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Solving bi-directional soliton equations in the KP hierarchy by gauge transformation

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Abstract: We present a systematic way to construct solutions of the $(n = 5)$-reduction of the BKP and CKP hierarchies from the general τ function $\tau^{(n+k)}$ of the KP hierarchy. We obtain the one-soliton, two-soliton, and periodic solution for the bi-directional Sawada-Kotera (bSK), the bi-directional Kaup-Kupershmidt (bKK) and also the bi-directional Satsuma-Hirota (bSH) equation. Different solutions such as left- and right-going solitons are classified according to the symmetries of the 5th roots of $e^{i\varepsilon}$. Furthermore, we show that the soliton solutions of the n-reduction of the BKP and CKP hierarchies with $n = 2j + 1$, $j = 1, 2, 3, \ldots$, can propagate along j directions in the $1 + 1$ space-time domain. Each such direction corresponds to one symmetric distribution of the nth roots of $e^{i\varepsilon}$. Based on this classification, we detail the existence of two-peak solitons of the n-reduction from the Grammian τ function of the sub-hierarchies BKP and CKP. If n is even, we again find two-peak solitons. Last, we obtain the "stationary" soliton for the higher-order KP hierarchy.

Key words. KP hierarchy – BKP hierarchy – CKP hierarchy – τ-function – gauge transformation – bSK equation – bKK equation – bSH equation – periodic solution – bidirectional soliton

1. Introduction

The Kadomtsev-Petviashvili (KP) hierarchy is of central interest for integrable systems and includes several well-known partial differential equations such as the Korteweg-de Vries (KdV) and the KP equation. With pseudo-differential Lax operator L given as [1–3]

$$L = \partial + u_2\partial^{-1} + u_3\partial^{-2} + \cdots , \tag{1.1}$$

the corresponding generalized Lax equation

$$\frac{\partial L}{\partial t_n} = [B_n, L], \quad n = 1, 2, 3, \cdots , \tag{1.2}$$

gives rise to the infinite number of partial differential equations (PDEs) of the KP hierarchy with dynamical variables $\{u_i(t_1, t_2, t_3, \cdots)\}$ with $i = 2, 3, 4, \cdots$. Here $B_n = \sum_{i=0}^{n} b_{n,i}\partial^i \equiv (L^n)_+$ denotes the *differential* part of L^n and in following we will use $L^n_- \equiv L^n - B_n$ to denote the *integral* part.

sub-hierarchy	Lax operator	example equation
BKP [1,4]	$L^* = -\partial L \partial^{-1}$	SK [6,7], bSK [10,11]
CKP [4]	$L^* = -L$	KK [8,9], bKK [10,11]
n-thKdV [5]	$L^n_- = 0$	KdV [12], Boussinesq-type [2], SH [13] $n = 2, 3, 4$
constrained KP(cKP) [17,21]	$L = \partial + \phi\partial^{-1}\psi$	YO [14], MKdV [15], NLS [16]

Table 1. Examples of sub-hierarchies of the KP hierarchy, Lax operators used to construct them and resulting equations. The symbol $*$ indicates the conjugation, for example, $\partial^* = -\partial$. There are some abbreviations used in Table: Sawada-Kotera (SK), bi-directional Sawada-Kotera (bSK), Kaup-Kupershmidt (KK), bi-directional Kaup-Kupershmidt (bKK), Satsuma-Hirota (SH), Yajima-Oikawa (YO), Modified KdV (MKdV), Non-linear Schrödinger (NLS).

The simplest nontrivial PDE constructed from (1.2) is the KP equation given as

$$\frac{\partial}{\partial x}\left(4\frac{\partial u_2}{\partial t_3} - 12u_2\frac{\partial u_2}{\partial x} - \frac{\partial^3 u_2}{\partial x^3}\right) - 3\frac{\partial^2 u_2}{\partial t_2} = 0 \quad . \tag{1.3}$$

In Table 1 we show the Lax operator and corresponding $(1+1)$-dimensional examples of sub-hierarchies of the KP hierarchy. An alternative way to express the KP hierarchy is given by the Zakharov-Shabat (ZS) equation [22],

$$\frac{\partial B_n}{\partial t_m} - \frac{\partial B_m}{\partial t_n} + [B_n, B_m] = 0, \quad m, n = 2, 3, 4, \cdots \quad . \tag{1.4}$$

The *eigenfunction* ϕ and the *adjoint eigenfunction* ψ of the KP hierarchy associated with Eq. (1.4) are defined by

$$\frac{\partial \phi}{\partial t_n} = B_n \phi, \quad \frac{\partial \psi}{\partial t_n} = -B_n^* \psi, \tag{1.5}$$

where $\phi = \phi(\lambda; \bar{t})$ and $\psi = \psi(\lambda; \bar{t})$ and $\bar{t} = (t_1, t_2, \cdots)$.

The n-reduction of the KP hierarchy corresponds to the situation $L^n_- = 0$ such that $L^n = B_n = \partial^n + v_{n-2}\partial^{n-2} + \cdots + v_1\partial + v_0$. Then the v_i, $i = 0, 1, \cdots, n-2$, are independent of $(t_n, t_{2n}, t_{3n}, \cdots)$. In this way the Lax pair of the $(1+1)$-dimensional integrable system can be found. Well-known examples of such n-reductions include the 4-reduction of the KP hierarchy [13] with Lax pair

$$(\partial_x^4 + 4u\partial_x^2 + 4u_x\partial_x + 2u_{xx} + 4u^2 + v)\phi = \lambda\phi, \tag{1.6}$$

$$\partial_t \phi = (\partial_x^3 + 3u\partial_x + \frac{3}{2}u_x)\phi, \quad t_1 = x, \quad t_3 = t, \tag{1.7}$$

corresponding to the Satsuma-Hirota (SH) equation [13]

$$-4u_t + 12uu_x + u_{xxx} + 3v_x = 0, \quad 2v_t + 6uv_x + v_{xxx} = 0. \tag{1.8}$$

Furthermore, eliminating v in the above equations, we can obtain a 6th order equation $(u = z_x)$

$$-8z_{tt} + z_{xxxxxx} - 2z_{xxxt} + 18z_x z_{xxxx} + 36z_{xx}z_{xxx} + 72z_x^2 z_{xx} = 0, \tag{1.9}$$

which has been called bi-directional Satsuma-Hirota (bSH) equation [23]. Naturally, there also exist n-reductions of the BKP and CKP hierarchies. For example, the 5-reduction of the BKP hierarchy with $u = u_2$ is given as

$$\left[\partial_x^5 + 5u\partial_x^3 + 5u_x\partial_x^2 + (5u^2 + \frac{10}{3}u_{xx} + \frac{5}{3}z_t)\partial_x\right]\phi = \lambda\phi, \tag{1.10}$$

$$\partial_t \phi = (\partial_x^3 + 3u\partial_x)\phi, \quad u = z_x, t_3 = t, t_1 = x, \tag{1.11}$$

which is the Lax pair corresponding to bi-directional Sawada-Kotera (bSK) equation [10,11]

$$\left(z_{xxxxx} + 15z_x z_{xxx} + 15z_x^3 - 15z_x z_t - 5z_{xxt}\right)_x - 5z_{tt} = 0. \tag{1.12}$$

The 5-reduction of the CKP hierarchy ($u = u_2$) with Lax pair [10, 11]

$$\left[\partial_x^5 + 5u\partial_x^3 + \frac{15}{2}u_x\partial_x^2 + (5u^2 + \frac{35}{6}u_{xx} + \frac{5}{3}z_t)\partial_x + 5uu_x + \frac{5}{3}u_{xxx} + \frac{5}{6}u_t\right]\phi = \lambda\phi, \tag{1.13}$$

$$\partial_t\phi = (\partial_x^3 + 3u\partial_x + \frac{3}{2}u_x)\phi, \quad u = z_x, t_3 = t, t_1 = x, \tag{1.14}$$

gives the bi-directional Kaup-Kupershmidt (bKK) equation

$$\left(z_{xxxxx} + 15z_xz_{xxx} + 15z_x^3 - 15z_xz_t - 5z_{xxt} + \frac{45}{4}z_{xx}^2\right)_x - 5z_{tt} = 0. \tag{1.15}$$

An essential characteristic of the KP hierarchy is the existence of the τ-function and all dynamical variables $\{u_i\}$, $i = 2, 3, \ldots,$ can be constructed from it [1, 2], e.g.,

$$u_2 = \frac{\partial^2}{\partial x^2}\log\tau, \tag{1.16}$$

$$u_3 = \frac{1}{2}\left(-\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x\partial t_2}\right)\log\tau, \tag{1.17}$$

$$\vdots$$

So it is a central task to construct the τ-function in order to solve the nonlinear PDEs associated with the KP hierarchy. In the following, we will show that ϕ and ψ play a key role in this construction.

Gauge transformations [24, 25] offer an efficient route towards the construction of the τ function of the KP hierarchy. In Ref. [26] two kinds of such a gauge transformation have been proposed, namely,

$$T_D(\phi) = \phi\partial\phi^{-1}, \quad T_I(\psi) = \psi^{-1}\partial^{-1}\psi \tag{1.18}$$

resulting in a very general and universal τ function (see Eq. (3.17) of [26] and also $IW_{k,n}$ in [27]). The *determinant representation* of the gauge transformation operators with $(n+k)$ steps is given in Ref. [27]. In particular, the Grammian τ function [28] of the KP hierarchy can be generated by an iteration of the transformation [26, 29, 30]. This is straightforwardly understood from Chau's τ function and the determinant representation [27] if we impose a restriction on the generating functions of the gauge transformation. Grammian τ function have also been used to solve the reduction of the constrained BKP and CKP hierarchies [31–33, 36].

There are two issues that arise when one wants to study the solutions of the $(1+1)$-dimensional solitons equations given by the n-reduction of the BKP and CKP hierarchies. The first is how it retain the restrictions, i.e. $L^* = -\partial L\partial^{-1}$for BKP and $L^* = -L$ for CKP, for the transformed Lax operators $L^{(1)} = TLT^{-1}$. In other words, the problem is how to obtain the τ-functions $\tau_{\text{BKP}}^{(n+k)}$and $\tau_{\text{CKP}}^{(n+k)}$ from the general τ-function $\tau^{(n+k)} = IW_{k,n}\tau^{(0)}$ with the gauge transformation T_{n+k} of the KP hierarchy. Here $\tau^{(0)}$ is the initial value of the τ-function of the KP hierarchy. Also, the generating functions ϕ_i, ψ_i of the gauge transformation will be complex-valued and related to the n-th roots of $e^{i\varepsilon}$. The second issue therefore is how to choose generating functions $\phi_i = \phi(\lambda_i; x, t)$ and $\psi_i = \psi(\mu_i; x, t))$ such that $\tau_{\text{BKP}}^{(n+k)}$ and $\tau_{\text{CKP}}^{(n+k)}$ correspond to a *physical* τ-function $\hat{\tau}_{Eq}^{(n+k)}$, which is real and positive on the full (x, t) plane.

In fact, the bKK and bSK equations have been introduced recently by Dye and Parker [10, 11] when looking for the bidirectional soliton analogues of the Sawada-Kotera (SK) [6, 7] and Kaup-Kupershmidt (KK) [8, 9] equations. The Lax pairs of bKK and bSK related similarly as the Lax pairs of KdV and Boussineq equation, thus ensuring their integrability. Both bKK and bSK equation have a *bidirectional* soliton solution [10, 11] which have been obtained by the Hirota bilinear method [37]. The profile of the bKK solitons depend on their direction of propagation. The right-going solitons of bKK are standard one-peak solitons, but the left-going solitons have two peaks. Very recently, Verhoeven and Musette [23] have plotted the bi-directional solitons for the bKK and bSH equation based on the Grammian τ function.

In this paper, we want to study why the 5-reduction of the BKP and CKP hierarchies have bidirectional soliton solutions, whereas their 3-reduction does not. As a first step, we will therefore exhibit the relationship between the periodic, left-going and right-going solitons of the 5-reduction and the 5-th roots of $e^{i\varepsilon}$. In order to do so, we derive the τ functions of the BKP and CKP hierarchies in sections 2-4. The explicit formulas of the corresponding τ-functions for solitons as well as for the periodic solutions of bSK and bKK are given and the two-peak soliton is discussed in detail. In section 5, we will prove that no two-peak solitons exist for the bSH equation. The one-peak soliton has bi-directional motion and we also obtain the periodic and two-soliton solutions. In section 6, we will discuss the lower and higher-order reductions of BKP and CKP hierarchies and also the n =even-reductions of the KP hierarchy. We will show that the soliton of the $(2j+1)$-reduction of BKP and CKP hierarchies can move along j directions $(j = 1, 2, \dots)$, investigate the relationship with the symmetric distribution of the $(2j+1)$-th roots of $e^{i\varepsilon}$. In particular, we will obtain the "stationary" soliton for the higher reduction of the KP hierarchy. For the higher-order equation and even-reduction of KP hierarchy, we can again find a two-peak soliton.

2. τ functions for BKP and CKP hierarchies

Let us first define the generalized Wronskian determinant

$$IW_{k,n} \equiv IW_{k,n}(g_k^{(0)}, g_{k-1}^{(0)}, \cdots, g_1^{(0)}; f_1^{(0)}, f_2^{(0)}, \cdots, f_n^{(0)})$$

$$= \begin{vmatrix} \int g_k^{(0)}\cdot f_1^{(0)} & \int g_k^{(0)}\cdot f_2^{(0)} & \int g_k^{(0)}\cdot f_3^{(0)} & \cdots & \int g_k^{(0)}\cdot f_n^{(0)} \\ \int g_{k-1}^{(0)}\cdot f_1^{(0)} & \int g_{k-1}^{(0)}\cdot f_2^{(0)} & \int g_{k-1}^{(0)}\cdot f_3^{(0)} & \cdots & \int g_{k-1}^{(0)}\cdot f_n^{(0)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \int g_1^{(0)}\cdot f_1^{(0)} & \int g_1^{(0)}\cdot f_2^{(0)} & \int g_1^{(0)}\cdot f_3^{(0)} & \cdots & \int g_1^{(0)}\cdot f_n^{(0)} \\ f_1^{(0)} & f_2^{(0)} & f_3^{(0)} & \cdots & f_n^{(0)} \\ f_{1,x}^{(0)} & f_{2,x}^{(0)} & f_{3,x}^{(0)} & \cdots & f_{n,x}^{(0)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ (f_1^{(0)})^{(n-k-1)} & (f_2^{(0)})^{(n-k-1)} & (f_3^{(0)})^{(n-k-1)} & \cdots & (f_n^{(0)})^{(n-k-1)} \end{vmatrix} \tag{2.1}$$

In particular, $IW_{0,n} = W_n\left(f_1^{(0)}, f_2^{(0)}, \cdots, f_n^{(0)}\right)$ with $\left(f_i^{(0)}\right)^{(k)} = \dfrac{\partial^k f_i^{(0)}}{\partial x^k}$ is the usual Wronskian determinant of functions $\{f_1^{(0)}, f_2^{(0)}, \cdots, f_n^{(0)}\}$. We shall also use the abbreviation $\int f = \int f dx$ with integration constant equal to zero.

Lemma 1 [26, 27]. *The τ function of the KP hierarchy generated by the gauge transformation T_{n+k} is given as*

$$\tau^{(n+k)} = IW_{k,n}(\psi_k^{(0)}, \psi_{k-1}^{(0)}, \cdots, \psi_1^{(0)}; \phi_1^{(0)}, \phi_2^{(0)}, \cdots, \phi_n^{(0)})\tau^{(0)}, \tag{2.2}$$

where $\left(\phi_i^{(0)}, \psi_j^{(0)}\right) = \left(\phi(\lambda_i;\bar{t}), \psi(\mu_j;\bar{t})\right)$ are solutions of Eq. (1.5) with initial value $\tau^{(0)}$ for the τ-function and the initial values of the $\{u_i\}$ are $\{u_i^{(0)}\}$.

Let us now discuss how to reduce the $\tau^{(n+k)}$ in (2.2) to the τ function of the BKP hierarchy. The key problem is how to keep the restriction $(L^{(n+k)})^* = -\partial L^{(n+k)}\partial^{-1}$ under the gauge transformation T_{n+k} [27]. It should be noted that $\bar{t} = (t_1, t_3, t_5, \cdots)$ in BKP hierarchy.

Proposition 1 [30, 39].

1. *The Lax operator transforms as* $L^{(n+k)} = T_{n+k} L T_{n+k}^{-1}$ *under the gauge transformation* T_{n+k} *with* $n = k$ *and generating functions* $\psi_i^{(0)} = \phi_{i,x}^{(0)}$ *for* $i = 1, 2, \cdots, n$.
2. *The* τ *function* $\tau_{\mathrm{BKP}}^{(n+n)}$ *of the* BKP *hierarchy is*

$$\tau_{\mathrm{BKP}}^{(n+n)} = IW_{n,n}(\phi_{n,x}^{(0)}, \phi_{n-1,x}^{(0)}, \cdots, \phi_{1,x}^{(0)}; \phi_1^{(0)}, \phi_2^{(0)}, \cdots, \phi_n^{(0)})$$
$$= \begin{vmatrix} \int \phi_{n,x}^{(0)} \cdot \phi_1^{(0)} & \int \phi_{n,x}^{(0)} \cdot \phi_2^{(0)} & \int \phi_{n,x}^{(0)} \cdot \phi_3^{(0)} & \cdots & \int \phi_{n,x}^{(0)} \cdot \phi_{n-1}^{(0)} & \frac{1}{2}(\phi_n^{(0)})^2 \\ \int \phi_{n-1,x}^{(0)} \cdot \phi_1^{(0)} & \int \phi_{n-1,x}^{(0)} \cdot \phi_2^{(0)} & \int \phi_{n-1,x}^{(0)} \cdot \phi_3^{(0)} & \cdots & \frac{1}{2}(\phi_{n-1}^{(0)})^2 & \int \phi_{n-1,x}^{(0)} \cdot \phi_n^{(0)} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \int \phi_{2,x}^{(0)} \cdot \phi_1^{(0)} & \frac{1}{2}(\phi_2^{(0)})^2 & \int \phi_{2,x}^{(0)} \cdot \phi_3^{(0)} & \cdots & \int \phi_{2,x}^{(0)} \cdot \phi_{n-1}^{(0)} & \int \phi_{2,x}^{(0)} \cdot \phi_n^{(0)} \\ \frac{1}{2}(\phi_1^{(0)})^2 & \int \phi_{1,x}^{(0)} \cdot \phi_2^{(0)} & \int \phi_{1,x}^{(0)} \cdot \phi_3^{(0)} & \cdots & \int \phi_{1,x}^{(0)} \cdot \phi_{n-1}^{(0)} & \int \phi_{1,x}^{(0)} \cdot \phi_n^{(0)} \end{vmatrix} \tau_{\mathrm{BKP}}^{(0)} \quad (2.3)$$

Proof. 1. It is clear that a single step of the gauge transformations T_D or T_I can not keep the restriction. So we use

$$T \equiv T_{1+1} = T_I\left(\psi_1^{(1)}\right) \cdot T_D\left(\phi_1^{(0)}\right) \quad (2.4)$$

such that the lax operator is $L^{(2)} = TLT^{-1}$. Let us check whether it satisfies the required restriction

$$(L^{(2)})^* = -\partial L^{(2)} \partial^{-1} \quad (2.5)$$

which means in terms of T that

$$T_D\left(\psi_1^{(1)}\right) T_I\left(\phi_1^{(0)}\right) \partial = \partial T_I\left(\psi_1^{(1)}\right) T_D\left(\phi_1^{(0)}\right). \quad (2.6)$$

Based on the determinant representation of T [27] we see from (2.6) that

$$\text{r.h.s} = \partial - \left(\frac{\phi_1^{(0)}}{\int \phi_1^{(0)} \psi_1^{(0)}}\right)_x \partial^{-1} \psi_1^{(0)} - \frac{\phi_1^{(0)} \psi_1^{(0)}}{\int \phi_1^{(0)} \psi_1^{(0)}}, \quad (2.7)$$

$$\text{l.h.s} = \partial + \left(\frac{\psi_1^{(0)}}{\int \phi_1^{(0)} \psi_1^{(0)}}\right) \partial^{-1} \phi_{1,x}^{(0)} - \frac{\phi_1^{(0)} \psi_1^{(0)}}{\int \phi_1^{(0)} \psi_1^{(0)}}. \quad (2.8)$$

This implies $\psi_1^{(0)} = \phi_{1,x}^{(0)}$. So we have seen that in order to keep the restriction of the Lax operator, we have to regard $T = T_{1+1}$ as basic building block in iteration of the gauge transformations T_{n+k}. In particular,

$$T_{2+2} = T_I\left(\phi_{2,x}^{(3)}\right) T_D\left(\phi_2^{(2)}\right) T_I\left(\phi_{1,x}^{(1)}\right) T_D\left(\phi_1^{(0)}\right),$$
$$T_{3+3} = T_I\left(\phi_{3,x}^{(5)}\right) T_D\left(\phi_3^{(4)}\right) T_I\left(\phi_{2,x}^{(3)}\right) T_D\left(\phi_2^{(2)}\right) T_I\left(\phi_{1,x}^{(1)}\right) T_D\left(\phi_1^{(0)}\right),$$

and so on such that $k = n$ and $\psi_i^{(0)} = \phi_{i,x}^{(0)}$ for $i = 1, 2, \ldots, n$.

