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A note on state space representations of locally stationary wavelet time series

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Abstract

In this note we show that the locally stationary wavelet process can be decomposed into a sum of signals, each of which following a moving average process with time-varying parameters. We then show that such moving average processes are equivalent to state space models with stochastic design components. Using a simple simulation step, we propose a heuristic method of estimating the above state space models and then we apply the methodology to foreign exchange rates data.

Some key words: wavelets, Haar, locally stationary process, time series, state space, Kalman filter.

1 Introduction

Nason *et al.* (2000) define a class of locally stationary time series making use of non-decimated wavelets. Let $\{y_t\}$ be a scalar time series, which is assumed to be locally stationary, or stationary over ceratin intervals of time (regimes), but overall non-stationary. For more details on local stationarity the reader is referred to Dahlhaus (1997), Nason *et el.* (2000), Francq and Zakoan (2001), and Mercurio and Spokoiny (2004). For example, Figure 1 shows the nonstationary process considered in Nason *et al.* (2000), which is the concatenation of 4 stationary moving average processes, but each with different parameters. We can see that within each of the four regimes, the process is weakly stationary, but overall the process is non-stationary.

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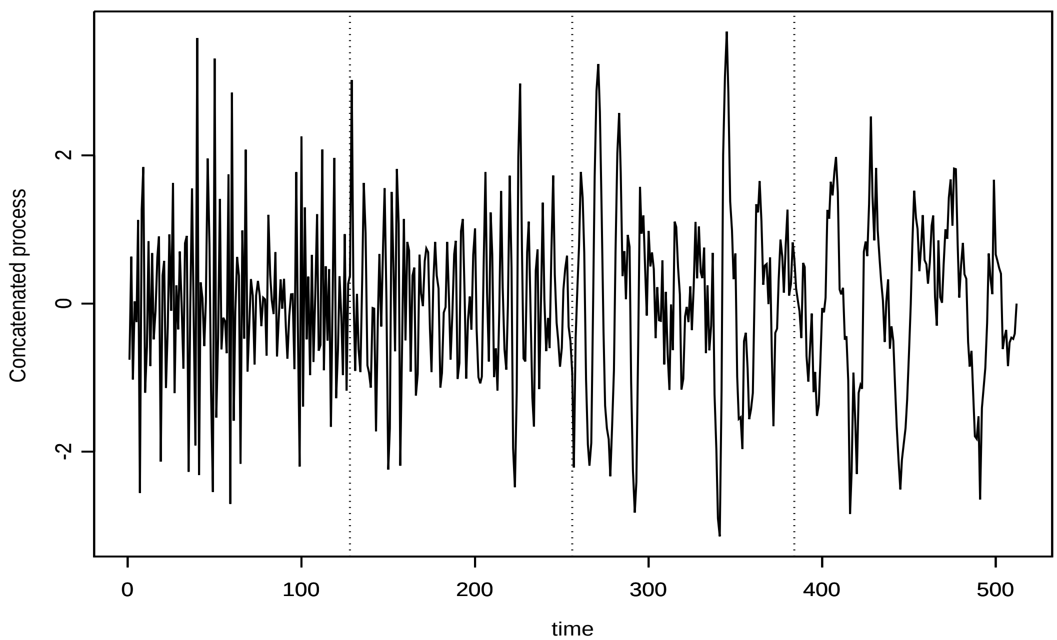

Figure 1: Concatenation of four MA time series with different parameters. Overall the process is not stationary. The dotted vertical lines indicate the transition between one MA process and the next.

The locally stationary wavelet (LSW) process is a doubly indexed stochastic process, defined by

$$y_t = \sum_{j=-J}^{-1} \sum_{k=0}^{T-1} w_{jk} \psi_{j,t-k} \xi_{jk}, \tag{1}$$

where ξ_{jk} is a random orthonormal increment sequence (below this will be iid Gaussian) and $\{\psi_{jk}\}_{j,k}$ is a discrete non-decimated family of wavelets for $j = -1, -2, \ldots, -J$, $k = 0, \ldots, T-1$, based on a mother wavelet $\psi(t)$ of compact support. Denote with $I_A(x)$ the indicator function, i.e. $I_A(x) = 1$, if $x \in A$ and $I_A = 0$, otherwise. The simplest class of wavelets are the Haar wavelets, defined by

$$\psi_{jk} = 2^{j/2} I_{\{0,\ldots,2^{-j-1}-1\}}(k) - 2^{j/2} I_{\{2^{-j-1},\ldots,2^{-j}-1\}}(k),$$

for $j \in \{-1, -2, \ldots, -J\}$ and $k \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, where $j = -1$ is the finest scale. It is also assumed that $E(\xi_{jk}) = 0$, for all j and k and so y_t has zero mean. The orthonormality assumption of $\{\xi_{jk}\}$ implies that $\text{Cov}(\xi_{jk}, \xi_{\ell m}) = \delta_{j\ell}\delta_{km}$, where δ_{jk} denotes the Kronecker delta, i.e. $\delta_{jj} = 1$ and $\delta_{jk} = 0$, for $j \neq k$.

