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A natural space of functions for the Ruelle operator theorem

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Abstract. We study a new space, $R(X)$, of real-valued continuous functions on the space X of sequences of zeros and ones. We show exactly when the Ruelle operator theorem holds for such functions. Any g -function in $R(X)$ has a unique g -measure and powers of the corresponding transfer operator converge. We also show $\text{Bow}(X, T) \neq W(X, T)$ and relate this to the existence of bounded measurable coboundaries, which are not continuous coboundaries, for the shift on the space of bi-sequences of zeros and ones.

0. Introduction

We study a family of continuous functions on the space, $X = \prod_0^\infty \{0, 1\}$, of sequences $x = (x_n)_0^\infty$ of zeros and ones. This family, $R(X)$, is well behaved with respect to the Ruelle operator theorem (also called the Ruelle–Perron–Frobenius theorem). This theorem concerns the Ruelle transfer operator \mathcal{L}_φ on the Banach space $C(X)$ of real-valued continuous functions on X . With suitable assumptions on $\varphi \in C(X)$ there is a number $\lambda > 0$ and some $h \in C(X)$ with $h > 0$ and $\mathcal{L}_\varphi h = \lambda h$, some probability measure ν on X with $\mathcal{L}_\varphi^* \nu = \lambda \nu$, and, for all $f \in C(X)$, $\mathcal{L}_\varphi^n f / \lambda^n$ converges, in the sup norm on $C(X)$, to $(\int f d\nu)h$. Also $\mu_\varphi = h\nu$ turns out to be the unique equilibrium state of φ with respect to the shift transformation T on X . When φ is in our space $R(X) \subset C(X)$ we obtain necessary and sufficient conditions for the existence of such an eigenfunction h , and we show that the existence of h forces the rest of the Ruelle operator theorem to hold. Moreover, if $\varphi \in R(X)$ and an eigenfunction h exist, then $g = e^\varphi h / \lambda h \circ T \in R(X)$ and also $\log g \in R(X)$. This allows us to reduce the study of certain $\varphi \in R(X)$ to that of g -functions in $R(X)$. The space $R(X)$ includes the functions studied by Hofbauer [Ho]. These include examples of functions of the type devised by Fisher, without unique equilibrium states [Fi].

In §1 we define our space $R(X)$ and obtain necessary and sufficient conditions for a function $\varphi \in R(X)$ to be in the space $\text{Bow}(X, T)$, necessary and sufficient conditions for $\varphi \in R(X)$ to be in $W(X, T)$, and necessary and sufficient conditions for $\varphi \in R(X)$ to be

a coboundary. The spaces $\text{Bow}(X, T)$, $W(X, T)$ and $\text{Cob}(X, T)$ are important in the study of transfer operators and equilibrium states. We give examples from $R(X)$ of functions in $\text{Bow}(X, T)$ but not in $W(X, T)$. This type of example can be modified to show that $\text{Bow}(X, T) \setminus W(X, T)$ is non-empty for any non-trivial subshift of finite type $T : X \rightarrow X$.

In §2 we study those members of $R(X)$ which are g -functions for the shift T . Each such g has a unique g -measure, which we describe. Also if \mathcal{L} denotes the transfer operator of $\log g$, then, for all $f \in C(X)$, $\mathcal{L}^n f$ converges uniformly on X to a constant $\mu(f)$ as $n \rightarrow \infty$. This result had been proved for a smaller class than $R(X)$ as part of the thesis of Hulse [Hu].

In §3 we investigate the Ruelle operator theorem for $\varphi \in R(X)$. In Theorem 3.1 we obtain necessary and sufficient conditions for the existence of a positive eigenfunction for \mathcal{L}_φ . These turn out to be necessary and sufficient for the whole of the conclusion of the Ruelle operator theorem. If $\varphi \in R(X) \cap \text{Bow}(X, T)$ the necessary and sufficient conditions hold. We give examples of $\varphi \in R(X)$ where these conditions do not hold.

In §4 we use $R(X)$ to obtain a class of continuous functions on the two-sided shift space $\widehat{X} = \{0, 1\}^Z$ which are bounded measurable coboundaries but not continuous coboundaries for the shift S on \widehat{X} .

We now explain our notation and terminology. Let $X = \prod_0^\infty \{0, 1\}$ be the full one-sided shift space with symbols 0 and 1 and let $T : X \rightarrow X$ denote the one-sided shift transformation. Points of X are sequences $x = (x_n)_0^\infty$ of zeros and ones. The topology on X is the direct product of the discrete topology on $\{0, 1\}$. If $i \geq 0$, $j \geq 1$ and $a_0, \dots, a_{j-1} \in \{0, 1\}$ then ${}_i[a_0 \dots a_{j-1}]_{i+j-1}$ or ${}_i[a_0 \dots a_{j-1}]$ denote the set $\{x = (x_n)_0^\infty \mid x_{k+i} = a_k, 0 \leq k \leq j-1\}$. Such a set is called a cylinder set based at coordinate i . All cylinder sets are finite unions of cylinder sets based at coordinate zero, and these form a basis for the topology. Note that $T^{-i} {}_0[a_0 \dots a_{j-1}] = {}_i[a_0 \dots a_{j-1}]$. A metric on X with this topology is given by: if $x \neq y$, $d(x, y) = 1/(j+1)$ if j is the smallest non-negative integer with $x_j \neq y_j$.

If $j \geq 1$ and $a_0, \dots, a_{j-1} \in \{0, 1\}$ then, if $x \in X$, $a_0 \dots a_{j-1}x$ denotes the point $z = (z_n)_0^\infty$ of X with $z_i = a_i$ for $0 \leq i \leq j-1$ and $z_{i+j} = x_i$ for $i \geq 0$. If $j \geq 1$ then 0^jx is the point $z = (z_n)_0^\infty$ with $z_i = 0$, $0 \leq i \leq j-1$, and $z_{j+i} = x_i$ for $i \geq 0$. The point 0^∞ is the sequence with all entries zero and if $j \geq 1$ and $a_0, \dots, a_{j-1} \in \{0, 1\}$ then $a_0 \dots a_{j-1}0^\infty$ is the point $z = (z_n)$ with $z_n = a_n$, $0 \leq n \leq j-1$, and $z_{j+i} = 0$ for $i \geq 0$. If $j \geq 1$ and $a_0, \dots, a_{j-1} \in \{0, 1\}$ then $(a_0 \dots a_{j-1})^\infty$ is the point $z = (z_n)_0^\infty$ with $z_{mj+i} = a_i$ for $0 \leq i \leq j-1$ and $m \geq 0$. Such points are exactly the points $z \in X$ with $T^j z = z$.

Let $C(X)$ denote the Banach space of all real-valued continuous functions on X , equipped with the supremum norm. Continuity properties of a function $f : X \rightarrow \mathbb{R}$ can often be expressed using the sequence of numbers $\{v_n(f)\}_1^\infty$ defined by

$$v_n(f) = \sup\{f(x) - f(y) \mid x, y \in X \text{ and } x_i = y_i \text{ for } 0 \leq i \leq n-1\}.$$

For example $f \in C(X)$ if and only if $v_n(f) \rightarrow 0$.

We let $M(X)$ denote the space of all probability measures on the Borel subsets of X , equipped with the weak*-topology, and let $M(X, T)$ denote the non-empty subset of T -invariant members of $M(X)$. We say that $\tau \in M(X)$ has support X if $\tau(U) > 0$ for

every non-empty open set U . If $\varphi \in C(X)$ we let $P(T, \varphi)$ denote the pressure of T at φ (see [W1]), and let $T_n\varphi$ be the function $\sum_{i=0}^{n-1} \varphi \circ T^i$. The Ruelle operator of $\varphi \in C(X)$ will be denoted by $\mathcal{L}_\varphi : C(X) \rightarrow C(X)$, so that $(\mathcal{L}_\varphi f)(x) = \sum e^{\varphi(y)} f(y)$ where the sum is over all $y \in T^{-1}x$. Hence $(\mathcal{L}_\varphi f)(x) = e^{\varphi(0x)} f(0x) + e^{\varphi(1x)} f(1x)$.

The dual operator \mathcal{L}_φ^* always has an eigenmeasure in $M(X)$, i.e. there exist $\nu \in M(X)$ and $\lambda > 0$ with $\mathcal{L}_\varphi^* \nu = \lambda \nu$ (see [W2]).

We consider two spaces of functions which are important in studying equilibrium states. These spaces can be defined for a general continuous transformation $T : X \rightarrow X$ of a compact metric space. We say that $\varphi \in C(X)$ belongs to $\text{Bow}(X, T)$ if there exist $\delta > 0$, $C > 0$ with the property that whenever $n \geq 1$ and $x, y \in X$ satisfy $d(T^i x, T^i y) < \delta$ for all $0 \leq i \leq n - 1$ then $|(T_n\varphi)(x) - (T_n\varphi)(y)| \leq C$ (see [Bow, W4, W5, W6]). We say that $\varphi \in C(X)$ belongs to $W(X, T)$ if for all $\epsilon > 0$ there exists $\delta > 0$ with the property that whenever $n \geq 1$ and $x, y \in X$ satisfy $d(T^i x, T^i y) < \delta$ for all $0 \leq i \leq n - 1$ then $|(T_n\varphi)(x) - (T_n\varphi)(y)| < \epsilon$ (see [Bou, W5, W6]). Clearly $W(X, T) \subset \text{Bow}(X, T)$. For the one-sided shift $T : X \rightarrow X$ on the space $X = \prod_0^\infty \{0, 1\}$, which we are studying in this paper, we have $\varphi \in \text{Bow}(X, T)$ if and only if $\varphi \in C(X)$ and there exists $p \geq 0$ with $\sup_{n \geq 1} v_{n+p}(T_n\varphi) < \infty$. This latter condition is equivalent to $\sup_{n \geq 1} v_n(T_n\varphi) < \infty$. Also $\varphi \in W(X, T)$ if and only if $\sup_{n \geq 1} v_{n+p}(T_n\varphi) \rightarrow 0$ as $p \rightarrow \infty$.

In [W3] the author showed that, for a topologically mixing subshift of finite type, if $\varphi \in W(X, T)$ then the Ruelle operator theorem holds (that is, there exist $\lambda > 0, \nu \in M(X)$, and $h \in C(X)$ with $h > 0$ and $\int h d\nu = 1$ such that $\mathcal{L}_\varphi h = \lambda h$, $\mathcal{L}_\varphi^* \nu = \lambda \nu$ and, for all $f \in C(X)$,

$$\frac{(\mathcal{L}_\varphi^n f)(x)}{\lambda^n} \rightrightarrows h(x) \int f d\nu,$$

where \rightrightarrows denotes uniform convergence on X , φ has a unique equilibrium state μ_φ and (T, μ_φ) has a Bernoulli natural extension. Here $\mu_\varphi = h\nu$, and μ_φ is the unique g -measure for the g -function $g(x) = e^{\varphi(x)} h(x) / \lambda h(Tx)$. In [W4], the author considered these questions for $\varphi \in \text{Bow}(X, T)$ and proved a weakened version of the Ruelle operator theorem. Each $\varphi \in \text{Bow}(X, T)$ has a unique equilibrium state μ_φ and (T, μ_φ) has a Bernoulli natural extension [W6].

We shall also use the space of continuous coboundaries. If $T : X \rightarrow X$ is any continuous transformation of a compact metric space then the space of continuous coboundaries for T is $\text{Cob}(X, T) = \{f \in C(X) \mid \exists l \in C(X) \text{ with } f = l \circ T - l\}$. Such a function l is called a cobounding function for f . We have $\text{Cob}(X, T) \subset W(X, T)$. Coboundaries are important in the study of equilibrium states.

1. The space $R(X)$

We now define the space $R(X)$ of functions on $X = \prod_{n=0}^\infty \{0, 1\}$. A function $\varphi \in C(X)$ is in the space $R(X)$ if it is defined in the following way: there are four convergent sequences of real numbers $(a_n)_2^\infty \rightarrow a$, $(b_n)_1^\infty \rightarrow b$, $(c_n)_2^\infty \rightarrow c$, $(d_n)_1^\infty \rightarrow d$ and for all $z \in X$, for all $p \geq 2$, for all $q \geq 1$, $\varphi(0^p 1z) = a_p$, $\varphi(01^q 0z) = b_q$, $\varphi(1^p 0z) = c_p$, $\varphi(10^q 1z) = d_q$, $\varphi(0^\infty) = a$, $\varphi(01^\infty) = b$, $\varphi(1^\infty) = c$ and $\varphi(10^\infty) = d$. So at a point with initial symbol 0 the value of φ is a_p if the initial block of zeros has length $p \geq 2$, but if the initial zero is

immediately followed by a block of ones of length $q \geq 1$ the value of φ is b_q . Similarly if the initial symbol is 1.

The space $R(X)$ is a vector subspace of $C(X)$ and $\varphi \in R(X)$ if and only if $e^\varphi \in R(X)$.

We now characterize the spaces $R(X) \cap \text{Bow}(X, T)$ and $R(X) \cap W(X, T)$ and show that they differ.

THEOREM 1.1. *Let $\varphi \in R(X)$ be defined by the sequences $(a_p)_2^\infty \rightarrow a$, $(b_q)_1^\infty \rightarrow b$, $(c_p)_2^\infty \rightarrow c$, $(d_q)_1^\infty \rightarrow d$ as above. Then we have the following:*

- (i) $\varphi \in \text{Bow}(X, T)$ if and only if $\sum_{n=2}^\infty (a_n - a)$ and $\sum_{n=2}^\infty (c_n - c)$ both have bounded sequences of partial sums;
- (ii) $\varphi \in W(X, T)$ if and only if $\sum_{n=2}^\infty (a_n - a)$ and $\sum_{n=2}^\infty (c_n - c)$ are both convergent;
- (iii) $\varphi \in \text{Cob}(X, T)$ if and only if $b_1 + d_1 = 0$ and, for all $p \geq 2$, $b_p + d_1 + \sum_{i=2}^p c_i = 0$ and $d_q + b_1 + \sum_{i=2}^p a_i = 0$.

When these conditions hold the cobounding function $k \in C(X)$ has the form $k((0^q 1z)) = \alpha_q$, $q \geq 1$, $z \in X$, $k((1^q 0z)) = \beta_q$, $q \geq 1$, $z \in X$, $k(0^\infty) = \alpha$, $k(1^\infty) = \beta$ where $\alpha_q \rightarrow \alpha$, $\beta_q \rightarrow \beta$.