2. According to the determinant of T_{n+k} [27] and $\tau^{(n+k)}$ [26, 27] with $k = n$ and $\psi_i^{(0)} = \phi_{i,x}^{(0)}$, $i = 1, 2, \cdots, n$, $\tau_{\mathrm{BKP}}^{(n+n)}$ can be obtained directly from $\tau^{(n+k)}$ as in Lemma 1. □

For the CKP hierarchy, we have again $\bar{t} = (t_1, t_3, t_5, \ldots)$ and the restriction is $(L^{(n+k)})^* = -L^{(n+k)}$.

Proposition 2 [30, 39].

1. *The appropriate gauge transformation* T_{n+k} *is given by* $n = k$ *and generating functions* $\psi_i^{(0)} = \phi_i^{(0)}$ *for* $i = 1, 2, \ldots, n$.

2. The τ function $\tau_{\rm CKP}^{(n+n)}$ of the CKP *hierarchy has the form*

$$\tau_{\rm CKP}^{(n+n)} = IW_{n,n}(\phi_n^{(0)}, \phi_{n-1}^{(0)}, \cdots, \phi_1^{(0)}; \phi_1^{(0)}, \phi_2^{(0)}, \cdots, \phi_n^{(0)})$$
$$= \begin{vmatrix} \int \phi_n^{(0)} \cdot \phi_1^{(0)} & \int \phi_n^{(0)} \cdot \phi_2^{(0)} & \int \phi_n^{(0)} \cdot \phi_3^{(0)} & \cdots & \int \phi_n^{(0)} \cdot \phi_{n-1}^{(0)} & \int \phi_n^{(0)} \cdot \phi_n^{(0)} \\ \int \phi_{n-1}^{(0)} \cdot \phi_1^{(0)} & \int \phi_{n-1}^{(0)} \cdot \phi_2^{(0)} & \int \phi_{n-1}^{(0)} \cdot \phi_3^{(0)} & \cdots & \int \phi_{n-1}^{(0)} \cdot \phi_{n-1}^{(0)} & \int \phi_{n-1}^{(0)} \cdot \phi_n^{(0)} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \\ \int \phi_2^{(0)} \cdot \phi_1^{(0)} & \int \phi_2^{(0)} \cdot \phi_2^{(0)} & \int \phi_2^{(0)} \cdot \phi_3^{(0)} & \cdots & \int \phi_2^{(0)} \cdot \phi_{n-1}^{(0)} & \int \phi_2^{(0)} \cdot \phi_n^{(0)} \\ \int \phi_1^{(0)} \cdot \phi_1^{(0)} & \int \phi_1^{(0)} \cdot \phi_2^{(0)} & \int \phi_1^{(0)} \cdot \phi_3^{(0)} & \cdots & \int \phi_1^{(0)} \cdot \phi_{n-1}^{(0)} & \int \phi_1^{(0)} \cdot \phi_n^{(0)} \end{vmatrix} \tau_{\rm CKP}^{(0)} \quad (2.9)$$

Proof. 1. Similar to the BKP hierarchy we have to try the two-step gauge transformation

$$T \equiv T_{1+1} = T_I\left(\psi_1^{(1)}\right) T_D\left(\phi_1^{(0)}\right). \quad (2.10)$$

With $L^{(2)} = TLT^{-1}$, the restriction $\left(L^{(2)}\right)^* = -L^{(2)}$ then implies

$$T_D\left(\psi_1^{(1)}\right) T_I\left(\phi_1^{(0)}\right) = T_I\left(\psi_1^{(1)}\right) T_D\left(\phi_1^{(0)}\right). \quad (2.11)$$

Based on the determinant representation of T [27], we find from (2.11) that

$$\text{r.h.s} = \partial - \frac{\phi_1^{(0)}}{\int \phi_1^{(0)} \psi_1^{(0)}} \partial^{-1} \psi_1^{(0)}, \quad (2.12)$$

$$\text{l.h.s} = \partial - \frac{\psi_1^{(0)}}{\int \phi_1^{(0)} \psi_1^{(0)}} \partial^{-1} \phi_1^{(0)}. \quad (2.13)$$

Then $\psi_1^{(0)} = \phi_1^{(0)}$. Again, we have to regard $T = T_I\left(\phi_1^{(1)}\right) T_D\left(\phi_1^{(0)}\right)$ as basic building block such that

$$T_{2+2} = T_I\left(\phi_2^{(3)}\right) T_D\left(\phi_2^{(2)}\right) T_I\left(\phi_1^{(1)}\right) T_D\left(\phi_1^{(0)}\right),$$
$$T_{3+3} = T_I\left(\phi_3^{(5)}\right) T_D\left(\phi_3^{(4)}\right) T_I\left(\phi_2^{(3)}\right) T_D\left(\phi_2^{(2)}\right) T_I\left(\phi_1^{(1)}\right) T_D\left(\phi_1^{(0)}\right),$$

so $k = n$ and $\psi_i^{(0)} = \phi_i^{(0)}$ for $i = 1, 2, \ldots, n$.

2. According to the determinant of T_{n+k} [27] and $\tau^{(n+k)}$ [26, 27] with $k = n$ and $\psi_i^{(0)} = \phi_i^{(0)}$, $i = 1, 2, \cdots, n$, $\tau_{\rm CKP}^{(n+n)}$ is obtained directly from $\tau^{(n+k)}$ in Lemma 1. □

In fact, we can let $\psi_i^{(0)} = c_i \phi_{i,x}^{(0)}$ (or $\psi_i^{(0)} = c_i \phi_i^{(0)}$) with constants c_i. However, the new $\tau_{\rm BKP}^{(n+n)}$ (or $\tau_{\rm CKP}^{(n+n)}$) associated with $\psi_i^{(0)} = c_i \phi_{i,x}^{(0)}$ (or $\psi_i^{(0)} = c_i \phi_i^{(0)}$) is equivalent to the ones in Proposition 1 (or Proposition 2). Although Refs. [30, 39] have results similar to our Propositions 1 and 2, our approach is more direct and simpler for the construction $\tau_{\rm BKP}^{(n+n)}$ and $\tau_{\rm CKP}^{(n+n)}$. If the initial values of dynamical variables $\{u_i\}$ of BKP(CKP) hierarchy are zero, then the equations in (1.5) of $\left(\phi_i^{(0)}, \psi_j^{(0)}\right) = \left(\phi(\lambda_i; \bar{t}), \psi(\mu_j; \bar{t})\right)$ become more simpler as

$$\frac{\partial \phi(\lambda; \bar{t})}{\partial t_n} = \left(\partial_x^n \phi(\lambda; \bar{t})\right), \quad \bar{t} = (t_1, t_3, t_5, \cdots), \quad (2.14)$$

$$\frac{\partial \psi(\mu; \bar{t})}{\partial t_n} = (-1)^{n+1} \left(\partial_x^n \psi(\mu; \bar{t})\right), \quad \bar{t} = (t_1, t_3, t_5, \cdots), \quad (2.15)$$

and $\tau_{\rm BKP}^{(0)} = 1 (\tau_{\rm CKP}^{(0)} = 1)$. Last, we note that for the generalized KP (gKP) hierarchy with Lax operator $\hat{L} = L^n$, $n = 2, 4, 6, 8, \ldots$, and $\hat{L}^* = \hat{L}$, the τ function $\tau_{\rm gKP}^{(n+k)}$ generated by gauge transformations T_{n+k} has the same form as for the CKP hierarchy. This result will afford a simple way to construct the τ function of bSH equation in Section 5.

3. Soliton solutions of the bSK equation

As pointed out in the introduction, there are two steps en route from a τ function $\tau^{(n+k)}$ generated by the gauge transformations T_{n+k} of the KP hierarchy to the τ function of equations as the n-reduction of BKP or CKP hierarchies. The second step is to build physical τ functions from the complex-valued $\tau^{(n+n)}_{\text{BKP}}$ and $\tau^{(n+n)}_{\text{CKP}}$ constructed in the last section. In the following Sections, we will illustrate our approach by computing the τ function for the 5-reduction of BKP and CKP, i.e., for the bSK and bKK equations.

The 5-reduction of the BKP hierarchy is the bSK equation (1.12). Assume for the initial value $u = 0$ in Eqs. (1.10) and (1.11), then $\phi_i^{(0)} = \phi(\lambda_i; x, t)$ are solutions of

$$\partial_x^5\phi(\lambda_i; x, t) = \lambda_i\phi(\lambda_i; x, t), \quad \frac{\partial\phi(\lambda_i; x, t)}{\partial t} = \left(\partial_x^3\phi(\lambda_i; x, t)\right). \tag{3.1}$$

So proposition 1 with $\tau^{(0)}_{\text{BKP}} = 1$ implies that the τ function of bSK is given as follows.

Proposition 3. *The τ function of the* bSK *equation generated by T_{n+n} from initial value* 1 *is*

$$\tau^{(n+n)}_{\text{bSK}} = IW_{n,n}\left(\phi^{(0)}_{n,x}, \phi^{(0)}_{n-1,x}, \cdots, \phi^{(0)}_{1,x}; \phi^{(0)}_1, \phi^{(0)}_2, \cdots, \phi^{(0)}_n\right) \tag{3.2}$$

and the solution of the bSK *equation generated by T_{n+n} from initial value* 0 *is*

$$u = \partial_x^2\left(\log\tau^{(n+n)}_{\text{bSK}}\right) \tag{3.3}$$

Here $\phi_i^{(0)} = \phi(\lambda_i; x, t)$ are solutions of Eq. (3.1).

In general, this τ function $\tau^{(n+n)}_{\text{bSK}}$ for bSK is complex and related to 5-th roots of $e^{i\varepsilon}$. We have to find the real and non-zero τ function from it such that u in Eq. (3.3) is a real and smooth solution of bSK. This is main task of this section. We start by analysing the solution $\phi(\lambda; x, t)$ of Eq. (3.1)and make the universal ansatz

$$\phi(\lambda; x, t) = \sum_{j=1}^{5} A_j e^{xp_j + tp_j^3}, \quad \text{with} \quad p_j^5 = \lambda. \tag{3.4}$$

Here $p_j = k\exp\left(\frac{\varepsilon+2\pi j}{5}i\right)$, $k^5 = |\lambda|, k \in \mathbb{R}$, $0 \le \varepsilon < 2\pi$ and $j = 0, 1, 2, 3, 4$. There are two important ingredients which we can use to find the desired solution. The first is that the 5-th roots $\varepsilon_j = \exp\left(\frac{\varepsilon+2\pi j}{5}i\right)$ of $e^{i\varepsilon}$ are distributed uniformly on the unit circle in $\mathbb{C}$. So for a suitable value of ε there exist combinations of p_j's which are symmetric upon reflection on the x-axes; similarly for the y-axes for other values of ε. The second ingredient is that τ_{bSK} and $\exp(\alpha x + \beta t)\,\tau_{\text{bSK}}$ will imply the same solution u since $u = \partial_x^2\log\tau_{\text{bSK}}$. Here, α and β are arbitrary, complex constants. Therefore we can obtain the desired real and smooth solutions of the bSK if τ_{bSK} can be expressed as $\tau_{\text{bSK}} = e^{\alpha x+\beta t}\hat{\tau}_{\text{bSK}} \approxeq \hat{\tau}_{\text{bSK}}$, in which $\hat{\tau}_{\text{bSK}}$ is a real and nonzero function although τ_{bSK} is complex. We call $\hat{\tau}_{\text{bSK}}$ the *physical* τ function for the bSK equation. Based on the above arguments, let us make the refined ansatz

$$\phi(\lambda_1; x, t) = A_1 e^{p_1x+p_1^3t} + B_1 e^{q_1x+q_1^3t}, \quad p_1 = k_1e^{i\varepsilon_1}, q_1 = -k_1e^{-i\varepsilon_1}, k_1^5 = |\lambda_1|, k_1 \in \mathbb{R}, \tag{3.5}$$

or

$$\phi(\lambda_1; x, t) = A_1 e^{p_1x+p_1^3t} + B_1 e^{q_1x+q_1^3t}, \quad p_1 = k_1e^{i\varepsilon_1}, q_1 = k_1e^{-i\varepsilon_1}, k_1^5 = |\lambda_1|, k_1 \in \mathbb{R}, \tag{3.6}$$

and in the next step we need to fix the ratio $\dfrac{B_1}{A_1}$. We stress that the above analysis is also true for the derivation of the bKK equation.

Proposition 4. *Define* $\xi_1 = xk_1\cos\varepsilon_1 + tk_1^3\cos 3\varepsilon_1$. *Then the physical* τ *function of* bSK *generated by* T_{1+1} *is*

$$\hat{\tau}_{\mathrm{bSK}}^{(1+1)} = e^{2\xi_1} + \left(\frac{B_1}{A_1}\right)^2 e^{-2\xi_1} + 2\left(\frac{B_1}{A_1}\right) \tag{3.7}$$

and the corresponding one soliton $u = \partial_x^2 \log \hat{\tau}_{\mathrm{bSK}}^{(1+1)}$ *is given as*

$$u = \frac{16\left(\frac{B_1}{A_1}\right)^2 k_1^2\cos^2\varepsilon_1}{\left(e^{-2\xi_1} + \left(\frac{B_1}{A_1}\right)^2 e^{-2\xi_1} + 2\left(\frac{B_1}{A_1}\right)\right)^2}\ . \tag{3.8}$$

Here $\frac{B_1}{A_1} > 0$. *The velocity of the moving soliton is* $v = -k_1^2\dfrac{\cos 3\varepsilon_1}{\cos\varepsilon_1}$ *and can be both positive and negative depending on the choice of* ε_1. *Specifically, we have* $v_- = v|_{\varepsilon_1=\frac{\pi}{10}} < 0$ *and* $v_+ = v|_{\varepsilon_1=\frac{3\pi}{10}} > 0$.

Proof.

$$\tau_{\mathrm{bSK}}^{(1+1)} = \left(\phi_1^{(0)}\right)^2 = A_1^2 e^{2i(xk_1\sin\varepsilon_1 + tk_1^3\sin 3\varepsilon_1)}\left[e^{2\xi_1} + \left(\frac{B_1}{A_1}\right)^2 e^{-2\xi_1} + 2\left(\frac{B_1}{A_1}\right)\right] \tag{3.9}$$

and $\phi_1^{(0)} = \phi(\lambda_1; x, t)$ defined by Eq. (3.5). □

Let us point out a relation between the distribution of the 5-th roots of $e^{i\varepsilon}$ and the direction of movement for the soliton.

1. $\left(e^{i\varepsilon_1}|_{\varepsilon_1=\pi/10}, -e^{-i\varepsilon_1}|_{\varepsilon_1=\pi/10}\right) \longrightarrow$ one distribution of 5-th roots of $e^{i\varepsilon} \longrightarrow (p_1 = k_1e^{i\varepsilon_1}|_{\varepsilon_1=\pi/10}$, $q_1 = -k_1e^{-i\varepsilon_1}|_{\varepsilon_1=\pi/10})$ in Eq. (3.5)$\longrightarrow v|_{\varepsilon_1=\pi/10} < 0$, left-going soliton u in Eq. (3.8) ;
2. $\left(e^{i\varepsilon_1}|_{\varepsilon_1=3\pi/10}, -e^{-i\varepsilon_1}|_{\varepsilon_1=3\pi/10}\right) \longrightarrow$ another distribution of 5-th roots of $e^{i\varepsilon} \longrightarrow (p_1 = k_1e^{i\varepsilon_1}|_{\varepsilon_1=3\pi/10}$, $q_1 = -k_1e^{-i\varepsilon_1}|_{\varepsilon_1=3\pi/10})$ in Eq. (3.5) $\longrightarrow v|_{\varepsilon_1=3\pi/10} > 0$, right-going soliton u in Eq. (3.8) .

We can see from Eq. (3.8) that the one-soliton of bSK has only one peak in its profile. The process of generating a two-soliton by T_{2+2} is more complicated.

Lemma 2. *With* $\phi(\lambda_1; x, t)$ *and* $\phi(\lambda_2; x, t)$ *as in Eq. (3.5), and using Proposition 3,* $\tau_{\mathrm{bSK}}^{(2+2)}$ *is given by*

$$\begin{aligned}
\tau_{\mathrm{bSK}}^{(2+2)} &= A_1^2A_2^2 e^{2i\left[x(k_1\sin\varepsilon_1 + k_2\varepsilon_2) + t(k_1^3\sin 3\varepsilon_1 + k_2^3\sin 3\varepsilon_2)\right]}\times \\
&\Bigg\{\frac{(4z_1 - f_1)e^{2(\xi_1+\xi_2)}}{4\left[k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\right]^2} + \frac{(4z_1^* - f_1)e^{-2(\xi_1+\xi_2)}}{4\left[k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\right]^2}\left(\frac{B_1}{A_1}\right)^2\left(\frac{B_2}{A_2}\right)^2 \\
&+ \frac{-(4z_3 + f_3)e^{2(\xi_1-\xi_2)}}{4\left[k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\right]^2}\left(\frac{B_2}{A_2}\right)^2 + \frac{-(4z_1^* + f_3)e^{-(2\xi_1-\xi_2)}}{4\left[k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\right]^2}\left(\frac{B_1}{A_1}\right)^2 \\
&+ \frac{(2iz_5 - f_5)e^{2\xi_1}}{2\left[k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\right]\left[k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\right]}\left(\frac{B_2}{A_2}\right) \\
&+ \frac{(-2iz_5^* - f_5)e^{-2\xi_1}}{2\left[k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\right]\left[k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\right]}\left(\frac{B_1}{A_1}\right)^2\left(\frac{B_2}{A_2}\right) \\
&+ \frac{(-2iz_7 - f_5)e^{2\xi_2}}{2\left[k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\right]\left[k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\right]}\left(\frac{B_1}{A_1}\right) \\
&+ \frac{(2iz_7^* - f_5)e^{-2\xi_2}}{2\left[k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\right]\left[k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\right]}\left(\frac{B_1}{A_1}\right)\left(\frac{B_2}{A_2}\right)^2
\end{aligned}$$

$$+\frac{-(k_1^2+k_2^2)^2}{2\left[k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\right]\left[k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\right]}\left(\frac{B_1}{A_1}\right)\left(\frac{B_2}{A_2}\right)\Bigg\} \tag{3.10}$$

Here the z_i*,* $i = 1,3,5,7$ *are given in Appendix A and* $f_1 = \left[k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\right]^2$*,* $f_3 = \left[k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\right]^2$ *and* $f_5 = \sqrt{f_1 f_3}$*, as well as* $\xi_i = xk_i\cos\varepsilon_i + tk_i^3\cos 3\varepsilon_i$ *for* $i = 1$ *and* 2*.*

We now need to find a suitable solution of $\frac{B_i}{A_i}, i = 1,2$ such that the summation of terms in the $\{\}$ bracket of Eq. (3.10) is a positive function on the whole (x,t) plane. The following two lemmas is useful. Let $z_1' = 4z_1 - f_1, z_3' = 4z_3 + f_3, z_5' = 2iz_5 - f_5, z_7' = -2iz_7 - f_5$, and $z_i, i = 1,3,5,7$ given in Appendix A.

