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Quantum Field Theory and Jet Phenomena

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## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acknowledgements</td>
<td>1</td>
</tr>
<tr>
<td>Contents</td>
<td>2</td>
</tr>
<tr>
<td>Abstract</td>
<td>3</td>
</tr>
<tr>
<td>Background</td>
<td>4</td>
</tr>
<tr>
<td>CERN</td>
<td>4</td>
</tr>
<tr>
<td>Introduction to Jets</td>
<td>4</td>
</tr>
<tr>
<td>Visualization Studies of Advanced Jet Tagging Algorithms</td>
<td>7</td>
</tr>
<tr>
<td>From Classical Physics to Relativistic Quantum Field Theory</td>
<td>12</td>
</tr>
<tr>
<td>Classical Fields</td>
<td>12</td>
</tr>
<tr>
<td>1st Quantization</td>
<td>12</td>
</tr>
<tr>
<td>1st Quantization Runs into Trouble</td>
<td>13</td>
</tr>
<tr>
<td>Relativistic Quantum Field Theory; 2nd Quantization to the Rescue</td>
<td>14</td>
</tr>
<tr>
<td>The Canonical Quantization Machine</td>
<td>15</td>
</tr>
<tr>
<td>Feynman Calculus</td>
<td>18</td>
</tr>
<tr>
<td>John Wheeler's Scattering Matrix</td>
<td>18</td>
</tr>
<tr>
<td>The Feynman Propagator</td>
<td>18</td>
</tr>
<tr>
<td>Wick's Expansion</td>
<td>19</td>
</tr>
<tr>
<td>Feynman Diagrams</td>
<td>21</td>
</tr>
<tr>
<td>Some Extra (Important) Notes</td>
<td>23</td>
</tr>
<tr>
<td>Finding the QED $H_I$ from Symmetry and Unification</td>
<td>24</td>
</tr>
<tr>
<td>Gauge Invariance and Spontaneous Broken Symmetry</td>
<td>24</td>
</tr>
<tr>
<td>Applying Symmetries</td>
<td>25</td>
</tr>
<tr>
<td>Unification: A Final Product</td>
<td>25</td>
</tr>
<tr>
<td>Making This Useful: Finally Finding $H_I$ for QED Interactions</td>
<td>25</td>
</tr>
<tr>
<td>A Jet Calculation</td>
<td>26</td>
</tr>
<tr>
<td>Calculating $M$ Using the Feynman Rules</td>
<td>26</td>
</tr>
<tr>
<td>Casimir's Trick</td>
<td>27</td>
</tr>
<tr>
<td>Calculating Scattering Cross Sections</td>
<td>28</td>
</tr>
<tr>
<td>Conclusion</td>
<td>30</td>
</tr>
<tr>
<td>References</td>
<td>31</td>
</tr>
</tbody>
</table>
Abstract

This honors thesis studies graduate-level quantum field theory including Feynman diagrams and Feynman calculus in order to connect experimental results with the theoretical background. It discusses results of a research experience for undergraduates through Duke University’s high energy physics program regarding jet phenomena and explains the inadequacy of quantum mechanics in predicting particle interactions. It follows the canonical method of building a relativistic quantum field theory and describes the process for building one from Langrangians. It addresses the non-interacting part of the theory using the canonical quantization and uses the Feynman propagator and Wick’s theorem to address the interacting part of the theory and write it as Feynman diagrams. It then reconnects the theory to jets with a detailed following of an established jet calculation using the Feynman calculus. The final numerical computation establishes the goal of this work.
Background

I participated in a research experience for undergraduates (REU) program through Duke University in the summer of 2015. As part of the high energy physics (HEP) group, I was responsible for working on the project of visualization studies for advanced jet tagging algorithms under the direction of Dr. Ayana Arce. This project required me to learn algorithms used in HEP data analysis as well as C++ and the programming language used in ROOT. As part of this program, I travelled to CERN in Geneva, Switzerland to work with data taken from the ATLAS detector.

CERN

CERN is the European Organization for Nuclear Research and is a multinational collaboration for the study of high energy physics. Currently there are one hundred thirteen countries that collaborate on experiments. Figure 1 shows an aerial view of CERN and its location with respect to the French-Swiss border [1].

![Aerial view of CERN](image.png)

Figure 1: An aerial picture of CERN including the location of the active projects including ALICE, ATLAS, CMS and LHCb.

Located on the border of France and Switzerland, the accelerator is 27km in circumference, making it the largest in the world. As of the last set of updates and innovations, the Large Hadron Collider can now accelerate individual protons to speeds faster than 0.9999c, causing collisions of 13 trillion electron volts [1]. With this enormous amount of energy, new particles have been created including heavy particles like the top and bottom quarks, the Higgs boson [2], and most recently the pentaquark [3].

One of the four main detectors at CERN as shown in Figure 2 is the ATLAS detector, which is a general purpose particle physics detector made to test the predictions of the standard model. This detector, completed in 2008, is a massive cylinder - weighing 7000 tons and measuring 46m in length and 25m in diameter - which is buried underground and made of four parts. An inner detector measures the momentum of each charged particle that passes through. Various calorimeters detect the energy deposited by all particles that pass and a muon spectrometer focuses on the momenta of muons [4].

Introduction to Jets

Jets occur due to the extremely high levels of energy at the point where the particles are fast enough to fall in the realm of special relativity yet are small enough to exhibit quantum mechanical behavior. They occur
immediately after particle collisions of very fast, small particles and result in the creation of many particles, some of which are often heavier than the total original mass.

CERN describes this process using fruit in Figure 3. It shows two small strawberries with high speed and kinetic energy colliding together. Out of the collision come heavier, slower fruits like apples, bananas and pears. The jets in this picture would be clusters of the produced fruit with similar angles of trajectory.

The conservation of energy allows for these occurrences. Einstein, from his postulates of Special Relativity (1905) derived the total energy given by

\[ E_T = \gamma mc^2 \]  

which using his definition of momentum, \( p = \gamma mv \) where \( \gamma = \sqrt{1 - \frac{v^2}{c^2}} \) can be used to rewrite \( E_T \) in a way that emphasizes the momentum of particles given by

\[ E^2 = p^2 c^2 + m^2 c^4 \]  

where \( p \) is the momentum of an object, \( m \) is the mass and \( c \) is the speed of light. This shows that the total energy of particles has two parts - momentum and mass - and explains how the final total mass can be greater if the final total momentum is smaller.

These equations suggested that mass and energy were in fact related and therefore could be interchanged given the right conditions. Relating this to Figure 3, the rest mass of the products can be related to the total energy of the incident strawberries like

\[ p_{\text{strawberries}}^2 c^2 + m_{\text{strawberries}}^2 c^4 = m_{\text{fruit out}}^2 c^4. \]  

To make the example easier the final products were at rest and therefore \( \gamma = 1 \), but essentially this is how larger and heavier particles are made at CERN.

Jets occur after the collision due to the strong nuclear force and can be displayed using Feynman diagrams.

Figure 4 shows a collision of two particles (in this case the strawberries of the fruit collision). These small strawberries combine to form a single product before decaying into smaller pieces. These pieces then cause a chain reaction that creates many more pieces until the pieces start to recombine into larger fruits.

These collisions can be run with any particle combination with sufficient energy. Electrons are easier to run whereas heavier protons require more energy from the accelerator and larger particles require even more. Converting the fruit example into particle lingo, the strawberries are an electron and a positron. During the collision they annihilate to form a photon of energy equal to the energy of the collision which quickly decays into a quark and antiquark. This follows the rules of quantum electrodynamics (QED) and can be written

\[ e^- + e^+ \rightarrow \gamma \rightarrow q + \bar{q}. \]  

5
Figure 3: CERN uses this picture to illustrate particle collisions at relativistic energies. If two strawberries are given enough speed, when they collide, there will be enough energy to produce new fruits like bananas, pears, apples and many more strawberries. Groups of these fruits and acorns projected in similar directions are grouped together in jets.

The second part of the collision includes the rest of the diagram. The pieces that were created are quarks and antiquarks. The rest of the jet is controlled by the rules of quantum chromodynamics (QCD) which as the name suggests, deals with color. Every observable object must be color neutral, but these objects are made of quarks which have a color charge of either red, blue or green (the antiquarks are colored anti-red, anti-blue and anti-green). Unlike any of the other forces, the strong force which dictates QCD gets stronger as distance increases. As the quark and antiquark move apart, the distance between colored particles increases and soon it is easier to create a particle-antiparticle pair than it is for the distance to increase. It does so by emitting a gluon (curly line) which decays into a quark and antiquark. This is all part of a chain reaction that continues until all particles are incorporated in color neutral mesons (quark and antiquark) and baryons (3 quarks or 3 antiquarks) through a process called hadronization.

In summary, with high energy collisions between an electron and a positron, a photon is created followed by a quark and antiquark which in turn through QCD create jets of mesons and baryons.

It is impossible to "see" the massive particles created by the collision, however their existence can be predicted by the jets that can be "seen" by the detectors. The properties of the original particles that decayed to form the jets are the same as the overall properties of their corresponding jets. The overall properties of each jet are determined by the sum of the properties of all the particles in the jet. For this reason, it is important to make sure every particle is associated with its correct jet. This may seem easy if an event display looked like something from Griffiths' book (Figure 5(a)) but this only happens in ideal conditions and often times the event display looks more like the one in Figure 5(b) with many more jets and little definition between them. In both of these figures it is hard to visualize the jets because a jet is most often defined by its proximity to particles in the pseudorapidity-azimuthal ($\eta$-$\phi$) plane, which is essentially the flattened lateral surface of the cylindrical detector. When viewed in a manner that shows this plane, the jets become much more noticeable.

