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DEFORMATIONS OF FUNCTIONS AND F-MANIFOLDS

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Abstract

We study deformations of functions on isolated singularities. A unified proof of the equality of Milnor and Tjurina numbers for functions on isolated complete intersections singularities and space curves is given. As a consequence, the base space of their miniversal deformations is endowed with the structure of an F-manifold, and we can prove a conjecture of V. Goryunov, stating that the critical values of the miniversal unfolding of a function on a space curve are generically local coordinates on the base space of the deformation.

1. *Introduction*

The theory of Frobenius manifolds plays a central role in mirror symmetry, after the construction by Givental and Barannikov [**2**] of an isomorphism between the quantum cohomology of $\mathbb{CP}^n$ and the base space of the miniversal deformation of the linear function $f = x_1 + \ldots + x_{n+1}$ on the divisor $D := \{x_1 \ldots x_{n+1} = 1\}$. There are now a number of conjectures stating similar isomorphisms between quantum cohomology rings of algebraic varieties and unfoldings of functions on affine varieties. In this paper we propose a singularity theory framework in which at least one of the ingredients making up the definition of Frobenius manifolds, namely the multiplication, can be naturally defined. This structure is known as an F-*manifold* [**10**, **11**].

A seemingly inescapable feature of this construction is that the multiplication is not defined on the whole tangent sheaf of the base space, but only on a certain subsheaf, that of logarithmic vector fields to the discriminant. Contrary to those Frobenius manifolds constructed from unfoldings of isolated hypersurface singularities, our construction does contain some promising candidates for mirrors of algebraic varieties.

The main result of this paper can be stated as follows.

Theorem 1.1. *Let $f\colon (X,x) \longrightarrow (\mathbb{C},0)$ be a function-germ with an isolated singularity on an isolated complete intersection or a space curve. Then the sheaf $\Theta(-\log\Delta)$ of logarithmic vector fields of the discriminant of its miniversal deformation is in a natural manner a (logarithmic) F-manifold. Moreover, each stratum of the logarithmic stratification of the base space inherits this structure.*

The content of the paper is as follows. First we provide a construction of the miniversal deformation of a function on a singular variety. We define a morphism closely related to the Kodaira–Spencer map that will be used to define the

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multiplication. Secondly, we state a condition that ensures the equality of the dimension of the miniversal base space (Tjurina number) and the number of critical points of an unfolding of f in the smooth fibre of the deformation. We then show that the condition holds for functions on isolated complete intersection singularities and (reduced) space curves. This provides a unified treatment to the ($\mu = \tau$)-type results of V. Goryunov [**7**] in the case of functions on isolated complete intersection singularities, and of D. van Straten and D. Mond in the case of functions on space curves [**15**]. Our methods are closer to those of [**15**]. To finish, we prove that the multiplication satisfies an integrability condition, making it into a logarithmic F-manifold.

Before going into the technical details, we would like to work out a relatively simple example in which a full Frobenius structure can be constructed, namely that of the function $f = x^p + y^q$ on the ordinary double point $X\colon xy = 0 \hookrightarrow \mathbb{C}^2$. This case is closely related to the construction of Frobenius manifolds on Hurwitz spaces by B. Dubrovin [**5, 17**], although as we are also collapsing the curve, a new structure on the discriminant is made apparent. The aim of this example is firstly to guide the reader through the rest of the paper and secondly to show how indeed our construction contains some interesting examples in mirror symmetry. It appears to be known among specialists that the resulting Frobenius manifold is the mirror of the orbifold $\mathbb{CP}(p, q)$.