Lemma 3. *For* $z_i', i = 1,3,5,7$ *there exist relations*

$$-z_1'z_3' = (z_5')^2; \quad -z_1'(z_3')^* = (z_7')^2. \tag{3.11}$$

Lemma 4. *Let* $\frac{B_1}{A_1} = \frac{z_1'}{z_7'}, \frac{B_2}{A_2} = \frac{z_1'}{z_5'}$*, and* $g_2 = \frac{|z_1'|^2}{|z_3'|^2},\ g_6 = g_8 = \frac{|z_1'|^2}{|z_5'|^2},\ g_9 = \frac{1}{|z_3'|}$ *then*

$$-\frac{z_3'}{z_1'}\left(\frac{B_2}{A_2}\right)^2 = 1, \qquad -\frac{(z_3')^*}{z_1'}\left(\frac{B_1}{A_1}\right)^2 = 1, \tag{3.12}$$

$$\frac{(z_1')^*}{z_1'}\left(\frac{B_1}{A_1}\right)^2\left(\frac{B_2}{A_2}\right)^2 = g_2, \qquad \frac{-1}{z_1'}\left(\frac{B_1}{A_1}\right)\left(\frac{B_2}{A_2}\right) = g_9, \tag{3.13}$$

$$\frac{(z_5')^*}{z_1'}\left(\frac{B_1}{A_1}\right)^2\left(\frac{B_2}{A_2}\right) = g_6, \qquad \frac{(z_7')^*}{z_1'}\left(\frac{B_1}{A_1}\right)\left(\frac{B_2}{A_2}\right)^2 = g_8, \tag{3.14}$$

hold.

The $\frac{B_i}{A_i}, i = 1,2$ in Lemma 4 are what we are looking for, and then the *physical* τ function $\hat{\tau}_{\mathrm{bSK}}^{(2+2)}$ is give by following proposition.

Proposition 5.

$$\begin{aligned}
&\hat{\tau}_{\mathrm{bSK}}^{(2+2)} = \\
&\Bigg\{\frac{e^{2(\xi_1+\xi_2)}}{4\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)^2} + \frac{g_2 e^{-2(\xi_1+\xi_2)}}{4\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)^2} \\
&+ \frac{e^{2(\xi_1-\xi_2)}}{4\big(k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)^2} + \frac{e^{-(2\xi_1-\xi_2)}}{4\big(k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)^2} \\
&+ \frac{e^{2\xi_1}}{2\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)\big(k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)} \\
&+ \frac{g_6 e^{-2\xi_1}}{2\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)\big(k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)} \\
&+ \frac{e^{2\xi_2}}{2\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)\big(k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)} \\
&+ \frac{g_8 e^{-2\xi_2}}{2\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)\big(k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)}
\end{aligned}$$

$$+\frac{(k_1^2+k_2^2)^2 g_9}{2\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)\big(k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)}\Big\}, \tag{3.15}$$

the two soliton solution is $u=(\partial_x^2\log\hat\tau_{\rm bSK}^{(2+2)})$. *In particular,* $\varepsilon_1=\varepsilon_2=\frac{\pi}{10}$ *results in two overtaking solitons and moving in negative direction;* $\varepsilon_1=\varepsilon_2=\frac{3\pi}{10}$ *produces two overtaking solitons and moving in positive direction;* $\varepsilon_1=\frac{\pi}{10},\varepsilon_2=\frac{3\pi}{10}$ *results in two head-on solitons.*

We have plotted the one solitons in Fig. 1 associated with parameters $A_1=B_1=1,k_1=1$ of Eq. (3.8). The two solitons in Proposition 5 are shown in Fig. 2 associated with parameters $k_1=2$ and $k_2=1.3$.

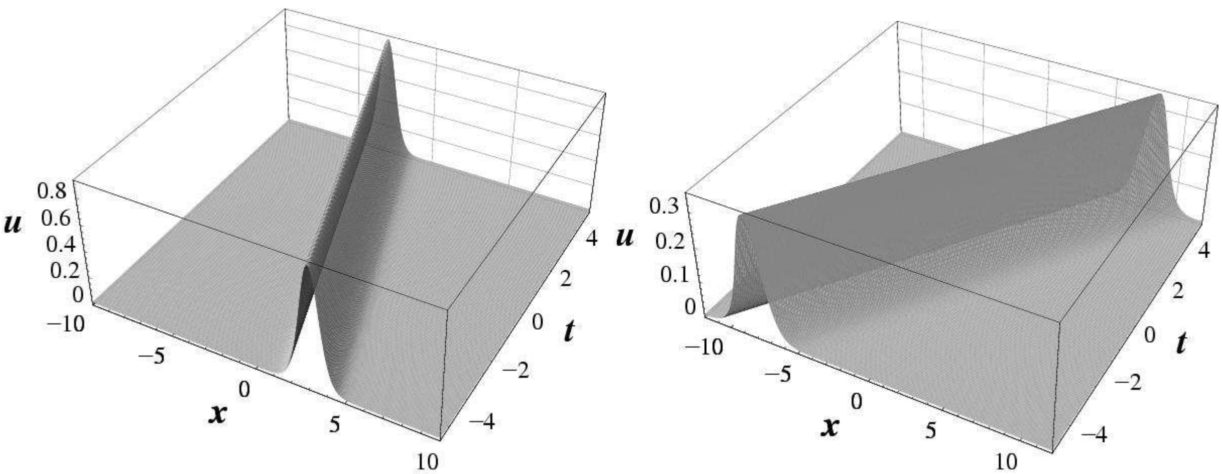

Fig. 1. Single left- and right-going solitons for the bSK equation (1.12) : $\varepsilon_1=\dfrac{\pi}{10}$ (left), $\varepsilon_1=\dfrac{3\pi}{10}$ (right).

4. Periodic and soliton solutions of bKK equation

The 5-reduction of the CKP hierarchy yields the bKK equation Eq. (1.15). Let the initial value be $u=0$ in Eqs. (1.13) and (1.14), then $\phi_i^{(0)}=\phi(\lambda_i;x,t)$ are solutions of

$$\partial_x^5\phi(\lambda_i;x,t)=\lambda_i\phi(\lambda_i;x,t),\quad \frac{\partial\phi(\lambda_i;x,t)}{\partial t}=(\partial_x^3\phi(\lambda_i;x,t)). \tag{4.1}$$

So the Proposition 2 implies the τ function of bKK equation.

Proposition 6. *The* τ *function of the* bKK *equation generated by* T_{n+n} *from initial value* 1 *is*

$$\begin{aligned}\tau_{\rm bKK}^{(n+n)}&=IW_{n,n}(\phi_n^{(0)},\phi_{n-1}^{(0)},\cdots,\phi_1^{(0)};\phi_1^{(0)},\phi_2^{(0)},\cdots,\phi_n^{(0)})\\ &=\begin{vmatrix}\int\phi_n^{(0)}\cdot\phi_1^{(0)} & \int\phi_n^{(0)}\cdot\phi_2^{(0)} & \int\phi_n^{(0)}\cdot\phi_3^{(0)} & \cdots & \int\phi_n^{(0)}\cdot\phi_{n-1}^{(0)} & \int\phi_n^{(0)}\cdot\phi_n^{(0)}\\ \int\phi_{n-1}^{(0)}\cdot\phi_1^{(0)} & \int\phi_{n-1}^{(0)}\cdot\phi_2^{(0)} & \int\phi_{n-1}^{(0)}\cdot\phi_3^{(0)} & \cdots & \int\phi_{n-1}^{(0)}\cdot\phi_{n-1}^{(0)} & \int\phi_{n-1}^{(0)}\cdot\phi_n^{(0)}\\ \vdots & \vdots & \vdots & \cdots & \vdots & \\ \int\phi_2^{(0)}\cdot\phi_1^{(0)} & \int\phi_2^{(0)}\cdot\phi_2^{(0)} & \int\phi_2^{(0)}\cdot\phi_3^{(0)} & \cdots & \int\phi_2^{(0)}\cdot\phi_{n-1}^{(0)} & \int\phi_2^{(0)}\cdot\phi_n^{(0)}\\ \int\phi_1^{(0)}\cdot\phi_1^{(0)} & \int\phi_1^{(0)}\cdot\phi_2^{(0)} & \int\phi_1^{(0)}\cdot\phi_3^{(0)} & \cdots & \int\phi_1^{(0)}\cdot\phi_{n-1}^{(0)} & \int\phi_1^{(0)}\cdot\phi_n^{(0)}\end{vmatrix}\end{aligned} \tag{4.2}$$

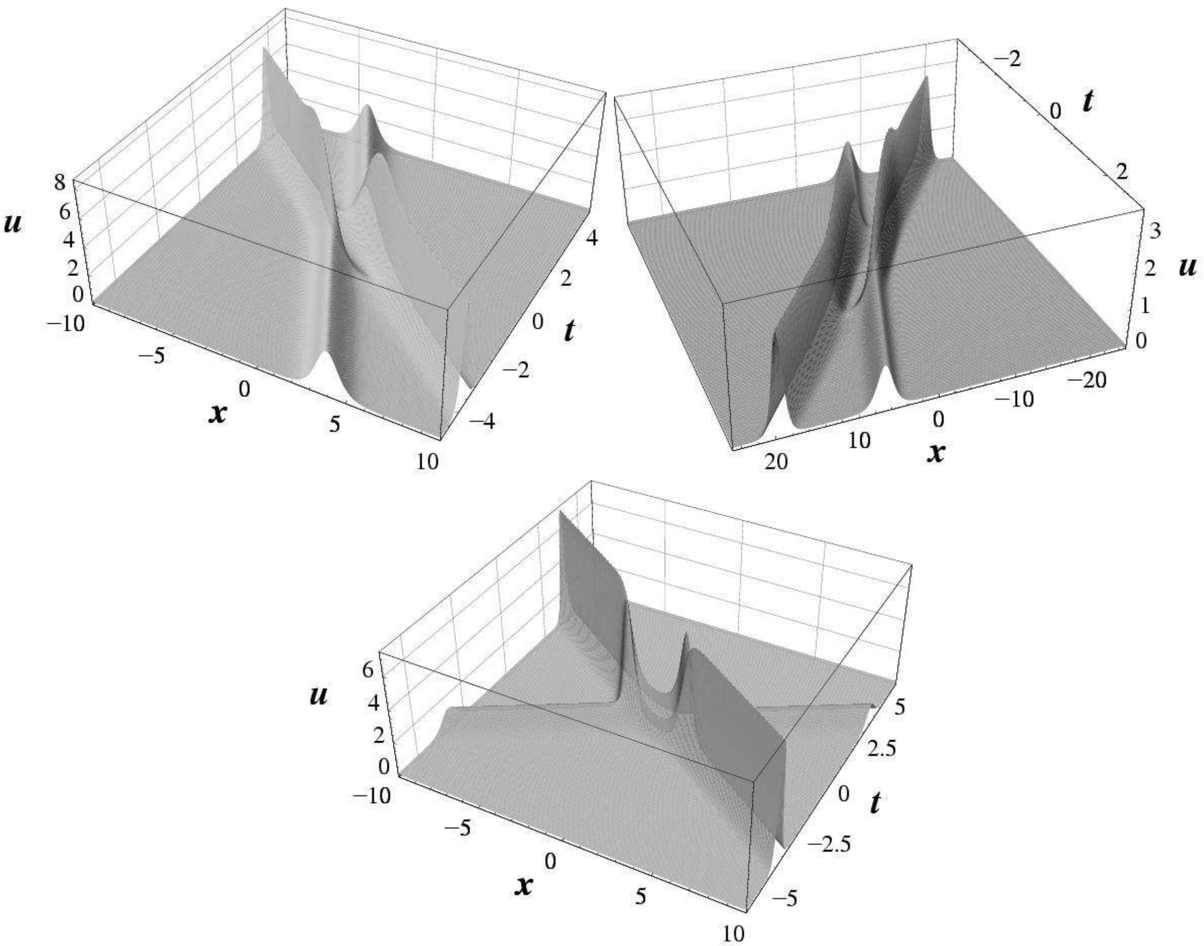

Fig. 2. Two left- and right-going as well as head-on colliding solitons for the bSK equation (1.12): $\varepsilon_1 = \varepsilon_2 = \frac{\pi}{10}$ (left), $\varepsilon_1 = \varepsilon_2 = \frac{3\pi}{10}$ (right) and $\varepsilon_1 = \frac{\pi}{10}, \varepsilon_2 = \frac{3\pi}{10}$ (collision).

and the solution u of the bKK *from initial value zero is*

$$u = \left(\partial_x^2 \log \tau_{\mathrm{bKK}}^{(n+n)}\right). \tag{4.3}$$

Here $\phi_i^{(0)} = \phi(\lambda_i; x, t)$ *are solutions of Eq. (4.1).*

As before, $\tau_{\mathrm{bKK}}^{(n+n)}$ is complex and related to the 5-th roots of $e^{i\varepsilon}$ and again we have to find a *physical* τ function $\hat{\tau}_{\mathrm{bKK}}^{(n+n)}$ such that u in Eq. (4.3) is real and smooth solution includes solitons and periodic solutions. The case of $n = 1$ and $n = 2$ will be discussed in detail. Similar to the bSK equation, we should assume the solutions of Eq. (4.1) as

$$\phi(\lambda_1; x, t) = A_1 e^{p_1 x + p_1^3 t} + B_1 e^{q_1 x + q_1^3 t}, \quad p_1 = k_1 e^{i\varepsilon_1}, q_1 = -k_1 e^{-i\varepsilon_1}, k_1^5 = |\lambda_1|, k_1 \in \mathbb{R}, \tag{4.4}$$

or

$$\phi(\lambda_1; x, t) = A_1 e^{p_1 x + p_1^3 t} + B_1 e^{q_1 x + q_1^3 t}, \quad p_1 = k_1 e^{i\varepsilon_1}, q_1 = k_1 e^{-i\varepsilon_1}, k_1^5 = |\lambda_1|, k_1 \in \mathbb{R}, \tag{4.5}$$

to extract *physical* τ function $\hat{\tau}_{\mathrm{bKK}}^{(n+n)}$ from $\tau_{\mathrm{bKK}}^{(n+n)}$. At first, we would like to give the two simple cases which are generated by the gauge transformations T_{1+1}.

Proposition 7. *Let* $\xi_1 = xk_1\cos\varepsilon_1 + tk_1^3\cos 3\varepsilon_1$, $\frac{B_1}{A_1} = ie^{-i\varepsilon_1}$ *and* $\phi_1^{(0)} = \phi(\lambda_1; x, t)$ *defined by Eq. (4.4), then the physical* τ *function of* bKK *extracted from* $\tau_{\mathrm{bKK}}^{(n+n)}\Big|_{n=1}$ *is*

$$\hat{\tau}_{\mathrm{bKK}}^{(1+1)} = e^{2\xi_1} + e^{-2\xi_1} + \frac{2}{\sin\varepsilon_1} \tag{4.6}$$

and the corresponding one soliton $u = \left(\partial_x^2 \log \hat{\tau}_{\mathrm{bKK}}^{(1+1)}\right)$ *is*

$$u = \frac{4k_1^2\left(\cos\varepsilon_1\right)^2\left(1 + \dfrac{\cosh 2\xi_1}{\sin\varepsilon_1}\right)}{\left(\cosh 2\xi_1 + \dfrac{1}{\sin\varepsilon_1}\right)^2} \tag{4.7}$$

with $\varepsilon_1 = \dfrac{\pi}{10}$ *or* $\dfrac{3\pi}{10}$. *The velocity of the soliton is* $v = -k_1^2\dfrac{\cos 3\varepsilon_1}{\cos\varepsilon_1}$. *In particular, the left-going soliton have two peaks in its profile and the negative speed* $v_- = v|_{\varepsilon_1=\frac{\pi}{10}}$; *the right-going soliton have only one peak and positive speed* $v_+ = v|_{\varepsilon_1=\frac{3\pi}{10}}$.

Proof. Taking $\phi_1^{(0)} = \phi(\lambda_1; x, t)$ of Eq. (4.4) and $n = 1$ back into Proposition 6, the straightforward calculation leads to

$$\tau_{\mathrm{bKK}}^{(1+1)} = \int\left(\phi_1^{(0)}\right)^2 = \frac{A_1^2 e^{2i(xk_1\sin\varepsilon_1 + tk_1^3\sin 3\varepsilon_1)}}{2p_1}\left[e^{2\xi_1} + e^{-2\xi_1} + \frac{2}{\sin\varepsilon_1}\right]. \tag{4.8}$$

Here $\xi_1 = xk_1\cos\varepsilon_1 + tk_1^3\cos 3\varepsilon_1$. □

If let $\phi_1^{(0)} = \phi(\lambda_1; x, t)$ defined by Eq. (4.5), then we can get a periodic solution as following proposition.

Proposition 8. *Let* $\eta_1 = xk_1\sin\varepsilon_1 + tk_1^3\sin 3\varepsilon_1$, *and* $\phi_1^{(0)} = \phi(\lambda_1; x, t)$ *defined by Eq. (4.5),* $A_1 = B_1 = 1$ *in* $\phi_1^{(0)}$, *then the physical* τ *function of* bKK *extracted from* $\tau_{\mathrm{bKK}}^{(n+n)}\Big|_{n=1}$ *is*

$$\hat{\tau}_{\mathrm{bKK}}^{(1+1)} = \frac{1}{\cos\varepsilon_1} + \cos(2\eta_1 - \varepsilon_1) \tag{4.9}$$

and the corresponding solution

$$u = \left(\partial_x^2 \log \hat{\tau}_{\mathrm{bKK}}^{(1+1)}\right) = \frac{-4k_1^2\sin^2\varepsilon_1\left(\dfrac{\cos(2\eta_1 - 2\varepsilon_1)}{\cos\varepsilon_1} + 1\right)}{\left(\dfrac{1}{\cos\varepsilon_1} + \cos(2\eta_1 - \varepsilon_1)\right)^2} \tag{4.10}$$

is periodic. Here $\varepsilon_1 = \dfrac{2\pi}{10}$ *or* $\dfrac{4\pi}{10}$. *The velocity for the solution is* $v = -k_1^2\dfrac{\sin 3\varepsilon_1}{\sin\varepsilon_1}$. *If* $\varepsilon_1 = \frac{2\pi}{10}$, u *in Eq. (4.10) is a left-going periodic wave. If* $\varepsilon_1 = \frac{4\pi}{10}$, u *in Eq. (4.10) is a right-going periodic wave.*

Proof.

$$\tau_{\rm bKK}^{(1+1)} = \int \left(\phi_1^{(0)}\right)^2 = \frac{e^{2(xk_1\cos\varepsilon_1 + tk_1^3\cos 3\varepsilon_1)}}{4k_1}\left[\frac{1}{\cos\varepsilon_1} + \cos(2\eta_1 - \varepsilon_1)\right]. \tag{4.11}$$

Here $\eta_1 = xk_1\sin\varepsilon_1 + tk_1^3\sin 3\varepsilon_1$, and $\phi_1^{(0)} = \phi(\lambda_1; x, t)$ is defined by Eq. (4.5) □

Based on Propositions 7 and 8, we can find following corresponding relationship between symmetrical distributions of 5-th roots of $e^{i\varepsilon}$ and moving direction of solutions.

