PREScribing SCALAR CURVATURE

LIU HONG
(B.Sc., Peking University)

A THESIS SUBMITTED FOR THE
Degree of Doctor of Philosophy

Supervisor
Professor Xingwang Xu

DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF SINGAPORE

2015
Declaration

I hereby declare that the thesis is my original work and it has been composed by myself in its entirety.
I have duly acknowledged all the sources of information which have been used in the thesis.
This thesis has also not been submitted for any degree in any university previously.

Hong, Liu
October 26, 2015
Acknowledgements

I would like to thank all people who have helped and inspired me through my thesis.

First and foremost I would like to express the deepest appreciation and sincerest gratitude to my supervisor, Professor Xingwang Xu, who has supported me during my doctoral study with his patience and knowledge. Without his guidance, encouragement and persistent help, this dissertation would not have been possible. I am grateful to be his student, and his perpetual energy and enthusiasm in research enabled me to develop a good understanding of research.

I would like to thank Professor G. Tian, Professor P. Yang, Professor A. Chang, Professor S. T. Yau, Professor R. Schoen and Professor J. Case for their kind help and encouragement. I thank Professor Fei Han, Professor Lei Zhang, Dr. Hengfei Lu, Dr. Feng Zhou, Dr. Hong Zhang, Dr. Ruilun Cai, Dr. Jize Yu, Dr. Cailhua Luo, Dr. Yuke Li, Dr. Ran Wei, Dr. Liuqin Yang, Dr. Ying Cui, Dr. Yufei Zhao, Dr. Ruixiang Zhang, Dr. Xiaowei Jia and his girlfriend Master Ya Sun, Dr. Chen Yang and his wife Master Dan Zheng, Dr. Yan Wang and his boyfriend Dr. Xiaofei Zhao, Dr. Jinjong Yu and his girlfriend Master Yile Li, Dr. Hawk, Master Yuchen Wang, Miss Andrea, Dr. Yuke Li, Master Qiong Liu, Miss Jiayu Zhu and Miss Xueliang Hu for their assistance, stories and jokes.

I would like to thank my grandmother for her spiritual support. I would like to thank Mr. Bo Yang, Ms. Zhao Han, Miss Mengqi Hong and their families for their aid in my most difficult time.

Finally, I would like to thank my father Heng Long Hong and mother Liu Ying Xie for giving my life in the first place for unconditional support and love.
# Contents

Acknowledgements ........................................... i

Summary ......................................................... iv

List of Symbols ............................................... v

1 Introduction ................................................ 1

2 Elementary estimates and long time existence ....... 3

3 Reduction ..................................................... 18

4 The case $u_{\infty} = 0$ ....................................... 20

5 Convergence .................................................. 26
Summary

The prescribing scalar curvature problem originated from the classical Yamabe problem. Since Yamabe constant is a conformal invariant, we can divide conformal closed manifolds into positive, negative and zero Yamabe cases. The negative Yamabe case is well understood. So we focus on and solve the positive case here.
## Symbol

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>a conformal closed manifold</td>
</tr>
<tr>
<td>$n$</td>
<td>dimension of $N$</td>
</tr>
<tr>
<td>$\chi(N)$</td>
<td>Euler characteristic of $N$</td>
</tr>
<tr>
<td>$S_g, S$</td>
<td>scalar curvature</td>
</tr>
<tr>
<td>$[g_0]$</td>
<td>conformal class of $g_0$</td>
</tr>
<tr>
<td>$Y(N, [g_0])$</td>
<td>Yamabe constant</td>
</tr>
<tr>
<td>$E_g[u]$</td>
<td>Yamabe energy</td>
</tr>
<tr>
<td>$S^n$</td>
<td>standard sphere</td>
</tr>
<tr>
<td>$\exp_p$</td>
<td>exponential map from $p$</td>
</tr>
<tr>
<td>$T_p(N)$</td>
<td>tangent space at $p$</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>Laplacian operator</td>
</tr>
<tr>
<td>$\nabla$</td>
<td>gradient operator</td>
</tr>
<tr>
<td>$2^*$</td>
<td>$\frac{2n}{n-2}$</td>
</tr>
<tr>
<td>$Vol$</td>
<td>volume</td>
</tr>
<tr>
<td>$\partial$</td>
<td>partial differential operator</td>
</tr>
<tr>
<td>$o(1)$</td>
<td>tending to zero</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

Prescribing scalar curvature problem is a fundamental problem in modern geometry. This problem in diffeomorphic class is well understood from the work of J. L. Kazdan and F. W. Warner [15],[16]. For closed 2-manifolds, Kazdan and Warner proved in 1975 [15] that the obvious sign condition demanded by the Gauss-Bonnet Theorem:

(a) $K$ positive somewhere if $\chi(N) > 0$,
(b) $K$ changes sign unless $K \equiv 0$ if $\chi(N) = 0$,
(c) $K$ negative somewhere if $\chi(N) < 0$,

is sufficient for a smooth function $K$ on a given closed 2-manifold $N$ to be the Gaussian curvature of some metric on $N$. In case of dimension $n \geq 3$, they proved if $f$ is negative somewhere, then $f$ could be the scalar curvature of some smooth metric in [16]. Furthermore, if $N$ admits a metric with positive scalar curvature, any smooth function could be a scalar curvature of some Riemannian metric (see [16]). They also proved if $N$ is non-compact of dimension $n \geq 3$ diffeomorphic to an open submanifold of some closed manifold, then every smooth function could be a scalar curvature (see [16]).

So we consider the prescribing scalar curvature problem in conformal class which is more subtle and plays a crucial role in modern conformal geometry. The Yamabe constant of a conformal manifold $(N, [g_0])$ is defined as

$$Y(N, [g_0]) = \inf_{g \in [g_0]} \frac{\int_N S_g d\mu_g}{(\int_N d\mu_g)^{\frac{n-2}{n}}} ,$$

where $S_g$ is the scalar curvature of $g$ and $\mu_g$ is the volume density of the metric $g$. The Yamabe
energy of a function $u$ on $(N, g)$ of class $H^1$ is defined as

$$E_g[u] = \int_N (c_n |\nabla u|^2 + S_g u^2) d\mu_g \cdot \frac{Vol(N, g)}{Vol(N)}$$

$c_n = \frac{4(n - 1)}{n - 2}.$

Since Yamabe constant is a conformal invariant, we can divide conformal manifolds into positive, negative and zero Yamabe cases. The negative Yamabe case is easier and well understood. So we focus on positive Yamabe case in this paper. In 1976, T. Aubin proved a perturbation theorem with non-sphere condition. Namely, if $(N, g)$ with positive Yamabe constant is not conformal to standard sphere, then there is $\kappa > 1$ depending on $(N, g)$ such that any smooth positive function $f$ satisfying $\sup f \geq \kappa \inf f$ is the scalar curvature of a conformal metric (see [1]). In 1986, J. F. Escobar and R. M. Schoen proved in [12] that for given closed 3-manifold of positive scalar curvature which is not conformally equivalent to standard sphere, any smooth function which is positive somewhere could be the scalar curvature of a conformal metric. They treated the case of dimension $n \geq 4$ with locally conformally flat restriction.


During 2005-2007, S. Brendle provided a new proof of Yamabe problem by constructing a family of test functions in [3] and [4] which is reminiscent of Schoen’s proof of Yamabe problem [19]. Both proofs study the role of Green’s functions on manifolds and rely on Yau’s positive mass theorem (see [22]).

Since Q. A. Ngo and X. Xu [17] have solved the zero Yamabe case, we treat the non-sphere positive Yamabe case here as the last piece of the jigsaw puzzle. Our main result is:

**Theorem 1.** Given a closed conformal manifold $(N, [g_0])$ which is not conformally equivalent to the round sphere and satisfies $Y(N, [g_0]) > 0$. If $n \geq 3$, then any positive smooth function on $N$ could be the scalar curvature of some $g \in [g_0]$.

It seems that G. Bianchi and E. Egnel [7] have constructed a counter example to Theorem 1. But we have clarified their case in [14].
Chapter 2

Elementary estimates and long time existence

From now on, we always assume \((N, [g_0])\) is a manifold satisfying the conditions in Theorem 1. Let \(f\) be a positive smooth function on \(N\). We can choose \(g_0\) in the conformal class such that the scalar curvature \(S_0\) of \(g_0\) is a positive constant. Since \(N\) is compact, \(0 < m \leq f \leq M\). We write one conformal metric \(g\) as \(g = u^{\frac{4}{n-2}}g_0\). To find the desired metric we need to find a positive solution of the following equation

\[-c_n \Delta u + S_0 u = f u^{\frac{n+2}{n-2}}.\]

Consider the scalar curvature flow

\[u_t = \frac{n-2}{4} [\alpha(t) f - S(t)] u\]

where \(S(t)\) is the scalar curvature of \(g(t)\) which has standard formula

\[S = u^{-\frac{n+2}{n-2}} (-c_n \Delta u + S_0 u)\]

where \(c_n = \frac{4(n-1)}{n-2}\), \(\Delta = \Delta_{g_0}\) and \(S_0\) is the scalar curvature of \(g_0\). By multiplying a constant, we can assume

\[\int_N f d\mu_{g_0} = 1.\]
From now on, the integration is always on the whole manifold \( N \). First let us define

\[
E[u] := \int (c_n|\nabla u|^2 + S_0u^2) d\mu = \int Sd\mu_g
\]

where \( d\mu_g \) is the volume density of \( g \) and \( d\mu = d\mu_{g_0}, \cdot | = | \cdot |_{g_0} \).

