5-13-2014

Dynamics of Climate Change: Explaining Glacier Retreat Mathematically

Robert Guillette

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Dynamics of Climate Change: Explaining Glacier Retreat Mathematically

Robert Guillette

Submitted in Partial Completion of the
Requirements for Commonwealth Honors in Mathematics

Bridgewater State University

May 13, 2014

Dr. Irina Seceleanu, Thesis Director
Dr. Kevin Rion, Committee Member
Dr. Mahmoud El-Hashash, Committee Member
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1 Glacier Retreat

1.1 Introduction to Glacier Retreat

Glaciers currently cover 10% of the Earth’s land, with glacial ice blanketing over 15 million km$^2$ of the planet. However, as the world’s climate continues to undergo monumental shifts, these mobile masses of ice are being dramatically affected.

![Mountain Glacier Changes Since 1970](image)

**Figure 1: Mountain glacier changes since 1970.**

Glaciers across the globe have been increasingly losing mass over the last century. Figure 1 shows the average yearly thinning rate since 1970 of mountain glaciers around the world. The colors yellow and red represent thinning of glacier ice, while blue represents thickening. From the map we can observe that there has been considerable melting of glaciers occurring in areas such as northwest North America, Greenland, the Andes, Alps, Himalayas, and New Zealand. More specifically, Carrara & McGimsey (1981) report that by 1980 the Glacier National park in Montana had lost over two thirds of its estimated 150 glaciers in 1850, while the remaining glaciers have suffered a significant reduction in area. The Muir glacier, situated in Glacier Bay National Park, Alaska, has also undergone a dramatic retreat between the years 1941 and 2004 as we can observe from figure 2. The figure visually compares the Muir glacier in 1941 to its state in 2004. Field & Molnia report that the glacier retreated more than 12 km and thinned by more

![Evolution of Muir glacier.](image)

**Figure 2: Evolution of Muir glacier.**
than 800 meters in this time frame. Moreover, between 1919 and 2006, 17 of the 523 glaciers in the central and southern Canadian Rocky Mountains disappeared, while glacier cover for the area decreased by 520 – 660 km², which represents 35 – 45% of the area’s total glacier cover (Tennant, 2012).

Glaciers are extremely sensitive to changes in climate, and so there are many studies that link temperature, precipitation, and insolation to glacier retreat (March & O’Neel, 2011; Peduzzi, Herold, & Silverio, 2010; Robson, 2012). The sensitivity glaciers exhibit toward climatic factors make these mobile ice masses “excellent barometers of climate change” (Hall & Fagre, 2003). Therefore, studying the relationship between the retreat of glaciers and changes in climate is of utmost importance, as glacier retreat signals a shifting climate.

1.2 Consequences of Glacier Retreat

The increasing rate of the retreat of glaciers worldwide is a serious issue, as negative repercussions for both humans and animals follow from the rapid disappearance of glaciers. Of the world’s population, fifty percent of people are indirectly supported by glaciers, while ten percent are completely dependent on them for life support (Beniston, Bradley, & Diaz, 1997). The main resource glaciers provide to people is that of a water supply, which is created from melted snow and ice runoff. This water supply is used for drinking water, as well as for irrigation purposes. The runoff of melted snow and ice is also harnessed as a creator of hydroelectric power, a renewable source of energy which accounted for about 16.1% of global electricity consumption in 2010 (“Use and Capacity of Global Hydropower”, n.d.). As glaciers provide important resources critical to sustaining life, the rapid disappearance of these ice masses threatens the very lives they support. For example, the habitat of snow leopards is closely linked with glaciers. The area in which this now endangered species lives lies between the tree and snow line of a mountain. As glaciers retreat, the snow leopard is forced to move up the mountain where there is less food, causing a life-threatening problem for the animal. Alteration of delicate ecological systems and loss of habitat for numerous species, severe reduction of water supplies for irrigation and drinking supplies, loss of hydroelectric power sources, and rising sea levels are major problems that await us if glaciers continue to melt at a staggering pace.

1.3 Literature Review

Numerous studies have been performed to study the connection between changes in climatic factors and glacier retreat.

Letreguilly (1988) analyzes the relationship between the mass balance (annual net loss/gain of snow/ice of a glacier) of three western Canadian glaciers and temperature and precipitation. The three glaciers studied were Peyto, Place and Sentinel. A multiple linear regression analysis using data from 1966 – 1984 of mass balance on summer temperature and winter precipitation was performed for Place and Sentinel. The two climate variables explained
67.24% \((r = 0.82)\) and 79.21% \((r = 0.89)\) of the variation in the mass balance of Place and Sentinel glaciers, respectively. Regression equations were also used to reconstruct the past mass balance records for each of the three glaciers.

Anderson et al. (2006) created a model to study past changes and predict future changes in the mass balance of Franz Josef Glacier, located in New Zealand. Specifically, a degree-day model which allows for the calculation of accumulation, ablation, and net mass balance from climate variables was constructed. Using 111 years of data (1984 – 2005), the correlation between the modeled mass balance of Franz Josef Glacier and annual, winter, and summer temperature and precipitation was measured. Summer temperature was found to be strongly correlated with mass balance \((r = 0.88)\), whereas the annual precipitation correlated better with mass balance \((r = 0.41)\) than either of the seasonal amounts of precipitation. Future mass balance was then predicted until the year 2100 using projected changes in global mean annual temperature, found in a report from the Intergovernmental Panel on Climate Change, in the degree-day model.

Bitz & Battisti (1999) study the relationship between the mass balance of six glaciers along the northwestern coast of North America and temperature, precipitation, and local climatic phenomena. In particular, the relationship between the mass balance of the South Cascade Glacier, located in Washington, and two local climatic phenomena, the El Nino Southern Oscillation (ENSO) and the decadal ENSO-like variability, is examined. A multiple linear regression analysis of mass balance of South Cascade glacier on the Cold Tongue (CT) index and the Global Residual (GR) index (indexes used as a measure of the El Nino Southern Oscillation (ENSO) and the decadal ENSO-like variability, respectively) explains 40% of the variation in net winter mass balance and 36% of the variation in net annual mass balance.

Lefauconnier & Hagen (1990) analyze the connection between multiple climatic factors and the mass balance of Broggerbreen glacier located in Svalbard. Using 19 years of data, a strong correlation \((r = 0.90)\) was found between mass balance and winter precipitation and positive summer and autumn temperatures. When including summer insolation in the regression and using 8 years of data, a stronger correlation \((r = 0.98)\) was found between mass balance and the three climatic factors.

Hoffman et al. (2007) use a regression equation to estimate 55 years of cumulative mass balance data for Andrews glacier. The link between the mass balance of this Rocky Mountain glacier and temperature and precipitation is also examined. Using data from 1957 – 1964, the mass balance of the glacier was discovered to be highly correlated with May – October temperatures \((r = 0.93)\) and April – June precipitation \((r = 0.79)\).

From these studies, we learn that a considerable amount of variation in the mass balance of a glacier is due to a variety of climatic factors, with temperature and precipitation being the most correlated variables with the changes in ice mass.
1.4 Description of Research Question

As the earth’s climate is currently undergoing significant changes, glaciers around the world are retreating at a staggering pace. This rapid melting of glacial ice will have serious consequences that impact people and animals on a global scale. In order to mitigate the severity of these consequences, we must first better understand the relationship between climate change and glacier retreat. In this thesis, we construct a mathematical model of glacier retreat representing how changes in climatic factors, such as temperature and precipitation, affect the ice mass of a glacier. We perform a multiple linear regression using data for the Midtfonna glacier, located in Norway, to study the effects of temperature, precipitation, local climatic phenomena, wind speed, and insolation on the glacier’s total area. Our goal is to determine what proportion of the total variation in the glacier’s ice mass over the past decades can be explained by the five climatic factors.