Figure 6 shows another event display at a similar detector called CMS [6]. In this case, the jets are seen much more clearly as the cluster of particles surrounding each of the large bars extending from the detector. It is easy to see in an event display like this that the boundaries of each jet are not well defined and some particles from one jet may be mistaken as being part of a different jet. Since the properties of created
Figure 4: This is a plausible example of a particle collision that results in two outgoing jets. The labels were added to relate this to Figure 3.

Particles are predicted by the overall properties of the jets that are caused by their decay, it is extremely important to assign each particle to its proper jet. Otherwise, the overall properties will be incorrect and the experimental prediction for the originally created particle will be off.

Visualization Studies of Advanced Jet Tagging Algorithms

Due to this need for getting the most accurate assignment of particles, physicists have developed algorithms to determine which particles belong to which jets. The first algorithms were called cone algorithms and attempted to fit a fraction $1-\epsilon$ of the total energy of the event into two jets with half angle $\delta$ [7]. Although these algorithms have been further developed to include iteration, progressive removal and split-merge, algorithms commonly used in high energy physics (HEP) experiments are the $k_t$, anti-$k_t$ and Cambridge-Aachen algorithms. Before describing how each works, it is important to describe the new $\eta$-$\phi$ coordinate system used in these algorithms.

The ATLAS detector is a cylindrical detector surrounded by calorimeters and special detectors designed to capture specific particles. Because of its shape it would make sense to use cylindrical coordinates, yet because of the nature of HEP experiments, many if not all of the particles are relativistic, therefore it was necessary to define a set of coordinates that are invariant under a Lorentz transformation. In order to do this, the $z$ coordinate was replaced with rapidity ($y$) which can be described as the inverse hyperbolic tangent of the ratio of momentum in the $z$ axis to the total energy.

$$y = \tanh^{-1} \left( \frac{p_z c}{E} \right)$$

By this definition, particles with a momentum perpendicular to the beam have a rapidity of $y = 0$ while particles along the beam axis have a rapidity of $y = \pm 1$. 
Figure 5: Two event displays of a particle collision. a) An $e^-e^+$ collision causes two jets in opposite directions [5]. b) One of the event displays from the ATLAS detector. This shows both an azimuthal as well as a rapidity cross section [4].

Rapidity works well on paper, yet runs into problems when working with real data. In high energy experiments, many of the particles are highly relativistic. When particles are highly relativistic, it may be hard to measure the total momentum vector making rapidity difficult to measure. A more convenient approximation at highly relativistic speed is called pseudorapidity ($\eta$) and is approximated by assuming that the $p^2c^2$ fraction of the energy is much larger than the $m^2c^4$ fraction. By starting with the first definition of rapidity and rewriting E using Einstein’s relation from special relativity, $pc$ is factored out and the rest is approximated using a binomial expansion. Using trigonometric identities, pseudorapidity can be written as a function of the angle of trajectory as opposed to the momentum vector.

$$\eta = -\ln\left(\frac{\tan \frac{\theta}{2}}{2}\right)$$

This is more commonly used in HEP experiments than rapidity and has a range of $-\infty$ to $\infty$ where the angle perpendicular to the beam is still $\eta = 0$.

Like cylindrical coordinates, the second coordinate is considered the azimuthal angle ($\phi$) and has a range of $2\pi$. The radial coordinate is not considered very often because the calorimeters only measure on the surface of the cylindrical detector and do not take distance from the origin into account.

Now that we have defined the $\eta$-$\phi$ coordinate system, we can return to the different algorithms used to tag jets. Unlike the cone algorithms which worked by picking a seed particle and drawing a cone around it and calling it a jet, the algorithms used today are called sequential recombination algorithms and work by finding the smallest distance $d$ between two particles in the $\eta$-$\phi$ plane and recombining them to make a new particle that the two original particles could have decayed from. This process would continue until all particles were recombined as far as allowed by the parameters set and then the final particles are considered the jets along with all particles that went into them. All three algorithms work off of the same three equations:

$$d_{ij} = \min(p^2_{ti}, p^2_{tj}) \frac{\Delta R^2_{ij}}{R^2}$$

$$\Delta R^2 = (y_i - y_j)^2 + (\phi_i - \phi_j)^2$$

$$d_{iB} = p^2_{ti}$$

where $R_{ij}$ is the distance between particles in the $\eta$-$\phi$ plane, $R$ is a set value, $p_{ti}$ and $p_{tj}$ are the momenta of particles $i$ and $j$ and $d_{iB}$ is the same calculation as $d_{ij}$ except it uses the beam instead of a second particle. $\rho$ is a value specific to the algorithm where $\rho = 1$ for the $k_t$ algorithm, $\rho = 0$ for the Cambridge/Aachen algorithm and $\rho = -1$ for the anti-$k_t$ algorithm. Each is then evaluated using the rules [7]:

1. Work out all of the $d_{ij}$ and $d_{iB}$. 

8
Figure 6: An event display produced by CMS is shown in three dimensions. The green and grey internal lines are the paths of the particles through the detector while the blue and red bars are showing the energy that was deposited in each calorimeter of the detector. These energies correspond to the sum of the momentum of each particle that passed through each calorimeter.

2. Find the minimum of the $d_{ij}$ and $d_{iB}$.

3. If it is a $d_{ij}$, recombine i and j into a single new particle and return to step 1.

4. Otherwise if it is a $d_{iB}$, declare i to be a final jet and remove it from the list of particles. Return to step 1.

5. Stop when no particles remain.

(a) Event 1

(b) Event 2

Figure 7: ROOT is used to represent the collision and scattering in three dimensions. Particles belonging to the same jets are displayed in the same colors.

The difference in these algorithms lies in the hardness of the particles that each starts with. Hardness is a relative comparison of total energy where harder particles have more energy and softer particles have less energy. Because the value of $\rho$ is different for each algorithm, each one forms jets differently. For example the $k_t$ algorithm favors softer clusters of particles and then recombines the clusters later with harder particles whereas the anti-$k_t$ algorithm starts with harder particles and moves towards the softer ones. The Cambridge/Aachen algorithm is unique in the fact that it uses a value of $\rho = 0$ so it is energy
independent. This causes it to give abnormally shaped jets that are based solely on the distance of particles in the $\eta\phi$ plane.

Figure 8: Lego plots are a way to view an event in the $\eta\phi$ plane. The height of the bars are proportional to energy. Figure 8(b) shows the same event in a more three-dimensional manner.

The Duke HEP department most frequently uses the Cambridge/Aachen and anti-$k_t$ algorithms in their experiments. Using FastJet, a module containing all of the current jet tagging algorithms, in conjunction with ROOT, I was able to write programs that could read data collected from the ATLAS detector from proton collisions and model the events in various ways.

After learning about the different jet tagging algorithms, the first part of my project was to use ROOT’s THelix class to display the paths of each particle from the point of collision to the detector. The magnitude of each line in Figure 7 is proportional to the energy each particle had as it was detected by ATLAS. Other options can show the event in the $\eta\phi$ plane like the lego plots in Figure 8. Many of the plots that show collisions use this form of plot due to its ease of interpretation.

An important part of the project was to create ghost plots of each event. The purpose of a ghost plot is to track the path of an algorithm as it moves through each particle in the event. It works by creating a grid of very low energy “ghost” particles and adding them to the particles of the event. When a program reads the data of the combined ghost particle grid and the actual particles, it will add a value to the bin it is in every time it finds a particle with energy of that magnitude. By giving it a value of one to the number of ghost particles present and setting it to a color scale, you can follow the algorithm and see why each gives a different output.

Figure 9: Two different algorithms were used to make a ghost plot. Red are particles weighted heaviest (found by the algorithm first) while blue was found last.

As you can see in Figure 9(a), the Cambridge/Aachen algorithm started in the relative middle of the clusters and then worked its way out until it could find no more real particles close enough to add to the jet. The anti-$k_t$ algorithm in Figure 9(b) defined an outer ring and then filled in the middle.

My thesis is about understanding quantum field theory to the point where I can do calculations related to jet processes at CERN that I worked with for my REU. The quantum field theory included lectures and
discussions from Dr. Deveney using three main sources: Quantum Field Theory for the Gifted Amateur [8], the Harvard 2012 Lecture Notes [9] and Introduction to Elementary Particles [5]. The calculation is the $e^- + e^+ \rightarrow \gamma \rightarrow q + \bar{q}$ which is from Griffiths. In order, this thesis will discuss:

- The 1st and 2nd quantizations and how relativistic QFT is needed to describe interactions and fields on the scales of individual particles and high energies

- Fermi’s golden rule and how it uses the value of $|M|$ to solve the cross section of

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|M|^2}{(E_1 + E_2)^2} \frac{|p_f|}{|p_i|}$$  \hspace{1cm} (10)

- The dependence of $|M|$ on John Wheeler’s scattering matrix and the Dyson Expansion of $\hat{S}$

$$|M| = \langle q_1 q_2 | \hat{S} | p_2 p_1 \rangle$$  \hspace{1cm} (11)

- The dependence of $\hat{S}$ on the interacting part of the Hamiltonian, $\mathcal{H}_I$.

$$\hat{S} = T \left[ e^{i \int d^4x \mathcal{H}_I} \right]$$  \hspace{1cm} (12)

- The use of Wick’s Theorem to convert the Dyson Expansion to Feynman diagrams

- The use of special relativity, gauge invariance, spontaneous broken symmetry and local symmetries to derive the Lagrangian of the unified Electroweak theory.

- The determination of the QED $\mathcal{H}_I$ from the full Lagrangian

- The use of Casimir’s Trick with Feynman calculus to calculate the absolute cross section $\sigma$ of the initial QED interaction of a jet collision.