The factor \( \alpha(t) \) is chosen such that

\[
0 = \frac{d}{dt} \int f d\mu_g = \frac{d}{dt} \int f u^{2^*} d\mu_0 = 2^* \int f u^{\frac{n+2}{n-2}} u_t d\mu
\]

where \( 2^* = \frac{2n}{n-2} \) is the critical exponent and \( \nabla = \nabla_{g_0} \). Hence the natural choice for this factor is

\[
\alpha(t) = \frac{\int f Sd\mu_g}{\int f^2d\mu_g}
\]

For \( p \geq 1 \), we define

\[
F_p(g(t)) = \int |\alpha f - S|^p d\mu_g
\]

**Lemma 1.** If a positive smooth function \( u \) satisfies

\[
u_t = \frac{n-2}{4} (\alpha f - S) u, \\
S = u^{\frac{n+2}{n-2}} (-c_n \Delta u + S_0 u),
\]

then

\[
\frac{d}{dt} E[u] = \frac{d}{dt} E_f[u] = -\frac{n-2}{2} F_2.
\]

**Proof.**

\[
\frac{d}{dt} E_f[u] = \frac{dE}{d\mu_g} \frac{\nabla u}{\nabla u^{\frac{n-2}{n+2}} u_t d\mu} - 2E[u] \int f u^{\frac{n+2}{n-2}} u_t d\mu
\]

Since

\[
\frac{d}{dt} (|\nabla u|^2) = 2(\nabla u, \nabla u_t)
\]

\[
= 2 \text{div}(u_t \nabla u) - 2u_t \Delta u,
\]
we have
\[
\frac{dE[u]}{dt} = \frac{d}{dt} \int (c_n |\nabla u|^2 + S_0 u^2) d\mu \\
= 2 \int (-c_n \Delta u + S_0 u) u_t d\mu ,
\]
\[
\frac{d}{dt} E_f[u] = 2 \left( \int f d\mu_g \right) \frac{2-n}{n} \int (-c_n \Delta u + S_0 u - \alpha f u^{\frac{n+2}{2}}) u_t d\mu \\
= -2 \left( \int f d\mu_g \right) \frac{2-n}{n} \int (\alpha f - S) u^{\frac{n+2}{2}} u_t d\mu \\
= -\frac{n-2}{2} \left( \int f d\mu_g \right) \frac{2-n}{n} \int (\alpha f - S)^2 d\mu_g \\
= -\frac{n-2}{2} \int (\alpha f - S)^2 d\mu_g .
\]
The identity holds because \( E_f[u] = E[u] \) in our flow.

Hence \( E_f[u(t)] \) is decreasing. For any initial function \( u(0) = u_0 \in H^1(N, g_0) \), we have
\[
E[u(t)] = E_f[u(t)] \left( \int f d\mu_g \right)^{\frac{n-2}{2}} \leq C E_f[u_0] < \infty , C = C(N, u_0) .
\]

**Lemma 2.** There exists a constant \( \alpha_1 \) depending on \( f \) and \( u_0 \) such that
\[
|\alpha(t)| \leq \alpha_1 .
\]

**Proof.**
\[
|\alpha(t)| = \left| \int f S d\mu_g \right| \\
\leq C \left| \int f S d\mu_g \right| \left( \int f d\mu_g \right)^2 \\
\leq C \left| \int f S d\mu_g \right| ,
\]
\[
\int f S d\mu_g = -\frac{c_n}{2} \int (\Delta f) u^2 d\mu + c_n \int f |\nabla u|^2 d\mu + \int f S_0 u^2 d\mu .
\]

Thus,
\[
|\alpha(t)| \leq C E[u] \leq C E[u_0] .
\]
Chapter 2. Elementary Estimates and Long Time Existence

Lemma 3. \[(\alpha f - S)_t = (n-1)\Delta_g(\alpha f - S) + S(\alpha f - S) + \alpha' f .\]

Proof. By the formula
\[S = u^{-\frac{n+2}{n-2}}(-c_n\Delta u + S_0 u),\]
we have
\[S_t = u^{-\frac{n+2}{n-2}}(-c_n\Delta u_t + S_0 u_t) - \frac{n+2}{n-2}u^{-\frac{2n}{n-2}}u_t(-c_n\Delta u + S_0 u)\]
\[= \frac{n-2}{4}u^{-\frac{n+2}{n-2}}\{ -(n-1)u\Delta(\alpha f - S)+2(\nabla(\alpha f - S), \nabla u) + (\alpha f - S)\Delta u \}
+ \frac{n+2}{4}(\alpha f - S)S\]
\[= -\frac{n-2}{4}(\alpha f - S)S + \frac{n-2}{4}u^{-\frac{n+2}{n-2}}(\alpha f - S)(-c_n\Delta u + S_0 u) - (n-1)u^{-\frac{n+2}{n-2}}(\alpha f - S)\Delta u + 2(\nabla(\alpha f - S), \nabla u) .\]

Since
\[\Delta_g = u^{-\frac{2n}{n-2}}\det(g_0)^{\frac{n}{2}}\partial_j(u^2\det(g_0)^{\frac{1}{2}}g_0^{ij}\partial_i)\]
\[= u^{-\frac{n}{n-2}}\Delta + 2u^{-\frac{n+2}{n-2}}(\nabla u, \nabla u),\]
we have
\[S_t = -(\alpha f - S)S - (n-1)\Delta_g(\alpha f - S) ,\]
\[(\alpha f - S)_t = (n-1)\Delta_g(\alpha f - S) + (\alpha f - S)S + \alpha' f .\]

Since the case of dimension three has been proved by Escobar and Schoen, we always assume \(n \geq 4\) from now on.

Lemma 4. \(F_2\) is bounded.

Proof.
\[\frac{df}{dt} = \frac{d}{dt} \int_{\Sigma} \alpha f(S) |\partial_t g| d\mu_g \leq \int_{\Sigma} \alpha f(S) |\partial_t g| d\mu_g + \int_{\Sigma} \alpha f(S)^2 d\mu_g \]
\[\leq \frac{n}{2} \int_{\Sigma} \alpha f(S)^2 d\mu_g \quad \text{since} \ Y(N) > 0 \]
\[\leq CF_2\]
for some constant \(C > 0\). Thus,
\[F_2(g(t)) \leq F_2(g(0)) + C \int_0^t F_2(g(s)) d\mu_g(s) \]
\[ \leq F_2(g(0)) + \frac{2C}{n-2} E[u_0]. \]

In later estimates, we will constantly utilize positive Yamabe condition.

**Lemma 5.** There exists a constant \( C \) such that \( \alpha' \leq C F_2^{\frac{1}{2}} \).

**Proof.**

\[ |\alpha'| = (\int f^2 d\mu_g)^{-1} |\frac{d}{dt} \int f S d\mu_g - \alpha \frac{d}{dt} \int f^2 d\mu_g| \]
\[ \leq C |\frac{d}{dt} \int f S d\mu_g - \alpha \frac{d}{dt} \int f^2 d\mu_g| \]
\[ \leq C| - (n-1) \int f \Delta_g (\alpha f - S) d\mu_g + \frac{n-2}{2} \int f S (\alpha f - S) d\mu_g - \frac{n\alpha}{2} \int f^2 (\alpha f - S) d\mu_g| \]
\[ \leq C| - (n-1) \int (\Delta_g f)(\alpha f - S) d\mu_g| + C_1 F_1 + C_2 F_2 + C_3 F_2^{\frac{1}{2}}. \]

By Cauchy inequality and boundness of \( F_2 \), we have

\[ |\alpha'| \leq C \int (\Delta_g f)(\alpha f - S) d\mu_g| + C F_2^{\frac{1}{2}}. \]

Moreover,

\[ \int (\Delta_g f)(\alpha f - S) d\mu_g = \int (\alpha f - S)(\Delta f) u^{-\frac{4}{n-2}} d\mu_g + 2 \int (\alpha f - S)(\nabla u, \nabla f) u^{-\frac{n+2}{n-2}} d\mu_g. \]

By Cauchy inequality,

\[ |\int (\alpha f - S)(\Delta f) u^{-\frac{4}{n-2}} d\mu_g| \leq (\sup |\Delta f|)(\int u^{-\frac{2(n-4)}{n-2}} d\mu)^{\frac{1}{2}} F_2^{\frac{1}{2}}. \]

Since \( \frac{2(n-4)}{n-2} < \frac{2n}{n-2} \), \( \int u^{-\frac{2(n-4)}{n-2}} d\mu \) is bounded.

\[ |\int (\alpha f - S)(\nabla u, \nabla f) u^{-\frac{n+2}{n-2}} d\mu_g| \leq \sup(|\nabla f|)(\int |\nabla u|^2 u^{-\frac{4}{n-2}} d\mu)^{\frac{1}{2}} F_2^{\frac{1}{2}}. \]

Hence we only need to bound \( \int |\nabla u|^2 u^{-\frac{4}{n-2}} d\mu. \)

If \( n = 6 \),

\[ \int |\nabla u|^2 u^{-\frac{4}{5}} d\mu = \int |\nabla u|^2 u^{-1} d\mu. \]
\[ = - \int \frac{\Delta u}{\sqrt{u}} \sqrt{u} \ln u \, d\mu \]
\[ \leq \left( \int (\Delta u)^2 u^{-1} \, d\mu \right)^{\frac{1}{2}} \left( \int u(\ln u)^2 \, d\mu \right)^{\frac{1}{2}}. \]

Since \( \frac{2 \times 6}{6-2} = 3 \), we have
\[ \int u(\ln u)^2 \, d\mu = \int_{u \geq 1} u(\ln u)^2 \, d\mu + \int_{u < 1} u(\ln u)^2 \, d\mu \]
\[ \leq \int u^3 \, d\mu + \int \frac{4}{e^2} \, d\mu \]
\[ \leq C. \]

Since \( S = u^{-2}(S_0 u - 5 \Delta u) \), we have
\[ \int (\Delta u)^2 u^{-1} \, d\mu = \frac{1}{25} \int (S_0^2 u - 2S_0 Su^2 + S^2 u^3) \, d\mu. \]

Since \( \int u^3 \, d\mu = 1, \int (\alpha f - S)^2 u^3 \, d\mu \) and \( \alpha \) are bounded, we know the above term is bounded too and the desired estimate follows.

If \( n \neq 6 \),
\[ \int (S - S_0 u^{-\frac{4}{n-2}}) u^2 \, d\mu = \frac{4(n-1)(n-6)}{(n-2)^2} \int |\nabla u|^2 u^{-\frac{4}{n-2}} \, d\mu. \]

So the result follows from
\[ \int S_0 u^{-\frac{2n-8}{n-2}} \, d\mu \leq C, \]
\[ \int S u^2 \, d\mu \leq \left( \int S^2 \, d\mu_{\infty} \right)^{\frac{1}{2}} \left( \int u^{\frac{2(n-4)}{n-2}} \, d\mu \right)^{\frac{1}{2}} \leq C. \]

Now we are able to give a lower bound of the scalar curvature \( S \).

