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## SOME PROPERTIES OF TOP GRADED LOCAL COHOMOLOGY MODULES

MORDECHAI KATZMAN AND RODNEY Y. SHARP

## 0. Introduction

Let  $R = \bigoplus_{d \in \mathbb{N}_0} R_d$  be a positively graded commutative Noetherian ring which is standard in the sense that  $R = R_0[R_1]$ , and set  $R_+ := \bigoplus_{d \in \mathbb{N}} R_d$ , the irrelevant ideal of  $R$ . (Here,  $\mathbb{N}_0$  and  $\mathbb{N}$  denote the set of non-negative and positive integers respectively;  $\mathbb{Z}$  will denote the set of all integers.) Let  $M = \bigoplus_{d \in \mathbb{Z}} M_d$  be a non-zero finitely generated graded  $R$ -module. This paper is concerned with the behaviour of the graded components of the graded local cohomology modules  $H_{R_+}^i(M)$  ( $i \in \mathbb{N}_0$ ) of  $M$  with respect to  $R_+$ .

It is known (see [BS, 15.1.5]) that there exists  $r \in \mathbb{Z}$  such that  $H_{R_+}^i(M)_d = 0$  for all  $i \in \mathbb{N}_0$  and all  $d \geq r$ , and that  $H_{R_+}^i(M)_d$  is a finitely generated  $R_0$ -module for all  $i \in \mathbb{N}_0$  and all  $d \in \mathbb{Z}$ .

The first part (§1) of this paper deals with the case in which  $R = R_0[U_1, \dots, U_s]/I$ , where  $U_1, \dots, U_s$  are indeterminates of degree one, and  $I \subset R_0[U_1, \dots, U_s]$  is a homogeneous ideal. The main theorem of that section is that for  $d \geq s$ , all the associated primes of  $H_{R_+}^s(R)_{-d}$  contain a certain ideal of  $R_0$  called the “content” of  $I$  (see Definition 1.3.) This result provides an affirmative answer, in a special case, to a question discussed by M. Brodmann and M. Hellus in [BH], namely, whether it is true that there exists  $d_0 \in \mathbb{Z}$  such that either  $H_{R_+}^i(M)_{-d} = 0$  for all  $d > d_0$  or  $H_{R_+}^i(M)_{-d} \neq 0$  for all  $d > d_0$ . In [BH, Lemma (4.2)], Brodmann and Hellus showed that the answer is affirmative for all values of  $i$  when  $R_0$  is semi-local and of dimension not exceeding 1; in §1 below, we show that the answer is affirmative when  $R_0$  is a domain,  $M = R$  and  $R_+$  can be generated by  $i$  homogeneous elements of degree 1.

There are instances when all the homogeneous components  $H_{R_+}^i(M)_r$  ( $r \in \mathbb{Z}$ ) have finite length (for example, when  $R_0$  is Artinian) and we may then define the *i*-th cohomological Hilbert function of  $M$ , denoted by  $h_M^i : \mathbb{Z} \rightarrow \mathbb{N}_0$ , by  $h_M^i(r) = \text{length}_{R_0} H_{R_+}^i(M)_r$  for all  $r \in \mathbb{Z}$ . When  $R_0$  is Artinian this function agrees with a polynomial for all  $r \ll 0$  (see [BS, Theorem 17.1.9]). In §§2,3 of this paper we construct examples which show that this result need not be true when  $R_0$  is not Artinian.

## 1. The vanishing of top local cohomology modules

Throughout this section  $R_0$  will denote an arbitrary commutative Noetherian domain. We set  $S = R_0[U_1, \dots, U_s]$  where  $U_1, \dots, U_s$  are indeterminates of degree one, and  $R = S/I$  where  $I \subset R_0[U_1, \dots, U_s]$  is a graded ideal. For  $t \in \mathbb{Z}$ , we shall denote by  $(\bullet)(t)$  the  $t$ -th shift functor (on the category of graded  $R$ -modules and homogeneous homomorphisms).

For any multi-index  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(s)}) \in \mathbb{Z}^s$  we shall write  $U^\lambda$  for  $U_1^{\lambda^{(1)}} \dots U_s^{\lambda^{(s)}}$  and we shall set  $|\lambda| = \lambda^{(1)} + \dots + \lambda^{(s)}$ .

**1.1. Lemma.** *Let  $I$  be generated by homogeneous elements  $f_1, \dots, f_r \in S$ . Then there is an exact sequence of graded  $S$ -modules and homogeneous homomorphisms*

$$\bigoplus_{i=1}^r H_{S_+}^s(S)(-\deg f_i) \xrightarrow{(f_1, \dots, f_r)} H_{S_+}^s(S) \longrightarrow H_{R_+}^s(R) \longrightarrow 0.$$

*Proof.* The functor  $H_{S_+}^s$  is right exact and the natural equivalence between  $H_{S_+}^s$  and  $(\bullet) \otimes_S H_{S_+}^s(S)$  (see [BS, 6.1.8 & 6.1.9]) actually yields a homogeneous  $S$ -isomorphism

$$H_{S_+}^s(S)/(f_1, \dots, f_r)H_{S_+}^s(S) \cong H_{S_+}^s(R).$$

To complete the proof, just note that there is an isomorphism of graded  $S$ -modules  $H_{S_+}^s(R) \cong H_{R_+}^s(R)$ , by the Graded Independence Theorem [BS, 13.1.6].  $\square$

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We can realize  $H_{S_+}^s(S)$  as the module  $R_0[U_1^-, \dots, U_s^-]$  of inverse polynomials described in [BS, 12.4.1]: this graded  $R_0$ -module has end  $-s$ , and, for each  $d \geq s$ , its  $(-d)$ -th component is a free  $R_0$ -module with base  $\mathcal{B}(d) := (U^\lambda)_{-\lambda \in \mathbb{N}^s, |\lambda| = -d}$ . We combine this realisation with the previous lemma to find a presentation of each homogeneous component of  $H_{R_+}^s(R)$  as a cokernel of a matrix with entries in  $R_0$ .

