J. Fluid Mech. (2007), *vol.* 578, *pp.* 495–507.
doi:10.1017/S0022112007005137 Printed in the United Kingdom

Parametrization of global attractors, experimental observations, and turbulence

JAMES C. ROBINSON

Mathematics Institute, University of Warwick, Coventry CV4 7AL, UK

(Received 12 April 2006 and in revised form 13 December 2006)

This paper is concerned with rigorous results in the theory of turbulence and fluid flow. While derived from the abstract theory of attractors in infinite-dimensional dynamical systems, they shed some light on the conventional heuristic theories of turbulence, and can be used to justify a well-known experimental method.

Two results are discussed here in detail, both based on parametrization of the attractor. The first shows that any two fluid flows can be distinguished by a sufficient number of point observations of the velocity. This allows one to connect rigorously the dimension of the attractor with the Landau–Lifschitz 'number of degrees of freedom', and hence to obtain estimates on the 'minimum length scale of the flow' using bounds on this dimension. While for two-dimensional flows the rigorous estimate agrees with the heuristic approach, there is still a gap between rigorous results in the three-dimensional case and the Kolmogorov theory.

Secondly, the problem of using experiments to reconstruct the dynamics of a flow is considered. The standard way of doing this is to take a number of repeated observations, and appeal to the Takens time-delay embedding theorem to guarantee that one can indeed follow the dynamics 'faithfully'. However, this result relies on restrictive conditions that do not hold for spatially extended systems: an extension is given here that validates this important experimental technique for use in the study of turbulence.

Although the abstract results underlying this paper have been presented elsewhere, making them specific to the Navier–Stokes equations provides answers to problems particular to fluid dynamics, and motivates further questions that would not arise from within the abstract theory itself.

1. Introduction

The results discussed here are based on the treatment of certain partial differential equations as dynamical systems. Such an approach converts the original problem, defined on a physical domain, into a system evolving on an abstract infinite-dimensional phase space. It is therefore no surprise that many of the results obtained in this way are themselves abstract.

Thus, although one of the prime motivations for the development of this theory has been its relevance to questions in fluid dynamics and turbulence via its application to the Navier–Stokes equations, it has often not addressed directly the physical systems it seeks to understand. The aim of the work presented here is to provide more 'concrete' results that are valid in the original physical domain.

In the abstract setting, the results described in this paper give possible parametrizations of the attractor of an infinite-dimensional system. In terms of fluid flows

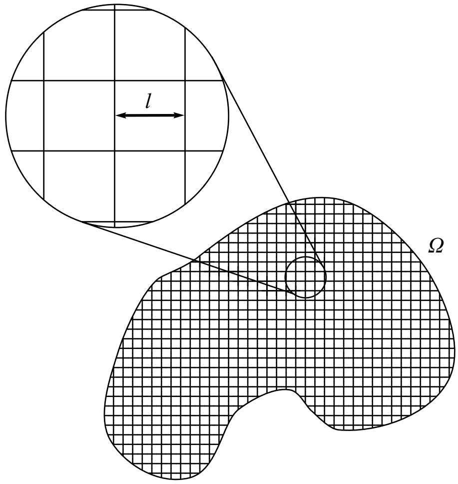

FIGURE 1. The Landau–Lifshitz definition of the number of degrees of freedom, N, involves dividing the domain Ω that contains the fluid into boxes whose sides have length l, the 'minimum length scale of the flow'. For a two-dimensional domain it follows that $N \sim l^{-2}$.

they provide finite sets of realizable experimental observations via which the dynamics of a fluid can be followed faithfully: enough point observations of the velocity or a sufficiently long time series of 'almost any' repeated observation.

That the dynamics can be followed by point observations of the velocity can be used to establish a rigorous connection between certain notions from the classical theory of turbulent flows and the mathematical theory of attractors for infinite-dimensional systems; while the analytical justification of the use of time series in spatially-extended systems legitimizes a popular experimental technique.

2. Landau–Lifshitz degrees of freedom

Using dimensional analysis, Landau & Lifshitz (1959) introduced a heuristic notion of the 'number of degrees of freedom' in a turbulent flow which has since been extensively applied. The result of their argument is that if l is 'the minimum length scale of the flow' (a quantity also arrived at via dimensional analysis), then the number of degrees of freedom of the flow is the number of boxes of side l needed to fill the domain Ω that contains the fluid (see figure 1).

In three-dimensional turbulence the 'minimum length scale' is usually identified as the Kolmogorov dissipative scale l_{K}, while in two-dimensional turbulence it is most naturally taken to be the Kraichnan length scale l_χ (both described in more detail later). One would therefore expect the 'number of degrees of freedom' of a three-dimensional flow to be proportional to l_{K}^{-3}, and that of a two-dimensional flow proportional to l_χ^{-2}.

A more abstract definition of the 'number of degrees of freedom' of a flow is provided by the dimension of the global attractor of the corresponding mathematical model. Identifying this dimension, d, with the Landau–Lifshitz definition implies that in two-dimensional flows one should expect $l_\chi \sim d^{-1/2}$, and in three dimensions $l_{\mathrm{K}} \sim d^{-1/3}$. Attractors and their dimension are now discussed in more detail.