Our second objective for this thesis is to create a method to predict the evolution of a glacier over time by using different climate scenarios projecting future temperature and precipitation. Given that one-sixth of the world’s population depends on glacier ice and snow melt for its water supply, a mathematical model predicting the future of glaciers can help people adapt to the realities of a changing climate. To predict the future of a glacier, we perform a multiple regression using available data for glacier area, temperature, and precipitation to obtain a prediction equation. We then use this equation to extrapolate past data to predict the area of the glacier based on future values for temperature and precipitation. We apply this method to predict the future of the Midtfonna glacier. Using two scenarios for projected changes in temperature and precipitation over the course of the next century from the Intergovernmental Panel on Climate Change’s 2013 report on climate change (IPCC, 2013), we find the estimated year for when the Midtfonna glacier will disappear under each scenario.

1.4.1 Quantifying Glacier Retreat

The focus of this thesis is to study changes in a glacier’s ice mass, and so we now turn to analyzing different ways to quantify the retreat of a glacier. The size of a glacier can be measured and expressed using a variety of methods, each differing in the frequency and cost of measurements. The four types of measurements we will introduce in this section are: total glacier volume, total glacier area, terminus point fluctuation, and mass balance. Total glacier volume ($km^3$) and total glacier area ($km^2$) both provide an overall view of the glacier but require the combined use of satellite photographs and measurements on the ground to calculate. Since it is very costly to make these measurements on a consistent basis, total glacier volume and total glacier area are usually not systematically available for many glaciers across the globe. Moreover, given that these two measurements offer an overall picture of the glacier, they can be used in prediction models to study the future evolution of the glacier and to calculate when the glacier will disappear.
Another measurement for glacier retreat is found by tracking the lowest point of a glacier over time to measure the so called *terminus point fluctuation*. Finally, we can also measure the annual variation of a glacier’s mass by subtracting the amount of melted snow/ice in the summer from the accumulated snow/ice in the winter. This measurement is called *mass balance* and constitutes the object of many studies in the literature on glacier retreat as this data is readily available for many glaciers. Calculating the mass balance of a glacier is easier than calculating a glacier’s total area since only a person on the ground with some basic equipment is necessary to measure the depth of snow/ice of a glacier, whereas a combination of satellite photos and ground measurements are required to find total area. However, mass balance only provides a yearly account of a glacier’s snow/ice budget, whereas the total area gives an overall measurement of a glacier and can be used to predict a glacier’s future.

1.4.2 Quantifying Climatic Factors

Numerous climatic factors contribute to the retreat of glaciers, in particular temperature, precipitation, local climatic phenomena, wind speed, and insolation play a significant role in the melting of glaciers. These five climatic factors can be quantified in various ways. We can compute an annual, seasonal, or monthly average for temperature and precipitation. In addition, we can also consider the average of positive summer and autumn temperatures or the total number of days that the temperature is above 0°C, as above freezing temperatures lead to snow and ice melt. Moreover, these averages can be a moving or a regular average. Since the thickness of glacial ice helps determine the rate at which it melts, and snow takes time to compress into ice, the data might show a delay in the effects of temperature and precipitation on a glacier. Thus, in our analysis we can account for a delay in the effects of temperature and precipitation on the ice mass of the glacier, because changes in the ice mass of a glacier due to temperature or precipitation may take a few years until they are reflected in the glacier data.

Local climatic phenomena are cyclical weather patterns capable of causing significant changes to climatic factors such as temperature and precipitation. Examples of local climatic phenomena are: North Atlantic Oscillation, North Pacific Oscillation, Atlantic Multidecadal Oscillation, Pacific Decadal Oscillation, El Niño/La Niña, and monsoons. To measure a local climatic phenomenon, an index value representing the strength of the particular cyclical weather system is used. Depending on the location of the glacier the model is being applied to, there may or may not be a local climatic phenomenon that influences the climate near the glacier.

Wind speed can also be quantified in a multitude of ways. We will focus on the *highest mean wind value*, which represents the highest mean of wind speed over a ten minute interval. An annual, seasonal, or monthly average can be computed for wind speed. Insolation can be measured by a few different instruments, each responsible for collecting a specific type of solar radiation. The measurement of insolation that we will focus on is the *global horizontal irradiance*, which represents the total amount of radiation received from the sun as measured on a surface parallel to the ground. An annual, seasonal, or monthly average can be computed for
insolation. In the explanation of our regression model in chapter 3, we make recommendations about how to best quantify these climatic factors for use in the model.

In chapter 2, we present an overview of the statistical tools used for creating our model of glacier retreat. In particular, we introduce simple and multiple linear regression. Moreover, we explain how to perform a hypothesis test, which is used to check whether or not there exists a linear relationship between the random variable and the independent variables included in the regression. We also outline how to measure the strength of the linear relationship between the random variable and the independent variables, and discuss how to determine if regression assumptions have been violated.

In chapter 3, we outline our model of glacier retreat. We discuss what variables we chose to include in the model, as well as the steps necessary to apply the model to study the effects of climatic factors on a glacier. Furthermore, we present our results obtained from applying our model to the Midtfonna glacier in Norway.

In chapter 4, we outline two methods to predict the evolution of a glacier over time, as well as how to apply each method to a specific glacier in order to predict its future. Moreover, we present our results obtained from applying both methods to predict the future of the Midtfonna glacier.

In chapter 5, we present the conclusions of our study on the relationship between glacier retreat and changes in climatic factors. Contained within this chapter is a summary of our results obtained from applying our model of glacier retreat to study the climate’s effects on Midtfonna glacier, as well as the predictions we made about the future of Midtfonna glacier. We conclude by reflecting on the repercussions of glacier retreat and emphasizing that our mathematical model, along with our two methods for predicting a glacier’s future, can be instrumental in helping people prepare for the consequences of climate change.

2 Methodology

2.1 Linear Regression

2.1.1 Introduction to Linear Regression

Linear regression is a statistical tool that helps us analyze relationships between various components of a complex system, and develop methods of prediction for the output of the system. Among the variables of this system, we distinguish between a response variable \( y \) that depends on one or more independent variables \( x_1, x_2, \ldots, x_k \). A regression equation is a prediction equation that is used to predict the value of a random variable \( y \) based on a value for one or more independent variables. In the case of a regression using only one independent variable, known as a simple linear regression, the prediction equation will have the form:
\( y = \alpha + \beta x \), where \( y \) is the random variable and \( x \) is the independent variable. The coefficients \( \alpha \) and \( \beta \) are called the regression coefficients and are parameters. Since \( \alpha \) and \( \beta \) represent parameters, we must estimate their values from the sample data. We will denote the estimate for \( \alpha \) as \( a \), and \( \beta \) as \( b \). Once we find \( a \) and \( b \), we are able to construct the fitted regression line, the line of best fit through the data points which will be used to make estimates of \( y \) based on a value for \( x \). The fitted regression line has the form: \( \hat{y} = a + bx \), where \( \hat{y} \) represents the predicted value of \( y \) given by the regression line. Figure 3 shows a graph showing the difference between the theoretical regression line, \( y = \alpha + \beta x \), and the fitted regression line, \( \hat{y} = a + bx \). The symbol \( \hat{y} \) is used to distinguish between the predicted value and an actual observed experimental value \( y \) for some value \( x \).