Assume first that  $I$  is generated by one homogeneous element  $f$  of degree  $\delta$ . For any  $d$  we have, in view of Lemma 1.1, a graded exact sequence

$$R_0[U_1^-, \dots, U_s^-]_{-d-\delta} \xrightarrow{\phi_d} R_0[U_1^-, \dots, U_s^-]_{-d} \longrightarrow H_{R_+}^s(R)_{-d} \longrightarrow 0.$$

The map of free  $R_0$ -modules  $\phi_d$  is given by multiplication on the left by a  $\binom{d-1}{s-1} \times \binom{d+\delta-1}{s-1}$  matrix which we shall denote later by  $M(f; d)$ .

In the general case, where  $I$  is generated by homogeneous elements  $f_1, \dots, f_r \in S$ , it follows from Lemma 1.1 that the  $R_0$ -module  $H_{R_+}^s(R)_{-d}$  is the cokernel of a matrix  $M(f_1, \dots, f_r; d)$  whose columns consist of all the columns of  $M(f_1, d), \dots, M(f_r, d)$ .

Consider a homogeneous  $f \in S$  of degree  $\delta$ . We shall now describe the matrix  $M(f; d)$  in more detail and to do so we start by ordering the bases of the source and target of  $\phi_d$  as follows. For any  $\lambda, \mu \in \mathbb{Z}^s$  with negative entries we declare that  $U^\lambda < U^\mu$  if and only if  $U^{-\lambda} <_{\text{Lex}} U^{-\mu}$  where “ $<_{\text{Lex}}$ ” is the lexicographical term ordering in  $S$  with  $U_1 > \dots > U_s$ . We order these bases, and by doing so also the columns and rows of  $M(f; d)$ , in ascending order.

**1.2. Lemma.** *Let  $f \neq 0$  be a homogeneous element in  $S$ . Then, for all  $d \geq s$ , the matrix  $M(f; d)$  has maximal rank.*

*Proof.* We prove the lemma by producing a non-zero maximal minor of  $M(f; d)$ . Write  $f = \sum_{\lambda \in \Lambda} a_\lambda U^\lambda$  where  $a_\lambda \in R_0 \setminus \{0\}$  for all  $\lambda \in \Lambda$  and let  $\lambda_0$  be such that  $U^{\lambda_0}$  is the minimal member of  $\{U^\lambda : \lambda \in \Lambda\}$  with respect to the lexicographical term order in  $S$ .

Let  $\delta$  be the degree of  $f$ . Each column of  $M(f; d)$  corresponds to a monomial  $U^\lambda \in \mathcal{B}(d + \delta)$ ; its  $\rho$ -th entry is the coefficient of  $U^\rho$  in  $fU^\lambda \in R_0[U_1^-, \dots, U_s^-]_{-d}$ .

Fix any  $U^\nu \in \mathcal{B}(d)$  and consider the column  $c_\nu$  corresponding to  $U^{\nu-\lambda_0} \in \mathcal{B}(d + \delta)$ . The  $\nu$ -th entry of  $c_\nu$  is obviously  $a_{\lambda_0}$ . Also, for any other  $\lambda_1 \in \Lambda$  with  $U^{\lambda_1} >_{\text{Lex}} U^{\lambda_0}$ , either  $\nu - \lambda_0 + \lambda_1$  has an entry in  $\mathbb{N}_0$ , in which case the corresponding term of  $fU^{\nu-\lambda_0} \in R_0[U_1^-, \dots, U_s^-]_{-d}$  is zero, or  $U^{\nu-\lambda_0+\lambda_1} < U^\nu$ . This last statement follows from the fact that if  $j$  is the first coordinate where  $\lambda_0$  and  $\lambda_1$  differ then  $\lambda_0^{(j)} < \lambda_1^{(j)}$  and so also  $-\nu^{(j)} > -\nu^{(j)} + \lambda_0^{(j)} - \lambda_1^{(j)}$ ; this implies that  $U^{-\nu+\lambda_0-\lambda_1} <_{\text{Lex}} U^{-\nu}$  and  $U^{\nu-\lambda_0+\lambda_1} < U^\nu$ . We have shown that the last non-zero entry in  $c_\nu$  occurs at the  $\nu$ -th row and is equal to  $a_{\lambda_0}$ . Consider the square submatrix of  $M(f; d)$  whose columns are the  $c_\nu$  ( $\nu \in \mathcal{B}(d)$ ); its determinant is clearly a power of  $a_{\lambda_0}$  and hence is non-zero.  $\square$

**1.3. Definition.** For any  $f \in R_0[U_1, \dots, U_s]$  write  $f = \sum_{\lambda \in \Lambda} a_\lambda U^\lambda$  where  $a_\lambda \in R_0$  for all  $\lambda \in \Lambda$ . For such an  $f \in R_0[U_1, \dots, U_s]$  we define the *content*  $c(f)$  of  $f$  to be the ideal  $\langle a_\lambda : \lambda \in \Lambda \rangle$  of  $R_0$  generated by all the coefficients of  $f$ . If  $J \subset R_0[U_1, \dots, U_s]$  is an ideal, we define its *content*  $c(J)$  to be the ideal of  $R_0$  generated by the contents of all the elements of  $J$ . It is easy to see that if  $J$  is generated by  $f_1, \dots, f_r$ , then  $c(J) = c(f_1) + \dots + c(f_r)$ .

**1.4. Lemma.** *Suppose that  $I$  is generated by homogeneous elements  $f_1, \dots, f_r \in S$ . Fix any  $d \geq s$ . Let  $t := \text{rank } M(f_1, \dots, f_r; d) = \binom{d-1}{s-1}$  and let  $I_d$  be the ideal generated by all  $t \times t$  minors of  $M(f_1, \dots, f_r; d)$ . Then  $c(I) \subseteq \sqrt{I_d}$ .*

*Proof.* It is enough to prove the lemma when  $r = 1$ ; let  $f = f_1$ . Write  $f = \sum_{\lambda \in \Lambda} a_\lambda U^\lambda$  where  $a_\lambda \in R_0 \setminus \{0\}$  for all  $\lambda \in \Lambda$ . Assume that  $c(I) \not\subseteq \sqrt{I_d}$  and pick  $\lambda_0$  so that  $U^{\lambda_0}$  is the minimal element in  $\{U^\lambda : \lambda \in \Lambda\}$  (with respect to the lexicographical term order in  $S$ ) for which  $a_\lambda \notin \sqrt{I_d}$ . Notice that the proof of Lemma 1.2 shows that  $U^{\lambda_0}$  cannot be the minimal element of  $\{U^\lambda : \lambda \in \Lambda\}$ .

