ b_3 & b_5 & b_3 & b_5 & b_4 \end{bmatrix}$$

shows that $\Delta_{13}I_3(M_3) \subseteq I_3(M_4)$. We conclude the proof by noticing that $\Delta_{45} = \sigma(\Delta_{12})$, $\Delta_{34} = \sigma(\Delta_{23})$ and $\Delta_{35} = \sigma(\Delta_{13})$. \square

Proposition 3.4. (a) $I_3(M_4) \subseteq I_2(M_5) \cap I_3(M_3)$.

(b) *The ideal $I_3(M_4)$ is unmixed of height 4.*

(c) *For any $P \in \text{Ass } I_3(M_4) \setminus \{I_3(M_3)\}$, $I_2(M_5) \subseteq P$.*

(d) $\text{Ass } I_3(M_4) = \{I_2(M_5), I_3(M_3)\}$.

Proof. (a) Both inclusions $I_3(M_4) \subseteq I_2(M_5)$ and $I_3(M_4) \subseteq I_3(M_3)$ are easy to verify.

(b) Recall from the proof of Corollary 3.2 that $I_3(M_3)$ has height 4. From (a) we deduce that $\text{ht } I_3(M_4) \leq \text{ht } I_3(M_4) = 4$, and since $I_3(M_4)R_{\Delta_{12}} = I_3(M_3)R_{\Delta_{12}}$, $\text{ht } I_3(M_4) \geq \text{ht } I_3(M_3)R_{\Delta_{12}} = 4$.

Take X to be the generic 4×4 matrix as in Theorem 1.1. That theorem tells us that $T/I_3(X)$ is Cohen-Macaulay and 12-dimensional. Also, if J is the ideal of T generated by the six elements $x_{12} - x_{21}, x_{13} - x_{22}, x_{14} - x_{23}, x_{32} - x_{41}, x_{33} - x_{42}, x_{34} - x_{43}$, we have $R/I_3(M_4) \cong P/(I_3(X) + J)$. But now

$$6 = \dim R/I_3(M_4) = \dim T/(I_3(X) + J)$$

and so the six generators of J form a system of parameters in the Cohen-Macaulay ring $T/I_3(X)$ and so $R/I_3(M_4) \cong T/(I_3(X) + J)$ is Cohen-Macaulay, hence unmixed.

(c) First notice that Proposition 3.3 implies that $\Delta_{12}, \Delta_{23}, \Delta_{13}, \Delta_{45}, \Delta_{34}, \Delta_{35} \in P$. We use the symmetry induced by σ , to reduce the problem to showing that $P \supset \{\Delta_{14}, \Delta_{24}, \Delta_{15}\}$.

Assume that $\Delta_{14} \notin P$:

$$\begin{cases} a_3\Delta_{14} + a_4\Delta_{13} + a_1\Delta_{34} = 0 \\ b_3\Delta_{14} + b_4\Delta_{13} + b_1\Delta_{34} = 0 \end{cases} \Rightarrow a_3, b_3 \in P$$

and modulo a_3, b_3

$$\begin{vmatrix} a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 \\ b_2 & b_3 & b_4 \end{vmatrix} \equiv \begin{vmatrix} a_2 & 0 & a_4 \\ b_1 & b_2 & 0 \\ b_2 & 0 & b_4 \end{vmatrix} = b_2\Delta_{24}, \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \end{vmatrix} \equiv \begin{vmatrix} a_1 & a_2 & 0 \\ a_2 & 0 & a_4 \\ b_2 & 0 & b_4 \end{vmatrix} = a_2\Delta_{24}.$$

If $\Delta_{24} \notin P$, we obtain $a_2, b_2 \in P$; now P also contains the minor

$$\begin{vmatrix} a_1 & a_3 & a_4 \\ a_2 & a_4 & a_5 \\ b_1 & b_3 & b_4 \end{vmatrix} \equiv a_4\Delta_{14} \pmod{(a_2, a_3, b_2, b_3)}$$

so $a_2, a_3, a_4, b_2, b_3 \in P$, $\text{ht } P \geq 5$ and we obtain a contradiction. So now we assume that $\Delta_{24} \in P$, we deduce from

$$\begin{cases} a_2\Delta_{14} + a_4\Delta_{12} + a_1\Delta_{24} = 0 \\ b_2\Delta_{14} + b_4\Delta_{12} + b_1\Delta_{24} = 0 \end{cases}$$

that $a_2, b_2 \in P$ as well, and obtain, as before, a contradiction. We deduce that $\Delta_{14} \in P$.

A similar argument shows that $\Delta_{24} \in P$.

Assume now that $\Delta_{15} \notin P$:

$$\begin{cases} a_3\Delta_{15} + a_5\Delta_{13} + a_1\Delta_{35} = 0 \\ b_3\Delta_{15} + b_5\Delta_{13} + b_1\Delta_{35} = 0 \\ a_4\Delta_{15} + a_5\Delta_{14} + a_1\Delta_{45} = 0 \\ b_4\Delta_{15} + b_5\Delta_{14} + b_1\Delta_{45} = 0 \end{cases} \Rightarrow a_3, b_3, a_4, b_4 \in P.$$

Modulo a_3, b_3, a_4, b_4 , the ideal $I_3(M_4)$ contains the non-zero element $a_5\Delta_{12}$ and, if that were to happen, $\text{ht } I_3(M_4) > 4$, a contradiction.