- The arrival at a final numerical computation that can be compared directly to a first order QED process for jets, completing my goal.
From Classical Physics to Relativistic Quantum Field Theory

Classical Fields

When solving the classical equations for the travelling wave on a string from Newton’s laws and the electric field of light from Maxwell’s equations, the solutions to both are field equations:

\[ \nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2} \]  \hspace{1cm} (13)

\[ \nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} \]  \hspace{1cm} (14)

The most general solution to both of these wave equations are inverse Fourier transforms:

\[ f(x,t) = (2\pi)^n \int \left( a(k)e^{-ikx} + a'(k)e^{ikx} \right) d^nk \]  \hspace{1cm} (15)

\[ \vec{E}(x,t) = (2\pi)^n \int \left( a(k)e^{-ikx} + a'(k)e^{ikx} \right) d^nk \hat{i} \]  \hspace{1cm} (16)

Mathematically, an inverse Fourier transform says that the solutions are continuum expansions that are projections onto a complete basis in function space. Here the function space and expansion is onto plane travelling waves designated as “k” space or wave number space where \( k = \frac{2\pi}{\lambda} \), where \( \lambda \) and therefore \( k \) runs continuously from \(-\infty\) to \( \infty \). The Fourier transform is the projection functions, \( a(k) \) and \( a'(k) \), that in wave number space depicts the amounts of each wavelength that goes into creating the wave solution.

The scalar field, \( f(x,t) \), differs from the vector field, \( \vec{E}(x,t) \), however both solutions are written in the same form, only one gives scalars and the other gives vectors. Therefore a field equation can be thought of as a machine or function where parameters are put in and it produces a scalar or a vector depending on the type of field.

1st Quantization

The 1st quantization is the view of physics that comes from the first half of the wave-particle duality that states that all particles behave like waves. It is characterized by the interpretation that all matter can be described by its DeBroglie wavelength

\[ \lambda_{\text{DeBroglie}} = \frac{\hbar}{p} \]  \hspace{1cm} (17)

In the 1900’s when classical physics (Newton’s laws, Maxwell’s equations and Maxwell-Boltzmann Thermodynamics) were inadequate to explain particles as just particles and waves as just waves, in particular the electron of the hydrogen atom, it became clear to physicists like Born, DeBroglie and Schrödinger that solving for the wave-like nature of the electron would fix the theoretical problems and lead to experimentally verifiable results.

The first step of this came in 1926 when Schrödinger derived the nonrelativistic wave equation of the electron in general matter given by

\[ -\frac{\hbar^2}{2m} \nabla^2 \Psi + V(r,t)\Psi = i\hbar \frac{\partial \Psi}{\partial t} \]  \hspace{1cm} (18)

where the left side is the non-relativistic total energy of the particle given by \( E_{\text{total}} = T + U = \frac{p^2}{2m} + V(r,t) \). The right hand side is the form of total energy that reflects the wave-like total energy \( E_{\text{total}} = \hbar \omega \) which is dependent on time. This expression is written in terms of differential operators (the ordinary space three-vector Laplacian \( \nabla^2 \) and the partial time derivative with respect to time) that act on plane wave basis solutions of the form \( e^{i(kx - \omega t)} \). As stated however, Schrödinger’s equation is non-relativistic. Schrödinger knew this at the time but had to abandon his fully relativistic version of the same equation because when he actually solved the relativistic version for free particles, it led to inconsistencies that were not understood at
the time - most notably the negative total energy states and negative probabilities. In the non-relativistic limit, Schrödinger’s equation is extraordinarily successful and is what is normally practiced in non-relativistic realms.

In an effort to unify the successful new quantum physics with the tenets of Special Relativity, Dirac united quantum mechanics and relativity by deriving the fully relativistic wave equation given by

\[(i\hbar \gamma^i \cdot \vec{\nabla} + \gamma^0 mc^2)\Psi = i\hbar \frac{\partial}{\partial t}\Psi\] (19)

Here the left hand side is now the final form of the relativistic total energy. With this, Dirac was able to find free particle solutions, however in order to combine quantum mechanics with Special Relativity, he had to introduce new space-time \(\gamma\) matrices. These were in four-vector notation so there were four components: \(\gamma^\mu = \gamma^0, \gamma^i\). This introduced 4 particle solutions accounting for what is now known to be the free particle and antiparticle states each with a spin up and a spin down. The Dirac equation still suffered from negative energy states which were, at the time, incorrectly interpreted as electron holes in an infinite negative energy sea. While that interpretation would later be refined by Feynman, the equation was correct and the relativistic quantum mechanical world now necessitated antiparticle states and spin projections of up and down.

The general inverse Fourier plane wave solution in equations 15 and 16 is also the same form as the general solution for a free particle using Schrödinger’s and Dirac’s equations. This similarity in form shows that quantum mechanical solutions are field equations as well.

In summary, the 1st quantization is wave-like interpretation of quantum mechanics. Using the Fourier transforms, all matter waves can take the form of an infinite number of waves with different wavelengths. It is ultimately a wave-like interpretation of matter waves.

1st Quantization Runs into Trouble

As mentioned earlier, Schrödinger encountered some difficulties with deriving the fully relativistic wave equation of the electron. In particular, his equation had solutions of free particles with negative energies and probabilities - both of which are not physically consistent. As it turns out, Schrödinger’s equation was in fact correct, however it is correct for bosons with integer spins like the photon as opposed to fermions with \(\frac{1}{2}\) spins like the electron. Dirac’s equation described fermions yet his too suffered from negative total energies. The first quantization cannot account for these negative energies by itself.

Another problem appears with the concept of the Compton wavelength. The Compton wavelength is the smallest wavelength (highest frequency and highest energy) that a massive particle can have. To solve for it, assume that the entire rest mass of the particle is converted to the energy of a photon:

\[mc^2 = hf\] (20)

where \(f\) can be written in terms of the wavelength such that

\[mc^2 = \frac{\hbar c}{\lambda}\] (21)

Solving for \(\lambda\) the equation becomes

\[\lambda = \frac{\hbar}{mc}.\] (22)

Once again, this wavelength corresponds to the highest energy photon a particle can have. However, if a box was drawn with a length of \(\lambda\) around the particle and the space begins to shrink (the same thing that happens at high energies when the box is probed at a smaller scale), the wavelength must decrease and cause the frequency and therefore energy to increase. Since the energy began with the rest energy of the particle, there is now more energy than the particle is made of which causes particle-antiparticle pairs to be created from the vacuum.

Physics on these scales require the possibility of multiple particles popping into and out of existence, but the 1st quantization has nothing that allows for this to happen. 1st quantization is a purely single particle theory and therefore cannot describe physics at its smallest scales.
Additionally, the Schrödinger and Dirac equations suffer from a deeper problem called causality. With just 1\textsuperscript{st} quantization, the wave function of a matter wave can leak into space-time where information can travel faster than the speed of light leading to non-causal influences. This is a disaster and cannot be correct because it would violate special relativity and lead to a long list of philosophically wrong results.

**Relativistic Quantum Field Theory; 2\textsuperscript{nd} Quantization to the Rescue**

The 2\textsuperscript{nd} quantization can be achieved in two separate ways: The path integral method and the canonical quantization method. Each has its advantages and disadvantages, but the canonical quantization method is more widely used and follows more closely what has already been set up in the 1\textsuperscript{st} quantization. According to Matthew Schwartz [9], "With a risk of oversimplifying things a little" the 2\textsuperscript{nd} quantization is changing the field equations from the 1\textsuperscript{st} quantization and replacing them with operator-valued fields.

To do this, all of the physics that is present in the wave functions will be encoded into the operators. This includes whether the particle is a fermion or a boson, the time dependence (the 1\textsuperscript{st} quantization was done in the Schrödinger picture where the time dependence was in the states while the 2\textsuperscript{nd} quantization is in the Heisenberg picture with the time dependence in the operators), special relativity and causality and will all be done through the commutation relations. 2\textsuperscript{nd} quantization operator fields therefore look like 1\textsuperscript{st} quantization fields except the Fourier integral projections are operator valued (designated with small hats) and given by:

\[
\hat{\phi}(x) = \int \frac{d^3 p}{(2\pi)^2 (2E_p)^{1/2}} \left( \hat{a} e^{-ipx} + \hat{a}^\dagger e^{ipx} \right)
\]

which is the prototype of all relativistic quantum fields in momentum space with all the subtle physics and interactions of the universe encoded in the different raising and lowering operators. The form however is the same for all of the corresponding field equations, just now operator valued.

Operator valued means that these Fourier integrals are operators that have been borrowed from the harmonic oscillator and either create (\(\hat{a}^\dagger\)) or destroy (\(\hat{a}\)) quantized particles in k space (or momentum space since \(p = \hbar k\)). This new particle-like interpretation means that instead of viewing quantum solutions as being infinite sums of all possible wavelengths (that go into the making of a matter wave), the matter wave is seen as the infinite sum of all possible harmonic oscillator-like quantized excitations in momentum space. In a harmonic oscillator, a quantized excitation is viewed as a quantized particle.

Now recalling the negative total energies, this can be resolved with an explanation Richard Feynman gave. Through a suggestion by his advisor, John Wheeler, Feynman came up with an integration that leads to physically calculable results and introduces a new idea: antiparticles. Feynman noted that the equations leading to negative total energies were exactly the same if time or charge were made negative. This way there would be a positive total energy, but the interpretation would be a particle moving backward in time or an antiparticle moving forwards in time. With the goal of computing measurable quantities, Feynman interpreted this to give Feynman propagators which take into account the retarded times (particles moving forwards in time) and the advanced times (particles moving backwards in time). The Feynman calculus that comes from these propagators is what connects theory to experiment by allowing us to calculate numerical values. Therefore 2\textsuperscript{nd} quantization is the interpretation of matter particles and antiparticles as an infinite number of excitations of the quantum field or an infinite number of particles. In this interpretation, particles are viewed as excitations of the quantum field itself, created by \(\hat{a}^\dagger\) and destroyed by \(\hat{a}\) that act on a vacuum state \(\lvert 0 \rangle\).