1. $(e^{i\varepsilon_1}, -e^{-i\varepsilon_1})|_{\varepsilon_1=\frac{\pi}{10}} \longrightarrow$ the first distribution of 5-th roots of $e^{i\varepsilon} \longrightarrow (p_1 = k_1e^{i\varepsilon_1}, q_1 = -k_1e^{-i\varepsilon_1})|_{\varepsilon_1=\frac{\pi}{10}}$ in Eq. (4.4) $\longrightarrow$ left-going two-peak soliton in Eq. (4.7);
2. $(e^{i\varepsilon_1}, -e^{-i\varepsilon_1})|_{\varepsilon_1=\frac{3\pi}{10}} \longrightarrow$ the second distribution of 5-th roots of $e^{i\varepsilon} \longrightarrow (p_1 = k_1e^{i\varepsilon_1}, q_1 = -k_1e^{-i\varepsilon_1})|_{\varepsilon_1=\frac{3\pi}{10}}$ in Eq. (4.4) $\longrightarrow$ right-going one-peak soliton in Eq. (4.7);
3. $(e^{i\varepsilon_1}, e^{-i\varepsilon_1})|_{\varepsilon_1=\frac{2\pi}{10}} \longrightarrow$ the third distribution of 5-th roots of $e^{i\varepsilon} \longrightarrow (p_1 = k_1e^{i\varepsilon_1}, q_1 = k_1e^{-i\varepsilon_1})|_{\varepsilon_1=\frac{2\pi}{10}}$ in Eq. (4.5) $\longrightarrow$ left-going periodic wave in Eq. (4.10);
4. $(e^{i\varepsilon_1}, e^{-i\varepsilon_1})|_{\varepsilon_1=\frac{4\pi}{10}} \longrightarrow$ the fourth distribution of 5-th roots of $e^{i\varepsilon} \longrightarrow (p_1 = k_1e^{i\varepsilon_1}, q_1 = k_1e^{-i\varepsilon_1})|_{\varepsilon_1=\frac{4\pi}{10}}$ in Eq. (4.5) $\longrightarrow$ right-going periodic wave in Eq. (4.10).

There are only four distributions of 5-th roots of $e^{i\varepsilon}$, which are symmetric respect with x-axes or y-axes. However, there exist several other pairs of roots in the above four distributions which will result in divergent solutions of bKK through the above procedure. For example, $p_1 = k_1e^{i\frac{13\pi}{10}}, q_1 = -k_1e^{-i\frac{13\pi}{10}}$ or $p_1 = k_1e^{i\frac{8\pi}{10}}, q_1 = k_1e^{-i\frac{8\pi}{10}}$.

Let us now concentrate on the two-peak soliton solution in Eq. (4.7).

Lemma 5. *Let $x \geq 1$, constant $a > 0$ and function*

$$y = y(x) = \frac{1 + \dfrac{x}{a}}{\left(x + \dfrac{1}{a}\right)^2}, \tag{4.12}$$

then
1) if $a > 1/2$, $\dfrac{\partial y}{\partial x} < 0$;
2) if $a = 1/2$, then $\dfrac{\partial y}{\partial x}|_{x=1} = 0$;
3) if $\frac{1}{2} > a > 0$, then there exists one point $x_1 > 1$ such that $\left.\dfrac{\partial y}{\partial x}\right|_{x=x_1} = 0$, x_1 is one extreme maximum point of y, and $\left.\dfrac{\partial y}{\partial x}\right|_{x=1} > 0$.

Proof. We have

$$y_x = \frac{\partial y}{\partial x} = \frac{\frac{1}{a}\left(\frac{1}{a} - 2a - x\right)}{\left(x + \frac{1}{a}\right)^3}. \tag{4.13}$$

Firstly, $y_x < 0$ if $a > 1/2$. Secondly, if $a = 1/2$, $y_x = 0$ when $x = 1$. At last, if $1/2 > a > 0$, there exist $x_1 > 1$ such that $y_x = 0$. Note that $y_x > 0$ if $x \in (1, x_1)$, $y_x < 0$ if $x > x_1$. So x_1 is one extreme maximum point of y. □

Proposition 9. *Let a, b, k be positive constants, $\xi = kx + ct, c \in \mathbb{R}$, for following kind of solution*

$$u = \frac{b\left(1 + \dfrac{\cosh 2\xi}{a}\right)}{\left(\cosh 2\xi + \dfrac{1}{a}\right)^2}, \tag{4.14}$$

1) if $a \geq 1/2$, u has one peak in its profile defined by $\xi = 0$;
2) if $0 < a < \frac{1}{2}$, then there exist two peaks in profile;
3) There exist no more than two peaks in a soliton give by Eq. (4.14).

Proof. By calculation, we have

$$u_x = \frac{2kb\sinh 2\xi\left(\dfrac{1}{a} - 2a - \cosh 2\xi\right)}{a\left(\cosh 2\xi + \dfrac{1}{a}\right)^3}.$$

According to the Lemma 5, we have
1) $a > 1/2$, there exist $\xi = 0$ such that $u_x = 0$ because $\sinh 2\xi|_{\xi=0} = 0$. Note $\left(\dfrac{1}{a} - 2a - \cosh 2\xi\right) < 0$.
2) $a = 1/2$, there exist $\xi = 0$ such that $u_x = 0$ because $\sinh 2\xi|_{\xi=0} = 0$ and $\left.\left(\dfrac{1}{a} - 2a - \cosh 2\xi\right)\right|_{\xi=0} = 0$. However, let $|\xi|$ be sufficiently small, we have $u_x < 0$ if $\xi > 0$ and $u_x > 0$ if $\xi < 0$. So $\xi = kx + ct = 0$ defines one extreme maximum line of $u(x,t)$ on (x,t) plane.
3) $1/2 > a > 0$, there exist $\xi = 0$ and $\xi_1 > 0$ and $\xi_2 = -\xi_1 < 0$ such that $u_x = 0$. But $u_x > 0$ if $\xi < -\xi_1$; $u_x < 0$ if $\xi \in (-\xi_1, 0)$; $u_x > 0$ if $\xi \in (0, \xi_1)$; $u_x < 0$ if $\xi > \xi_1$. So $\xi = kx + ct = 0$ defines one extreme minimal line on (x,t) plane; $0 < \xi_1 = kx + ct$ and $0 > -\xi_1 = -(kx + ct)$ define two extreme maximum lines on the (x,t) plane. Using $u \to 0$ if $|\xi| \to \infty$, conclusions are proven. □

Comparing Eq. (4.14) with Eq. (4.7) we get $a = \sin\varepsilon_1$, and then can understand why $\sin\varepsilon_1|_{\varepsilon_1=\pi/10}$ will lead to two peaks in one soliton of bKK but $\sin\varepsilon_1|_{\varepsilon_1=3\pi/10}$ will lead only to one peak in one soliton of bKK. On the other hand, one soliton solution of u in Eq. (4.7) have one peak or two peaks(maximum case) in its profile. According to analysis above, we can claim from the point of view of reduction in KP hierarchy that the existence of two peaks in the soliton is traced to three facts:

1. The Grammian τ function in Proposition 6 which determines the form of soliton in Eq. (4.7);
2. The order of n-reduction, i.e. $n \geq 5$ can produce two peaks soliton in KP hierarchy;
3. The phase ε_1 of n-th root($n \geq 5$) of $e^{i\varepsilon}$, such that $0 < a = \sin\varepsilon_1 < 1/2$.

Now we turn to the more complicated $\tau^{(2+2)}_{\text{bKK}}$ from Proposition 6, which generates the two soliton and periodic solution with two spectral parameters of bKK equation. The first case is the two soliton solution.

Lemma 6. *Let $\phi_i^{(0)} = \phi(\lambda_i; x, t), i = 1, 2$, defined by Eq.(4.4), $\xi_i = xk_i\cos\varepsilon_i + tk_i^3\cos 3\varepsilon_i, \eta_i = xk_i\sin\varepsilon_i + tk_i^3\sin 3\varepsilon_i, i = 1, 2$, then $\tau^{(n+n)}_{\text{bKK}}|_{n=2}$ gives out*

$$\begin{aligned}
\tau^{(2+2)}_{\text{bKK}} &= A_1^2A_2^2e^{2i(\eta_1+\eta_2)}\times \\
&\Bigg\{\frac{z_1^* e^{2(\xi_1+\xi_2)}}{4\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)^2} + \frac{z_1 e^{-2(\xi_1+\xi_2)}}{4\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)^2}\left(\frac{B_1}{A_1}\right)^2\left(\frac{B_2}{A_2}\right)^2 \\
&+ \frac{-z_3^* e^{2(\xi_1-\xi_2)}}{4ik_1k_2\big(k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)^2}\left(\frac{B_2}{A_2}\right)^2 + \frac{-z_3 e^{-(2\xi_1-\xi_2)}}{4ik_1k_2\big(k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)^2}\left(\frac{B_1}{A_1}\right)^2 \\
&+ \frac{z_2^* e^{2\xi_1}}{2ik_1k_2\sin\varepsilon_2\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)\big(k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)}\left(\frac{B_2}{A_2}\right) \\
&+ \frac{-z_2 e^{-2\xi_1}}{2ik_1k_2\sin\varepsilon_2\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)\big(k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)}\left(\frac{B_1}{A_1}\right)^2\left(\frac{B_2}{A_2}\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{z_4^* e^{2\xi_2}}{2ik_1k_2\sin\varepsilon_1\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)\big(k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)}\left(\frac{B_1}{A_1}\right)\\
&+ \frac{-z_4 e^{-2\xi_2}}{2ik_1k_2\sin\varepsilon_1\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)\big(k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)}\left(\frac{B_1}{A_1}\right)\left(\frac{B_2}{A_2}\right)^2\\
&+ \frac{-\big((k_1^2+k_2^2)^2-4k_1^2k_2^2(\cos^2\varepsilon_1\cos^2\varepsilon_2+\sin^2\varepsilon_1\sin^2\varepsilon_2)\big)}{2k_1k_2\sin\varepsilon_1\sin\varepsilon_2\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)\big(k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)}\left(\frac{B_1}{A_1}\right)\left(\frac{B_2}{A_2}\right)\Bigg\}
\end{aligned}
\tag{4.15}
$$

Here $z_i, i=1,2,3,4$, are given in Appendix B. z_i^* means the complex conjugation of z_i.

In order to extract physical τ function $\hat{\tau}_{\mathrm{bKK}}^{(2+2)}$, we need following two Lemmas for suitable $\frac{B_i}{A_i}, i=1,2$.

Lemma 7. *For $z_i, i=1,2,3,4$, are given in Appendix B, the following identities*

$$z_2^2 = z_1 z_3, \quad z_4^2 = z_1 z_3^* \tag{4.16}$$

hold.

Lemma 8. *Let $\frac{B_1}{A_1} = i\frac{z_1^*}{z_4^*}, \frac{B_2}{A_2} = i\frac{z_1^*}{z_2^*}$, and $g_5 = \frac{1}{|z_3|}$, $g_6 = g_8 = \frac{|z_1|^2}{|z_2|^2}$, $g_9 = \frac{|z_1|^4}{|z_2|^4}$, then*

$$-\frac{z_3^*}{z_1^*}\left(\frac{B_2}{A_2}\right)^2 = 1, \qquad -\frac{z_3}{z_1^*}\left(\frac{B_1}{A_1}\right)^2 = 1, \tag{4.17}$$

$$\frac{z_1}{z_1^*}\left(\frac{B_1}{A_1}\right)^2\left(\frac{B_2}{A_2}\right)^2 = g_9, \qquad \frac{-1}{z_1^*}\left(\frac{B_1}{A_1}\right)\left(\frac{B_2}{A_2}\right) = g_5, \tag{4.18}$$

$$\frac{z_2}{-iz_1^*}\left(\frac{B_1}{A_1}\right)^2\left(\frac{B_2}{A_2}\right) = g_8, \qquad \frac{(z_4)^*}{-iz_1^*}\left(\frac{B_1}{A_1}\right)\left(\frac{B_2}{A_2}\right)^2 = g_6 \tag{4.19}$$

hold.

Taking $\frac{B_i}{A_i}, i=1,2$, and relations in Lemma 8 back into Lemma 6, the physical τ function $\hat{\tau}_{\mathrm{bKK}}^{(2+2)}$ is obtained.

Proposition 10.

$$
\begin{aligned}
&\hat{\tau}_{\mathrm{bKK}}^{(2+2)} =\\
&\Bigg\{\frac{e^{2(\xi_1+\xi_2)}}{4k_1k_2\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)^2} + \frac{g_9 e^{-2(\xi_1+\xi_2)}}{4k_1k_2\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)^2}\\
&+ \frac{e^{2(\xi_1-\xi_2)}}{4k_1k_2\big(k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)^2} + \frac{e^{-(2\xi_1-\xi_2)}}{4k_1k_2\big(k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)^2}\\
&+ \frac{e^{2\xi_1}}{2k_1k_2\sin\varepsilon_2\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)\big(k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)}\\
&+ \frac{g_8 e^{-2\xi_1}}{2k_1k_2\sin\varepsilon_2\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)\big(k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)}\\
&+ \frac{e^{2\xi_2}}{2k_1k_2\sin\varepsilon_1\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)\big(k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)}
\end{aligned}
$$

$$+\frac{g_6 e^{-2\xi_2}}{2k_1k_2\sin\varepsilon_2\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)\big(k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)}$$
$$+\frac{g_5\big((k_1^2+k_2^2)^2-4k_1^2k_2^2(\cos^2\varepsilon_1\cos^2\varepsilon_2+\sin^2\varepsilon_1\sin^2\varepsilon_2)\big)}{k_1k_2\sin\varepsilon_1\sin\varepsilon_2\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)\big(k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)}\Bigg\} \tag{4.20}$$

The two solitons solution is $u=(\partial_x^2\log\hat\tau_{\mathrm{bKK}}^{(2+2)})$. *In particular,* $\varepsilon_1=\varepsilon_2=\frac{\pi}{10}$ *results in two overtaking soltions moving in negative direction;* $\varepsilon_1=\varepsilon_2=\frac{3\pi}{10}$ *results in two overtaking soltions moving in positive direction;* $\varepsilon_1=\frac{\pi}{10},\varepsilon_2=\frac{3\pi}{10}$ *results in head-on colliding two soltions.*

The second case is a periodic solution with two spectral parameters of bKK equation from Proposition 6.

Lemma 9. *Let* $\phi_i^{(0)}=\phi(\lambda_i;x,t), i=1,2$, *defined by Eq.(4.5),* $\xi_i=xk_i\cos\varepsilon_i+tk_i^3\cos3\varepsilon_i, \eta_i=xk_i\sin\varepsilon_i+tk_i^3\sin3\varepsilon_i(i=1,2)$, *then* $\tau_{\mathrm{bKK}}^{(n+n)}\Big|_{n=2}$ *gives*

$$\tau_{\mathrm{bKK}}^{(2+2)}=A_1^2A_2^2e^{2i(\xi_1+\xi_2)}\times$$
$$\Bigg\{\frac{z_1^*e^{2i(\eta_1+\eta_2)}}{4\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)^2}+\frac{z_1e^{-2i(\eta_1+\eta_2)}}{4\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)^2}\left(\frac{B_1}{A_1}\right)^2\left(\frac{B_2}{A_2}\right)^2$$
$$+\frac{z_3^*e^{2i(\eta_1-\eta_2)}}{4k_1k_2\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)^2}\left(\frac{B_2}{A_2}\right)^2+\frac{z_3e^{-i(2\eta_1-\eta_2)}}{4k_1k_2\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)^2}\left(\frac{B_1}{A_1}\right)^2$$
$$+\frac{z_2^*e^{2i\eta_1}}{2k_1k_2\cos\varepsilon_2\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)}\left(\frac{B_2}{A_2}\right)$$
$$+\frac{z_2e^{-2i\eta_1}}{2k_1k_2\cos\varepsilon_2\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)}\left(\frac{B_1}{A_1}\right)^2\left(\frac{B_2}{A_2}\right)$$
$$+\frac{z_4^*e^{2i\eta_2}}{2k_1k_2\cos\varepsilon_1\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)}\left(\frac{B_1}{A_1}\right)$$
$$+\frac{z_4e^{-2i\eta_2}}{2k_1k_2\cos\varepsilon_1\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)}\left(\frac{B_1}{A_1}\right)\left(\frac{B_2}{A_2}\right)^2$$
$$+\frac{\big((k_1^2+k_2^2)^2-4k_1^2k_2^2(\cos^2\varepsilon_1\cos^2\varepsilon_2+\sin^2\varepsilon_1\sin^2\varepsilon_2)\big)}{k_1k_2\cos\varepsilon_1\cos\varepsilon_2\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)}\left(\frac{B_1}{A_1}\right)\left(\frac{B_2}{A_2}\right)\Bigg\} \tag{4.21}$$

Here $z_i, i=1,2,3,4$, *are given in Appendix C.* z_i^* *indicates the complex conjugation of* z_i.

Similar to the two solitons solution of bKK equation, we need following two Lemmas to find suitable $\dfrac{B_i}{A_i}, i=1,2$, to extract physical $\hat\tau_{\mathrm{bKK}}^{(2+2)}$ from Eq. (4.21) for periodic solution.

Lemma 10. *For* $z_i, i=1,2,3,4$, *are given in Appendix C, the following identities*

$$z_2^2=z_1z_3,\quad z_4^2=z_1z_3^* \tag{4.22}$$

hold.

Lemma 11. *Let* $z_k=|z_k|e^{i\theta_k}, k=1,2,3,4$, *are given in Appendix C, and* $\dfrac{B_1}{A_1}=e^{-i\theta_2}, \dfrac{B_2}{A_2}=e^{-i\theta_4}$, *and* $g_2=\dfrac{|z_2|}{|z_1|}$, $g_3=\dfrac{|z_3|}{|z_1|}$, $g_4=\dfrac{|z_4|}{|z_1|}$, $g_5=\dfrac{1}{|z_1|}$, *then*

$$\frac{z_3^*}{z_1^*}\left(\frac{B_2}{A_2}\right)^2=\frac{z_3}{z_1^*}\left(\frac{B_1}{A_1}\right)^2=g_3, \tag{4.23}$$

$$\frac{z_1}{z_1^*}\left(\frac{B_1}{A_1}\right)^2\left(\frac{B_2}{A_2}\right)^2 = 1, \quad \frac{1}{z_1^*}\left(\frac{B_1}{A_1}\right)\left(\frac{B_2}{A_2}\right) = g_5, \tag{4.24}$$

$$\frac{z_2^*}{z_1^*}\frac{B_2}{A_2} = \frac{z_2}{z_1^*}\left(\frac{B_1}{A_1}\right)^2\left(\frac{B_2}{A_2}\right) = g_2, \quad \frac{z_4^*}{z_1^*}\frac{B_1}{A_1} = \frac{z_4}{z_1^*}\left(\frac{B_1}{A_1}\right)^2\left(\frac{B_2}{A_2}\right) = g_4 \tag{4.25}$$

hold.

We can get the physical τ function $\hat{\tau}_{\mathrm{bKK}}^{(2+2)}$ by taking $\dfrac{B_i}{A_i}, i = 1, 2$ and relations in Lemma 11 back into Lemma 9.