![Figure 3: Fitted regression line vs. theoretical regression line.](image)

2.1.2 Estimating the Regression Coefficients \( \alpha \) and \( \beta \)

Method of Least Squares

A residual, which we will denote as \( e \), is the difference between an observed value of \( y \) and its predicted counterpart \( \hat{y} \):

\[
e_i = y_i - \hat{y}_i
\]

For each data point, \( \{ (x_i, y_i); i = 1, n \} \) where \( n \) represents sample size, there exists an \( e_i \) such that \( y_i = a + bx_i + e_i \). A residual is the error in the fit of the model at the \( i^{th} \) data point. This error is represented in the theoretical regression line by \( \varepsilon_i \), so we have \( y_i = \alpha + \beta x_i + \varepsilon_i \). Figure 4 illustrates the difference between \( e_i \) and \( \varepsilon_i \). The residual sum of squares (SSE) is the sum of the squares of the residuals, and measures the variation in the data that is not explained by the independent variable \( x \).
Since we want the fitted regression line to fit through the data points as best as possible, we find $a$ and $b$, the estimators of the parameters $\alpha$ and $\beta$, so that the sum of the squares of the residuals is minimized. This is accomplished by performing the procedure known as the **method of least squares**. The residual sum of squares (SSE) is found by:

$$SSE = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

where $n$ represents sample size. If we differentiate SSE with respect to $a$ and $b$, we get:

$$\frac{\partial(SSE)}{\partial a} = -2\sum_{i=1}^{n} (y_i - a - bx_i)$$

$$\frac{\partial(SSE)}{\partial b} = -2\sum_{i=1}^{n} (y_i - a - bx_i)x_i$$

In order to obtain the following **normal equations**, which are solved simultaneously to find formulas for $a$ and $b$, we set the partial derivatives equal to zero and rearrange the terms to get:

$$na + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$$

$$a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i$$

From these two equations we derive the formulas for $a$ and $b$: 

**Figure 4: Difference between $e_i$ and $\epsilon_i$.**
Using these formulas we are able to find the estimates for the parameters \( \alpha \) and \( \beta \), and thus construct the fitted regression line \( \hat{y} = \alpha + \beta x \), which can be used to make predictions for \( y \) based on a value for \( x \). It is useful to know that \( b \) can be written alternatively as:

\[
b = \frac{S_{xy}}{S_{xx}}
\]

where \( S_{xy} \) and \( S_{xx} \) are defined as:

\[
S_{xy} = \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})
\]

\[
S_{xx} = (x_i - \bar{x})^2
\]

Regression, Error, and Total Sum of Squares

While the SSE measures the variation in the data that is not explained by the independent variable \( x \), the regression sum of squares (SSR) measures the amount of variation explained by using the regression with the independent variable \( x \). The SSR is found by:

\[
SSR = \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2 = \sum_{i=1}^{n} (a + bx_i - \bar{y})^2
\]

\[
= \left( \frac{S_{xy}}{S_{xx}} \right)^2 S_{xx} = \frac{(S_{xy})^2}{S_{xx}}
\]

The Total Sum of Squares (Total SS) measures the total variation in the random variable \( y \) and is easily found once we know the SSE and SSR. Before we introduce the equation for the Total SS, however, we will first re-write the SSE as follows:
\[ \text{SSE} = \sum_{i=1}^{n} (y_i - a - bx_i)^2 = \sum_{i=1}^{n} [(y_i - \bar{y}) - b(x_i - \bar{x})]^2 = \]
\[ = \sum_{i=1}^{n} (y_i - \bar{y})^2 - 2b\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) + b^2\sum_{i=1}^{n} (x_i - \bar{x})^2 = \]
\[ = S_{yy} - 2bS_{xy} + b^2S_{xx} = S_{yy} - bS_{xy} = S_{yy} - \frac{(S_{xy})^2}{S_{xx}} \]

where \( S_{yy} = \sum_{i=1}^{n} (y_i - \bar{y})^2 \). Now, the Total SS is given by:

\[ \text{Total SS} = \text{SSE} + \text{SSR} = S_{yy} - \frac{(S_{xy})^2}{S_{xx}} + \frac{(S_{xy})^2}{S_{xx}} = S_{yy} \]

Variance of Residuals

For the linear regression model \( y_i = \alpha + \beta x_i + \epsilon_i \), we assume that the mean of the residuals, \( \epsilon_1, \epsilon_2, \ldots, \epsilon_n \), is equal to zero. In addition to this assumption, we assume that each \( \epsilon_i \) has the same variance. The variance of the model error, denoted \( \sigma^2 \), reflects random variation around the regression line. Like \( \alpha \) and \( \beta \), \( \sigma^2 \) is a parameter that must be estimated. An estimate of \( \sigma^2 \) is given by:

\[ s^2 = \frac{\text{SSE}}{n-2} = \frac{\sum_{i=1}^{n} (y_i - \hat{y})^2}{n-2} = \frac{S_{yy} - bS_{xy}}{n-2} \]

2.2 Testing the Usefulness of the Linear Regression Model

2.2.1 Hypothesis Testing

When making assertions or conjectures concerning a population or a system, we need a way to test whether or not our hypothesis is correct. We achieve this by performing a hypothesis.

In order to perform a hypothesis test, we must first construct a statistical hypothesis, an assertion concerning one or more populations or systems. To test our statistical hypothesis, we create a null hypothesis, denoted \( H_0 \), and an alternative hypothesis, denoted \( H_1 \). The null hypothesis is stated in such a way so that it specifies the exact value for the parameter of a population or a system, whereas the alternative hypothesis permits the possibility of multiple values. For example, if the null hypothesis, \( H_0 \), states that parameter \( p = 0.5 \), then the alternative hypothesis, \( H_1 \), says that either \( p < 0.5 \), \( p > 0.5 \), or \( p \neq 0.5 \). When performing a hypothesis test,
the null hypothesis is assumed to be true, while the alternative hypothesis false. The alternative hypothesis is what we wish to prove and is set up as being the opposite of the null hypothesis. Therefore, the acceptance of \( H_1 \) follows from the rejection of \( H_0 \).

The next step in determining whether we accept or reject the null hypothesis is to calculate the test statistic, which is a value calculated from a sample of data and used as a numerical summary of the sample data. The test statistic is found by:

\[
t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}},
\]

where \( \mu_0 \) represents the hypothesized value of the parameter made in the null hypothesis. The test statistic, \( t \), falls into one of two regions: the acceptance region or the critical region, with the two regions being separated by a critical value. The critical value, denoted \( t_\alpha \), represents the last number observed when passing from the acceptance region into the critical region and is found using a \( t \)-table. If the test statistic falls into the acceptance region, then we do not reject \( H_0 \), the null hypothesis. However, if the test statistic falls into the critical region, then we do reject \( H_0 \), and thus, accept \( H_1 \), the alternative hypothesis. The critical region is also known as the level of significance, which is denoted by \( \alpha \) (not to be confused with the parameter \( \alpha \)), and represents the probability of mistakenly rejecting the null hypothesis when it is true. We choose the value of \( \alpha \) ourselves, and because of this, we are able to control the amount of risk involved in incorrectly rejecting the null hypothesis. Based on the level of significance, we use a \( t \)-table to find the corresponding \( t_\alpha \) for which the test statistic is compared to in order to determine whether it falls into the critical region. Also, depending on whether the alternative hypothesis is made up using the inequality \(<, >, \text{ or } \neq\), the location of the critical region will vary. Figure 5 shows the various locations for the critical region, represented in red. There are three different scenarios for performing a \( t \)-test: left-tailed, right-tailed, and two-tailed. In each case, the null hypothesis should be rejected when the test statistic falls into the critical region.

\[\text{Figure 5: Critical regions for various hypothesis tests.}\]

Another value we look to when deciding whether or not to reject the null hypothesis is the \( p \)-value for the test statistic. The \( p \)-value is a value between 0 and 1 representing the probability of obtaining a test statistic as extreme as the one observed assuming that the null hypothesis is true. For example, a \( p \)-value of 0.04 says that only 4% of the time the statistical process would produce a test statistic this extreme if the null hypothesis were true. Thus, if the \( p \)-
value is lower than a previously established level of significance (usually 0.05 or 0.01), we can confidently reject the null hypothesis and accept the alternative hypothesis.

2.2.2 Hypothesis Testing for the Linear Regression Model

To check whether the random variable $y$ is linearly related to the independent variable $x$, a hypothesis test can be performed to test whether or not the slope of the regression line is zero. The parameter $\beta$ represents the slope of the regression line. Therefore, if $\beta$ is equal to zero, there is no linear relationship between $y$ and $x$. So, to test the existence of a linear relationship between the two variables, a hypothesis test is performed on the parameter $\beta$ to see what the odds are that it is equal to zero. A statistical hypothesis for this scenario would be set up as follows, $H_0$: $\beta = 0$ and $H_1$: $\beta \neq 0$. Finding the test statistic for $\beta$ is accomplished using the following formula:

$$t = \frac{b - 0}{s / \sqrt{S_{xx}}}$$

where $b$ is the estimate of $\beta$. Since this is a two-tailed test, the absolute value of the test statistic must be greater than the corresponding $t_{\alpha/2}$ value from the $t$-table in order to reject $H_0$.