Note that when the equations in (iii) hold then $\sum_{i=2}^\infty a_i$ converges so $a = 0$. Similarly $c = 0$ when the equations in (iii) hold.

Note that the conditions for $\varphi \in \text{Bow}(X, T)$ and $\varphi \in W(X, T)$ do not involve the sequences $(b_n)_1^\infty$ and $(d_n)_1^\infty$. In the condition in (iii) once b_1 is chosen then $(b_i)_{i=2}^\infty$ and $(d_j)_{j=1}^\infty$ are determined in terms of b_1 , $(a_n)_2^\infty$ and $(c_n)_2^\infty$.

We prove Theorem 1.1 using the following lemma.

LEMMA 1.2. *Let $\varphi \in R(X)$ be defined by the sequences $(a_p)_2^\infty \rightarrow a$, $(b_q)_1^\infty \rightarrow b$, $(c_p)_2^\infty \rightarrow c$ and $(d_q)_1^\infty \rightarrow d$ as in Theorem 1.1. Then we have the following.*

- (i) For $n \geq 2$,

$$v_n(\varphi) = \sup\{\max(a_{n+t} - a_{n+s}, b_{n+t-1} - b_{n+s-1}, c_{n+t} - c_{n+s}, d_{n+t-1} - d_{n+s-1}) : s, t \geq 0\}.$$

Hence if

$$C_n = \sup\{\max(|a_j - a|, |b_{j-1} - b|, |c_j - c|, |d_{j-1} - d|) : j \geq n\}$$

then $C_n \leq v_n(\varphi) \leq 2C_n$.

- (ii) For $n, N \geq 2$,

$$v_{n+N}(T_n \varphi) = \max\left(\sup_{i, j \geq N} [(a_{i+1} + \dots + a_{i+n}) - (a_{j+1} + \dots + a_{j+n})], \sup_{i, j \geq N, 1 \leq k \leq n-1} [d_{k+i} - d_{k+j} + (a_{i+1} + \dots + a_{i+k}) - (a_{j+1} + \dots + a_{j+k})],\right)$$

$$\sup_{i,j \geq N} (b_i - b_j), \sup_{i,j \geq N} [(c_{i+1} + \dots + c_{i+n}) - (c_{j+1} + \dots + c_{j+n})],$$

$$\sup_{i,j \geq N, 1 \leq k \leq n-1} [b_{k+i} - b_{k+j} + (c_{i+1} + \dots + c_{i+k}) - (c_{j+1} + \dots + c_{j+k})], \sup_{i,j \geq N} (d_i - d_j).$$

Hence if $D_N = \sup_{i,j \geq N} (d_i - d_j)$, $B_N = \sup_{i,j \geq N} (b_i - b_j)$ and

$$A_{n,N} = \max \left(B_N, D_N, \sup_{i \geq N, 1 \leq k \leq n} |(a_{i+1} + \dots + a_{i+k}) - ka|, \sup_{i \geq N, 1 \leq k \leq n} |(c_{i+1} + \dots + c_{i+k}) - kc| \right)$$

then for $n, N \geq 2$

$$A_{n,N} - D_N - B_N \leq v_{n+N}(T_n \varphi) \leq 2A_{n,N} + D_N + B_N.$$

Proof. (i) Let $n \geq 2$ and let $x, y \in X$ have $(x_0, \dots, x_{n-1}) = (y_0, \dots, y_{n-1})$.

Suppose $x_0 = y_0 = 0$.

If $x, y \in {}_0[0^p 1]$ for some $p \geq 2$ then $\varphi(x) = \varphi(y)$, and if $x, y \in {}_0[01^q 0]$ for some $q \geq 1$ then $\varphi(x) = \varphi(y)$.

If $x \in {}_0[0^{n+t} 1]$ for some $t \geq 0$ and $y \in {}_0[0^{n+s} 1]$ for some $s \geq 0$ then $\varphi(x) - \varphi(y) = a_{n+t} - a_{n+s}$. If $x \in {}_0[0^{n+t} 1]$ for some $t \geq 0$ and $y = 0^\infty$ then $\varphi(x) - \varphi(y) = a_{n+t} - a$.

If $x \in {}_0[01^{n-1+t} 0]$ for some $t \geq 0$ and $y \in {}_0[01^{n-1+s} 0]$ for some $s \geq 0$ then $\varphi(x) - \varphi(y) = b_{n+t-1} - b_{n+s-1}$. If $x \in {}_0[01^{n-1+t} 0]$ and $y = (01)^\infty$ then $\varphi(x) - \varphi(y) = b_{n+t-1} - b$.

When $x_0 = y_0 = 1$ we get similar results and hence the expression in (i). The inequality involving C_n follows from the triangle inequality.

(ii) Let $n, N \geq 2$. Let $x, y \in X$ have $(x_0, \dots, x_{n+N-1}) = (y_0, \dots, y_{n+N-1})$.

Consider the case $x_{n-1} = 0 = y_{n-1}$; the case when $x_{n-1} = 1 = y_{n-1}$ is handled in a similar way. Consider firstly when $(x_{n-1}, x_n) = (0, 0) = (y_{n-1}, y_n)$.

Suppose $(x_0, \dots, x_{n-1}) = 0^n$. If $x \in {}_0[0^{n+i} 1]$ for some $i \geq N$ and $y \in {}_0[0^{n+j} 1]$ for some $j \geq N$ then

$$(T_n \varphi)(x) - (T_n \varphi)(y) = (a_{n+i} + \dots + a_{1+i}) - (a_{n+j} + \dots + a_{1+j}).$$

If $x \in {}_0[0^{n+i} 1]$ for some $i \geq N$ and $y = (0)^\infty$ then

$$(T_n \varphi)(x) - (T_n \varphi)(y) = (a_{n+i} + \dots + a_{1+i}) - na.$$

If $x \in {}_0[0^{n+i} 1]$ for some $1 \leq i \leq N - 1$ then $y \in {}_0[0^{n+i} 1]$ and $(T_n \varphi)(x) = (T_n \varphi)(y)$.

Suppose $x_r = 1$ for some $0 \leq r \leq n - 2$, so that $x \in {}_{n-1-k}[10^{k+i} 1]$ for some $1 \leq k \leq n - 1$ and $i \geq 1$ or $T^{n-1-k} x = (10)^\infty$. If $x \in {}_{n-1-k}[10^{k+i} 1]$ for some $1 \leq k \leq n - 1$ and $1 \leq i \leq N - 1$ then $y \in {}_{n-1-k}[10^{k+i} 1]$ and $(T_n \varphi)(x) = (T_n \varphi)(y)$. If $x \in {}_{n-1-k}[10^{k+i} 1]$ for some $1 \leq k \leq n - 1$ and some $i > N - 1$ then either $y \in {}_{n-1-k}[10^{k+j} 1]$ for some $j > N - 1$ and then

$$(T_n \varphi)(x) - (T_n \varphi)(y) = d_{k+i} - d_{k+j} + (a_{k+i} + \dots + a_{1+i}) - (a_{k+j} + \dots + a_{1+j}),$$

or $T^{n-1-k}y = (10^\infty)$ and then

$$(T_n\varphi)(x) - (T_n\varphi)(y) = d_{k+i} + (a_{k+i} + \dots + a_{1+i}) - d - (n-1)a.$$

If $T^{n-1-k}x = (10^\infty)$ then either $y \in_{n-1-k}[10^{k+j}1]$ for some $j > N-1$ and then

$$(T_n\varphi)(x) - (T_n\varphi)(y) = d + (n-1)a - d_{k+j} - (a_{k+j} + \dots + a_{1+j}),$$

or $x = y$.

Now consider when $(x_{n-1}, x_n) = (0, 1)$. Either $x \in_{n-1}[01^i0]$ for some $i \geq 1$, or $T^{n-1}x = (01^\infty)$. Suppose $x \in_{n-1}[01^i0]$ for some $i \geq 1$. If $i < N$ then $y \in_{n-1}[01^i0]$ and $(T_n\varphi)(x) = (T_n\varphi)(y)$. If $i \geq N$ then either $y \in_{n-1}[01^j0]$ for some $j \geq N$ and then $(T_n\varphi)(x) - (T_n\varphi)(y) = b_i - b_j$, or $T^{n-1}y = (01^\infty)$ and then $(T_n\varphi)(x) - (T_n\varphi)(y) = b_i - b$. If $T^{n-1}x = (01^\infty)$ then either $y \in_{n-1}[01^j0]$ for some $j \geq N$ and then $(T_n\varphi)(x) - (T_n\varphi)(y) = b - b_j$, or $y = x$.

The corresponding reasoning can be used when $x_{n-1} = 1 = y_{n-1}$ and we get the equality in (ii). The inequalities follow from the triangle inequality. \square

Proof of Theorem 1.1. Parts (i) and (ii) follow from Lemma 1.2(ii), since $\varphi \in \text{Bow}(X, T)$ means $\sup_{n \geq 1} v_{n+N}(T_n\varphi) < \infty$ for some $N \geq 2$ and $\varphi \in W(X, T)$ means $\sup_{n \geq 1} v_{n+N}(T_n\varphi) \rightarrow 0$ as $N \rightarrow \infty$.

We turn to the proof of part (iii). Suppose $\varphi \in \text{Cob}(X, T)$. If $T^n(x) = x$ then $T_n\varphi(x) = 0$. If we let $x = (01)^\infty$ then $\varphi((01)^\infty) + \varphi((10)^\infty) = 0$ so $b_1 + d_1 = 0$. Let $p \geq 2$ and let $x = (0^p1)^\infty$. Since $T^{p+1}(x) = x$ we have $(T_{p+1}\varphi)(x) = 0$. Hence $a_p + a_{p-1} + \dots + a_2 + b_1 + d_p = 0$. Similarly, taking $x = (1^p0)^\infty$ gives $c_p + c_{p-1} + \dots + c_2 + d_1 + b_p = 0$. Hence we get the equations in (iii).

Now suppose the equations in (iii) hold and we show $\varphi \in \text{Cob}(X, T)$. We have $a = 0 = c$. Let α_1 be any real number. Define α_p for $p \geq 2$ by $\alpha_p = \alpha_1 - \sum_{i=2}^p a_i = \alpha_1 + b_1 + d_p$, and define $\beta_q, q \geq 1$, by $\beta_q = \alpha_1 + b_q$. Then $\alpha_p \rightarrow \alpha_1 + b_1 + d$ and $\beta_q \rightarrow \alpha_1 + b$.

Define $k : X \rightarrow \mathbb{R}$ by $k((0^q1z)) = \alpha_q, q \geq 1, z \in X, k((1^q0z)) = \beta_q, k(0^\infty) = \alpha_1 + b_1 + d, k(1^\infty) = \alpha_1 + b$. Then $k \in C(X)$ and we show that $k(Tx) - k(x) = \varphi(x), x \in X$.

If $x \in_0[0^p1]$ with $p \geq 2$ then $k(Tx) - k(x) = \alpha_{p-1} - \alpha_p = a_p = \varphi(x)$. If $x \in_0[01^q0]$ with $q \geq 1$ then $k(Tx) - k(x) = \beta_q - \alpha_1 = b_q = \varphi(x)$.

For $x = (0^\infty), \varphi(0^\infty) = a = 0 = k(Tx) - k(x)$. When $x = (01)^\infty, k(Tx) - k(x) = \alpha_1 + b - \alpha_1 = b = \varphi(x)$.

If $x \in_0[1^p0]$ with $p \geq 2$ then $k(Tx) - k(x) = \beta_{p-1} - \beta_p = b_{p-1} - b_p = c_p = \varphi(x)$. If $x \in_0[10^q1]$ with $q \geq 2$ then $k(Tx) - k(x) = \alpha_q - \beta_1 = \alpha_1 - \beta_1 - \sum_{i=2}^q a_i = \alpha_1 - \beta_1 + d_q + b_1 = d_q = \varphi(x)$ by the definition of β_1 . If $x \in_0[10^q1]$ with $q = 1$ then $k(Tx) - k(x) = \alpha_1 - \beta_1 = -b_1 = d_1 = \varphi(x)$. When $x = (1^\infty), \varphi(x) = c = 0 = k(Tx) - k(x)$, and when $x = (10)^\infty, k(Tx) - k(x) = \alpha_1 + b_1 + d - \beta_1 = d = \varphi(x)$ by the definition of β_1 . Hence k is a cobounding function for φ .

The difference $k_1 - k_2$ of any two cobounding functions for φ is a T -invariant continuous function. Since T is topologically transitive, $k_1 - k_2$ is a constant, so any cobounding function has the form given. \square

COROLLARY 1.3. We have $W(X, T) \neq \text{Bow}(X, T)$.

Proof. Using Theorem 1.1 we can get examples of $\varphi \in \text{Bow}(X, T) \setminus W(X, T)$. Let $\sum_{n=2}^{\infty} a_n$ be a divergent series with a bounded sequence of partial sums and with $a_n \rightarrow 0$. For example we could take $a_n = \sin(\sqrt{n+1}) - \sin \sqrt{n}$. So if we take $\varphi \in R(X)$ to correspond to $(a_n)_2^{\infty}$ as above, $a = 0$, all $c_n = 0$, $c = 0$, and $(b_n), (d_n)$ to be any convergent sequences (say $b_n = 0 = d_n$ for all n), then $\varphi \in \text{Bow}(X, T)$. Clearly $\varphi \notin W(X, T)$ by Theorem 1.1. \square

We could choose $\sum_{n=2}^{\infty} (a_n - a)$ and $\sum_{n=2}^{\infty} (c_n - c)$ to be any series with bounded sequences of partial sums and $(b_n)_1^{\infty}$ and $(d_n)_1^{\infty}$ to be any convergent sequences. Then the corresponding $\varphi \in R(T)$ belongs to $\text{Bow}(X, T) \setminus W(X, T)$ as long as one of the above series is not convergent.