Proposition 11.

$$\begin{aligned}
&\hat{\tau}_{\mathrm{bKK}}^{(2+2)} = \\
&\Big\{\frac{2\cos 2(\eta_1+\eta_2)}{4\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)^2} + \frac{2g_3\cos 2(\eta_1-\eta_2)}{4k_1k_2\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)^2} \\
&+ \frac{2g_2\cos 2\eta_1}{2k_1k_2\cos\varepsilon_2\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)} \\
&+ \frac{2g_4\cos 2\eta_2}{2k_1k_2\cos\varepsilon_1\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)} \\
&+ \frac{g_5\big((k_1^2+k_2^2)^2-4k_1^2k_2^2(\cos^2\varepsilon_1\cos^2\varepsilon_2+\sin^2\varepsilon_1\sin^2\varepsilon_2)\big)}{k_1k_2\cos\varepsilon_1\cos\varepsilon_2\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big)\big(k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big)}\Big\}
\end{aligned} \tag{4.26}$$

The periodic solution with two parameters k_1 and k_2 is $u = \left(\partial_x^2 \log \hat{\tau}_{\mathrm{bKK}}^{(2+2)}\right)$. Furthermore, $\varepsilon_1 = \varepsilon_2 = \frac{2\pi}{10}$ results in two overtaking waves moving in negative direction ; $\varepsilon_1 = \varepsilon_2 = \frac{4\pi}{10}$ results in two overtaking waves moving in positive direction; $\varepsilon_1 = \frac{2\pi}{10}, \varepsilon_2 = \frac{4\pi}{10}$ results in two head-on colliding waves.

We have plotted soliton solutions of bKK in Fig. 3, and there periodic solutions with two spectral parameters in Fig. 4.

5. Periodic and soliton solutions of bSH equation

The τ function of the bSH equation is still in the form of a Grammian although the bSH equation does not belong to the CKP hierarchy, which is obtained in [38] through the Bäcklund transformation. Similar to the bKK equation, its τ function is in the form of Grammian, we can find τ function $\tau_{\mathrm{bSH}}^{(1+1)}$ and $\tau_{\mathrm{bSH}}^{(2+2)}$of bSH from Grammian τ function. Let the initial value be $u = 0$ in Eqs. (1.6) and (1.7), then $\phi_i^{(0)} = \phi(\lambda_i; x, t)$ are solutions of

$$\partial_x^4\phi(\lambda_i; x, t) = \lambda_i\phi(\lambda_i; x, t), \quad \frac{\partial\phi(\lambda_i; x, t)}{\partial t} = \left(\partial_x^3\phi(\lambda_i; x, t)\right). \tag{5.1}$$

Proposition 12 [38]. *The τ function of* bSH *equation generated by Bäcklund transformation from initial value $u = 0$ is*

$$\tau_{\mathrm{bSH}}^{(n+n)} = IW_{n,n}(\phi_n^{(0)}, \phi_{n-1}^{(0)}, \cdots, \phi_1^{(0)}; \phi_1^{(0)}, \phi_2^{(0)}, \cdots, \phi_n^{(0)})$$

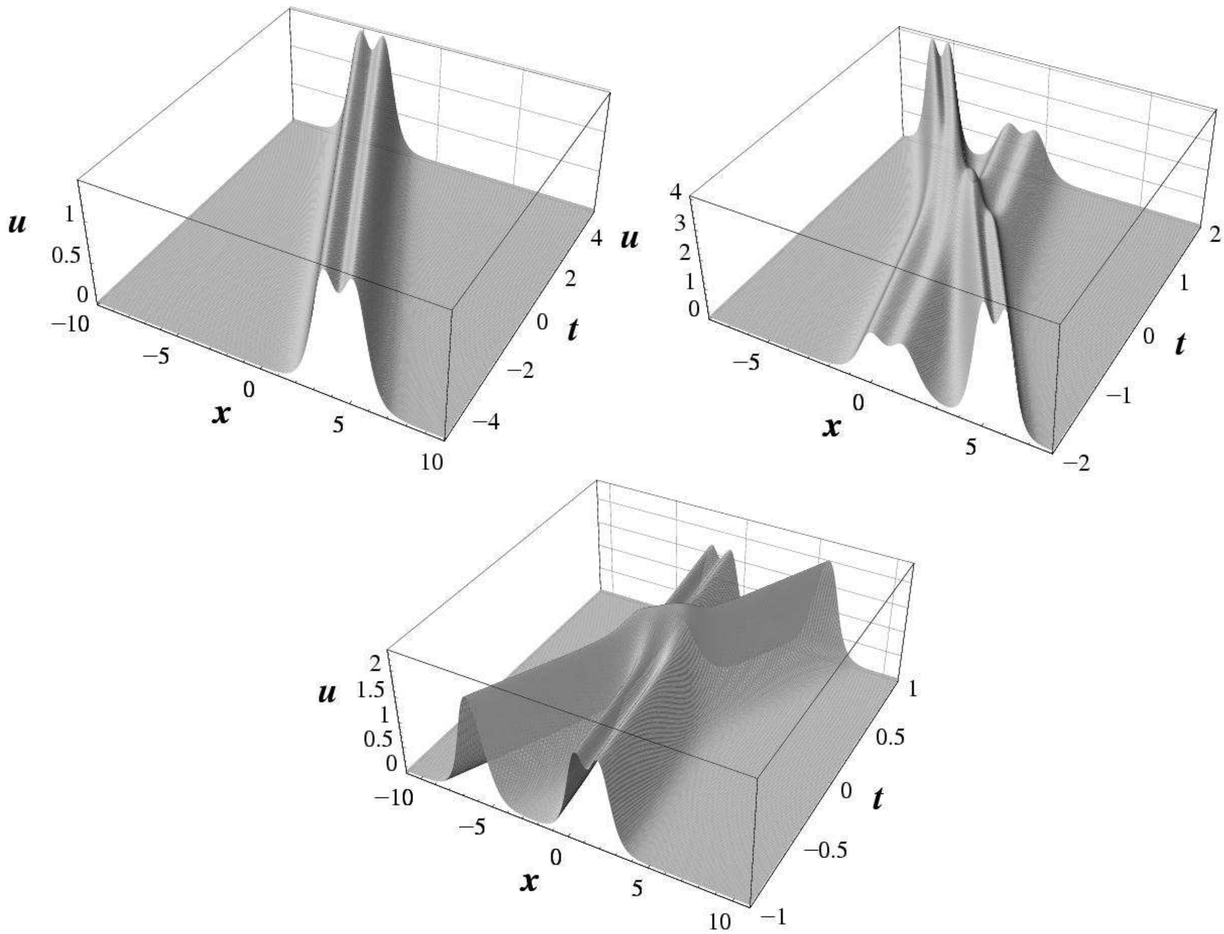

Fig. 3. Soliton solutions of the bKK equation (1.15). Top left: one left-going (two-peak) soliton when $\varepsilon_1 = \frac{\pi}{10}$ and $k_1 = 1.2$. Top right: two left-going soliton when $k_1 = 2, k_2 = 1.3, \varepsilon_1 = \varepsilon_2 = \frac{\pi}{10}$. Bottom: Head-on collision of left- and right-going solitons when $k_1 = 1.8, k_2 = 1.3, \varepsilon_1 = \frac{3\pi}{10}, \varepsilon_2 = \frac{\pi}{10}$.

$$= \begin{vmatrix} \int \phi_n^{(0)} \cdot \phi_1^{(0)} & \int \phi_n^{(0)} \cdot \phi_2^{(0)} & \int \phi_n^{(0)} \cdot \phi_3^{(0)} & \cdots & \int \phi_n^{(0)} \cdot \phi_{n-1}^{(0)} & \int \phi_n^{(0)} \cdot \phi_n^{(0)} \\ \int \phi_{n-1}^{(0)} \cdot \phi_1^{(0)} & \int \phi_{n-1}^{(0)} \cdot \phi_2^{(0)} & \int \phi_{n-1}^{(0)} \cdot \phi_3^{(0)} & \cdots & \int \phi_{n-1}^{(0)} \cdot \phi_{n-1}^{(0)} & \int \phi_{n-1}^{(0)} \cdot \phi_n^{(0)} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \\ \int \phi_2^{(0)} \cdot \phi_1^{(0)} & \int \phi_2^{(0)} \cdot \phi_2^{(0)} & \int \phi_2^{(0)} \cdot \phi_3^{(0)} & \cdots & \int \phi_2^{(0)} \cdot \phi_{n-1}^{(0)} & \int \phi_2^{(0)} \cdot \phi_n^{(0)} \\ \int \phi_1^{(0)} \cdot \phi_1^{(0)} & \int \phi_1^{(0)} \cdot \phi_2^{(0)} & \int \phi_1^{(0)} \cdot \phi_3^{(0)} & \cdots & \int \phi_1^{(0)} \cdot \phi_{n-1}^{(0)} & \int \phi_1^{(0)} \cdot \phi_n^{(0)} \end{vmatrix} \tag{5.2}$$

and the solution u of bSH *from initial value zero is*

$$u = \left(\partial_x^2 \log \tau_{\text{bSH}}^{(n+n)}\right) \tag{5.3}$$

Here $\phi_i^{(0)} = \phi(\lambda_i; x, t)$ are solutions of Eq. (5.1).

In fact, $\tau_{\text{bSH}}^{(n+n)}$ can be generated by gauge transformation $T_{n+k}|_{n=k}$. The Lax pair of bSH is

$$L_{\text{bSH}} = \partial_x^4 + 4u\partial_x^2 + 4u_x\partial_x + 2u_{xx} + 4u^2 + v, \quad M_{\text{bSH}} = \partial_x^3 + 3u\partial_x + \frac{3}{2}u_x,$$

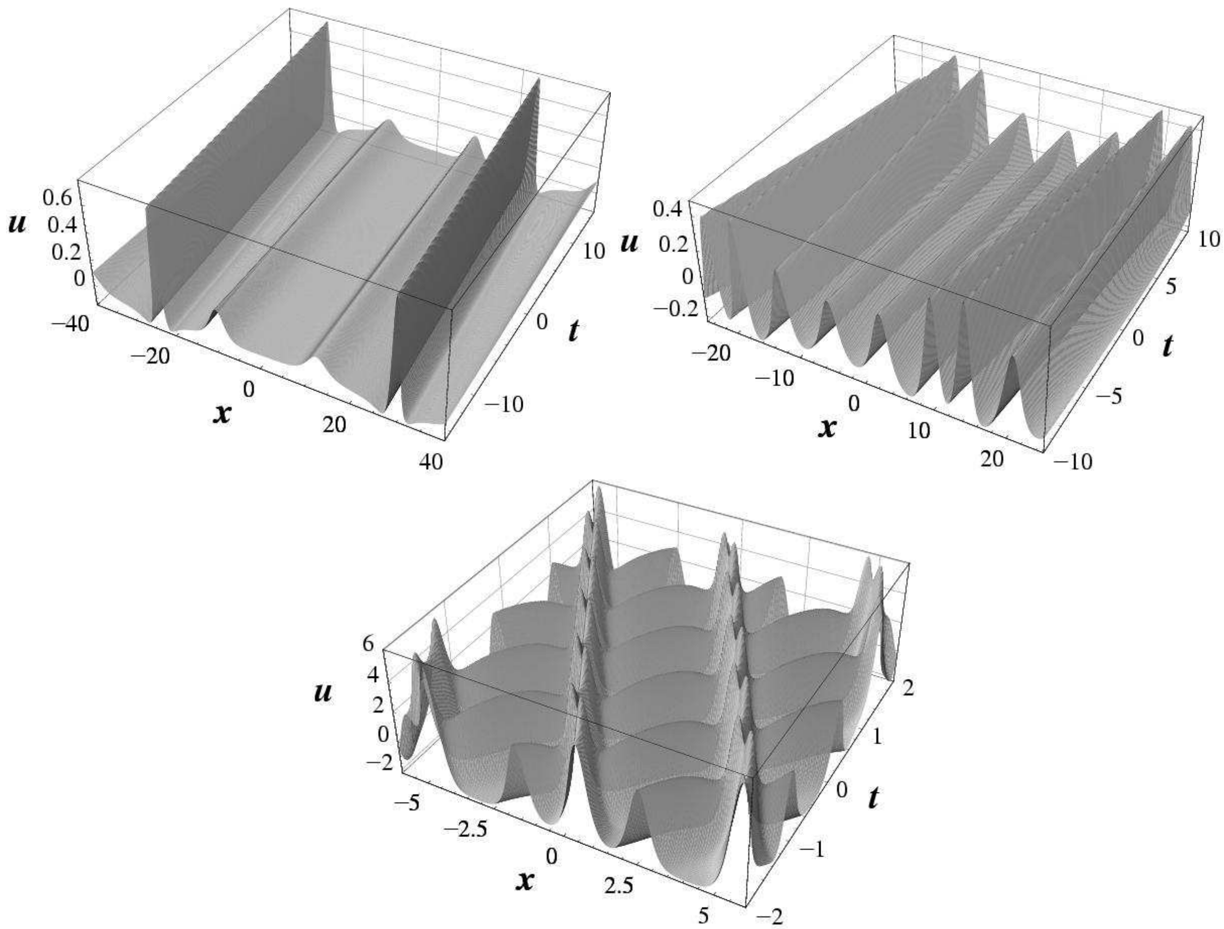

Fig. 4. Periodic solutions with two spectral parameters of bKK equation (1.15). Top left: left-going periodic solution with $k_1 = 0.2, k_2 = 0.3, \varepsilon_1 = \varepsilon_2 = \frac{2\pi}{10}$. Top right: right-going periodic solution when $k_1 = 0.4, k_2 = 0.5, \varepsilon_1 = \varepsilon_2 = \frac{4\pi}{10}$. Bottom: Collision of left- and right-going periodic solution when $k_1 = 1, k_2 = 1.5, \varepsilon_1 = \frac{2\pi}{10}, \varepsilon_2 = \frac{4\pi}{10}$.

and satisfy $L^*_{\mathrm{bSH}} = L_{\mathrm{bSH}}, M^*_{\mathrm{bSH}} = -M_{\mathrm{bSH}}$. Similar to the CKP hierarchy, let $T = T_{1+1} = T_I(\psi_1^{(1)})T_D(\phi_1^{(0)})$, and do gauge transformation $L^{(2)}_{\mathrm{bSH}} = TL_{\mathrm{bSH}}T^{-1}$. So $(L^{(2)}_{\mathrm{bSH}})^* = L^{(2)}_{\mathrm{bSH}}$ requires $T_D(\psi_1^{(1)})T_I(\phi_1^{(0)}) = T_I(\psi_1^{(1)})T_D(\phi_1^{(0)})$ as we have seen in CKP hierarchy. The remaining procedure is the same as the gauge transformation of the CKP hierarchy as well as the bKK equation. Of course, the generating functions $(\phi_i^{(0)}, \psi_i^{(0)}) = (\phi(\lambda_i; x, t), \psi(\lambda_i; x, t))$ satisfy Eq. (1.6) and Eq. (1.7) if the initial values are $u \neq 0, v \neq 0$, or Eq. (5.1) if the initial values are $u = 0, v = 0$.

Remark 1. We should note that $L_{\mathrm{bSH}}|_{v=0} = \partial_x^4 + 4u\partial_x^2 + 4u_x\partial_x + 2u_{xx} + 4u^2 = (\partial_x^2 + 2u)^2 = L^2_{\mathrm{KdV}}$. The Lax pair of the KdV equation is

$$L_{\mathrm{KdV}} = \partial_x^2 + 2u, \quad M_{\mathrm{KdV}} = \partial_x^3 + 3u\partial_x + \frac{3}{2}u_x.$$

$T_D(\phi_1^{(0)})$ generates a single soliton solution of the KdV from zero initial value. Here $\phi_1^{(0)} = \phi(\lambda_1; x, t)$ satisfy $L_{\mathrm{KdV}}\phi(\lambda_1; x, t) = \lambda_1\phi(\lambda_1; x, t)$ and $\dfrac{\partial\phi(\lambda_1; x, t)}{\partial t} = M_{\mathrm{KdV}}\phi(\lambda_1; x, t)$ simultaneously. The left-going multi-soliton can be produced by using repeated iteration of T_D.

In order to get real and smooth solutions, such as soliton and periodic solution, we should construct *physical* τ function $\hat{\tau}_{\rm bSH}$ from $\tau_{\rm bSH}^{(n+n)}$ which is complex and related to 4-th roots of $e^{i\varepsilon}$. The case of $n=1$ and $n=2$ will be discussed in detail. Let us start to discuss the single soliton with two directional propagation. To do this, similar to the above two sections, we should assume the solution of Eq. (5.1) as

$$\phi(\lambda_1;x,t)=A_1e^{p_1x+p_1^3t}+B_1e^{q_1x+q_1^3t},\quad p_1=k_1e^{i\varepsilon_1},q_1=-k_1e^{-i\varepsilon_1},k_1^4=|\lambda_1|,k_1\in\mathbb{R},\tag{5.4}$$

or

$$\phi(\lambda_1;x,t)=A_1e^{p_1x+p_1^3t}+B_1e^{q_1x+q_1^3t},\quad p_1=k_1e^{i\varepsilon_1},q_1=k_1e^{-i\varepsilon_1},k_1^4=|\lambda_1|,k_1\in\mathbb{R}.\tag{5.5}$$

For $\phi(\lambda_i;x,t),i=1,2$, the difference between here and above two sections is the $k_i^4=|\lambda_i|,i=1,2$, instead of $k_i^5=|\lambda_i|,i=1,2$. From Proposition 12 we can extract *physical* τ function $\hat{\tau}_{\rm bSH}^{(1+1)}$ from $\tau_{\rm bSH}^{(n+n)}\Big|_{n=1}$.

Proposition 13. *Let* $\xi_1=xk_1\cos\varepsilon_1+tk_1^3\cos 3\varepsilon_1$, $\frac{B_1}{A_1}=ie^{-\varepsilon_1}$, *and* $\phi_1^{(0)}=\phi(\lambda_1;x,t)$ *as defined by Eq. (5.4), then the physical* τ *function of* bSH *extracted from* $\tau_{\rm bSH}^{(n+n)}|_{n=1}$ *is*

$$\hat{\tau}_{\rm bSH}^{(1+1)}=e^{2\xi_1}+e^{-2\xi_1}+\frac{2}{\sin\varepsilon_1}\tag{5.6}$$

and the corresponding single soliton $u=\left(\partial_x^2\log\hat{\tau}_{\rm bSH}^{(1+1)}\right)$ *is*

$$u=\frac{4k_1^2(\cos\varepsilon_1)^2\left(1+\dfrac{\cosh 2\xi_1}{\sin\varepsilon_1}\right)}{\left(\cosh 2\xi_1+\dfrac{1}{\sin\varepsilon_1}\right)^2}.\tag{5.7}$$

Here $\varepsilon_1=\frac{\pi}{4}$. *The velocity of the soliton is* $v=-k_1^2\dfrac{\cos 3\varepsilon_1}{\cos\varepsilon_1}|_{\varepsilon_1=\frac{\pi}{4}}>0$.

Proof.

$$\tau_{\rm bSH}^{(1+1)}=\int(\phi_1^{(0)})^2=\frac{A_1^2e^{2i(xk_1\sin\varepsilon_1+tk_1^3\sin 3\varepsilon_1)}}{2p_1}\left(e^{2\xi_1}+e^{-2\xi_1}+\frac{2}{\sin\varepsilon_1}\right)\tag{5.8}$$

□

As we discussed in Remark 1, the left-going soliton can also be generated by T_D.