The 2\textsuperscript{nd} quantization goes further to solve the problems present in the 1\textsuperscript{st} quantization. It can also successfully describe the world down to the infinitely small. Unlike the 1\textsuperscript{st} quantization which falls apart when multiple particles are introduced, at high energies, 2\textsuperscript{nd} quantization intrinsically requires many particles and therefore provides an interpretation and computational ability that the 1\textsuperscript{st} quantization simply could not.

By using the 2\textsuperscript{nd} quantization, it is possible to fix all of the problems that were present in the first quantization. Therefore to get everything correct, fields must be operator-valued.
The Canonical Quantization Machine

This 2\textsuperscript{nd} quantization of a free particle can be achieved through a process called the second quantization machine. It is given such a title because it is machine-like in nature where it will take a classical field and transform it into an operator-valued relativistic quantum field through the same steps. Following Lancaster and Blundell, there are five steps to this machine [8]:

1. Find the Lagrangian density
2. Calculate the momentum density and the Hamiltonian density
3. Treat the fields and momentum density as operators and impose commutation relations
4. Rewrite the fields in terms of raising and lowering operators
5. Apply normal ordering

As an example of using these rules, we can find the second quantized massive scalar field.

A note on units and notation

There is another guideline that falls into these rules that can be considered rule zero. All of quantum field theory is done using natural units where $c = \hbar = 1$. This is because base units were created to describe what we as humans knew, however now that we know more, it would have made more sense to set $c$ and $\hbar$ equal to one and relate everything else to them. In natural units, the potential energy of an object can be written

$$E = m.$$ \hspace{1cm} (24)

How can this be true? Using dimensional analysis $E$ is written in units of $\frac{kg \cdot m^2}{s^2}$. Mass has units of kg so by multiplying mass by two factors of $c$ (units of $\frac{m}{s}$) the expression becomes

$$E = mc^2.$$ \hspace{1cm} (25)

The same thing can be said for kinetic energy and momentum:

$$E = p.$$ \hspace{1cm} (26)

In this case momentum has units of $\frac{kg \cdot m}{s}$. In order to get units consistent with energy, $p$ must be multiplied by one factor of $c$:

$$E = pc.$$ \hspace{1cm} (27)

The process of second quantizing a classical field also uses a new notation. Special Relativity works in four vectors, so the new four vector derivative is denoted and defined as $\partial_\mu = (\frac{\partial}{\partial t}, \nabla)$. Further, when working with fermions, solutions to the Dirac equation (electrons and positrons in the scattering calculation we are performing), Feynman introduced the slash notation. A slash across an operator signifies a space-time (four-vector) dot product of the Dirac $\gamma$ matrix with that operator. For example, $\partial_\mu = \gamma^\mu \cdot \partial_\mu$.

1. Find the Lagrangian density

In step one, we must find the Lagrangian density. This is derived from the Lagrangian which in its simplest definition is the difference between kinetic and potential energy. Unlike the laws of motion, this is a fundamental equation and, by using the Euler-Lagrange equation and the correct conditions, it will produce all of the equations of motion including Newton’s laws, the wave equation of both a string and light and Schrödinger’s equation. The Lagrangian is the integral over space of the Lagrangian density.

$$L = \int L \, d^3x.$$ \hspace{1cm} (28)
From here, we will be using the Lagrangian density. As stated, a Lagrangian must include the potential and kinetic energies. In the case of the massive scalar field, the Lagrangian density is

\[
\mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi(x) \right)^2 - \frac{1}{2} m^2 \left[ \phi(x) \right]^2
\]  

(29)

where the kinetic energy is contained in the first term which is dependent on its movement through space-time and the potential energy is in the second term which describes the energy needed for the field to exist in the vacuum at all.

2. Calculate the momentum density and the Hamiltonian density

The second step is to find the momentum density and use it to write the Hamiltonian density. According to Lancaster and Blundell, the momentum density is simply the four-vector version of the canonical momentum and is defined as

\[
\Pi^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)}.
\]  

(30)

Using this definition for the massive scalar field, the momentum density is calculated to be \( \partial^\mu \phi(x) \). The Hamiltonian density, like the momentum density, is calculated the same way as the Hamiltonian except in four-vector space.

\[
\mathcal{H} = \Pi^\mu(x) \partial_\mu \phi(x) - \mathcal{L}
\]

(31)

\[
\mathcal{H} = \frac{1}{2} |\partial_\mu \phi(x)|^2 + \frac{1}{2} |\nabla \phi(x)|^2 + \frac{1}{2} m^2 |\phi(x)|^2.
\]

(32)

The first term gives an expression for kinetic energy having to do with changes in the field over time whereas the second term is called a shear term and describes the energy from spatial changes in the field. The last term can be referred to as a mass term and describes the potential energy of the field simply existing.

3. Treat the fields and momentum density as operators and impose commutation relations

To put it simply, instead of the field being a function \( \phi \), it will now be an operator, \( \hat{\phi} \). Likewise, momentum, once referred to as \( \Pi \) will now be \( \hat{\Pi} \). These two take on the same commutation relation as \( \hat{x} \) and \( \hat{p} \) except in three dimensions.

\[
[\hat{\phi}(t, x), \hat{\Pi}^\mu(t, y)] = i\delta^{(3)}(x - y)
\]

(33)

This is the same concept as other commutation relations where the delta function means that the two operators commute unless the space coordinates are the same \( (x = y) \). Also, there are two types of each operator: a raising operator and a lowering operator. These will either raise or lower an excitation by one level at the state corresponding to the operator. In the case of the \( \hat{\phi} \), the states are in position space whereas the \( \hat{\Pi} \) states are in momentum space. Like the previous commutation relation, the raising and lowering versions of the same operator will commute at different spatial coordinates.

The transformation of the expressions into operators also causes the Hamiltonian to be transformed into operator \( \mathcal{H} \).

4. Rewrite the fields in terms of raising and lowering operators

The fourth step will take our new field \( \phi \) and write it in terms of creation and annihilation operators. As alluded to earlier,

\[
\hat{\phi} = \int \frac{d^3p}{(2\pi)^3(2E_p)^{1/2}} \left[ e^{i(kx + \omega t)} + e^{-i(kx + \omega t)} \right]
\]

(34)

Separating the time dependent part \( (e^{\pm i\omega t}) \) from the time independent part \( (e^{\pm ikx}) \), a more useful form of the field begins to emmerge.
\[ \hat{\phi} = \int \frac{d^3 p}{(2\pi)^{\frac{3}{2}} (2E_p)^{\frac{3}{2}}} \left[ (e^{i\omega t} e^{ikx} + e^{-i\omega t} e^{-ikx}) \right]. \] (35)

This can be described as two components of a basis set times a time dependent part. In this case the \(e^{ikx}\) and \(e^{-ikx}\) can be thought of as a raising and lowering operator while \(e^{\pm i\omega t}\) can be rewritten in terms of momentum to be \(e^{\pm ipx}\)

\[ \hat{\phi}(x) = \int \frac{d^3 p}{(2\pi)^{\frac{3}{2}} (2E_p)^{\frac{3}{2}}} \left[ \hat{a}_p e^{-ipx} + \hat{a}^\dagger_p e^{ipx} \right] \] (36)

which is exactly the prototype that was introduced in equation 23. This full quantum field solution can be recognized as the free particle solution with positive energy from quantum mechanics while incorporating the free particle solution with negative energies with particles going backwards in time which we interpret as antiparticles.

5. Apply normal ordering

Essentially, step four completes the machine however when using this field in a Hamiltonian, it will pick up a delta function which will give an infinite contribution to the energy. Normal ordering, which is simply rewriting the expression with all the raising or creation operators on the left and all the lowering or annihilation operators on the right, will allow us to rewrite this in a way that will remove the infinity without changing the rest of the field.
Feynman Calculus

Feynman calculus can be defined as the steps and procedures that give calculable theoretical results. The following sections follow Lancaster and Blundell [8] through the steps to get Feynman diagrams and numerical values for the amplitude \( \mathcal{M} \) which according to Fermi’s golden rule, shows how the initial states are connected to the final states via the interaction from the Lagrangian. Once it is understood that everything is in terms of relativistic quantum fields, the canonical quantization machine does not tell anything that isn’t already known. The beauty of QFT is its ability to explain and predict phenomena and particles in scenarios with interacting fields. Fermi’s golden rule states

\[
\text{rate of change} \propto |\mathcal{M}|^2 \times \text{(phase space)}. \tag{37}
\]

The phase space of an interaction is proportional to the size of the particle created as well as to the number of particles created. In other words, it is proportional to the entropy of the products of the interaction. Recalling Figure 3, if the two strawberries collided to make one watermelon, the entropy decreases and therefore the phase space is small. However, the actual products included an apple, a banana, a pear, a coconut and many small acorns. The entropy of these products is greater due to the larger number of smaller products. In this case, the phase space is much larger.

The amplitude \( \mathcal{M} \) is defined as

\[
|M| = |\langle q_1 q_2 \ldots q_n | \hat{S} | p_n \ldots p_2 p_1 \rangle| \tag{38}
\]

which is essentially the probability of multiple particles propagating from particle \( p_n \) to particle \( q_n \) (should there only be one particle \( p \) propagating to one particle \( q \), this amplitude would be denoted \( A \)). It is dictated by \( \hat{S} \) which stands for John Wheeler’s scattering matrix.