Fix any  $U^\nu \in \mathcal{B}(d)$  and consider the column  $c_\nu$  corresponding to  $U^{\nu-\lambda_0} \in \mathcal{B}(d + \delta)$ . The  $\nu$ -th entry of  $c_\nu$  is obviously  $a_{\lambda_0}$ . An argument similar to the one in the proof of Lemma 1.2 shows that, for any other  $\lambda_1 \in \Lambda$  with  $U^{\lambda_1} >_{\text{Lex}} U^{\lambda_0}$ , either  $\nu - \lambda_0 + \lambda_1$  has an entry in  $\mathbb{N}_0$ , in which case the corresponding term of  $fU^{\nu-\lambda_0} \in R_0[U_1^-, \dots, U_s^-]_{-d}$  is zero, or  $U^\nu > U^{\nu-\lambda_0+\lambda_1}$ .

Similarly, for any other  $\lambda_1 \in \Lambda$  with  $U^{\lambda_1} <_{\text{Lex}} U^{\lambda_0}$ , either  $\nu - \lambda_0 + \lambda_1$  has an entry in  $\mathbb{N}_0$ , in which case the corresponding term of  $fU^{\nu-\lambda_0} \in R_0[U_1^-, \dots, U_s^-]_{-d}$  is zero, or  $U^\nu < U^{\nu-\lambda_0+\lambda_1}$ .

We have shown that all the entries above the  $\nu$ -th row of  $c_\nu$  are in  $\sqrt{I_d}$ . Consider the matrix  $M$  whose columns are  $c_\nu$  ( $\nu \in \mathcal{B}(d)$ ) and let  $\overline{\phantom{x}} : R_0 \rightarrow R_0/\sqrt{I_d}$  denote the quotient map. We have

$$0 = \overline{\det(M)} = \det(\overline{M}) = \overline{a_{\lambda_0}}^{\binom{d-1}{s-1}}$$

and, therefore,  $a_{\lambda_0} \in \sqrt{I_d}$ , a contradiction.  $\square$

**1.5. Theorem.** *Suppose that  $I$  is generated by homogeneous elements  $f_1, \dots, f_r \in S$ . Fix any  $d \geq s$ . Then each associated prime of  $H_{R_+}^s(R)_{-d}$  contains  $c(I)$ . In particular  $H_{R_+}^s(R)_{-d} = 0$  if and only if  $c(I) = R_0$ .*

*Proof.* Recall that for any  $p, q \in \mathbb{N}$  with  $p \leq q$  and any  $p \times q$  matrix  $M$  of maximal rank with entries in any domain,  $\text{Coker } M = 0$  if and only if the ideal generated by the maximal minors of  $M$  is the unit ideal. Let  $M = M(f_1, \dots, f_r; d)$ , so that  $H_{R_+}^s(R)_{-d} \cong \text{Coker } M$ .

In view of Lemmas 1.2 and 1.4, the ideal  $c(I)$  is contained in the radical of the ideal generated by the maximal minors of  $M$ ; therefore, for each  $x \in c(I)$ , the localization of  $\text{Coker } M$  at  $x$  is zero; we deduce that  $c(I)$  is contained in all associated primes of  $\text{Coker } M$ .

To prove the second statement, assume first that  $c(I)$  is not the unit ideal. Since all minors of  $M$  are contained in  $c(I)$ , these cannot generate the unit ideal and  $\text{Coker } M \neq 0$ . If, on the other hand,  $c(I) = R_0$  then  $\text{Coker } M$  has no associated prime and  $\text{Coker } M = 0$ .  $\square$

**1.6. Corollary.** *Let the situation be as in 1.5. The following statements are equivalent:*

- (i)  $c(I) = R_0$ ;
- (ii)  $H_{R_+}^s(R)_{-d} = 0$  for some  $d \geq s$ ;
- (iii)  $H_{R_+}^s(R)_{-d} = 0$  for all  $d \geq s$ .

Consequently,  $H_{R_+}^s(R)$  is asymptotically gap-free in the sense of [BH, (4.1)].

It follows that, if  $T = \bigoplus_{n \in \mathbb{N}_0} T_n$  is any standard graded finitely generated  $R_0$ -algebra with  $T_0 = R_0$  and if the  $R_0$ -module  $T_1$  can be generated by  $s$  elements, then  $H_{T_+}^s(T)$  is asymptotically gap-free, because there is a homogeneous surjective ring homomorphism  $S \rightarrow T$ .

**1.7. Remark.** Theorem 1.5 cannot be extended to  $H_{R_+}^i(R)_{-d} = 0$  for all  $i < s$ . For example, take  $R_0 = \mathbb{C}[X_1, X_2]$ ,  $S = R_0[U_1, U_2]$  (where  $X_1, X_2, U_1, U_2$  are indeterminates) and

$$I = (U_1(X_1U_1 + X_2U_2), U_2(X_1U_1 + X_2U_2)) \subseteq S.$$

Then  $c(I) = (X_1, X_2)$  but  $H_{(U_1, U_2)}^0(S/I) \cong (X_1U_1 + X_2U_2)/I$  and it does not vanish when localized at  $X_1$ .

We consider the following consequence of our work to be interesting because of its relevance to associated primes of local cohomology modules.

**1.8. Corollary.** *The  $R$ -module  $H_{R_+}^s(R)$  has finitely many minimal associated primes, and these are just the minimal primes of the ideal  $c(I)R + R_+$ .*

*Proof.* Let  $r \in c(I)$ . By Theorem 1.5, the localization of  $H_{R_+}^s(R)$  at  $r$  is zero. Hence each associated prime of  $H_{R_+}^s(R)$  contains  $c(I)R$ . Such an associated prime must contain  $R_+$ , since  $H_{R_+}^s(R)$  is  $R_+$ -torsion.