The specific example we gave above was an example of the type studied by Hofbauer [Ho]. These are given by a sequence $(a_n)_0^{\infty}$ with $a_n \rightarrow a$ and we put $b_q = b = a_1$, for all $q \geq 1$, and $c_p = d_q = a_0 = c = d$, for all $p \geq 2, q \geq 1$. Hence $\varphi(0^k 1z) = a_k$ for $k \geq 0, z \in X$ and $\varphi(0^{\infty}) = a$. For these functions $\varphi \in \text{Bow}(X, T)$ if and only if $\sum_{n=0}^{\infty} (a_n - a)$ has a bounded sequence of partial sums and $\varphi \in W(X, T)$ if and only if $\sum_{n=0}^{\infty} (a_n - a)$ converges. (The condition $\varphi \in \text{Bow}(X, T)$ is the same as φ having a homogeneous measure in the sense of [Ho], so the condition above for $\varphi \in \text{Bow}(X, T)$ corrects the theorem of [Ho, p. 230] (see [W4].) For such a function $v_n(\varphi) = \sup_{i,j \geq n} (a_i - a_j), n \geq 2$, and $\sup_{i \geq n} |a_n - a| \leq v_n(\varphi) \leq 2 \sup_{i \geq n} |a_n - a|$ by Lemma 1.2. Note that, for all $f \in C(X), v_n(f) \geq 0$ and $v_n(f) \searrow 0$. Given any sequence $(u_n)_1^{\infty}$ with $u_n \geq 0$ and $u_n \searrow 0$ we can get φ of the above type with $v_n(\varphi) = u_n$ for all $n \geq 1$ by taking $a_n = u_n, n \geq 1$ and $a_0 = 0$.

For functions of this Hofbauer type we have $\sum_{n=1}^{\infty} (v_n(\varphi))^t < \infty$ if and only if $\sum_{n=1}^{\infty} (\sup_{i \geq n} |a_i - a|)^t < \infty$ so we can get for each $t > 0$ a function $\varphi \in W(X, T)$ with $\sum_{n=1}^{\infty} (v_n(\varphi))^t = \infty$ as follows. Let $a_n = (-1)^{n+1}/n^{1/t}, n \geq 1$. Then $a_n \rightarrow 0$, so $a = 0$, and $v_n(\varphi) = \sup_{i \geq n} |a_i| = 1/n^{1/t}$. Hence $\sum_{n=1}^{\infty} (v_n(\varphi))^t = \infty$. We have that $\sum_{n=1}^{\infty} a_n$ is convergent by the Leibnitz alternating series test, so $\varphi \in W(X, T)$. This shows that the classes studied in [JO] do not include all of $W(X, T)$.

The conditions for $\varphi \in R(X)$ to belong to $\text{Bow}(X, T)$ or $W(X, T)$ do not involve $(b_q)_1^{\infty}$ and $(d_q)_1^{\infty}$, whereas $v_n(\varphi)$ does involve these sequences.

2. The g -functions in $R(X)$

A g -function for $T : X \rightarrow X$ is a continuous $g : X \rightarrow (0, 1)$ satisfying $\sum_{y \in T^{-1}x} g(y) = 1$ for all $x \in X$. We can write this condition as $g(0x) + g(1x) = 1$ for all $x \in X$.

Let $G(X, T)$ denote the set of all g -functions for T . If $g \in G(X, T)$ we can define the continuous operator $\mathcal{L} : C(X) \rightarrow C(X)$ by $(\mathcal{L}f)(x) = \sum_{y \in T^{-1}x} g(y)f(y)$. Then $\mathcal{L}1 = 1, \|\mathcal{L}\| = 1$, and $\mathcal{L}U_T f = f$ for all $f \in C(X)$ where $U_T f = f \circ T$. We write $\mathcal{L}_{\log g}$ instead of \mathcal{L} to indicate which g is being used, and this fits in with the notation for the Ruelle operator. We say that $\mu \in M(X)$ is a g -measure if $\mathcal{L}^* \mu = \mu$. Such a measure always belongs to $M(X, T)$, and μ is a g -measure if and only if μ is an equilibrium state for $\log g$ (see [L, W2]). Since $P(T, \log g) = 0$ for $g \in G(X, T)$, this condition becomes $h_{\mu}(T) + \int \log g d\mu = 0$. All g -measures have support X (see [W2]).

We shall see in §3 that $g \in G(X, T) \cap R(X)$ arises naturally from the Ruelle operator theorem applied to certain functions in $R(X)$.

Note that if $g \in G(X, T)$ then $g \in R(X)$ if and only if $\log g \in R(X)$.

We have $g \in G(X, T) \cap R(X)$ if and only if there are sequences $(\gamma_p)_2^\infty \rightarrow \gamma$ and $(\delta_p)_2^\infty \rightarrow \delta$ for which some $c \in (0, 1)$ exists with $c \leq \gamma_p$, $\delta_p \leq 1 - c$ for all $p \geq 2$, and $g(0^p 1z) = \gamma_p$, $g(1^p 0z) = \delta_p$, for all $p \geq 2$, $z \in X$, $g(01^q 0z) = 1 - \delta_{q+1}$, $g(10^q 1z) = 1 - \gamma_{q+1}$ for all $q \geq 1$, $z \in X$, $g(0^\infty) = \gamma$, $g(1^\infty) = \delta$, $g(10^\infty) = 1 - \gamma$, and $g(01^\infty) = 1 - \delta$.

From Theorem 1.1 we have the following result.

THEOREM 2.1. *Let $g \in G(X, T) \cap R(X)$ be given in terms of $(\gamma_p)_2^\infty$ and $(\delta_p)_2^\infty$ as above. Then the following hold:*

- (i) $\log g \in \text{Bow}(X, T)$ if and only if there exists $A > 1$ with $A^{-1} \leq \gamma_2 \cdots \gamma_{1+n} / \gamma^n \leq A$ and $A^{-1} \leq \delta_2 \cdots \delta_{1+n} / \delta^n \leq A$ for all $n \geq 1$;
- (ii) $\log g \in W(X, T)$ if and only if $\sum_{n=2}^\infty \log(\gamma_n / \gamma)$ and $\sum_{n=2}^\infty \log(\delta_n / \delta)$ are both convergent.

We can get examples of $g \in R(X)$ with $\log g \in \text{Bow}(X, T) \setminus W(X, T)$ as follows. Let $\sum_{i=2}^\infty a_i$ be a non-convergent series with $a_i \rightarrow 0$, $|a_i| \leq 1$ for all i , and having a bounded sequence of partial sums. Such an example was given in §1. Choose $\gamma \in (0, e^{-1})$ and put $\gamma_p = \gamma e^{a_p}$, $p \geq 2$. Then $\gamma_p \rightarrow \gamma$, $\gamma e^{-1} \leq \gamma_p \leq \gamma e < 1$, for all $p \geq 2$. Since $\log(\gamma_p / \gamma) = a_p$ the series $\sum_{p=2}^\infty \log(\gamma_p / \gamma)$ is not convergent but has a bounded sequence of partial sums. We could choose a similar example for $(\delta_p)_2^\infty$ or we could put $\delta_p = 1/2$ for all $p \geq 2$ and then $\log g \in \text{Bow}(X, T) \setminus W(X, T)$ by Theorem 2.1.

In the proof of the next theorem we often use the following. If $g \in G(X, T)$, μ is a g -measure and ${}_0[a_0, \dots, a_n]$ is a cylinder set starting at coordinate 0, then

$$\begin{aligned} \mu({}_0[a_0, \dots, a_n]) &= \int \mathcal{X}_{0[a_0, \dots, a_n]} d\mu = \int \mathcal{L}^n \mathcal{X}_{0[a_0, \dots, a_n]} d\mu \\ &= \int g(a_0 \dots a_n x) g(a_1 \dots a_n x) \cdots g(a_n x) d\mu(x). \end{aligned}$$

Note that since $\mu \in M(X, T)$ we have $\mu({}_0[a_0, \dots, a_n]) = \mu({}_k[a_0, \dots, a_n])$ for all $k \geq 0$, so we can write $\mu([a_0, \dots, a_n])$ unambiguously.

We now show that each $g \in G(X, T) \cap R(X)$ has a unique g -measure and we describe this measure.

THEOREM 2.2. *Let $g \in G(X, T) \cap R(X)$ be defined by $(\gamma_p)_2^\infty$ and $(\delta_p)_2^\infty$ as above. There is a unique g -measure μ which is given as follows.*

For $k \geq 2$ let $\Gamma_k = \sum_{i=0}^\infty \gamma_k \cdots \gamma_{k+i}$ and $\Delta_k = \sum_{i=0}^\infty \delta_k \cdots \delta_{k+i}$. Then $\mu([0, 1]) = \mu([1, 0]) = 1 / (\Gamma_2 + \Delta_2 + 2)$, $\mu([0, 0]) = \Gamma_2 / (\Gamma_2 + \Delta_2 + 2)$, and $\mu([1, 1]) = \Delta_2 / (\Gamma_2 + \Delta_2 + 2)$. For $k \geq 3$, $\mu([0^k]) = \gamma_2 \cdots \gamma_{k-1} \Gamma_k / (\Gamma_2 + \Delta_2 + 2)$ and $\mu([1^k]) = \delta_2 \cdots \delta_{k-1} \Delta_k / (\Gamma_2 + \Delta_2 + 2)$. For $r \geq 1$ and $k_i, l_i \geq 1$ for $1 \leq i \leq r$,

$\mu([0^{k_1} 1^{l_1} 0^{k_2} \dots 0^{k_r} 1^{l_r}]) = i_{k_1} d_{l_1} c_{k_2} \dots c_{k_r} f_{l_r} / (\Gamma_2 + \Delta_2 + 2)$ where

$$i_k = \begin{cases} 1 & \text{if } k = 1, \\ \gamma_k \dots \gamma_2 & \text{if } k \geq 2, \end{cases} \quad c_k = \begin{cases} 1 - \gamma_2 & \text{if } k = 1, \\ (1 - \gamma_{k+1}) \gamma_k \dots \gamma_2 & \text{if } k \geq 2, \end{cases}$$

$$d_l = \begin{cases} 1 - \delta_2 & \text{if } l = 1, \\ (1 - \delta_{l+1}) \delta_l \dots \delta_2 & \text{if } l \geq 2, \end{cases} \quad f_l = \begin{cases} 1 & \text{if } l = 1, \\ \delta_l \dots \delta_2 & \text{if } l \geq 2, \end{cases}$$

and $\mu([0^{k_1} 1^{l_1} 0^{k_2} \dots 1^{l_{r-1}} 0^{k_r}]) = i_{k_1} d_{l_1} c_{k_2} \dots d_{l_{r-1}} i_{k_r} / (\Gamma_2 + \Delta_2 + 2)$. The μ -measure of blocks with initial entry 1 are given by the corresponding expressions.

Proof. Since a g -measure has no atoms

$$\mu([0, 1]) = \sum_{i=0}^{\infty} \mu([01^{1+i}0]) = (1 - \delta_2)\mu([10]) + (1 - \delta_3)\delta_2\mu([10]) + \dots = \mu([10]).$$

Also $\mu([00]) = \sum_{i=0}^{\infty} \mu([0^{2+i}1]) = \Gamma_2\mu([01])$ and, similarly, $\mu([11]) = \Delta_2\mu([01])$. Since $\mu([00]) + \mu([01]) + \mu([10]) + \mu([11]) = 1$ we have $\mu([01]) = 1/(\Gamma_2 + \Delta_2 + 2)$ and we get the expressions for $\mu([00])$ and $\mu([11])$.

Now let $k \geq 3$. Then

$$\mu([0^k]) = \sum_{i=0}^{\infty} \mu([0^{k+i}1]) = \sum_{i=0}^{\infty} \gamma_{k+i} \dots \gamma_2 \mu([01]) = \frac{\gamma_2 \dots \gamma_{k-1} \Gamma_k}{\Gamma_2 + \Delta_2 + 2}.$$

We get the corresponding expressions for $\mu([1^k])$.

To prove the expression for $\mu([0^{k_1} 1^{l_1} 0^{k_2} \dots 0^{k_r} 1^{l_r}])$ we use induction on r . Consider the case $r = 1$. We study $\mu([0^k 1^l])$. If $k = 1 = l$ we know that the stated expression is true. Let $k = 1$ and $l \geq 2$. Then

$$\mu([01^l]) = \sum_{i=0}^{\infty} \mu([01^{l+i}0]) = \sum_{i=0}^{\infty} (1 - \delta_{l+i+1}) \delta_{l+i} \dots \delta_2 \mu([10]) = \delta_l \dots \delta_2 \mu([10]).$$

Now let $k \geq 2, l = 1$. Then $\mu([0^k 1]) = \gamma_k \dots \gamma_2 \mu([01])$. Now if $k, l \geq 2$,

$$\begin{aligned} \mu([0^k 1^l]) &= \sum_{i=0}^{\infty} \mu([0^k 1^{l+i}0]) = \gamma_k \dots \gamma_2 \sum_{i=0}^{\infty} (1 - \delta_{l+i+1}) \delta_{l+i} \dots \delta_2 \mu([10]) \\ &= \gamma_k \dots \gamma_2 \delta_l \dots \delta_2 \mu([10]). \end{aligned}$$

Hence the statement holds for $r = 1$.

Now assume that the stated equalities hold for the natural number r and we shall show that they hold for $r + 1$.

Let $k_i, l_i \geq 1$ be given for $1 \leq i \leq r + 1$. If $k_1, l_1 \geq 2$ then

$$\begin{aligned} \mu([0^{k_1} 1^{l_1} 0^{k_2} \dots 0^{k_{r+1}} 1^{l_{r+1}}]) \\ = \gamma_{k_1} \dots \gamma_2 (1 - \delta_{l_1+1}) \delta_{l_1} \dots \delta_2 (1 - \gamma_{k_2+1}) \mu([0^{k_2} 1^{l_2} \dots 0^{k_{r+1}} 1^{l_{r+1}}]) \end{aligned}$$

and the required result follows by the induction assumption.

If $k_1 \geq 2$ and $l_1 = 1$ then

$$\mu([0^{k_1} 1^{l_1} 0^{k_2} \dots 0^{k_{r+1}} 1^{l_{r+1}}]) = \gamma_{k_1} \dots \gamma_2 (1 - \delta_2) (1 - \gamma_{k_2+1}) \mu([0^{k_2} 1^{l_2} \dots 0^{k_{r+1}} 1^{l_{r+1}}])$$

and the required result follows by the induction assumption.