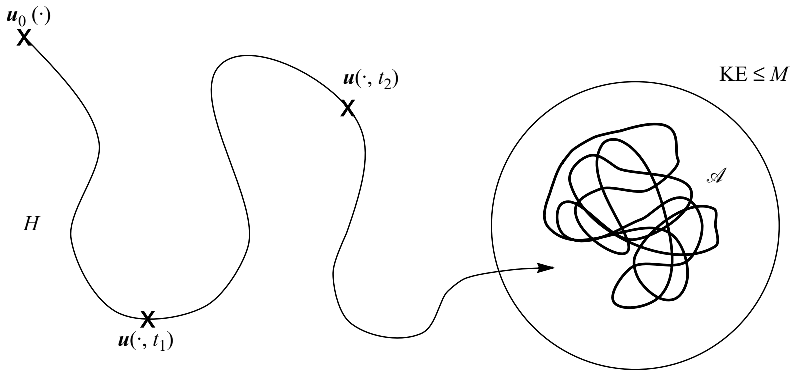

FIGURE 2. As $u(x, t)$ changes in time, the function $u(\cdot, t)$ traces out a path in the phase space H. Eventually any trajectory enters a bounded region in which the kinetic energy is no larger than M, and tends towards the global attractor $\mathscr{A}$.

3. Global attractors in the Navier–Stokes equations

A mathematically rigorous approach to the theory of turbulence begins with the Navier–Stokes equations. Since the question of existence and uniqueness of solutions for the three-dimensional case is still a major unsolved problem (see Doering & Gibbon 1995, for example), fully rigorous results can only be obtained for the two-dimensional case. However, the results presented here hold in the three-dimensional situation under the assumption that the equations are well-posed.

The presence of boundaries complicates the mathematical treatment, and for this reason the theory is often developed in the first instance for periodic boundary conditions (cf. Temam 1985: '... it is interesting to consider another boundary condition which has no physical meaning'). However, all the theory that will now be described works in the case of Dirichlet boundary conditions with a little additional work; when there are significant differences these will be mentioned.

3.1. *The Navier–Stokes equations as a dynamical system*

The two-dimensional incompressible Navier–Stokes equations

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad \text{and} \qquad \nabla \cdot \boldsymbol{u} = 0, \tag{3.1}$$

with periodic boundary conditions on $Q = [0, L]^2$, can be recast as a dynamical system evolving on the phase space H of all admissible flow fields with finite kinetic energy,

$$H = \left\{ \boldsymbol{u}\colon \int_Q |\boldsymbol{u}(\boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x} < +\infty, \ \nabla \cdot \boldsymbol{u} = 0, \ \boldsymbol{u} \text{ periodic on } Q \text{ with zero average} \right\}.$$

(The zero average condition, which corresponds to zero total momentum, is mathematically convenient but can be relaxed.) A single 'point' in the phase space corresponds to a single state of the system (the flow field throughout the domain Q). As $\boldsymbol{u}(\boldsymbol{x}, t)$ changes in time, the function $\boldsymbol{u}(\cdot, t)$ moves in the phase space H, see figure 2.

Before proceeding it should be noted that although in many senses the model considered in (3.1) is very general, it also has some peculiar features. First, although periodic boundary conditions are mathematically convenient, by removing any boundaries they also remove the possibility of applying any forcing at the boundaries. Therefore in order for the fluid to develop any persistent motion it must be forced by a theoretically convenient but physically somewhat mysterious body force $\boldsymbol{f}$. (Even in the case of a domain with boundaries, for simplicity the mathematical theory usually retains the body force and imposes $\boldsymbol{u}=0$ on the boundaries.) Furthermore, in order for the solutions to define a conventional dynamical system, $\boldsymbol{f}$ is required to be independent of time. Both these problems can be circumvented by appropriate extensions of the theory (e.g. Miranville & Wang 1997), but the treatment here will deal only with the simplest situation.

3.2. *Dissipation and global attractors*

A simple calculation, based on multiplying the governing equation by $\boldsymbol{u}$ and integrating by parts, taking into account that the flow field is divergence-free, shows that the kinetic energy of any solution of (3.1) is ultimately bounded by a constant independent of the initial condition:

$$\limsup_{t\to\infty} \frac{1}{2} \int_Q |\boldsymbol{u}(\boldsymbol{x},t)|^2 \, \mathrm{d}\boldsymbol{x} \leqslant M := \frac{L^2}{8\pi^2 \nu} \int_Q |\boldsymbol{f}(\boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x}. \tag{3.2}$$

Such a bound guarantees that the long-term dynamics takes place within a bounded region of the phase space H. Along with a similar asymptotic bound for the enstrophy ($\int_Q |\mathrm{curl}\, \boldsymbol{u}|^2 \, \mathrm{d}\boldsymbol{x}$) this enables standard theory (see Temam 1988 or Robinson 2001) to be used to show that there is an even smaller set on which the asymptotic dynamics occurs: the *global attractor* $\mathscr{A}$ (see figure 2 again). Every solution of (3.1) approaches $\mathscr{A}$ as $t\to\infty$ (so $\mathscr{A}$ is clearly 'an attractor'), and $\mathscr{A}$ is invariant, in that every solution that starts with $\boldsymbol{u}(0)\in\mathscr{A}$ remains in $\mathscr{A}$ for all $t\geqslant 0$. It therefore makes sense to talk about the dynamics 'restricted to the attractor', and this has a good claim to be a formal definition of 'the asymptotic dynamics' of the original equation.