The parameters w_{jk} are the amplitudes of the LSW process. The quantity w_{jk} characterizes the amount of each oscillation, $\psi_{j,t-k}$ at each scale, j, and location, k (modified by the random amplitude, ξ_{jk}). For example, a large value of w_{jk} indicates that there is a chance (depending on ξ_{jk}) of an oscillation, $\psi_{j,t-k}$, at time t. Nason *et al.* (2000) control the evolution of the statistical characteristics of y_t by coupling w_{jk} to a function $W_j(z)$ for $z \in (0,1)$ by $w_{jk} = W_j(k/T) + \mathcal{O}(T^{-1})$. Then, the smoothness properties of $W_j(z)$ control the possible rate of change of w_{jk} as a function of k, which consequently controls the evolution of the statistical properties of y_t. The smoother $W_j(z)$ is, as a function of z, the slower that y_t can evolve. Ultimately, if $W_j(z)$ is a constant function of z, then y_t is weakly stationary.

The non-stationarity in the above studies is better understood as local-stationarity so that the w_{jk}'s are close to each other. To elaborate on this, if $w_{jk} = w_j$ (time invariant), then y_t would be weakly stationary. The attractiveness of the LSW process, is its ability to consider time-changing w_{jk}'s.

Nason *et al.* (2000) define the evolutionary wavelet spectrum (EWS) to be $S_j(z) = |W_j(z)|^2$ and discuss methods of estimation. Fryzlewicz *et al.* (2003) and Fryzlewicz (2005) modify the LSW process to forecast log-returns of non-stationary time series. These authors analyze daily FTSE 100 time series using the LSW toolbox. Fryzlewicz and Nason (2006) estimate the EWS by using a fast Haar-Fisz algorithm. Van Bellegem and von Sachs (2008) consider adaptive estimation for the EWS and permit jump discontinuities in the spectrum.

In this paper we show that the process y_t can be decomposed into a sum of signals, each of which follows a moving average process with time-varying parameters. We deploy a heuristic approach for the estimation of the above moving average process and an example, consisting of foreign exchange rates, illustrates the proposed methodology.

2 Decomposition at scale j

The LSW process (1) can be written as

$$y_t = \sum_{j=-J}^{-1} x_{jt}, \tag{2}$$

where

$$x_{jt} = \sum_{k=0}^{T-1} w_{jk} \psi_{j,t-k} \xi_{jk}. \tag{3}$$

For computational simplicity and without loss in generality, we omit the minus sign of the scales $(-J, \ldots, -1)$ so that the summation in equation (2) is done from $j = 1$ (scale -1) until $j = J$ (scale $-J$).

Using Haar wavelets, we can see that at scale 1, we have from (3) that $x_{1t} = \psi_{1,0}w_{1t}\xi_{1t} + \psi_{1,-1}w_{1,t-1}\xi_{1,t-1}$, since there are only 2 non-zero wavelet coefficients. Then we can re-write (3) as $x_{1t} = \alpha_{1t}^{(0)}\xi_{1t} + \alpha_{1t}^{(1)}\xi_{1,t-1}$, which is a moving average process of order one, with time-varying parameters $\alpha_{1t}^{(0)}$ and $\alpha_{1t}^{(1)}$. This process can be referred to as TVMA(1) process.

