1 Introduction

In the field of operations management, the risk averse decision problem has been extensively studied. For a risk averse corporation, one problem to consider is how to manage risks versus the expected return of inventory activities. This question is often formulated as a mean-variance type decision problem; several papers such as [4], [12] discuss the optimal operational decision for this problem. A non financial corporation can be exposed to various sources of risk, which can be subsumed into two types: financial risk and non-financial risk. The financial risk comes from financial market and hence can be hedged, to some extent, using financial instruments. The non-financial risk is assumed to be independent of the financial market, and hence cannot be hedged through financial trading. This naturally leads to a financial hedging problem in an incomplete market. A recent line of research addresses incorporating hedging in operations management, in particular, the financial department of a non-financial corporation can trade in financial market to hedge risks arising in operational activities. This kind of problem leads to making financial and operational decision simultaneously. Different inventory models with hedging have been proposed, see for instance [1], [2] and [6].

Since the non-financial risk cannot be hedged through financial trading, we are dealing with a problem of financial hedging in an incomplete market. This is a widely studied field in mathematical finance. A classical approach to this problem is to control the hedging error by a quadratic criterion. This is mathematically equivalent to solving an optimal investment problem for a mean-variance type objective function. From an operations management point of view, an attractive feature of this approach is its high degree of tractability. We refer to [11] for a thorough overview on the quadratic hedging literature.

For big corporations, the hedging decision is taken care of by the financial department, while different operational departments’ decisions are usually naturally inter-connected. Moreover, it is usually the case that it is inefficient and expensive for each operational department to communicate and exchange the decision. We propose a decision model such that the operational decision can be made given the global information from financial control and without other operational branches’ decisions. In other words, our main contribution is that we achieve a separation result of a multiple operational decision problem.
In the special case of a single product operational decision problem, our work closely follows Caldenty and Haugh [1] who propose a dynamic hedging strategy for the profits of a risk-averse corporation when these profits are correlated with returns in the financial markets. We depart from their framework by considering a slightly different mean-variance type objective function. Most importantly, this allows us to extend their results to a multi-product problem which we can solve by a separation theorem.

One example of financial risk that can affect a corporation’s profits is inflation. We will use the model in [7] to describe a market of inflation-related financial securities. This setting enables us to characterize an inflationary economy. We discuss a classical newsvendor problem in which the demand is a positive function depending on inflation. As will be proved in section 3.3, the high inflation leads to a malinvestment in inventory decision without hedging opportunity. Our interest is to provide a solution to the malinvestment in operational decisions under monetary inflation economy. However, like in [1], our main results in section 3 are formulated for a generic financial asset, and thus can be applied in the context of other sources of financial risk.

The paper is organized as follows. In section 2 we introduce the financial market model and the inventory model, and formulate the hedging problem for multiple operational decision. Our main results are stated in section 3, where we solve the problem via separation. We conclude the paper with numerical examples in section 4. Proofs are collected in section 5.

2 Model and Problem Formulation

Fix a time horizon $T^* \in (0, \infty)$. Our set of states is given by the product probability space $(\Omega, \mathcal{F}, P) = (\Omega^W \times E, \mathcal{F}^W \otimes \mathcal{E}, P^W \otimes P^E)$, where $(\Omega^W, \mathcal{F}^W, \mathcal{F}^W_t, P^W)$ and $(E, \mathcal{E}, \mathcal{E}_t, P^E)$ are two complete filtered probability spaces. In particular, $(\Omega^W, \mathcal{F}^W, \mathcal{F}^W_t, P^W)$ is a probability space endowed with Brownian motions $(W^*(t), W^*(t), W^*(t) : t \in [0, T^*])$ with correlations given by

$$dW^*_n(t)dW^*_r(t) = \rho_{nr}dt$$
$$dW^*_n(t)dW^*_I(t) = \rho_{nI}dt$$
$$dW^*_r(t)dW^*_I(t) = \rho_{rI}dt$$

The space $\mathcal{E}$ represents the additional source of randomness which affects the market, where $\{\mathcal{E}_t : t \in [0, T^*]\}$ is the standard filtration generated by the $N$-dimensional Brownian motion $B(t) = (B_1(t), \ldots, B_N(t)), t \in [0, T^*]$, independent of $\mathcal{F}^W_t$.

2.1 Financial market model

To analyze the impact of inflation risk on inventory management, we start by describing a market for inflation-related financial securities. We shall use the Heath-Jarrow-Morton type
term structure model by Jarrow and Yildirim [7] where the tradable assets in the market are a bank account, nominal zero-coupon treasury bonds, and the Treasury Inflation-Protected Securities (TIPS) zero-coupon bonds. The following notations for financial markets will be used in this paper:

- ’r’ for real, ’n’ for nominal.
- $P_n(t,T)$: time $t$ price of a nominal zero-coupon bond maturing at time $T$ in dollars.
- $I(t)$: time $t$ CPI inflation index, i.e. dollars per CPI unit.
- $P_r(t,T)$: time $t$ price of a real zero-coupon bond maturing at time $T$ in CPI units.
- $f_k(t,T)$: time $t$ nominal ($k=n$) and real ($k=r$) forward rates for date $T$, i.e.
  $$P_k(t,T) = \exp \left\{ \int_t^T f_k(t,u)du \right\} .$$
- $r_k(t) = f_k(t,t)$: the time $t$ nominal ($k=n$) and real ($k=r$) spot rate.
- $B_n(t)$: time $t$ money market account value, i.e.
  $$B_n(t) = \exp \left\{ \int_0^t r_n(v)dv \right\} .$$
- $P_{TIPS}(t,T)$: time $t$ TIPS zero-coupon bond maturing at time $T$, i.e.
  $$P_{TIPS}(t,T) = I(t)P_r(t,T)$$

Given the initial forward rate curve $f_k(0,T)$ with $T \in [0,T^*]$, $k \in \{r,n\}$, we assume that the nominal and real $T$-maturity forward rate evolves as:

$$df_n(t,T) = \alpha_n(t,T)dt + \sigma_n(t,T)dW_n(t)$$
$$df_r(t,T) = \alpha_r(t,T)dt + \sigma_r(t,T)dW_r(t)$$

for $0 \leq t \leq T \leq T^*$, where $\alpha_k(t,T)$ and $\sigma_k(t,T)$ are stochastic processes satisfying some technical measurability and integrability conditions.\footnote{The process $\alpha_k(t,T)$ is $\mathcal{F}_t$-adapted and jointly measurable with $\int_0^T |\alpha_k(t,T)|dt < \infty$ P-a.s. and $\sigma_k(t,T)$ satisfies $\int_0^T \sigma_k^2(t,T)dt < \infty$ P-a.s.}
The inflation index's evolution is given by

\[
\frac{dI(t)}{I(t)} = \mu_I(t)dt + \sigma_I(t)dW_I(t)
\]  

for \( t \in [0, T^*] \), where \( \mu_I(t) \) and \( \sigma_I(t) \) are stochastic processes satisfying some technical measurability and integrability conditions.\(^2\)

The financial market \( B_n(t), P_n(t, T), P_{TIPS}(t, T), 0 \leq t \leq T \leq T^* \), is arbitrage-free if there exists a probability measure \( Q \) equivalent to \( P^W \) on \((\Omega, \mathcal{F}^W)\) such that:

\[
\frac{P_n(t, T)}{B_n(t)}, \frac{P_{TIPS}(t, T)}{B_n(t)} \text{ are } Q-\text{local martingales for all } T \in [0, T^*]
\]

By Girsanov's theorem, given that \((W_n(t), W_r(t), W_I(t) : t \in [0, T])\) is a \( P \)-Brownian motion and that \( Q \) is a probability measure equivalent to \( P \), then there exists market prices of risk \((\lambda_n(t), \lambda_r(t), \lambda_I(t) : t \in [0, T])\) such that

\[
\tilde{W}_k(t) = W_k(t) - \int_0^t \lambda_k(s)ds \quad \text{for } k \in \{n, r, I\}
\]

are \( Q \)-Brownian motions.

The following proposition characterizes the necessary and sufficient conditions for the economy to be arbitrage-free.

**Proposition 1** \(\frac{P_n(t, T)}{B_n(t)}, \frac{P_{TIPS}(t, T)}{B_n(t)}\) are \( Q-\text{local martingales for all } T \in [0, T^*] \) if and only if the following conditions hold:

\[
\begin{align*}
\alpha_n(t, T) &= \sigma_n(t, T) \left( \int_t^T \sigma_n(t, s)ds - \lambda_n(t) \right) \\
\alpha_r(t, T) &= \sigma_r(t, T) \left( \int_t^T \sigma_r(t, s)ds - \sigma_I(t)\rho_{rI} - \lambda_r(t) \right) \\
\mu_I(t) &= r_n(t) - r_r(t) - \sigma_I(t)\lambda_I(t)
\end{align*}
\]

The proof can be found in the Appendix.

We will further specify the model parameter as:

\[
\begin{align*}
\sigma_I(t) &= \sigma_I \\
\sigma_k(t, T) &= \sigma_k \exp(-\alpha_k(T - t)), \quad k \in \{n, r\}
\end{align*}
\]

\(^2\)The process \( \mu_I(t) \) is \( \mathcal{F}_t \)-adapted with \( E[\int_0^T |\mu_I(t)|^2dt] < \infty \) and \( \sigma_I(t) \) is a deterministic function of time with \( \int_0^T \sigma_I^2(v)dv < \infty \) \( P \)-a.s.
where $\sigma_I, \sigma_n, \sigma_r$ and $a_n, a_r$ are constants. Under these assumptions, the bond prices and inflation index follow a lognormal model under the risk-neutral measure $Q$. The processes $\frac{P_n(t,T)}{B_n(t)}$, $\frac{P_{TIPS}(t,T)}{B_n(t)}$ are martingales under $Q$.

**Proposition 2** Under the risk neutral measure $Q$, the dynamics are:

\[
\begin{align*}
    df_n(t,T) &= -\frac{\sigma_n^2}{a_n}e^{-a_n(T-t)} (e^{-a_n(T-t)} - 1) dt + \sigma_n e^{-a_n(T-t)} d\tilde{W}_n(t) \\
    df_r(t,T) &= -\sigma_r e^{-a_r(T-t)} \left( \frac{\sigma_r}{a_r} (e^{-a_r(T-t)} - 1) - \sigma_I \rho_r) \right) dt + \sigma_r e^{-a_r(T-t)} d\tilde{W}_r(t) \\
    dI(t) &= [r_n(t) - r_r(t)] dt + \sigma_I d\tilde{W}_I(t) \\
    dP_n(t,T) &= r_n(t) dt + \frac{\sigma_n}{a_n}(e^{-a_n(T-t)} - 1)d\tilde{W}_n(t) \\
    dP_r(t,T) &= r_r(t) + \rho_{rI} \frac{\sigma_r}{a_r} (e^{-a_r(T-t)} - 1) dt + \frac{\sigma_r}{a_r} [e^{-a_r(T-t)} - 1] d\tilde{W}_r(t) \\
    dP_{TIPS}(t,T) &= r_n(t) dt + \sigma_I d\tilde{W}_I(t) + \frac{\sigma_r}{a_r}(e^{-a_r(T-t)} - 1)d\tilde{W}_r(t)
\end{align*}
\]  

The proof can be found in [7] Proposition 2.