More physically, the attractor consists of a collection of states of the system (i.e. a subset of H) that represents all possible asymptotic configurations given a particular forcing function $\boldsymbol{f}$. It is important to note that the attractor does not merely consist of a collection of stationary states, but potentially many entwined chaotic trajectories (as is the case with the well-known Lorenz attractor) whose time evolution one would hope should capture all the features of turbulence.

Indeed, it is tempting to identify the collection of states that make up the attractor with those that can occur in 'fully developed turbulence'. Although this interpretation is plausible, there are some caveats. The time scales involved in the production of experimental turbulence are relatively short, whereas one has to wait for an 'infinitely long time' for a solution to lie on the attractor. Furthermore the development of turbulence is often dependent on a spatial, rather than temporal, separation, as in the case of turbulence generated by a flow through a mesh. Nevertheless, this suggestive terminology will be adopted in what follows as a convenient shorthand.

3.3. *Estimating the dimension of the attractor*

A natural way to try to quantify the complexity of the attractor, and so of the possible dynamics that it can support, is to estimate its dimension. (Throughout this paper the 'dimension' is the upper box-counting dimension: if one denotes by $N(X,\epsilon)$ the maximum number of balls of radius ϵ needed to cover X, then $d_{\mathrm{box}}(X)= \limsup_{\epsilon\to 0} \log N(X,\epsilon)/\log(1/\epsilon)$. Essentially this extracts the exponent d from $N(X,\epsilon)\sim\epsilon^{-d}$. For more details see Falconer (1990) or Robinson (2001), for example.)

Constantin, Foias & Temam (1988) showed that the attractor for the two-dimensional Navier–Stokes equations in the case of periodic boundary conditions is a finite-dimensional set, with an explicit bound on its dimension d:

$$d \leqslant c G^{2/3} (1+\log G)^{1/3}, \tag{3.3}$$

where G is the dimensionless Grashof number,

$$G = \frac{L^2}{\nu^2}\left(\int_Q |\boldsymbol{f}(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}\right)^{1/2}.$$

In the case of Dirichlet boundary conditions the best currently available bound in terms of G is only linear, $d \leqslant cG$ (see Constantin, Foias & Temam 1985, for example).

It is important to note that because the estimate in (3.3) depends on $\boldsymbol{f}$ only via its square integral $\|\boldsymbol{f}\|_{L^2}^2 = \int_Q |\boldsymbol{f}(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}$, it does not take into account any detail of the structure of the forcing function. This implies that, with ν and L fixed, this estimate must be valid for every possible forcing function with the same value of $\|\boldsymbol{f}\|_{L^2}^2$, however simple or complicated $\boldsymbol{f}$ itself, or the resulting flow. There is therefore reason to hope that although the numerical values of the dimension that arise via (3.3) can be very large, in particular situations the real attractor dimension could be significantly smaller. Indeed, relatively simple examples show that it is possible to have an attractor that is a single steady state (and so zero-dimensional) when $\|\boldsymbol{f}\|_{L^2}^2$ is arbitrarily large.

It is possible to make other general estimates of the attractor dimension that do take into account some of the structure of $\boldsymbol{f}$. For example, one can define an alternative Grashof number G^* using the Fourier expansion of $\boldsymbol{f}$:

$$G^* = \frac{L}{\nu^2}\left(\sum_{\boldsymbol{k}\in\mathbb{Z}^2} |\boldsymbol{k}|^{-2}|\boldsymbol{f}_{\boldsymbol{k}}|^2\right)^{1/2} \qquad \text{when} \qquad \boldsymbol{f} = \sum_{\boldsymbol{k}\in\mathbb{Z}^2} \boldsymbol{f}_{\boldsymbol{k}}\,\mathrm{e}^{2\pi\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}/L},$$

and obtain the bound $d \leqslant cG^*$ (this observation is due independently to Tran, Shepherd & Cho 2004 and Robinson 2003). It is clear that one can keep

$$G = \frac{L^2}{\nu^2}\left(\sum_{\boldsymbol{k}\in\mathbb{Z}^2} |\boldsymbol{f}_{\boldsymbol{k}}|^2\right)^{1/2}$$

constant while letting $G^* \to 0$ by concentrating all the 'energy' in $\boldsymbol{f}$ at progressively higher wavenumbers. It is worth noting that in the case of Dirichlet boundary conditions $d \leqslant cG^*$ is the best bound currently available (after an equivalent definition of G^* that is also valid without recourse to the Fourier expansion of $\boldsymbol{f}$) since $G^* \leqslant G$ always.

Whatever estimate one has of the attractor dimension, it is tempting to identify it with the number of 'degrees of freedom' of the flow, but it is not immediately clear that this can be done rigorously, nor in fact exactly what these 'degrees of freedom' might be.

4. Parametrization of attractors

A parametrization of an attractor $\mathscr{A}$ can be thought of as a finite set of quantities, knowledge of which will differentiate between elements of $\mathscr{A}$. If these quantities have physical meaning then they provide a set of experimental measurements by which different physical states of the system can be distinguished.