**Lemma 6.**

\[ S - \alpha f \geq \gamma, \quad t \geq 0, \]

where \( \gamma := \min\{S_0 - \alpha(0)f, -\frac{2}{\sqrt{\pi}} \sqrt{\max\{0, \alpha' f + \alpha^2 f^2\}}, -\alpha f\}. \)

**Proof.** Define
\[ L = \partial_t - (n-1)\Delta_g + \alpha f - \gamma, \]
then
\[ LS = S_t - (n-1)\Delta_g S + (\alpha f - \gamma)S \]
\[ = S(S - \gamma) - (n-1)\Delta_g(\alpha f) . \]

By using comparison function \( w(t) = \alpha f + \gamma \), we have
\[ Lw = \alpha_t f - (n-1)\Delta_g(\alpha f) + (\alpha f - \gamma)(\alpha f + \gamma) \]
\[ \leq -\frac{1}{4}\gamma^2 - (n-1)\Delta_g(\alpha f) \]
\[ \leq S(S - \gamma) - (n-1)\Delta_g(\alpha f) \]
\[ \leq LS . \]

But
\[ w(0) \leq \alpha(0)f + \gamma \leq S_0 . \]

Thus we get a lower bound of \( S(t), t \geq 0 \), by the maximum principle for a linear parabolic differential equation.

The following estimates will help us to achieve the long time existence of the flow.

**Lemma 7.** For any \( T > 0 \), there exists \( C = C(T) \) such that
\[ C^{-1} \leq u(x,t) \leq C, \quad (x,t) \in N \times [0,T) . \]

**Proof.** From the equation of the flow and lemma 6, we know
\[ u(t) \leq e^{-\frac{\alpha f}{4}t}u(0), \quad t \in [0,T] . \]

Define a constant
\[ P = S_0 + \sup_{t \in [0,T]} \sup_{N}[-(\alpha f + \gamma)u^{\frac{4}{n-2}}] . \]

From Lemma 6 again, we have
\[ 0 \leq (S - \alpha f - \gamma)u^{\frac{n+2}{n-2}} \]
\[ = -c_n \Delta u + S_0 u - (\alpha f + \gamma)u^{\frac{n+2}{n-2}} \]
\[ \leq -c_n \Delta u + Pu . \]
Hence by the Corollary A.3 in [3], we have
\[ 1 = Vol(N, g_0) \leq C \inf_N u(\sup_N u)^{n+2}. \]

So the assertion follows. \(\square\)

Define
\[ \delta = \sup_{t\in[0,T]} \|\alpha f + \gamma\|_\infty + 1. \]

From Lemma 6, we have
\[ S + \delta \geq \alpha f + \gamma + |\alpha f + \gamma| + 1 \geq 1. \]

**Lemma 8.** For any \( p > 2 \), we have
\[
\frac{d}{dt} \int (S + \delta)^{p-1} d\mu_g = -\frac{4(p-2)(n-1)}{p-1} \int |\nabla_g (S + \delta)^{\frac{n+2}{2}}|^2 d\mu_g \\
+ (p-1)(p-2)(n-1) \int (S + \delta)^{p-3} (\nabla_g (S + \delta), \nabla_g (\alpha f + \delta))_g d\mu_g \\
- (\frac{n}{2} + 1 - p) \int (S + \delta)^{p-1} (S - \alpha f) d\mu_g - (p-1)\delta \int [(S + \delta)^{p-2} - (\alpha f + \delta)^{p-2}] (S - \alpha f) d\mu_g \\
- (p-1)\delta \int (\alpha f + \delta)^{p-2} (S - \alpha f) d\mu_g.
\]

**Proof.**
\[
\frac{d}{dt} \int (S + \delta)^{p-1} d\mu_g = \frac{d}{dt} \int (S + \delta)^{p-1} u^{\frac{n+2}{2}} u d\mu \\
= \frac{2n}{n-2} \int (S + \delta)^{p-1} u^{\frac{n+2}{2}} u d\mu + (p-1) \int (S + \delta)^{p-2} S_t d\mu_g \\
= \frac{n}{2} \int (S + \delta)^{p-1} (\alpha f - S) d\mu_g - (1-p) \int (S + \delta)^{p-2} (S - \alpha f) S d\mu_g - (p-1)(n-1) \int (S + \delta)^{p-2} \Delta_g (\alpha f - S) d\mu_g \\
= -\frac{n}{2} \int (S + \delta)^{p-1} (S - \alpha f) d\mu_g - (1-p) \int (S + \delta)^{p-2} (S - \alpha f)(S + \delta - \delta) d\mu_g \\
- (p-1)(n-1) \int (S + \delta)^{p-2} \text{div}_g (\nabla_g (\alpha f - S)) d\mu_g \\
= - (\frac{n}{2} + 1 - p) \int (S + \delta)^{p-1} (S - \alpha f) d\mu_g - (p-1)\delta \int (S + \delta)^{p-2} (S - \alpha f) d\mu_g.
\]
\[-(p-1)(n-1) \int \{ \text{div}_g[(S + \delta)^{p-2}\nabla_g(\alpha f - S)] - (p-2)(S+\delta)^{p-3}\langle \nabla_g(S+\delta), \nabla_g(\alpha f - S) \rangle_g \} d\mu_g \]

\[= -(\frac{n}{2} + 1 - p) \int (S + \delta)^{p-1}(S - \alpha f) d\mu_g - (p-1)\delta \int [(S+\delta)^{p-2} - (\alpha f + \delta)^{p-2}](S - \alpha f) d\mu_g \]

\[-(p-1)\delta \int (\alpha f + \delta)^{p-2}(S - \alpha f) d\mu_g + (p-1)(p-2)(n-1) \int (S+\delta)^{p-3}\langle \nabla_g(S+\delta), \nabla_g(\alpha f + \delta - S - \delta) \rangle_g d\mu_g. \]

Since

\[(S + \delta)^{p-3}\langle \nabla_g(S + \delta), \nabla_g(\alpha f + \delta) \rangle_g - (S + \delta)^{p-3}|\nabla_g(S + \delta)|^2_g \]

\[= (S + \delta)^{p-3}\langle \nabla_g(S + \delta), \nabla_g(\alpha f + \delta) \rangle_g - \frac{4}{(p-1)^2}|\nabla_g(S + \delta)|^2_g, \]

the identity follows. \(\square\)

For any \(p \geq 2\), we have

\[\frac{d}{dt} F_p = \frac{d}{dt} \int |\alpha f - S|^p d\mu_g = p \int |\alpha f - S|^{p-2}(\alpha f - S)(\alpha f - S)_t d\mu_g + \frac{n}{2} \int |\alpha f - S|^p(\alpha f - S) d\mu_g \]

\[= p \int |\alpha f - S|^{p-2}(\alpha f - S)(n-1)\Delta_g(\alpha f - S) + (\alpha f - S)S + \alpha' f] d\mu_g + \frac{n}{2} \int |\alpha f - S|^p(\alpha f - S) d\mu_g. \]

Since

\[|\alpha f - S|^{p-2}(\alpha f - S)\Delta_g(\alpha f - S) = \text{div}_g(|\alpha f - S|^{p-2}(\alpha f - S)\nabla_g(\alpha f - S)) \]

\[-\langle \nabla_g(|\alpha f - S|^{p-2}(\alpha f - S)), \nabla_g(\alpha f - S) \rangle_g, \]

\[\nabla_g(|\alpha f - S|^{p-2}(\alpha f - S)) = (p-1)|\alpha f - S|^{p-2}\nabla_g(\alpha f - S), \]

hence

\[\frac{d}{dt} \int |\alpha f - S|^p d\mu_g = -p(p-1)(n-1) \int |\alpha f - S|^{p-2}|\nabla_g(\alpha f - S)|^2_g d\mu_g + p \int |\alpha f - S|^pS d\mu_g \]

\[+p\alpha' \int f|\alpha f - S|^{p-2}(\alpha f - S) d\mu_g + \frac{n}{2} \int |\alpha f - S|^p(\alpha f - S) d\mu_g \]

\[= -p(p-1)(n-1) \int |\alpha f - S|^{p-2}|\nabla_g(\alpha f - S)|^2_g d\mu_g + p \int \alpha f|\alpha f - S|^p d\mu_g \]

\[p\alpha_t \int f|\alpha f - S|^{p-2}(\alpha f - S) d\mu_g + (\frac{n}{2} - p) \int |\alpha f - S|^p(\alpha f - S) d\mu_g. \]
Lemma 9. For any $p > \max\{\frac{n}{2}, 2\}$, we have

$$\frac{d}{dt} F_p + C_1 F_p^{\frac{2}{n-2}} \leq C_2 (F_p + F_p^{2p-n-2}),$$

where the positive constants $C_1, C_2$ are independent of $t$.