John Wheeler’s Scattering Matrix

The scattering matrix can be thought of as a matrix containing all of the interactions (or time-evolutions) that turn particles \( p_1, p_2, \ldots, p_n \) into particles \( q_1, q_2, \ldots, q_n \). A way to write this is

\[
\hat{S} = T \left[ e^{i \int_{-\infty}^{\infty} d^4x \hat{H}_I(x)} \right]. \tag{39}
\]

In this form the scattering matrix includes an integral of the Hamiltonian density \( \hat{H} \) which can be derived from the Lagrangian of the system. In a \( \phi^4 \) interaction, the Hamiltonian density would be written \( \frac{1}{4!} \phi(x)^4 \).

Unfortunately, this representation of the S-matrix is extremely complicated and almost impossible to calculate anything from. Since it is currently written in the form of an exponential, a Taylor Series expansion makes a very good approximation.

\[
\hat{S} = T \left[ 1 - i \int d^4 z \hat{H}_I(z) + \frac{(-i)^2}{2!} \int d^4 y d^4 w \hat{H}_I(y) \hat{H}_I(w) + \ldots \right] \tag{40}
\]

Later it will become apparent that each of these terms corresponds to an interaction. The first order terms are called tree level and contain no complexities whereas the higher ordered terms describe the same interactions but begin to introduce loops which play into the more precise measurements. These become quite overwhelming quite fast so only the zeroeth, first and second terms of the expansion have been included.

The T that precedes all this stands for time-ordered. This is something that comes from the definition of a Feynman propagator and must include both the retarded state (going forwards in time) and the advanced state (going backwards in time).

The Feynman Propagator

This propagator is a construct that describes how a particle propagates from point \( x^\mu \) to \( y^\mu \) in space time. As mentioned in the previous section, it is time ordered which means it includes the process going forwards in time as well as backwards.

This propagator is built from Green’s functions which are functions built to be the solution of a differential equation using an infinite sum of delta functions. Picture a pin art toy where the function is the shape of
your hand. The other side of the toy will display the shape of your hand as a sum of many little pins over a surface. The Green’s function is the equivalent of one of these pins. A complete solution will then be the integral of the Green’s function weighted by the original function.

One of the benefits of this is that it can be used to solve problems that are unsolvable without it. For example, one of the Green’s functions that has relevance in quantum mechanics is

$$G^+(x, y, E) = \sum_n \frac{i\phi_n(x)\phi_n(y)}{E - E_n}$$

(41)

which has poles or asymptotic behavior at energies where \(E = E_n\). In order to solve this, perturbation theory is used and Green’s functions play out very nicely in these cases. For instance, it helps to view this if the original Green’s function is written in the matrix-like form \((H - E)G = -1\) which means \(G = \frac{1}{E - H}\). Since perturbation theory is being used to find the full \(H\), an approximated \(H\) can be defined as the free, non-interacting part plus a small adjustment: \(H = H_0 + H_I\). Now the full Green’s function is written as

$$G = \frac{1}{E - H} = \frac{1}{E - H_0 - H_I}.$$  

(42)

There is a matrix identity that states that

$$\frac{1}{A+B} = \frac{1}{A} - \frac{1}{A}B\frac{1}{A} + \frac{1}{A}B\frac{1}{A}B\frac{1}{A} - \cdots$$  

(43)

By grouping \(E - H_0\) as \(A\) and \(H_I\) as \(B\), the Green’s function can be written as an expansion

$$G = \frac{1}{E - H_0 - H_I} = \frac{1}{E - H_0} - \frac{1}{E - H_0}H_I\frac{1}{E - H_0} + \frac{1}{E - H_0}H_I\frac{1}{E - H_0}H_I\frac{1}{E - H_0} - \cdots$$  

(44)

Although it was impossible to solve for the full Green’s function, by adding a free part and a perturbation, it could be written in terms of variables of known values. By defining \(\frac{1}{E - H_0}\) as \(G_0\) The final expression for \(G\) can be expressed as a picture.

Figure 10: The full Green’s function can be thought of as an expansion. As more \(H_I\) terms are added the contribution should decrease so the series converges on a value.

Another name for this full Green’s function is the Feynman propagator. As stated before, this is time ordered so the process is also possible in reverse (this makes the existence of antiparticle possible as well). These will be given values with the rules that accompany the corresponding Feynman diagrams.

**Wick’s Expansion**

Wick’s theorem must be applied to the expanded S-matrix. This creates understandable expressions which can be interpreted as interactions to be modeled by Feynman diagrams. Wick’s theorem goes as follows:
\[ T[\hat{A}\hat{B}\hat{C}...\hat{Z}] = N[\hat{A}\hat{B}\hat{C}...\hat{Z} + \text{all possible contractions of } \hat{A}\hat{B}\hat{C}...\hat{Z}] \]  

(45)

where each operator includes both a creation and annihilation operator \( \hat{A} = \hat{A} + \hat{A}^\dagger \). To get to this we must start with the vacuum expectation value (VEV) of a time ordered set. This can be written as

\[ \langle 0|T[\hat{A}\hat{B}\hat{C}...\hat{Z}]|0 \rangle. \]  

(46)

Although this may be difficult to analyze, a simpler, yet similar problem is the VEV of a normal ordered set

\[ \langle 0|N[\hat{A}\hat{B}\hat{C}...\hat{Z}]|0 \rangle. \]  

(47)

By definition, a normal ordered set has all of the \( \hat{A} \) on the right and all of the \( \hat{A}^\dagger \) on the left. By placing this in a vacuum, the annihilation operators will act on the ket to annihilate the vacuum and yield zero, while the creation operators act on the bra to do the same thing.

\[ \langle \psi|\hat{A}^\dagger = \hat{A}|\psi \rangle = 0 \]  

(48)

The next step is to consider a simple case of two operators \( \hat{A}\hat{B} \). To normal order the product, we must distribute each creation and annihilation operators and then normal order each piece.

\[ N[\hat{A}\hat{B}] = N[(\hat{A} + \hat{A}^\dagger)(\hat{B} + \hat{B}^\dagger)] \]  

(49)

\[ = N[\hat{A}\hat{B} + \hat{A}\hat{B}^\dagger + \hat{A}^\dagger\hat{B} + \hat{A}^\dagger\hat{B}^\dagger] \]  

(50)

\[ = \hat{A}\hat{B} + \hat{B}^\dagger\hat{A} + \hat{A}^\dagger\hat{B} + \hat{A}^\dagger\hat{B}^\dagger \]  

(51)

There is only one difference between the non-normal-ordered expansion and the normal ordered one. If we take the difference of the two, we get

\[ \hat{A}\hat{B} - N[\hat{A}\hat{B}] = \hat{A}\hat{B}^\dagger - \hat{B}^\dagger\hat{A} = [\hat{A}, \hat{B}^\dagger] \]  

(52)

which is a commutation relation. We can time order the product \( \hat{A}\hat{B} \) to get

\[ T[\hat{A}(x)\hat{B}(y)] = \begin{cases} \hat{A}(x)\hat{B}(y) & \text{if } x^0 > y^0, \\ \hat{B}(y)\hat{A}(x) & \text{if } x^0 < y^0. \end{cases} \]  

(53)

which means

\[ T[\hat{A}(x)\hat{B}(y)] - N[\hat{A}(x)\hat{B}(y)] = \begin{cases} [\hat{A}(x)\hat{B}^\dagger(y)] & \text{if } x^0 > y^0, \\ [\hat{B}(y)\hat{A}^\dagger(x)] & \text{if } x^0 < y^0. \end{cases} \]  

(54)

Because the VEV of any normal ordered set is zero, the normal ordered term in equation 54 evaluates to zero and drops out.

\[ T[\hat{A}(x)\hat{B}(y)] = \begin{cases} [\hat{A}(x)\hat{B}^\dagger(y)] & \text{if } x^0 > y^0, \\ [\hat{B}(y)\hat{A}^\dagger(x)] & \text{if } x^0 < y^0. \end{cases} \]  

(55)

This difference is denoted as a contraction

\[ \hat{A}\hat{B} = T[\hat{A}(x)\hat{B}(y)] - N[\hat{A}(x)\hat{B}(y)]. \]  

(56)

The contraction is a commutation which means the value of the contraction is a complex number. This allows us to insert it into a vacuum state and simplify it as a time ordered VEV.

\[ \hat{A}\hat{B} = \hat{A}\hat{B}|0\rangle = \langle 0|\hat{A}\hat{B}|0\rangle = \langle 0|T[\hat{A}\hat{B}]|0\rangle. \]  

(57)

We can now write the time ordered set in another way:
This is the simple version of Wick’s theorem, but the generalized theorem takes the form of equation 45. Keeping in mind that the VEV of any normal ordered operator set is zero, it can be concluded that if there are any free operators, the vacuum will be annihilated and the result will be zero. Using this conclusion, any odd-numbered set of operators will always evaluate to zero because a contraction can only connect two operators, always leaving one free operator. It has also been found that \( \phi^2 \) terms correspond to mass terms. These facts are why a \( \phi^4 \) interaction is the most basic interaction that we can find and why there are no odd-degree interactions.