If $k_1 = 1$ and $l_1 \geq 2$ then

$$\mu([0^{k_1} 1^{l_1} 0^{k_2} \dots 0^{k_{r+1}} 1^{l_{r+1}}]) = (1 - \delta_{l_1+1})\delta_{l_1} \cdots \delta_2(1 - \gamma_{k_2+1})\mu([0^{k_2} 1^{l_2} \dots 0^{k_{r+1}} 1^{l_{r+1}}])$$

and the required result follows by the induction assumption.

If $k_1 = 1 = l_1$ then

$$\mu([010^{k_2} \dots 0^{k_{r+1}} 1^{l_{r+1}}]) = (1 - \delta_2)(1 - \gamma_{k_2+1})\mu([0^{k_2} 1^{l_2} \dots 0^{k_{r+1}} 1^{l_{r+1}}])$$

and the required result follows by the induction assumption.

The formula for $\mu([0^{k_1} 1^{l_1} 0^{k_2} \dots 1^{l_{r-1}} 0^{k_r}])$ can be proved by induction in a similar way. □

COROLLARY 2.3. For $g \in G(X, T) \cap R(X)$ the unique g -measure μ is reversible, i.e.

$$\mu([a_0, a_1, \dots, a_{n-1}]) = \mu([a_{n-1}, a_{n-2}, \dots, a_0])$$

for all $a_0, a_1, \dots, a_{n-1} \in \{0, 1\}$, $n \geq 1$.

We can state this in terms of the natural extension $\hat{\mu}$ of μ to the two-sided shift space $\hat{X} = \prod_{-\infty}^{\infty} \{0, 1\}$. The measure $\hat{\mu}$ is determined by requiring that $\hat{\mu}(l[a_0, a_1, \dots, a_n]) = \mu([a_0, a_1, \dots, a_n])$ for all $l \in \mathbb{Z}$, $n \geq 0$, $a_0, a_1, \dots, a_n \in \{0, 1\}$. Here

$$(l[a_0, a_1, \dots, a_n]) = \{(x_i)_{-\infty}^{\infty} \in \hat{X} \mid x_{k+l} = a_k \ 0 \leq k \leq n\}.$$

If $\Phi : \hat{X} \rightarrow \hat{X}$ is the reversal map, defined by

$$\Phi(\dots, x_{-2}, x_{-1}, x_0^*, x_1, x_2, \dots) = (\dots, x_2, x_1, x_0^*, x_{-1}, x_{-2}, \dots)$$

then Corollary 2.3 means that $\hat{\mu} \circ \Phi = \hat{\mu}$. Here $*$ indicates the entry in the 0th position.

We now show that if $g \in G(X, T) \cap R(X)$ then, for all $f \in C(X)$, $\mathcal{L}_{\log g}^n f \xrightarrow{\text{unif}} \int f d\mu$, where μ is the unique g -measure. This has been proved in the cases when $\delta_p = \delta$ for all $p \geq 2$ by Hulse [Hu]. Here the symbol $\xrightarrow{\text{unif}}$ denotes that the convergence is uniform on X .

THEOREM 2.4. Let $g \in G(X, T) \cap R(X)$. For every $f \in C(X)$ there exists $c(f) \in \mathbb{R}$ with $\mathcal{L}_{\log g}^n f \xrightarrow{\text{unif}} c(f)$. In fact, $c(f) = \int f d\mu$ where μ is the unique g -measure.

Proof. We write \mathcal{L} instead of $\mathcal{L}_{\log g}$. Let g be defined using the sequences $(\gamma_n)_2^\infty$ and $(\delta_n)_2^\infty$. Since linear combinations of characteristic functions of cylinders based at coordinate zero, $\mathcal{X}_{0[w_0, w_1, \dots, w_{k-1}]}$, are dense in $C(X)$, it suffices to consider $f = \mathcal{X}_{0[w_0, w_1, \dots, w_{k-1}]}$, where $w = (w_0, w_1, \dots) \in X$.

Fix $w \in X$ and $k \geq 1$ and let $f = \mathcal{X}_{0[w_0, w_1, \dots, w_{k-1}]}$. For $n \geq 1$

$$\begin{aligned} (\mathcal{L}^{n+k} f)(x) &= \sum_{z \in T^{-(n+k)}x} g(z)g(Tz) \cdots g(T^{n+k-1}z)f(z) \\ &= \sum_{y_0, \dots, y_{n-1}} [g(y_0 \cdots y_{n-1}x) \cdots g(y_{n-1}x) \\ &\quad \times g(w_0 \cdots w_{k-1}y_0 \cdots y_{n-1}x) \cdots g(w_{k-1}y_0 \cdots y_{n-1}x)]. \end{aligned}$$

We first show that it suffices to consider only the two cases $w_0 = w_1 = \cdots = w_{k-1}$.

Assume that $w_{k-1} = 0$. If $w_0 = w_1 = \dots = w_{k-1} = 0$ then we need not consider further. So let $w_i = 1$ for some $i < k - 1$, and choose $i < k - 1$ so that $w_i = 1$ and $w_{i+1} = 0 = w_{i+2} = \dots = w_{k-1}$. Hence

$$[w_0, w_1, \dots, w_{k-1}] = [w_0, w_1, \dots, w_{i-1} 10^{k-i-1}].$$

If $0 \leq j < i$ then, by the definition of g , $g(w_j \dots w_i \dots w_{k-1} y_0 \dots y_{n-1} x)$ does not depend on $(y_0 \dots y_{n-1} x)$. Hence $\prod_{j=0}^{i-1} g(w_j \dots w_{k-1} y_0 \dots y_{n-1} x) = C$, a constant. Then,

$$(\mathcal{L}^{n+k} f)(x) = C \left[\sum_{y_0, \dots, y_{n-1}} g(y_0 \dots y_{n-1} x) \cdots g(y_{n-1} x) \times g(10^{k-i-1} y_0 \dots y_{n-1} x) \cdots g(0 y_0 \dots y_{n-1} x) \right].$$

But $g(10^{k-i-1} y_0 \dots y_{n-1} x) = 1 - g(0^{k-i} y_0 \dots y_{n-1} x)$ so

$$(\mathcal{L}^{n+k} f)(x) = C[(\mathcal{L}^{n+k-i-1} \mathcal{X}_{[0^{k-i-1}]})(x) - (\mathcal{L}^{n+k-i} \mathcal{X}_{[0^{k-i}]})(x)].$$

So when $w_{k-1} = 0$ it suffices to consider $(\mathcal{L}^{n+k-i-1} \mathcal{X}_{[0^{k-i-1}]})(x)$ and $(\mathcal{L}^{n+k-i} \mathcal{X}_{[0^{k-i}]})(x)$.

Now assume that $w_{k-1} = 1$. The corresponding argument shows that the convergence of $(\mathcal{L}^{n+k} f)(x)$ depends on that of $(\mathcal{L}^{n+k-i-1} \mathcal{X}_{[1^{k-i-1}]})(x)$ and $(\mathcal{L}^{n+k-i} \mathcal{X}_{[1^{k-i}]})(x)$.

So we only need to consider the cases when $f = \mathcal{X}_{[0^{1^k}]}$ and $f = \mathcal{X}_{[1^{1^k}]}$.

So now assume that $f = \mathcal{X}_{[0^{1^k}]}$. The case when $f = \mathcal{X}_{[1^{1^k}]}$ follows by symmetry.

Let $l \geq 1$, and we now show that $\mathcal{L}^n f$ is constant on ${}_0[0^l 1]$. Let $x \in {}_0[0^l 1]$. Then

$$(\mathcal{L}^n f)(x) = \sum_{y_0, \dots, y_{n-1}} g(y_0 \dots y_{n-1} x) \cdots g(y_{n-1} x) f(y_0 \dots y_{n-1} x).$$

If $n + l < k$ then $(y_0 \dots y_{n-1} x) \notin [0^k]$ so $(\mathcal{L}^n f)(x) = 0$.

If $k \leq n$ then $f(y_0 \dots y_{n-1} x) = 1$ if and only if $y_0 = 0 = \dots = y_{k-1}$ and then $(\mathcal{L}^n f)(x) = \sum_{y_k, \dots, y_{n-1}} g(0^k y_k \dots y_{n-1} x) \cdots g(y_{n-1} x)$ which is constant on ${}_0[0^l 1]$.

If $n < k \leq n + l$ then $f(y_0 \dots y_{n-1} x) = 1$ if and only if $y_0 = 0 = \dots = y_{n-1}$ and then $(\mathcal{L}^n f)(x) = \gamma_{n+l} \dots \gamma_{1+l}$.

Hence $(\mathcal{L}^n f)$ is constant on ${}_0[0^l 1]$ and we denote this value by $(\mathcal{L}^n f)([0^l 1])$.

Again let $l \geq 1$ and we now show that $\mathcal{L}^n f$ is constant on ${}_0[1^l 0]$:

$$(\mathcal{L}^n f)(x) = \sum_{y_0, \dots, y_{n-1}} g(y_0 \dots y_{n-1} x) g(y_1 \dots y_{n-1} x) \cdots g(y_{n-1} x) f(y_0 \dots y_{n-1} x).$$

If $n < k$ then $f(y_0 \dots y_{n-1} x) = 0$ so $(\mathcal{L}^n f)(x) = 0$.

If $n = k$ then $f(y_0 \dots y_{n-1} x) = 1$ if and only if $y_0 = 0 = \dots = y_{n-1}$ so $(\mathcal{L}^n f)(x) = \gamma_k \dots \gamma_2 (1 - \delta_{l+1})$.

If $k < n$ then $f(y_0 \dots y_{n-1} x) = 1$ if and only if $y_0 = 0 = \dots = y_{k-1}$ so $(\mathcal{L}^n f)(x) = \sum_{y_k, \dots, y_{n-1}} g(0^k y_k \dots y_{n-1} x) \cdots g(y_{n-1} x)$ which is constant on ${}_0[1^l 0]$.

Hence $(\mathcal{L}^n f)$ is constant on ${}_0[1^l 0]$ and we denote this value by $(\mathcal{L}^n f)([1^l 0])$.

We now show that if $x_0 = 0$ then for all $n \geq 1$

$$\begin{aligned} (\mathcal{L}^{n+k} f)(x) &= \left(\prod_{i=1}^n g(0^i x) \right) [(\mathcal{L}^n f)(0^n x) - (\mathcal{L}^n f)([10])] + (\mathcal{L}^{n+k-1} f)([10]) \\ &\quad + \sum_{i=1}^{n-1} \left(\prod_{j=1}^{n-i} g(0^j x) \right) [(\mathcal{L}^{k+i-1} f)([10]) - (\mathcal{L}^{k+i} f)([10])], \end{aligned} \quad (1)$$

where the final term is absent if $n = 1$.

We use induction on n . When $n = 1$ the right side of (1) becomes

$$\begin{aligned} g(0x)[(\mathcal{L}^k f)(0x) - (\mathcal{L}^k f)([10])] + (\mathcal{L}^k f)([10]) \\ = g(0x)(\mathcal{L}^k f)(0x) + g(1x)(\mathcal{L}^k f)(1x), \end{aligned}$$

which equals $(\mathcal{L}^{1+k} f)(x)$. Hence (1) holds for $n = 1$.

Assume that (1) holds for $n - 1$ and we shall prove it for n . Let $x_0 = 0$. Then

$$\begin{aligned} (\mathcal{L}^{n+k} f)(x) &= g(0x)(\mathcal{L}^{n+k-1} f)(0x) + g(1x)(\mathcal{L}^{n+k-1} f)(1x) \\ &= g(0x)[(\mathcal{L}^{n+k-1} f)(0x) - (\mathcal{L}^{n+k-1} f)([10])] + (\mathcal{L}^{n+k-1} f)([10]) \\ &= g(0x) \left[\left(\prod_{i=1}^{n-1} g(0^{i+1} x) \right) \{(\mathcal{L}^k f)(0^n x) - (\mathcal{L}^k f)([10])\} + (\mathcal{L}^{n+k-2} f)([10]) \right. \\ &\quad \left. + \sum_{i=1}^{n-2} \left(\prod_{j=1}^{n-1-i} g(0^{j+1} x) \right) \{(\mathcal{L}^{k+i-1} f)([10]) - (\mathcal{L}^{k+i} f)([10])\} \right. \\ &\quad \left. - (\mathcal{L}^{n+k-1} f)([10]) \right] + (\mathcal{L}^{n+k-1} f)([10]) \end{aligned}$$

using the induction assumption. Hence

$$\begin{aligned} (\mathcal{L}^{n+k} f)(x) &= \left(\prod_{i=1}^n g(0^i x) \right) [(\mathcal{L}^k f)(0^n x) - (\mathcal{L}^k f)([10])] + (\mathcal{L}^{n+k-1} f)([10]) \\ &\quad + \sum_{i=1}^{n-1} \left(\prod_{j=1}^{n-i} g(0^j x) \right) [(\mathcal{L}^{k+i-1} f)([10]) - (\mathcal{L}^{k+i} f)([10])]. \end{aligned}$$

Hence (1) holds for all $n \geq 1$ and all $x \in {}_0[0]$.

We next show that if $x_0 = 1$ then for all $n \geq 1$

$$\begin{aligned} (\mathcal{L}^{n+k} f)(x) &= \left(\prod_{i=1}^n g(1^i x) \right) [(\mathcal{L}^k f)(1^n x) - (\mathcal{L}^k f)([01])] + (\mathcal{L}^{n+k-1} f)([01]) \\ &\quad + \sum_{i=1}^{n-1} \left(\prod_{j=1}^{n-i} g(1^j x) \right) [(\mathcal{L}^{k+i-1} f)([01]) - (\mathcal{L}^{k+i} f)([01])], \end{aligned} \quad (2)$$

where the last term is absent if $n = 1$.

We use induction on n . When $n = 1$ the right side of (2) becomes

$$\begin{aligned} g(1x)[(\mathcal{L}^k f)(1x) - (\mathcal{L}^k f)([01])] + (\mathcal{L}^k f)([01]) \\ = g(1x)(\mathcal{L}^k f)(1x) + g(0x)(\mathcal{L}^k f)(0x), \end{aligned}$$

which equals $(\mathcal{L}^{1+k} f)(x)$ and so (2) holds for $n = 1$.