Proof. Since

$$c_n |\nabla_g (\alpha f - S)|^2_g = \frac{p^2(n-1)}{n-2} |\alpha f - S|^{p-2} |\nabla_g (\alpha f - S)|^2_g ,$$

so from the derivative formula we have

$$\frac{d}{dt} F_p = -\frac{(p-1)(n-2)}{p} \int (c_n |\nabla_g (\alpha f - S)|^2_g + S|\alpha f - S|^p) d\mu_g$$

$$+ [p + \frac{(p-1)(n-2)}{p}] \int S|\alpha f - S|^p d\mu_g + \frac{n}{2} \int |\alpha f - S|^{p} (\alpha f - S) d\mu_g + pa' \int f|\alpha f - S|^{p-2} (\alpha f - S) d\mu_g .$$

By lemma 5 and Hölder inequality, we have $|\alpha'| \leq CF_p^{\frac{1}{2}} \leq CF_p^\frac{1}{2}$, and thus

$$|pa' \int f|\alpha f - S|^{p-2} (\alpha f - S) d\mu_g| \leq C|\alpha'| F_{p-1}$$

$$\leq CF_p .$$

With the help of positive Yamabe condition, we have

$$\frac{d}{dt} F_p \leq -C(\int |\alpha f - S|^{\frac{np}{n-2}} d\mu_g)^{\frac{n-2}{n}} + CF_p + CF_{p+1} .$$

By Hölder inequality and Young inequality we deduce that

$$F_{p+1} = \int |\alpha f - S|^{p+1} d\mu_g$$

$$\leq (\int |\alpha f - S|^{\frac{np}{n-2}} d\mu_g)^{\frac{n-2}{np}} (\int |\alpha f - S|^p d\mu_g)^{\frac{2p-n-2}{2p}}$$

$$\leq \varepsilon (\int |\alpha f - S|^{\frac{np}{n-2}} d\mu_g)^{\frac{n-2}{n}} + C(\varepsilon)(\int |\alpha f - S|^p d\mu_g)^{\frac{2p-n-2}{2p-n}} .$$

By choosing suitable $\varepsilon$, the assertion follows. \qed

Lemma 10. For any $T > 0$, there exists $C = C(T)$ such that

$$F_{\frac{n^2}{2(n-2)}} \leq C(T), \quad t \in [0, T) .$$
Proof. Since \( S + \delta, \alpha f + \delta > 0 \) and \( p > 2 \), we deduce

\[
[(S + \delta)^{p-2} - (\alpha f + \delta)^{p-2}](S - \alpha f) = [(S + \delta)^{p-2} - (\alpha f + \delta)^{p-2}][(S + \delta) - (\alpha f + \delta)] \geq 0.
\]

So we can choose \( p = \frac{n+2}{2} > 2 \) in Lemma 8 to achieve

\[
d \frac{d}{dt} \int (S + \delta)^\frac{2}{n} d\mu_g \leq - \frac{4(n-1)(n-2)}{n} \int |\nabla_g(S + \delta)^\frac{2}{n} d\mu_g
\]

\[
+ \frac{n(n-1)(n-2)}{4} \int (S + \delta)^\frac{n-4}{\nu} \langle \nabla_g(S + \delta), \nabla_g(\alpha f + \delta) \rangle_g d\mu_g - \frac{n}{2} \delta \int (\alpha f + \delta)^\frac{n-2}{2}(S - \alpha f) d\mu_g.
\]

From Lemma 6 we have

\[
- \frac{n}{2} \delta \int (\alpha f + \delta)^\frac{n-2}{2}(S - \alpha f) d\mu_g \leq - \frac{n}{2} \delta \gamma \int (\alpha f + \delta)^\frac{n-2}{2} d\mu_g
\]

\[
\leq C(T) \int d\mu_g
\]

\[
\leq C(T).
\]

By Cauchy inequality and Young inequality we have

\[
| \int (S + \delta)^\frac{n-4}{\nu} \langle \nabla_g(S + \delta), \nabla_g(\alpha f + \delta) \rangle_g d\mu_g | = \frac{4}{n} | \int (S + \delta)^\frac{n-4}{\nu} \langle \nabla_g(S + \delta)^\frac{\nu}{2}, \nabla_g(\alpha f + \delta) \rangle_g d\mu_g |
\]

\[
\leq \frac{4}{n} \int (S + \delta)^\frac{n-4}{\nu} |\nabla_g(S + \delta)^\frac{\nu}{2}||\nabla_g(\alpha f + \delta)||_g d\mu_g
\]

\[
\leq \varepsilon \int |\nabla_g(S + \delta)^\frac{\nu}{2}|_{\nu}^2 d\mu_g + C(\varepsilon) \int |\nabla_g(\alpha f + \delta)|_{\nu}^2 d\mu_g + C(T, \varepsilon) \int (S + \delta)^\frac{\nu}{2} d\mu_g \]

\[
\leq \varepsilon \int |\nabla_g(S + \delta)^\frac{\nu}{2}|_{\nu}^2 d\mu_g + C(\varepsilon) \int |\nabla_g(\alpha f + \delta)|_{\nu}^2 d\mu_g + C(T, \varepsilon) \int (S + \delta)^\frac{\nu}{2} d\mu_g \]

We know from Lemma 7 that \( u \) is bounded on \([0, T]\). Hence by using Hölder inequality we can achieve

\[
| \int (S + \delta)^\frac{n-4}{\nu} \langle \nabla_g(S + \delta), \nabla_g(\alpha f + \delta) \rangle_g d\mu_g | \leq \varepsilon \int |\nabla_g(S + \delta)^\frac{\nu}{2}|_{\nu}^2 d\mu_g + C(T, \varepsilon) \int (S + \delta)^\frac{\nu}{2} d\mu_g \]

Set \( y(t) = \int (S + \delta)^\frac{2}{n} d\mu_g \). By choosing small \( \varepsilon \) we can get

\[
d \frac{dy}{dt} + \int |\nabla_g(S + \delta)^\frac{\nu}{2} d\mu_g \leq C_2(T)y^\frac{n-4}{n} + C_2(T).
\]

We claim that \( y \) is bounded by a constant depending on \( T \).

Since \( n \geq 4 \), \( (y^\frac{4}{n})' \leq C_2 + C_2y^\frac{4-n}{n} \). So \( y^\frac{4}{n} \geq 1 \Rightarrow (y^\frac{4}{n})' \leq 2C_2 \) and the claim follows.
From the definition of $y$ we know that $y$ has the volume of $N$ as a lower bound. Integrating the inequality
\[ \frac{dy}{dt} + \int |\nabla_g (S + \delta)^{\frac{n}{2}}| d\mu_g \leq C_2(T) y^{\frac{n-2}{n}} + C_2(T) \]
we get
\[ \int_0^T \int |\nabla_g (S + \delta)^{\frac{n}{2}}| d\mu_g dt \leq C_3(T). \]

Due to our positive Yamabe condition, we have
\[ (\int |S + \delta|^{\frac{n^2}{2(n-2)}} d\mu)^{\frac{n-2}{n}} \leq C \int (c_n |\nabla| S + \delta|^{\frac{n}{2}} + S_0 |S + \delta|^{\frac{n}{2}}) d\mu \leq C_4(T). \]

So
\[ \int_0^T (\int |S + \delta|^{\frac{n^2}{2(n-2)}} d\mu_g)^{\frac{n-2}{n}} dt \leq C_5(T) \int_0^T (\int |S + \delta|^{\frac{n^2}{2(n-2)}} d\mu)^{\frac{n-2}{n}} dt \leq C_6(T). \]

Since $S + \delta, \alpha f + \delta > 0$, we have
\[ |S - \alpha f| = |(S + \delta) - (\alpha f + \delta)| \leq \max\{|S + \delta|, C_7(T)\} \]
and
\[ |S - \alpha f|^p \leq |S + \delta|^p + |\alpha f + \delta|^p. \]

Since $\frac{n-2}{n} \leq 1$, we have
\[ (\int |S - \alpha f|^{\frac{n^2}{2(n-2)}} d\mu_g)^{\frac{n-2}{n}} \leq (\int |S + \delta|^{\frac{n^2}{2(n-2)}} d\mu_g)^{\frac{n-2}{n}} + (\int |\alpha f + \delta|^{\frac{n^2}{2(n-2)}} d\mu_g)^{\frac{n-2}{n}}. \]

Setting $p = \frac{n^2}{2(n-2)}$ in Lemma 9 we get
\[ (\log F^{\frac{n^2}{2(n-2)}})' \leq C + CF^{\frac{n-2}{2(n-2)}}. \]

So the assertion follows by integrating above inequality. \hfill \Box

By the argument of Proposition 2.6 in [3] or Lemma 2.11 in [11], we achieve the following inequality:

Lemma 11. For any $\lambda \in (0, \min\{\frac{4}{n}, 1\})$, $T > 0$, there exists $C = C(\lambda, T) > 0$ such that
\[ |u(x, t) - u(y, s)| \leq C[d_N(x, y)^{\lambda} + |t - s|^{\frac{n-2}{2}}], \]
for any $x, y \in N$, $t, s \in [0, T)$, $0 < |t - s| < 1$. 

CHAPTER 2. ELEMENTARY ESTIMATES AND LONG TIME EXISTENCE

So the long time existence of the scalar curvature flow follows from the standard result of parabolic equations. For example, one can read Theorem 8.3 and Theorem 8.4 in [24].

Lemma 12. \( \int_0^\infty F_2 dt < \infty \).

Proof. Since the flow has long time existence, from lemma 1 we know
\[
\int_0^\infty F_2 dt = \frac{2}{n-2} (E[u_0] - \lim_{t \to \infty} E[u(t)]) < \infty .
\]
Thus the assertion follows.

Lemma 13.
\( F_2 \to 0 \).

Proof. In the proof of lemma 4, we know \( \frac{d}{dt} F_2 \leq CF_2 \). Thus, for any \( t > t_\nu > 0 \), we have
\[
F_2(t) \leq F_2(t_\nu) + \int_{t_\nu}^\infty F_2 .
\]
By lemma 12, we can pick a time sequence \( t_\nu \to \infty \) such that \( F_2(t_\nu) \to 0 \) and the result follows.

Now we are going to achieve the convergence of \( \alpha(t) \).

Lemma 14.
\( \alpha = E[u] + o(1), \ t \to \infty. \)

Proof. Since
\[
\frac{d}{dt} Vol(N, g(t)) = \frac{n}{2} \int (\alpha f - S) d\mu_g ,
\]
by Lemma 13 and Cauchy inequality, we have \( \frac{d}{dt} Vol(N, g(t)) \to 0 \). We also have the identity
\[
\frac{d}{dt} Vol(N, g(t)) = \frac{n}{2} \alpha \int f d\mu_g - \frac{n}{2} E[u] = \frac{n}{2} (\alpha - E[u]) .
\]
So the result follows.