Functions on the double point. Let us consider a function germ $f = x^p + y^q$ on the A_1-singularity $X\colon xy = 0$. The miniversal deformation of f is given by the function $F = c + \sum_{i=1}^{p-1} a_i x^i + x^p + \sum_{i=1}^{q-1} b_i y^i + y^q$ on the fibration $\pi(x, y, a, b, c) = (xy, a, b, c)$, where $a = (a_{p-1}, \ldots, a_1)$ and $b = (b_{q-1}, \ldots, b_1)$ (see Corollary 2.3 and the paragraph below). We take the coordinates (ε, a, b, c) on the base space B of this deformation so that $\Delta\colon \{\varepsilon = 0\} \subset B$ is the (smooth) discriminant of π.

The result of the calculation that we are going to carry out is encapsulated in the following theorem. We remark that certain aspects of the proof, particularly the multiplication and the potentiality, will be evident only after the results given in the main body of this paper have been applied.

THEOREM 1.2. *The sheaf* $\Theta(-\log \Delta)$ *is naturally endowed with a multiplication* $\star$ *and a bilinear pairing* $\langle\, ,\rangle$ *satisfying the following conditions.*

(i) *The pairing* $\langle\, ,\rangle$ *is everywhere non-degenerate with respect to* $\Theta(-\log \Delta)$, *flat and compatible with* $\star$ *in the sense that*

$$\langle u \star v, w\rangle = \langle u, v \star w\rangle \qquad \text{for any } u, v, w \in \Theta(-\log \Delta).$$

(ii) *The multiplication* $\star$ *is commutative, associative and with unit* $\partial/\partial c$. *The unit is globally defined and flat.*

(iii) (*Potentiality*) *At every point* $p \in B \backslash \Delta$ *there exist a germ* $\Phi \in \mathcal{O}_{B,p}$ *and flat coordinates* $(y_1, \ldots, y_{p+q})$ *such that*

$$\frac{\partial^3 \Phi}{\partial y_i \partial y_j \partial y_k} = \left\langle \frac{\partial}{\partial y_i} \star \frac{\partial}{\partial y_j}, \frac{\partial}{\partial y_k} \right\rangle \qquad \text{for all } i, j, k = 1, \ldots, p+q.$$

(iv) *There exists a globally defined conformal Euler vector field* E; *that is:*

$$\mathrm{Lie}_E(\star) = \star \qquad \text{and} \qquad \mathrm{Lie}_E(\langle\, ,\rangle) = \langle\, ,\rangle.$$

Proof. The multiplication is defined by the following lifting process. Lift a vector field $u \in \Theta(-\log\Delta)$ to $\tilde{u}$ such that $t\pi(\tilde{u}) = u \circ \pi$. Differentiating F with respect to $\tilde{u}$, we obtain an element in the ring of germs $\mathcal{O}_{\mathfrak{X},0} = \mathbb{C}\{x, y, a, b, c\}$. We denote by $t'F(u)$ its class in the quotient $\mathcal{O}_{\mathfrak{X},0}/(H)$, where H is the Jacobian determinant

$$\frac{\partial(F, \pi_1 = xy)}{\partial(x,y)} = \sum_{i=1}^{p-1} i a_i x^i + p x^p - \sum_{i=1}^{q-1} i b_i y^i - q y^q.$$

It will be clear from later constructions (although it can also be checked directly) that the map $t'F$ so constructed is an isomorphism of $\mathcal{O}_{B,0}$-free modules of rank $p+q$. We use it to pull back the algebra structure on $\mathcal{O}_{\mathfrak{X},0}/(H)$ so defining a multiplication $\star$ in $\Theta(-\log\Delta)$.