Assume that (2) holds for $n - 1$ and we shall prove it for n . Let $x_0 = 1$. Then

$$\begin{aligned} (\mathcal{L}^{n+k} f)(x) &= g(1x)(\mathcal{L}^{n+k-1} f)(1x) + (1 - g(1x))(\mathcal{L}^{n+k-1} f)(0x) \\ &= g(1x)[(\mathcal{L}^{n+k-1} f)(1x) - (\mathcal{L}^{n+k-1} f)([01])] + (\mathcal{L}^{n+k-1} f)([01]) \\ &= g(1x) \left[\left(\prod_{i=1}^{n-1} g(1^{i+1}x) \right) \{(\mathcal{L}^k f)(1^n x) - (\mathcal{L}^k f)([01])\} + (\mathcal{L}^{n+k-2} f)([01]) \right. \\ &\quad \left. + \sum_{i=1}^{n-2} \left(\prod_{j=1}^{n-1-i} g(1^{j+1}x) \right) \{(\mathcal{L}^{k+i-1} f)([01]) - (\mathcal{L}^{k+i} f)([01])\} \right. \\ &\quad \left. - (\mathcal{L}^{n+k-1} f)([01]) \right] + (\mathcal{L}^{n+k-1} f)([01]) \end{aligned}$$

using the induction assumption. Hence

$$\begin{aligned} (\mathcal{L}^{n+k} f)(x) &= \left(\prod_{i=1}^n g(1^i x) \right) [(\mathcal{L}^k f)(1^n x) - (\mathcal{L}^k f)([01])] + (\mathcal{L}^{n+k-1} f)([01]) \\ &\quad + \sum_{i=1}^{n-1} \left(\prod_{j=1}^{n-i} g(1^j x) \right) [(\mathcal{L}^{k+i-1} f)([01]) - (\mathcal{L}^{k+i} f)([01])]. \end{aligned}$$

Hence (2) holds for all $n \geq 1$ and all $x \in {}_0[1]$.

We use (1) to show that if $(\mathcal{L}^n f)([10]) \rightarrow c(f)$ then $(\mathcal{L}^n f)(x) \rightarrow c(f)$ uniformly for $x \in {}_0[0]$. Assume that $(\mathcal{L}^n f)([10]) \rightarrow c(f)$.

By (1) we have

$$\begin{aligned} &(\mathcal{L}^{n+k} f)(x) - (\mathcal{L}^{n+k-1} f)([10]) \\ &= \left(\prod_{i=1}^n g(0^i x) \right) [(\mathcal{L}^k f)(0^n x) - (\mathcal{L}^k f)([10])] \\ &\quad + \sum_{i=1}^{n-1} \left(\prod_{j=1}^{n-i} g(0^j x) \right) [(\mathcal{L}^{k+i-1} f)([10]) - (\mathcal{L}^{k+i} f)([10])]. \end{aligned}$$

Note that $\left| \left(\prod_{j=1}^n g(0^j x) \right) [(\mathcal{L}^k f)(0^n x) - (\mathcal{L}^k f)([10])] \right| \leq 2(\sup g)^n \rightarrow 0$ as $n \rightarrow \infty$.

Given $\epsilon > 0$ choose N so that $\sum_{i=N}^{\infty} (\sup g)^i < \epsilon$ and so that $n \geq N$ implies $|(\mathcal{L}^{n+k-1} f)([10]) - (\mathcal{L}^{n+k} f)([10])| < \epsilon$.

For all $n \geq 2N$

$$\begin{aligned} &\left| \sum_{i=1}^{n-1} \left(\prod_{j=1}^{n-i} g(0^j x) \right) [(\mathcal{L}^{k+i-1} f)([10]) - (\mathcal{L}^{k+i} f)([10])] \right| \\ &\leq 2 \sum_{i=1}^N (\sup g)^{n-i} + \epsilon \sum_{i=N+1}^{n-1} (\sup g)^{n-i} \end{aligned}$$

$$\begin{aligned} &\leq 2 \sum_{q=N}^{\infty} (\sup g)^q + \epsilon \sum_{p=1}^{\infty} (\sup g)^p \\ &< \epsilon \left(2 + \sum_{p=1}^{\infty} (\sup g)^p \right). \end{aligned}$$

Therefore $|(\mathcal{L}^{n+k} f)(x) - (\mathcal{L}^{n+k-1} f)([10])| \rightarrow 0$ as $n \rightarrow \infty$, uniformly on ${}_0[0]$.

Similarly (2) implies that if $(\mathcal{L}^{n+k-1} f)([01])$ converges then $(\mathcal{L}^{n+k} f)(x)$ converges to the same limit uniformly for $x \in {}_0[1]$.

So consider $(\mathcal{L}^{n+k} f)([10])$.

By (2) we have

$$\begin{aligned} &(\mathcal{L}^{n+k} f)([10]) \\ &= \left(\prod_{i=1}^n g(1^{i+1}0) \right) [(\mathcal{L}^k f)([1^{n+1}0]) - (\mathcal{L}^k f)([01])] + (\mathcal{L}^{n+k-1} f)([01]) \\ &\quad + \sum_{i=1}^{n-1} \left(\prod_{j=1}^{n-i} g(1^{1+j}0) \right) [(\mathcal{L}^{k+i-1} f)([01]) - (\mathcal{L}^{k+i} f)([01])] \\ &= \left(\prod_{i=1}^n \gamma_{i+1} \right) [(\mathcal{L}^k f)([1^{n+1}0]) - (\mathcal{L}^k f)([01])] + (\mathcal{L}^{n+k-1} f)([01]) \\ &\quad + \sum_{i=1}^{n-1} \left(\prod_{j=1}^{n-i} \gamma_{j+1} \right) [(\mathcal{L}^{k+i-1} f)([01]) - (\mathcal{L}^{k+i} f)([01])] \\ &= \left(\prod_{j=2}^{n+1} \gamma_j \right) (\mathcal{L}^k f)([1^{n+1}0]) + \sum_{i=0}^{n-2} (\mathcal{L}^{k+i} f)([01]) \left(\prod_{j=2}^{n-i} \gamma_j \right) (1 - \gamma_{n+1-i}) \\ &\quad + (\mathcal{L}^{k+n-1} f)([01]) (1 - \gamma_2). \end{aligned}$$

Similarly, using (1) we have

$$\begin{aligned} (\mathcal{L}^{n+k} f)([01]) &= \left(\prod_{j=2}^{n+1} \delta_j \right) (\mathcal{L}^k f)([0^{n+1}1]) + \sum_{i=0}^{n-2} (\mathcal{L}^{k+i} f)([10]) \left(\prod_{j=2}^{n-i} \delta_j \right) (1 - \delta_{n+1-i}) \\ &\quad + (\mathcal{L}^{k+n-1} f)([10]) (1 - \delta_2). \end{aligned}$$

For $n \geq 0$ put $u_n = (\mathcal{L}^{n+k} f)([01])$ and $v_n = (\mathcal{L}^{n+k} f)([10])$. Then

$$v_n = \beta_n + \alpha_1 u_{n-1} + \alpha_2 u_{n-2} + \dots + \alpha_n u_0 \quad \text{for } n \geq 1,$$

where $\beta_n = \left(\prod_{j=2}^{n+1} \gamma_j \right) (\mathcal{L}^k f)([1^{n+1}0]) > 0$ for $n \geq 1$, $\alpha_1 = 1 - \gamma_2 > 0$ and for $n \geq 2$, $\alpha_n = \left(\prod_{j=2}^n \gamma_j \right) (1 - \gamma_{n+1})$.

Note that $\sum_{n=1}^{\infty} \alpha_n = 1$ and $0 < \beta_n \leq (\sup_j \gamma_j)^{n-1}$ so $\sum \beta_n < \infty$.

If we let $\alpha'_n = 1 - \delta_2$, $\alpha'_n = \left(\prod_{j=2}^n \delta_j \right) (1 - \delta_{n+1})$ for $n \geq 2$, and $\beta'_n = \left(\prod_{j=2}^{n+1} \delta_j \right) (\mathcal{L}^k f)([0^{n+1}1]) > 0$ then

$$u_n = \beta'_n + \alpha'_1 v_{n-1} + \dots + \alpha'_n v_0 \quad \text{for } n \geq 1.$$

If we put $\beta_0 = v_0, \alpha_0 = 0$ and if we let $A(s) = \sum_{n=0}^{\infty} \alpha_n s^n, B(s) = \sum_{n=0}^{\infty} \beta_n s^n, U(s) = \sum_{n=0}^{\infty} u_n s^n, V(s) = \sum_{n=0}^{\infty} v_n s^n$ then we have $V(s) = B(s) + A(s)U(s)$. Note that $A(1) = \sum_{n=0}^{\infty} \alpha_n = 1$ and $B(1) = \sum_{n=0}^{\infty} \beta_n < \infty$.

Similarly $U(s) = B'(s) + A'(s)V(s)$ where $\beta'_0 = u_0, \alpha'_0 = 0, A'(s) = \sum_{n=0}^{\infty} \alpha'_n s^n$ and $B'(s) = \sum_{n=0}^{\infty} \beta'_n s^n$.

Then we have

$$\begin{aligned} U(s) &= B'(s) + A'(s)[B(s) + A(s)U(s)] \\ &= (B'(s) + A'(s)B(s)) + A'(s)A(s)U(s). \end{aligned}$$

This gives a renewal equation for (u_n) of the form

$$u_n = b_n + a_0 u_n + a_1 u_{n-1} + \dots + a_n u_0 \quad \text{for } n \geq 0,$$

where b_n is the coefficient of s^n in $B'(s) + A'(s)B(s)$ and a_n is the coefficient of s^n in $A'(s)A(s)$. Hence $\sum_{n=0}^{\infty} b_n = B'(1) + A'(1)B(1) = B'(1) < \infty$ and $\sum_{n=0}^{\infty} a_n = A'(1)A(1) = 1$ so by the renewal theorem [Fe, p. 291] we have $u_n \rightarrow \sum_{i=0}^{\infty} b_i / \sum_{i=0}^{\infty} i a_i$.

Similarly

$$V(s) = (B(s) + A(s)B'(s)) + A(s)A'(s)V(s)$$

so

$$v_n = b'_n + a_0 v_n + a_1 v_{n-1} + \dots + a_n v_0 \quad \text{for } n \geq 0,$$

where b'_n is the coefficient of s^n in $B(s) + A(s)B'(s)$. Hence

$$\sum_{i=0}^{\infty} b'_i = B(1) + A(1)B'(1) = B(1) + B'(1) = \sum_{i=0}^{\infty} b_i$$

and the renewal theorem gives $v_n \rightarrow \sum_{i=0}^{\infty} b_i / \sum_{i=0}^{\infty} i a_i$.

Hence $(\mathcal{L}^{n+k} f)([01])$ and $(\mathcal{L}^{n+k} f)([10])$ converge to the same limit, $c(f)$, so $(\mathcal{L}^{n+k} f)(x)$ converges uniformly to $c(f)$. Therefore $(\mathcal{L}^n f)(x)$ converges uniformly to $c(f)$.

If μ is a g -measure then integrating $\mathcal{L}^n f \rightrightarrows c(f)$ with respect to μ gives $c(f) = \int f d\mu$ for all $f \in C(X)$. This gives another way of showing that there is a unique g -measure. \square

The convergence $\mathcal{L}^n f \rightrightarrows \int f d\mu$ gives several properties of μ . One is that T is an exact endomorphism with respect to μ (i.e. all sets in the σ -algebra $\bigcap_{n=0}^{\infty} T^{-n} \mathcal{B}(X)$ have μ -measure 0 or 1, where $\mathcal{B}(X)$ is the σ -algebra of Borel subsets of X) [W3].

One can obtain examples of g -functions with $\mathcal{L}^n f$ converging uniformly to a constant but $\log g \notin \text{Bow}(X, T)$ as follows. Let $\gamma, \delta \in (0, 1)$ and for $p \geq 2$ put $\gamma_p = p\gamma/(p+1), \delta_p = \delta$. The corresponding g is in $R(X)$ so we get the convergence by Theorem 2.4. However $\log g \notin \text{Bow}(X, T)$ by Theorem 2.1 since $\gamma_2 \dots \gamma_{1+n} / \gamma^n = 2/(n+2)$.

3. Ruelle operator theorem for functions in $R(X)$

In this section we investigate exactly when $\varphi \in R(X)$ satisfies the Ruelle operator theorem for $T : X \rightarrow X$.

For $\varphi \in C(X)$ the Ruelle operator $\mathcal{L}_\varphi : C(X) \rightarrow C(X)$ is defined by

$$(\mathcal{L}_\varphi f)(x) = \sum_{y \in T^{-1}x} e^{\varphi(y)} f(y) = e^{\varphi(0x)} f(0x) + e^{\varphi(1x)} f(1x).$$

To say the Ruelle operator theorem holds for φ means that there exist $\lambda \in \mathbb{R}, \lambda > 0, h \in C(X), h > 0, v \in M(X)$ with $\mathcal{L}_\varphi h = \lambda h$ and $\mathcal{L}_\varphi^* v = \lambda v$, and if we normalize h so that $v(h) = 1$ then for all $f \in C(X)$,

$$\frac{\mathcal{L}_\varphi^n f}{\lambda^n} \rightrightarrows v(f)h.$$

We shall give necessary and sufficient conditions for $\varphi \in R(X)$ to satisfy the Ruelle operator theorem. This turns out to be equivalent to the existence of a positive eigenfunction h . When these conditions hold then

$$g = \frac{e^\varphi h}{\lambda h \circ T} \in G(X, T) \cap R(X),$$

and since

$$\varphi - \log g = \log \lambda + \log h \circ T - \log h$$

the unique equilibrium state for φ is the unique g -measure for g . Also λ is given as the solution to an equation.

THEOREM 3.1. *Let $\varphi \in R(X)$ be defined by the sequences $(a_p)_2^\infty \rightarrow a, (b_q)_1^\infty \rightarrow b, (c_p)_2^\infty \rightarrow c$ and $(d_q)_1^\infty \rightarrow d$ as in §1. The following statements are pairwise equivalent.*

- (i) *There exists $h \in C(X), h > 0$, and a real number $\lambda > 0$ with $\mathcal{L}_\varphi h = \lambda h$.*
- (ii) *We have*

$$\frac{1}{e^{2 \max(a,c)}} \left[e^{d_1} + \sum_{j=1}^\infty e^{d_{1+j}} \frac{e^{a_2+\dots+a_{1+j}}}{e^{j \max(a,c)}} \right] \left[e^{b_1} + \sum_{j=1}^\infty e^{b_{1+j}} \frac{e^{c_2+\dots+c_{1+j}}}{e^{j \max(a,c)}} \right] > 1,$$

where the left side could be ∞ .