To simplify the problem, we will fix the time horizon $T$ of our inventory management problem, use $P_{TIPS}(\cdot, T)$ as numeraire and immediately pass to quantities discounted with $P_{TIPS}(\cdot, T)$. This means that $P_{TIPS}(\cdot, T)$ has (discounted) price 1 at all times and the discounted nominal bond price is $X(\cdot) := P_n(\cdot, T)/P_{TIPS}(\cdot, T)$. The following proposition characterizes the dynamic of the discounted nominal bond:

**Proposition 3** Let $X(t) = P_n(t,T)/P_{TIPS}(t,T)$ be the discounted nominal bond process using the same maturity TIPS as numeraire, its price process under the measure $P^W$ is

\[
\frac{dX(t)}{X(t)} = \mu(t)dt + \sigma(t) dW(t)
\]

where $W(t)$ is a $P^W$-Brownian motion defined as

\[
W(t) = \int_0^t \frac{1}{\sigma(s)} \left( \sum_{k=n,r} \frac{\sigma_k}{a_k} (e^{-a_k(T-s)} - 1) dW_k(s) - \sigma_I dW_I(s) \right)
\]  

(14)
and $\mu(t)$ and $\sigma(t)$ are defined as

$$
\mu(t) = -\lambda_n(t) \frac{\sigma_n}{a_n} (e^{-a_n(T-t)} - 1) + \lambda_r(t) \frac{\sigma_r e^{a_r(T-t)}}{a_r} - \rho_{nr} \frac{\sigma_n \sigma_r}{a_n a_r} (e^{-a_n(T-t)} - 1) - \rho_{nr} \frac{\sigma_n}{a_n} (e^{-a_n(T-t)} - 1)
$$

$$
\sigma(t)^2 = \frac{\sigma_n^2}{a_n^2} (e^{-a_n(T-t)} - 1)^2 + \frac{\sigma_r^2}{a_r^2} (e^{-a_r(T-t)} - 1)^2 - 2 \rho_{nr} \frac{\sigma_n \sigma_r}{a_n a_r} (e^{-a_n(T-t)} - 1)(e^{-a_r(T-t)} - 1) + 2 \rho_{nr} \frac{\sigma_r \sigma_I}{a_r^2} (e^{-a_r(T-t)} - 1) - 2 \rho_{nr} \frac{\sigma_n \sigma_I}{a_n a_r} (e^{-a_n(T-t)} - 1)(e^{-a_r(T-t)} - 1)
$$

Equation (15)

Equation (16)

The proof can be found in Appendix.

In [7], the authors described in detail the procedure to estimate parameters $a_k, k \in \{n, r\}$, $\sigma_k, k \in \{n, r, I\}$ and correlations $\rho_{nI}, \rho_{rI}, \rho_{rI}$ from three different data sets: Treasury bond data, TIPS prices, and CPI-U data. For our application, we also need to know the parameters $\lambda_k, k \in \{n, r\}$, or equivalently, estimate the drifts of the financial assets, which is a complicated problem in econometrics. We leave this practical issue an open question for now.

### 2.2 Inventory model and financial hedging

In this section, we will introduce the inventory model, propose a method of hedging in financial market, and introduce the optimization criterion of the hedging problem.

#### 2.2.1 Inventory model

We consider a classical single period, multi-products newsvendor model for the inventory. There are $N$ different products. At time $t = 0$ the operation manager makes the inventory decision $\gamma = (\gamma_1, \ldots, \gamma_N)$, which is a vector control, to satisfy a stochastic demand $D(T) = (D_1(T), \ldots, D_N(T))$. At time $T$, the demand is realized. For any product $j, j = 1, \ldots, N$, the net profit at time $T$ will be

$$
H_T(\gamma_j) = R_j(T) \min\{D_j(T), \gamma_j\} + s_j(T)(\gamma_j - D_j(T))^+ - q_j(T)(D_j(T) - \gamma_j)^+ - p_j(0) \frac{P(T)}{P(0)} \gamma_j
$$

$$
= (R_j(T) - s_j(T))D_j(T) + s_j(T)\gamma_j - (R_j(T) + q_j(T) - s_j(T))(D_j(T) - \gamma_j)^+ - p_j(0) \frac{P(T)}{P(0)} \gamma_j
$$

with $\gamma_j$ the corresponding operational decision for product $j$, $R_j$ is the unit retail price, $s_j$ is the salvage value of unsold unit, $q_j$ is the backordered cost per unit of unsatisfied demand, $p_j$ is the unit purchase price, and $P(t)$ is the price of the financial asset used as numeraire (or accounting). Notice that the purchasing occurs at time 0 and the retail activities is realized
at time $T$.

Under an economy with monetary inflation, we can expect that both price and demand will be affected by the inflation index. In particular, we consider a product whose demand depends on the level of the inflation index, and nominal prices of goods increase with the index. The model we have for the price is that for any time $t$, the price equals a fundamental price multiplied by the inflation index. That is, for $j = 1, \ldots, N$:

$$
\begin{align*}
R_j(t) &= R_j(0)I(t) \\
p_j(t) &= p_j(0)I(t) \\
s_j(t) &= s_j(0)I(t) \\
q_j(t) &= q_j(0)I(t)
\end{align*}
$$

where $R_j(0), p_j(0), s_j(0)$ and $q_j(0)$ are constants satisfying $R_j(0) > \frac{p(0)}{\Pi_{TIPS(0,T)}} > s_j(0)$. We further assume that the demand is a power function of inflation linked price:

$$
D_j(t) = a_j e^{-b_j \log R_j(t) + c_j B_j(t)}
$$

with constants $a_j > 0$ and $b_j, c_j \in \mathbb{R}$, and $B_j(t)$ is the jth element of $B(t) = (B_1(t), \ldots, B_N(t))$, an $N$-dimensional Brownian motion independent of $W(t)$ denoting the non-financial noise. By our assumption, there are two sources of randomness in the demand process: risky financial asset$^3$ and a non-financial noise. As stated in model setup, the filtration $\mathcal{F}_t^W \otimes \mathcal{E}_t$, $t \in [0, T^*]$ represents the evolution of observable information in the model. We suppose that the non-financial noise $B_j(t), j = 1, \ldots, N$ are considered observable. For example, $B_j(t)$ can be macroeconomic indicators such as rate of unemployment and industry situation.

The total payoff function of the corporation is to sum over all products:

$$
H_T(\gamma) = \sum_{j=1}^{N} H_T(\gamma_j)
$$

### 2.2.2 Hedging in the financial market

Consider a market consisting of a riskless and a risky asset with prices $P(t)$ and $S(t)$. We express all value and price processes in terms of the riskless asset $P$ as numeraire. In particular, in numeraire $P$, the price of the riskless asset $P$ itself is equal to 1, and the price of the risky asset $S$ is given by $X(t) = \frac{S(t)}{P(t)}$. We assume that $X(t)$ satisfies the SDE:

$$
\frac{dX(t)}{X(t)} = \mu(t)dt + \sigma(t)dW(t)
$$

$^3$the CPI index
where $\mu(t)$ and $\sigma(t)$ are given in proposition 3.

We further assume that the mean-variance trade-off $\eta(t) := \mu(t)/\sigma(t)$ is a bounded and deterministic function. In our application, we take $P(t) = P_{TIPS}(t, T)$ and $S(t) = P_n(t, T)$, so inflation-adjusted time $T$-dollars are interpreted as riskless asset and nominal time $T$-dollars as the risky asset.

With everything expressed in inflation adjusted dollars, the corresponding payoff in discounted units is

$$H^D_T(\gamma_j) = \frac{H_T(\gamma_j)}{P_{TIPS}(T, T)} = (R_j(0) - s_j(0))D_j(T) - (R_j(0) + q_j(0) - s_j(0))(D_j(T) - \gamma_j)^+$$

$$+ s_j(0)\gamma_j - p_j(0)\frac{\gamma_j}{P_{TIPS}(0, T)} \tag{18}$$

where we used $P_{TIPS}(T, T) = I(T)$ and parameters defined in (17).

The total discounted payoff is

$$H^D_T(\gamma) = \sum_{j=1}^n H^D_T(\gamma_j)$$

Define the set of self-financing trading strategies $\Theta$ to be the collection of $F^W_0 \otimes \mathcal{E}$-predictable processes $(\theta_t)_{0 \leq t \leq T}$ such that

$$E\left[\int_0^T \theta_t^2 X(t)^2 dt\right] < \infty \tag{19}$$

$\theta_t$ denotes the number of shares in the risky asset $X(t)$ held at time $t$. The (discounted) gain process $G_t(\theta)$ associated with trading strategy $\theta \in \Theta$ is defined by

$$G_t(\theta) := \int_0^t \theta_s dX(s) \quad \text{for all } t \in [0, T]$$

Consider a risk-averse nonfinancial corporation that operates during $[0, T]$. It earns discounted profit $H^D_T$ which depends on an operating strategy $\gamma \in \Gamma$, and gains $G_T(\theta)$ from trading in the financial market. We let $\Gamma$ be the set of $F^W_0 \otimes \mathcal{E}$-predictable policies $\gamma = (\gamma_1, \ldots, \gamma_N)$ with $N$ components. $H^D_T$ is an $F^W_T \otimes \mathcal{E}_T$-measurable random variable. Since $\sigma(X(t) | 0 \leq t \leq T) \not\subseteq F^W_T \otimes \mathcal{E}_T$, the market is now incomplete. In other words, there is risk in the stochastic demands $D_j(t)$ (modeled by $B(t)$) which can not be hedged by trading in the financial market $X(t)$. 

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Starting with an initial wealth \( W_0 \), the corporation makes an operational decision and implements a self-financing hedging strategy. As the result of operational and financial activities, the final discounted wealth at time \( T \) will be

\[
Y_T^{(\gamma, \theta)} := \hat{W}_0 + H_T^D(\gamma) + G_T(\theta)
\]

where \( \hat{W}_0 = \frac{W_0}{P_{TIPS}(0, T)} \) is the discounted initial wealth. For fixed risk aversion parameter \( \kappa > 0 \), we are interested in the optimal solution of the problem

\[
U = \max_{\gamma \in \Gamma, \theta \in \Theta} \left( E \left[ Y_T^{(\gamma, \theta)} \right] - \kappa Var \left[ Y_T^{(\gamma, \theta)} \right] \right)
\]

Before we proceed to the main result of the paper, we would like to compare our setup with [1]. There are four main differences. First of all, we consider a multi-product problem while a single product model is explored in [1]. Secondly, different demand model is used. Compared to the linear demand model in section 3, [1], we have a nonlinear demand model which does not allow negative value, and characterize the impact of inflation. Thirdly, we will depart from criterion (20) and consider (26) as the dual criterion, and the later problem is the criterion considered in [1]. In the end, [1] develops solution for both complete and incomplete information model, according to the assumption that whether the non-financial noise is observable or not. We only assume that the non-financial noise is observable, which corresponds to the complete information scenario in [1].