Proposition 14. *Let* $\xi_1=(xk_1\cos\varepsilon_1+tk_1^3\cos 3\varepsilon_1)|_{\varepsilon_1=0}$, *and* $\phi_1^{(0)}=\phi(\lambda_1;x,t)|_{\varepsilon_1=0}$ *as defined by Eq. (5.4), then the physcial* τ *function of* bSH *generated by* $T_D(\phi_1^{(0)})$ *is*

$$\hat{\tau}_{\rm bSH}^{(1)}=1+\frac{A_1}{B_1}e^{2\xi_1}\tag{5.9}$$

and the corresponding single soliton $u=\left(\partial_x^2\log\hat{\tau}_{\rm bSH}^{(1)}\right)$ *is*

$$u=\frac{4k_1^2\dfrac{A_1}{B_1}}{\left(e^{-\xi_1}+\dfrac{A_1}{B_1}e^{\xi_1}\right)^2}.\tag{5.10}$$

Here $\frac{A_1}{B_1}>0$. *The velocity of the soliton is* $v=-k_1^2<0$.

Proof.

$$\tau_{\rm bSH}^{(1)}=\phi_1^{(0)}=B_1\left(e^{-\xi_1}+\frac{A_1}{B_1}e^{\xi_1}\right) \tag{5.11}$$

It can be clarified by $(u=0,v=0)\xrightarrow{T_D(\phi_1^{(0)})}(u^{(1)}\neq 0,v^{(1)}=0)$, and then $\tau_{\rm bSH}^{(1)}=\phi_1^{(0)}$. □

On the other hand, if $\phi_1^{(0)}=\phi(\lambda_1;x,t)$ as defined by Eq. (5.5), then we can get periodic solution from Proposition 12.

Proposition 15. *Let* $\eta_1=xk_1\sin\varepsilon_1+tk_1^3\sin 3\varepsilon_1$, $A_1=B_1=1$ *in* $\phi_1^{(0)}$, *then the physical* τ *function of* bSH *equation for periodic solution extracted from* $\tau_{\rm bSH}^{(n+n)}|_{n=1}$ *is*

$$\hat\tau_{\rm bSH}^{(1+1)}=\frac{1}{\cos\varepsilon_1}+\cos(2\eta_1-\varepsilon_1) \tag{5.12}$$

and the corresponding peroidic solution $u=\left(\partial_x^2\log\hat\tau_{\rm bSH}^{(1+1)}\right)$ *is*

$$u=\frac{-4k_1^2\sin^2\varepsilon_1\left(\dfrac{\cos(2\eta_1-2\varepsilon_1)}{\cos\varepsilon_1}+1\right)}{(\dfrac{1}{\cos\varepsilon_1}+\cos(2\eta_1-\varepsilon_1))^2}. \tag{5.13}$$

Here $\varepsilon_1=\frac{\pi}{4}$. *The velocity of the solution is* $v=-k_1^2\dfrac{\sin 3\varepsilon_1}{\sin\varepsilon_1}|_{\varepsilon_1=\frac{\pi}{4}}<0$.

Proof.

$$\tau_{\rm bSH}^{(1+1)}=\int(\phi_1^{(0)})^2=\frac{e^{2(xk_1\cos\varepsilon_1+tk_1^3\cos 3\varepsilon_1)}}{4k_1}\left(\frac{1}{\cos\varepsilon_1}+\cos(2\eta_1-\varepsilon_1)\right). \tag{5.14}$$

□

There are some relationship between the distributions of 4-th roots of $e^{i\varepsilon}$ and moving direction of solutions.

1. $(e^{i\varepsilon_1},-e^{-i\varepsilon_1})|_{\varepsilon_1=0}$ the first distribution of 4-th roots of $e^{i\varepsilon}$ $-->$ $(p_1=k_1e^{i\varepsilon_1},q_1=-k_1e^{-i\varepsilon_1})|_{\varepsilon_1=0}$ in Eq. (5.4) $-->$ left-going soliton in Eq. (5.10);
2. $(e^{i\varepsilon_1},-e^{-i\varepsilon_1})|_{\varepsilon_1=\frac{\pi}{4}}$ the second distribution of 4-th roots of $e^{i\varepsilon}$ $-->$ $(p_1=k_1e^{i\varepsilon_1},q_1=-k_1e^{-i\varepsilon_1})|_{\varepsilon_1=\frac{\pi}{4}}$ in Eq. (5.4) $-->$ right-going soliton in Eq. (5.7);
3. $(e^{i\varepsilon_1},e^{-i\varepsilon_1})|_{\varepsilon_1=\frac{\pi}{4}}$ the third distribution of 4-th roots of $e^{i\varepsilon}$ $-->$ $(p_1=k_1e^{i\varepsilon_1},q_1=k_1e^{-i\varepsilon_1})|_{\varepsilon_1=\frac{\pi}{4}}$ in Eq. (5.5) $-->$ left-going periodic wave in Eq. (5.13).

In the above discussion, we know the right-going soliton and left-going periodic wave of of bSH have the completely same form with the bKK equation, except $\varepsilon_1=\pi/4$ instead of $\varepsilon_1=\pi/10$ and $\varepsilon_1=3\pi/10$. The reason is that the τ function of two equations is in the same Grammian of generating functions $\phi_i^{(0)}$, and generating functions $\phi_i^{(0)}$ for two equations satisfy analogous linear partial differential equations with constant coefficients, i.e. Eq. (4.1) for bKK equation, Eq. (5.1) for bSH equation. These relations between bKK and bSH are still true for their two soliton and two parameters periodic solutions.

Proposition 16. *The two right-going solitons are given*

$$u=\left(\partial_x^2\log\hat\tau_{\rm bSH}^{(2+2)}\right) \tag{5.15}$$

in which $\hat\tau_{\rm bSH}^{(2+2)}$ *is*

$$\hat\tau_{\rm bSH}^{(2+2)}=\hat\tau_{\rm bKK}^{(2+2)}|_{\varepsilon_1=\varepsilon_2=\pi/4} \tag{5.16}$$

and $\hat\tau_{\rm bKK}^{(2+2)}$ *is given by Proposition 10.*

Proposition 17. *The right-going periodic wave with two spectral parameters k_1 and k_2 is given by*

$$u = \left(\partial_x^2 \log \hat{\tau}_{\rm bSH}^{(2+2)}\right) \tag{5.17}$$

in which $\hat{\tau}_{\rm bSH}^{(2+2)}$ is

$$\hat{\tau}_{\rm bSH}^{(2+2)} = \hat{\tau}_{\rm bKK}^{(2+2)}|_{\varepsilon_1=\varepsilon_2=\pi/4} \tag{5.18}$$

and $\hat{\tau}_{\rm bKK}^{(2+2)}$ is given by Proposition 11.

According to the analysis in Remark 1, two left-going solitons of bSH equation can be generated by a chain of gauge transformations $(u=0, v=0) \xrightarrow{T_D(\phi_1^{(0)})} \left(u^{(1)} \neq 0, v^{(1)}=0\right) \xrightarrow{T_D(\phi_2^{(1)})} \left(u^{(2)} \neq 0, v^{(2)}=0\right)$(using the notation of [27]), $\phi_i^{(0)} = \phi(\lambda_i; x, t)|_{\varepsilon_1=0}, i=1,2$, are defined by Eq. (5.4). Their τ function of bSH generated by $T_2 = T_D(\phi_2^{(1)})T_D(\phi_1^{(0)})$ is

$$\tau_{\rm bSH}^{(2)} = \begin{vmatrix} \phi_1^{(0)} & \phi_2^{(0)} \\ \phi_{1,x}^{(0)} & \phi_{2,x}^{(0)} \end{vmatrix}. \tag{5.19}$$

From $\tau_{\rm bSH}^{(2)}$ we can obtain the *physical* τ function $\hat{\tau}_{\rm bSH}^{(2)}$ and two soliton solution.

Proposition 18. *Let $\phi_i^{(0)} = \phi(\lambda_i; x, t)|_{\varepsilon_1=0}$ are defined by Eq. (5.4), $\xi_i = k_i x + k_i^3 t, i=1,2$. If $\dfrac{B_1}{A_1} > 0, \dfrac{B_2}{A_2} < 0, k_2 > k_1$, then the physical τ function $\hat{\tau}_{\rm bSH}^{(2)}$ is given by*

$$\begin{aligned} \hat{\tau}_{\rm bSH}^{(2)} &= (k_2-k_1)e^{\xi_1+\xi_2} - \frac{B_1}{A_1}\frac{B_2}{A_2}(k_1+k_2)e^{-(\xi_1+\xi_2)} \\ &\quad -(k_2-k_1)\frac{B_2}{A_2}e^{\xi_1-\xi_2} + \frac{B_1}{A_1}(k_1+k_2)e^{-(\xi_1-\xi_2)}. \end{aligned} \tag{5.20}$$

The two soliton solution is $u = \left(\partial_x^2 \log \hat{\tau}_{\rm bSH}^{(2)}\right)$, which is left-going.

The collision of two soliton is generated by gauge transformation chain $(u=0, v=0) \xrightarrow{T_D(\phi_1^{(0)})} \left(u^{(1)} \neq 0, v^{(1)}=0\right) \xrightarrow{T_I(\psi_2^{(2)})T_D(\phi_2^{(1)})} \left(u^{(2)} \neq 0, v^{(2)} \neq 0\right)$, $\phi_1^{(0)} = \phi(\lambda_1; x, t)|_{\varepsilon_1=0}$ is defined by Eq. (5.4), $\psi_2^{(0)} = \phi_2^{(0)}$, $\phi_2^{(0)} = \phi(\lambda_2; x, t)$ is defined by Eq. (5.4). The corresponding τ function of bSH is

$$\tau_{\rm bSH}^{(2+1)} = \begin{vmatrix} \int \psi_2^{(0)}\phi_1^{(0)} & \psi_2^{(0)}\phi_2^{(0)} \\ \phi_1^{(0)} & \phi_2^{(0)} \end{vmatrix} \overset{\psi_2^{(0)}=\phi_2^{(0)}}{=\!=\!=} \begin{vmatrix} \int \phi_2^{(0)}\phi_1^{(0)} & \phi_2^{(0)}\phi_2^{(0)} \\ \phi_1^{(0)} & \phi_2^{(0)} \end{vmatrix}. \tag{5.21}$$

Taking $\phi_i^{(0)}, i=1,2$, back into Eq. (5.21), we have its explicit expression as following Lemma.

Lemma 12. *Let $\xi_1 = xk_1 + tk_1^3, \xi_2 = xk_2\cos\varepsilon_2 + tk_2^3\cos 3\varepsilon_2, \eta_2 = xk_2\sin\varepsilon_2 + tk_2^3\sin 3\varepsilon_2, z_i = c_i + d_i, i = 1,3,5$, are given in Appendix D.*

$$\begin{aligned} &\tau_{\rm bSH}^{(2+1)} = e^{2i\eta_2}A_2^2A_1 \times \\ &\Big\{\frac{z_1^* e^{\xi_1+2\xi_2}}{2k_2\left(k_1^2+k_2^2+2k_1k_2\cos\varepsilon_2\right)} - \frac{z_1 e^{-\xi_1-2\xi_2}}{2k_2\left(k_1^2+k_2^2+2k_1k_2\cos\varepsilon_2\right)}\left(\frac{B_2}{A_2}\right)^2\left(\frac{B_1}{A_1}\right) \end{aligned}$$

$$
\begin{aligned}
&+\frac{z_3^* e^{-\xi_1+2\xi_2}}{2k_2\left(k_1^2+k_2^2-2k_1k_2\cos\varepsilon_2\right)}\left(\frac{B_1}{A_1}\right)-\frac{z_3 e^{\xi_1-2\xi_2}}{2k_2\left(k_1^2+k_2^2-2k_1k_2\cos\varepsilon_2\right)}\left(\frac{B_2}{A_2}\right)^2\\
&+\frac{z_5 e^{\xi_1}}{ik_2\sin\varepsilon_2\left(k_1^2+k_2^2+2k_1k_2\cos\varepsilon_2\right)\left(k_1^2+k_2^2-2k_1k_2\cos\varepsilon_2\right)}\left(\frac{B_2}{A_2}\right)\\
&+\frac{z_5^* e^{-\xi_1}}{ik_2\sin\varepsilon_2\left(k_1^2+k_2^2+2k_1k_2\cos\varepsilon_2\right)\left(k_1^2+k_2^2-2k_1k_2\cos\varepsilon_2\right)}\left(\frac{B_2}{A_2}\right)\left(\frac{B_1}{A_1}\right)\Big\}. \qquad (5.22)
\end{aligned}
$$

Lemma 13. *For* $z_i = |z_i|e^{\theta_i}(i = 1, 3, 5)$, *the following identities*

$$z_1 z_5 = z_3 z_5^*, \quad e^{i(\theta_3-\theta_1)} = e^{2i\theta_3} \qquad (5.23)$$

are true.

Lemma 14. *Let* $z_i, i = 1, 3, 5$, *are given by Appendix D, if* $\frac{B_1}{A_1} = \frac{z_1^*}{z_3^*}$, $\frac{B_2}{A_2} = i\frac{z_1^*}{z_5}$, $g_2 = g_4 = \frac{|z_1|^2}{|z_5|^2}$, $g_6 = 1$, *then*

$$g_2 = -\frac{z_1}{z_1^*}\left(\frac{B_2}{A_2}\right)^2\left(\frac{B_1}{A_1}\right), \quad g_4 = -\frac{z_3}{z_1^*}\left(\frac{B_2}{A_2}\right)^2, \; g_6 = \frac{z_5^*}{iz_1^*}\left(\frac{B_2}{A_2}\right)\left(\frac{B_1}{A_1}\right), \qquad (5.24)$$

hold.

With the help of Lemmata 13 and 14, we deduce the *physical* τ function of colliding two soliton of bSH equation from Lemma 12.

Proposition 19. *Let* g_2, g_4 *are given in Lemma 14, then*

$$
\begin{aligned}
\hat{\tau}^{(2+1)}_{\mathrm{bSH}} = &\Big\{\frac{e^{\xi_1+2\xi_2}}{2k_2\left(k_1^2+k_2^2+2k_1k_2\cos\varepsilon_2\right)}+\frac{g_2 e^{-\xi_1-2\xi_2}}{2k_2\left(k_1^2+k_2^2+2k_1k_2\cos\varepsilon_2\right)}\\
&+\frac{e^{-\xi_1+2\xi_2}}{2k_2\left(k_1^2+k_2^2-2k_1k_2\cos\varepsilon_2\right)}+\frac{g_4 e^{\xi_1-2\xi_2}}{2k_2\left(k_1^2+k_2^2-2k_1k_2\cos\varepsilon_2\right)}\\
&+\frac{e^{\xi_1}}{k_2\sin\varepsilon_2\left(k_1^2+k_2^2+2k_1k_2\cos\varepsilon_2\right)\left(k_1^2+k_2^2-2k_1k_2\cos\varepsilon_2\right)}\\
&+\frac{e^{-\xi_1}}{k_2\sin\varepsilon_2\left(k_1^2+k_2^2+2k_1k_2\cos\varepsilon_2\right)\left(k_1^2+k_2^2-2k_1k_2\cos\varepsilon_2\right)}\Big\} \qquad (5.25)
\end{aligned}
$$

We have plotted the two soliton solutions of bSH equation in Fig. 5, and periodic solutions with one spectral parameter and with two spectral parameters of the same equation in Fig. 6.

6. Lower and Higher order reductions

In this section, we want to discuss the general character of soliton equation from lower order to higher order in one same sub-hierarchy. The purpose is to show the relation between propagation of soliton on (x,t) plane and the order of Lax pair, and show the difference between the lower reduction and higher reduction. Let Lax pair of soliton equation is (L, M), which defines $\phi(\lambda; x, t)$ by

$$L\phi(\lambda; x, t) = \lambda\phi(\lambda; x, t), \quad \frac{\partial\phi(\lambda; x, t)}{\partial t} = M\phi(\lambda; x, t). \qquad (6.1)$$

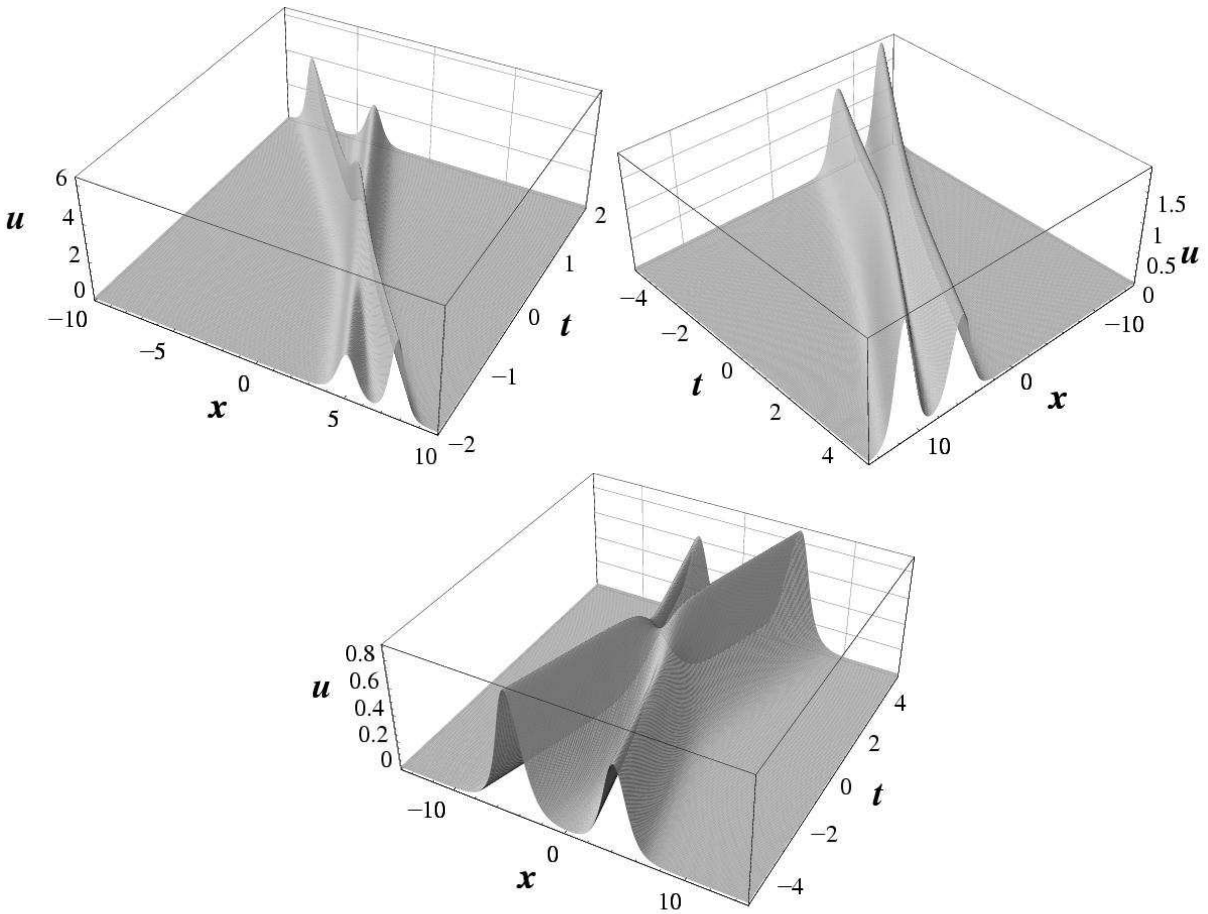

Fig. 5. Two left- and right-going as well as head-on colliding solitons for the bSH equation (1.9). Parameters are chosen as: $A_1 = B_1, A_2 = 2, B_2 = -1, k_1 = 1.5, k_2 = 2$ (left); $k_1 = 1.5, k_2 = 1.3, \varepsilon_1 = \varepsilon_2 = \frac{\pi}{4}$ (right); $k_1 = 0.8, k_2 = 0.9, \varepsilon_2 = \frac{\pi}{4}$ (collision).