- (iii) *There exists $h \in C(X), h > 0$, and a real number $\lambda > 0$ with $\mathcal{L}_\varphi h = \lambda h$ and h has the following form: there exist sequences $(\alpha_q)_1^\infty$ and $(\beta_q)_1^\infty$ with $\alpha_q \rightarrow \alpha, \beta_q \rightarrow \beta, h(0^q 1z) = \alpha_q, q \geq 1, h(1^q 0w) = \beta_q, q \geq 1, h(0^\infty) = \alpha$ and $h(1^\infty) = \beta$.*
- (iv) *There exists $h \in C(X), h > 0, \lambda > 0$ with $\mathcal{L}_\varphi h = \lambda h$ and there exists $v \in M(X)$ with $\mathcal{L}_\varphi^* v = \lambda v$ and, for all $f \in C(X), (\mathcal{L}_\varphi^n f)(x)/\lambda^n \rightrightarrows h(x)v(f)$ as $n \rightarrow \infty$.*

When φ satisfies the statements above and h is given in (iii) then $g = e^\varphi h/\lambda h \circ T$ is a g -function for T and $g \in R(X)$. Hence φ has a unique equilibrium state which is the unique g -measure.

Note that (iv) says that the Ruelle operator theorem holds for φ .

We shall use the following lemmas in the proof of Theorem 3.1. We use the notation from Theorem 3.1.

LEMMA 3.2. *The power series $\sum_{j=1}^\infty e^{d_{1+j}} e^{a_2+\dots+a_{1+j}} x^j$ has radius of convergence e^{-a} .*

Proof. We have $\sqrt[n]{e^{d_{1+n}} e^{a_2+\dots+a_{1+n}}} \rightarrow e^a$ since $d_{1+n}/n \rightarrow 0$ and $(a_2 + \dots + a_{1+n})/n \rightarrow a$. □

LEMMA 3.3. Let $\varphi \in R(X)$. We can find $\rho > \max(e^a, e^c)$ with

$$\rho^{-2} \left[e^{d_1} + \sum_{j=1}^{\infty} e^{d_{1+j}} \frac{e^{a_2+\dots+a_{1+j}}}{\rho^j} \right] \left[e^{b_1} + \sum_{j=1}^{\infty} e^{b_{1+j}} \frac{e^{c_2+\dots+c_{1+j}}}{\rho^j} \right] < 1.$$

Proof. Let

$$F(\rho) = \left[e^{d_1} + \sum_{j=1}^{\infty} e^{d_{1+j}} \frac{e^{a_2+\dots+a_{1+j}}}{\rho^j} \right] \left[e^{b_1} + \sum_{j=1}^{\infty} e^{b_{1+j}} \frac{e^{c_2+\dots+c_{1+j}}}{\rho^j} \right].$$

By Lemma 3.2 if $\rho_0 > \max(e^a, e^c)$ then $F(\rho) < \infty$. But $\rho > \rho_0$ implies that $F(\rho) < F(\rho_0)$ so $\rho^{-2}F(\rho) < \rho^{-2}F(\rho_0) < 1$ for large enough ρ . \square

LEMMA 3.4. Statement (ii) in Theorem 3.1 is equivalent to the existence of $\lambda > \max(e^a, e^c)$ with

$$\frac{1}{\lambda^2} \left[e^{d_1} + \sum_{j=1}^{\infty} e^{d_{1+j}} \frac{e^{a_2+\dots+a_{1+j}}}{\lambda^j} \right] \left[e^{b_1} + \sum_{j=1}^{\infty} e^{b_{1+j}} \frac{e^{c_2+\dots+c_{1+j}}}{\lambda^j} \right] = 1.$$

Proof. Let $G(\rho) = \rho^{-2}F(\rho)$, where F is defined in the proof of Lemma 3.3. By Lemma 3.3 there is $\rho_0 > \max(e^a, e^c)$ with $G(\rho_0) < 1$.

If statement (ii) holds then $G(\max(e^a, e^c)) > 1$. If $G(\max(e^a, e^c)) < \infty$ then on the interval $[\max(e^a, e^c), \rho_0]$ G is continuous and, by the intermediate value theorem, there is some $\lambda \in (\max(e^a, e^c), \rho_0)$ with $G(\lambda) = 1$.

Suppose $G(\max(e^a, e^c)) = \infty$. By Lemma 3.2, $G(\rho) < \infty$ for all $\rho > \max(e^a, e^c)$. If $G(\rho) \leq 1$ for all $\rho > \max(e^a, e^c)$ then, for all $J \geq 1$,

$$\rho^{-2} \left[e^{d_1} + \sum_{j=1}^J e^{d_{1+j}} \frac{e^{a_2+\dots+a_{1+j}}}{\rho^j} \right] \left[e^{b_1} + \sum_{j=1}^J e^{b_{1+j}} \frac{e^{c_2+\dots+c_{1+j}}}{\rho^j} \right] \leq 1$$

for all $\rho > \max(e^a, e^c)$. Then

$$e^{-2 \max(a,c)} \left[e^{d_1} + \sum_{j=1}^J e^{d_{1+j}} \frac{e^{a_2+\dots+a_{1+j}}}{e^{j \max(a,c)}} \right] \left[e^{b_1} + \sum_{j=1}^J e^{b_{1+j}} \frac{e^{c_2+\dots+c_{1+j}}}{e^{j \max(a,c)}} \right] \leq 1$$

for all $J \geq 1$ so $G(\max(e^a, e^c)) \leq 1$, a contradiction. So we can choose $\rho_1 \in (\max(e^a, e^c), \rho_0)$ with $1 < G(\rho_1) < \infty$ and the intermediate value theorem, applied to G restricted to $[\rho_1, \rho_0]$, gives some $\lambda \in (\rho_1, \rho_0)$ with $G(\lambda) = 1$.

If there exists $\lambda > \max(e^a, e^c)$ with $G(\lambda) = 1$ then $G(\max(e^a, e^c)) > G(\lambda) = 1$ so statement (ii) of Theorem 3.1 holds. \square

We now turn to the proof of the theorem.

Proof of Theorem 3.1. (i) \Rightarrow (ii) Let $h \in C(X)$, $h > 0$, and let $\lambda > 0$ satisfy $\mathcal{L}_\varphi h = \lambda h$. We shall show that

$$1 \leq \frac{1}{\lambda^2} \left[e^{d_1} + \sum_{j=1}^{\infty} e^{d_{1+j}} \frac{e^{a_2+\dots+a_{1+j}}}{\lambda^j} \right] \left[e^{b_1} + \sum_{j=1}^{\infty} e^{b_{1+j}} \frac{e^{c_2+\dots+c_{1+j}}}{\lambda^j} \right]$$

and $\lambda > \max(e^a, e^c)$.

We have $e^{\varphi(0x)}h(0x) + e^{\varphi(1x)}h(1x) = \lambda h(x)$. Put $x = (0^{q+j}1z)$, $q \geq 1, j \geq 0, z \in X$ to get

$$e^{a_{q+j+1}}h(0^{q+j+1}1z) + e^{d_{q+j}}h(10^{q+j}1z) = \lambda h(0^{q+j}1z).$$

Multiply this equation by $e^{a_{q+1}+\dots+a_{q+j}}/\lambda^j$ if $j \geq 1$, and by 1 if $j = 0$, and sum over j from 0 to n to get

$$\frac{e^{a_{q+1}+\dots+a_{q+n+1}}}{\lambda^n}h(0^{q+n+1}1z) + e^{d_q}h(10^q1z) + \sum_{j=1}^n e^{d_{q+j}} \frac{e^{a_{q+1}+\dots+a_{q+j}}}{\lambda^j}h(10^{q+j}1z) = \lambda h(0^q1z).$$

The right side of this equation is independent of n and both terms on the left side are non-negative. Therefore

$$\sum_{j=1}^{\infty} e^{d_{q+j}} \frac{e^{a_{q+1}+\dots+a_{q+j}}}{\lambda^j}h(10^{q+j}1z) < \infty$$

and since $\inf h > 0$ we have

$$\sum_{j=1}^{\infty} e^{d_{q+j}} \frac{e^{a_{q+1}+\dots+a_{q+j}}}{\lambda^j} < \infty.$$

Hence $e^{a_{q+1}+\dots+a_{q+j}}/\lambda^j \rightarrow 0$ as $j \rightarrow \infty$. Therefore

$$e^{d_q}h(10^q1z) + \sum_{j=1}^{\infty} e^{d_{q+j}} \frac{e^{a_{q+1}+\dots+a_{q+j}}}{\lambda^j}h(10^{q+j}1z) = \lambda h(0^q1z), \tag{3}$$

$q \geq 1, z \in X$.

By Lemma 3.2 we have $\lambda \geq e^a$. From $(\mathcal{L}_\varphi h)(x) = \lambda h(x)$ with $x = 0^\infty$ we have $e^a h(0^\infty) + e^d h(10^\infty) = \lambda h(0^\infty)$, so $e^a < \lambda$ since $h > 0$. Similarly we have

$$e^{b_q}h(01^q0w) + \sum_{j=1}^{\infty} e^{b_{q+j}} \frac{e^{c_{q+1}+\dots+c_{q+j}}}{\lambda^j}h(01^{q+j}0w) = \lambda h(1^q0w) \tag{4}$$

and $\lambda > e^c$.

By (3) and (4) with $q = 1$ we have

$$\begin{aligned} \lambda^2 h(01z)h(10w) &= \left[e^{d_1}h(101z) + \sum_{j=1}^{\infty} e^{d_{1+j}} \frac{e^{a_2+\dots+a_{1+j}}}{\lambda^j}h(10^{1+j}1z) \right] \\ &\quad \times \left[e^{b_1}h(010w) + \sum_{j=1}^{\infty} e^{b_{1+j}} \frac{e^{c_2+\dots+c_{1+j}}}{\lambda^j}h(01^{1+j}0w) \right]. \end{aligned}$$

Choose z, w so that $h(01z) = \sup_{y \in X} h(01y)$ and $h(10w) = \sup_{x \in X} h(10x)$. Then

$$\lambda^2 \leq \left[e^{d_1} + \sum_{j=1}^{\infty} e^{d_{1+j}} \frac{e^{a_2+\dots+a_{1+j}}}{\lambda^j} \right] \left[e^{b_1} + \sum_{j=1}^{\infty} e^{b_{1+j}} \frac{e^{c_2+\dots+c_{1+j}}}{\lambda^j} \right].$$

Since $\lambda > \max(e^a, e^c)$ this implies (ii).

(ii) \Rightarrow (iii) By Lemma 3.4 choose $\lambda > \max(e^a, e^c)$ with

$$\frac{1}{\lambda^2} \left[e^{d_1} + \sum_{j=1}^{\infty} e^{d_{1+j}} \frac{e^{a_2+\dots+a_{1+j}}}{\lambda^j} \right] \left[e^{b_1} + \sum_{j=1}^{\infty} e^{b_{1+j}} \frac{e^{c_2+\dots+c_{1+j}}}{\lambda^j} \right] = 1.$$

Let $\alpha > 0$ and define β by

$$\beta = \frac{\alpha e^b (\lambda - e^a)}{e^d (\lambda - e^c) \lambda} \left[e^{d_1} + \sum_{j=1}^{\infty} e^{d_{1+j}} \frac{e^{a_2+\dots+a_{1+j}}}{\lambda^j} \right].$$

For $q \geq 1$ define α_q and β_q by

$$\alpha_q = \frac{\alpha (\lambda - e^a)}{\lambda e^d} \left[e^{d_q} + \sum_{j=1}^{\infty} e^{d_{q+j}} \frac{e^{a_{q+1}+\dots+a_{q+j}}}{\lambda^j} \right],$$

$$\beta_q = \frac{\beta (\lambda - e^c)}{\lambda e^b} \left[e^{b_q} + \sum_{j=1}^{\infty} e^{b_{q+j}} \frac{e^{c_{q+1}+\dots+c_{q+j}}}{\lambda^j} \right].$$

We show that $\alpha_q \rightarrow \alpha$ as $q \rightarrow \infty$. Let

$$u_q = \sum_{j=1}^{\infty} e^{d_{q+j}} \frac{e^{a_{q+1}+\dots+a_{q+j}}}{\lambda^j}$$

which is finite since $\lambda > e^a$. Since $a_n \rightarrow a$ we have $a_n < a + \epsilon$ for n sufficiently large, so for q sufficiently large

$$u_q \leq e^{\sup(d_n)} \sum_{j=1}^{\infty} \left(\frac{e^{a+\epsilon}}{\lambda} \right)^j.$$

Hence $\bar{u} = \limsup_{n \rightarrow \infty} (u_n) < \infty$ and since $u_q = (e^{a_{q+1}}/\lambda)[e^{d_{q+1}} + u_{q+1}]$ we have $\bar{u} = (e^a/\lambda)[e^d + \bar{u}]$ so that $\bar{u} = e^{a+d}/(\lambda - e^a)$.

Similarly $\underline{u} = \liminf_{n \rightarrow \infty} (u_n) = e^{a+d}/(\lambda - e^a)$ so $u_q \rightarrow e^{a+d}/(\lambda - e^a)$ and $\alpha_q \rightarrow \alpha$.

Similarly $\beta_q \rightarrow \beta$.

Define $h : X \rightarrow \mathbb{R}$ by $h(0^q 1z) = \alpha_q, q \geq 1, z \in X, h(1^q 0z) = \beta_q, q \geq 1, z \in X, h(0^\infty) = \alpha$ and $h(1^\infty) = \beta$. Then $h > 0$ and $h \in C(X)$.

We shall now show that $(\mathcal{L}_\varphi h)(x) = \lambda h(x)$.

Note that $\beta_1 = \alpha(\lambda - e^a)/e^d$ since

$$\begin{aligned} \beta_1 &= \frac{\beta (\lambda - e^c)}{\lambda e^b} \left[e^{b_1} + \sum_{j=1}^{\infty} e^{b_{1+j}} \frac{e^{c_2+\dots+c_{1+j}}}{\lambda^j} \right] \\ &= \frac{\beta (\lambda - e^c)}{e^b} \frac{\lambda}{[e^{d_1} + \sum_{j=1}^{\infty} e^{d_{1+j}} e^{a_2+\dots+a_{1+j}}/\lambda^j]} \\ &= \frac{\alpha (\lambda - e^a)}{e^d} \end{aligned}$$

by the definitions of λ and β .