3 Hedging of multiple products

In Caldentey and Haugh [1], the single product hedging problem has been discussed. However, such results cannot be extended to multiple products problem, since there are correlations between product’s payoff function. As a result, in multi-product inventory management problems, we are interested in separating the global optimization problem by individual products, hence simplify the problem to multiple single product problems.

In the previous section, the optimization problem (20) has been defined. It involves optimizing over operational and financial decisions. Instead of finding the optimal controls simultaneously, we first fix the operational control \( \gamma \in \Gamma \) and consider

\[
U^\gamma = \sup_{\theta \in \Theta} \left( E \left[ Y_T^{(\gamma, \theta)} \right] - \kappa Var \left[ Y_T^{(\gamma, \theta)} \right] \right)
\]

This problem can be reformulated as follows. Let

\[
B^\gamma(m) = \inf_{\theta \in \Theta} \left\{ Var \left[ Y_T^{(\gamma, \theta)} \right] \mid E \left[ Y_T^{(\gamma, \theta)} \right] = m \right\}, \quad \text{for each } m \in \mathbb{R}
\]

Then

\[
U^\gamma = \sup_{m \in \mathbb{R}} (m - \kappa B^\gamma(m))
\]
On the other hand, define
\[ A_\gamma^T(\lambda) = \inf_{\theta \in \Theta} E \left[ (Y_T^{(\gamma, \theta)} - \lambda)^2 \right] \quad \text{for each } \lambda \in \mathbb{R} \] (24)

We have the following theorem states that the auxiliary problem is conjugate to the mean-variance problem.

**Theorem 1** With \( A_\gamma^T(\lambda) \) and \( B^\gamma(m) \) defined as in (24) and (22), we have
\[ B^\gamma(m) = \sup_{\lambda \in \mathbb{R}} (A^\gamma(\lambda) - (m - \lambda)^2) \] (25)

and with \( \lambda_m \) the optimizer in (25), the optimal control in \( B^\gamma(m) \) is equal to the optimal control in \( A^\gamma(\lambda) \) with \( \lambda = \lambda_m \).

The proof can be found in the Appendix.

By theorem 1, to solve the optimization problem (21), it suffices to find the optimal solution of the dual problem \( A_\gamma^T(\lambda) \). It turns out that \( A_\gamma^T(\lambda) \) is the auxiliary process of the mean-variance hedging problem, which is a classical mathematical finance topic. We are going to introduce the mean-variance hedging problem in the follow section, and use it to solve the dual problem (21).

### 3.1 Mean-variance hedging problem and Föllmer-Schweizer decomposition

In this section, we are going to consider an alternative objective function to the optimal hedging issue, which has been widely studied as the mean-variance hedging problem in mathematical finance. We are going to prove that the dual problem \( A_\gamma^T(\lambda) \) is in fact the auxiliary process of the mean-variance hedging problem, which can be explicitly expressed, and then show that the expression of \( B^\gamma(m) \) can be found correspondingly. Finally, we are going to state the separation result of the multi-product problem.

Instead of considering the optimization problem (20), the mean-variance hedging problem (24) arises from maximizing the expected quadratic utility of terminal wealth, where the utility function is defined as \( u(w) = w - lw^2 \). Indeed, the problem
\[ \max_{(\gamma, \theta) \in \Gamma \times \Theta} E[u(\hat{W}_0 + H_T^D(\gamma) + G_T(\theta))] \] (26)
is equivalently to
\[ \min_{(\gamma, \theta) \in \Gamma \times \Theta} E \left[ \left( \hat{W}_0 + H_T^D(\gamma) + G_T(\theta) - \lambda \right)^2 \right] \]

---

4 see Theorem 3 below
with \( \lambda = \frac{1}{2l} \). To solve this, we first fix \( \gamma \in \Gamma \) and consider the optimization problem

\[
\min_{\theta \in \Theta} E \left[ \left( \hat{W}_0 + H_T^D(\gamma) + G_T(\theta) - \lambda \right)^2 \right] \quad (27)
\]

Given the assumption that \( \eta(t) \) is a bounded and deterministic function, the solution of (27) can be found using the minimal equivalent martingale measure (MEMM), see Föllmer and Schweizer [5], defined by

\[
\frac{d\hat{P}}{dP} := \exp \left\{ \int_0^T \eta(t)dW(t) - \frac{1}{2} \int_0^T \eta^2(t)dt \right\} \quad (28)
\]

By Girsanov’s theorem, both \( X \) and \( B \) are square-integrable martingale under \( \hat{P} \). We will use \( \hat{E}[\cdot] \) to denote the expectation under \( \hat{P} \). The following theorem is the key result in mean-variance hedging. It has been established in a number of setups by various authors; we refer to [3] for a treatment of the mean-variance hedging problem in a general semimartingale model and a discussion of the literature on this problem. The version we are using here is due to Schweizer [9].

**Theorem 2** For any \( \mathcal{F}_T \)-measurable claim \( H_T^D(\gamma_j) \in \mathcal{L}^p(P) \), \( j = 1, \ldots, N \) for some \( p > 2 \), there is a hedging strategy, \( \vartheta^{(\gamma_j)} \), and a process \( \delta^{(\gamma_j)} \in \mathcal{L}^2(P) \), such that \( H_T^D(\gamma_j) \) admits the decomposition

\[
H_T^D(\gamma_j) = V^{(\gamma_j)}_0 + \int_0^T \vartheta^{(\gamma_j)}_dX(t) + \int_0^T \delta^{(\gamma_j)}_t dB_j(t) \quad (29)
\]

where \( V^{(\gamma_j)}_0 := \hat{E}[H_T^D(\gamma_j)] \).

As a result, \( H_T^D(\gamma) = \sum_{j=1}^N H_T^D(\gamma_j) \) admits the decomposition

\[
H_T^D(\gamma) = V^{(\gamma)}_0 + \int_0^T \vartheta^{(\gamma)}_dX(t) + \int_0^T \delta^{(\gamma)}_t dB(t) \quad (30)
\]

with

\[
V^{(\gamma)}_0 = \sum_{j=1}^N V^{(\gamma_j)}_0 \quad (31)
\]

\[
\vartheta^{(\gamma)}_t = \sum_{j=1}^N \vartheta^{(\gamma_j)}_t \quad (32)
\]

\[
\delta^{(\gamma)}_t = \sqrt{\sum_{j=1}^N \left( \delta^{(\gamma_j)}_t \right)^2} \quad (33)
\]

\[
B(t) = \int_0^t \frac{1}{\delta^{(\gamma)}_s} \sum_{j=1}^N \delta^{(\gamma_j)} s dB_j(s) \quad (34)
\]
with \( \overline{B}(t) \) a Brownian motion under \( P \) and \( \hat{P} \).

In addition, the optimal strategy, \( \theta^* \), that solves (27) is given by \( \theta^* = \Phi(G_t^*) \) where \( G_t^* \) solves the SDE

\[
dG_t^* = -\Phi(G_t^*)dX(t)
\]  

(35)

where \( G_0^* = 0 \), \( \Phi(G_t^*) = \psi_t^{(\gamma)} + \mu(t)/(\sigma(t)^2X(t))(V_t^{(\gamma)} + G_t^* + \hat{W}_0 - \lambda) \), and \( V_t^{(\gamma)} \) is the intrinsic value process defined by

\[
V_t^{(\gamma)} := \mathbb{E}[H^D_t(\gamma)|F_t] = V_0^{(\gamma)} + \int_0^t \vartheta_s^{(\gamma)} dX(s) + \int_0^t \delta_s^{(\gamma)} d\overline{B}(s)
\]  

(36)

The decomposition (36) is known as the Galtchouk-Kunita-Watanabe (GKW) decomposition of \( V_t^{(\gamma)} \) under \( \hat{P} \) with respect to \( X \).

**Remark:** The decomposition in the theorem is also known as the Föllmer-Schweizer decomposition of \( H^D_t(\gamma) \) with respect to the semimartingale \( X \). In particular, when the price of discounted risky asset \( X \) is a martingale, as in our model, the Föllmer-Schweizer decomposition coincides with the Galtchouk-Kunita-Watanabe (GKW) decomposition under \( P \).

The proof can be found in the Appendix.

Once we obtain the optimal hedging strategy for fixed \( \gamma \), the optimal operation strategy can be found by the auxiliary process, which is given in the following theorem

**Theorem 3** Define the auxiliary process

\[
A_t^{(\gamma)} := E[(V_t^{(\gamma)} + G_t^* + \hat{W}_0 - \lambda)^2]
\]  

(37)

and \( K_t = \int_0^t \eta(s)^2 ds \), then \( A_t^{(\gamma)} \) is given by

\[
A_T^{(\gamma)}(\lambda) = e^{-K_T} \left( (\hat{W}_0 + V_0^{(\gamma)} - \lambda)^2 + \int_0^T e^{K_s} E \left[ (\delta_s^{(\gamma)})^2 \right] ds \right)
\]  

(38)

\[
= e^{-K_T} \left( (\hat{W}_0 + V_0^{(\gamma)} - \lambda)^2 + \int_0^T e^{K_s} \sum_{j=1}^N E \left[ (\delta_s^{(\gamma_j)})^2 \right] ds \right)
\]  

(39)

By virtue of this result, the problem of finding the optimal operational decision reduces to finding the intrinsic value \( V_0^{(\gamma)} \) and the non-financial noise term \( \delta_t^{(\gamma)} \) in the decomposition (36), and then minimizing (38) over \( \gamma \).

On the other hand, we find that the auxiliary process \( A_T^{(\gamma)} \) is equal to the value in problem (24), since the optimizer exists, so we can replace the \( \inf \) and \( \sup \) with \( \min \) and \( \max \) in (24),
(21), (22) and (25). Moreover, we can find the optimizer $\lambda_{opt}$ and $m_{opt}$ in (25) and (23), and this will allow us to separate the multi-product problem by product.

The following theorem states the optimal control solution to the multi-product problem. It will allow us to separate the original optimization problem (20) by product, and can be considered the main result of this paper.

**Theorem 4** The optimizer and the corresponding optimal value of problem (25) is

$$\lambda_m = \frac{m - e^{-K_T}(\hat{W}_0 + V_0^{(\gamma)})}{1 - e^{-K_T}}$$

$$B^\gamma(m) = \frac{e^{-K_T}(\hat{W}_0 + V_0^{(\gamma)} - m)^2 + e^{-K_T} \int_0^T e^{K_s} \sum_{j=1}^N E[(\delta_\gamma^j)^2] \, ds}{1 - e^{-K_T}}$$

The optimizer $m_{opt}$ of problem (23) is given by

$$m_{opt} = \frac{1}{2\kappa} \frac{1 - e^{-K_T}}{e^{-K_T}} + \hat{W}_0 + V_0^{(\gamma)}$$

and the optimal value in problem (21) is

$$U^\gamma = \hat{W}_0 + \frac{1}{4\kappa}(e^{K_T} - 1) + V_0^{(\gamma)} - \kappa e^{-K_T} \int_0^T e^{K_s} \sum_{j=1}^N E[(\delta_\gamma^j)^2] \, ds$$

$$= \hat{W}_0 + \frac{1}{4\kappa}(e^{K_T} - 1) + \sum_{j=1}^N V_0^{(\gamma)} - \kappa e^{-K_T} \int_0^T e^{K_s} \sum_{j=1}^N E[(\delta_\gamma^j)^2] \, ds$$

Finally the optimal control $\gamma$ in (20) can be found by maximizing (43) over $\gamma$.