Lemma 15. For any \( p \in [2, \frac{n}{2}] \),
\[
\frac{d}{dt} F_p \leq CF_p .
\]
Proof. By the derivative formula in the proof of lemma 9 and the inequalities
\[
|p\alpha f - S|^{p-2}(\alpha f - S)\,d\mu_g \leq C F_p^{\frac{1}{2}} F_{p-1} \leq CF_p
\]
\[
\alpha f - S \leq -\gamma ,
\]
we deduce the desired inequality. \hfill \square

**Lemma 16.** If \(2 \leq p < \frac{n}{2}\) and \(F_p\) is integrable over \((0, \infty)\), then \(F_{p+1}\) is also integrable over \((0, \infty)\).

**Proof.** By the formula
\[
(n - p) \int |\alpha f - S|^p(S - \alpha f)\,d\mu_g = p \int \alpha f|\alpha f - S|^p\,d\mu_g
\]
\[
+ p\alpha' \int f|\alpha f - S|^{p-2}(\alpha f - S)\,d\mu_g - \frac{d}{dt} F_p - \frac{4(p-1)(n-1)}{p} \int |\nabla|\alpha f - S|^2\,d\mu_g ,
\]
we have
\[
\int_0^T \int |\alpha f - S|^p(S - \alpha f)\,d\mu_g \,dt \leq C \int_0^T [F_p + \frac{2}{n-2p} F_p(0)]\,dt \leq C .
\]
Since \(\alpha f - S \leq -\gamma\), we have
\[
\int_0^T F_{p+1}\,dt \leq \int_0^T \int |\alpha f - S|^p(S - \alpha f + 2|\gamma|)\,d\mu_g \,dt \leq C
\]
for any \(T > 0\). As \(C\) is independent of \(T\), the result follows. \hfill \square

Now we have the following fundamental result.

**Lemma 17.** For any \(p \in [1, \frac{n}{2}]\), \(F_p \to 0\).

**Proof.** By lemma 12, lemma 16 and Hölder inequality, \(F_p\) is integrable for any \(p \in [2, \frac{n}{2}]\), and the result follows from the same argument in the proof of lemma 13. \hfill \square

Define
\[
\alpha_\infty := \lim_{t \to \infty} \alpha(t) ,
\]
\[
E_\infty := \lim_{t \to \infty} E_f(t) ,
\]
\[
E_f := \inf \{ E_f[u] : u \in H_1, u \text{ is not constantly } 0 \} .
\]
Since $E[u]$ is monotone, $E_\infty$ clearly exists and $\alpha_\infty = E_\infty$ by lemma 14.

**Corollary 1.** For any $p \in [1, \frac{n}{2}]$, $\int |S - \alpha_\infty f|^pd\mu_g \to 0$.

*Proof.* It is direct from lemma 14, lemma 17 and the inequality

$$|S - \alpha_\infty f|^p \leq 2^{p-1}(|S - \alpha f|^p + |\alpha - \alpha_\infty|^p|f|^p).$$
Chapter 3

Reduction

The classical result of T. Aubin in *Nonlinear Analysis on Manifolds. Monge-Ampere equations* 131, shows that:

**Lemma 18.** If $N$ is not a standard sphere, then $E_f \leq Y(\sup f)^{-\frac{n-2}{n}} := n(n-1)\omega_n^\frac{2}{n} (\sup f)^{-\frac{n-2}{n}}$. Moreover, if $E_f < n(n-1)\omega_n^\frac{2}{n} (\sup f)^{-\frac{n-2}{n}}$, the prescribing scalar curvature problem of $f$ has a smooth positive solution.

Therefore, from now on, we only need to consider the case where $E_f = n(n-1)\omega_n^\frac{2}{n} (\sup f)^{-\frac{n-2}{n}}$. For abbreviation, let $u_\nu = u(t_\nu)$, $\lambda_\nu = \lambda_{u(t_\nu)}$, $\alpha(g_\nu) = \alpha_\nu$ and $g_\nu = g(t_\nu)$. We have the following compactness result from Brendle [3]:

**Lemma 19** (Bubble Decomposition).

After passing to a subsequence if necessary, we can find an $l \in \mathbb{N}$, a smooth function $u_\infty$ and a sequence of $m$-tuples $(x^*_{k,\nu}, \varepsilon^*_k)_{1 \leq k \leq l}$ such that:

1. $\frac{4(n-1)}{n-2} \Delta u_\infty - S_0 u_\infty + \alpha_\infty f u_\infty^{\frac{n+2}{n-2}} = 0$,

where $\alpha_\infty$ is the limit of the subsequence.

2. For any $i \neq j$, we have

\[ d\left(\frac{x^*_{i,\nu}, x^*_{j,\nu}}{\varepsilon^*_i}, \frac{x^*_{j,\nu}}{\varepsilon^*_j}\right)_{\nu \to \infty} \to \infty. \]

3. $\left\| u_\nu - u_\infty - \sum_{k=1}^{l} \bar{u}(x^*_{k,\nu}, \varepsilon^*_k)\right\|_{H^1(N)} \to 0$.
where \( \tilde{u}_{(p, \epsilon)} \) are a family of test functions satisfying

\[
\lim_{\epsilon \to 0} \epsilon^{\frac{n-2}{2}} \tilde{u}_{(p, \epsilon)}(\exp_p(\epsilon \xi)) = \left(\frac{1}{1 + |\xi|^2}\right)^{\frac{n-2}{2}}
\]

for all \( p \in N \) and \( \xi \in T_pN \).

Proof. Since \( E_f[u(t)] \) is decreasing, \( \{u_\nu\} \) is bounded in \( H^1(N) \). By passing to a subsequence, \( \alpha_\nu \) tends to a limit \( \alpha_\infty \) and \( u_\infty \) is defined as the weak limit of \( \{u_\nu\} \). Then the argument from Struwe’s proof of Proposition 2.1 in [20] works. The first and third results are from Struwe [20], and the second statement is due to Bahri and Coron [5].

The recent work of Q. A. Ngo and X. Xu, [17] also gives a proof of the above lemma in the special case of scalar curvature flow. A useful corollary is that, when \( E[u_0] < 2^{\frac{n}{2}} Y(\sup f)^{-\frac{n-2}{2}} \), the bubble number \( l \leq 1 \). The reason is that, if \( l \geq 2 \), then for sufficiently small \( \epsilon > 0 \), we have contradictory inequalities \( \frac{l}{2}(\sup f)^{\frac{n-2}{2}} < l(\frac{Y}{\alpha_\infty})^{\frac{n}{2}} \leq \limsup_{t \to \infty} \int_{B(x_j, \epsilon)} f^{\frac{n}{2}} d\mu_g \leq (\sup f)^{\frac{n}{2}-1} \limsup_{t \to \infty} \int f d\mu_g \leq (\sup f)^{\frac{n}{2}} \). We also refer the reader to [17] for the detailed computation.

By the classical strong maximum principle (see Theorem 8.19 in [13]) we can deduce the following proposition from

\[-\frac{4(n-1)}{n-2} \Delta u_\infty + S_0 u_\infty = \alpha_\infty f u_\infty^{\frac{n+2}{2}} \geq 0 .\]

**Proposition 1.** If \( u_\infty \) vanishes at one point, then it vanishes everywhere.

From now on, as \( u_\infty \) itself is a solution in case \( u_\infty > 0 \), we only consider the case \( u_\infty = 0 \).
Chapter 4

The case $u_\infty = 0$

From now on, we restrict the initial data $u(0)$ to satisfy $E_f[u(0)] < 2^{\frac{2}{n}}E_f$. Thus, the bubble number $l$ in the bubble decomposition could only be 1 noticing that $E_f[u(t)]$ is decreasing w.r.t. $t, u_\infty = 0$ and the bubble decomposition is $H_1$ convergence. For every time sequence $t_\nu$, let $x^{\nu}$ be one maximum point of $u(t_\nu)$. Since $N$ is compact, by passing to a subsequence, we can assume that $x^{\nu} \to x_\infty \in N$. Set

$$k_\nu = u_\nu(x^{\nu})^{\frac{2}{n-2}}$$

$$\varepsilon^{\nu} = \frac{1}{k_\nu}$$

where $u_\nu = u(t_\nu)$. We choose the normal coordinate system $\{\xi\}$ w.r.t. $g_0$ which means

$$(g_0)_{ij} = \delta_{ij} + O(|\xi|^2),$$

$$dV_{g_0} = (1 + O(|\xi|^2))d\xi.$$  

Thus,

$$\Delta_{g_0} = \Delta + b_i \partial_i + d_{ij} \partial_{ij}^2$$

where

$$b_i = O(|\xi|^2),$$

$$d_{ij} = O(|\xi|^2).$$

Now we follow the method from Q. A. Ngo and X. Xu [17] to construct the bubble decomposition explicitly. First of all, we pick a smooth cut-off function on $\mathbb{R}^n$ for $\delta > 0$ such
that

\[ 0 \leq \eta_\delta \leq 1, \]

\[ \chi_{B(0,\delta)} \leq \eta_\delta \leq \chi_{B(0,2\delta)}, \]

\[ |\nabla \eta_\delta| \leq \frac{C}{\delta}. \]

\[ \eta_\nu(\xi) := \eta_\delta\left(\frac{\xi}{\kappa_\nu}\right). \]

We could write down the standard bubble as

\[ V_\nu(\xi) = \eta_\nu(\xi)\frac{n(n-1)}{\alpha_\infty f(x_\infty)} \frac{2\varepsilon^{\nu}}{(\varepsilon^{\nu})^2 + |\xi - \xi^{\nu}|^2} \]