When $x = 0^\infty$,

$$(\mathcal{L}_\varphi h)(0^\infty) = e^{\varphi(0^\infty)} h(0^\infty) + e^{\varphi(10^\infty)} h(10^\infty) = e^a \alpha + e^d \beta_1 = \lambda \alpha = \lambda h(0^\infty).$$

Note that, for $q \geq 1$, $e^{a_{q+1}}\alpha_{q+1} + e^{d_q}\beta_1 = \lambda\alpha_q$, since

$$\begin{aligned} \lambda\alpha_q &= \frac{\alpha(\lambda - e^a)}{e^d} \left[e^{d_q} + \frac{e^{a_{q+1}}}{\lambda} \left\{ e^{d_{q+1}} + \sum_{j=1}^{\infty} e^{d_{q+1+j}} \frac{e^{a_{q+2}+\dots+a_{q+1+j}}}{\lambda^j} \right\} \right] \\ &= \beta_1 e^{d_q} + e^{a_{q+1}}\alpha_{q+1}. \end{aligned}$$

Now when $x = (0^q 1z)$, $q \geq 1$, $z \in X$,

$$(\mathcal{L}_\varphi h)(0^q 1z) = e^{a_{q+1}}\alpha_{q+1} + e^{d_q}\beta_1 = \lambda\alpha_q = \lambda h(0^q 1z).$$

Similarly $(\mathcal{L}_\varphi h)(x) = \lambda h(x)$ when $x = 1^\infty$ and $x = (1^q 0w)$, $q \geq 1$, $w \in X$.

(iii) \Rightarrow (iv) Let h be as in (iii) and put $g = e^\varphi h / \lambda h \circ T$. Then $g \in G(X, T) \cap R(X)$.

By Theorem 2.4, $(\mathcal{L}_{\log g}^n f)(x) \xrightarrow{\lambda^n} \mu(f)$ for all $f \in C(X)$ where μ is the unique g -measure. Hence for all $f \in C(X)$

$$\frac{(\mathcal{L}_\varphi^n f)(x)}{\lambda^n} \xrightarrow{\lambda^n} h(x)\mu(f/h).$$

Let $v(f) = \mu(f/h)$ and we have $\mathcal{L}_\varphi^* v = \lambda v$.

Clearly (iv) implies (i).

This completes the proof of Theorem 3.1 □

COROLLARY 3.5. *Let $\varphi \in R(X)$ satisfy the statements in Theorem 3.1. There is only one number $\lambda > 0$ that satisfies statement (i) and it is that number $\lambda > \max(e^a, e^c)$ satisfying*

$$\frac{1}{\lambda^2} \left[e^{d_1} + \sum_{j=1}^{\infty} e^{d_{1+j}} \frac{e^{a_2+\dots+a_{1+j}}}{\lambda^j} \right] \left[e^{b_1} + \sum_{j=1}^{\infty} e^{b_{1+j}} \frac{e^{c_2+\dots+c_{1+j}}}{\lambda^j} \right] = 1.$$

We have $\lambda = e^{P(T, \varphi)}$. The function h satisfying statement (i) is unique up to scalar multiples. There is a unique $v \in M(X)$ with $\mathcal{L}_\varphi^* v = \lambda v$.

Proof. In the proof of Theorem 3.1 we showed that the number λ given above satisfies $\mathcal{L}_\varphi h = \lambda h$ for a certain continuous $h > 0$, and that, for all $f \in C(X)$, $(\mathcal{L}_\varphi^n f)(x) / \lambda^n \xrightarrow{\lambda^n} h(x)v(f)$. If also $\mathcal{L}_\varphi l = \tau l$ for some number $\tau > 0$ and some $l \in C(X)$ with $l > 0$ then $(\tau/\lambda)^n l(x) \xrightarrow{\lambda^n} h(x)v(l)$. Since $h(x)v(l) > 0$ we have $\tau = \lambda$ and $l(x) = h(x)v(l)$. If $\sigma \in M(X)$ satisfies $\mathcal{L}_\varphi^* \sigma = \lambda \sigma$ then integrating $(\mathcal{L}_\varphi^n f)(x) / \lambda^n \xrightarrow{\lambda^n} h(x)v(f)$ with respect to σ gives $\sigma(f) = \sigma(h)v(f)$ for all $f \in C(X)$. Putting $f = 1$ gives $\sigma(h) = 1$ and $\sigma = v$.

Since $(1/n) \log(\mathcal{L}_\varphi^n 1)(x) \xrightarrow{\lambda^n} P(T, \varphi)$ (see [W4, Theorem 1.3]) we have $P(T, \varphi) = \log \lambda$. □

We now show that if $\varphi \in R(X) \cap \text{Bow}(X, T)$ then the Ruelle operator theorem holds for φ .

COROLLARY 3.6. *Let $\varphi \in R(X) \cap \text{Bow}(X, T)$. Then statement (ii) of Theorem 3.1 holds so there exists $h \in C(X)$, $h > 0$ with $\mathcal{L}_\varphi h = \lambda h$, where $\lambda = e^{P(X, \varphi)}$, and $v \in M(X)$ with $\mathcal{L}_\varphi^* v = \lambda v$ and, for all $f \in C(X)$, $(\mathcal{L}_\varphi^n f)(x) / \lambda^n \xrightarrow{\lambda^n} h(x)v(f)$.*

The measure μ given by $\mu(f) = v(hf)$ is the unique equilibrium state for φ .

Proof. From Theorem 1.1 there exists $K > 0$ so that

$$|a_2 + \dots + a_{1+j} - ja| \leq K \quad \text{and} \quad |c_2 + \dots + c_{1+j} - jc| \leq K$$

for all $j \geq 1$. Therefore $e^{-K} e^{aj} \leq e^{a_2+\dots+a_{1+j}}$ and $e^{-K} e^{cj} \leq e^{c_2+\dots+c_{1+j}}$ for all $j \geq 1$. Hence

$$\begin{aligned} & \frac{1}{e^{2\max(a,c)}} \left[e^{d_1} + \sum_{j=1}^{\infty} e^{d_{1+j}} \frac{e^{a_2+\dots+a_{1+j}}}{e^{j\max(a,c)}} \right] \left[e^{b_1} + \sum_{j=1}^{\infty} e^{b_{1+j}} \frac{e^{c_2+\dots+c_{1+j}}}{e^{j\max(a,c)}} \right] \\ & \geq \frac{e^{\inf d_i} e^{\inf b_i}}{e^{2\max(a,c)}} \left[1 + e^{-K} \sum_{j=1}^{\infty} \left(\frac{e^a}{e^{\max(a,c)}} \right)^j \right] \left[1 + e^{-K} \sum_{j=1}^{\infty} \left(\frac{e^c}{e^{\max(a,c)}} \right)^j \right] = \infty. \end{aligned}$$

Hence statement (ii) of Theorem 3.1 holds. □

COROLLARY 3.7. Let $\varphi \in R(X)$ be defined using the sequences $(a_p)_2^\infty, (b_q)_1^\infty, (c_p)_2^\infty$ and $(d_q)_1^\infty$ as in §1. If $(a_p)_2^\infty, (b_q)_2^\infty, (c_p)_2^\infty$ and $(d_q)_2^\infty$ satisfy

$$\left[\sum_{j=1}^{\infty} e^{d_{1+j}} \frac{e^{a_2+\dots+a_{1+j}}}{e^{j\max(a,c)}} \right] \left[\sum_{j=1}^{\infty} e^{b_{1+j}} \frac{e^{c_2+\dots+c_{1+j}}}{e^{j\max(a,c)}} \right] \geq e^{2\max(a,c)},$$

then for all choices of b_1 and d_1 an eigenfunction $h > 0$ exists. If

$$\left[\sum_{j=1}^{\infty} e^{d_{1+j}} \frac{e^{a_2+\dots+a_{1+j}}}{e^{j\max(a,c)}} \right] \left[\sum_{j=1}^{\infty} e^{b_{1+j}} \frac{e^{c_2+\dots+c_{1+j}}}{e^{j\max(a,c)}} \right] < e^{2\max(a,c)},$$

then for some choices of b_1 and d_1 an eigenfunction $h > 0$ exists and for the other choices of b_1 and d_1 no positive eigenfunction exists.

Note that one or both of the sums above could be ∞ . This is the case when $\varphi \in \text{Bow}(X, T)$.

Proof. Statement (ii) of Theorem 3.1 says

$$[e^{d_1} + S_1][e^{b_1} + S_2] > e^{2\max(a,c)}, \tag{5}$$

where

$$S_1 = \sum_{j=1}^{\infty} e^{d_{1+j}} \frac{e^{a_2+\dots+a_{1+j}}}{e^{j\max(a,c)}} \quad \text{and} \quad S_2 = \sum_{j=1}^{\infty} e^{b_{1+j}} \frac{e^{c_2+\dots+c_{1+j}}}{e^{j\max(a,c)}}.$$

If $S_1 S_2 \geq e^{2\max(a,c)}$ then (5) is true for all choices of b_1 and d_1 .

If $S_1 S_2 < e^{2\max(a,c)}$ then (5) holds for some choices of b_1 and d_1 and fails for other choices. □

The following result deals with the class of functions studied by Hofbauer [Ho]. He studied the case when $a = 0$.

THEOREM 3.8. Let $(a_n)_0^\infty$ be a convergent sequence of real numbers with $(a_n) \rightarrow a$, and let $\varphi \in C(X)$ be defined by $\varphi(0^k 1z) = a_k$ for $k \geq 0, z \in X$ and $\varphi(0^\infty) = a$. Then there exist $h \in C(X)$ with $h > 0$ and $\mathcal{L}_\varphi h = \lambda h$ for some real number $\lambda > 0$ if and only if $\sum_{i=0}^{\infty} e^{a_0+a_1+\dots+a_i-(i+1)a} > 1$.

When this holds $\lambda = e^{P(T,\varphi)} > \max(a, a_0)$ and is given by

$$\sum_{j=0}^{\infty} \frac{e^{a_0+a_1+\dots+a_j}}{\lambda^{1+j}} = 1.$$

When $\sum_{i=0}^{\infty} e^{a_0+a_1+\dots+a_i-(i+1)a} > 1$ the unique equilibrium state for φ is the unique g -measure for the g -function given by: $g(01^q 0z) = 1 - e^{a_0}/\lambda$, for all $q \geq 1, z \in X$, and $g(0^p 1z) = D_p/(1 + D_p)$ for $p \geq 2, z \in X$ where

$$D_p = \sum_{i=0}^{\infty} \frac{e^{a_p+\dots+a_{p+i}}}{\lambda^{i+1}},$$

$g(0^\infty) = e^a/\lambda$ and $g(01^\infty) = 1 - e^{a_0}/\lambda$.

When $\sum_{i=0}^{\infty} e^{a_0+a_1+\dots+a_i-(i+1)a} > 1$ we have, for all $f \in C(X)$,

$$\frac{(\mathcal{L}_\varphi^n f)(x)}{\lambda^n} \rightrightarrows h(x)v(f)$$

where v is the unique member of $M(X)$ with $\mathcal{L}_\varphi^* v = \lambda v$.

Proof. In the notation of Theorem 3.1 $b_q = b = a_1$ for all $q \geq 1$ and $c_p = c = d_q = d = a_0$ for all $p \geq 2, q \geq 1$. Statement (ii) of Theorem 3.1 becomes

$$\frac{e^{a_0+a_1}}{e^{2 \max(a, a_0)}} \left[1 + \sum_{j=1}^{\infty} \frac{e^{a_2+\dots+a_{1+j}}}{e^{j \max(a, a_0)}} \right] \left[1 + \sum_{j=1}^{\infty} \left(\frac{e^{a_0}}{e^{\max(a, a_0)}} \right)^j \right] > 1.$$

If $a_0 \geq a$ the second series diverges to ∞ so the above inequality holds.

If $a_0 < a$ the above inequality becomes

$$e^{a_0+a_1-2a} \left[1 + \sum_{j=1}^{\infty} \frac{e^{a_2+\dots+a_{1+j}}}{e^{ja}} \right] \frac{1}{1 - e^{a_0-a}} > 1.$$

This is equivalent to

$$e^{a_0-a} + e^{a_0+a_1-2a} + \sum_{j=1}^{\infty} e^{a_0+a_1+a_2+\dots+a_{1+j}-(2+j)a} > 1.$$

Therefore, by Theorem 3.1, a positive continuous eigenfunction h exists for \mathcal{L}_φ if and only if

$$\sum_{i=0}^{\infty} e^{a_0+\dots+a_i-(i+1)a} > 1.$$

When this condition holds Corollary 3.5 shows that $\lambda = e^{P(T,\varphi)} > \max(e^a, e^{a_0})$ and

$$\frac{e^{a_0}}{\lambda^2} \left[1 + \sum_{j=1}^{\infty} \frac{e^{a_2+\dots+a_{1+j}}}{\lambda^j} \right] e^{a_1} \left[\frac{\lambda}{\lambda - e^a} \right] = 1.$$

The last equation becomes

$$\frac{e^{a_0}}{\lambda} + \frac{e^{a_0+a_1}}{\lambda} + \sum_{j=1}^{\infty} \frac{e^{a_0+\dots+a_{1+j}}}{\lambda^{2+j}} = 1.$$

From the proof of Theorem 3.1 the eigenfunction, h , for \mathcal{L}_φ has the following form. Let $\alpha > 0$. Let $\beta = \alpha(\lambda - e^a)/e^{a_0}$. For $q \geq 1$ let $\alpha_q = (\alpha(\lambda - e^a)/\lambda)[1 + D_{q+1}]$ and $\beta_q = \beta$.