With this theorem, we finally achieved separation for the multi-product problem which is stated in the following corollary.

**Corollary 1** With $U^\gamma$ defined as in (21), the problem

$$\max_{\gamma} U^\gamma$$

is equivalent to solving

$$\max_{\gamma_j} \left( V_0^{(\gamma_j)} - \kappa e^{-K_T} \int_0^T e^{K_s} E[(\delta_\gamma^j)^2] \, ds \right)$$

for each $j = 1, \ldots, N$. 

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The following theorem proves that the problem above is well-defined.

**Theorem 5** There exist optimal solution $\gamma_j$ for problem (45).

The proof can be found in the Appendix.

Armed with the existence of the optimal operation strategy $\gamma_j$, problem (45) can be solved numerically after we obtain $V_t^{(\gamma_j)}$ and $\delta_t^{(\gamma_j)}$ via the F-S decomposition. The following theorem provides this decomposition in explicit form.

**Theorem 6** The intrinsic value of discounted profit $V_t^{(\gamma_j)} = \hat{E}[H_T^D(\gamma_j)|F_t]$ from product $j$ is given by

$$V_t^{(\gamma_j)} = \left[ (R_j(0) - s_j(0))N_t^{(j)} + s_j(0)\gamma_j - (R_j(0) + q_j(0) - s_j(0))M_t^{(\gamma_j)} - \frac{P_j(0)\gamma_j}{P_{TIPS}(0,T)} \right]$$

for all $t \in [0,T]$, and it has the Galtchouk-Kunita-Watanabe decomposition

$$V_t^{(\gamma_j)} = \hat{E}[H_T^D(\gamma_j)|F_t] = V_0^{(\gamma_j)} + \int_0^t \dot{\psi}_s^{(\gamma_j)} dX(s) + \int_0^t \dot{\delta}_s^{(\gamma_j)} dB_j(s)$$

where

$$\dot{\psi}_t^{(\gamma_j)} = \frac{b_j}{X(t)} J_j(t) L^{(\gamma_j)}(t)$$

$$\dot{\delta}_t^{(\gamma_j)} = c_j J_j(t) L^{(\gamma_j)}(t)$$

$$J_j(t) = a_j \exp(-b_j \log R_j(0) + b_j \log X(t) + c_j B_j(t))$$

$$L^{(\gamma_j)}(t) = -(R_j(0) + q_j(0) - s_j(0))F_j(t)\Phi \left( \frac{\mu_{z}^{j}(t) + \log \frac{j(t)}{\gamma_j}}{\sigma_{z}^{j}(t)} + \sigma_{z}^{j}(t) \right)$$

$$+(R_j(0) - s_j(0))F_j(t)$$

$$F_j(t) = e^{\mu_{z}^{j}(t) + \frac{1}{2}\sigma_{z}^{j}(t)^2}$$

$$M^{(\gamma_j)}(t) = J_j(t)F_j(t)\Phi \left( \frac{\mu_{z}^{j}(t) + \log \frac{j(t)}{\gamma_j}}{\sigma_{z}^{j}(t)} + \sigma_{z}^{j}(t) \right)$$

$$- \gamma_j \Phi \left( \frac{\mu_{z}^{j}(t) + \log \frac{j(t)}{\gamma_j}}{\sigma_{z}^{j}(t)} \right)$$

$$N_t^{(j)} = J_j(t)F_j(t)$$

and $\mu_{z}^{j}(t) = b_j \mu_Y(t)$, $\sigma_{z}^{j}(t)^2 = b_j^2 \sigma_Y^2(t) + c_j^2 (T - t)$ with
\[ \sigma^2_Y(t) = \sum_{k=n,r} \sigma_k^2 \left( \frac{1}{2a_k} (1 - e^{-2a_k(T-t)}) - \frac{2}{a_k} (1 - e^{-a_k(T-t)}) + T - t \right) + \sigma_I^2(T-t) \]

\[ -2\rho_{nr} \sigma_n \sigma_r \frac{1}{a_n + a_r} \left[ (1 - e^{-(a_n+a_r)(T-t)}) \right] - \sum_{k=n,r} \frac{1}{a_k} (1 - e^{-a_k(T-t)}) + (T-t) \]

\[ \frac{1}{a_k} (1 - e^{-a_k(T-t)}) -(T-t) \]

\[ = \int_t^T \sigma(s)^2 ds \]

\[ \mu_Y(t) = -\frac{1}{2} \sigma_Y^2(t) \]

and \( \Phi(\cdot), \phi(\cdot) \) is the CDF and pdf of standard normal random variable respectively.

The proof can be found in the Appendix.

**Remark:** Notice that in (45) the drifts of the financial assets only enters via the risk aversion parameter \( \kappa e^{-kT} \) and \( e^{Ks} \). As we stated before, the evaluation of drifts of financial assets is complicated, while in problem (45) it corresponds to evaluating the risk aversion, which is also a complicated issue in econometrics.

The optimal hedging strategy \( \theta^* = \theta^*(\gamma) \) in (20) can now be computed by solving (21) with the optimal \( \gamma = (\gamma_1, \ldots, \gamma_N) \). This is achieved by using the duality in theorem 1 and the optimal control given in theorem 2. Combining these results, we find that \( \theta^*(\gamma) \) is given in feedback form by

\[ \theta_t^*(\gamma) = -\left( \varphi_t^{(\gamma)} + \mu(t)/\sigma(t)^2 X(t) \right) \left( V_t^{(\gamma)} + G_t^* + \tilde{W}_0 - \frac{1}{2\kappa} e^{K_T} - V_0^{(\gamma)} \right) \]

where \( G_t^* \) is the solution of the SDE:

\[ dG_t^* = -\left[ \varphi_t^{(\gamma)} + \mu(t)/\sigma(t)^2 X(t) \right] \left( V_t^{(\gamma)} + G_t^* - \frac{1}{2\kappa} e^{K_T} - V_0^{(\gamma)} \right) dX(t) \]

where \( \varphi^{(\gamma)} = \sum_{j=1}^N \varphi^{(\gamma_j)} \). The solution of this SDE can be expressed in terms of a stochastic integral with respect to \( X \). We will discuss how to solve it numerically in the next section.

### 3.2 Approximation of mean-variance hedging strategy

According to theorem 2, we need to solve a stochastic differential equation in order to obtain the optimal hedging strategy, which requires numerical techniques for stochastic differential
equations. In practice, a strategy which can be quickly and easily calculated is desirable. Hence we introduce an approximation hedging strategy.

We are going to suppress the product index \( j \) from this point, the argument will apply to any product. First recall that the optimal mean-variance hedging gain process for the discounted problem satisfies the stochastic differential equation:

\[
dG^*_t = - \left[ \varphi_t^*(\gamma) + \mu_t/(\sigma(t)^2 X_t) (V_t(\gamma) + G^*_t + \hat{W}_0 - \frac{1}{2\kappa} e^{K_T} - V_0^\gamma) \right] dX_t \tag{60}
\]

with \( G^*_0 = 0 \). The optimal hedging strategy is then given by

\[
\theta_t^*(\gamma) = - \left( \varphi_t^*(\gamma) + \mu_t/(\sigma(t)^2 X_t) (V_t(\gamma) + G^*_t + \hat{W}_0 - \frac{1}{2\kappa} e^{K_T} - V_0^\gamma) \right) \tag{61}
\]

To avoid solving an SDE for each step, we propose an approximation hedging strategy. The following theorem gives the approximation and evaluates the quality of the approximation by considering the expected squared difference of the gain processes.

**Theorem 7** Consider the approximation strategy

\[
\tilde{\theta}_t(\gamma) = - \left( \varphi_t^*(\gamma) + \mu_t/(\sigma(t)^2 X_t) (V_t(\gamma) + G^*_t + \hat{W}_0 - \frac{1}{2\kappa} e^{K_T} - V_0^\gamma) \right) \tag{62}
\]

and the gain process under the approximation strategy

\[
\tilde{G}_t = \int_0^t \tilde{\theta}_s(\gamma) dX(s) \tag{63}
\]

If \( |\eta(t)| \leq \epsilon_1, |\sigma(t)| \leq \epsilon_2 \), we have

\[
E[(G^*_t - \tilde{G}_t)^2] < \epsilon_1^2 (1 + t\epsilon_1^2) t^2 \Upsilon_t^* \tag{64}
\]

where \( \Upsilon_t^* = \sup_{u \in [0,t]} E[u^2] \).

The proof can be found in the Appendix.

In practice, our conjecture is that the optimal hedging strategy can be approximated with smaller error via a forward finite difference method. Consider \( m \) discrete time points in \([0,T]\). For any \( i = 1, \ldots, m \) we are interested in solving

\[
G^*_t - G^*_{t-1} = - \left[ \varphi_{ti}^*(\gamma) + \mu_{ti}/(\sigma(t)^2 X_{ti}) (V_{ti}(\gamma) + G^*_{ti-1} + \hat{W}_0 - \frac{1}{2\kappa} e^{K_T} - V_0(\gamma)) \right] (X(t_i) - X(t_{i-1})) \tag{65}
\]

the difference between this approach and (60) is that the observed gain process of the optimal hedging strategy is used to replace \( G^*_t \) on the right hand side of the SDE. This reduces the difficulties of solving nonhomogeneous linear SDE (60). It also worth mentioning that the numerical approach is a two-dimensional procedure which yields the optimal hedging strategy \( \theta_t^* \) and gain process \( G^*_t \) simultaneously. In fact, we can obtain both values as in (61) for each time step.
3.3 Comparison of optimal inventory decisions of hedging and non-hedging

In this section, we compare the optimal inventory decision when the financial instrument for hedging is not available with the hedging case. We will show that if there is no inflation-protected financial instrument, in a high inflation economy, the investor tends to purchase as much inventory as possible to preserve the value. In other words, inflation distorts the inventory decision and causes a malinvestment. On the other hand, hedging enables the operational department to make the correct decision while the financial department takes care of inflation.

In the following theorem, we consider a single product case without loss of generality.

**Theorem 8** There exist a critical value $\mu_I^*$ such that for all $\mu_I > \mu_I^*$, the optimal inventory decision with hedging is less than the optimal inventory decision without hedging, that is, $\gamma^*_H < \gamma^*_NH$.

**Proof.** If there is no inflation-protected financial instrument, at time 0, the corporation purchases inventory with unit price $p(0)$, and the riskless asset in this case is bank account, so at $T$, the present value of purchase cost is $p(0)\gamma B_n(0)$. On contrary, with hedging opportunity, the riskless asset we consider in (18) is the TIPS. As a result, the non-hedging discounted payoff is

$$H_D^T(\gamma) = (R(0) - s(0))D(T) - (R(0) + q(0) - s(0))(D(T) - \gamma)^+ + s(0)\gamma - p(0)\frac{\gamma B_n(0)}{P_{TIPS}(T,T)}$$

and the objective function under this case is

$$\max_{\gamma} \left( E[H_D^T] - \kappa Var[H_D^T] \right)$$

As proved in [12], theorem 2.4, the variance function is bounded in $\gamma \in [0, +\infty)$, also notice that

$$\lim_{\mu_I \to +\infty} E[\frac{B_n(0)}{P_{TIPS}(T,T)}] = 0$$

hence $E[H_D^T]$ is an increasing function of $\gamma$ for $\mu_I$ sufficiently large. That is, the optimal inventory decision without hedging $\gamma^*_NH \to \infty$ as $\mu_I \to \infty$.