On the other hand,  $H_{R_+}^s(R)_{-s} \cong R_0/c(I)$  and  $H_{R_+}^s(R)_i = 0$  for all  $i > -s$ ; therefore there is an element of the  $(-s)$ -th component of  $H_{R_+}^s(R)$  that has annihilator (over  $R$ ) equal to  $c(I)R + R_+$ . All the claims now follow from these observations.  $\square$

**1.9. Remark.** In [H, Conjecture 5.1], C. Huneke put forward the conjecture that every local cohomology module (with respect to any ideal) of a finitely generated module over a local Noetherian ring has only finitely many associated primes. Recently, this conjecture was settled by the first author, who gave a counterexample in [K, Corollary 1.3]. Corollary 1.8 provides a little evidence in support of the weaker conjecture that every local cohomology module (with respect to any ideal) of a finitely generated module over a local Noetherian ring has only finitely many *minimal* associated primes.

## 2. Cohomological Hilbert functions which are not of reverse polynomial type

In this section, we shall use the terminology used in the final paragraph of the Introduction; also, we follow the terminology of [BS, 17.1.1(vi)], and say that a function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$  is of *reverse polynomial type* if and only if there exists a polynomial  $P \in \mathbb{Q}[T]$  such that  $f(r) = P(r)$  for all  $r \ll 0$ . In this and the next section, we are interested in situations where  $H_{R_+}^i(M)_{-d}$  has finite length as an  $R_0$ -module for all  $d \in \mathbb{Z}$ . In such situations, one can ask whether the  $i$ -th cohomological Hilbert function  $h_M^i$  of  $M$  is of reverse polynomial type: it always is when  $R_0$  is Artinian, by [BS, Theroem 17.1.9]. In this and the next section, we shall study such situations where  $R_0$  is not Artinian, and we shall present an example in which  $h_M^i$  is of reverse polynomial type, and examples (in which  $R_0$  is a polynomial ring over a field) in which  $h_M^i$  is not of reverse polynomial type: in this section we give an example over each field  $\mathbb{Z}/p\mathbb{Z}$ , where  $p$  is a prime number; in the next section, we give an example over an arbitrary field of characteristic zero.

In this section we study the cohomological Hilbert functions arising from an example first studied by A. Singh in [S]; the example was also studied in [BKS, §2], where the asymptotic behaviour of the sets of associated primes of the graded components of one of its local cohomology modules were investigated.

In this section, we shall use several results from [BKS, §2], and unexplained terminology will be found in that section. Throughout this section,  $L$  will denote either a field or a principal ideal domain; let  $R_0 = L[X, Y, Z]$  and  $R = R_0[U, V, W]/(XU + YV + ZW)$ , where  $X, Y, Z, U, V, W$  are independent indeterminates over  $L$ ; we also assign degree 0 to  $X, Y, Z$  and degree 1 to  $U, V, W$ . Denote by  $R_+$  the ideal of  $R$  generated by the images of  $U, V, W$ .

During the course of the section, we shall have occasion to take  $L$  to be  $\mathbb{Q}$  and  $\mathbb{Z}/p\mathbb{Z}$ , where  $p$  is a prime number. The way in which Proposition 2.8 of [BKS] is formulated means that the calculations in that result can be used in these cases.

Notice that Theorem 1.5 implies that, when  $L$  is a field, for any  $d \geq 3$ , the vector space  $H_{R_+}^3(R)_{-d}$  over  $L$  has finite dimension, and that  $\dim_L H_{R_+}^3(R)_{-d} = \text{length}_{R_0} H_{R_+}^3(M)_{-d}$ .

**2.1. Definition.** Suppose that  $L$  is a field. For all  $d \in \mathbb{Z}$  we define  $h_L(d) = \dim_L H_{R_+}^3(R)_d$ . We abbreviate  $h_{\mathbb{Q}}$  by  $h_0$ , and  $h_{\mathbb{Z}/p\mathbb{Z}}$ , for a prime number  $p$ , by  $h_p$ . Thus  $h_0$  and  $h_p$  are cohomological Hilbert functions.

**2.2. Lemma.** *Let  $d \in \mathbb{N}$  with  $d \geq 3$ . Consider the matrix*

$$T_d := \begin{bmatrix} A_{d-2} & XI_{d-2} & 0 & \dots & 0 \\ 0 & A_{d-3} & XI_{d-3} & 0 & \dots & 0 \\ 0 & 0 & A_{d-4} & XI_{d-4} & & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & A_1 & XI_1 \end{bmatrix}$$

of [BKS, Lemma 2.2]. *The induced monomorphism*

$$L[Y, Z]^{\binom{d-1}{2}} / \left( \text{Im } T_d \cap L[Y, Z]^{\binom{d-1}{2}} \right) \longrightarrow \text{Coker } T_d$$

*is an  $L$ -isomorphism. Consequently, there is an  $L$ -isomorphism  $H_{R_+}^3(R)_{-d} \cong \text{Coker } H_d$ , where*

$$H_d := \begin{bmatrix} A_{d-2} & 0 & 0 & \dots & 0 \\ 0 & A_{d-3}A_{d-2} & 0 & \dots & 0 \\ 0 & 0 & A_{d-4}A_{d-3}A_{d-2} & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & & A_1A_2 \dots A_{d-2} \end{bmatrix},$$

*as in [BKS, Theorem 2.4(ii)].*

*Proof.* The first part follows from the fact that, for all integers  $i$  with  $1 \leq i \leq \binom{d-1}{2}$ , the image of  $T_d$  contains a vector  $X\mathbf{e}_i + \sum_{i < j \leq \binom{d-1}{2}} \alpha_j \mathbf{e}_j$ , where each  $\alpha_j$  is in  $L[Y, Z]$ ; we can use these to show that any member of  $L[X, Y, Z]^{\binom{d-1}{2}}$  differs by an element of  $\text{Im } T_d$  from some element of  $L[Y, Z]^{\binom{d-1}{2}}$ .

The second part follows from the facts that the columns of  $H_d$  generate  $\text{Im } T_d \cap L[Y, Z]^{\binom{d-1}{2}}$  (by [BKS, Theorem 2.4(ii)]) and that, by [BKS, Lemma 2.2], there is an  $L[X, Y, Z]$ -isomorphism  $\text{Coker } T_d \cong H_{R_+}^3(R)_{-d}$ .  $\square$

**2.3. Proposition.**

- (i) For any prime number  $p$  and for all  $d \geq 3$ , we have  $h_0(-d) \leq h_p(-d)$ ; equality occurs if and only if

$$p \notin \Pi(d-2) := \left\{ p : p \text{ is a prime factor of } \binom{d-2}{i} \text{ for some } i \in \{1, \dots, d-2\} \right\}.$$

- (ii) We have  $h_0(-d) = d(d-1)^2(d-2)/12$  for all  $d \geq 3$ . Thus the cohomological Hilbert function  $h_0$  is of reverse polynomial type.