Then $h(0^q 1z) = \alpha_q$, $h(1^q 0z) = \beta$, $q \geq 1$, $z \in X$, and $h(0^\infty) = \alpha$ and $h(1^\infty) = \beta$. Then the corresponding g -function is $g = e^\varphi h/\lambda h \circ T$ so $g(0^p 1z) = e^{a_p} \alpha_p/\lambda \alpha_{p-1} = D_p/(1 + D_p)$ for all $p \geq 2$, $z \in X$, $g(01^q 0z) = e^{a_1} \alpha_1/\lambda \beta$ for all $q \geq 1$, $z \in X$, $g(1^p 0z) = a_0/\lambda$ for all $p \geq 2$, $z \in X$, and $g(10^q 1z) = 1/(1 + D_{q+1})$ for all $q \geq 1$, $z \in X$. \square

We can get functions of Hofbauer type for which \mathcal{L}_φ has no continuous eigenfunction $h > 0$ as follows. Suppose a_1, a_2, \dots satisfy $a_n \rightarrow a$ and $\sum_{j=1}^\infty e^{a_1 + \dots + a_j - ja} < \infty$. Then choose a_0 so that

$$e^{a_0 - a} \left(1 + \sum_{j=1}^\infty e^{a_1 + \dots + a_j - ja} \right) \leq 1.$$

Examples are given by choosing $s > 1$ and, for $n \geq 1$,

$$a_n = s \log \left(\frac{n}{n+1} \right).$$

Then

$$1 + \sum_{j=1}^\infty e^{a_1 + \dots + a_j - ja} = \sum_{i=1}^\infty \frac{1}{i^s}.$$

4. Coboundaries for the two-sided shift

We can use the space $R(X)$ to obtain examples of functions on the two-sided shift space $\hat{X} = \prod_{-\infty}^\infty \{0, 1\}$ which are not continuous coboundaries, with respect to the shift $S : \hat{X} \rightarrow \hat{X}$, but are bounded measurable coboundaries. Points of \hat{X} are bisequences $\hat{x} = (x_n)_{-\infty}^\infty$ of zeros and ones and the homomorphism S is defined by $S\hat{x} = (y_n)_{-\infty}^\infty$ where $y_n = x_{n+1}$ for all $n \in \mathbb{Z}$.

Let $\text{Cob}(\hat{X}, S) = \{F \in C(\hat{X}) \mid \exists H \in C(\hat{X}) \text{ with } F = H \circ S - H\}$ be the space of continuous coboundaries, and let $\text{Cob}_{\text{BM}}(\hat{X}, S) = \{F \in C(\hat{X}) \mid \exists H : \hat{X} \rightarrow \mathbb{R} \text{ which is bounded and Borel measurable with } F = H \circ S - H\}$ be the space of bounded measurable coboundaries. If $F = HS - H$ then H is called a cobounding function for F . Similarly we can define $\text{Cob}(X, T)$ and $\text{Cob}_{\text{BM}}(X, T)$.

We have $\text{Cob}(\hat{X}, S) \subset \text{Cob}_{\text{BM}}(\hat{X}, S)$ and $\text{Cob}(X, T) \subset \text{Cob}_{\text{BM}}(X, T)$, and for the one-sided shift $T : X \rightarrow X$ Quas [Q] has shown that $\text{Cob}(X, T) = \text{Cob}_{\text{BM}}(X, T)$ but $\text{Cob}(\hat{X}, S) \neq \text{Cob}_{\text{BM}}(\hat{X}, S)$.

We show how we can use $\varphi \in R(X) \cap (\text{Bow}(X, T) \setminus W(X, T))$ to get members of $\text{Cob}_{\text{BM}}(\hat{X}, S) \setminus \text{Cob}(\hat{X}, S)$.

We use the following well-known characterization of the members of $\text{Cob}_{\text{BM}}(X, T)$ for a continuous transformation $T : X \rightarrow X$ of a compact metric space (see [KH, p. 102] where sup should be replaced by lim sup or lim inf).

THEOREM 4.1. *Let T be a continuous transformation of a compact metric space X . Let $f \in C(X)$. Then $f \in \text{Cob}_{\text{BM}}(X, T)$ if and only if there exists $K > 0$ such that $|(T_n f)(x)| \leq K$ for all $x \in X$, for all $n \geq 1$. When this condition holds $l(x) = -\limsup_{n \rightarrow \infty} (T_n f)(x)$ is a cobounding function.*

We now return to the shift maps $T : X \rightarrow X$ and $S : \hat{X} \rightarrow \hat{X}$.

LEMMA 4.2. *Let $\varphi \in R(X)$, let $n \geq 1$ and choose $x_i \in \{0, 1\}$ for $0 \leq i \leq n - 1$. Then $(T_n\varphi)((x_0 \dots x_{n-1})^\infty) = (T_n\varphi)((x_{n-1} \dots x_0)^\infty)$.*

Proof. Let φ be defined by the sequences $(a_p)_2^\infty, (b_q)_1^\infty, (c_p)_2^\infty$ and $(d_q)_1^\infty$ as in §1. Let

$$A_k = \begin{cases} 1 & \text{if } k = 1, \\ a_k a_{k-1} \dots a_2 & \text{if } k \geq 2, \end{cases} \quad \text{and} \quad C_l = \begin{cases} 1 & \text{if } l = 1, \\ c_l c_{l-1} \dots c_2 & \text{if } l \geq 2. \end{cases}$$

Let $x_0 = 0$.

If $x_0 \dots x_{n-1} = 0^{k_1} 1^{l_1} \dots 0^{k_r} 1^{l_r}$ with $k_i, l_i \geq 1, 1 \leq i \leq r$, then

$$(T_n\varphi)((x_0 \dots x_{n-1})^\infty) = A_{k_1} b_{l_1} C_{l_1} d_{k_2} \dots C_{l_r} d_{k_1}$$

and

$$(T_n\varphi)((x_{n-1} \dots x_0)^\infty) = C_{l_1} d_{k_{r-1}} A_{k_{r-1}} \dots A_{k_1} b_{l_r},$$

so the result holds.

If $x_0 \dots x_{n-1} = 0^{k_1} 1^{l_1} \dots 0^{k_r} 1^{l_r} 0^{k_{r+1}}$ then

$$(T_n\varphi)((x_0 \dots x_{n-1})^\infty) = A_{k_1} b_{l_1} C_{l_1} d_{k_2} \dots C_{l_r} d_{k_1+k_{r+1}} a_{k_1+k_{r+1}} \dots a_{1+k_1}$$

and

$$(T_n\varphi)((x_{n-1} \dots x_0)^\infty) = A_{k_{r+1}} b_{l_r} C_{l_r} \dots d_{k_1+k_r} a_{k_1+k_{r+1}} \dots a_{1+k_{r+1}},$$

so the result holds. Similar calculations deal with the cases when $x_0 = 1$. □

Let $\Phi : \hat{X} \rightarrow \hat{X}$ be the reversal map of \hat{X} , defined by $\Phi(\hat{x}) = \hat{y}$ where $y_n = x_{-n}$ for all $n \in \mathbb{Z}$. Let $\pi : \hat{X} \rightarrow X$ be the natural projection, given by $\pi((x_n)_{-\infty}^\infty) = (x_j)_0^\infty$.

THEOREM 4.3. *Let $\varphi \in R(X)$. Then the following hold:*

- (i) $\varphi \in \text{Bow}(X, T)$ if and only if $\varphi \circ \pi - \varphi \circ \pi \circ \Phi \in \text{Cob}_{\text{BM}}(\hat{X}, S)$;
- (ii) $\varphi \in W(X, T)$ if and only if $\varphi \circ \pi - \varphi \circ \pi \circ \Phi \in \text{Cob}(\hat{X}, S)$.

Proof. Let $\varphi \in R(X)$.

(i) Let $\varphi \in R(X) \cap \text{Bow}(X, T)$. We want to find a constant K so that $|S_n(\varphi \circ \pi - \varphi \circ \pi \circ \Phi)(\hat{x})| \leq K$ for all $n \geq 1, \hat{x} \in \hat{X}$, and then we can use Theorem 4.1.

Let C be the constant occurring in the Bowen condition so that if $x, y \in X, n \geq 1$, and $x_i = y_i, 0 \leq i \leq n - 1$, then $|(T_n\varphi)(x) - (T_n\varphi)(y)| \leq C$.

Let $\hat{x} = (x_j)_{-\infty}^\infty \in \hat{X}$. Let $n \geq 1$. Then we have

$$\begin{aligned} S_n(\varphi \circ \pi - \varphi \circ \pi \circ \Phi)(\hat{x}) &= (T_n\varphi)(x_0 x_1 x_2 \dots) - (T_n\varphi)(x_n x_{n-1} \dots x_1 x_0 x_{-1} x_{-2} \dots) \\ &= (T_n\varphi)(x_0 x_1 x_2 \dots) - (T_n\varphi)((x_0 \dots x_{n-1})^\infty) \\ &\quad + (T_n\varphi)((x_0 \dots x_{n-1})^\infty) - (T_n\varphi)((x_{n-1} \dots x_0)^\infty) \\ &\quad + (T_n\varphi)((x_{n-1} \dots x_0)^\infty) - (T_n\varphi)(x_{n-1} \dots x_1 x_0 x_{-1} x_{-2} \dots) \end{aligned}$$

so $|S_n(\varphi \circ \pi - \varphi \circ \pi \circ \Phi)(\hat{x})| \leq 2C$ by Lemma 4.2. Hence $\varphi \circ \pi - \varphi \circ \pi \circ \Phi \in \text{Cob}_{\text{BM}}(\hat{X}, S)$ by Theorem 4.1.

Now let $\varphi \circ \pi - \varphi \circ \pi \circ \Phi \in \text{Cob}_{\text{BM}}(\hat{X}, S)$. Then there exists K such that $|S_n(\varphi \circ \pi - \varphi \circ \pi \circ \Phi)(\hat{x})| \leq K$ for all $n \geq 1, \hat{x} \in \hat{X}$. Let $x, y \in X$ and $x_i = y_i, 0 \leq i \leq n - 1$. Choose $y_j = 0 = x_j$ for all $j < 0$ to form $\hat{x} = (x_i)_{-\infty}^{\infty}$ and $\hat{y} = (y_i)_{-\infty}^{\infty} \in \hat{X}$. Then we have

$$\begin{aligned} (T_n\varphi)(x) - (T_n\varphi)(y) &= (T_n\varphi)(x) - (T_n\varphi)(x_{n-1} \dots x_1 x_0 x_{-1} x_{-2} \dots) \\ &\quad + (T_n\varphi)(x_{n-1} \dots x_1 x_0 x_{-1} x_{-2} \dots) - (T_n\varphi)(y) \\ &= S_n(\varphi \circ \pi - \varphi \circ \pi \circ \Phi)(\hat{x}) - S_n(\varphi \circ \pi - \varphi \circ \pi \circ \Phi)(\hat{y}). \end{aligned}$$

Hence $|(T_n\varphi)(x) - (T_n\varphi)(y)| \leq 2K$, and $\varphi \in \text{Bow}(X, T)$.

(ii) Let $\varphi \in R(X) \cap W(X, T)$. Since

$$(S_n(\varphi \circ \pi \circ \Phi))(\hat{x}) = (T_n\varphi)(x_n x_{n-1} \dots x_1 x_0 x_{-1} x_{-2} \dots)$$

we have $\varphi \circ \pi \circ \Phi \in W(X, T)$ so there exists $\varphi_+ \in C(X)$ such that $\varphi \circ \pi \circ \Phi - \varphi_+ \circ \pi \in \text{Cob}(\hat{X}, S)$ (see [Bou]). By (i) $\varphi \circ \pi - \varphi \circ \pi \circ \Phi \in \text{Cob}_{\text{BM}}(\hat{X}, S)$ so $\varphi \circ \pi - \varphi_+ \circ \pi \in \text{Cob}_{\text{BM}}(\hat{X}, S)$. By Theorem 4.1 applied to S and T we have $\varphi - \varphi_+ \in \text{Cob}_{\text{BM}}(X, T)$, so $\varphi - \varphi_+ \in \text{Cob}(X, T)$ by [Q]. Hence $\varphi \circ \pi \circ \Phi - \varphi \circ \pi \in \text{Cob}(\hat{X}, S)$.

Now let $\varphi \circ \pi - \varphi \circ \pi \circ \Phi = FS - F$ where $F \in C(\hat{X})$. We show that $\sup_{n \geq 1} v_{n+N}(T_n\varphi) \rightarrow 0$ as $N \rightarrow \infty$.

Let $n \geq 1$ and $N \geq 1$ and let $x = (x_j)_0^{\infty}, y = (y_j)_0^{\infty} \in X$ have $x_j = y_j, 0 \leq j \leq n + N - 1$. Let $x_i = 0 = y_i$ for all $i \leq -1$ to obtain $\hat{x} = (x_j)_{-\infty}^{\infty}$ and $\hat{y} = (y_j)_{-\infty}^{\infty} \in \hat{X}$. Then

$$\begin{aligned} (T_n\varphi)(x) - (T_n\varphi)(y) &= S_n(\varphi \circ \pi - \varphi \circ \pi \circ \Phi)(\hat{x}) - S_n(\varphi \circ \pi - \varphi \circ \pi \circ \Phi)(\hat{y}) \\ &= F(S^n \hat{x}) - F(\hat{x}) - F(S^n \hat{y}) + F(\hat{y}) \\ &= F(\dots \overset{*}{x}_n \dots x_{n+N-1} x_{n+N} \dots) - F(\dots \overset{*}{y}_n \dots y_{n+N-1} y_{n+N} \dots) \\ &\quad - [F(\dots \overset{*}{x}_0 \dots x_{n+N-1} x_{n+N} \dots) - F(\dots \overset{*}{y}_0 \dots y_{n+N-1} y_{n+N} \dots)] \\ &\leq v_N(F) + v_{n+N}(F) \leq 2v_N(F). \end{aligned}$$

Hence $\sup_{n \geq 1} v_{n+N}(T_n\varphi) \leq 2v_N(F)$ so $\varphi \in W(X, T)$.

This completes the proof of Theorem 4.3. □

We can get members of $\text{Cob}_{\text{BM}}(\hat{X}, S) \setminus \text{Cob}(X, S)$ as follows.

COROLLARY 4.4. *Let $\varphi \in R(X)$. Then $\varphi \in \text{Bow}(X, T) \setminus W(X, T)$ if and only if $\varphi \circ \pi - \varphi \circ \pi \circ \Phi \in \text{Cob}_{\text{BM}}(\hat{X}, S) \setminus \text{Cob}(\hat{X}, S)$.*

Examples of functions in $R(X) \cap (\text{Bow}(X, T) \setminus W(X, T))$ are given in §1.

Results of this type, in a more general setting, will appear in another paper.

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