On the other hand, we have proved in theorem 5 that the optimizer $\gamma^*_H$ of problem (45) exists and is finite. As a result, with other parameters the same, as $\mu_I$ increases, the optimal inventory decision $\gamma^*_NH$ increases to $+\infty$ while $\gamma^*_H$ remains unchanged, hence there is a critical value $\mu_I^*$ such that for all $\mu_I > \mu_I^*$, $\gamma^*_H < \gamma^*_NH$. 

$\blacksquare$
4 Numerical example

In this section we demonstrate the multiproduct separation result via a numerical example. In particular, we are interested in demonstrating the impact of hedging on demands which correlate the inflation index differently.

The following inventory parameter values were used for the example.

\[ R_0 = \$600, \ p_0 = \$500, \ s_0 = \$200, \ b_0 = \$300, \ \kappa = 0.2, \ T = 2 \text{ years} \]

We used the calibration result in [7] for the following financial market parameter values.

\[ a_n = 0.013398, \ a_r = 0.014339, \ \sigma_n = 0.0566, \ \sigma_r = 0.0299 \]
\[ \rho_{nI} = 0.01482, \ \rho_{nr} = 0.06084, \ \rho_{rI} = -0.032127 \]

Furthermore, we assume

\[ \alpha_n = 0.1, \ \alpha_r = 0.02, \ r_n(0) = 0.2, \ r_r(0) = 0, I_0 = 1 \]

Finally, we used the CPI parameter \( \sigma_I = 0.1874 \).

We consider two products with different correlations with CPI, that is \( b_1 > b_2 \), with \( b_1 = 0.2, \ b_2 = 0.9 \). Moreover, to demonstrate the hedging effect, we require that at time \( T \), the realized demands have the same distribution, which leads to the same optimal inventory decision without hedging. Practically, this can be done by fixing \( a_1, b_1, b_2 \) and \( c_1 \), and calculate \( a_2, c_2 \) via

\[
\begin{align*}
a_2 &= \exp(\log a_1 - (b_1 - b_2) \log R(0) + (b_1 - b_2) \mu_Y(0)) \\
c_2 &= \sqrt{(b_1^2 - b_2^2) \sigma_Y^2(0)} / T + c_1^2
\end{align*}
\]

We varied the drift of CPI by changing \( \lambda_I \). To mimic a high inflation economy, small \( \lambda_I \) value is required. The observation from the experiment is that

- For both products, and for sufficiently high inflation, the optimal inventory decision with hedging becomes smaller than the one without the hedging.
- The impact on the optimal inventory decision as inflation increases is weaker in product 1 compared to product 2.

The first observation verifies theorem 8, it shows that the malinvestement will occur under a high inflation economy while the application of hedging avoids it. The second observation shows that the as the inflation level changes, the optimal inventory decision with hedging changes, moreover, the product with higher dependence on inflation has more significant change on the optimal inventory decision.
<table>
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<th>Hedging</th>
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Table 1: Optimal inventory decision and objective function value for different products.

References


5 Appendix

Proof of proposition 1.

Proof. According to the fundamental theorem of asset pricing, any finite subfamily of the market is arbitrage-free if there exists $Q \approx P$ such that all $\frac{P_n(t,T)}{B_n(t)}$ and all $\frac{P_{TIPS}(t,T)}{B_n(t)}$ are $Q$-local martingales.

Suppose there exists such $Q$ as above. By Itô’s representation theorem and Girsanov’s theorem, there exist predictable processes $\lambda_k(t), k \in \{n, r, I\}$ such that

$$d\tilde{W}_k(t) = dW_k(t) - \lambda_k(t)dt, \quad k \in \{n, r, I\}$$

are $Q$-Brownian motions. Itô’s lemma yields

$$d\left(\frac{P_n(t,T)}{B_n(t)}\right) = \frac{1}{B_n(t)} (dP_n(t,T) - P_n(t,T)r_n(t)dt)$$

$$= \frac{P_n(t,T)}{B_n(t)} \left( \left[ \int_t^T \alpha_n(t,s)ds + \frac{1}{2} \left( \int_t^T \sigma_n(t,s)ds \right)^2 + f_n(t,t) - r_n(t) \right] dt 
- \left[ \int_t^T \sigma_n(t,s)ds \right] (d\tilde{W}_n(t) + \lambda_n(t)dt) \right)$$

The processes are $Q$-local martingale for all maturities $T \geq t$ if and only if the drifts vanish, i.e.

$$\alpha_n(t,T) = \sigma_n(t,T) \left( \int_t^T \sigma_n(t,s)ds - \lambda_n(t) \right)$$

noticing that $r_n(t) = f_n(t,t)$. 


Similarly, Itô’s lemma also yields

\[
d \left( \frac{P_{TIPS}(t,T)}{B_n(t)} \right) = \frac{P_{TIPS}(t,T)}{B_n(t)} \left[ -\int_t^T \alpha_r(t,s) ds + \frac{1}{2} \left( \int_t^T \sigma_r(t,s) ds \right)^2 + f_r(t,t) - r_n(t) \right] dt
- \left[ \int_t^T \sigma_r(t,s) ds \right] \left( \tilde{d}W_r(t) + \lambda_r(t) dt \right) + \sigma_I(t) \left( d\tilde{W}_r(t) + \lambda_I(t) dt \right)
+ \mu_I(t) dt - \int_t^T \sigma_r(t,s) ds \cdot \sigma_I(t) \rho_{rI} dt
\]

And the processes are \( Q \)-local martingales for all \( T \geq t \) if and only if the drifts vanish, i.e.

\[
\alpha_r(t,T) = \sigma_r(t,T) \left( \int_t^T \sigma_r(t,s) ds - \sigma_I(t) \rho_{rI} - \lambda_r(t) \right)
\]
\[
\mu_I(t) = r_n(t) - r_r(t) - \sigma_I(t) \lambda_I(t)
\]

\[\blacksquare\]

**Proof of proposition 3.**

*Proof.* We discount the nominal bond by TIPS, by Itô’s lemma and proposition 2, the discounted nominal bond price is given by

\[
d \left( \frac{P_a(t,T)}{P_{TIPS}(t,T)} \right) = \frac{P_a(t,T)}{P_{TIPS}(t,T)} \left[ \frac{\sigma_n}{a_n} \left( e^{-a_n(T-t)} - 1 \right) d\tilde{W}_n(t) - \sigma_I d\tilde{W}_I(t) - \frac{\sigma_r}{a_r} \left( e^{-a_r(T-t)} - 1 \right) d\tilde{W}_r(t) \right]
- \left[ \rho_n \frac{\sigma_n}{a_n} \left( e^{-a_n(T-t)} - 1 \right) + \rho_n \frac{\sigma_n \sigma_r}{a_n a_r} \left( e^{-a_r(T-t)} - 1 \right) \left( e^{-a_n(T-t)} - 1 \right) - \sigma_I^2 t \right]
- \frac{\sigma_r^2}{a_r^2} \left( e^{-a_r(T-t)} - 1 \right)^2 dt
\]

\[
= \left[ \frac{\sigma_n}{a_n} \left( e^{-a_n(T-t)} - 1 \right) d\tilde{W}_n(t) - \sigma_I d\tilde{W}_I(t) - \frac{\sigma_r}{a_r} \left( e^{-a_r(T-t)} - 1 \right) d\tilde{W}_r(t) \right]
- \left[ \lambda_n(t) \frac{\sigma_n}{a_n} \left( e^{-a_n(T-t)} - 1 \right) - \lambda_I(t) \sigma_I - \lambda_r(t) \frac{\sigma_r}{a_r} \left( e^{-a_r(T-t)} - 1 \right) \right] dt
- \left[ \rho_n \sigma_I \frac{\sigma_n}{a_n} \left( e^{-a_n(T-t)} - 1 \right) + \rho_n \frac{\sigma_n \sigma_r}{a_n a_r} \left( e^{-a_r(T-t)} - 1 \right) \left( e^{-a_n(T-t)} - 1 \right) \right]
- \frac{\sigma_I^2}{a_r^2} \left( e^{-a_r(T-t)} - 1 \right)^2 dt
\]
Let
\[ \mu(t) = -\lambda_n(t) \frac{\sigma_n}{a_n} (e^{-a_n(T-t)} - 1) + \sigma_I \lambda_I(t) + \frac{\sigma_r}{a_r} (e^{-a_r(T-t)} - 1) \lambda_r(t) \]
\[ -\rho_n I \frac{\sigma_n}{a_n} (e^{-a_n(T-t)} - 1) - \rho_n r \frac{\sigma_n \sigma_I}{a_n a_r} (e^{-a_r(T-t)} - 1)(e^{-a_n(T-t)} - 1) + \sigma_I^2 + \frac{\sigma_r^2}{a_r^2} (e^{-a_r(T-t)} - 1)^2 \]
\[ \sigma(t)^2 = \frac{\sigma_n^2}{a_n^2} (e^{-a_n(T-t)} - 1)^2 + \sigma_I^2 + \frac{\sigma_r^2}{a_r^2} (e^{-a_r(T-t)} - 1)^2 \]
\[ -2 \rho_n I \frac{\sigma_n \sigma_I}{a_n} (e^{-a_n(T-t)} - 1) + 2 \rho_n r \frac{\sigma_I}{a_r} (e^{-a_r(T-t)} - 1) - 2 \rho_n r \frac{\sigma_n \sigma_I}{a_n a_r} (e^{-a_n(T-t)} - 1)(e^{-a_r(T-t)} - 1) \]
and
\[ W(t) = \int_0^t \frac{1}{\sigma(s)} \left( \frac{\sigma_n}{a_n} (e^{-a_n(T-s)} - 1) dW_n(s) - \sigma_I dW_I(s) - \frac{\sigma_r}{a_r} (e^{-a_r(T-s)} - 1) dW_r(s) \right) \]

Then \( W(t) \) is a \( P \)-Brownian motion. Hence we have rewritten the discounted process \( X(t) \) with respect to a one-dimension \( P \)-Brownian motion. \( \blacksquare \)

**Proof of theorem 1**

*Proof.* Recall that
\[ A^{(\gamma)}(\lambda) = \inf_{\theta \in \Theta} E \left[ (Y_T^{(\gamma,\theta)} - \lambda)^2 \right] \]
\[ B^{(\gamma)}(m) = \inf_{\theta \in \Theta} \left\{ \text{Var} \left[ Y_T^{(\gamma,\theta)} \right] : E \left[ Y_T^{(\gamma,\theta)} \right] = m \right\} , \ m \in \mathbb{R} \]

(67)

(68)

We want to prove that
\[ A^{(\gamma)}(\lambda) = \inf_m \left[ B^{(\gamma)}(m) + (m - \lambda)^2 \right] \ \lambda \in \mathbb{R}, \]
\[ B^{(\gamma)}(m) = \sup_{\lambda} \left[ A^{(\gamma)}(\lambda) - (m - \lambda)^2 \right] , \ m \in \mathbb{R} \]

(69)

(70)

and \( \forall \ m \in \mathbb{R} \), the optimal control of \( B^{(\gamma)}(m) \) is equal to the optimal control in (70).