This is illustrated in figure 3: on the left is a schematic version of the attractor $\mathscr{A}$ sitting within the infinite-dimensional phase space H, while on the right is a reconstruction A of the attractor in a finite-dimensional space ($\mathbb{R}^N$) obtained using a finite set of observations (represented by $L\boldsymbol{u} = (l_1\boldsymbol{u}, \ldots, l_N\boldsymbol{u})$ in the figure). The key property is that distinct points on the attractor produce different sets of measurements,

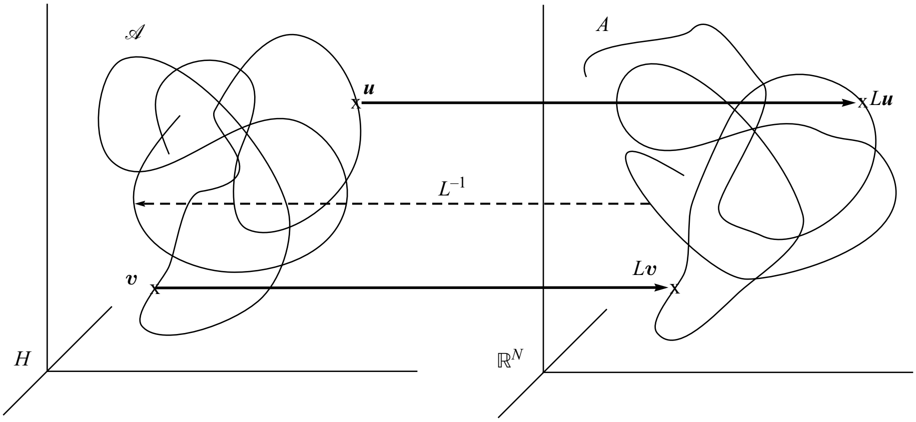

FIGURE 3. $l_1 \boldsymbol{u}, \ldots, l_N \boldsymbol{u}$ are a finite set of scalar measurements of $\boldsymbol{u}$, represented in the figure by $L\boldsymbol{u}$, that serve to differentiate between elements of the attractor. One can obtain a parametrization of the attractor in terms of these observations via the inverse map L^{-1}.

and hence different points in the right-hand picture. If one reverses this process, and instead starts with the values of the observations, these provide a parametrization of the possible states lying in the attractor via the inverse map L^{-1}.

4.1. *Abstract parametrization*

A system whose behaviour is asymptotically determined by the dynamics on a finite-dimensional attractor does, in some sense at least, have a finite number of degrees of freedom. This is guaranteed by an abstract result that any d-dimensional set can be parametrized by $2d+1$ variables. In the context of the two-dimensional Navier–Stokes equations it can be stated as follows (cf. figure 3):

THEOREM 4.1 (HUNT & KALOSHIN 1996 + FRIZ & ROBINSON 1999). *Let d be the dimension of the attractor, and take $N > 2d$. Then measurement of almost every choice of N scalar linear observables $\{l_1, \ldots, l_N\}$, i.e. N linear maps from H into $\mathbb{R}$, is sufficient to distinguish elements of $\mathscr{A}$: more explicity, if $\boldsymbol{u}, \boldsymbol{v} \in \mathscr{A}$ then*

$$l_j(\boldsymbol{u}) = l_j(\boldsymbol{v}) \quad \text{for} \quad j = 1, \ldots, N \quad \Rightarrow \quad \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{v}(\boldsymbol{x}) \quad \text{for all} \quad \boldsymbol{x} \in Q. \tag{4.1}$$

If f is C^∞ then for any $0 < \alpha < 1 - (2d/N)$ this remains true even if one requires that

$$c|L\boldsymbol{u} - L\boldsymbol{v}|^\alpha \geqslant |\boldsymbol{u} - \boldsymbol{v}| \tag{4.2}$$

for some $c > 0$, where $L\boldsymbol{u} = (l_1 \boldsymbol{u}, \ldots, l_N \boldsymbol{u})$.

(The contribution of Friz & Robinson (1999) to the theorem as stated is small but significant, namely a simplification of the range of α for which (4.2) is valid when $\boldsymbol{f}$ is C^∞.)

The theorem guarantees that different elements of the attractor, i.e. different fully developed states, can be distinguished using a finite number of measurements. These 'measurements' are scalar linear functionals of the flow field, so possible candidates would be, for example, the value of the second component of the velocity at a particular point $\boldsymbol{z} \in Q$,

$$l(\boldsymbol{u}) = u_2(\boldsymbol{z}),$$

or a local average of the first velocity component near a given point $\boldsymbol{y} \in Q$,

$$l(\boldsymbol{u}) = \int_Q \varphi(\boldsymbol{x} - \boldsymbol{y}) u_1(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$$

for some appropriate function φ. However, there is no guarantee that such physical observations are covered by the theorem, since it is an abstract result and only guarantees that measurement of 'almost every' choice of N observation functions will suffice. (One can make an appropriate definition of what 'almost every' means in this context using the notion of 'prevalence'; details are given in Hunt & Kaloshin's paper.)

The result as it stands is therefore of little practical use, and the main reason for stating it fully is to emphasize its abstraction: the set of parameters it allows ('most' choices of N independent linear functionals on H) is entirely abstract and there is no physical interpretation of what these 'degrees of freedom' might be. However, it can be used as a fundamental ingredient in the proof of two results that do provide physical sets of parameters, and in this context the information contained in (4.2) concerning the smoothness of the parametrization provided by L^{-1} is extremely useful.