Set \( \tau_n = \frac{n(n-2)}{n(n-2)+2} \) and \( \delta_\nu = (\varepsilon^{\nu})^{\tau_n} \). Given a pair \((x, \varepsilon) \in M \times [0, \infty)\), we define the test functions as the approximation of the bubble by

\[ V_{(x,\varepsilon)}(\xi) = \chi_{(x,\delta_\nu)}\frac{n(n-1)}{\alpha_\infty f(x_\infty)} \frac{2\varepsilon^{\nu}}{(\varepsilon^{\nu})^2 + |\xi - \xi^{\nu}|^2} \]

\[ := \chi_{(x,\delta_\nu)}\frac{n(n-1)}{\alpha_\infty f(x_\infty)} \frac{2\varepsilon^{\nu}}{(\varepsilon^{\nu})^2 + |\xi - \xi^{\nu}|^2} \]

where \( \xi^{\nu} \) is the coordinate of \( x \). Lemma 5.8, 5.9, 5.10 and 5.11 from [22] imply that \( u_\nu - v_\nu \to 0 \) in \( H^1 \) sense. By the Sobolev imbedding \( H^1(N) \hookrightarrow L^{\frac{2n}{n-2}}(N) \), we have

\[ 1 = \lim_{\nu} \int f u_\nu^{2n} \, d\mu \]

\[ = \lim_{\nu} \int f V_\nu^{2n-2} \, d\mu \]

\[ = \lim_{\nu} \int f V_\nu^{2n-2} \, d\mu \]

\[ = \lim_{\nu} \int_{B(x,2\delta_{\nu})} f\left(\frac{n(n-1)}{\alpha_\infty f(x_\infty)}\right)^\frac{n-2}{4} \frac{2\varepsilon^{\nu}}{(\varepsilon^{\nu})^2 + |\xi - \xi^{\nu}|^2} \, d\xi \]

\[ = \lim_{\nu} \int_{B(x,2\delta_{\nu})} f\left(\frac{n(n-1)}{\alpha_\infty f(x_\infty)}\right)^\frac{n-2}{4} \frac{2\varepsilon^{\nu}}{(\varepsilon^{\nu})^2 + |\xi|} \, d\xi \]

\[ = f(x_\infty)\frac{n(n-1)}{\alpha_\infty f(x_\infty)} \frac{2}{n} \omega_n. \]

Thus,

\[ \alpha_\infty = f(x_\infty)^{\frac{n}{n-1}} n(n-1)^{\frac{n}{n-1}} \omega_n^2 = f(x_\infty)^{\frac{n}{n-1}} Y. \]
CHAPTER 4. THE CASE $U_\infty = 0$

For every $\nu \in \mathbb{N}$, we define $\mathcal{A}_\nu = \{(x, \varepsilon, \gamma) \in (M \times \mathbb{R}^+ \times \mathbb{R}^+) : d(x, x') \leq \varepsilon^\nu, \frac{1}{2} \leq \frac{\varepsilon}{\varepsilon^\nu} \leq 2, \frac{1}{2} \leq \gamma \leq 2\}$. Letting the coordinate of $x_\nu$ be $\xi_\nu$, as in $[3]$, we can find a 3-tuple $(x_\nu, \varepsilon_\nu, \gamma_\nu) \in \mathcal{A}_\nu$ such that

$$E[u_\nu - \gamma_\nu V(x_\nu, \varepsilon_\nu)] \leq E[u_\nu - \gamma V(x, \varepsilon)]$$

for all $(x, \varepsilon, \gamma) \in \mathcal{A}_\nu$. Since $\alpha$ is bounded, by passing to a subsequence we assume that $\alpha_\nu \to \alpha_\infty$, and thus we have $E_\nu \to E_\infty$.

**Lemma 20.**

$$\|u_\nu - \gamma_\nu V(x_\nu, \varepsilon_\nu)\|_{H^1} \xrightarrow{\nu \to \infty} 0.$$  

**Proof.** The assertion follows from the definition of $(x_\nu, \varepsilon_\nu, \gamma_\nu)$. \qed

**Lemma 21.** We have

$$d(x_\nu, x') \leq o(1)\varepsilon^\nu,$$

$$\frac{\varepsilon_\nu}{\varepsilon^\nu} = 1 + o(1),$$

$$\gamma_\nu = 1 + o(1).$$

In particular, $(x_\nu, \varepsilon_\nu, \gamma_\nu)$ is an interior point of $\mathcal{A}_\nu$ for sufficiently large $\nu$.

**Proof.** We have

$$\|\gamma_\nu V(x_\nu, \varepsilon_\nu) - V(x', \varepsilon')\|_{H^1} \leq \|u_\nu - \gamma_\nu V(x_\nu, \varepsilon_\nu)\|_{H^1} + \|u_\nu - V(x', \varepsilon')\|_{H^1} = o(1).$$

Thus the assertion follows from lemma 20. \qed

Let us decompose $u_\nu$ into

$$u_\nu = v_\nu + w_\nu,$$

where

$$v_\nu = \gamma_\nu V(x_\nu, \varepsilon_\nu).$$

From Lemma 20, we know $E[w_\nu] = o(1)$.

**Lemma 22.** (1) We have

$$|\int V^{n+2} w_\nu d\mu| \leq o(1)(\int |w_\nu|^{n-2} d\mu)^{\frac{n-2}{2n}}.$$
CHAPTER 4. THE CASE $U_\infty = 0$

(2) We have
\[ |\int V_{\frac{n+2}{n}}^n \frac{\nu^2 - d(x, x)^2}{\nu^2} w_{\nu} d\mu| \leq o(1) \left( \int |w_{\nu}|^{\frac{n}{n-2}} d\mu \right)^{\frac{n-2}{2n}}. \]

(3) We have
\[ |\int V_{\frac{n+2}{n}}^n \frac{\nu \exp^{-1}(x)}{\nu^2 + d(x, x)^2} w_{\nu} d\mu| \leq o(1) \left( \int |w_{\nu}|^{\frac{n}{n-2}} d\mu \right)^{\frac{n-2}{2n}}. \]

Proof. By the definition of $(x, \nu, \gamma)$, we have
\[ \int (c_n (\nabla V_{(x, \nu)}), \nabla w_{\nu}) + S_0 V_{(x, \nu)} w_{\nu}) d\mu = 0, \]
hence
\[ \int [c_n \Delta V_{(x, \nu)}] w_{\nu} d\mu = 0. \]

From the estimate
\[ \|c_n \Delta V_{(x, \nu)} - S_0 V_{(x, \nu)}\| L^{\frac{2n}{n-2}} = o(1), \]
we can conclude that
\[ |\int V_{\frac{n+2}{n}}^n w_{\nu} d\mu| \leq o(1) \|w_{\nu}\| L^{\frac{2n}{n-2}}. \]

This proves (1). The results in (2) and (3) follow from (1) and Lemma 18. \( \square \)

Lemma 23. If $\nu$ is sufficiently large, then we have
\[ \frac{n+2}{n-2} \alpha_\infty \int v_{\nu}^{\frac{4}{n-2}} w_{\nu}^2 f d\mu \leq (1 - c) \int (c_n |\nabla w_{\nu}|^2 + S_0 w_{\nu}^2) d\mu \]
for some positive constant $c$ independent of $\nu$.

Proof. By the definition of $v_{\nu}$ and lemma 21, we have
\[ \int |v_{\nu}^{\frac{4}{n-2}} - V_{(x, \nu)}^{\frac{4}{n-2}}|^2 d\mu = o(1). \]

Therefore, we only need to prove that
\[ \frac{n+2}{n-2} \alpha_\infty \int V_{\frac{n+2}{n}}^n w_{\nu}^2 f d\mu \leq (1 - c) \int (c_n |\nabla w_{\nu}|^2 + S_0 w_{\nu}^2) d\mu \]
for some positive constant $c$. 
Suppose this is not true. Upon rescaling, we obtain a sequence of functions \( \{ \tilde{w}_\nu : \nu \in \mathbb{N} \} \) such that
\[
\int (c_n |\nabla \tilde{w}_\nu|^2 + S_0 \tilde{w}_\nu^2) d\mu = 1
\]
and
\[
\lim_{\nu \to \infty} \frac{n + 2}{n - 2} \alpha_\infty \int V_{(x, \xi)}^{\frac{n}{n-2}} \tilde{w}_\nu^2 f d\mu \geq 1.
\]
Note that \( \int |\tilde{w}_\nu|^{\frac{2n}{n-2}} d\mu \leq Y(N)^{-\frac{n}{n-2}}. \) Take a sequence \( \{ N_\nu \} \) such that \( N_\nu \to \infty, N_\nu \xi_\nu \to 0. \) Let \( \Omega_\nu = B_{N_\nu \xi_\nu}(x_\nu) \setminus B_{N_\nu \xi_\nu}(x_\nu). \) We have
\[
\lim_{\nu \to \infty} \int_{\Omega_\nu} V_{(x, \xi)}^{\frac{n}{n-2}} \tilde{w}_\nu^2 d\mu > 0,
\]
and
\[
\lim_{\nu \to \infty} \int_{\partial \Omega_\nu} (c_n |\nabla \tilde{w}_\nu|^2 + S_0 \tilde{w}_\nu^2) d\mu \leq \lim_{\nu \to \infty} \frac{n + 2}{n - 2} \alpha_\infty \int V_{(x, \xi)}^{\frac{n}{n-2}} \tilde{w}_\nu^2 f d\mu
\]
We now define a sequence of functions \( \hat{w}_\nu : TM_{x_\nu} \to \mathbb{R} \) by
\[
\hat{w}_\nu(\xi) = \frac{1}{\nu} \tilde{w}_\nu(\exp_{x_\nu}(\xi)), \quad \xi \in TM_{x_\nu}.
\]
This sequence satisfies
\[
\lim_{\nu \to \infty} \int_{\{ \xi \in TM_{x_\nu} : |\xi| \leq N_\nu \}} c_n |\nabla \hat{w}_\nu(\xi)|^2 d\xi \leq 1
\]
and
\[
\lim_{\nu \to \infty} \int_{\{ \xi \in TM_{x_\nu} : |\xi| \leq N_\nu \}} |\hat{w}_\nu(\xi)|^{\frac{2n}{n-2}} d\xi \leq Y(N)^{-\frac{n}{n-2}}.
\]
Hence, if we take the weak limit as \( \nu \to \infty, \) then we obtain a function \( \hat{w} : \mathbb{R}^n \to \mathbb{R} \) such that
\[
\int \left( \frac{1}{1 + |\xi|^2} \right)^2 \hat{w}(\xi)^2 d\xi > 0,
\]
and
\[
\int |\nabla \hat{w}(\xi)|^2 d\xi \leq n(n + 2) \int \left( \frac{1}{1 + |\xi|^2} \right)^2 \hat{w}(\xi)^2 d\xi.
\]
Moreover, it follows from Lemma 18 that
\[
\int \left( \frac{1}{1 + |\xi|^2} \right)^{\frac{n+2}{2}} \hat{w}(\xi) d\xi = 0,
\]
\[
\int \left( \frac{1}{1 + |\xi|^2} \right)^{\frac{n+2}{2}} 1 - |\xi|^2 \hat{w}(\xi) d\xi = 0.
\]
CHAPTER 4. THE CASE $U_\infty = 0$

\[ \int \left( \frac{1}{1 + |\xi|^2} \right)^{\frac{n+2}{2}} \frac{\xi}{1 + |\xi|^2} \hat{w}(\xi) d\xi = 0. \]