To define the metric, we consider the relative dualising form $\alpha = dx \wedge dy/d\pi_1$ and use it to identify $\mathcal{O}_{\mathfrak{X},0}$ with $\omega_{\mathfrak{X}/B,0}$. Hence we have $dF = H\alpha$, and we consider the Grothendieck residue pairing on $\omega_{\mathfrak{X}/B,0}/\mathcal{O}_{\mathfrak{X},0}\langle dF\rangle$. We use $t'F$ to define a multiplicatively invariant non-degenerate bilinear pairing on $\Theta(-\log\Delta)$. For $u, v \in \Theta(-\log\Delta)_b$, this is explicitly given by

$$\langle u, v\rangle = \int_{\partial X_b} \frac{t'F(u)t'F(v)}{H}\alpha,$$

with ∂X_b the boundary of an appropriate representative of the fibre $\pi^{-1}(b)$. For $b \in B\backslash\Delta$, the fibre X_b is a smooth rational curve with two points deleted, say ∞_1 and ∞_2, corresponding to $x = 0$ and $y = 0$. Hence the pairing can be expressed as

$$\langle u, v\rangle = -\operatorname{Res}_{\infty_1} \frac{t'F(u)t'F(v)}{H}\alpha - \operatorname{Res}_{\infty_2} \frac{t'F(u)t'F(v)}{H}\alpha. \tag{1.1}$$

If we take the free basis of $\Theta(-\log\Delta)$ given by $\varepsilon\,\partial/\partial\varepsilon$ and the rest of the coordinate vector fields, the decomposition (1.1) allows us to express the matrix of $\langle\,,\rangle$ as a sum, each summand corresponding to the residues at a point. A direct calculation, necessary for what follows, shows that the matrix is given by

$$\begin{pmatrix} 0 & 0 & 0 & 4^{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & M_{\infty_1} & 0 \\ 4^{-1} & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 4^{-1} \\ 0 & M_{\infty_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4^{-1} & 0 & 0 & 0 \end{pmatrix} \tag{1.2}$$

where

$$M_{\infty_1} = \begin{pmatrix} 2b_2 & 3b_3 & 4b_4 & \dots & (q-1)b_{q-1} & q \\ 3b_3 & 4b_4 & \dots & \dots & q & 0 \\ 4b_4 & \dots & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ q & \dots & \dots & \dots & \dots & 0 \end{pmatrix}^{-1},$$

and analogously for M_{∞_2}.

To show that the pairing is indeed flat, we compute flat coordinates. Let $b = (\varepsilon_0, a_0, b_0, c_0) \in B\backslash\Delta$. As F has a pole of order p at ∞_2, we can find a local coordinate u at ∞_2 such that $F = u^{-p}$. On the other hand, the function xu is holomorphic and not vanishing at ∞_2. Fixing a branch of log, we can expand it as a power series:

$$\log xu = t_0 + t_1 u + \ldots + t_{p-1}u^{p-1} + O(u^p).$$

Arguing as above, we find a coordinate v such that $F = v^{-q}$ around ∞_2, and a series

$$\log yv = s_0 + s_1 v + \ldots + s_{q-1}v^{q-1} + O(v^q).$$

Write $t = (t_1, \ldots, t_{p-1})$ and $s = (s_1, \ldots, s_{q-1})$. The interested reader can check, by the series expansion of $x = u^{-1}\exp(\sum_{i\geqslant 0} t_i u^i)$ and analogously for y, the following claim.

Claim. *The functions* ($\varepsilon' = \log\varepsilon, t, s, c$) *form a coordinate system. The functions* t *and* s *depend, respectively, only on* a *and only on* b.

We can now show that $\langle\, ,\rangle$ has a constant matrix in these coordinates. Let us take, for example, $\partial/\partial t_i$. We have

$$\begin{gathered}\frac{1}{x}\frac{\partial x}{\partial t_i} = u^i, \qquad \frac{1}{y}\frac{\partial y}{\partial t_i} = -u^i,\\ \frac{\partial F}{\partial x}\frac{\partial x}{\partial t_i} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t_i} = \left(x\frac{\partial F}{\partial x} - y\frac{\partial F}{\partial y}\right)u^i = Hu^i\end{gathered} \tag{1.3}$$

As the functions t depend only on a, according to (1.2) we need only to look at the residues at ∞_2. Hence

$$\begin{aligned}\left\langle \frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j}\right\rangle &= -\operatorname{Res}_{\infty_2}\frac{(Hu^i)(Hu^j)}{H}\alpha = -\operatorname{Res}_{\infty_2} u^{i+j}H\alpha\\ &= -\operatorname{Res}_{\infty_2} u^{i+j}dF = \operatorname{Res}_{u=0} pu^{i+j-(p+1)}du = p\delta_p^{i+j}.\end{aligned} \tag{1.4}$$

A similar calculation, together with the orthogonality relations between a and b (and hence between t and s) deduced from (1.2), proves that $\langle\, ,\rangle$ is flat.