*Proof.* We begin by considering the situation where  $L$  is an arbitrary field. As in [BKS, Proposition 2.8], we consider  $L[Y, Z]$  as an  $\mathbb{N}_0^2$ -graded ring in which  $L[Y, Z]_{(0,0)} = L$  and  $\deg Y^i Z^j = (i+j, i)$ . We also endow the free  $L[Y, Z]$ -module

$$L[Y, Z]^r = L[Y, Z]\mathbf{e}_1 \oplus \cdots \oplus L[Y, Z]\mathbf{e}_r$$

with a structure as  $\mathbb{N}_0^2$ -graded module over the  $\mathbb{N}_0^2$ -graded ring  $L[Y, Z]$  in such a way that  $\deg \mathbf{e}_i = (0, i)$  for  $i = 1, \dots, r$ . Except where specified, we shall use this bigrading henceforth in this proof.

Lemma 2.2 now implies that  $h_L(-d) = \dim_L \text{Coker } H_d$  for all  $d \geq 3$ ; hence, in the notation of [BKS, Proposition 2.8], for such a  $d$  we have

$$h_L(-d) = \sum_{r=1}^{d-2} \dim_L \text{Coker } Q_{r,d-1}.$$

We now use [BKS, Proposition 2.8(ii)] to calculate  $g_L(r, r+k) := \dim_L \text{Coker } Q_{r,r+k}$  for  $r, k \in \mathbb{N}$ .

It follows from [BKS, Proposition 2.8] that  $(L[Y, Z]^r)_{(i,j)} = 0$  whenever  $j > i+r$ , that  $\text{Im } Q_{r,r+k}$  is a graded submodule of  $L[Y, Z]^r$  and that  $(\text{Im } Q_{r,r+k})_{(i,j)} = (L[Y, Z]^r)_{(i,j)}$  whenever  $i \geq 2k+r$ . Hence

$$\dim_L \text{Coker } Q_{r,r+k} = \sum_{i=0}^{2k+r-1} \sum_{j \in \mathbb{N}} \dim_L (L[Y, Z]^r)_{(i,j)} - \sum_{i=0}^{2k+r-1} \sum_{j \in \mathbb{N}} \dim_L (\text{Im } Q_{r,r+k})_{(i,j)}.$$

The first term on the right-hand side is equal to  $r$  times the number of monomials  $Y^\tau Z^\omega$  (where  $\tau, \omega \in \mathbb{N}_0$ ) of total (ordinary) degree not exceeding  $2k+r-1$ . Therefore

$$\sum_{i=0}^{2k+r-1} \sum_{j \in \mathbb{N}} \dim_L (L[Y, Z]^r)_{(i,j)} = r \sum_{i=0}^{2k+r-1} (i+1) = \frac{r(2k+r)(2k+r+1)}{2}.$$

Furthermore, it also follows from [BKS, Proposition 2.8] that  $(\text{Im } Q_{r,r+k})_{(i,j)} = 0$  whenever  $i < k$ , while if  $k \leq i \leq 2k+r-1$ , then  $\dim_L (\text{Im } Q_{r,r+k})_{(i,j)}$  is equal to the rank of a submatrix of  $\tilde{Q}_{r,r+k}$  (defined in [BKS, Proposition 2.8]) made up of the (consecutive) columns of that matrix numbered

$$\max\{j+k-i, 1\}, \max\{j+k-i, 1\}+1, \dots, \min\{j, r+k\}.$$

This rank depends on the characteristic of  $L$ : when  $L = \mathbb{Q}$ , each such submatrix has maximal rank (by [BKS, Corollary 2.12]), whereas when  $L = \mathbb{Z}/p\mathbb{Z}$  for a prime number  $p$ , such a submatrix may not have maximal rank. Therefore  $g_{\mathbb{Q}}(r, r+k) \leq g_{\mathbb{Z}/p\mathbb{Z}}(r, r+k)$ , and equality holds if and only if, over  $\mathbb{Z}/p\mathbb{Z}$ , all submatrices of  $\tilde{Q}_{r,r+k}$  formed by consecutive columns have maximal rank. Let  $p$  denote a prime number. Now [BKS, Corollary 2.14] implies that when  $p \notin \Pi(r+k-1)$  the above submatrices do all have maximal rank. Since

$$h_L(-d) = \sum_{r=1}^{d-2} \dim_L \text{Coker } Q_{r,d-1} \quad \text{for all } d \geq 3,$$

it follows that  $h_0(-d) \leq h_p(-d)$  for all  $d \geq 3$ , and that equality holds if  $p \notin \Pi(d-2)$ . However, when  $p \in \Pi(d-2)$ , some of the  $1 \times 1$  submatrices of

$$\tilde{Q}_{1,d-1} = \left[ \binom{d-2}{0}, \binom{d-2}{1}, \dots, \binom{d-2}{\gamma}, \dots, \binom{d-2}{1}, \binom{d-2}{0} \right]$$

will vanish, and for such  $p$  and  $d$  we must have  $h_0(d) < h_p(d)$ .