4.2. *Parametrization by point values and the Landau–Lifshitz theory*

For the two-dimensional Navier–Stokes equations, Foias & Temam (1984) showed that there exists a separation δ such that if $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k$ are a finite collection of points such that

$$\text{for each } \boldsymbol{x} \in Q, \qquad |\boldsymbol{x} - \boldsymbol{x}_j| < \delta \quad \text{for some } j \in \{1, \ldots, k\},$$

then for any two solutions $\boldsymbol{u}(\boldsymbol{x}, t)$ and $\boldsymbol{v}(\boldsymbol{x}, t)$,

$$\max_j |\boldsymbol{u}(\boldsymbol{x}_j, t) - \boldsymbol{v}(\boldsymbol{x}_j, t)| \to 0 \qquad \text{as} \qquad t \to \infty$$

implies that

$$\sup_{x \in Q} |\boldsymbol{u}(\boldsymbol{x}, t) - \boldsymbol{v}(\boldsymbol{x}, t)| \to 0 \qquad \text{as} \qquad t \to \infty.$$

They called such a set a collection of 'determining nodes'. Although this is a striking result, knowledge of the solution at these nodes *for all* $t \geqslant 0$ is required in order to distinguish two solutions. However, they also conjectured that solutions *on the attractor* should be determined by a collection of *instantaneous* point values of the velocity.

It is a proof of this conjecture that rigorously connects the abstract theory of attractors and the heuristic theory of Landau & Lifshitz: the number of points required is proportional to the attractor dimension d, and hence (in two dimensions) to the minimal length scales like $d^{-1/2}$. Originally proved by Friz & Robinson (2001) for periodic boundary conditions, the most general such result is currently due to Kukavica & Robinson (2004) and extends its applicability to Dirichlet boundary conditions and observations distributed in time as well as space. The simplest possible statement is given here:

THEOREM 4.2 (FRIZ & ROBINSON 2001; KUKAVICA & ROBINSON 2004). *Let f be a real analytic function and choose $k \geqslant 16d(\mathscr{A}) + 1$, where d is the dimension of the attractor. Then for almost every set $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\}$ of k points in Q, different elements of the attractor can be distinguished by point measurements of the velocity at these k points:*

$$\boldsymbol{u}(\boldsymbol{x}_j) = \boldsymbol{v}(\boldsymbol{x}_j) \quad \textit{for} \quad j = 1, \ldots, k \;\Rightarrow\; \boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{v}(\boldsymbol{x}) \quad \textit{for all} \quad \boldsymbol{x} \in Q. \tag{4.3}$$

Furthermore the point values of $\boldsymbol{u}$ at $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\}$ parametrize $\mathscr{A}$.

When $\boldsymbol{f}$ is analytic the attractor consists of analytic functions (Foias & Temam 1989). Since these are C^∞ the result of Theorem 4.1 allows for parametrization of $\mathscr{A}$ with α as close to one as required. In essence the proof of Theorem 4.2 uses such a parametrization along with the limitations that exist on the zero sets of analytic functions. (To understand why analyticity is important, consider the set of differences of elements of the attractor, $W=\{\boldsymbol{a}_1-\boldsymbol{a}_2\colon\ \boldsymbol{a}_1,\boldsymbol{a}_2\in\mathscr{A}\}$. Suppose that $\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k\}$ are k points for which (4.3) does not hold: then there is a non-zero element $\boldsymbol{w}\in W$ for which $\boldsymbol{w}(\boldsymbol{x}_j)=0$ for $j=1,\ldots,k$, i.e. $\boldsymbol{w}$ has k simultaneous zeros. Since the structure of the zero sets of analytic functions is controlled, there cannot be many collections of such points that have so many simultaneous zeros. Unlike the other results discussed here, the analyticity of $\boldsymbol{f}$ appears to be necessary to obtain the result, and is not only included to state the theorem more simply.)

The theorem does not require that the k points are evenly spaced throughout the fluid domain, although this would seem the most natural choice. If they are chosen in this way then this yields one point in every box with sides of length $l_{\mathrm{R}}=Ld^{-1/2}$, and the result says that knowing the velocity at each of these points determines the velocity throughout the entire domain. In a rigorous way, therefore, this (i) shows that there is a 'smallest effective length scale', so that if the flow is resolved to this degree it is 'fully resolved'; (ii) gives a reason for dividing the domain into small boxes whose sides are the 'smallest scale'; and thus (iii) ties together the 'degrees of freedom' of Landau & Lifshitz with the theoretical attractor dimension.

4.2.1. *The minimum length scale and Kraichnan's theory of two-dimensional turbulence*

By using the bound on the two-dimensional attractor dimension in (3.3), it follows that

$$l_{\mathrm{R}}/L\sim G^{-1/3}(1+\log G)^{-1/6}. \tag{4.4}$$

This is of the same order as the Kraichnan length scale l_χ (Kraichnan 1967): denoting by $\langle\cdot\rangle$ the space-time average, l_χ is formed from the viscosity ν and the average viscous enstrophy dissipation $\chi=\nu\langle|\boldsymbol{\Delta u}|^2\rangle$,

$$l_\chi=\nu^{1/2}\chi^{-1/6}.$$

Using a standard estimate on solutions of the two-dimensional Navier–Stokes equations (e.g. Temam 1988), it follows that $\chi\leqslant\|\boldsymbol{f}\|_{L^2}^2L^{-2}\nu^{-1}$, and so

$$l_\chi/L\sim G^{-1/3}.$$

Thus (4.4), whose derivation is now entirely rigorous, agrees with the Kraichnan length to within logarithmic corrections. (That $d\sim l_\chi^{-2}$ for the two-dimensional attractor is an old observation (e.g. Gibbon & Titi 1997); the rigour is new.)