Notice
\[ E[(Y_T^{\lambda,\theta} - \lambda)^2] = \text{Var}[Y_T^{(\gamma,\theta)}] + (E[Y_T^{(\gamma,\theta)}] - \lambda)^2 \]

(71)

By definition of \( B^{(\gamma)}(m) \), for each \( \epsilon > 0 \) we can find \( \theta^* \in \Theta \) with controlled diffusion \( Y^{\gamma,\theta^*} \), such that \( E[Y_T^{\gamma,\theta^*}] = m \) and \( \text{Var}[Y_T^{\gamma,\theta^*}] \leq B^{(\gamma)}(m) + \epsilon \) i.e.
\[ E[(Y_T^{\gamma,\theta^*} - \lambda)^2] \leq B^{(\gamma)}(m) + (m - \lambda)^2 + \epsilon \]

(72)

and hence
\[ A^{(\gamma)}(\lambda) = \inf_{\theta \in \Theta} E[(Y_T^{\gamma,\theta^*} - \lambda)^2] \leq B^{(\gamma)}(m) + (m - \lambda)^2 \ \forall m, \lambda \in \mathbb{R} \]

(73)
On the other hand, for $\lambda \in \mathbb{R}$, let $\hat{\theta}_{\lambda} \in \Theta$ with controlled diffusion $\hat{Y}_T^{\gamma, \theta, \lambda}$, and optimal control for $A^{\gamma}(\lambda)$.

Set $m_{\lambda} = E[\hat{Y}_T^{\gamma, \theta, \lambda}]$.

\[
A^{\gamma}(\lambda) = Var[\hat{Y}_T^{\gamma, \theta, \lambda}] + (m_{\lambda} - \lambda)^2 \geq B^{\gamma}(m_{\lambda}) + (m_{\lambda} - \lambda)^2 \tag{74}
\]

Combine (73) and (74)

\[
A^{\gamma}(\lambda) = \inf_m [B^{\gamma}(m) + (m - \lambda)^2] = B^{\gamma}(m_{\lambda}) + (m_{\lambda} - \lambda)^2 \tag{75}
\]

and $\hat{\theta}_{\lambda}$ is the solution to $B^{\gamma}(m_{\lambda})$.

Also, since $X \mapsto Var[X]$ is convex in $X$, the function $B^{\gamma}(m)$ is convex in $m$, and since

\[
\frac{(\lambda^2 - A^{\gamma}(\lambda))}{2} = \sup_m \left[ m\lambda - \frac{B^{\gamma}(m) + m^2}{2} \right] \tag{77}
\]

the function $\lambda \mapsto \frac{\lambda^2 - A^{\gamma}(\lambda)}{2}$ is the Fenchel-Legendre transform of the convex function $m \mapsto \frac{(B^{\gamma}(m) + m^2)}{2}$. We then have the duality relation

\[
\frac{(B^{\gamma}(m) + m^2)}{2} = \sup_\lambda \left[ m\lambda - \frac{(\lambda^2 - A^{\gamma}(\lambda))}{2} \right] \tag{78}
\]

and hence (70):

\[
B^{\gamma}(m) = \sup_\lambda [A^{\gamma}(\lambda) - (m - \lambda)^2] \tag{79}
\]

Finally, $\forall m \in \mathbb{R}$, let $\lambda_m \in \mathbb{R}$ be the argument maximum of $B^{\gamma}(m)$ in (70), then $m$ is an argument minimum of $A^{\gamma}(\lambda)$ in (70). Since

\[
m \mapsto B^{\gamma}(m) + (m - \lambda)^2
\]

is strictly convex,

this argument minimum is unique, so $m = m_{\lambda_m} = E[\hat{Y}_T^{\gamma, \theta, \lambda_m}]$. Hence

\[
B^{\gamma}(m) = A^{\gamma}(\lambda_m) + (m - \lambda_m)^2 = E[\hat{Y}_T^{\gamma, \theta, \lambda_m}]^2 + \left( E[\hat{Y}_T^{\gamma, \theta, \lambda_m}] - \lambda_m \right)^2 = Var[\hat{Y}_T^{\gamma, \theta, \lambda_m}]
\]

i.e. $\hat{\theta}_{\lambda_m}$ is a solution to $B^{\gamma}(m)$. ■
Proof of theorem 3.

Proof. Define the process $N_t^{(\gamma)} := (V_t^{(\gamma)} + G_t^* + \hat{W}_0 - \lambda)^2$. Using Itô’s lemma:

$$dN_t^{(\gamma)} = 2(V_t^{(\gamma)} + G_t^* + \hat{W}_0 - \lambda)(dV_t^{(\gamma)} + dG_t^*) + d < V^{(\gamma)} + G^*, V^{(\gamma)} + G^*>_t.$$  

Using the definition of $V_t^{(\gamma)}$ and $G_t^*$, we obtain

$$N_t^{(\gamma)} = N_0^{(\gamma)} + 2\int_0^t (V_s^{(\gamma)} + G_s^* + \hat{W}_0 - \lambda)(\delta_s^{(\gamma)} d\mathbb{B}(s) - \eta_s(V_s^{(\gamma)} + G_s^* + \hat{W}_0 - \lambda)dW(s))$$

Taking expectations, canceling all martingale terms, and using Fubini’s theorem with the deterministic mean-variance assumption, we obtain

$$A_t^{(\gamma)} = E[N_t^{(\gamma)}] = E[N_0^{(\gamma)}] + \int_0^t (E[\delta_s^{(\gamma)^2}] - \eta_s^2 A_s^{(\gamma)})ds$$

this implies the ODE:

$$\frac{d}{dt}A_t^{(\gamma)} + \eta_t^2 A_t^{(\gamma)} = E[\delta_t^{(\gamma)^2}]$$

Finally, use the integrating factor $K_t = \exp(\int_0^t \eta_s^2 ds)$ and the boundary condition $A_0^{(\gamma)} = (V_0^{(\gamma)} + \hat{W}_0 - \lambda)^2$ to obtain the desired result. ■

Proof of theorem 4.

Proof. By theorem 1 we have

$$B^\gamma(m) = \min_{\theta} \left\{ \text{Var}[\hat{W}_0 + H_T^\gamma + G_T(\theta)] \mid E[\hat{W}_0 + H_T^\gamma + G_T(\theta)] = m \right\}$$

$$= \max_{\lambda} (A^\gamma(\lambda) - (m - \lambda)^2)$$

The maximum is achieved for

$$0 = \frac{\partial}{\partial \lambda} A^\gamma(\lambda) + 2(m - \lambda)$$

where

$$A^\gamma(\lambda) = e^{-K_T} \left( (\hat{W}_0 + V_0^{(\gamma)} - \lambda)^2 + \int_0^T e^{K_u} \sum_{j=1}^n E[(\delta_u^{(\gamma)})^2] \right)$$
So the optimal condition is

\[-2e^{-K_T}(\hat{W}_0 + V_0^{(\gamma)} - \lambda) + 2(m - \lambda) = 0\]

\[\Leftrightarrow (1 - e^{-K_T})\lambda = m - e^{-K_T}(\hat{W}_0 + V_0^{(\gamma)})\]

\[\Leftrightarrow \lambda^* = \frac{m - e^{-K_T}(\hat{W}_0 + V_0^{(\gamma)})}{1 - e^{-K_T}}\]

Plug the result in the problem to obtain

\[B^{\gamma^*}(m) = A^\gamma(\lambda_m^*) - (m - \lambda_m^*)^2\]

\[= e^{-K_T} \left( \frac{W_0 + V_0^{(\gamma)} - m}{1 - e^{-K_T}} \right)^2 - \left( \frac{e^{-K_T}(\hat{W}_0 + V_0^{(\gamma)} - m)}{1 - e^{-K_T}} \right)^2 + e^{-K_T} \int_0^T e^{K_u} \sum_{j=1}^n E[(\delta_{\gamma,j}^{(\gamma)})^2]du\]

This proves the first part of the theorem.

To solve the problem

\[U^\gamma = \max_m (m - \kappa B^{\gamma}(m))\]

note that the first order condition is

\[1 - \kappa \frac{\partial}{\partial m} B^{\gamma}(m) = 0\]

\[\Leftrightarrow 2(m^* - \hat{W}_0 - V_0^{(\gamma)}) \frac{e^{-K_T}}{1 - e^{-K_T}} = \frac{1}{\kappa}\]

\[\Leftrightarrow m^* = \frac{1}{2\kappa} \frac{e^{-K_T}}{1 - e^{-K_T}} + \hat{W}_0 + V_0^{(\gamma)}\]

\[\Leftrightarrow (m^* - \hat{W}_0 - V_0^{(\gamma)})^2 = \frac{1}{4\kappa^2} \left( \frac{e^{-K_T}}{1 - e^{-K_T}} \right)^2\]

That is

\[U^\gamma = m^* - \kappa B^{\gamma}(m^*)\]

\[= \frac{1}{2\kappa} \frac{e^{-K_T}}{1 - e^{-K_T}} + \hat{W}_0 + V_0^{(\gamma)} - \frac{1}{4\kappa} \frac{e^{-K_T}}{1 - e^{-K_T}} - \kappa e^{-K_T} \int_0^T e^{K_u} \sum_{j=1}^n E[(\delta_{\gamma,j}^{(\gamma)})^2]du\]

\[= \hat{W}_0 + \sum_{j=1}^N V_0^{(\gamma,j)} + \frac{1}{4\kappa}(e^{K_T} - 1) - \kappa e^{-K_T} \int_0^T e^{K_u} \sum_{j=1}^n E[(\delta_{\gamma,j}^{(\gamma)})^2]du\]

This finishes the second part of the theorem. ■

Proof of theorem 5.

Proof. For simplicity, we suppress the dependence of product \(j\) for the proof.

The problem we consider is

\[
\max_{\gamma} \left( V_0^{(\gamma)} - \kappa e^{-K_T} \int_0^T e^{K_s} E[(\delta_{\gamma,s}^{(\gamma)})^2]ds \right) \tag{80}
\]
where \( V_0^{(\gamma)} \) and \( \delta_s^{(\gamma)} \) is given in theorem 6, both amounts are independent of \( \mu_I \).

The objective function above in general is not a concave function of \( \gamma \), we want to prove the existence by proving the concavity of \( V_0^{(\gamma)} \) and the variance part is bounded with respect to \( \gamma \).

First observe that

\[
\frac{dV_0^{(\gamma)}}{d\gamma} = s - \frac{p(0)}{P_{TIPS}(0, T)} + (R(0) + q(0) - s(0))\Phi\left(\frac{\mu_z(t) + \log \frac{J(t)}{\gamma}}{\sigma_z(0)}\right)
\]

\[
\frac{d^2V_0^{(\gamma)}}{d\gamma^2} = -(R(0) + q(0) - s(0)) \frac{1}{\gamma} \phi\left(\frac{\mu_z(0) + \log \frac{J(t)}{\gamma}}{\sigma_z(0)}\right) < 0
\]

where \( \phi(\cdot) \) and \( \Phi(\cdot) \) is the pdf and CDF of standard normal distribution, which proves that \( V_0^{(\gamma)} \) is a concave function of \( \gamma \).