This can be extended outside of simple, two operator sets to include for instance four operators. Wick’s theorem suggests that the VEV of a time ordered set will reduce to the sum of the products of the separate VEV’s of the individual contractions. For example

\[
\langle 0 | T [\hat{A} \hat{B} \hat{C} \hat{D}] | 0 \rangle = \langle 0 | T [\hat{A} \hat{B} \hat{C}] | 0 \rangle \langle 0 | T [\hat{D}] | 0 \rangle + \langle 0 | T [\hat{A}] | 0 \rangle \langle 0 | T [\hat{B}] | 0 \rangle \langle 0 | T [\hat{C}] | 0 \rangle \langle 0 | T [\hat{D}] | 0 \rangle
\]

where

\[
\langle 0 | T [\hat{A} \hat{B} \hat{C}] | 0 \rangle = \langle 0 | T [\hat{A}] | 0 \rangle \langle 0 | T [\hat{B}] | 0 \rangle \langle 0 | T [\hat{C}] | 0 \rangle
\]

\[
\langle 0 | T [\hat{A}] | 0 \rangle = \langle 0 | T [\hat{B}] | 0 \rangle \langle 0 | T [\hat{A}] | 0 \rangle \langle 0 | T [\hat{B}] | 0 \rangle
\]

\[
\langle 0 | T [\hat{A}] | 0 \rangle = \langle 0 | T [\hat{B}] | 0 \rangle \langle 0 | T [\hat{A}] | 0 \rangle \langle 0 | T [\hat{B}] | 0 \rangle
\]

Feynman Diagrams

Using Wick’s theorem with John Wheeler’s scattering matrix and the Dyson expansion, it is now possible to build Feynman diagrams that include all of the physics required to do more advanced calculations. Using the interaction Hamiltonian of a \( \phi^4 \) field \( \frac{x^4}{4!} \phi(x)^4 \), the expansion of the S-matrix is written as

\[
\hat{S} = T \left[ 1 - \frac{i\lambda}{4!} \int d^4 z \hat{\phi}(z)^4 + \frac{(-i)^2 \lambda^2}{2!4!} \int d^4 y d^4 w \hat{\phi}(y)^4 \hat{\phi}(w)^4 + ... \right].
\]

The terms in front of the integrals involving \( \lambda \) are just complex numbers and have no effect on the form of the diagrams. The important part of this expansion is what is inside the integrals. The goal of Wicks theorem, the Dyson expansion and Feynman diagrams is to solve a probability amplitude (either \( \mathcal{M} \) for multiple particles or \( \mathcal{A} \) for one particle) so this expansion is put into the expression for the amplitude and it causes a term expansion as well.

\[
\mathcal{A} = \langle q | \hat{S} | p \rangle = \langle 0 | \hat{a}_q \hat{S} \hat{a}^\dagger_q | 0 \rangle
\]

\[
= CT \left[ \langle 0 | \hat{a}_q \hat{a}^\dagger_p | 0 \rangle + \frac{-i\lambda}{4!} \langle 0 | \hat{a}_q \hat{\phi}(z)^4 \hat{a}^\dagger_p | 0 \rangle + \frac{(-i)^2 \lambda^2}{2!4!} \langle 0 | \hat{a}_q \hat{\phi}(y)^4 \hat{\phi}(w)^4 \hat{a}^\dagger_p | 0 \rangle + ... \right]
\]

In this equation, \( C \) represents the constant of \( C = (2\pi)^3 (2E_p)^\frac{1}{2} (2E_q)^\frac{1}{2} \) which comes from the relativistic normalization of states \( | p \rangle \) and \( | q \rangle \). From here, this long equation will be broken down into separate parts.

The first part is the first term, \( \langle 0 | \hat{a}_q \hat{a}^\dagger_p | 0 \rangle \) and can be considered the \( \mathcal{A}^{(0)} \) term. Using Wick’s theorem we can contract \( \hat{a}_q \) and \( \hat{a}^\dagger_p \) to show that this term does not go to zero but in fact forms a delta function \( \delta^3(q-p) \) which we can remember as a property of \( \langle q | p \rangle \). This makes logical sense because there is no interaction in this term, therefore the particle coming in as \( p \) cannot change before going out as \( q \). This can be represented by a line.

The second part of the expansion can be called the \( \mathcal{A}^{(1)} \) and is the first order interaction. We need to expand \( \phi(z)^4 = \hat{\phi}(z) \hat{\phi}(z) \hat{\phi}(z) \hat{\phi}(z) \) to use Wick’s theorem. Remembering that any sets of contractions that leave any operators not paired will evaluate to zero, we will skip right to any sets where all operators are contracted.
expressions including the second part of the expansion. This can now be written by contractions to get

$$\langle 0|\hat{a}_q\hat{\phi}(z)\hat{\phi}(z)\hat{\phi}^\dagger(z)\hat{a}^\dagger_p|0\rangle = \langle 0|T[\hat{a}_q\hat{\phi}(z)\hat{\phi}(z)]\hat{a}^\dagger_p|0\rangle + \langle 0|T[\hat{a}_q\hat{\phi}(z)\hat{\phi}(z)\hat{\phi}(z)]\hat{a}^\dagger_p|0\rangle + ... \quad (66)$$

To understand this, we must be careful because we are only contracting identical $\hat{\phi}(z)$’s, many of these expansions are identical. If you completely write out the full expansion you will conclude that there will be three identical expressions including the first part of the expansion in equation 66 and twelve identical expressions including the second part of the expansion. This can now be written by contractions to get

$$\langle 0|\hat{a}_q\hat{\phi}(z)\hat{\phi}(z)\hat{\phi}(z)\hat{a}^\dagger_p|0\rangle = 3\langle 0|\hat{a}_q\hat{a}^\dagger_p|0\rangle \langle 0|T[\hat{\phi}(z)\hat{\phi}(z)]|0\rangle \langle 0|T[\hat{\phi}(z)\hat{\phi}(z)]|0\rangle$$

$$+ 12\langle 0|\hat{a}_q\hat{\phi}(z)|0\rangle \langle 0|T[\hat{\phi}(z)\hat{\phi}(z)]|0\rangle \langle 0|\hat{\phi}(z)\hat{a}^\dagger_p|0\rangle \quad (67)$$

Contraction with a creation or annihilation operator are no longer time ordered leaving only the contractions with two $\phi$ terms to be time ordered. Since there are two added parts to this, we can expect two diagrams to complete the $A^{(1)}$ term. The $\langle 0|\hat{a}_q\hat{a}^\dagger_p|0\rangle$ form the $A^{(0)}$ term is equivalent to $\delta^{(3)}(q-p)$ so the contraction in the first half of the expansion will give a line. The second and third contractions in the first half are describing the $\hat{\phi}$ field interacting with itself at a point $z$. Since there are no creation or annihilation operators, there is no in or out particles. Now comes the second half of the expansion:

$$\langle 0|\hat{a}_q\hat{\phi}(z)|0\rangle \langle 0|T[\hat{\phi}(z)\hat{\phi}(z)]|0\rangle \langle 0|\hat{\phi}(z)\hat{a}^\dagger_p|0\rangle. \quad (68)$$

We can read this term piece by piece and since each contraction is multiplied, we can assume that each part is part of the same diagram. Working from right to left, the $\langle 0|\hat{\phi}(z)\hat{a}^\dagger_p|0\rangle$ can be interpreted as a particle $p$ coming in and interacting with the field $\hat{\phi}$ at point $z$ (in space-time coordinates). This is drawn as a line labeled $p$ coming into a point $z$. The next term, $\langle 0|T[\hat{\phi}(z)\hat{\phi}(z)]|0\rangle$, can be interpreted as the field $\hat{\phi}$ interacting with itself at point $z$ which is drawn as a loop beginning and ending at point $z$. The last term, $\langle 0|\hat{\phi}(z)\hat{a}^\dagger_p|0\rangle$, can be thought of as an interaction with the $\hat{\phi}$ field at point $z$ where a particle $q$ is created and leaves. This is drawn as a line coming out of a point $z$ labeled $q$.

Figure 12: The full $A^{(1)}$ term of the $\phi^4$ interaction is a sum of three diagrams. The first half of the expansion yielded a diagram of the delta function (Figure 12(a)) with a field interacting with itself twice at point $z$ (Figure 12(b)). The second half of the expansion yielded a diagram with $p$ propagating to $q$ at point $z$ including an interaction with the $\hat{\phi}$ field as well as the field interacting with itself (Figure 12(c)).

The third part is going to take much more effort to work through due to their being two $H_I$’s. This means there will be two points of interaction and the possibilities increase rapidly. To start, each $\phi^4$ term must be written separately as in the $A^{(1)}$ term. This must then be contracted using Wick’s theorem.

$$A^{(2)} = \langle 0|\hat{a}_q\hat{\phi}(y)\hat{\phi}(y)\hat{\phi}(y)\hat{\phi}(w)\hat{\phi}(w)\hat{\phi}(w)\hat{a}^\dagger_p|0\rangle \quad (69)$$
This will result in a similar set of contractions, however now that there are two spatial coordinates, there will be two vertices in each diagram. In total there will be six new diagrams that will have more internal lines, but will end in the same manner of a particle p entering and particle q exiting.

We have now created Feynman diagrams from the math using Wick’s theorem. Given all of this it is important to point out that it is all dependent on \( H_f \) so it is important to get the Lagrangian correct. For this reason, the Lagrangian and Hamiltonian for jet collisions will need to include both QED and QCD terms. There are now specific rules that apply to these diagrams to produce numbers and integrals that will allow us to calculate probability amplitudes for measurements like particle lifetimes and cross sections.

**Renormalization**

Now that there is an easy way to produce the numbers and amplitudes we need to do calculations, it is important to point out some key notes.