To finish, we prove the last claim. The Euler vector field corresponds to the class of F in $\mathcal{O}_{\mathfrak{X},x}$. It is given by

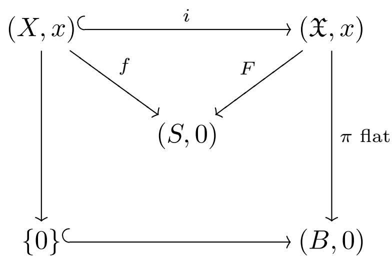
$$E = \left(\frac{1}{p} + \frac{1}{q}\right)\varepsilon\frac{\partial}{\partial\varepsilon} + \sum_{i=1}^{p-1}\left(\frac{p-i}{p}\right)a_i\frac{\partial}{\partial a_i} + \sum_{i=1}^{q-1}\left(\frac{q-i}{q}\right)b_i\frac{\partial}{\partial b_i} + c\frac{\partial}{\partial c}. \tag{1.5}$$

Giving weights $1/p + 1/q$ to the variable ε, $(p-i)/p$ to a_i, $(q-i)/q$ to b_i and 1 to c, we see that a polynomial $h(\varepsilon, a, b, c)$ is quasi-homogeneous of degree d if and only if $\operatorname{Lie}_E(h) = d\cdot h$. From (1.2) we see that the entry in the position ij of $M_{\infty_2}^{-1}$ is (if not constant) quasi-homogeneous of degree $(i+j-p)/p$. Likewise, the entry in the position ij of $M_{\infty_1}^{-1}$ is (if not constant) quasi-homogeneous of degree $(i+j-q)/q$. This proves the claim.

The remaining statements in Theorem 1.2 both follow from Proposition 5.2. The multiplication $\star$ satisfies an integrability condition, and the class of F in the Jacobian algebra (that is, the vector field E) is an Euler vector field (see Definition 5.1 for the precise definition). The integrability condition satisfied by $\star$, together with the flatness of the identity, also guarantees the existence of the potential Φ with the desired properties (see [**10**, Theorem 2.15 and Remark 2.17]). □

2. *Versal deformations of functions on isolated singularities*

Given a reduced analytic variety (X, x) and a germ $f \in \mathfrak{m}_{X,x}$, we will say that f has an isolated singularity if there exists a representative $f\colon U \longrightarrow S$ onto the complex line S such that $U\backslash\{x\}$ is smooth and f is submersive at any point of $U\backslash\{x\}$. The deformation problem with which we will be concerned is referred to as *deformations of* X *over* S; that is, we will consider diagrams such as the following.



The notions of induced diagrams, and pull-back or isomorphism of diagrams, are defined in the customary fashion through maps on the base spaces, keeping the complex line $(S,0)$ fixed. This deformation theory is sometimes denoted by $\mathrm{Def}(X/S)$ and from a purely algebraic point of view it corresponds to the study of the deformations of $\mathcal{O}_{X,x}$ as an $\mathcal{O}_{S,0}$ algebra.