In a similar way, for any scale $j = 1, \dots, J$, we can write

$$x_{jt} = \psi_{j,0}w_{jt}\xi_{jt} + \psi_{j,-1}w_{j,t-1}\xi_{j,t-1} + \cdots + \psi_{j,-2^j+1}w_{j,t-2^j+1}\xi_{j,t-2^j+1}$$

so that we obtain the TVMA($2^j - 1$) process

$$x_{jt} = \alpha_{jt}^{(0)}\xi_{jt} + a_{jt}^{(1)}\xi_{j,t-1} + \cdots + a_{jt}^{(2^j-1)}\xi_{j,t-2^j+1}, \tag{4}$$

where $\alpha_{jt}^{(\ell)} = \psi_{j,-\ell}w_{j,t-\ell}$, for all $\ell = 0, 1, \dots, 2^j - 1$ and $j = 1, \dots, J$. Thus the process y_t is the sum of J TVMA processes. However, we note that not all time-varying parameters $a_{jt}^{(\ell)}$ ($\ell = 0, 1, \dots, 2^j - 1$) are independent, since, for a fixed j, they are all functions of the $\{w_{jt}\}$ series.

We advocate that w_{jt} is a signal and as such we treat it as an unobserved stochastic process. Indeed, from the slow evolution of w_{jt}, we can postulate that $w_{jt} - w_{j,t-1} \approx 0$, which motivates a random walk evolution for w_{jt} or $w_{jt} = w_{j,t-1} + \zeta_{jt}$, where ζ_{jt} is a Gaussian white noise, i.e. $\zeta_{jt} \sim N(0, \sigma_j^2)$, for σ_j^2 a known variance, and ζ_{jt} is independent of ζ_{kt}, for all $j \neq k$. The magnitude of the differences between $w_{j,t-1}$ and w_{jt} can be controlled by σ_j^2 and this controls on the degree of evolution of w_{jt} as a function of t and hence on y_t through (2).

At scale 1 we can write x_{1t} as

$$x_{1t} = \psi_{1,0}w_{1t}\xi_{1t} + \psi_{1,-1}w_{1,t-1}\xi_{1,t-1} = (\psi_{1,0}\xi_{1t} + \psi_{1,-1}\xi_{1,t-1})w_{1,t-1} + \psi_{1,0}\zeta_{1t}\xi_{1t},$$

where we have used $w_{1t} = w_{1,t-1} + \zeta_{1t}$. Likewise at scale 2 we have

$$\begin{aligned} x_{2t} &= \psi_{2,0}w_{2t}\xi_{2t} + \psi_{2,-1}w_{2,t-1}\xi_{2,t-1} + \psi_{2,-2}w_{2,t-2}\xi_{2,t-2} + \psi_{2,-3}w_{2,t-3}\xi_{2,t-3} \\ &= (\psi_{2,0}\xi_{2t} + \psi_{2,-1}\xi_{2,t-1} + \psi_{2,-2}\xi_{2,t-2} + \psi_{2,-3}\xi_{2,t-3})w_{2,t-3} \\ &\quad + \psi_{2,0}\zeta_{2,t-2}\xi_{2t} + \psi_{2,0}\zeta_{2,t-1}\xi_{2t} + \psi_{2,0}\zeta_{2t}\xi_{2t} \\ &\quad + \psi_{2,-1}\zeta_{2,t-2}\xi_{2,t-1} + \psi_{2,-1}\zeta_{2,t-1}\xi_{2,t-1} \\ &\quad + \psi_{2,-2}\zeta_{2,t-2}\xi_{2,t-2}, \end{aligned}$$

where we have used $w_{2,t-2} = w_{2,t-3} + \zeta_{2,t-2}$, $w_{2,t-1} = w_{2,t-3} + \zeta_{2,t-2} + \zeta_{2,t-1}$ and $w_{2t} = w_{2,t-3} + \zeta_{2,t-2} + \zeta_{2,t-1} + \zeta_{2t}$.

In general we observe that at any scale $j = 1, \dots, J$ we can write

$$x_{jt} = \sum_{k=0}^{2^j-1} \psi_{j,-k}\xi_{j,t-k}w_{j,t-2^j+1} + \sum_{k=0}^{2^j-2}\sum_{m=k}^{2^j-2} \psi_{j,-k}\xi_{j,t-k}\zeta_{j,t-m}, \quad t = 2^j, 2^j + 1, \dots, \tag{5}$$

where the w_{jt}'s follow the random walk

$$w_{j,t-2^j+1} = w_{j,t-2^j} + \zeta_{j,t-2^j+1}, \quad \zeta_{j,t-2^j+1} \sim N(0, \sigma_j^2). \tag{6}$$

3 A state space representation

For estimation purposes one could use a time-varying moving average model in order to estimate $\{w_{jk}\}$ in (4). Moving average processes with time-varying parameters are useful models for locally stationary time series data, but their estimation is more difficult that that of time-varying autoregressive processes (Hallin, 1986; Dahlhaus, 1997). The reason for this is that the time-dependence of the moving average coefficients may result in identifiability problems. The consensus is that some restrictions of the parameter space of the time-varying coefficients should be applied; for more details the reader is referred to the above references as well as to Triantafyllopoulos and Nason (2007).