Furthermore

\[
\lim_{\gamma \to 0} \delta_t^{(\gamma)} = cJ(t)(-(R(0) + q(0) - s(0))F(t) + (R(0) - s(0))F(t)) = -cJ(t)q(0)F(t)
\]

\[
\lim_{\gamma \to +\infty} \delta_t^{(\gamma)} = cJ(t) \left(-(R(0) + q(0) - s(0))F(t)\Phi\left(\frac{\mu_z(t)}{\sigma_z(t)}\right) + (R(0) - s(0))F(t)\right)
\]

notice that \( E[J(t)^2] < \infty \), which implies that \( E[(\delta_s^{(\gamma)})^2] \) is bounded for any time \( s \) as \( \gamma \in [0, +\infty) \).

We also need

\[
\lim_{\gamma \to +\infty} \frac{dV_0^{(\gamma)}}{d\gamma} = s - \frac{p(0)}{P_{TIPS}(0, T)} < 0
\]

where the last inequality is due to the assumption.

Up to this point, we have proved that problem (45)is composed from a concave function and a bounded function of \( \gamma \), hence we have proved that the problem (45) is well-defined, the optimizer \( \gamma^* \) exists and is finite.

The following Lemmas will be used in proving theorem 6. In particular, Lemma 1 is contributed to proof of Lemma 2, which will be the building blocks to the proof of theorem 6.

**Lemma 1** Under the MEMM \( \hat{P} \), the discounted nominal bond \( X(t) \) is a \( \hat{P} \)-local martingale with dynamic

\[
\frac{dX(t)}{X(t)} = \frac{\sigma_n}{a_n} \left(e^{-a_n(T-t)} - 1\right) d\hat{W}_n(t) - \sigma_I d\hat{W}_I(t) - \frac{\sigma_r}{a_r} \left(e^{-a_r(T-t)} - 1\right) d\hat{W}_r(t)
\]
where \( \hat{W}_k(t), k \in \{n, I, r\} \) are \( \hat{P} \)-Brownian motions defined as

\[
\begin{align*}
\hat{W}_I(t) &= \hat{W}_I(t) - \int_0^t \left( \sigma_I + \rho_{rI} \sigma_r \left( e^{-a_r(T-s)} - 1 \right) \right) ds \\
\hat{W}_n(t) &= \hat{W}_n(t) - \int_0^t \left( \rho_{nI} \sigma_I + \rho_{nr} \sigma_r \left( e^{-a_r(T-s)} - 1 \right) \right) ds \\
\hat{W}_r(t) &= \hat{W}_r(t) - \int_0^t \left( \sigma_I \rho_{rI} + \sigma_r \left( e^{-a_r(T-s)} - 1 \right) \right) ds
\end{align*}
\]

**Proof of lemma 1.**

**Proof.** First notice the dynamic of \( X(t) \) under risk-neutral measure \( Q \) is:

\[
\frac{dX(t)}{X(t)} = \left[ \frac{\sigma_n}{a_n} \left( e^{-a_n(T-t)} - 1 \right) d\hat{W}_n(t) - \sigma_I d\hat{W}_I(t) - \frac{\sigma_r}{a_r} \left( e^{-a_r(T-t)} - 1 \right) d\hat{W}_r(t) \right] \\
- \left[ \rho_{nI} \sigma_I \frac{\sigma_n}{a_n} \left( e^{-a_n(T-t)} - 1 \right) + \rho_{nr} \sigma_n \sigma_r \frac{\sigma_r}{a_r} \left( e^{-a_r(T-t)} - 1 \right) \left( e^{-a_n(T-t)} - 1 \right) \\
- \frac{\sigma_I^2}{a_I^2} + \frac{\sigma_r^2}{a_r^2} \left( e^{-a_r(T-t)} - 1 \right)^2 \right] dt
\]

According to Lévy’s theorem, \( \hat{W}_k(t), k \in \{n, I, r\} \) defined as in the lemma will be \( \hat{P} \) Brownian motions, and \( X(t) \) is a \( \hat{P} \)-local martingales. ■

**Lemma 2** Under the MEMM measure \( \hat{P} \), \( Y(t) = \log X(t) \) has dynamic

\[
dY(t) = d \log X(t) = \left[ \rho_{nI} \frac{\sigma_n}{a_n} \left( e^{-a_n(T-t)} - 1 \right) + \rho_{nr} \frac{\sigma_n \sigma_r}{a_n a_r} \left( e^{-a_n(T-t)} - 1 \right) \left( e^{-a_r(T-t)} - 1 \right) - \rho_{rI} \frac{\sigma_r \sigma_I}{a_r} \left( e^{-a_r(T-t)} - 1 \right) \\
- \frac{\sigma_I^2}{2} \left( e^{-a_n(T-t)} - 1 \right)^2 + \frac{\sigma_r^2}{2} \left( e^{-a_r(T-t)} - 1 \right)^2 \right] dt \\
+ \frac{\sigma_n}{a_n} \left( e^{-a_n(T-t)} - 1 \right) d\hat{W}_n(t) - \sigma_I d\hat{W}_I(t) - \frac{\sigma_r}{a_r} \left( e^{-a_r(T-t)} - 1 \right) d\hat{W}_r(t)
\]

So given \( \mathcal{F}_t \), \( Y(T) - Y(t) \) is a normally distributed random variable with mean \( \mu_Y(t) \) and variance \( \sigma_Y^2(t) \) defined in theorem 5.

Moreover, we have

\[
\sigma_Y^2(t) = \int_t^T \sigma(s)^2 ds \tag{81}
\]

**Proof of lemma 2.**

**Proof.** The dynamic of \( Y(t) \) is a direct consequence of Itô’s lemma.

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By Lemma 1:

\[ Y(T) - Y(t) = \log X(T) - \log X(t) \]

\[ = \int_t^T \left[ \rho_{tI} \sigma_{tI} \left( e^{-a_n(T-s)} - 1 \right) + \rho_{tr} \frac{\sigma_{tI} \sigma_{r}}{a_{n} a_{r}} \left( e^{-a_n(T-s)} - 1 \right) \left( e^{-a_{r}(T-s)} - 1 \right) \right. \]

\[ - \rho_{rt} \frac{\sigma_{r} \sigma_{I}}{a_{r}} \left( e^{-a_n(T-s)} - 1 \right) \left( \frac{1}{2} \sigma_{tI}^2 - \frac{1}{2} \sigma_{n}^2 \left( e^{-a_n(T-s)} - 1 \right)^2 - \frac{1}{2} \sigma_{r}^2 \left( e^{-a_{r}(T-s)} - 1 \right)^2 \right] ds \]

\[ + \int_t^T \left[ \sigma_{n} \left( e^{-a_n(T-s)} - 1 \right) d\tilde{W}_n(s) - \sigma_{I} d\tilde{W}_I(s) - \frac{\sigma_{r}}{a_{r}} \left( e^{-a_{r}(T-s)} - 1 \right) d\tilde{W}_r(s) \right] \]

which is a normally distributed random variable given \( F_t \) with mean \( \mu_Y(t) \) and variance \( \sigma^2_Y(t) \).

To prove (81), recall that by assumption

\[ \frac{dX(t)}{X(t)} = \mu(t) dt + \sigma(t) dW(t) \]

so

\[ dY(t) = d \log X(t) = (\mu(t) - \frac{1}{2} \sigma(t)^2) dt + \sigma(t) dW(t) \]

Girsanov theorem then implies (81).

**Proof of theorem 6.**

*Proof.* For each product \( j, j = 1, \ldots, N \), the intrinsic value of the discounted payoff is

\[ V^{(\gamma_j)}_t = \hat{E}[H^{(\gamma_j)} | F_t] \]

\[ = \hat{E} \left[ (R_j(0) - s_j(0)) D_j(T) + s_j(0) \gamma_j - (R_j(0) + q_j(0) - s_j(0))(D_j(T) - \gamma_j)^+ - \frac{p_j(0) \gamma_j}{P_{TIPS}(0, T)} \right] \]

Let

\[ M^{(\gamma_j)}_t = \hat{E}[D_j(T) - \gamma_j]^+ | F_t] \]

\[ N^{(\gamma_j)}_t = \hat{E}[D_j(T) | F_t] \]

Notice that the demand

\[ D_j(T) = a_j e^{-b_j \log R_j(T) + c_j B_j(T)} \]

\[ = a_j e^{-b_j \log R_j(0) - b_j \log |T| + c_j B_j(T)} \]

\[ = a_j \exp \left( -b_j \log R_j(0) - b_j \log \frac{P_{TIPS}(T, T)}{P_n(T, T)} + c_j B_j(T) \right) \]

\[ = a_j \exp (-b_j \log R_j(0) + b_j \log X(T) + c_j B_j(T)) \]

\[ 28 \]
Let \( J_j(t) = a_j e^{-b_j \log R_j(0) + b_j \log X(t) + c_j B_j(t)} \). Conditioning on \( \mathcal{F}_t \), we have

\[
M_t^{\gamma_j} = \hat{E}[(D_j(T) - \gamma_j)^+ | \mathcal{F}_t] = \hat{E} \left[ (a_j e^{-b_j \log R_j(0) + b_j \log X(t) + c_j B_j(T) - B_j(t)} - \gamma_j)^+ | \mathcal{F}_t \right] \\
= \hat{E} \left[ (J_j(t)e^{b_j (\log X(T) - \log X(t)) + c_j (B_j(T) - B_j(t))} - \gamma_j)^+ | \mathcal{F}_t \right]
\]

Let \( Z_j(t, T) = b_j (\log X(T) - \log X(t)) + c_j (B_j(T) - B_j(t)) = b_j (Y(T) - Y(t)) + c_j (B_j(T) - B_j(t)) \), which is a normal random variable with mean \( \mu_j^2(t) \) and variance \( \sigma_j^2(t)^2 \). By Lemma 2, we have \( \mu_j^2(t) = b_j \mu_j^2(t) \) and \( \sigma_j^2(t)^2 = b_j^2 \sigma_j^2(t)^2 + c_j^2 (T - t) = b_j^2 \int_0^t \sigma(s)^2 ds + c_j^2 (T - t) \) by Lemma 2.

We can calculate the conditional expectation

\[
M^{(\gamma_j)}(t) = J_j(t) F_j(t) \Phi(\frac{\mu_j^2(t) + \log \frac{J_j(t)}{\gamma_j}}{\sigma_j^2(t)} + \sigma_j^2(t)) - \gamma_j \Phi(\frac{\mu_j^2(t) + \log \frac{J_j(t)}{\gamma_j}}{\sigma_j^2(t)})
\]

with \( \Phi(\cdot) \) being the CDF of standard normal distribution and \( F_j(t) = e^{\mu_j^2(t)} + \frac{1}{2} \sigma_j^2(t)^2 \).