4.2.2. *The minimum length scale and Kolmogorov's theory of three-dimensional turbulence*

The abstract approach adopted here remains valid in the three-dimensional case if one assumes that the equations are well-posed. Constantin *et al.* (1985) showed that such a 'regularity' assumption implies the existence of a global attractor which is once again a finite-dimensional set; the attractor still consists of analytic functions if $\boldsymbol{f}$ is analytic (Foias & Temam 1989).

There are some numerical calculations in particular situations that give an indication of the possible attractor dimension: while Grappin & Léorat (1991) reported relatively small values for the dimension in a somewhat artificial setting (akin to (3.1)), the computations of turbulent Poiseuille flow at a Reynolds number of around 3200 performed by Keefe, Moin & Kim (1992) suggest an attractor dimension of approximately 780; although this is large, with the advent of increased computing power this is no longer prohibitively so.

However, the attractor dimension cannot be bounded rigorously, since it depends on quantities for which no estimate exists (such estimates would provide a proof of the existence and uniqueness of regular solutions for all time). Nevertheless, the dimension of the attractor can still be related to the Kolmogorov dissipative scale. In Kolmogorov's 1941 theory of three-dimensional turbulence, the minimum length scale in the flow, l_{K}, is formed (again via dimensional analysis) from the viscosity ν and the average viscous energy dissipation $\epsilon = \nu\langle|\boldsymbol{\omega}|^2\rangle$, where $\boldsymbol{\omega} = \mathrm{curl}\,\boldsymbol{u}$ is the vorticity:

$$l_{\mathrm{K}} = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}.$$

Gibbon & Titi (1997) show that

$$d(\mathscr{A}) \sim (L/l_{\mathrm{K}})^{4.8}.$$

The unwelcome exponent of 4.8 (it would ideally be 3) was avoided by Constantin *et al.* (1985), who replaced l_{K} with a 'Kolmogorov-like' dissipation length $l_{\tilde{\mathrm{K}}}$ based not on ϵ but on

$$\tilde{\epsilon} = \nu\left\langle|\boldsymbol{\omega}|^{5/2}\right\rangle^{4/5},$$

for which $d(\mathscr{A}) \sim (L/l_{\tilde{\mathrm{K}}})^3$.

It is of course possible that $l_{\tilde{\mathrm{K}}}$ is the more relevant physical quantity, but l_{K} has tradition on its side. It would be interesting to devise a numerical experiment in which this distinction could be tested.

This gap between the dimension estimate and the Kolmogorov theory is an important unresolved problem, and is one situation in which the improvement of a mathematical estimate would have interesting physical consequences.

5. Parametrization by repeated observations: the Takens theorem

Although theoretically significant, Theorem 4.2 has the practical drawback that a large number of simultaneous observations are needed in order to follow the flow. A method that is more practicable, and often used in experiments, is to make the same observation at equally spaced time intervals.

In a 1981 paper, "Detecting strange attractors in turbulence", Takens proved his celebrated 'time-delay embedding theorem': for a generic smooth system $\boldsymbol{x}(t)$ evolving on a smooth d-dimensional manifold M, the dynamics of solutions can be followed faithfully by taking k time-delayed copies of a 'generic measurement' $h: M \to \mathbb{R}$

$$h(\boldsymbol{x}), \quad h(\boldsymbol{x}(T)), \quad \ldots, \quad h(\boldsymbol{x}(kT)), \tag{5.1}$$

with $k \geqslant 2d$. Although the conclusions of his theorem are strong, so are its assumptions, which are hard to verify in general and may in fact fail in a number of practical applications. The requirement that the dynamics take place on a compact finite-dimensional manifold is very restrictive, and *a priori* excludes the application of the result to the infinite-dimensional dynamical system that arises from the Navier–Stokes equations. This means that the Takens Theorem provides no rigorous justification for the use of time-delay reconstruction for data from experiments in fluid dynamics.