(d) Both $I_3(M_3)$ and $I_2(M_5)$ are height-4 primes so they are minimal primes of $I_3(M_4)$ hence $\{I_3(M_3), I_2(M_5)\} \subseteq \text{Ass } I_3(M_4)$. Now (c) implies $\text{Ass } I_3(M_4) = \{I_3(M_3), I_2(M_5)\}$.

□

Proposition 3.5. *Write*

$$\delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

We have $\delta I_2(M_5) \subseteq I_3(M_4)$.

Proof. Since

$$\begin{aligned} \begin{vmatrix} a_1 & a_1 & a_2 & a_3 \\ a_2 & a_2 & a_3 & a_4 \\ a_3 & a_3 & a_4 & a_5 \\ b_1 & b_1 & b_2 & b_3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 0 & a_1 & a_2 & a_3 \\ a_2^2 - a_1a_3 & a_2 & a_3 & a_4 \\ a_2a_3 - a_1a_4 & a_3 & a_4 & a_5 \\ \Delta_{12} & b_1 & b_2 & b_3 \end{vmatrix} = 0 \Rightarrow \\ -(a_2^2 - a_1a_3) \begin{vmatrix} a_1 & a_2 & a_3 \\ a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 \end{vmatrix} + (a_2a_3 - a_1a_4) \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 \end{vmatrix} - \delta\Delta_{12} = 0 \end{aligned}$$

and

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_4 \\ a_2 & a_3 & a_5 \\ b_1 & b_2 & b_4 \end{vmatrix} - \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \end{vmatrix} \in I_3(M_4)$$

we see that $\delta\Delta_{12} \in I_3(M_4)$. A similar argument shows that $\delta\Delta_{13}, \delta\Delta_{23} \in I_3(M_4)$. By symmetry, i.e., an application of σ , we also obtain $\delta\Delta_{45}, \delta\Delta_{35}, \delta\Delta_{34} \in I_3(M_4)$.

Rather more mysteriously,

$$\delta\Delta_{14} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 \end{vmatrix} (a_3a_5 - a_4^2) + \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \end{vmatrix} (a_2a_5 - a_3a_4) + \begin{vmatrix} a_1 & a_3 & a_4 \\ a_2 & a_4 & a_5 \\ b_1 & b_3 & b_4 \end{vmatrix} (a_3^2 - a_2a_4)$$

and hence also $\delta\sigma(\Delta_{14}) = \delta\Delta_{25} \in I_3(M_4)$;

$$\begin{aligned} \delta\Delta_{15} &= - \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 \end{vmatrix} (a_3a_5 - a_4^2) + \begin{vmatrix} a_1 & a_2 & a_3 \\ a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 \end{vmatrix} (a_2a_5 - a_3a_4) \\ &+ \begin{vmatrix} a_1 & a_2 & a_4 \\ a_2 & a_3 & a_5 \\ b_2 & b_3 & b_5 \end{vmatrix} (a_1a_5 - a_3^2) + \begin{vmatrix} a_1 & a_3 & a_4 \\ a_2 & a_4 & a_5 \\ b_1 & b_3 & b_4 \end{vmatrix} (a_2a_4 - a_1a_5) + \begin{vmatrix} a_1 & a_3 & a_4 \\ a_2 & a_4 & a_5 \\ b_2 & b_4 & b_5 \end{vmatrix} (a_2a_3 - a_1a_4); \end{aligned}$$

and

$$\delta\Delta_{24} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 \end{vmatrix} (a_3a_5 - a_4^2) + \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \end{vmatrix} (a_2a_5 - a_3a_4) + \begin{vmatrix} a_1 & a_3 & a_4 \\ a_2 & a_4 & a_5 \\ b_1 & b_3 & b_4 \end{vmatrix} (a_3^2 - a_2a_4).$$

□

Theorem 3.6. *The primary decomposition of $I_3(M_4)$ is given by $I_2(M_5) \cap I_3(M_3)$.*

Proof. Proposition 3.4 shows that $\text{Ass } I_3(M_4) = \{I_2(M_5), I_3(M_3)\}$; let $q_1 \cap q_2$ be the primary decomposition of $I_3(M_4)$ where q_1 is associated to $I_2(M_5)$ and q_2 is associated to $I_3(M_3)$.

Let δ be as in Proposition 3.5;

$$q_1 = I_3(M_4)R_{I_2(M_5)} \cap R \supseteq I_3(M_4)R_\delta \cap R \supseteq I_2(M_5)$$

hence $q_1 = I_2(M_5)$.

An application of Proposition 3.3 yields

$$q_2 = I_3(M_4)R_{I_3(M_3)} \cap R \supseteq I_3(M_4)R_{\Delta_{12}} \cap R \supseteq I_3(M_3)$$

hence $q_2 = I_3(M_3)$.

□

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