As in any deformation theory, we have the powerful theory of the cotangent cohomology modules at our disposal. Given any holomorphic map $h\colon A \longrightarrow B$ between analytic spaces, and an $\mathcal{O}_A$-module M, we will denote by $T^i_{A/B}(M)$ the ith cotangent cohomology group with coefficients in M (see, for example, [**12**, **13**]). In the absolute case, where B reduces to a point, it is customary to write $T^i_A(M)$. If M is just $\mathcal{O}_A$, then the notation is further simplified to $T^i_{A/B}$. Another piece of notation which we will use is the following. The 0th cotangent cohomology $T^0_{A/B}(M)$ is simply the module of relative vector fields with coefficients in M; that is, $\Theta_{A/B}\otimes_A M$. We use this notation (and those derived from it, like $T^0_B(\mathcal{O}_A) = \Theta(h)$), to be in line with the long-established tradition in singularity theory.

Most of the usefulness of the cotangent modules, as for any cohomology theory, resides in the long exact sequences derived from short exact sequences of modules. In the particular case of the cotangent cohomology modules this is, if possible, even more so. We obtain long exact sequences, not only from short exact sequences of modules but also from homomorphisms of the base rings (a neat review of the properties that we will use can be found in [**3**]). Going back to our function $f\colon (X,x) \longrightarrow (S,0)$, there are two sequences of special relevance. The first is obtained by considering the problem of deforming (X,x) alone. If $\pi\colon (\mathfrak{X},x) \longrightarrow (B,0)$ is a (flat) deformation of (X,x), it is the *Zariski–Jacobi long exact sequence* associated to the ring homomorphism $\mathbb{C} \longrightarrow \mathcal{O}_{B,0} \longrightarrow \mathcal{O}_{\mathfrak{X},x}$. It begins

$$0 \longrightarrow T^0_{\mathfrak{X}/B,x} \longrightarrow T^0_{\mathfrak{X},x} \longrightarrow T^0_{B,0}(\mathcal{O}_{\mathfrak{X},x}) \longrightarrow T^1_{\mathfrak{X}/B,x} \longrightarrow T^1_{\mathfrak{X},x} \longrightarrow \cdots. \tag{2.1}$$

The composite of $\Theta_{B,0} \longrightarrow \Theta(\pi)_x$ with the connecting homomorphism of (2.1) is the *Kodaira–Spencer map* of the deformation. Its kernel is the submodule of *liftable vector fields*, and we will denote it by $\mathcal{L}_{\pi,0}$. In many interesting cases it coincides with those vector fields tangent to the discriminant of π.

If we now consider an extension F of f to the total space $(\mathfrak{X},x)$, we can write $\varphi = (\pi, F)$. The second sequence is also a Zariski–Jacobi sequence, this time corresponding to $\mathcal{O}_{B,0} \longrightarrow \mathcal{O}_{S\times B,0} \longrightarrow \mathcal{O}_{\mathfrak{X},x}$:

$$0 \longrightarrow T^0_{\mathfrak{X}/S\times B,x} \longrightarrow T^0_{\mathfrak{X}/B,x} \longrightarrow T^0_{S\times B/B,0}(\mathcal{O}_{\mathfrak{X},x}) \longrightarrow T^1_{\mathfrak{X}/S\times B,x} \longrightarrow T^1_{\mathfrak{X}/B,x} \cdots \tag{2.2}$$

As before, we will be specially interested in a kernel, this time that of the map $T^1_{\mathfrak{X}/S\times B,x} \longrightarrow T^1_{\mathfrak{X}/B,x}$. We will denote it by $M_{\varphi,x}$. In fact, this module is readily described in more familiar terms using the exactness of (2.2). If $tF\colon \Theta_{\mathfrak{X},x} \longrightarrow \Theta(F)_x$ denotes the tangent map of F, we have

$$M_{\varphi,x} = \frac{\Theta(F)_x}{tF(\Theta_{\mathfrak{X}/B,x})}. \tag{2.3}$$

After all these clarifications, we can state the main lemma of this section. The proof is so straightforward that it can safely be left to the reader. It neatly separates the problem of finding a versal deformation of a function on a singular germ into: firstly, versally deforming (X,x) and, secondly, versally unfolding f.