(ii) Next, in the case where  $L = \mathbb{Q}$ , we calculate  $\sum_{i=0}^{2k+r-1} \sum_{j \in \mathbb{N}} \dim_{\mathbb{Q}} (\text{Im } Q_{r,r+k})_{(i,j)}$ . The comments in the preceding paragraph show that

$$\sum_{i=0}^{2k+r-1} \sum_{j \in \mathbb{N}} \dim_{\mathbb{Q}} (\text{Im } Q_{r,r+k})_{(i,j)} = \sum_{i=k}^{2k+r-1} \sum_{j \in \mathbb{N}} \dim_{\mathbb{Q}} (\text{Im } Q_{r,r+k})_{(i,j)},$$

and that, for an integer  $i$  with  $k \leq i \leq 2k + r - 1$ , the sum  $\sum_{j \in \mathbb{N}} \dim_{\mathbb{Q}}(\text{Im } Q_{r,r+k})_{(i,j)} =: T_i$  is equal to the sum of the ranks of the submatrices of  $Q_{r,r+k}$  obtained by selecting consecutive columns numbered by the sets of integers in the list

$$\{1\}, \{1, 2\}, \dots, \{1, 2, \dots, c\}, \{2, 3, \dots, c+1\}, \dots, \{r+k-c+1, r+k-c+2, \dots, r+k\},$$

$$\{r+k-c+2, r+k-c+3, \dots, r+k\}, \dots, \{r+k-1, r+k\} \text{ and } \{r+k\},$$

where  $c = i - k + 1$ . Since each such submatrix has maximal rank (by [BKS, Corollary 2.12]) and has  $r$  rows, it follows that

$$\begin{aligned} \sum_{i=k}^{2k+r-1} T_i &= \sum_{c=1}^r (k+r-c+1)c + \sum_{c=r+1}^{k+r} (k+r-c+1)r + 2 \sum_{c=1}^r \sum_{l=1}^{c-1} l + 2 \sum_{c=r+1}^{k+r} \left( \sum_{l=1}^r l + \sum_{l=r+1}^{c-1} r \right) \\ &= \frac{(k+r)^2(k+r+1)}{2} - \frac{k^2(k+1)}{2}. \end{aligned}$$

Hence

$$\begin{aligned} g_{\mathbb{Q}}(r, r+k) &= \dim_{\mathbb{Q}} \text{Coker } Q_{r,r+k} \\ &= \sum_{i=0}^{2k+r-1} \sum_{j \in \mathbb{N}} \dim_{\mathbb{Q}}(L[Y, Z]^r)_{(i,j)} - \sum_{i=0}^{2k+r-1} \sum_{j \in \mathbb{N}} \dim_{\mathbb{Q}}(\text{Im } Q_{r,r+k})_{(i,j)} \\ &= \frac{r(2k+r)(2k+r+1)}{2} - \frac{(k+r)^2(k+r+1)}{2} + \frac{k^2(k+1)}{2} \\ &= \frac{rk(k+r)}{2}. \end{aligned}$$

Finally, we return to our cohomological Hilbert function: for all  $d \geq 3$ ,

$$\begin{aligned} h_0(-d) &= \sum_{r=1}^{d-2} \dim_{\mathbb{Q}} \text{Coker } Q_{r,d-1} \\ &= \sum_{r=1}^{d-2} \frac{r(d-1-r)(d-1)}{2} \\ &= \frac{d(d-1)^2(d-2)}{12}. \end{aligned}$$

□

#### 2.4. Theorem.

(i) For any prime number  $p$  both the sets

$$\{d : d \geq 3, h_p(-d) = h_0(-d)\} \quad \text{and} \quad \{d : d \geq 3, h_p(-d) > h_0(-d)\}$$

are infinite.

(ii) None of the cohomological Hilbert functions  $h_p$  is of reverse polynomial type for any prime number  $p$ .

(iii) If  $p$  and  $q$  are different prime numbers then the set  $\{d : d \geq 3, h_p(-d) > h_q(-d)\}$  is infinite.

*Proof.* (i) This follows from Proposition 2.3(i) and [BKS, Lemma 2.16].

(ii) Let  $p$  be any prime number. If  $h_p(r) = P(r)$  for some polynomial  $P \in \mathbb{Q}[T]$  and for all  $r \ll -3$ , then  $P(r) = h_0(r)$  for infinitely many  $r \ll -3$  by part (i). Thus, by Proposition 2.3(ii), we must have  $P = T(T-1)^2(T-2)/12$ , so that the set  $\{d : d \geq 3, h_p(-d) > h_0(-d)\}$  is finite. But this contradicts part (i).

(iii) Assume now that  $p$  and  $q$  are different prime numbers. By Proposition 2.3(i), it is enough to show that  $\{j \in \mathbb{N} : p \in \Pi(j), q \notin \Pi(j)\}$  is infinite. The proof of Lemma 2.16 in [BKS] shows that  $q \notin \Pi(q^k - 1)$  for all  $k \geq 1$ . On the other hand, if  $p$  divides  $a \in \mathbb{N}$  then  $p$  divides  $\binom{a}{1} = a$ , so that  $p \in \Pi(a)$ ; it is therefore enough to show that  $p$  divides  $q^k - 1$  for infinitely many  $k \in \mathbb{N}$ . Let  $\alpha$  be the order of  $q$  in the multiplicative group of  $\mathbb{Z}/p\mathbb{Z}$ . For all  $\beta \in \mathbb{N}$  we have

$$q^{\alpha p^\beta} - 1 \equiv q^\alpha - 1 \equiv 0 \pmod{p}.$$

□

### 3. An example in characteristic zero

In §2, we provided, for each prime number  $p$ , an example of a cohomological Hilbert function of a standard positively graded finitely generated algebra over the field  $\mathbb{Z}/p\mathbb{Z}$  that fails to be of reverse polynomial type. In this section, we provide an example over a field of characteristic 0 that exhibits similar behaviour.

Fix  $K$  to be any field of characteristic zero. Let  $R_0 = K[X, Y]$  and let  $S = R_0[U, V]$ , where  $X, Y, U, V$  are independent indeterminates over  $K$ . Define a grading on  $S$  by declaring that  $\deg X = \deg Y = 0$  and  $\deg U = \deg V = 1$ . Let  $f = 2X^2V^2 + 2XYUV + Y^2U^2$  and let  $R = S/fS$ . Notice that  $f$  is homogeneous and hence  $R$  is graded. Let  $S_+$  be the ideal of  $S$  generated by  $U$  and  $V$  and let  $R_+$  be the ideal of  $R$  generated by the images of  $U$  and  $V$ .