2.3 Measuring the Strength of the Relationship

In determining the usefulness of a regression model, it is helpful to know how well the random and independent variables in the model correlate with each other, as well as how much of the variation in the random variable $y$ is explained by the linear regression of $y$ on $x$. The answer to both of these questions is found by performing a correlation analysis, which leads to a value known as the correlation coefficient.

To figure out the strength of the correlation between the random and independent variables in the model are, we compute the sample correlation coefficient $r$, which is an estimate of the linear association between the two variables $x$ and $y$:

$$r = b \frac{S_{xx}}{\sqrt{S_{yy} S_{yy}}} = \frac{S_{xy}}{\sqrt{S_{xx} S_{yy}}}, \quad \text{for } -1 \leq r \leq 1$$

The closer the absolute value of $r$ is to 1, the stronger the linear association between the random variable $y$ and the independent variable $x$.

Learning the proportion of variation in $y$ that is explained by the linear relationship between $x$ and $y$ is found by computing the sample coefficient of determination $r^2$. To find this value we square the sample correlation coefficient $r$: 

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The sample coefficient of determination $r^2$ is a number between 0 and 1; the closer the $r^2$ is to 1, the larger the proportion of total variation in the random variable $y$ explained by the independent variable $x$. From the above equation, we see that the amount of variation in $y$ explained by the linear relationship between $x$ and $y$, the $r^2$, is the ratio of the amount of variation explained by using the regression with the independent variable $x$ (SSR) to the total variation in the random variable $y$ (Total SS).

2.4 Checking the Regression Assumptions

Once we have constructed a regression model from a set of data, we must check that the following assumptions hold. The first assumption is that the residuals, the $e_i$’s for $i = 1, n$, are random variables. This is verified by looking at a residual plot. The horizontal axis on a residual plot represents predicted values of $y$. For each $\hat{y}_i$, the corresponding residual is plotted. If the model is a good fit to the data, the plot should show the residuals randomly scattered around zero. In other words, the residual plot should be free of any patterns. Figure 6 shows a residual plot free of patterns.

The second assumption that we must check is that our data is normally distributed. This is verified by viewing the normal probability plot. A normal probability plot is graphed by plotting each residual against its expected value if it had come from a normal distribution. If the residuals are normally distributed, the plot will appear as a straight line with positive slope. Figure 7 is an example of a normal probability plot showing a normal distribution of data.
2.5 Confidence and Prediction Intervals

Among the several reasons for constructing a linear regression equation is to make predictions about values of the random variable given a value for the independent variable. Using the equation $\hat{y} = a + bx$, there are two ways in which we can make a prediction for the random variable; we can predict the mean response of $y$ at $x = x_0$, or we can predict a single value $y_0$ at $x = x_0$.

To assist us in making predictions about $Ey$, the mean response of $y$ at $x = x_0$, we construct a confidence interval. Using a level of significance $\alpha$, a $(1 - \alpha)100\%$ confidence interval for the mean response of $y$ at $x = x_0$ is given by:

$$\hat{y} - t_{\alpha/2} \sqrt{\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}}} < Ey < \hat{y} + t_{\alpha/2} \sqrt{\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}}}.$$  

With this confidence interval, we can say that the mean of the response values at a point $x = x_0$ will be within this interval with $(1 - \alpha)100\%$ certainty.

To assist us in making predictions about a single value $y_0$ at $x = x_0$, we create a prediction interval. Using a level of significance $\alpha$, a $(1 - \alpha)100\%$ prediction interval for a single response $y_0$ at $x = x_0$ is given by:

$$\hat{y} - t_{\alpha/2} \sqrt{\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}}} < y_0 < \hat{y} + t_{\alpha/2} \sqrt{\frac{1}{n} + \frac{(x_0 - \bar{x})^2}{S_{xx}}}.$$  

With this prediction interval, we can say that a single value $y_0$ at a point $x = x_0$ will fall within this interval with $(1 - \alpha)100\%$ certainty. As it is more difficult to accurately predict a single value of $y_0$ at $x = x_0$ than it is to predict the mean response of $y$ at $x = x_0$, the width of the prediction interval is larger than that of the confidence interval.
2.6 Multiple Linear Regression

2.6.1 Introduction to Multiple Linear Regression

Creating an accurate model of a complex system using regression usually requires the inclusion of more than one independent variable in the regression model. A multiple regression model is obtained when multiple independent variables are used in a regression model to predict the response variable. If this model is linear in the coefficients, then it is a multiple linear regression model. For \( k \) independent variables \( x_1, x_2, \ldots, x_k \) and the random variable \( y \), the prediction equation will have the form:

\[
y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_k x_k
\]

where each regression coefficient \( \beta_i \) is a parameter that must be estimated. We will denote the estimate of each parameter by \( b_i \). The estimated response is given by the sample regression equation:

\[
\hat{y} = b_0 + b_1 x_1 + \ldots + b_k x_k
\]

As we saw with a simple linear regression, using just one independent variable gives us a regression in the form of a line. Using two independent variables will give us a regression plane, a regression in the form of a plane, illustrated by figure 8. We note that for a value of \( x_1 \) and \( x_2 \), the height of the plane represents the predicted response of \( y \) at \( (x_1, x_2) \).

Figure 8: Graph of a regression plane through data points (red dots).
Finding estimates for each parameter $\beta_i$ amounts to using the method of least squares. However, as computations are quite involved, the standard way to find such estimates is through the use of computer software.

### 2.6.2 Analysis of Variance

For a multiple linear regression, the error sum of squares (SSE), regression sum of squares (SSR), and total sum of squares (Total SS) are calculated in the same way as in the case of a simple linear regression. When using computer software to perform a multiple linear regression, an analysis of variance table (ANOVA table) is usually printed out by the computer and holds important information regarding the regression. An example of an ANOVA table is shown in table 1.

#### Table 1: ANOVA table.

Looking at the section labeled “ANOVA” in table 1, we can learn what the SSE, SSR, and Total SS for the regression are. The column labeled “SS” holds the sum of squares information. The regression sum of squares is found at the intersection of the SS column and the row labeled “Regression”; the error sum of squares is found at the intersection of the SS column and the row labeled “Residual; the total sum of squares is found at the intersection of the SS column and the row labeled “Total.”

#### 2.6.3 Hypothesis Testing

To check whether or not the random variable has a linear relationship with the independent variables in a multiple linear regression model we will perform two different hypothesis tests, both of which involve testing the parameters $\beta_1, \beta_2, \ldots, \beta_k$, where $k$ represents the number of independent variables included in the model. First, we test the parameters together to see if there is a good chance they all equal zero, meaning there is no linear relationship between
the random and independent variables. Our statistical hypothesis for this case is set up as the following: $H_0: \beta_i = 0$ for all $i = 1, k$. Similar to the case of a simple linear regression, we look to the $p$-value in order to decide whether or not to reject the null hypothesis. If the $p$-value for the test statistic is less than a pre-determined level of significance, we reject the null hypothesis. The $p$-value for the test statistic can be found on the ANOVA at the intersection of the “Significance F” column and “Regression” row.

Secondly, we perform hypothesis tests to check whether or not there is a linear relationship between the random variable and each of the independent variables given that all other independent variables are already in the regression. This tests whether or not the inclusion of an additional independent variable into the regression improves the prediction of the random variable. In this case, for each $i = 1, 5$ we set up an individual statistical hypothesis for the parameter $\beta_i$ with the null hypothesis $H_0: \beta_i = 0$. Thus, for each $i = 1, 5$ we obtain a $p$-value corresponding to the parameter $\beta_i$. If the $p$-value for the parameter $\beta_i$ is less than 0.05, we reject $H_0$ and conclude that there is a significant linear relationship between the random variable and the $i^{th}$ independent variable given that all other independent variables are already in the model. Each of these $p$-values can also be found on the summary output printed out by the computer software used to perform the regression.