LEMMA 2.1. *There is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{L}_{\pi,0} & \longrightarrow & \Theta_{B,0} & \longrightarrow & T^1_{\mathfrak{X}/B,x} & \longrightarrow & T^1_{\mathfrak{X},x} \\ & & \downarrow{\scriptstyle -t'F} & & \downarrow & & \Vert{\scriptstyle \mathbb{I}} & & \\ 0 & \longrightarrow & M_{\varphi,x} & \longrightarrow & T^1_{\mathfrak{X}/S\times B,x} & \longrightarrow & T^1_{\mathfrak{X}/B,x} & \longrightarrow & 0 \end{array}$$

where $t'F$ is defined as follows: for $u \in \mathcal{L}_{\pi,0}$, let $\tilde{u} \in \Theta_{\mathfrak{X},x}$ be a lift of u. Then $t'F(u)$ is the class of $tF(\tilde{u})$ in $M_{\varphi,x}$.

REMARK 2.2. The vertical arrow in the middle is the Kodaira–Spencer map of the map φ, understood as a deformation of $f\colon (X,x) \longrightarrow (S,0)$. It is the composite of $\Theta_{B,0} \longrightarrow \Theta(\pi)_x$ with the connecting homomorphism of the Zariski–Jacobi sequence derived from $\mathcal{O}_{S,0} \longrightarrow \mathcal{O}_{S\times B,0} \longrightarrow \mathcal{O}_{\mathfrak{X},x}$.

We deduce the following criterion for versality.

COROLLARY 2.3. *A deformation $\varphi = (F,\pi)$ of $f\colon (X,x) \longrightarrow (S,0)$ is versal if and only if π is versal as a deformation of (X,x) and $t'F$ is surjective.*

Versal deformations can now be easily constructed from a versal deformation π of (X,x). We take $f_1,\dots,f_l$ generators of the vector space $\operatorname{coker} t'f/\mathfrak{m}_{B,0}(\operatorname{coker} t'f)$ and consider the function $F = f + a_1 f_1 + \ldots + a_l f_l$, adding new parameters $a_1,\dots,a_l$. Requiring that π be miniversal and $f_1,\dots,f_l$ be a basis, we will obtain a miniversal deformation. We will later see examples where this can be explicitly carried out.

3. *Milnor and Tjurina numbers*

An important feature of unfoldings of isolated singularities on smooth spaces is the conservation of the Milnor number. This invariant can be defined, among other ways, as the length of the Jacobian $\mathcal{O}_{\mathbb{C}^{n+1},0}/(\partial f/\partial x_1,\dots,\partial f/\partial x_{n+1})$. It is therefore both the number of non-degenerate critical points of a generic unfolding and the minimal number of parameters needed to versally unfold f. From this latter point of view, it could also be called the *Tjurina number* of the deformation problem defined by right equivalence of functions.

In our situation, even if the singularity (X,x) is smoothable and we can speak of non-degenerate critical points of an unfolding, we might have a different number of those in non-isomorphic Milnor fibres. An example of this phenomenon is provided by the linear section $f = x_0 + x_1 + x_2 + x_3 + x_4$ on the germ $(X,0)$ of the cone over the rational normal curve of degree 4 (see [**16**]). On the other hand, we do have a well-defined Tjurina number as the dimension of the vector space of first-order infinitesimal deformations, namely the length $\tau(X/S)$ of $T^1_{X/S,x}$. The next proposition tells us of the conditions under which the Tjurina number indeed coincides with the number of non-degenerate critical points in every generic deformation.