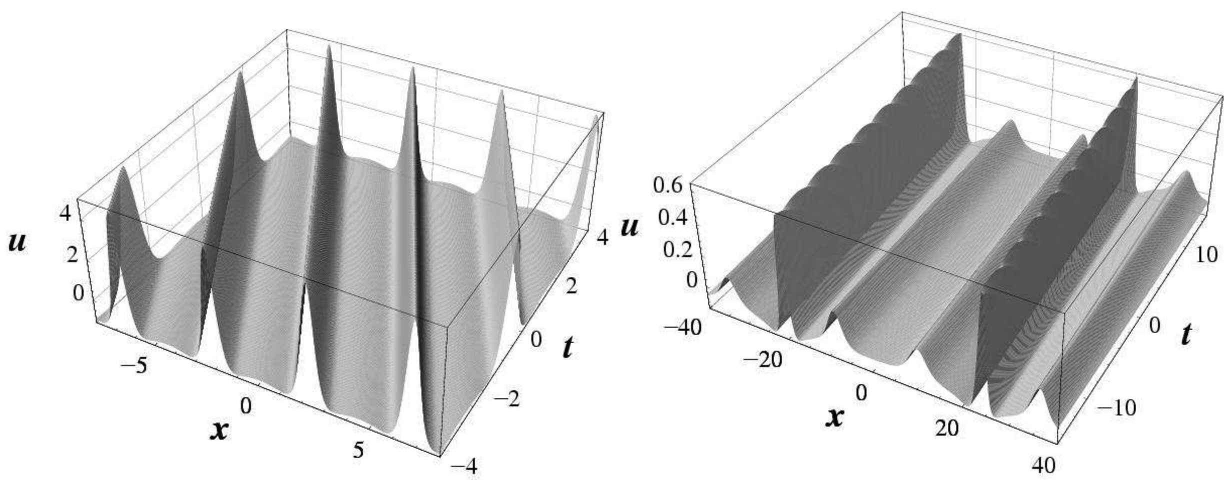

Fig. 6. Left-going periodic solutions with one (left) and two (right) spectral parameters for the bSH equation (1.9). Parameters: $k_1 = 1, \varepsilon_1 = \frac{\pi}{4}$(left); $k_1 = 0.2, k_2 = 0.3, \varepsilon_1 = \varepsilon_2 = \frac{\pi}{4}$(right).

There are some examples of n-reduction of the KP hierarchy. For the BKP hierarchy,

$$\left.\begin{array}{lllll} \text{Lax pair} & \text{3-reduction} & \text{5-reduction} & \text{7-reduction} & \text{9-reduction} \\ \text{L} & B_3 & B_5 & B_7 & B_9 \\ \text{M} & \widetilde{B}_5 & B_3 & B_3 & B_3 \\ \text{Equation} & \text{SK} & \text{bSK} & \text{higher order} & \text{higher order} \end{array}\right\}. \tag{6.2}$$

Here

$$\widetilde{B}_5 = \partial_x^5 + 5u\partial_x^3 + 5u_x\partial_x^2 + \Big(5u^2 + \frac{10}{3}u_{xx}\Big)\partial_x, \tag{6.3}$$

$$\text{SK [6,7]}: \quad 9u_t + 45u^2u_x + u_{xxxxx} + 15uu_{xxx} + 15u_xu_{xx} = 0. \tag{6.4}$$

B_5 and B_3 are given by Eqs. (1.10) and (1.11). For the CKP hierarchy,

$$\left.\begin{array}{lllll} \text{Lax pair} & \text{3-reduction} & \text{5-reduction} & \text{7-reduction} & \text{9-reduction} \\ \text{L} & B_3 & B_5 & B_7 & B_9 \\ \text{M} & \widetilde{B}_5 & B_3 & B_3 & B_3 \\ \text{Equation} & \text{KK} & \text{bKK} & \text{higher order} & \text{higher order} \end{array}\right\}. \tag{6.5}$$

Here

$$\widetilde{B}_5 = \partial_x^5 + 5u\partial_x^3 + \frac{15}{2}u_x\partial_x^2 + \Big(5u^2 + \frac{35}{6}u_{xx}\Big)\partial_x + 5uu_x + \frac{5}{3}u_{xxx}, \tag{6.6}$$

$$\text{KK [8,9]}: \quad 9u_t + 45u^2u_x + u_{xxxxx} + 15uu_{xxx} + \frac{75}{2}u_xu_{xx} = 0, \tag{6.7}$$

B_5 and B_3 are given in Eqs. (1.13) and (1.14). There are several even-reductions of the KP hierarchy as following,

$$\left.\begin{array}{lllll} \text{Lax pair} & \text{2-reduction} & \text{4-reduction} & \text{6-reduction} & \text{8-reduction} \\ \text{L} & B_2 & B_4 & B_6 & B_8 \\ \text{M} & B_3 & B_3 & B_3 & B_3 \\ \text{Equation} & \text{KdV} & \text{bSH} & \textit{higher order} & \textit{higher order} \end{array}\right\}. \tag{6.8}$$

Now we start to discuss the BKP hierarchy.

Lemma 15. *Let* $\widetilde{\xi_1} = xk_1\cos\varepsilon_1 + tk_1^5\cos 5\varepsilon_1$, *then* $\hat{\tau}_{\text{SK}}^{(1+1)}$ *is expressed by*

$$\hat{\tau}_{\text{SK}}^{(1+1)} = \hat{\tau}_{\text{bSK}}^{(1+1)}|_{\xi_1 -> \widetilde{\xi}_1} \tag{6.9}$$

and the corresponding single soliton is $u = \Big(\partial_x^2 \log\hat{\tau}_{\text{SK}}^{(1+1)}\Big)$. *The velocity of soliton* $v_- = -k_1^4\dfrac{\cos 5\varepsilon_1}{\cos\varepsilon_1}\Big|_{\varepsilon_1=\frac{\pi}{6}} =$ $k_1^4 > 0$. *Here* $\hat{\tau}_{\text{bSK}}^{(1+1)}$ *is given by Proposition 4.*

Proof. Because the SK equation and bSK equation belong to the same sub-hierarchy BKP, so the results of bSK are also hold by SK equation only if we replace ξ_1 in bSK by $\widetilde{\xi_1} = xk_1\cos\varepsilon_1 + tk_1^5\cos 5\varepsilon_1$. For SK equation, the generating functions $\phi_i^{((0))} = \phi(\lambda_i; x, t)$ of gauge transformation satisfy

$$\partial_x^3\phi(\lambda; x, t) = \lambda\phi(\lambda; x, t), \quad \frac{\phi(\lambda; x, t)}{\partial t} = \partial_x^5\phi(\lambda; x, t), \tag{6.10}$$

which are different with Eq. (3.1) for bSK equation. So $k_1^3 = |\lambda_1|$. This difference determines replacement in Eq. (6.9). Of course, similar to the bSK, we also should assume the solutions of Eq. (6.10) be the form of

$$\phi(\lambda_1; x, t) = A_1e^{p_1x+p_1^5t} + B_1e^{q_1x+q_1^5t}, p_1 = k_1e^{i\varepsilon_1}, q_1 = -k_1e^{-i\varepsilon_1}, k_1^3 = |\lambda_1|, k_1 \in \mathbb{R}, \tag{6.11}$$

or

$$\phi(\lambda_1;x,t)=A_1e^{p_1x+p_1^5t}+B_1e^{q_1x+q_1^5t},p_1=k_1e^{i\varepsilon_1},q_1=k_1e^{-i\varepsilon_1},k_1^3=|\lambda_1|,k_1\in\mathbb{R}. \quad (6.12)$$

Taking the generating functions $\phi_i^{((0))}$ in Eq. (6.11) back into the Proposition 1, then we can extract $\hat{\tau}_{\rm SK}^{(1+1)}$ from $\tau_{\rm SK}^{(1+1)}$. The relation $\hat{\tau}_{\rm SK}^{(1+1)}=\hat{\tau}_{\rm bSK}^{(1+1)}|_{\xi_1->\widetilde{\xi}_1}$ is given by comparison. □

In particular, there are two distributions of roots of third-order of $e^{i\varepsilon}$ on circle, which is symmetric with respect to y-axes. However, they are corresponding to same single soliton solution.

1. $\left(e^{i\frac{\pi}{6}},-e^{-i\frac{\pi}{6}}\right)$ one distribution of 3-order root of $e^{i\varepsilon}$ on unit circle $\longrightarrow$ $\left(p_1=k_1e^{i\frac{\pi}{6}},q_1=-k_1e^{-i\frac{\pi}{6}}\right)$ in Eq. (6.11) $\longrightarrow$ a single soliton in Lemma 15.
2. $\left(e^{i\frac{11\pi}{6}},-e^{i\frac{-11\pi}{6}}\right)$ one distribution of 3-order root of $e^{i\varepsilon}$ on unit circle $\longrightarrow$ $\left(p_1=k_1e^{i\frac{11\pi}{6}},q_1=-k_1e^{-i\frac{11\pi}{6}}\right)$ in Eq. (6.11) $\longrightarrow$ one soliton as 1.

Lemma 16. *The higher order equations of the* BKP *hierarchy are defined by Eq. (6.2). For the n-reduction equation of the* BKP *hierarchy* (nBKP), $n=2j+1, j=3,4,5,\cdots$, *and let* $\widetilde{\xi}_{mp}=xk_m\cos\varepsilon_p+tk_m^3\cos3\varepsilon_p, k_m^n=k_m^{2j+1}=|\lambda_m|$, *then the physical* τ *function of the* nBKP *generated by* T_{1+1} *is*

$$\hat{\tau}_{\rm nBKP}^{(1+1)}=\hat{\tau}_{\rm bSK}^{(1+1)}|_{\xi_1->\widetilde{\xi}_{1p}}, \quad (6.13)$$

and the corresponding single soliton of the nBKP *is* $u=(\partial_x^2\log\hat{\tau}_{\rm nBKP}^{(1+1)})$. *Here* $\varepsilon_p=\frac{2p-1}{2n}\pi=\frac{2p-1}{4j+2}\pi, p=1,2,3,\cdots,j$, $\hat{\tau}_{\rm bSK}^{(1+1)}$ *is given by Proposition 4. So the single soliton can move along* j *directions in (x,t) plane, which are given by* $\widetilde{\xi}_{1p}=0$ *associated with* j*-value of* ε_p *given before.*

Proof. Comparing the nBKP with the bSK equation, the main change here is the Lax pair (L,M). The Lax pair of the nBKP defines the generating functions $\phi_i^{((0))}=\phi(\lambda_i;x,t)$ are slight different as

$$\partial_x^n\phi(\lambda;x,t)=\lambda\phi(\lambda;x,t),\quad \frac{\phi(\lambda;x,t)}{\partial t}=\partial_x^3\phi(\lambda;x,t) \quad (6.14)$$

and then we assume

$$\phi(\lambda_1;x,t)=A_1e^{p_1x+p_1^3t}+B_1e^{q_1x+q_1^3t},p_1=k_1e^{i\varepsilon_p},q_1=-k_1e^{-i\varepsilon_p},k_1^n=|\lambda_1|,k_1\in\mathbb{R}, \quad (6.15)$$

or

$$\phi(\lambda_1;x,t)=A_1e^{p_1x+p_1^3t}+B_1e^{q_1x+q_1^3t},p_1=k_1e^{i\varepsilon_p},q_1=k_1e^{-i\varepsilon_p},k_1^n=|\lambda_1|,k_1\in\mathbb{R}. \quad (6.16)$$

In order to avoid the divergence of u, we only take $0<\varepsilon_p<\frac{\pi}{2}$, and then $\varepsilon_p=\frac{2p-1}{2n}\pi=\frac{2p-1}{4j+2}\pi, p=1,2,3,\cdots,j$. This change results to the emergence of $\widetilde{\xi}_{mp}=xk_m\cos\varepsilon_p+tk_m^3\cos3\varepsilon_p, k_m^n=k_m^{2j+1}=|\lambda_m|$. The $\hat{\tau}_{\rm nBKP}^{(1+1)}$ and single soliton solution $u=(\partial_x^2\log\hat{\tau}_{\rm nBKP}^{(1+1)})$ can be derived directly from the Proposition 1 and the generating functions $\phi_1^{((0))}$ in Eq. (6.15) associated with λ_1 for the gauge transformation. Further, for a given p, $\widetilde{\xi}_{1p}=0$ determines one moving direction of the single soltion on (x,t) plane, then the single soliton solution have j directions for propagation because $p=1,2,\cdots,j$. □

From Lemmata 15, 16 and the results of the bSK equation, we have

Proposition 20.

1. *The single soliton* $u = \left(\partial_x^2 \log \hat{\tau}^{(1+1)}_{\text{nBKP}}\right)$ *of the* nBKP *equation,* $n = 2j+1, j = 2, 3, 4, \cdots$ *, can move along a direction defined by* $\widetilde{\xi}_{1p} = 0$ *on (x,t) plane for a given p.*
2. $\left(e^{i\varepsilon_p}, -e^{-i\varepsilon_p}\right)$ *one distribution of n-th order roots of* $e^{i\varepsilon}$ *on circle* $\longrightarrow$ $\left(p_1 = k_1 e^{i\varepsilon_p}, q_1 = -k_1 e^{-i\varepsilon_p}\right)$ *in Eq. (6.15)* $\longrightarrow$ *The single soliton moves along a line* $\widetilde{\xi}_{1p} = 0$ *on (x, t) plane. Here* $\varepsilon_p \in \left\{\frac{\pi}{4j+2}, \frac{3\pi}{4j+2}, \frac{5\pi}{4j+2}, \cdots, \frac{(2j-1)\pi}{4j+2}\right\}$.
3. *For a given* $n = 2j+1$*, the single soliton of the* nBKP *have j directions to propagate on (x,t) plane, which are defined* $\widetilde{\xi}_{1p} = 0, p = 1, 2, 3, \cdots j$.

Note that the result of $j = 1$ in above Proposition is given by Lemma 15.

Now we turn to the lower and higher reductions of the CKP hierarchy. Similar to the discussion of the BKP hierarchy in this section, we can obtain parallel results in the CKP hierarchy, so we write out the results without proof in the following to save space.

Lemma 17. *Let* $\widetilde{\xi_1} = xk_1 \cos\varepsilon_1 + tk_1^5 \cos 5\varepsilon_1$*, then* $\hat{\tau}^{(1+1)}_{\text{KK}}$ *can be expressed by*

$$\hat{\tau}^{(1+1)}_{\text{KK}} = \hat{\tau}^{(1+1)}_{\text{bKK}}|_{\xi_1 -> \widetilde{\xi}_1} \tag{6.17}$$

and the corresponding single soliton is $u = \left(\partial_x^2 \log \hat{\tau}^{(1+1)}_{\text{KK}}\right)$*. The velocity of soliton is* $\hat{v}_- = -k_1^4 \frac{\cos 5\varepsilon_1}{\cos\varepsilon_1}\Big|_{\frac{\pi}{6}} = k_1^4 > 0$*. Here* $\hat{\tau}^{(1+1)}_{\text{bKK}}$ *is given by Proposition 7.*

Lemma 18. *The higher order equation of* CKP *defined by Eq. (6.5). For n-reduction of* CKP *hierarchy* (nCKP)*,* $n = 2j+1, j = 3, 4, 5, \cdots$*. Let* $\widetilde{\xi}_{mp} = xk_m \cos\varepsilon_p + tk_m^3 \cos 3\varepsilon_p, k_m^n = k_m^{2j+1} = |\lambda_m|$*, then the* τ *function of the* nCKP *generated by* T_{1+1} *is*

$$\hat{\tau}^{(1+1)}_{\text{nCKP}} = \hat{\tau}^{(1+1)}_{\text{bKK}}|_{\xi_1 -> \widetilde{\xi}_{1p}}, \tag{6.18}$$

and the corresponding single soliton of the nCKP *equation is* $u = \left(\partial_x^2 \log \hat{\tau}^{(1+1)}_{\text{nCKP}}\right)$*. Here* $\varepsilon_p = \frac{p}{2n}\pi = \frac{p}{4j+2}\pi, p = 1, 2, 3, \cdots, j$ *, and* $\hat{\tau}^{(1+1)}_{\text{bKK}}$ *is given by Proposition 7. So the single soliton can move along j directions on (x, t) plane, which are given by* $\widetilde{\xi}_{1p} = 0$ *associated with j-value of* ε_p *given before.*

Using the Lemmata 17, 18 and results for the bKK equation, we get

Proposition 21.

1. *The single soliton* $u = \left(\partial_x^2 \log \hat{\tau}^{(1+1)}_{\text{nCKP}}\right)$ *of the* nCKP*,* $n = 2j+1, j = 2, 3, 4, \cdots$*, can move along a direction defined by* $\widetilde{\xi}_{1p} = 0$ *on (x, t) plane for a given p.*
2. $(e^{i\varepsilon_p}, -e^{-i\varepsilon_p})$ *one distribution of n-th order roots of* $e^{i\varepsilon}$ *on circle* $\longrightarrow$ $(p_1 = k_1 e^{i\varepsilon_p}, q_1 = -k_1 e^{-i\varepsilon_p})$ *in Eq. (6.15)* $\longrightarrow$ *the single soliton moves along a line* $\widetilde{\xi}_{1p} = 0$ *on (x, t) plane. Here* $\varepsilon_p \in \left\{\frac{\pi}{4j+2}, \frac{3\pi}{4j+2}, \frac{5\pi}{4j+2}, \cdots, \frac{(2j-1)\pi}{4j+2}\right\}$.
3. *For a given* $n = 2j+1$*, the single soliton of the* nCKP *can move along j directions on (x,t) plane, which are defined by* $\widetilde{\xi}_{1p} = 0, p = 1, 2, 3, \cdots, j$.
4. *In particular, if* $0 < \varepsilon_p < \pi/6$*,* $u = \left(\partial_x^2 \log \hat{\tau}^{(1+1)}_{\text{nCKP}}\right)$ *is a two-peak soliton.*

In above Proposition, the case of $j=1$ is given by Lemma 17. This Proposition shows there exist several single two-peak solitons for nCKP if $n \geq 11$.

Corollary 1. *There are two single two-peak solitons for 11-reduction of* CKP *hierarchy, i. e.* 11CKP *equation,* $u_{11\mathrm{CKP}} = \left(\partial_x^2 \log \hat{\tau}_{\mathrm{bKK}}^{(1+1)}\right)$*, in which* $\varepsilon_1 = \pi/22$ *and* $\varepsilon_1 = 3\pi/22$ *respectively. Here* $\hat{\tau}_{\mathrm{bKK}}^{(1+1)}$ *is given in Eq. (4.6)*

We have plotted it out in Fig. 7 with $k_1 = 0.8$.