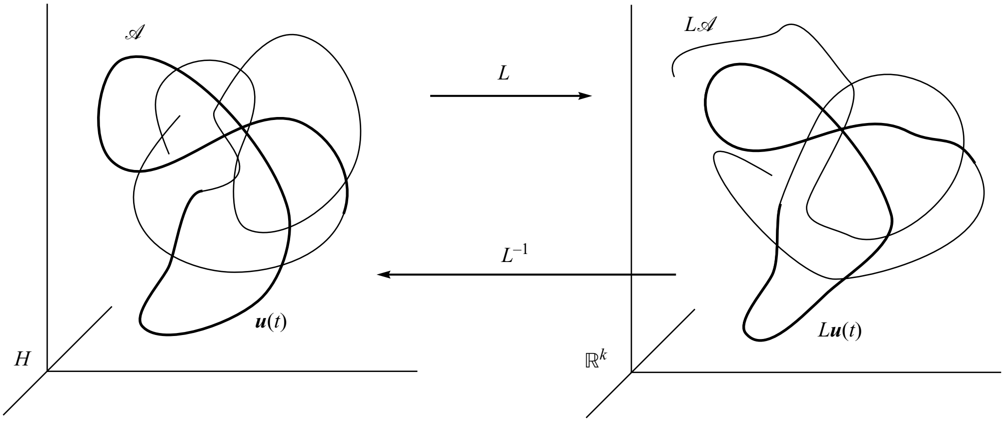

FIGURE 4. A trajectory $\boldsymbol{u}(t)$ within the attractor $\mathscr{A}$ (marked in bold) can be followed as $L\boldsymbol{u}(t)$ within the set $L\mathscr{A}$. Since $L\mathscr{A}$ lies within the finite-dimensional space $\mathbb{R}^k$ the problem then becomes one of reconstructing the dynamics on a subset of a finite-dimensional phase space. The smoothness of L^{-1} given in (5.2) allows one to adapt the proof of Sauer *et al.* (1993) to treat the resulting finite-dimensional dynamical system.

Sauer, Yorke & Casdagli (1993) proved a variant of the Takens theorem by replacing M with an attractor of dimension d and allowing dynamical systems that are only Lipschitz continuous rather than smooth, but their result was still only valid for attractors of finite-dimensional dynamical systems.

5.1. *An infinite-dimensional version of the Takens theorem*

If one is really to 'detect strange attractors in turbulence', a version of the Takens Theorem valid in the infinite-dimensional setting is required. The following result, stated for the two-dimensional Navier–Stokes equations with smooth forcing, is a particular version of a more general abstract theorem.

THEOREM 5.1 (ROBINSON 2005). *Let f be C^∞ and d the dimension of the Navier–Stokes attractor $\mathscr{A}$. Choose $k \geqslant 2d$ and $T > 0$ such that the sets $\mathscr{A}_p$ of all periodic orbits of periods pT satisfy $d_f(\mathscr{A}_p) < p/2$ for all $p = 1, \ldots, k+1$. Then almost every*† *Lipschitz observation map $h : H \to \mathbb{R}$ makes the k-fold delay map*

$$\boldsymbol{u} \mapsto \{h(\boldsymbol{u}), h(\boldsymbol{u}(T)), h(\boldsymbol{u}(2T)), \ldots, h(\boldsymbol{u}(kT))\}$$

one-to-one between $\mathscr{A}$ and its image.

A major advantage of taking such a set of observations is that every new observation (beyond the kth) yields information on the dynamical evolution of the system, since the k observations corresponding to $\boldsymbol{u}(T)$ are simply

$$\{h(\boldsymbol{u}(T)), h(\boldsymbol{u}(2T)), \ldots, h(\boldsymbol{u}(kT)), h(\boldsymbol{u}((k+1)T))\}.$$

Although the generalization of the Takens Theorem to infinite dimensions has been an open problem for a number of years, its proof is surprisingly simple: it is sketched here with the main ideas illustrated in figure 4. Theorem 4.1 guarantees that for each $0 < \theta < 1$ there is a linear map $L : H \to \mathbb{R}^k$ (for some k) such that

$$|L^{-1}\boldsymbol{x} - L^{-1}\boldsymbol{y}| \leqslant c|\boldsymbol{x} - \boldsymbol{y}|^\theta \qquad \text{for all} \qquad \boldsymbol{x}, \boldsymbol{y} \in L\mathscr{A} \tag{5.2}$$

† As in theorem 4.2, 'almost every' has to be taken in the sense of 'prevalence', see Hunt, Sauer & Yorke (1992) for more details.

for some $c > 0$. If $\boldsymbol{u}_0 \in \mathscr{A}$ then by considering $L\boldsymbol{u}(t) \in L\mathscr{A} \subset \mathbb{R}^k$ one can follow the dynamics of solutions lying in the attractor within a subset of a finite-dimensional space. Although the resulting finite-dimensional dynamical system is not Lipschitz (the case considered by Sauer *et al.*) it follows from (5.2) that it is Hölder continuous in such a way that it is simple to adapt the finite-dimensional proof of Sauer *et al.* (1993), and with a little care Theorem 5.1 follows.

5.2. *Minimal periods in infinite-dimensional systems: applicability of Theorem 5.1*

In Theorem 5.1 the conditions on the dimension of the sets of periodic orbits, which (following Sauer *et al.*) have replaced the genericity assumptions of Takens, imply that if the theorem is to apply then there can be no periodic orbits of periods T or $2T$, since any periodic orbit will have dimension at least 1.

It is therefore useful to have a result that guarantees the non-existence of periodic orbits with small periods, and this is provided by a generalization to infinite dimensions of an old theorem of Yorke (1969): every periodic orbit of an ordinary differential equation $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ has period at least $2\pi/L$, where L is the Lipschitz constant of $\boldsymbol{f}$ (i.e. when $|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})| \leqslant L|\boldsymbol{x} - \boldsymbol{y}|$).