We will study the graded components of  $H_{R_+}^2(R)$  by exploiting the fact that this local cohomology module is homogeneously isomorphic to  $H_{S_+}^2(S)/fH_{S_+}^2(S)$ , and that  $H_{S_+}^2(S)$  can be realised as the module  $R_0[U^-, V^-]$  of inverse polynomials. Thus, for all  $d \in \mathbb{Z}$ , the  $(-d)$ -th graded component  $H_{R_+}^2(R)_{-d}$  of  $H_{R_+}^2(R)$  is isomorphic to the cokernel of the  $R_0$ -homomorphism

$$f_d : R_0[U^-, V^-]_{-d-2} \longrightarrow R_0[U^-, V^-]_{-d}$$

given by multiplication by  $f$ , as described in Section 1. Note that  $H_{R_+}^2(R)_r = 0$  for all  $r > -2$ , and that Theorem 1.5 shows that, for all  $d$ , the  $R_0$ -module  $H_{R_+}^2(R)_{-d}$  has finite length

$$h_R^2(-d) := \text{length}_{R_0} H_{R_+}^2(R)_{-d} = \dim_K H_{R_+}^2(R)_{-d};$$

Thus  $h_R^2 : \mathbb{Z} \rightarrow \mathbb{N}_0$  is a cohomological Hilbert function, as explained at the beginning of the last section. Our aim in this section is to show that  $h_R^2$  is not of reverse polynomial type.

If we use the ordering of bases described in Section 1 (with  $U > V$ ) for both the source and target of each  $f_d$ , we can see that each  $f_d$  ( $d \geq 2$ ) is given by multiplication on the left by the  $(d-1) \times (d+1)$  tridiagonal matrix

$$C_{d-1} := \begin{bmatrix} 2X^2 & 2XY & Y^2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2X^2 & 2XY & Y^2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2X^2 & 2XY & Y^2 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 2X^2 & 2XY & Y^2 \end{bmatrix}.$$

We also write

$$\tilde{C}_{d-1} := \begin{bmatrix} 2 & 2 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2 & 2 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 2 & 2 & 1 \end{bmatrix},$$

the result of evaluation of  $C_{d-1}$  at  $X = Y = 1$ .

#### 3.1. Lemma.

(i) For each  $n \in \mathbb{N}$ , let  $\tilde{D}_n$  be the  $n \times n$  tridiagonal matrix

$$\begin{bmatrix} 2 & 1 & 0 & 0 & \dots & 0 \\ 2 & 2 & 1 & 0 & \dots & 0 \\ 0 & 2 & 2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 2 & 2 \end{bmatrix}$$

obtained by removing the first and last columns of  $\tilde{C}_n$ . Then  $\det \tilde{D}_n = 0$  if and only if  $n \equiv 3 \pmod{4}$ .



- (ii) Any submatrix of  $\tilde{C}_n$  consisting of consecutive columns has maximal rank except for the matrix obtained from  $\tilde{C}_n$  by removing its first and last columns when  $n \equiv 3 \pmod{4}$ . This exceptional submatrix has rank  $n - 1$ .

*Proof.* (i) Write  $\Delta_n = \det \tilde{D}_n$  for all  $n \geq 1$ . We have  $\Delta_1 = 2$  and  $\Delta_2 = 2$ ; for  $n \geq 3$  we can expand the determinant  $\Delta_n$  by the first row of  $\tilde{D}_n$  to obtain  $\Delta_n = 2\Delta_{n-1} - 2\Delta_{n-2}$ . So for all  $n \geq 3$  we can write

$$\begin{bmatrix} \Delta_n \\ \Delta_{n-1} \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta_{n-1} \\ \Delta_{n-2} \end{bmatrix}$$

and by induction

$$\begin{bmatrix} \Delta_n \\ \Delta_{n-1} \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

The  $2 \times 2$  rational matrix  $\begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix}$  has complex eigenvalues  $\sqrt{2}e^{i\pi/4}$  and  $\sqrt{2}e^{-i\pi/4}$ , and by diagonalizing it we see that, for  $n \geq 3$ ,

$$\begin{aligned} \begin{bmatrix} \Delta_n \\ \Delta_{n-1} \end{bmatrix} &= \begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \\ &= (\sqrt{2})^{n-2} \begin{bmatrix} 1+i & 1-i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{i(n-2)\pi/4} & 0 \\ 0 & e^{-i(n-2)\pi/4} \end{bmatrix} \begin{bmatrix} -i & 1+i \\ i & 1-i \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= (\sqrt{2})^{n-2} \begin{bmatrix} 1+i & 1-i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{i(n-2)\pi/4} \\ e^{-i(n-2)\pi/4} \end{bmatrix} \\ &= (\sqrt{2})^{n-2} \begin{bmatrix} (1+i)e^{i(n-2)\pi/4} + (1-i)e^{-i(n-2)\pi/4} \\ e^{i(n-2)\pi/4} + e^{-i(n-2)\pi/4} \end{bmatrix}. \end{aligned}$$

Hence  $\Delta_n = 0$  if and only if  $n \equiv 3 \pmod{4}$ .

(ii) The matrix obtained from  $\tilde{C}_n$  by removal of its first and last columns is  $\tilde{D}_n$ , and it is straightforward to check that all other selections of  $n$  or fewer consecutive columns from  $\tilde{C}_n$  are linearly independent.  $\square$

**3.2. Theorem.** *The cohomological Hilbert function  $h_R^2 : \mathbb{Z} \rightarrow \mathbb{N}_0$  is such that, for all  $d \geq 2$ ,*

$$h_R^2(-d) = \begin{cases} d^2 - 1 & \text{if } d \not\equiv 0 \pmod{4}, \\ d^2 & \text{if } d \equiv 0 \pmod{4}. \end{cases}$$

Hence  $h_R^2$  is not of reverse polynomial type.

*Proof.* Fix an integer  $d \geq 2$ . The length  $h_R^2(-d)$  is  $\dim_K \text{Coker } C_{d-1}$ . In order to calculate this dimension, we consider  $K[X, Y]$  as an  $\mathbb{N}_0^2$ -graded ring in which  $K[X, Y]_{(0,0)} = K$  and  $\deg X^i Y^j = (i + j, j)$  for all  $(i, j) \in \mathbb{N}_0^2$ . For convenience, we set  $d - 1 =: n$ . Turn the free  $R_0$ -module

$$R_0^n = K[X, Y]\mathbf{e}_1 \oplus \cdots \oplus K[X, Y]\mathbf{e}_n$$

into an  $\mathbb{N}_0^2$ -graded module over the  $\mathbb{N}_0^2$ -graded ring  $R_0 = K[X, Y]$  in such a way that  $\deg \mathbf{e}_i = (0, i)$  for  $i = 1, \dots, n$ . All references to gradings in the rest of this proof refer to this  $\mathbb{N}_0^2$ -grading. Note that  $(R_0^n)_{(i,j)} = 0$  whenever  $j > n + i$ .