In this section we use a heuristic approach for the estimation of the above models. First we recast model (5)-(6) into state space form. To end this we write

$$x_{jt} = A_{jt} w_{j,t-2^j+1} + \nu_{jt}, \tag{7}$$

where $A_{jt} = \sum_{k=0}^{2^j-1} \psi_{j,-k}\xi_{j,t-k}$ and $\nu_{jt} = \sum_{k=0}^{2^j-2}\sum_{m=k}^{2^j-2} \psi_{j,-k}\xi_{j,t-k}\zeta_{j,t-m}$, for $t = 2^j, 2^j+1, \ldots$. In addition we assume that ξ_{jt}^i is independent of ζ_{js}^i, for $i = 1, 2$ and for any t, s, so that

$$\nu_{jt} \sim N\left\{0, \sigma_j^2 \sum_{k=0}^{2^j-1} \psi_{j,-k}^2 (2^j - k - 1)\right\}. \tag{8}$$

Equations (7), (6), (8) define a state space model for x_{jt} and by defining $A_t = (A_{1t}, \ldots, A_{Jt})'$ and by noting that ν_{jt} is independent of ν_{kt}, for any $j \neq k$, we obtain by (2) a state space model for y_t, which essentially is the superposition of J state space models of the form of (7), (6), (8), each being a state space model for each scale $j = 1, \ldots, J$.

Given a set of data $y^T = \{y_1, \ldots, y_T\}$, a heuristic way to estimate $\{w_{jt}\}$, is to simulate independently all ξ_{jt} from $N(0,1)$, thus to obtain simulated values for A_{jt} and then, conditional on A_t, to apply the Kalman filter to the state space model for y_t. This procedure will give simulations from the posterior distributions of w_{jt} and also from the predictive distributions of $y_{t+h}|y^t$. The estimator of w_{jt} and the forecast of y_{t+h} are conditional on the simulated values of $\{\xi_{jt}\}$. For competing simulated sequences $\{\xi_{jt}\}$ the performance of the above estimators/forecasts can be judged by comparing the respective likelihood functions (which are easily calculable by the Kalman filter) or by comparing the respective posterior and forecast densities (by using sequential Bayes factors). Another means of model performance may be the computation of the mean square forecast error.

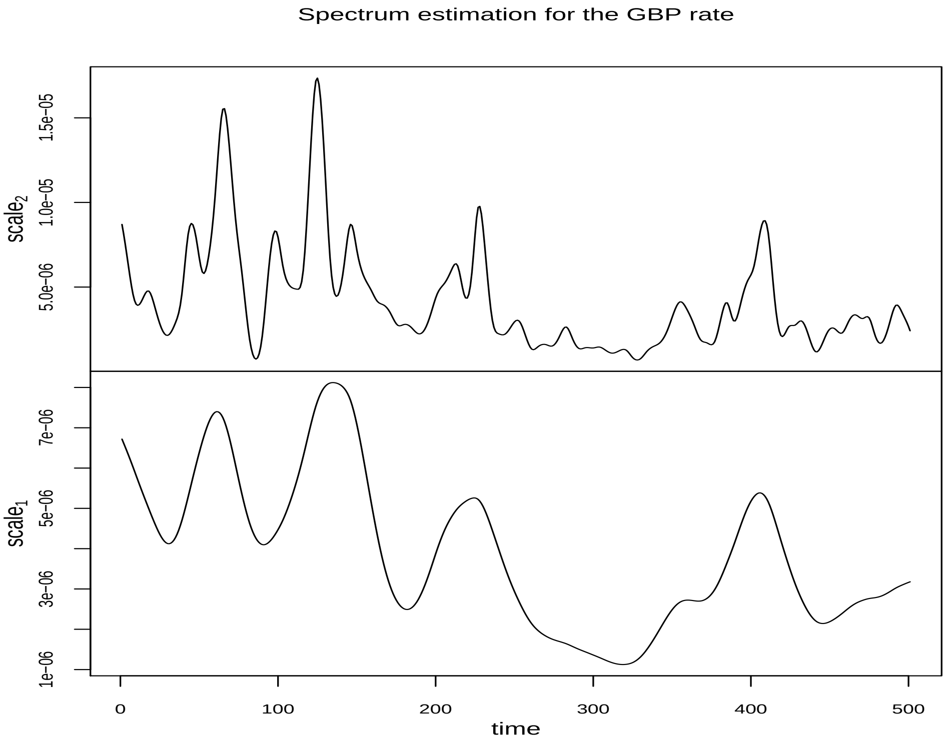

Figure 2: Simulated values of posterior estimates of $\{S_{jt} = w_{jt}^2\}$, for $\{y_{1t}\}$ (GBP rate). Shown are simulations of $\{S_{1t}\}$ and $\{S_{2t}\}$, corresponding to scales 1 and 2.