Currently, the method of Feynman diagrams is the easiest way to calculate the values needed, however there is a flaw in the interaction itself. The interaction \( (\phi^4) \) is based off of a given Lagrangian that is specific to the force that is mediating the interaction. If left by itself, many attempts to solve the diagrams using Feynman calculus will result in answers of infinity. In order to prevent this, Griffiths states that the Lagrangian must be re-normalized. This is done by adding terms to the Lagrangian that correspond to the terms that are already present. For instance if the Lagrangian is

\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \phi(x))^2 - \frac{1}{2} m^2 \phi(x)^2 - \frac{i\lambda}{4!} \phi(x)^4,
\]

(70)
terms corresponding to \( (\partial_\mu \phi(x))^2, \phi(x)^2 \) and \( \phi(x)^4 \) would be added to create a new Lagrangian:

\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \phi(x))^2 - \frac{1}{2} m^2 \phi(x)^2 - \frac{i\lambda}{4!} \phi(x)^4 + A(\partial_\mu \phi(x))^2 + B\phi(x)^2 + C\phi(x)^4.
\]

(71)

By doing this, the infinities that are encountered will be paired with negative infinities and they will cancel out leaving only finite terms that stand for observable properties. For example in this case the like terms can be combined to give

\[
B + m^2 \rightarrow m_{\text{renormalized}}
\]

(72)

where the original mass term is combined with the gauge field to give a new, renormalized mass term which yields observable and finite values. The same thing can go for the coupling (or interaction) strength term \( C \):

\[
C + \lambda \rightarrow \lambda_{\text{renormalized}}
\]

(73)

where the gauge field corresponding to the interaction term combines with the original interaction term to give a new renormalized coupling strength that is observable in nature.

In summary, the original form of the Lagrangian never included an absolute knowledge of mass or interacting strength so when trying to make it apply to our coordinate system, the diagrams would result in infinities. Renormalization essentially amounts to finding the actual coordinate system that reflects the world we live in and then removes the problems by having infinities cancel each other, leaving finite numbers. However obscure this may seem, computationally it is one of the best models humans have ever developed.

For example, in a great American Scientist article by Brian Hayes, this point is triumphantly celebrated. Here Hayes details the current theoretical limit of the gyromagnetic ratio of the electron \( (g_e) \) derived with near 1000 Feynman diagrams. He points to the latest experimental data which measures \( g_e \) to be \( 2.0023193043718 \pm 0.0000000000075 \) is the single most precise experimental measurement and yet is still consistent with the theoretical prediction [10]. If any doubt that the cancellation of infinities by renormalization is not on to something good, this result removes that notion.
Finding the QED $\mathcal{H}_I$ from Symmetry and Unification

At this point Feynman calculus has been developed to the point where it can compute cross sections given an $\mathcal{H}_I$. Up until now it has used the $\phi^4$ interaction with is used for introducing Feynman calculus. In order to calculate the cross section of $e^- + e^+ \rightarrow \gamma \rightarrow q + \bar{q}$, we must find the QED $\mathcal{H}_I$. Following Lancaster and Blundell [8], we will find the expression for $\mathcal{H}_I$ using more encompassing ideas that are fundamental to QFT and the Standard Model including symmetry and unification.

A significant piece to this whole puzzle is the concept of symmetry how it applies to the Lagrangian of a quantum field. In physics there are four forces: Electromagnetic, weak force, strong force and gravity. The first three make up the Standard Model whereas gravity, the one that we see occur every day does not fit. It is also interesting to point out that the standard model includes all three of these forces in one unified Lagrangian that includes every particle and its mass as well as every interaction that could possibly occur between such particles.

In order to build a theory, one must start with a field. This can technically be any field as shown in the math leading up to this section. Certain assumptions must be made to start off, for instance in a massive scalar field, one must assume there is a scalar field with mass. This however is unsatisfying because in essence, the result of assuming a scalar field with mass is a Lagrangian that includes the properties of a scalar field and a mass term. It says nothing new.

There is however much more that goes into the building of a theory including making all physical theories locally symmetric (gauge theory).

Gauge Invariance and Spontaneous Broken Symmetry

The first thing to look at is the invariance of a Lagrangian. QFT is a theory built to preserve special relativity which claims that the universe is symmetric. In other words, there is no absolute inertial frame of reference which is still while all other frames are moving. This can be expanded to say there are no absolute positions, angles times or even phases. Up until now the first three of these have been preserved, but not the phase. In order to maintain the local symmetry of the field, the phase invariance must be factored into the Lagrangian. This is done using gauge fields which add a factor to the field which ensures that

$$L(\partial_\mu \phi) = L(\partial_\mu e^{i\alpha} \phi).$$

(74)

If this were not true, then the Lagrangians built from the field and the interactions built from those would be dependent on an absolute phase. By doing this, a new gauge field $\hat{A}^2$ is created, but it is massless.

The only way to get mass however is to include spontaneous broken symmetry. This says that at one point before $10^{-12}$, the universe was globally symmetric but when the universe cooled to below its critical temperature around $10^{16}$ K, symmetry was broken and a vacuum state was created and particles gained mass. Imagine the universe as a particle sitting at the top of a sombrero shaped potential curve. While the universe was globally symmetric, everything was the same and there was no mass. The ball is in an unstable equilibrium where nothing can be approximated because the higher ordered terms would not converge to a number but would instead go to infinity. Once the energy from the temperature of the ball falls to below the potential curve, the particle drops to a new equilibrium that is locally symmetric, but there is no way of telling where the particle will drop.

To relate this to the gauge fields, Noether’s theorem states that whenever there is a break in symmetry, a field is created. This field represents something called a Goldstone boson. Simplifying this process tremendously, by combining the Goldstone boson from spontaneous broken symmetry with the gauge field from gauge invariance, many physicists say that the gauge field eats the Goldstone boson and becomes $m^2 \hat{A}^2$ which shows that it acquires mass.

The result of this process is a massive field that is manifestly Lorentz invariant. This can then be followed to get the mass of the Higgs boson and the coupling strength of the Higgs mechanism. The same can be said for all fields including QED, GWS and QCD.
Applying Symmetries

As was stated earlier, special relativity implies local symmetry. However, like many other things, there are levels of symmetry where each progressive level implies something new and therefore causes something new.

The first level of symmetry that is applied to the model is called a U(1) symmetry (where U stands for unitary) and it represents the Abelian symmetry of phase as was discussed in the previous section. The symmetry can be described mathematically as

$$\Psi \rightarrow \Psi e^{i\alpha(x)}$$

and is true for both fermions and bosons and gives rise to a term $F_{\mu\nu}$ which is involved in QED.

The second level is called an SU(2) symmetry (S stands for special) and it is non-Abelian. This is only for fermions and comes from the iso-spin property of the proton, neutron, electron and neutrino. The symmetry can be written as

$$\Psi \rightarrow \Psi e^{\frac{i}{2}\tau^\alpha(x)}$$

and produces a term $G_{\mu\nu}$ which is involved in the weak force.

The third symmetry is an SO(3) symmetry (O stands for orthogonal) and is also non-Abelian. This symmetry is written

$$\Phi \rightarrow \Phi - \theta(x) \times \Phi$$

which will give rise to terms used in the strong force used in quantum chromodynamics (QCD).

Unification: A Final Product

As many feel is necessary, physics must be unified in one single theory of everything. Although this has not been achieved yet, steps towards it have been made through Unification (the unifying of QED and the weak force into electroweak theory (GWS)) by demanding both U(1) and SU(2) symmetries. By beginning with the assumption of the presence of a massless $e^-$ field and a massless $\nu$ field, demanding local symmetries and then adding the Higgs field and interaction, the Lagrangian for the Electroweak theory becomes

$$\mathcal{L} = i\bar{e} \gamma^\mu \partial_\mu e + i\bar{\nu} e \gamma^\mu \partial_\mu \nu - g \sin \theta_W \bar{e} \gamma^\mu e A_\mu$$

$$+ \frac{g}{\cos \theta_W} \left( \sin^2 \theta_W \bar{e} R \gamma^\mu e_R - \frac{1}{2} \cos (2\theta_W) \bar{e} L \gamma^\mu e_L + \frac{1}{2} \bar{\nu} e \gamma^\mu \nu e \right) Z_\mu$$

$$+ \frac{g}{\sqrt{2}} \left[ \bar{\nu} e \gamma^\mu L W^\mu - \bar{e} L \gamma^\mu \nu e W^\mu \right].$$

In this Lagrangian, all of the particles and their proper masses as well as their coupling strengths during interactions can be found. This includes an $e^-$ and $e^+$ with mass, a $\nu$ and $\bar{\nu}$ with no mass, a $\gamma$ with no mass, and a $Z^0$, $W^+$ and $W^-$ with mass as well as how they all interact with each other. Still unconnected was the Lagrangian for QCD:

$$\mathcal{L}_{QCD} = \bar{\psi}(i\gamma_\mu D^\mu - m)\psi - \frac{1}{2} \text{Tr}(G_{\mu\nu}G^{\mu\nu})$$.

This was the first step. The unification effort then continued with Grand Unification (the unifying of Electroweak theory and QCD) by demanding the SO(3) symmetry. This is currently the best that physicists have been able to accomplish. Unfortunately, gravity has not fit into any of these efforts.

Making This Useful: Finally Finding $\mathcal{H}_I$ for QED Interactions

It is important to point out that in equation 78 the QED coupling strength term is $-g \sin \theta_W \bar{e} \gamma^\mu e A_\mu$ where $A_\mu$ is the electromagnetic field term that mediates the electromagnetic force between the electrons and the quarks via a photon. This will form the $\mathcal{H}_I$ that is needed to calculate John Wheeler’s scattering matrix, which through Wick’s theorem produces Feynman diagrams and the rules to interpret them. From this point, Griffiths begins and the Feynman calculus is calculating $|M|$ to fill into Fermi’s golden rule.
A Jet Calculation

In working with the bigger ideas in QFT including symmetry and unification, we have arrived at the $\mathcal{H}_I$ we need to complete the jet calculation. Although the collisions at CERN were proton-proton collisions, an $e^-e^+$ scattering provides much more straightforward system to do analytical calculations of jet events. The electron is fundamental and involves only the electron field whereas the proton is a composite system made of three quarks and therefore would require even more field theoretical tools.

![Feynman Diagram](image)

Figure 13: This is a Feynman diagram of an $e^-e^+$ collision that produces a photon, decays into quarks and will eventually produce jets. Positive time is going right. The outgoing quarks will combine into pairs and triplets to create color neutral particles through a process known as hadronization. The particles with similar angles of projection are grouped together to form jets.