Similarly,

\[
N_t^{(j)} = \hat{E}[D_j(T)| \mathcal{F}_t] = J_j(t) \hat{E}[e^{Z_j(t,T)}| \mathcal{F}_t] = J_j(t) F_j(t)
\]

Hence the intrinsic value of discounted profit for product \( j \) is

\[
V_t^{(\gamma_j)} = (R_j(0) - s_j(0)) N_t^{(\gamma_j)} + s_j(0) \gamma_j - (R_j(0) + q_j(0) - s_j(0)) M_t^{(\gamma_j)} - p_j(0) \frac{\gamma_j}{\hat{P}_{\text{TIPS}}(0, T)}
\]

So the decomposition with respect to \( X(t) \) can be obtained by Itô’s formula and finally we have the desired result.

The following proposition is dedicated to the proof of theorem 6.

**Proposition 4** Let \( \nu_t = \psi_t^{(\gamma_j)}(t) X(t) + \eta(t) (V_t + \hat{W}_0 - \frac{1}{2\kappa} e^{K_T} - V_0^{(\gamma_j)}) \), assuming \( |\eta(t)| \leq \epsilon_1 \), \( |\sigma(t)| \leq \epsilon_2 \), then there exists \( \nu_t^{*} = \sup_{u \in [0, t]} E[\nu_u^2] \)

**Proof of proposition 4.**
Proof. First recall from the proof of theorem 5 that
\[ \psi_t(\gamma) = \frac{b}{X(t)} J(t) L(t) \]

Hence
\[ v_t = b\sigma(t)J(t)L(t) + \eta(t)(V_t + \hat{W}_0 - \frac{1}{2\kappa} e^{\kappa t} - V_0^{(\gamma)}) \]
\[ v_t^2 = b^2\sigma(t)^2 J^2(t)L^2(t) + \eta^2(t)(V_t + \hat{W}_0 - \frac{1}{2\kappa} e^{\kappa t} - V_0^{(\gamma)})^2 + 2b\sigma(t)J(t)L(t)\eta(t)(V_t + \hat{W}_0 - \frac{1}{2\kappa} e^{\kappa t} - V_0^{(\gamma)}) \]

We want to find the upper bound for \(|L(t)|\) and \(|V_t|\), notice we have assumption \(R(0) > p(0) > s(0)\), hence for any \(u \in [0, t)\),
\[ |L(u)| \leq |(R(0) + q(0) - s(0))F(u)\Phi_{z,u}(\log \frac{J(u)}{\gamma})| + |(R(0) - s(0))F(u)| \]
\[ \leq (R(0) + q(0) - s(0))F(u) + (R(0) - s(0))F(u) =: \overline{L}(u) \]

and
\[ |V(u)| \leq |(R(0) - s(0))J(u)F(u)| + |s(0)\gamma| + |(R(0) + q(0) - s(0))J(u)F(u)\Phi_{z,u}(\log \frac{J(u)}{\gamma} - \sigma_z(u)^2)| + |(R(0) + q(0) - s(0))\gamma \Phi_{z,u}(\log \frac{J(u)}{\gamma})| + |p(0)\frac{\gamma}{P_{TIPS}(0, T)}| \]
\[ \leq (R(0) - s(0))J(u)F(u) + s(0)\gamma + (R(0) + q(0) - s(0))(J(u)F(u) + \gamma) + p(0)\frac{\gamma}{P_{TIPS}(0, T)} =: \overline{V}(t) \]

Let
\[ \Delta_1(u) = (R(0) + q(0) - s(0))F(u) + (R(0) - s(0))F(u) \]
\[ \Delta_2(u) = (2R(0) + q(0) - 2s(0))F(u) \]
\[ \Delta_3(u) = s(0)\gamma + (R(0) + q(0) - s(0))\gamma + p(0)\gamma P_{TIPS}(0, T) + \hat{W}_0 + \frac{1}{2\kappa} e^{\kappa u} + V_0^{(\gamma)} \]

Then
\[ E[v_u^2] \leq E[b^2\sigma(u)^2 J^2(u)\overline{L}(u)^2 + \eta^2(u)\overline{V}(u)^2 + 2b|\sigma(u)\eta(u)J(u)\overline{L}(u)\overline{V}(u)] \]
\[ = b^2\sigma(u)^2 E[J^2(u)\Delta_1(u)^2] + \eta^2(u) E[(\Delta_2(u)J(u) + \Delta_3(u))^2] + 2|\sigma(u)\eta(u)J(u)\Delta_1(u)J(u) + \Delta_3(u))] \]
\[ = (b^2\sigma(u)^2 \Delta_1(u)^2 + \eta^2(u)\Delta_2(u)^2 + 2b\eta(u)\sigma(u)\Delta_1(u)\Delta_2(u)) E[J^2(u)] + 2(\eta(u)^2 \Delta_2(u)\Delta_3(u) + |b|\eta(u)\sigma(u)\Delta_1(u)\Delta_3(u)) E[J(u)] + \eta^2(u)\Delta_3(u)^2 \]
To calculate the expectation, it is suffice to calculate $E[J(u)]$ and $E[J^2(u)]$:

\[
E[J(u)] = ae^{-b \log R(0)+\mu_z(u)+\frac{1}{2}\sigma_z(u)^2} \\
E[J^2(u)] = a^2e^{-2b \log R(0)+2\mu_z(u)+2\sigma_z(u)^2}
\]

Also notice that for any $u \in [0,t]$

\[
\sigma_z(u)^2 = b^2\sigma_Y(u)^2 + c^2(T-u) = b^2 \int_0^u \sigma(s)^2 ds + c^2(T-u) \leq b^2 te_2^2 + c^2 T =: \sigma_z^*(t)^2 \\
\mu_z(u) = b \mu_Y(u) = -b \frac{1}{2}\sigma_Y(u)^2 \leq \frac{1}{2} \int_0^u \sigma(s)^2 ds \leq \frac{1}{2} te_2^2 =: \mu_z^*(t) \\
F(u) = e^{b \mu_z(u)+\frac{1}{2}\sigma_z(u)^2} \leq e^{b \mu_z^*(t)+\frac{1}{2}\sigma_z^*(t)^2} =: F^*(t) \\
\Delta_1(u) \leq (R(0) + q(0) - s(0))F^*(t) + (R(0) - s(0))F^*(t) =: \Delta_1^*(t) \\
\Delta_2(u) \leq (2R(0) + q(0) - 2s(0))F^*(t) =: \Delta_2^*(t) \\
\Delta_3(u) \leq s(0) \gamma + (R(0) + q(0) - s(0)) \gamma + p(0) \frac{\gamma}{P_{TIPS}(0,T)} + \tilde{W}_0 + \frac{1}{2\kappa}e^{\epsilon_1 t} + V_0^{(\gamma)} =: \Delta_3^*(t)
\]

Hence

\[
E[v_u^2] \leq (b^2 \epsilon_2^2 \Delta_1^*(t)^2 + c^2 \Delta_2^*(t)^2 + 2|b| \epsilon_1 \epsilon_2 \Delta_1^*(t) \Delta_2^*(t))a^2e^{2(-b \log R(0)+\mu_z^*(t)+\sigma_z^*(t)^2)} + 2(\epsilon_1^2 \Delta_2^*(t) \Delta_3^*(t) + |b| \epsilon_1 \epsilon_2 \Delta_1^*(t) \Delta_3^*(t))ae^{-b \log R(0)} F^*(t) + \epsilon_1^2 \Delta_3^*(t)^2 \\
=: \Upsilon_t
\]

with $\Upsilon_t$ is bounded, hence $\Upsilon_t^* \leq \Upsilon_t$ is also bounded.

This finishes the proof.

\textbf{Proof of theorem 7.}

\textit{Proof.} We compute

\[
E[(G_t^*-\tilde{G}_t)^2] \\
= E\left[\left(\int_0^t \frac{\mu(s)}{\sigma(s)^2} G_s^* dX(s)\right)^2\right] \\
= E\left[\left(\int_0^t \frac{\mu(s)}{\sigma(s)^2} G_s^*(\mu(s)X(s)ds + \sigma(s)X(s)dW(s))\right)^2\right] \\
\leq 2E\left[\left(\int_0^t \frac{\mu(s)^2}{\sigma(s)^2} G_s^2 ds\right)^2\right] + 2E\left[\int_0^t \frac{\mu(s)^2}{\sigma(s)^2} G_s^2 ds\right] \\
\leq 2E\left[\int_0^t \frac{\mu(s)^4}{\sigma(s)^2} G_s^2 ds\right] + 2E\left[\int_0^t \frac{\mu(s)^2}{\sigma(s)^2} G_s^2 ds\right] \\
\leq 2\epsilon_1^2 (1 + te_1^2) \int_0^t E[G_s^2] ds
\]
Here we used the inequality \((a + b)^2 \leq 2(a^2 + b^2)\), Itô isometry, and Jensen’s inequality. Notice that \(G^*_t\) is the solution of a linear stochastic differential equation, which is given by

\[
G^*_t = -Z_t \int_0^t \frac{v_u}{Z_u} \bigl(2\eta(u)du + dW(u)\bigr)
\]

where

\[
Z_t = \exp \left( -\frac{3}{2} \int_0^t \eta(u)^2 du - \int_0^t \eta(u)dW(u) \right)
\]
\[
v_t = \vartheta^{(\gamma)}(t)X(t) + \eta(t)(V_t + \tilde{W}_0 - \lambda)
\]

Let \(\frac{dP}{d\bar{P}} = e^{-\int_0^t 2\eta(u)dW(u) - \int_0^t 2\eta(u)^2 du}\). Then \(d\bar{W}(t) = 2\eta(t)dt + dW(t)\) is a \(\bar{P}\)-Brownian motion by Girsanov’s theorem. Hence

\[
E\left[ G^*_t^2 \right] = E\left[ e^{-\int_0^t 2\eta(u)dW(u) - \int_0^t 2\eta(u)^2 du} \left( \int_0^t \frac{v_u}{Z_u} \bigl(2\eta(u)du + dW(u)\bigr) \right)^2 \right]
\]

\[
= e^{-\int_0^t \eta(u)^2 du} E\left[ \left( \int_0^t \frac{v_u}{Z_u} d\bar{W}(u) \right)^2 \right]
\]

\[
= e^{-\int_0^t \eta(u)^2 du} E\left[ \int_0^t \frac{v_u^2}{Z_u^2} du \right]
\]

\[
= e^{-\int_0^t \eta(v)^2 dv} \int_0^t e^{\int_0^v \eta(u)^2 du} E\left[ e^{-\int_0^u 2\eta(u)dW(u) - \int_0^u 2\eta(u)^2 dv} v_u^2 \right] du
\]

\[
= \int_0^t e^{-\int_0^v \eta(v)^2 dv} E\left[ v_u^2 \right] du
\]

Let \(\Upsilon^*_t = \sup_{u \in [0,t]} E[\nu_u^2]\) as proved in proposition 4. In combination with the last estimate we obtain

\[
E[(G^*_t - \tilde{G}_t)^2] \leq 2\epsilon_1^2(1 + t\epsilon_1^2) \int_0^t \int_0^s e^{-\int_0^u \eta(v)^2 dv} E\left[ \nu_u^2 \right] du ds
\]

\[
\leq \epsilon_1^2(1 + t\epsilon_1^2)t^2 \Upsilon^*_t
\]

\[\blacksquare\]