2.6.4 Measuring the Strength of the Relationship

In order to find out how well a regression model fits a set data, a correlation analysis is performed. The coefficient of determination $R^2$, a measure of the proportion of the total variation in the random variable $y$ that is explained by the regression of $y$ on $x_1, x_2, \ldots, x_k$ is found in the same manner as in the case of a simple linear regression (the $R^2$ value is given by the ratio $SSR/Total SS$). However, as more independent variables are added to the regression, the $r^2$ increases regardless of whether or not the additional variable actually contributes to the variation in the random variable. To find the true amount of variation of the random variable that is explained by the independent variables, we look at the adjusted $r^2$, which is the un-inflated $r^2$ adjusted for the number of independent variable used in the regression.

2.6.5 Checking Regression Assumptions

There are three assumptions of regression that need to be checked for a multiple linear regression. Two of the assumptions are the same as in the case of a simple linear regression: the residuals are random variables, and the data is normally distributed. To verify these assumptions we follow the same procedure outlined for a simple linear regression. In order to verify the third assumption of a multiple linear regression, we must check that the independent variables included in the regression are not highly correlated. Existence of highly correlated independent variables in a regression is called multicollinearity, a topic we cover next.
Multicollinearity: Consequences, Tests, and Remedies

If two or more independent variables in a multiple regression are highly correlated, we say that the regression exhibits multicollinearity. The existence of multicollinear variables in a regression model causes the standard errors and variances of the regression coefficients to increase. Consequently, coefficients for some of the independent variables can be estimated incorrectly. This means that some of the independent variables may seem to have little impact in the prediction of the random variable, when, in reality, those same independent variables are significantly linearly related to the random variable. An important statistic of regression that multicollinearity does not affect, however, is the $R^2$ value, the amount of total variation in the random variable explained by the independent variables. Therefore, if the goal is to find the amount of variation in the random variable explained by the independent variables, then whether or not multicollinearity exists within the model is of no concern.

To test for multicollinearity we calculate the variance inflation factor (VIF), which is a measure of the increase in variance of an estimated regression coefficient due to collinearity. We find a variance inflation factor for each coefficient. The VIF for the coefficient of the $i^{th}$ independent variable, $x_i$, is given by:

$$VIF_i = \frac{S^2_i (n - 1)SE^2_i}{MSE^2}$$

where $S^2_i$ is the standard deviation of the $i^{th}$ independent variable, $n$ is sample size, $SE^2_i$ is the standard error of the $i^{th}$ independent variable, and $MSE^2$ is the mean square of residuals. Variance inflation factors fall within the range 1 to infinity. If $x_i$ has a VIF of 1, then there is no linear relationship between $x_i$ and any of the other independent variables included in the regression. The farther away from 1 the variance inflation factor is for $x_i$, the more correlated $x_i$ is with at least one of the other independent variables. An independent variable with a VIF ≥ 5 means the variable is strongly linearly related to one or more independent variables, and hence a high degree of multicollinearity exists within the model.

If the VIF of any of the independent variables is equal to or greater than 5, we should consider trying to remedy the multicollinearity. There are various methods for reducing multicollinearity in a regression. One method involves removing a variable from the regression. If two independent variables both have a VIF ≥ 5, then one of the variables can be removed to reduce the high degree of multicollinearity. To select which variable to keep in the regression, select the variable that gives the highest $R^2$ when the other is removed. This procedure can also be performed if two or more variables all have a significant VIF. Another approach deals with obtaining more data. Having more data available to use in the regression allows for better estimates of the coefficients, which helps neutralize the effects of multicollinearity. Centering the data of the multicollinear variables can also help alleviate the high correlation between the
independent variables. As a final suggestion, since multicollinearity does not affect the $R^2$, one can choose to leave the model unchanged.

3 Constructing and Applying the Mathematical Model of Glacier Retreat

In this thesis, we study and explain the relationship between various climatic factors and the ice mass of a glacier. To accomplish this goal, we construct a mathematical model of glacier retreat representing how various climatic factors affect a glacier’s ice mass. In section 2, we described the statistical tools we utilized to create the model of glacier retreat. We will now explain how to use the tools of multiple linear regression to set up and apply the model to a specific glacier.

3.1 Constructing the Model

3.1.1 Variables Included in the Model

The variables we chose to include in our model of glacier retreat are summarized in table 2. The random variable is glacier ice mass, while the independent variables are the climatic factors: temperature, precipitation, local climatic phenomena, wind speed, and insolation. We chose to include these climatic factors because they have been shown to be highly correlated with changes in the ice mass of a glacier (Anderson et al., 2006; Bitz & Battisti, 1999; Letreguilly, 1988).

<table>
<thead>
<tr>
<th>Variables Included In The Model</th>
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</thead>
<tbody>
<tr>
<td>$y$ - Random Variable</td>
</tr>
<tr>
<td>$x_1$ - Independent Variable</td>
</tr>
<tr>
<td>Temperature (°C)</td>
</tr>
<tr>
<td>$x_2$ - Precipitation (mm/m²)</td>
</tr>
<tr>
<td>$x_3$ - Local Climatic Phenomena</td>
</tr>
<tr>
<td>$x_4$ - Wind Speed (m/s)</td>
</tr>
<tr>
<td>$x_5$ - Insolation (kwh/m²/day)</td>
</tr>
</tbody>
</table>

Table 2: Summary of variables included in the model of glacier retreat.

Glacier ice mass can be quantified in a variety of ways. Depending on the data available and purpose for applying the model, one can choose to use total glacier volume, total glacier area, terminus point, or mass balance as a measure of a glacier’s ice mass in the model. Mass balance is better correlated to the climatic factors than total glacier area. This can be explained by noting that the rate at which a glacier’s area will shrink under the influence of climatic factors depends on the relative thickness of the ice, whereas mass balance only measures the annual net loss/gain of snow and ice, making it much more sensitive to the climatic factors. However, glacier area is a better instrument to use for predicting the future of a glacier as it gives us an overall picture of a glacier’s size (as opposed to mass balance which only offers the yearly ice budget of a glacier rather than a measure of its total ice mass).
Temperature can be measured in a variety of ways such as mean annual temperature or mean summer temperature (months designated “summer” will vary depending on the location of the glacier the model is being applied to). As the majority of melting of a glacier’s ice occurs in the summer months when temperatures are at their peak, summer temperature is better correlated to variations in the ice mass of a glacier than yearly temperature. Therefore, we recommend summer temperature be used in the regression model. Also, because the thickness of a glacier’s ice helps determine the rate at which it melts, a delay in the effects of temperature on the glacier may need to be accounted for.

Similarly, precipitation can be measured in a variety of ways such as mean annual precipitation or mean winter precipitation (months designated “winter” will vary depending on the location of the glacier the model is being applied to). As the majority of snow accumulation and formation of a glacier’s ice occurs in the winter months, winter precipitation is better correlated to variations in the ice mass of a glacier than yearly precipitation. Therefore, we recommend winter precipitation be used in the regression model. Also, because snow takes time to compress into ice, a delay in the effects of precipitation on the glacier may need to be accounted for.

Local climatic phenomena are cyclical weather patterns capable of causing significant changes to climatic factors such as temperature and precipitation. Examples of local climatic phenomena are: North Atlantic Oscillation, North Pacific Oscillation, Atlantic Multidecadal Oscillation, Pacific Decadal Oscillation, El Niño/La Niña, and monsoons. To measure a local climatic phenomenon, an index value representing the strength of the particular cyclical weather system is used. Depending on the location of the glacier the model is being applied to, there may or may not be a local climatic phenomenon that influences the climate near the glacier.