Using a result of O. Rey, we conclude that $\hat{w} = 0$ (see [18], Appendix D, pp. 49-51). This is a contradiction.
Chapter 5

Convergence

Lemma 24. Let \( \{t_\nu\} \) be a time sequence tending to infinity. For sufficiently large \( \nu \), there exists a constant \( C \) such that

\[
E[u_\nu] - \alpha_\infty \leq C \left( \int |S(g_\nu) - \alpha_\infty f|^\frac{2n}{n+2} \, d\mu_{g_\nu} \right)^{\frac{n+2}{n}}.
\]

Proof. Using the identity

\[
S(g_\nu) = -u_\nu^{-\frac{n+2}{n-2}} (c_n \Delta u_\nu - S_0 u_\nu)
\]

we obtain

\[
E[u_\nu] = \int (c_n |\nabla u_\nu|^2 + S_0 u_\nu^2) \, d\mu
\]

\[
= \int (c_n |\nabla v_\nu|^2 + S_0 v_\nu^2) \, d\mu + 2 \int u_\nu^{\frac{n+2}{n-2}} S(g_\nu) w_\nu \, d\mu - \int (c_n |\nabla w_\nu|^2 + S_0 w_\nu^2) \, d\mu
\]

\[
= E[v_\nu] + 2 \int u_\nu^{\frac{n+2}{n-2}} [S(g_\nu) - \alpha_\infty f] w_\nu \, d\mu
\]

\[
- \int (c_n |\nabla w_\nu|^2 + S_0 w_\nu^2 - \frac{n + 2}{n - 2} \alpha_\infty f v_\nu^{\frac{1}{n-2}} w_\nu^2 ) \, d\mu
\]

\[
+ \alpha_\infty \int [ - \frac{n + 2}{n - 2} v_\nu^{\frac{1}{n-2}} w_\nu^2 + 2 (v_\nu + w_\nu) \frac{n+2}{n-2} w_\nu] f \, d\mu.
\]

Using Hölder inequality, we obtain

\[
|\int u_\nu^{\frac{n+2}{n-2}} [S(g_\nu) - \alpha_\infty f] w_\nu \, d\mu| \leq \left( \int u_\nu^{\frac{2n}{n-2}} |S(g_\nu) - \alpha_\infty f|^\frac{2n}{n+2} \, d\mu \right)^{\frac{n+2}{n}} \left( \int |w_\nu|^\frac{2n}{n-2} \, d\mu \right)^{\frac{n-2}{n}}
\]

\[
\leq \left( \int |S(g_\nu) - \alpha_\infty f|^\frac{2n}{n+2} \, d\mu_{g_\nu} \right)^{\frac{n+2}{n}} \left( \int |w_\nu|^\frac{2n}{n-2} \, d\mu_{g_\nu} \right)^{\frac{n-2}{n}}
\]

26
It follows from Lemma 23 that
\[
\int (c_n |\nabla w_\nu|^2 + S_0 w_\nu^2 - \frac{n + 2}{n - 2} \alpha_\infty f v_\nu^{-\frac{4}{n-2}} w_\nu^2) d\mu \geq c \int (c_n |\nabla w_\nu|^2 + S_0 w_\nu^2) d\mu ,
\]
hence
\[
\int (c_n |\nabla w_\nu|^2 + S_0 w_\nu^2 - \frac{n + 2}{n - 2} \alpha_\infty f v_\nu^{-\frac{4}{n-2}} w_\nu^2) d\mu \geq c(\int |w_\nu|^{\frac{2n}{n-2}} d\mu)^{\frac{n-2}{n}} .
\]
By the Cauchy inequality,
\[
\left( \int u_\nu^{\frac{2n}{n-2}} |S(g_\nu) - \alpha_\infty f|^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{n}} \leq C \left( \int u_\nu^{\frac{2n}{n-2}} |S(g_\nu) - \alpha_\infty f|^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{n}} + \frac{c}{2} \left( \int |w_\nu|^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{n}} .
\]
Moreover,
\[
E[v_\nu] - \alpha_\infty = E_f[v_\nu](\int f v_\nu^{-\frac{2n}{n-2}} d\mu)^{\frac{n-2}{n}} - \alpha_\infty
\]
\[
= (E_f[v_\nu] - \alpha_\infty)(\int f v_\nu^{-\frac{2n}{n-2}} d\mu)^{\frac{n-2}{n}} + \alpha_\infty[(\int f v_\nu^{-\frac{2n}{n-2}} d\mu)^{\frac{n-2}{n}} - 1]
\]
\[
\leq (E_f[v_\nu] - \alpha_\infty)(\int f v_\nu^{-\frac{2n}{n-2}} d\mu)^{\frac{n-2}{n}} + \frac{2n}{n} \alpha_\infty[\int f v_\nu^{-\frac{2n}{n-2}} d\mu - 1] .
\]
As calculated in [17] 7.3, we have the estimate
\[
E_f[v_\nu] = E_f[V_{x_\nu, \varepsilon_\nu}]
\]
\[
\leq \alpha_\infty + (\varepsilon_\nu \delta_\nu^{-1}) C[-1 + O(\varepsilon_\nu^{-4})] + O(\delta_\nu^2) .
\]
By lemma 21, we have
\[
E_f[v_\nu] \leq \alpha_\infty + C \varepsilon_\nu^{2n(n-2)+4} [-1 + O(\varepsilon_\nu^{-4})] + O(\varepsilon_\nu^{2n(n-2)+4}) .
\]
Thus, for sufficiently large \( \nu \),
\[
E_f[v_\nu] \leq \alpha_\infty - C \varepsilon_\nu^{\frac{2(n-2)}{n-2}+4} + O(\varepsilon_\nu^{\frac{2n(n-2)}{n-2}}) \leq 0 .
\]
Now we only need to estimate the term
\[
\alpha_\infty \int \left[ - \frac{n + 2}{n - 2} v_\nu^{-\frac{4}{n-2}} w_\nu^2 + 2(v_\nu + w_\nu)^{\frac{n+2}{n-2}} w_\nu + \frac{n - 2}{n} v_\nu^{\frac{2n}{n-2}} - \frac{n - 2}{n} (v_\nu + w_\nu)^{\frac{2n}{n-2}} \right] d\mu - \frac{c}{2} \left( \int |w_\nu|^{\frac{2n}{n-2}} d\mu \right)^{\frac{n-2}{n}} .
\]
We have a pointwise estimate due to Brendle [3] page 180,

\[
| - \frac{n + 2}{n - 2} v^{n-2}_\nu w^2 + 2(v_\nu + w_\nu) \frac{n+2}{n} w_\nu + \frac{n - 2}{n} v^{2n}_{n-2} - \frac{n - 2}{n} (v_\nu + w_\nu)^{\frac{2n}{n-2}} - \frac{n - 2}{n} w^{\frac{2n}{n-2}}_\nu | 
\leq C |v_\nu|^{\max(0, \frac{n-6}{n-2})} |w_\nu|^{\min(\frac{2n}{n-2}, 3)} .
\]

Therefore,

\[
| - \frac{n + 2}{n - 2} v^{n-2}_\nu w^2 + 2(v_\nu + w_\nu) \frac{n+2}{n} w_\nu + \frac{n - 2}{n} v^{2n}_{n-2} - \frac{n - 2}{n} (v_\nu + w_\nu)^{\frac{2n}{n-2}} |
\leq C |v_\nu|^{\max(0, \frac{n-6}{n-2})} |w_\nu|^{\min(\frac{2n}{n-2}, 3)} + \frac{n - 2}{n} |w^{\frac{2n}{n-2}}_\nu | ,
\]

\[
\alpha_\infty \int [- \frac{n + 2}{n - 2} v^{n-2}_\nu w^2 + 2(v_\nu + w_\nu) \frac{n+2}{n} w_\nu + \frac{n - 2}{n} v^{2n}_{n-2} - \frac{n - 2}{n} (v_\nu + w_\nu)^{\frac{2n}{n-2}}] f d\mu 
\leq C \int |v_\nu|^{\max(0, \frac{n-6}{n-2})} |w_\nu|^{\min(\frac{2n}{n-2}, 3)} d\mu + \frac{n - 2}{n} \int |w^{\frac{2n}{n-2}}_\nu | d\mu .
\]

If \( n \geq 6, \)

\[
C \int |v_\nu|^{\max(0, \frac{n-6}{n-2})} |w_\nu|^{\min(\frac{2n}{n-2}, 3)} d\mu + \frac{n - 2}{n} \int |w^{\frac{2n}{n-2}}_\nu | d\mu 
= C \int |w_\nu|^\frac{2n}{n-2} d\mu + \frac{n - 2}{n} \int |w^{\frac{2n}{n-2}}_\nu | d\mu 
= o(\int |w_\nu|^\frac{2n}{n-2} d\mu)^{\frac{n-2}{n}} .
\]

If \( n < 6, \)

\[
C \int |v_\nu|^{\max(0, \frac{n-6}{n-2})} |w_\nu|^{\min(\frac{2n}{n-2}, 3)} d\mu + \frac{n - 2}{n} \int |w^{\frac{2n}{n-2}}_\nu | d\mu 
= C \int |v_\nu|^\frac{n-6}{n-2} |w_\nu|^3 d\mu + \frac{n - 2}{n} \int |w^{\frac{2n}{n-2}}_\nu | d\mu 
\leq C(\int |w_\nu|^\frac{2n}{n-2} d\mu)^{\frac{n-6}{2n-2}} (\int |w_\nu|^\frac{2n}{n-2} d\mu)^{\frac{3(n-2)}{2n}} + \frac{n - 2}{n} \int |w^{\frac{2n}{n-2}}_\nu | d\mu 
\leq C(\int |w_\nu|^\frac{2n}{n-2} d\mu)^{\frac{3(n-2)}{2n}} + \frac{n - 2}{n} \int |w^{\frac{2n}{n-2}}_\nu | d\mu .
\]