The argument is based on a significantly simpler new proof of Yorke's finite-dimensional result (albeit in a slightly weaker form). Since Yorke's useful result does not appear to be widely known, this simple proof (inspired by Kukavica 1994) is worth giving here.

THEOREM 5.2. *Any periodic orbit of the equation $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ (with $\boldsymbol{x} \in \mathbb{R}^n$), where $\boldsymbol{f}$ has Lipschitz constant L, has period $T \geqslant 1/L$.*

Proof. Fix $\tau > 0$ and set $\boldsymbol{v}(t) = \boldsymbol{x}(t) - \boldsymbol{x}(t - \tau)$. Then

$$\boldsymbol{v}(t) - \boldsymbol{v}(s) = \int_s^t \dot{\boldsymbol{v}}(r)\,\mathrm{d}r.$$

Integrating both sides with respect to s from 0 to T gives

$$T\boldsymbol{v}(t) = \int_0^T \left(\int_s^t \dot{\boldsymbol{v}}(r)\,\mathrm{d}r \right) \mathrm{d}s$$

and so

$$T|\boldsymbol{v}(t)| \leqslant \int_0^T \int_0^T |\dot{\boldsymbol{v}}(r)|\,\mathrm{d}r\,\mathrm{d}s \leqslant T \int_0^T |\dot{\boldsymbol{v}}(r)|\,\mathrm{d}r,$$

i.e.

$$\begin{aligned} |\boldsymbol{x}(t) - \boldsymbol{x}(t-\tau)| \leqslant \int_0^T |\dot{\boldsymbol{v}}(s)|\,\mathrm{d}s &= \int_0^T |\boldsymbol{f}(\boldsymbol{x}(s)) - \boldsymbol{f}(\boldsymbol{x}(s-\tau))|\,\mathrm{d}s \\ &\leqslant L \int_0^T |\boldsymbol{x}(s) - \boldsymbol{x}(s-\tau)|\,\mathrm{d}s. \end{aligned}$$

Therefore

$$\int_0^T |\boldsymbol{x}(t) - \boldsymbol{x}(t-\tau)|\,\mathrm{d}t \leqslant LT \int_0^T |\boldsymbol{x}(s) - \boldsymbol{x}(s-\tau)|\,\mathrm{d}s,$$

and it follows that if $LT < 1$ then

$$\int_0^T |\boldsymbol{x}(t) - \boldsymbol{x}(t-\tau)|\,\mathrm{d}t = 0.$$

Thus $\boldsymbol{x}(t) = \boldsymbol{x}(t - \tau)$ for all $\tau > 0$, i.e. $\boldsymbol{x}(t)$ is constant. □

The infinite-dimensional result (Robinson & Vidal-López 2004) has a similar statement, but is couched in terms of semilinear evolution equations, of which the Navier–Stokes equations provide an example (see Henry 1981, for example). Rather than state the general result here, it seems better simply to state its consequence for the two-dimensional Navier–Stokes equations with periodic boundary conditions which has been discussed throughout the paper: in this case the minimal period of any periodic orbit is

$$c\nu^{-1}\lambda_1^{-1}G^{-2}(1+\log G)^{-1}.$$

In fact this result was obtained earlier by Kukavica (1994) using equation-specific methods; he also showed that for the three-dimensional Navier–Stokes equations the minimal period is bounded below by $c\nu^{-1}\lambda_1^{-1}G^{-4}$, where now the appropriate definition of the dimensionless Grashof number is $G = L^{3/2}\|\boldsymbol{f}\|_{L^2}/\nu^2$; under the assumption of regularity of solutions this allows Theorem 5.1 to be applied in the three-dimensional case also.

6. Conclusion

It is hoped that the results reported here go some way towards demonstrating the worth of the abstract dynamical systems approach for problems concerning fluid flow and the theory of turbulence.

While the question of existence and uniqueness of solutions for the three-dimensional Navier–Stokes equations is the most important open problem in the field, there are other interesting questions that needs resolving.

One issue is the very large dimensions that standard estimates produce. As discussed in § 3.3, it is likely that this is due to the fact that the bounds generally neglect any structural properties of the forcing function $\boldsymbol{f}$, instead making use only of gross overall features such as

$$\int_Q |\boldsymbol{f}(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}.$$

If the results presented here are to be practicable then situations in which the attractor dimension is lower need to be found. In particular more effort needs to be invested in finding good bounds for physically relevant flows, or their two-dimensional analogues.

Another important open problem, already highlighted in § 4.2.2, is to improve the bounds on the attractor dimension in the three-dimensional case (under the assumption of regularity) to coincide with that derived from the classical Kolmogorov dissipative scale.

More generally, a greater emphasis on the derivation of physically significant consequences of the existence of finite-dimensional attractors for important mathematical models is desirable.

Finally it should be remarked that, since the underlying results are proved in an abstract setting, they are in fact applicable to a wide range of examples in addition to the Navier–Stokes equations, including the pattern-formation Ginzburg–Landau and Kuramoto–Sivashinsky equations.

I would like to take the opportunity here to acknowledge the influence of John Gibbon on my research. Since the early years of my PhD he has been consistently inspiring and supportive, and I very much appreciate his help and friendship.

Sergei Chernyshenko read through an earlier version of this paper and made a number of helpful comments, for which I am most grateful. I am currently a Royal

Society University Research Fellow, and would like to thank The Society for all their support.

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