Wind speed is included in the model, as wind is capable of removing snow from windward slopes. The removal of snow scours the ice on the slope, which causes it to become more reflective of the sun. Therefore, wind is able to affect both the accumulation and ablation of snow and ice.

Insolation is the last variable we have included in our model. Ice will melt faster with greater solar exposure. The unit used to measure insolation is kilowatt hours per square meter per day (kwh/m²/day), which represents the amount of solar energy that strikes a square meter of the earth’s surface over the course of a day.

3.1.2 Description of the Model

The first step in setting up the model for a glacier is to perform a multiple linear regression of the glacier’s ice mass, the random variable, on the five climatic factors, the independent variables, and obtain a regression equation of the form:

\[ y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_5 x_5 \]
where y represents the glacier’s ice mass, \( x_1, x_2, \ldots, x_5 \) represent the climatic factors in table 2, and \( \beta_0, \beta_1, \ldots, \beta_5 \) represent the regression coefficients.

Once the linear regression equation has been set up, we test whether or not there is a significant linear relationship between the variables. To do this, we perform a hypothesis test where the null hypothesis is \( H_0: \beta_i = 0 \) for all \( i = 1,5 \), meaning there is no linear relationship between the glacier’s ice mass and the climatic factors with the significance level \( \alpha = 0.05 \). We then find the \( p \)-value for the test statistic. If the \( p \)-value is less than 0.05, we reject \( H_0 \) and conclude that there is a significant linear relationship between the glacier’s ice mass and the five climatic factors.

Now, while the aforementioned hypothesis test establishes whether or not there is a significant linear relationship between the glacier’s ice mass and all the climatic factors together, we also perform hypothesis tests to check whether or not there is a significant linear relationship between the glacier’s ice mass and each of the climatic factors given that all other climatic factors are already in the model. This tests whether or not the inclusion of an additional independent variable into the regression improves the prediction of the random variable. Thus, for each \( i = 1,5 \), we perform an individual hypothesis test, where the null hypothesis \( H_0 \) is \( \beta_i = 0 \), meaning there is no linear relationship between the glacier’s ice mass and the \( i \)th climatic factor. For each individual hypothesis test, we find the \( p \)-value for the test statistic with the significance level \( \alpha = 0.05 \). If the \( p \)-value is less than 0.05, we reject \( H_0 \) and conclude that there is a significant linear relationship between the glacier’s ice mass and the \( i \)th climatic factor.

Next, we measure the strength of the linear relationship. We do this by calculating three values: the sample correlation coefficient \( r \), the sample coefficient of determination \( r^2 \), and the adjusted \( r^2 \). The sample correlation coefficient \( r \) is a number between -1 and 1, which is an indicator of linear association between the random variable and the independent variables. The closer the absolute value of \( r \) is to 1, the stronger the linear association between the glacier’s ice mass and the climatic factors. The sample coefficient of determination \( r^2 \) is a number between 0 and 1 representing the proportion of total variation in the random variable that is explained by the independent variables. However, as more independent variables are added to the regression, the \( r^2 \) increases regardless of whether or not the additional variable actually contributes to the variation in the random variable. To find the true amount of variation of the random variable that is explained by the independent variables, we look at the adjusted \( r^2 \), which is the un-inflated \( r^2 \) adjusted for the number of independent variable used in the regression. The closer the adjusted \( r^2 \) is to 1, the larger the proportion of total variation in the glacier’s ice mass explained by the climatic factors.

The final step in our model is to check the assumptions of the regression. We will determine if there is high correlation among the independent variables (multicollinearity), if the regression model is a good fit to the data, and if the data is normally distributed. To test for multicollinearity, for each regression coefficient we calculate the variance inflation factor (VIF), a measure of the increase in variance of an estimated regression coefficient due to collinearity. If
all of the VIFs are less than 5, then we conclude that there is not a high degree of multicollinearity in the model. However, if one or more of the VIFs are greater than or equal to 5, then we must decide what to do in order to reduce the multicollinearity between independent variables.

To check whether or not the regression model is a good fit to the data, we set up the residual plot. A residual plot free of any patterns indicates that the model is a good fit for the data. In order to check whether or not the data is normally distributed, we view the normal probability plot. A normal probability plot showing the data following a straight line with positive slope indicates a normal distribution of data. Once these assumptions of the regression are verified, and the previous steps of the model have been completed, we are able to verify the usefulness of the regression model.

3.2 Applying the Model

We applied our model of glacier retreat to study the Midtfonna glacier, located in Folgefonna National Park in Norway. Midtfonna is the smallest of three glaciers that make up the Folgefonna glacier (see figure 9), the other two being Nordfonna glacier and Sorfonna glacier. We chose to perform a multiple regression analysis on the Midtfonna glacier because this glacier’s ice mass exhibits the most extreme response to changes in climatic factors. The random variable we chose to include in the regression was total glacier area, while the independent variables we included were the climatic factors: summer temperature (July and August mean), winter precipitation (October – April mean), North Atlantic Oscillation index (December – March mean), highest mean wind value (annual mean), and summer insolation (July and August mean). The North Atlantic Oscillation is a local climatic phenomenon that
affects Europe, among other areas, and causes cyclical precipitation increases and milder summers to occur in Norway.

We chose to perform two regression analyses. One analysis includes insolation, while the other does not. The reason for this is that we had insolation data for a much shorter time span compared to the data we had for the other climatic factors. The regression analysis that does not include insolation uses 50 years of data spanning the years 1962 – 2011, while the regression analysis that includes insolation uses only 21 years of data spanning the years 1985 – 2005.

We now discuss the analysis that does not include insolation first. Table 3 is the regression summary output obtained from performing a multiple linear regression using 50 years of data for the Midtfonna glacier. The sample correlation coefficient \( r \) is represented in the table by the “Multiple R.” From viewing the table, we find that the sample correlation coefficient \( r \) is 0.876. This tells us that there is a strong linear association between Midtfonna’s total glacier area and the climatic factors included in the regression. We are also able to see that the adjusted \( r^2 \) is quite high at 0.746. Therefore, 74.6% of the total variation in Midtfonna’s area is explained by summer temperature, winter precipitation, NAO index, and wind speed. The \( p \)-value of the regression is 1.01*10^{-13}, which tells us that at least one of the independent variables, summer temperature, winter precipitation, NAO index, or wind speed, is contributing significant information to the prediction of glacier area. The individual \( p \)-values corresponding to each of the independent variables (0.0062, 8.75*10^{-6}, 0.0124, 3.99*10^{-8}) show that all four variables add important information to the prediction of glacier area in the presence of the other ones already in the model.

To check if multicollinearity exists in the regression model, we computed the variance inflation factor for each independent variable. Each VIF can be found in table 3 in the column labeled “VIF.” The variance inflation factors for summer temperature and wind speed at 1.224 and 1.174, respectively, show very slight multicollinearity. The VIFs for winter precipitation and

<table>
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<th>Regression Statistics</th>
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<tbody>
<tr>
<td>Multiple R</td>
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<tr>
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<tr>
<td>Adjusted R Square</td>
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<table>
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<tr>
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<th>Standard Error</th>
<th>t Stat</th>
<th>P-value</th>
<th>VIF</th>
</tr>
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<tr>
<td>Intercept</td>
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<td>2.39016141</td>
<td>17.4476787</td>
<td>1.18418E-21</td>
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<td>Temperature</td>
<td>-0.410693454</td>
<td>0.143285726</td>
<td>-2.866255174</td>
<td>0.006295705</td>
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<td>Precipitation</td>
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<td>0.014028203</td>
<td>-5.014715756</td>
<td>8.7907E-06</td>
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<tr>
<td>NAO Index</td>
<td>1.142103091</td>
<td>0.438773794</td>
<td>2.60294281</td>
<td>0.01247288</td>
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<tr>
<td>Wind</td>
<td>-2.104659717</td>
<td>0.318853614</td>
<td>-6.600708364</td>
<td>3.99289E-08</td>
</tr>
</tbody>
</table>

Table 3: Summary output for regression using 50 years of data (insolation not included).
NAO index at 3.307 and 3.06, respectively, show a higher degree of multicollinearity. Though the variance inflation factors for winter precipitation and NAO index show that the two variables are moderately linearly related, both VIFs are under 5, the threshold for severe multicollinearity. To eliminate this redundancy in the model, we could remove the NAO index as a variable from our model. However, our goal for this regression is to identify what percentage of the total variation in the glacier’s total area is explained by the climatic factors, so, since multicollinearity does not affect the $R^2$, we will keep the NAO index in the model. However, as a consequence the regression coefficients of winter precipitation and NAO index are unstable and therefore difficult to predict.