Since \( \frac{3n-6}{2n} > \frac{n-2}{n} \), the desired inequality holds for sufficiently large \( \nu. \)

\[\square\]

**Corollary 2.** For any \( c > 0 \) and \( 0 < \gamma < 1 \) there exists \( t_0 > 0 \) such that, for any \( t > t_0, \)

\[
E[u] - \alpha_\infty \leq c(\int |S(g) - \alpha_\infty f|^{\frac{2n}{n-2}} d\mu_g)^{\frac{n+2}{2n}(1+\gamma)} .
\]
CHAPTER 5. CONVERGENCE

Proof. If this is not true, there exists a $\gamma$ and a time sequence $\{t_\nu\}$ such that

$$E[u_\nu] - \alpha_\infty > c(\int |S(g_\nu) - \alpha_\infty f|^{\frac{2n}{n+2}} d\mu_{g_\nu})^{\frac{n+2}{2n}}(1+\gamma).$$

Meanwhile, for sufficiently large $\nu$, we have

$$E[u_\nu] - \alpha_\infty \leq C(\int |S(g_\nu) - \alpha_\infty f|^{\frac{2n}{n+2}} d\mu_{g_\nu})^{\frac{n+2}{2n}}(1+\gamma).$$

So we deduce

$$\left(\int |S(g_\nu) - \alpha_\infty f|^{\frac{2n}{n+2}} d\mu_{g_\nu}\right)^{\frac{n+2}{2n}}(1+\gamma) > \frac{C}{C}$$

which is contradictory to corollary 1.

Lemma 25. For any $0 < \gamma < 1$, there exists $t_0 > 0$ and $C > 0$ such that, for any $t > t_0$,

$$E[u] - \alpha_\infty \leq C(\int |S - \alpha f|^{\frac{2n}{n+2}} d\mu_{g_\nu})^{\frac{n+2}{2n}}(1+\gamma).$$

Proof. We have

$$\left(\int |S(g_\nu) - \alpha_\infty f|^{\frac{2n}{n+2}} d\mu_{g_\nu}\right)^{\frac{n+2}{2n}}(1+\gamma) = \|S - \alpha_\infty f\|_{L^{\frac{2n}{n+2}}}^{1+\gamma}$$

$$\leq C\|S - \alpha f\|_{L^{\frac{2n}{n+2}}}^{1+\gamma} + C\|\alpha - \alpha_\infty\|_{L^{\frac{2n}{n+2}}}^{1+\gamma}.$$

Since

$$\alpha - \alpha_\infty = \alpha - E[u] + E[u] - \alpha_\infty$$

$$= \int (\alpha f - S) d\mu_{g} + E[u] - \alpha_\infty,$$

we have

$$\|\alpha - \alpha_\infty\|_{L^{\frac{2n}{n+2}}}^{1+\gamma} \leq CF_1^{1+\gamma} + C(E[u] - \alpha_\infty)^{1+\gamma},$$

$$(E[u] - \alpha_\infty)^{1+\gamma} \leq c^{1+\gamma}(\int |S(g_\nu) - \alpha_\infty f|^{\frac{2n}{n+2}} d\mu_{g_\nu})^{\frac{n+2}{2n}}(1+\delta)(1+\gamma),$$

$$(E[u] - \alpha_\infty)^{1+\gamma} = o\left((\int |S(g_\nu) - \alpha_\infty f|^{\frac{2n}{n+2}} d\mu_{g_\nu})^{\frac{n+2}{2n}}(1+\gamma)\right).$$

Thus the desired result follows.

By a similar argument in corollary 2, we can deduce
Corollary 3. For any $c > 0$ and $0 < \gamma < 1$ there exists $t_0 > 0$ such that, for any $t > t_0$,  

$$E[u] - \alpha_\infty \leq c(s) \int \frac{2n}{n+2} (S - \alpha f)^{\frac{n+2}{2n}} (1+\gamma) \, dt.$$

Proposition 2.

$$\int_0^\infty \left( \int u(t) \frac{2n}{n+2} (S - \alpha f)^2 \, d\mu \right)^{\frac{1}{2}} \, dt \leq C.$$

Proof. It follows from corollary 3 that there exists $0 < \gamma < 1, t_0 > 0$ and $C > 0$ such that, for any $t > t_0$,

$$E[u(t)] - \alpha_\infty \leq C\left( \int u(t) \frac{2n}{n+2} |S - \alpha f|^{\frac{n+2}{2n}} \, d\mu \right)^{\frac{n+2}{2n}} (1+\gamma) \, dt.$$

It follows from Lemma 1 that

$$\frac{d}{dt} \left( E[u(t)] - \alpha_\infty \right) \leq -cF_2 \leq -cF_2 \frac{n+2}{2n} \leq -c \left( E[u(t)] - \alpha_\infty \right)^{\frac{2}{1+\gamma}}, t \geq t_0.$$

Hence, for any $t > t_0$,

$$\frac{d}{dt} \left( E[u(t)] - \alpha_\infty \right)^{\frac{1}{1+\gamma}} \geq c,$$

and

$$(E[u(t)] - \alpha_\infty)^{-\frac{1}{1+\gamma}} \geq ct.$$

$$E[u(t)] - \alpha_\infty \leq Ct^{-\frac{1}{1+\gamma}}.$$

It follows from Hölder inequality that, for any $T > t_0$,

$$\int_T^{2T} \left( \int u(t) \frac{2n}{n+2} (S - \alpha f)^2 \, d\mu \right)^{\frac{1}{2}} \, dt \leq [T \int_T^{2T} \int u(t) \frac{2n}{n+2} (S - \alpha f)^2 \, d\mu \, dt]^{\frac{1}{2}} \leq C[T(E[u(2T)] - E[u(T)])]^{\frac{1}{2}} \leq C[T(E[u(2T)] - \alpha_\infty)]^{\frac{1}{2}} \leq CT^{-\frac{1}{1+\gamma}}.$$

Finally, by picking $L \in \mathbb{N}$ such that $2^L > t_0$, we have

$$\int_0^\infty \left( \int u(t) \frac{2n}{n+2} (S - \alpha f)^2 \, d\mu \right)^{\frac{1}{2}} \, dt = \int_0^{2^L} \left( \int u(t) \frac{2n}{n+2} (S - \alpha f)^2 \, d\mu \right)^{\frac{1}{2}} \, dt$$

$$+ \sum_{k=L}^{\infty} \int_{2^k}^{2^{k+1}} \left( \int u(t) \frac{2n}{n+2} (S - \alpha f)^2 \, d\mu \right)^{\frac{1}{2}} \, dt.$$
CHAPTER 5. CONVERGENCE

\[ \leq C + C \sum_{k=L}^{\infty} 2^{-\frac{k}{n-1}} \leq C. \]

From Proposition 2, we know that the volume does not concentrate:

**Proposition 3.** Given any \( \eta_0 > 0 \), we can find some \( r > 0 \) such that

\[ \int_{B_x(r)} u(t)^{\frac{2n}{n-2}} d\mu \leq \eta_0, \text{ for all } x \in N, \ t \geq 0. \]

**Proof.** Choose \( T > 0 \) such that

\[ \int_T^\infty \left( \int u(t)^{\frac{2n}{n-2}} (S - \alpha f)^2 d\mu \right)^{\frac{1}{2}} dt \leq \frac{\eta_0}{n}. \]

From the long time existence of the flow, we can choose \( r > 0 \) such that

\[ \int_{B_x(r)} u(t)^{\frac{2n}{n-2}} d\mu \leq \eta_0, \text{ for all } x \in N, \ t \in [0,T]. \]

Then, for \( t \geq T \) we have

\[ \int_{B_x(r)} u(t)^{\frac{2n}{n-2}} d\mu \leq \int_{B_x(r)} u(T)^{\frac{2n}{n-2}} d\mu + \frac{n}{2} \int_T^\infty \left( \int u(t)^{\frac{2n}{n-2}} (S - \alpha f)^2 d\mu \right)^{\frac{1}{2}} dt \leq \eta_0. \]

**Proposition 4.** There are positive constants \( c, C \) independent of \( t \) such that

\[ c \leq u(t) \leq C, \text{ for all } t \geq 0. \]

**Proof.** Fix \( \frac{n}{2} < q < p < \frac{n+2}{2} \). It follows from Lemma 2 and Lemma 14 that

\[ \int |S|^p d\mu_g \leq C \]

where \( C \) is a positive constant independent of \( t \). By Proposition 3, we can find a constant
CHAPTER 5. CONVERGENCE

$r > 0$ independent of $t$ such that

$$\int_{B_x(r)} d\mu_g \leq \eta_0 , \text{ for all } x \in N, \ t \geq 0 .$$

It follows from Hölder inequality that

$$\int |S|^q d\mu_g \leq \left( \int_{B_x(r)} d\mu_g \right)^{\frac{p}{p-q}} \left( \int |S|^p d\mu_g \right)^{\frac{q}{p}} .$$

Hence, if we choose $\eta_0$ small enough, then we have

$$\int_{B_x(r)} u(t)^q d\mu \leq \eta_1 , \text{ for all } x \in N, \ t \geq 0 ,$$

where $\eta_1$ is the constant in Proposition A.1 [3] and we can conclude that $u(t)$ is uniformly bounded from above. From Corollary A.3 [3] again, we know that $u(t)$ is uniformly bounded w.r.t. $\|(x,t)\|_{\infty}$.

Thus, under the assumption that the weak limit $u_\infty = 0$, we have proved $u$ converges strongly to a nonzero limit which is a contradiction. Thus the weakly limit is never zero under small initial energy condition which proves Theorem 1.
Bibliography