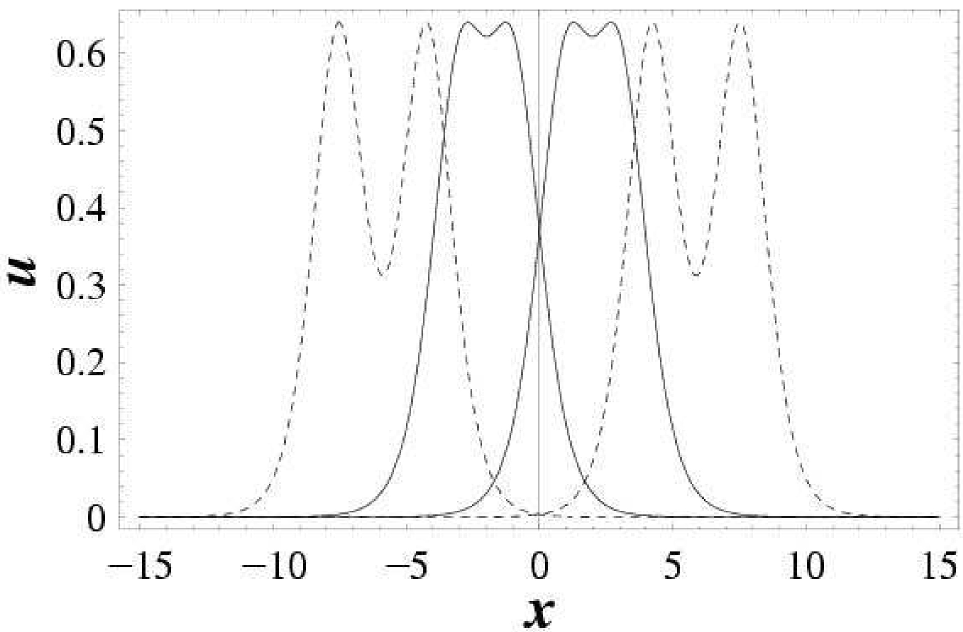

Fig. 7. Left-going two-peak soliton with dashed line ($\varepsilon_1 = \pi/22$) is faster, left-going two-peak soliton with full line ($\varepsilon_1 = 3\pi/22$). The left is plotted when $t = 10$, the right is plotted when $t = -10$

We have known that $a = 1/2$ in Lemma 5 and Proposition 9 is one crucial point to exist one-peak soliton or two-peak soliton. It is more interesting that $a = 1/2$ will lead to "stationary" soliton of higher reductions of the BKP and the CKP hierarchy, which is not moving on (x,t) plane. When $\xi_1|_{\varepsilon_1=\pi/6} = (k_1 x \cos\varepsilon_1 + t k_1^3 \cos 3\varepsilon_1|)_{\varepsilon_1=\pi/6} = (k_1 x \cos\varepsilon_1)$, ξ_1 is independent with t. So u is independent with t by taking this ξ_1 into Proposition 4 and Proposition 7.

Corollary 2.

1. *There exists "stationary" single soliton for the 9-reduction of* BKP *hierarchy, which is* $u_{9\mathrm{BKP}} = \left(\partial_x^2 \log \hat{\tau}_{\mathrm{bSK}}^{(1+1)}\right)|_{\varepsilon_1=3\pi/18}$*. Here* $\hat{\tau}_{\mathrm{bSK}}^{(1+1)}$ *is given by Proposition 4;*
2. *There exists "stationary" single soliton for the 9-reduction of* CKP *hierarchy, which is* $u_{9\mathrm{CKP}} = \left(\partial_x^2 \log \hat{\tau}_{\mathrm{bKK}}^{(1+1)}\right)|_{\varepsilon_1=3\pi/18}$*. Here* $\hat{\tau}_{\mathrm{bKK}}^{(1+1)}$ *is given by Proposition 7.*

We have plotted out "stationary" soliton for the 9-reduction of CKP in Fig. 8 when $k_1 = 1$.

Corollary 3. *There is single two-peak soliton* $u = \left(\partial_x^2 \log \hat{\tau}_{bSH}^{(1+1)}\right)|_{\varepsilon_1=\pi/8}$ *for 8-reduction of the* KP *hierarchy; there is "stationary" single one-peak soliton* $u = \left(\partial_x^2 \log \hat{\tau}_{\mathrm{bSH}}^{(1+1)}\right)|_{\varepsilon_1=\pi/6}$ *for the 6-reduction of the* KP *hierarchy. Here* $\hat{\tau}_{\mathrm{bSH}}^{(1+1)})$ *is given by Proposition 13.*

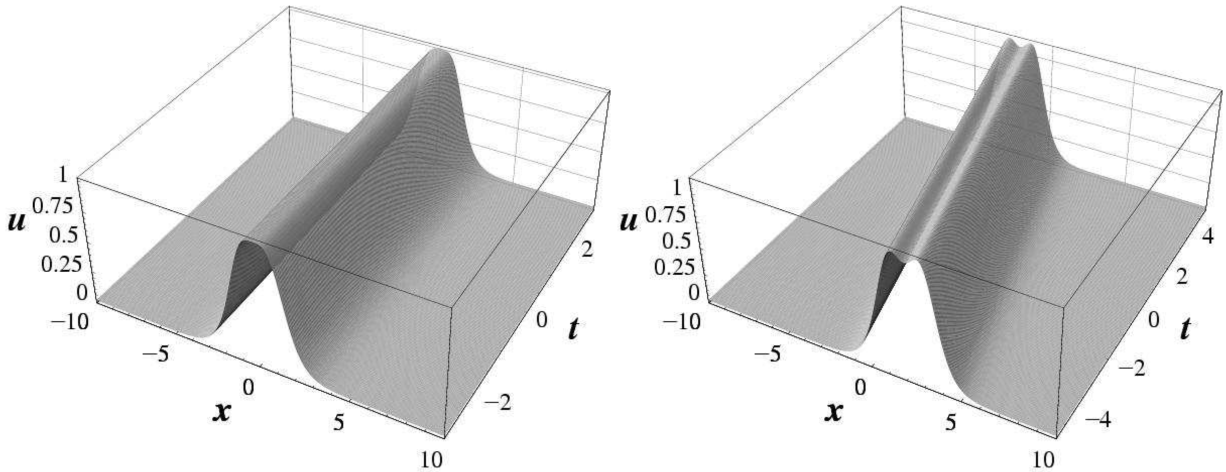

Fig. 8. Left: "stationary" soliton for 9-reduction of CKP, Right: Single two-peak soliton for 8-reduction of KP.

The two-peak soliton for the 8-reduction of KP hierarchy is plotted in Fig. 8 when $k_1 = 1$. For our best knowledge, this is first time to report the even-reduction of the KP hierarchy also has two-peak soliton solution. The possession of two-peak soliton solution is not sole property of CKP hierarchy.

7. Conclusions and Discussions

We have presented a systematic way in which to obtain the solution of the n-reduction ($n = 4, 5$) from the general τ function of the KP hierarchy. Our approach is based on the determinant representation of gauge transformations T_{n+k} [27] and $\tau^{(n+k)}$ [26]. It may be summarized as follows:
$\tau^{(n+k)} \xrightarrow{\text{constraints of generating functions and } k=n} \tau^{(n+k)}_{\text{BKP}}$ (or $\tau^{(n+k)}_{\text{CKP}}$) $\xrightarrow{5-\text{reduction}} \tau^{(n+k)}_{\text{Eq}}|_{k=n}(\text{Eq} = \text{bSK}, \text{bKK})$ $\xrightarrow{\text{assume the form of } \phi_i \text{and find suitable} \frac{B_i}{A_i}}$ efficient τ function $\hat{\tau}^{(n+k)}_{\text{Eq}}|_{k=n=1,2}$. We have applied this approach to various equations. The one soliton, two soliton and periodic solution are constructed for bSK, bKK and bSH. We show the corresponding relation between the distribution of 5-th (or 4th) roots of $e^{i\varepsilon}$ on the unit circle and several types of solutions (left-going one soliton, right-going one soliton, left/right-going periodic solutions). We also show the reason for the existence of the two-peak soliton. Furthermore, the lower reduction and higher reduction of BKP, CKP, and the even-reductions are explored by this method. Our results show that the soliton of the n-reduction (with $n = 2j + 1$, $j = 1, 2, 3, \ldots$) of BKP and CKP can move alone j directions, which are defined by $\widetilde{\xi}_{1p} = 0$. Each direction corresponds to one symmetry distribution of n-th roots of $e^{i\varepsilon}$ on the unit circle. This supplies a very natural explanation why the 5-reduction BKP (or CKP) has bi-directional solitons whereas the 3-reduction of BKP (or CKP) has only single-directional solitons. At last, the two-peak soliton is not a monopolizing phenomena of only the CKP hierarchy. Rather, we find that the higher-order even-reduction of KP also exhibits two-peak solitons and we elucidate the criterion for its existence from the Grammian τ function. At the same time, we show there is not three and more peak soliton from Grammian τ function. The "stationary" soliton for higher order reduction of KP hierarchy is also obtained.

We think that it is possible to construct an N-soliton solution of the bSK, bKK and bSH equations by this approach. Namely, there exist suitable $\dfrac{B_i}{A_i}$ ($i = 1, 2, \cdots, N$) such that we can find a *physical* τ function $\hat{\tau}^{(N+N)}|_{\text{Eq}}$ for these equations from a complex-valued $\tau^{(N+N)}|_{\text{Eq}}$, which is symmetric because we have assumed generating functions ϕ_i in Eq. (3.5) and Eq. (3.6) with symmetric form. Here Eq = bSK, bKK, bSH. Additionally, it is worthy to discuss the phase shift in the collision of one-peak soliton and two-peak soliton. Furthermore, it is possible to construct solutions for bSK, bKK and bSH from constant initial value $u = constant \neq 0$, which is parallel to present results.

Upon completion of this work, Prof. V. Sokolov kindly pointed out Ref. [40] where Eqs. (1.8, 6.4, 6.7) and their Lax operators as well as the Lax operator $L = \partial + u\partial^{-1}u$ for KdV equation have been obtained for the first time.

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A. bSK equation

Take $z_k = c_k + id_k (k = 1, 3, 5, 7)$

$$\begin{aligned} c_1 =\ & k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big[k_1^2\cos 2\varepsilon_1 + k_2^2\cos 2\varepsilon_2 + 2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big] \\ & + k_1k_2\sin(\varepsilon_1+\varepsilon_2)\big[k_1^2\sin 2\varepsilon_1 + k_2^2\sin 2\varepsilon_2 + 2k_1k_2\sin(\varepsilon_1+\varepsilon_2)\big] \end{aligned} \tag{A.1}$$

$$\begin{aligned} d_1 =\ & -k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big[k_1^2\sin 2\varepsilon_1 + k_2^2\sin 2\varepsilon_2 + 2k_1k_2\sin(\varepsilon_1+\varepsilon_2)\big] \\ & + k_1k_2\sin(\varepsilon_1+\varepsilon_2)\big[k_1^2\cos 2\varepsilon_1 + k_2^2\cos 2\varepsilon_2 + 2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big] \end{aligned} \tag{A.2}$$

$$\begin{aligned} c_3 =\ & k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big[k_1^2\cos 2\varepsilon_1 + k_2^2\cos 2\varepsilon_2 - 2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big] \\ & + k_1k_2\sin(\varepsilon_1-\varepsilon_2)\big[k_1^2\sin 2\varepsilon_1 - k_2^2\sin 2\varepsilon_2 - 2k_1k_2\sin(\varepsilon_1-\varepsilon_2)\big] \end{aligned} \tag{A.3}$$

$$\begin{aligned} d_3 =\ & -k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big[k_1^2\sin 2\varepsilon_1 - k_2^2\sin 2\varepsilon_2 - 2k_1k_2\sin(\varepsilon_1-\varepsilon_2)\big] \\ & + k_1k_2\sin(\varepsilon_1+\varepsilon_2)\big[k_1^2\cos 2\varepsilon_1 + k_2^2\cos 2\varepsilon_2 - 2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big] \end{aligned} \tag{A.4}$$

$$\begin{aligned} c_5 =\ & 2k_1k_2\sin\varepsilon_2\big[\cos\varepsilon_1\big(k_1^2\cos 2\varepsilon_1 - k_2^2 - 2k_1k_2\sin\varepsilon_1\sin\varepsilon_2\big)\big] \\ & + 2k_1k_2\sin\varepsilon_2\big[2k_1\cos\varepsilon_1\sin\varepsilon_1\big(k_1\sin\varepsilon_1 + k_2\sin\varepsilon_2\big)\big] \end{aligned} \tag{A.5}$$

$$\begin{aligned} d_5 =\ & -2k_1k_2\sin\varepsilon_2\big[2k_1\cos^2\varepsilon_1\big(k_1\sin\varepsilon_1 + k_2\sin\varepsilon_2\big)\big] \\ & + 2k_1k_2\sin\varepsilon_2\big[\sin\varepsilon_1\big(k_1^2\cos 2\varepsilon_1 - k_2^2 - 2k_1k_2\sin\varepsilon_1\sin\varepsilon_2\big)\big] \end{aligned} \tag{A.6}$$

$$\begin{aligned} c_7 =\ & 2k_1k_2\sin\varepsilon_1\big[\cos\varepsilon_2\big(k_1^2 - k_2^2\cos 2\varepsilon_2 + 2k_1k_2\sin\varepsilon_1\sin\varepsilon_2\big)\big] \\ & - 2k_1k_2\sin\varepsilon_1\big[2k_2\cos\varepsilon_2\sin\varepsilon_2\big(k_1\sin\varepsilon_1 + k_2\sin\varepsilon_2\big)\big] \end{aligned} \tag{A.7}$$

$$\begin{aligned} d_7 =\ & 2k_1k_2\sin\varepsilon_1\big[2k_2\cos^2\varepsilon_2\big(k_1\sin\varepsilon_1 + k_2\sin\varepsilon_2\big)\big] \\ & + 2k_1k_2\sin\varepsilon_1\big[\sin\varepsilon_2\big(k_1^2 - k_2^2\cos 2\varepsilon_2 + 2k_1k_2\sin\varepsilon_1\sin\varepsilon_2\big)\big] \end{aligned} \tag{A.8}$$

B. bKK equation(two solitons)

Take $z_k = c_k + id_k (k = 1, 2, 3, 4)$

$$\begin{aligned} c_1 =\ & \cos(\varepsilon_1+\varepsilon_2)\big[k_1^2 + k_2^2 + 2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big]^2 \\ & - 4k_1k_2\big[k_1\cos 2\varepsilon_1 + k_2\cos 2\varepsilon_2 + 2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big] \end{aligned} \tag{B.1}$$

$$\begin{aligned} d_1 =\ & \sin(\varepsilon_1+\varepsilon_2)\big[k_1^2 + k_2^2 + 2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big]^2 \\ & - 4k_1k_2\big[k_1^2\sin 2\varepsilon_1 + k_2^2\sin 2\varepsilon_2 + 2k_1k_2\sin(\varepsilon_1+\varepsilon_2)\big] \end{aligned} \tag{B.2}$$

$$c_2 = \cos\varepsilon_1\big[k_1^2 + k_2^2 + 2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big]\big[k_1^2 + k_2^2 - 2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big]$$

$$
\begin{aligned}
&\quad -4k_1k_2\sin\varepsilon_2\big(k_1^2\sin 2\varepsilon_1+2k_1k_2\sin\varepsilon_2\cos\varepsilon_1\big) &\text{(B.3)}\\
d_2 &= \sin\varepsilon_1\big[k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big]\big[k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big]\\
&\quad +4k_1k_2\sin\varepsilon_2\big(k_1^2\cos 2\varepsilon_1-k_2^2-2k_1k_2\sin\varepsilon_1\sin\varepsilon_2\big) &\text{(B.4)}\\
c_3 &= \cos(\varepsilon_1-\varepsilon_2)\big[k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big]^2\\
&\quad +4k_1k_2\big[k_1^2\cos 2\varepsilon_1+k_2^2\cos 2\varepsilon_2-2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big] &\text{(B.5)}\\
d_3 &= \sin(\varepsilon_1-\varepsilon_2)\big[k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big]^2\\
&\quad +4k_1k_2\big[k_1^2\sin 2\varepsilon_1-k_2^2\sin 2\varepsilon_2-2k_1k_2\sin(\varepsilon_1-\varepsilon_2)\big] &\text{(B.6)}\\
c_4 &= \cos\varepsilon_2\big[k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big]\big[k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big]\\
&\quad -4k_1k_2\sin\varepsilon_1\big(k_2^2\sin 2\varepsilon_2+2k_1k_2\sin\varepsilon_1\cos\varepsilon_2\big) &\text{(B.7)}\\
d_4 &= \sin\varepsilon_2\big[k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big]\big[k_1^2+k_2^2-2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big]\\
&\quad +4k_1k_2\sin\varepsilon_1\big(-k_1^2+k_2^2\cos 2\varepsilon_2-2k_1k_2\sin\varepsilon_1\sin\varepsilon_2\big) &\text{(B.8)}
\end{aligned}
$$

C. bKK equation(periodic solutions)

Take $z_k=c_k+id_k(k=1,2,3,4)$

$$
\begin{aligned}
c_1 &= \cos(\varepsilon_1+\varepsilon_2)\big[k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big]^2\\
&\quad -4k_1k_2\big[k_1^2\cos 2\varepsilon_1+k_2^2\cos 2\varepsilon_2+2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big] &\text{(C.1)}\\
d_1 &= \sin(\varepsilon_1+\varepsilon_2)\big[k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big]^2\\
&\quad -4k_1k_2\big[k_1^2\sin 2\varepsilon_1+k_2^2\sin 2\varepsilon_2+2k_1k_2\sin(\varepsilon_1+\varepsilon_2)\big] &\text{(C.2)}\\
c_2 &= \cos\varepsilon_1\big[k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big]\big[k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big]\\
&\quad -4k_1k_2\cos\varepsilon_2\big(k_1^2\cos 2\varepsilon_1+k_2^2+2k_1k_2\cos\varepsilon_1\cos\varepsilon_2\big) &\text{(C.3)}\\
d_2 &= \sin\varepsilon_1\big[k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big]\big[k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big]\\
&\quad -8k_1^2k_2\cos\varepsilon_2\sin\varepsilon_1\big(k_1\cos\varepsilon_1+k_2\cos\varepsilon_2\big) &\text{(C.4)}\\
c_3 &= \cos(\varepsilon_1-\varepsilon_2)\big[k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big]^2\\
&\quad -4k_1k_2\big[k_1^2\cos 2\varepsilon_1+k_2^2\cos 2\varepsilon_2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big] &\text{(C.5)}\\
d_3 &= \sin(\varepsilon_1-\varepsilon_2)\big[k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big]^2\\
&\quad -4k_1k_2\big[k_1^2\sin 2\varepsilon_1-k_2^2\sin 2\varepsilon_2+2k_1k_2\sin(\varepsilon_1-\varepsilon_2)\big] &\text{(C.6)}\\
c_4 &= \cos\varepsilon_2\big[k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big]\big[k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big]\\
&\quad -4k_1k_2\cos\varepsilon_1\big[k_1^2+k_2^2\cos 2\varepsilon_2+2k_1k_2\cos\varepsilon_1\cos\varepsilon_2\big] &\text{(C.7)}\\
d_4 &= \sin\varepsilon_2\big[k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1-\varepsilon_2)\big]\big[k_1^2+k_2^2+2k_1k_2\cos(\varepsilon_1+\varepsilon_2)\big]\\
&\quad -8k_1k_2^2\cos\varepsilon_1\sin\varepsilon_2\big(k_1\cos\varepsilon_1+k_2\cos\varepsilon_2\big) &\text{(C.8)}
\end{aligned}
$$

D. bSH equation

Take $z_k=c_k+id_k(k=1,3,5)$

$$
\begin{aligned}
c_1 &= 2k_2(k_2\cos\varepsilon_2+k_1)-(k_1^2+k_2^2+2k_1k_2\cos\varepsilon_2)\cos\varepsilon_2 &\text{(D.1)}\\
d_1 &= 2k_2^2\sin\varepsilon_2-(k_1^2+k_2^2+2k_1k_2\cos\varepsilon_2)\sin\varepsilon_2 &\text{(D.2)}\\
c_3 &= 2k_2(k_2\cos\varepsilon_2-k_1)-(k_1^2+k_2^2-2k_1k_2\cos\varepsilon_2)\cos\varepsilon_2 &\text{(D.3)}\\
d_3 &= 2k_2^2\sin\varepsilon_2-(k_1^2+k_2^2-2k_1k_2\cos\varepsilon_2)\sin\varepsilon_2 &\text{(D.4)}\\
c_5 &= 2k_2^2(k_2^2+k_1^2)\sin^2\varepsilon_2-(k_1^2+k_2^2+2k_1k_2\cos\varepsilon_2)(k_1^2+k_2^2-2k_1k_2\cos\varepsilon_2) &\text{(D.5)}\\
d_5 &= k_2^2\sin\varepsilon_2\big[2k_1(k_1^2+k_2^2)-4k_1k_2^2\cos^2\varepsilon_2\big] &\text{(D.6)}
\end{aligned}
$$

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