We verify the assumption of regression by viewing the residual plot (figure 10) and normal probability plot (figure 11). The residual plot is free of any patterns, meaning that the model is a good fit for the data; the normal probability plot shows data following a straight line, indicating a normal distribution of data.

![](image1.png) ![](image2.png)

Figure 10: Residual plot

Figure 11: Normal probability plot

Table 4 shows the regression summary output obtained from performing a multiple linear regression using 21 years of data for Midtfonna glacier. When we include insolation as an

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<tr>
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<tbody>
<tr>
<td>Multiple R</td>
</tr>
<tr>
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<td>Adjusted R Square</td>
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<td>Regression</td>
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<table>
<thead>
<tr>
<th>Coefficients</th>
<th>Standard Error</th>
<th>t Stat</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
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<td>6.307001415</td>
</tr>
<tr>
<td>Temperature</td>
<td>-0.627686993</td>
<td>0.130206158</td>
<td>-4.820716651</td>
</tr>
<tr>
<td>Precipitation</td>
<td>-0.069455514</td>
<td>0.019506829</td>
<td>-3.560574303</td>
</tr>
<tr>
<td>NAO</td>
<td>2.950447141</td>
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<td>5.564576969</td>
</tr>
<tr>
<td>Wind</td>
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<td>0.474251036</td>
<td>-2.95846565</td>
</tr>
<tr>
<td>Insolation</td>
<td>1.494641999</td>
<td>0.346932734</td>
<td>4.308160788</td>
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</table>

Table 4: Summary output for regression using 21 years of data (insolation included).
independent variable in the regression, we notice that the adjusted $r^2$ increases to 0.827. All of the $p$-values are also less than the significance level of 0.05. Viewing the residual plot (figure 12) and normal probability plot (figure 13), we verify the assumptions of the regression analysis.

From applying our model of glacier retreat to study the Midtfonna glacier, we have shown that the climatic factors: summer temperature, winter precipitation, the North Atlantic Oscillation, wind speed, and summer insolation, are strong predictors of the glacier’s total area.

4 Predicting a Glacier’s Future

Our second goal for this thesis is to use regression analysis to predict the future of a glacier. We accomplish this by two methods, each differing in the way the ice mass of a glacier is predicted. The first method involves performing a time series regression to study how the ice mass of a glacier changes with the passage of time. The second method entails using data from future prediction scenarios for temperature and precipitation in a multiple regression equation to predict when a glacier will disappear. For each method, we first outline the process of how the future of a glacier is predicted, and then present our results from applying the described method to predict the future of Midtfonna glacier.

4.1 Predicting a Glacier’s Future Using Time Series Regression

4.1.1 Description of Time Series Regression

The first method we employ to predict the future of a glacier’s ice mass is through the use of a time series regression. A time series is a sequence of data points ordered in time; in a time series regression, we study how the ice mass of a glacier changes with the passage of time. The data required to perform this type of regression are measurements of the ice mass of a glacier and the corresponding years that the measurements were taken. We perform a linear regression with the ice mass of a glacier as the random variable and the year as the independent variable to obtain a prediction equation of the form:
where \( y \) represents the glacier’s ice mass, \( x \) represents the year, and \( \alpha \) and \( \beta \) represent the regression coefficients. We can use this prediction equation to find the estimated ice mass of the glacier for a given year by inputting the desired year into the equation and viewing the output. To determine the estimated year the glacier will disappear, we start with the year of the last available data point as the input for the equation. We then incrementally increase the year by 1 until the output of the equation, representing the estimated ice mass of the glacier, is less than or equal to 0. Thus, using the prediction equation we obtain an estimate for the year when the glacier will seize to exist.

However, to be able to make a prediction for the glacier’s area with a desired probability (probability of 1- \( \alpha \), where \( \alpha \) is the significance level), we need to compute a \((1 - \alpha)100\%\) prediction interval estimating the actual future value of the random variable \( y \). A prediction interval allows us to say that a single value for the random variable \( y \) at a point \( x = x_0 \) will fall within the interval with \((1 - \alpha)100\%\) probability, where \( \alpha \) is the significance level. Therefore, for a significance level of \( \alpha = 0.05 \), a 95% prediction interval tells us that the glacier’s ice mass will take values in the interval with 0.95 probability.

### 4.1.2 Predicting Midtfonna Glacier with a Time Series Regression

Now that we have discussed how to predict the future of a glacier using a time series regression, we will now present our results from using this method to predict the future of Midtfonna glacier. To estimate when Midtfonna glacier will disappear, we performed a time series regression with data from the years 1962 – 2011 for Midtfonna’s total glacier area.

![Graph of the time series regression line.](image)

Figure 14 shows a graph of the regression line in blue obtained from the time series regression. Based on the time series regression, we estimate that Midtfonna will disappear by the year 2078.
The graph also contains the observed values of Midtfonna’s total glacier area, as well as a 95% prediction interval in red that we computed. With 95% confidence we predict that Midtfonna glacier will disappear sometime between the years 2055 and 2118.

4.2 Predicting a Glacier’s Future Using Multiple Regression with Climate Scenarios

4.2.1 Description of Multiple Regression with Climate Scenarios

The second method we employ to predict the future of a glacier is to incorporate scenarios for future temperature and precipitation in a regression equation. We first perform a multiple regression where a glacier’s ice mass is the random variable; the year, summer temperature, and winter precipitation, are the independent variables to obtain a prediction equation of the form:

\[ y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 \]

where \( y \) represents the glacier’s ice mass, \( x_1 \) represent the year, \( x_2 \) represents summer temperature, \( x_3 \) represents winter precipitation, and \( \beta_0, \beta_1, \beta_2, \beta_3 \) represent the regression coefficients. The year is included as an independent variable so that we can use it as a counter to keep track of what year we are predicting the glacier’s ice mass for with the prediction equation.

The second step of this method involves extracting the necessary data for temperature and precipitation from the available climate scenarios to use in the prediction equation. Suppose we have a 2012 scenario projecting a temperature increase of 3°C by 2100. To obtain the value of temperature for each year in the 2012–2100 interval, we must distribute the increase of 3°C over the time span of the projection. We break up the projected increase of 3°C into equally sized increments by dividing 3°C by the number of years in the span 2012–2100, which gives us an increase of 0.034°C per year. Obtaining the value of precipitation for each year in the 2012–2100 interval is found in a similar way, but with an additional step. Suppose we have a 2012 scenario projecting a precipitation increase of 10% by 2100. Since the precipitation amount for the year 2011 might be an outlier, we will instead compute the average precipitation for the past decade (2001–2011). The average is found using 10 years of precipitation data in order to smooth out any outliers that may exist in the data and obtain a fair estimation of the “normal” amount of precipitation for the current time period. We then calculate 10% of this average and divide it by the number of years in the interval 2012–2100 to find the increase in precipitation per year.

Now that we have obtained the projected yearly increase in temperature and precipitation, our next step is to calculate an average for both of these climatic factors that will be used as starting values in the prediction equation; we will call these averages our baselines. Continuing our example, because the projected changes in temperature and precipitation start from the year 2012, we will want to find the average summer temperature and winter precipitation for the decade 2001–2011. The two averages are calculated using 10 years of temperature and
precipitation data in order to smooth out any outliers that may exist in the data and obtain a fair estimation of the “normal” values of temperature and precipitation for the current time period. Since the baseline for precipitation was already found when we calculated the increase in precipitation per year, all that remains to be found is the baseline for temperature.