For each  $j = 1, \dots, n + 2$ , let  $\mathbf{c}_j$  denote the  $j$ -th column of  $C_n$ , and note that  $\mathbf{c}_j$  is homogeneous of degree  $(2, j)$ . Thus  $\text{Im } C_n$ , the  $R_0$ -submodule of  $R_0^n$  generated by the columns of  $C_n$ , is graded; hence  $\text{Coker } C_n$  is graded, too. Note that  $(\text{Im } C_n)_{(i,j)} = 0$  whenever  $i < 2$ .

It is not hard to see that the vectors

$$X^2\mathbf{e}_1, X^3\mathbf{e}_2, \dots, X^{n+1}\mathbf{e}_n \quad \text{and} \quad Y^2\mathbf{e}_n, Y^3\mathbf{e}_{n-1}, \dots, Y^{n+1}\mathbf{e}_1$$

are in the image of  $C_n$ . Hence  $(\text{Im } C_n)_{(i,j)} = (R_0^n)_{(i,j)}$  whenever  $i > n + 1$ .

These observations lead to the conclusion that

$$\dim_K \text{Coker } C_n = \sum_{i=0}^{n+1} \sum_{j \in \mathbb{N}} \dim_K (K[X, Y]^n)_{(i,j)} - \sum_{i=2}^{n+1} \sum_{j \in \mathbb{N}} \dim_K (\text{Im } C_n)_{(i,j)}.$$

The first term on the right-hand side is equal to  $n$  times the number of monomials  $X^\tau Y^\omega$  (where  $\tau, \omega \in \mathbb{N}_0$ ) of total (ordinary) degree not exceeding  $n + 1$ . Therefore

$$\sum_{i=0}^{n+1} \sum_{j \in \mathbb{N}} \dim_K (K[X, Y]^n)_{(i,j)} = n \sum_{i=0}^{n+1} (i+1) = \frac{n(n+2)(n+3)}{2}.$$

Now choose an integer  $i$  with  $2 \leq i \leq n + 1$ . Our observations above show that, for any  $j \in \mathbb{N}$ ,

$$(\mathrm{Im} C_n)_{(i,j)} = \sum_{\sigma=\max\{2+j-i,1\}}^{\min\{j,n+2\}} KX^{i-2-j+\sigma} Y^{j-\sigma} \mathbf{c}_\sigma.$$

Thus  $\dim_K(\mathrm{Im} C_n)_{(i,j)}$  is equal to the rank of the submatrix of  $\tilde{C}_n$  made up of the (consecutive) columns of that matrix numbered

$$\max\{2+j-i, 1\}, \max\{2+j-i, 1\} + 1, \dots, \min\{j, n+2\}.$$

It follows that  $\sum_{j \in \mathbb{N}} \dim_K(\mathrm{Im} C_n)_{(i,j)}$  is equal to the sum of the ranks of the submatrices of  $\tilde{C}_n$  obtained by selecting consecutive columns numbered by the sets of integers in the list

$$\{1\}, \{1, 2\}, \dots, \{1, 2, \dots, i-1\}, \{2, 3, \dots, i\}, \dots, \{n-i+4, n-i+5, \dots, n+2\}, \\ \{n-i+5, n-i+6, \dots, n+2\}, \dots, \{n+1, n+2\} \text{ and } \{n+2\}.$$

By Lemma 3.1(ii), all these submatrices have maximal rank, except when  $n \equiv 3 \pmod{4}$  and  $i = n + 1$ , when they all have maximal rank except for that corresponding to the choice  $\{2, 3, \dots, n+1\}$ , which has rank  $n - 1$  rather than  $n$ .

It follows that, unless  $n \equiv 3 \pmod{4}$  and  $i = n + 1$ ,

$$\sum_{j \in \mathbb{N}} \dim_K(\mathrm{Im} C_n)_{(i,j)} = 2 \sum_{\alpha=1}^{i-2} \alpha + (n-i+4)(i-1) \\ = (i-2)(i-1) + (n-i+4)(i-1) = (i-1)(n+2);$$

in the exceptional case, the sum is one less than that given by the above formula. Hence, unless  $n \equiv 3 \pmod{4}$ ,

$$\dim_K \mathrm{Coker} C_n = \sum_{i=0}^{n+1} \sum_{j \in \mathbb{N}} \dim_K (K[X, Y]^n)_{(i,j)} - \sum_{i=2}^{n+1} \sum_{j \in \mathbb{N}} \dim_K(\mathrm{Im} C_n)_{(i,j)} \\ = \frac{n(n+2)(n+3)}{2} - \sum_{i=2}^{n+1} (i-1)(n+2) \\ = \frac{n(n+2)(n+3)}{2} - \frac{n(n+1)(n+2)}{2} = n(n+2);$$

when  $n \equiv 3 \pmod{4}$ , we have  $\dim_K \mathrm{Coker} C_n = n(n+2) + 1$ .

Thus, since  $n = d - 1$ , we have shown that

$$h_R^2(-d) = \dim_K \mathrm{Coker} C_{d-1} = \begin{cases} d^2 - 1 & \text{if } d \not\equiv 0 \pmod{4}, \\ d^2 & \text{if } d \equiv 0 \pmod{4}. \end{cases}$$

If  $h_R^2(r)$  were to agree with  $P(r)$  for a polynomial  $P \in \mathbb{Q}[T]$  for all  $r \ll 0$ , then  $P$  would have to be both  $T^2 - 1$  and  $T^2$ , which is, of course, absurd.  $\square$

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(Katzman) DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF SHEFFIELD, HICKS BUILDING, SHEFFIELD S3 7RH, UNITED KINGDOM, *Fax number*: 0044-114-222-3769  
*E-mail address*: [M.Katzman@sheffield.ac.uk](mailto:M.Katzman@sheffield.ac.uk)

(Sharp) DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF SHEFFIELD, HICKS BUILDING, SHEFFIELD S3 7RH, UNITED KINGDOM, *Fax number*: 0044-114-222-3769  
*E-mail address*: [R.Y.Sharp@sheffield.ac.uk](mailto:R.Y.Sharp@sheffield.ac.uk)