The remaining sections follow Griffiths treatment explicitly [5]. The first step to calculating the cross section is to focus on the initial interaction. In this case, it is $e^- + e^+ \rightarrow \gamma \rightarrow q + \bar{q}$. Fermi's golden rule is now applied to this interaction.

Calculating $\mathcal{M}$ Using the Feynman Rules

The first part of Fermi’s golden rule is the amplitude $|\mathcal{M}|$. Since this first interaction is basic QED, those Feynman rule’s are applied to the diagram.

The first rule of labeling states that all external lines are labeled with momentum $p_n$ as well as spin $s_n$. Internal lines are labeled with momentum $q_n$ (since there is only one internal line, its momentum is labeled $q$). Arrows are also added to discern particle or antiparticle and to preserve the flow of the diagram.

The second rule deals with the contribution of each external line. Griffiths states this specifically for electron and positron, but when comparing this to quarks and antiquarks, the same general rule applies:
Figure 14: The tree level diagram of the initial interaction of the collision. All of the lines and particles have been labeled as demanded by the first Feynman rule.

Particles:
\[
\begin{aligned}
\text{Incoming:} & \quad u \\
\text{Outgoing:} & \quad \bar{u}
\end{aligned}
\]

Antiparticles:
\[
\begin{aligned}
\text{Incoming:} & \quad \bar{\nu} \\
\text{Outgoing:} & \quad \nu
\end{aligned}
\]

The third rule states that for every vertex, a factor of \( ig_\gamma \gamma^\mu \) is added to the expression, where \( g_\gamma \) is called the coupling constant and is related to the charge of the positron. This must also be expanded to add a \( Q \) which stands for quark charge at the vertex where a quark/antiquark is created.

The fourth rule regards propagators (internal lines). Since there is only a photon in this case, the factor that is added is \( -ig_{\mu \nu} q^2 \).

Conserving momentum at each vertex, the fifth rule calls for a delta function of the form \( (2\pi)^4 \delta^4(k_1 + k_2 + k_3) \) for every vertex.

The sixth rule will integrate over all internal momenta, so in this case only over the momentum \( q \) of the photon. This will give a factor of \( \frac{d^4q}{(2\pi)^4} \).

Last but not least, the delta functions are canceled. Before this step, there will be a large delta function corresponding to the overall conservation of momentum of the entire interaction. By removing it (this is possible because the overall momentum will be conserved), what is left is equal to \( iM \). In this case,

\[
M = \frac{Qg_\gamma^2}{(p_1 + p_2)^2} [\bar{\nu}(p_2)\gamma^\mu u(p_1)] [\bar{u}(p_4)\gamma^\mu \nu(p_3)]
\]  

where the \( p_n \)'s are just there to show which factor corresponds with which external momentum.

**Casmir’s Trick**

From here, the spin of the particles must be accounted for to make it an accurate problem. Most often however, experiments do not care about spin and use a beam of particles with random spin. To compute this, the average of all spins would give \( \langle |M|^2 \rangle \) and is actually much easier to calculate than the individual amplitudes.

Because QED involves fermions, there are spinors in the value for \( M \) which can be hard to deal with. Casmir’s trick (named because Casmir was the first to use it) is a process that removes the spinors and simplifies them into a matrix multiplication and taking the trace (sum of all diagonal terms) of the resulting matrix.

In the case of \( e^- + e^+ \rightarrow \gamma \rightarrow q + \bar{q} \), the \( M \) was calculated to be equation 80. Since \( M \) is a probability density, the useful value will be \( |M|^2 \), which would look like

\[
|M|^2 \propto [\bar{\nu}(p_2)\gamma^\mu u(p_1)] [\bar{u}(p_4)\gamma^\mu \nu(p_3)] [\bar{\nu}(p_2)\gamma^\mu u(p_1)]^* [\bar{u}(p_4)\gamma^\mu \nu(p_3)]^*.
\]  

In short, Casmir’s trick will rewrite this as the product of two trace expressions
where \( m \) is the mass of the electron and \( M \) is the mass of the quark.

There are trace theorems that can help with problems like this. Griffiths goes through many of them, but the ones important for this problem are

1. \( \text{tr}[A + B] = \text{tr}[A] + \text{tr}[B] \)
2. The trace of the product of an odd number of \( \gamma \) matrices is zero
3. \( \text{tr}[\gamma^\mu \gamma^\nu] = 4g^{\mu\nu} \)
4. \( \text{tr}[\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma] = 4(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}). \)

When using these rules it must be remembered that \( p_n = \gamma^\mu(p_n) \) so every \( p_n \) also counts as a gamma matrix. It is also important to note that when the superscripts of \( g \) (i.e. \( g^{\lambda\sigma} \)) both refer to a momentum \( \gamma \) matrix, it is written as the dot product of the two.

Using these trace theorems, \(|\mathcal{M}|^2\) can be written as

\[
|\mathcal{M}|^2 = 8 \left[ \frac{Qg^2}{(p_1 + p_2)^2} \right]^2 \left[ (p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) + (mc^2)(p_3 \cdot p_4) + (Mc^2)(p_1 \cdot p_2) + 2(mc^2)(Mc^2) \right].
\]  \tag{83}

Because of the presence of dot products, it is possible to express this in a more useful way by writing it in terms of the incident electron’s energy \((E)\) and the angle \( \theta \) between the outgoing quark and incident electron.

\[
|\mathcal{M}|^2 = Q^2g^4 \left\{ 1 + \frac{(mc^2)^2}{E^2} + \frac{(Mc^2)^2}{E^2} + \left[ 1 - \left( \frac{mc^2}{E} \right)^2 \right]^2 \right\} \cos \theta
\]  \tag{84}

This expression can then be used in other equations to calculate things like decay rate and cross sections.

**Calculating Scattering Cross Sections**

Keeping with Griffiths’ treatment, the rate of change of the cross section is given by

\[
\frac{d\sigma}{d\Omega} = \left( \frac{\hbar c}{8\pi} \right)^2 \frac{S|\mathcal{M}|^2}{|p_f|} \frac{|p_i|}{(E_1 + E_2)^2}
\]  \tag{85}

where \( d\Omega = \sin \theta d\theta d\phi \) and \( S \) is a statistical factor that goes like \( \frac{1}{j} \) for each group of \( j \) identical particles in the final state. By plugging in the newly calculated value of \(|\mathcal{M}|^2\) and then integrating over \( \theta \) and \( \phi \), an expression for the total cross section is calculated.

\[
\sigma = \frac{\pi Q^2}{3} \left( \frac{\hbar c \alpha}{E} \right)^2 \sqrt{1 + \frac{(mc^2)^2}{E^2}} \left[ 1 + \frac{1}{2} \left( \frac{mc^2}{E} \right)^2 \right] \left[ 1 + \frac{1}{2} \left( \frac{Mc^2}{E} \right)^2 \right] \]
\]  \tag{86}

This equation can be summed up by pointing out that until the energy of the incident electron surpasses the \( Mc^2 \), the energy of the outgoing quark, the square root will cause these cross sections to be either zero or imaginary, showing that it is impossible for this to occur until you have enough energy to create the quark-antiquark pair. Once \( E \) has far surpassed \( Mc^2 \) the cross section goes like

\[
\sigma = \frac{\pi}{3} \left( \frac{Q\hbar c \alpha}{E} \right)^2.
\]  \tag{87}
Figure 15: As predicted by the theory, the horizontal lines show a step-like plot of the ratio where a step is made as the E of the incident electron increases into the range of a new quark. The first line shows the ratio of about 2 when in the range of only u, d and s quarks. The second line shows a ratio of about $\frac{10}{3}$ in the range between the c and b quarks. The third line is the ratio of $\frac{11}{3}$ for E's above the energy of a b quark. The t quark is not included because it was not discovered until 11 years after Griffiths published the book.

This can then be compared to other things, for example the cross section of this collision in making $\mu^- - \mu^+$ pairs. Based on the energy of the collision, more quarks will contribute to the ratio by a factor of

$$R(E) = 3 \sum Q_i^2$$

where the 3 stands for the three colors of quarks and Q stands for the electromagnetic charge of each individual quark.

The final result given by equations 87 through 88, represent the electron scattering into a quark and an antiquark using QED $\mathcal{H}_I$. In addition, it was assumed that quarks come in three color charges (equation 88). These equations are then compared, again as done in Griffiths, to Figure 15 [5] and show excellent agreement with experiment.

This experimental agreement to quarks with color charge suggests that color charged quarks are the sources for the jets. The next step to take would be to consider the QCD $\mathcal{H}_I$ to compute explicit jet phenomena to compare to real experimental.
Conclusion

The purpose of this thesis was to explain some of the theory behind the high energy physics experiments that were being conducted at CERN as well as to follow a jet calculation using Feynman calculus. Beginning with a description of the work I had done at CERN, we were introduced to the concepts of relativistic jet phenomena. To explain this, we explored classical fields and the first quantization of quantum mechanics in which all particles acted like waves. We found that this was inadequate to describe phenomena like particle jets so we discussed the second quantization in which all waves act like particles and introduced relativistic quantum field theory. Our next step was to investigate the theory that is used to build easy to understand Feynman diagrams that model particle interactions including the Feynman propagator, John Wheeler’s scattering matrix and Wick’s theorem. In order to do these calculations for the first part of the jet collision, we needed to find $\mathcal{H}_I$ for QED. This was done by applying local symmetries, gauge invariance, spontaneous broken symmetry and interaction with the Higgs field to give the full Lagrangian of the Electroweak theory and pulling out the QED interaction term. Finally, we used Feynman calculus and rules of QED to follow Griffiths through a calculation regarding an electron-positron collision that forms particle jets.
Bibliography