At this point we have all the data we need to use the prediction equation to predict the future of a glacier using scenarios projecting future temperature and precipitation. Again, we will continue to use our example to explain the next steps to predict the future of the glacier. In the following, we let $T_0$ represent the baseline value for temperature; $\Delta T$ the increase in temperature per year; $P_0$ the baseline value for precipitation; $\Delta P$ the increase in precipitation per year; $i$ the year, and $j$ the number of years between $i$ and the first year before which the projections for temperature and precipitation began (in our example, $j = i – 2011$ because the projections begin with 2012). The estimated ice mass of a glacier at year $i$, for $i$ in the interval 2012 – 2100, is found by entering the following values into the prediction equation: the input for year is $i$; the input for temperature is $T_0 + j \Delta T$; the input for precipitation is $P_0 + j \Delta P$. Starting with $x_1 = i = 2012$ and the corresponding values for temperature and precipitation in the prediction equation, the evolution of the glacier over time is found by incrementally increasing $i$ by 1 and viewing the output of the prediction equation. This process of incrementally increasing $i$ can be continued until the estimated ice mass of the glacier reaches zero at year $i^*$ or the last year that temperature and precipitation were projected for is reached, which in our example is 2100. Thus, we either obtain the year $i^*$ when the glacier will completely disappear, or we obtain the predicted value of the glacier’s total area in 2100.

4.4 Predicting Midtfonna Glacier using a Multiple Regression with Climate Scenarios

To predict the future of Midtfonna glacier based on scenarios for future temperature and precipitation, we first performed a multiple regression using 50 years of available data spanning 1962 – 2011 for total glacier area, temperature, and precipitation. Included in the regression as the random variable was Midtfonna’s total glacier area, while the independent variables were the year, summer temperature, summer temperature squared, winter precipitation, and winter precipitation squared. Because the year is such a strong predictor for the total glacier area of Midtfonna, the effects of summer temperature and winter precipitation on the glacier are overshadowed. To fix this, we made summer temperature and winter precipitation more prominent predictors by introducing the squared terms into the regression. From this multiple regression we obtained the following prediction equation:

$$y = 291.497 - 0.149x_1 + 0.084x_2 - 0.017x_2^2 + 0.241x_3 - 0.0007x_3^2$$

where $y$ represents Midtfonna’s total glacier area, $x_1$ represents the year, $x_2$ represents summer temperature, and $x_3$ represents winter precipitation.
For this equation, we used two climate scenarios projecting future temperature and precipitation changes until the year 2100. These two scenarios (see figures 15 and 16) were obtained from the Intergovernmental Panel on Climate Change’s 2013 report on climate change (IPCC, 2013). In scenario 1, see figures 15 and 16, the annual mean surface temperature and annual mean precipitation in Norway, the home to Midtfonna glacier, is projected to increase by 1.5°C and 5%, respectively. In scenario 2, the annual mean surface temperature and annual mean precipitation in Norway is projected to increase by 4.5°C and 15%, respectively.

To use the data from these two climate scenarios in the prediction equation, we must distribute the projected changes in temperature and precipitation over the 2012 – 2100 interval. Breaking up the projected temperature changes for scenario 1 and scenario 2, we divide the 1.5°C and 4.5 °C projected increases by the number of years in the 2012 – 2100 interval to get an increase of 0.017°C and 0.051°C, respectively, per year. To be able to break up the projected precipitation increases into increments, we must first find the baseline for precipitation, meaning we have to find the average winter precipitation over the interval 2001 – 2011. We calculated this to be 180 mm/m² per year. Now we compute the per year increase in precipitation for scenario 1 and scenario 2 by dividing 5% and 15% of 180 by the number of years in the interval 2012 – 2100, which gives us an increase of 0.1 mm/m² and 0.3 mm/m², respectively, per year.
Our next step involves finding the baselines for summer temperature and winter precipitation, the values for temperature and precipitation that will be incrementally increased when predicting the future of Midtfonna glacier with the prediction equation. The baseline for winter precipitation has already been established at 180 mm/m². By calculating the average summer temperature for the years 2001 – 2011, we found the baseline for summer temperature to be 8°C.

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Table 5: Climate scenario data.

We will now present our results from using the climate data from each scenario in the prediction equation. Under scenario 1, with a yearly increase of 0.017°C in temperature and 0.1 mm/m² in precipitation, we estimate that the Midtfonna glacier will disappear by the year 2086. Under scenario 2, with a more significant yearly increase of 0.051°C in temperature and 0.3 mm/m² in precipitation, we estimate that the Midtfonna glacier will disappear sooner by the year 2079.

5 Conclusions

The climate of the earth is continually changing. As the earth’s climate presently undergoes major changes, glaciers around the world are retreating at a remarkable pace. Serious consequences impacting people and animals on a global scale await us if the increasing rate at which glacial ice is melting is sustained. If we wish to attenuate the severity of these consequences, we must better understand the relationship between climate change and glacier retreat.

In this thesis we accomplished two goals. In order to better understand the effects of climate change on glaciers, we constructed a mathematical model of glacier retreat representing how changes in climatic factors, such as temperature and precipitation, affect the ice mass of a glacier. We applied our model to study the Midtfonna glacier, located in Norway. Using multiple linear regression we studied the effects of temperature, precipitation, the North Atlantic Oscillation, wind speed, and insolation on the total area of the glacier, and found that within our model these factors explained 82.7% of the total variation in Midtfonna glacier’s area. By adapting the variables in the regression to reflect the geographic location of a glacier, our model can also be applied to other glaciers to determine what proportion of total variation in a glacier’s ice mass is explained by the five climatic factors.

Our second goal for this thesis was to create a method to predict the evolution of a glacier over time by using different climate scenarios projecting future temperature and precipitation. We found that a glacier’s future could be predicted this way by performing a multiple regression
using available data for glacier area, temperature, and precipitation. From this regression, we obtain a prediction equation that enables us to extrapolate the past data to predict the area of a glacier based on future values for temperature and precipitation. We applied this method to predict the evolution of Midtfonna glacier using two climate scenarios from the Intergovernmental Panel on Climate Change’s 2013 report on climate change (IPCC, 2013). For each scenario, we found the estimated year when the Midtfonna glacier will completely melt. Under scenario 1, with a local projected increase in temperature and precipitation by 2100 of 1.5°C and 5%, respectively, Midtfonna glacier is estimated to disappear by the year 2086. Under scenario 2, with a local projected increase in temperature and precipitation by 2100 of 4.5°C and 15%, respectively, Midtfonna glacier is estimated to disappear by the year 2079. Our method for predicting the evolution of a glacier can be applied to other glaciers provided the necessary climate and glacier data, outlined in section 4.2.1, is available.

Our model of glacier retreat shows that climate change and the retreat of glaciers are inextricably linked. As the earth’s climate currently undergoes significant shifts, the rate at which glaciers retreat is accelerating. The rapid melting of glaciers around the world is a serious issue, as negative repercussions for both humans and animals follow from the rapid disappearance of glaciers. Alteration of delicate ecological systems and loss of habitat for numerous species, severe reduction of water supplies for irrigation and drinking supplies, loss of hydroelectric power sources, and rising sea levels are major problems that await us if glaciers continue to melt at an increasing rate. The methods we have presented for predicting the future of glaciers can help people prepare for the disappearance of an important source of life and adapt to the realities of a changing climate.

References


## Midtfonna Glacier and Climatic Factor Data

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Midtfonna Total Area data from “A Remote Sensing Investigation into the evolution of Folgefonna Glacier over the last 150 years” Benjamin A. Robson
Temperature, Precipitation, and Wind Speed data from eklima.met.no
North Atlantic Oscillation Index data from https://climatedataguide.ucar.edu/sites/default/files/climate_index_files/nao_station_monthly.txt