We illustrate this approach by considering foreign exchange rates data. The data are collected in daily frequency from 3 January 2006 to and including 31 December 2007 (considering trading days there are 501 observations). We consider two exchange rates: US dollar with British pound (GBP rate) and US dollar with Euro (EUR rate). After we transform the data to the log scale, we propose to use the LSW process in order to obtain estimates of the spectrum process $\{S_{jt} = w_{jt}^2\}$, for each scale j. We form the vector $y_t = (y_{1t}, y_{2t})'$, where y_{1t} is the log-return value of GBP and y_{2t} is the log-return value of EUR. For each series $\{y_{1t}\}$ and $\{y_{2t}\}$, respectively, Figures 2 and 3 show simulations of the posterior spectrum $\{S_{jt}\}$, for scales 1 and 2. The smoothed estimates of these figures are achieved by first computing the smoothed estimates using the Kalman filter and then applying a standard Spline method (Green and Silverman, 1994). We note that, for the data set considered in this paper, the estimates of Figures 2 and 3 are less smooth than those produced by the method of Nason *et al.* (2000). However, a higher degree of smoothness in our estimates can be achieved by considering small values of the variance σ_j^2, which controls the smoothness of the shocks in the random walk of the w's.

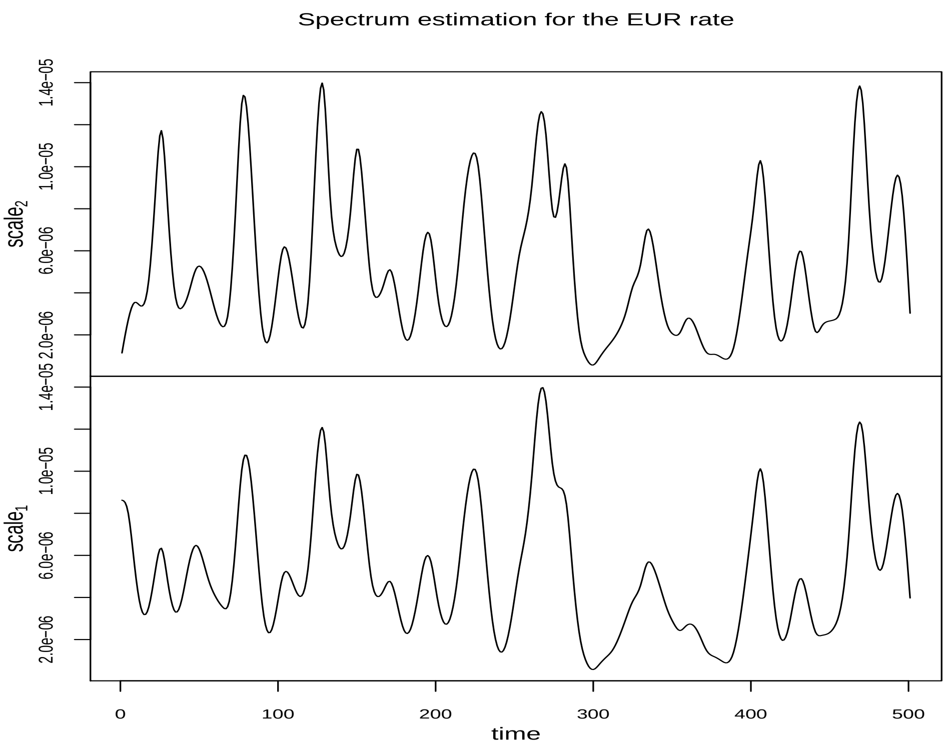

Figure 3: Simulated values of posterior estimates of $\{S_{jt} = w_{jt}^2\}$, for $\{y_{2t}\}$ (EUR rate). Shown are simulations of $\{S_{1t}\}$ and $\{S_{2t}\}$, corresponding to scales 1 and 2.

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