PROPOSITION 3.1. *Let $\varphi = (F,\pi)\colon (\mathfrak{X},0) \longrightarrow (S \times B, 0)$ be a one-parameter deformation of f. Assume that the following* extendability condition *is satisfied.*

$$\text{Any vector field tangent to the fibres of } f \text{ can be extended to a vector field tangent to the fibres of } \varphi. \tag{3.1}$$

Then both $T^1_{\mathfrak{X}/S\times B,x}$ and $M_{\varphi,x}$ are free $\mathcal{O}_{B,0}$-modules. Moreover, if $T^2_{X,x} = 0$ and the generic fibre of π is smooth, their ranks coincide.

Proof. Let y be a parameter in $(B,0)$. The exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathfrak{X},x} \xrightarrow{\cdot y} \mathcal{O}_{\mathfrak{X},x} \longrightarrow \mathcal{O}_{X,x} \longrightarrow 0$$

induces a long exact sequence:

$$\begin{aligned} 0 \longrightarrow \Theta_{\mathfrak{X}/S\times B,x} &\xrightarrow{\cdot y} \Theta_{\mathfrak{X}/S\times B,x} \longrightarrow \Theta_{X/S,x} \\ &\longrightarrow T^1_{\mathfrak{X}/S\times B,x} \xrightarrow{\cdot y} T^1_{\mathfrak{X}/S\times B,x} \longrightarrow T^1_{X/S,x} \longrightarrow \dots \end{aligned} \tag{3.2}$$

The condition (3.1) implies that the map $\Theta_{\mathfrak{X}/S\times B,x} \longrightarrow \Theta_{X/S,x}$ is surjective, and hence

$$T^1_{\mathfrak{X}/S\times B,x} \xrightarrow{\cdot y} T^1_{\mathfrak{X}/S\times B,x}$$

is injective. Therefore $T^1_{\mathfrak{X}/S\times B,x}$ and $M_{\varphi,x}$ are flat over $\mathbb{C}\{y\}$, and hence free.

For the second statement, we first show that the condition $T^2_{X,x} = 0$ also implies that $T^2_{X/S,x} = 0$. Associated to $\mathbb{C} \longrightarrow \mathcal{O}_{S,0} \longrightarrow \mathcal{O}_{X,x}$ we have a long exact sequence:

$$\dots \longrightarrow T^i_{X/S,x} \longrightarrow T^i_{X,x} \longrightarrow T^i_S(\mathcal{O}_{X,x}) \longrightarrow T^{i+1}_{X/S,x} \longrightarrow \dots .$$

As $(S,0)$ is smooth, $T^i_S(\mathcal{O}_{X,x}) = 0$ for $i \geqslant 1$, so that $T^i_{X/S,x} = T^i_{X,x}$ for $i \geqslant 2$. Finally, if the generic fibre of π is a smooth, then $T^2_{\mathfrak{X}/S\times B,x}$ is annihilated by a power of the maximal ideal $\mathfrak{m}_{B,0}$, and hence it is Artinian. The exact sequence (3.2) then contains the following short exact sequence:

$$0 \longrightarrow T^1_{\mathfrak{X}/S\times B,x} \xrightarrow{\cdot y} T^1_{\mathfrak{X}/S\times B,x} \longrightarrow T^1_{X/S,x} \longrightarrow 0.$$

It follows that $\operatorname{rk} T^1_{\mathfrak{X}/S\times B,x} = \dim_{\mathbb{C}} T^1_{X/S,x}$. To see that this is also the rank of $M_{\varphi,x}$, we write one more exact sequence:

$$0 \longrightarrow M_{\varphi,x} \longrightarrow T^1_{\mathfrak{X}/S\times B,x} \longrightarrow T^1_{\mathfrak{X}/B,x} \longrightarrow 0,$$

and we notice that $T^1_{\mathfrak{X}/B,x}$ is supported at x. □

REMARK 3.2. For a smooth fibre $X_b = \pi^{-1}(b)$, the module $(\pi_* M_\varphi)_b$ is the sum of local Jacobian algebras at the critical points of $f_b := F|_{X_b}$. Its rank is the sum of the local Milnor numbers μ_i at each of the critical points. Therefore, under the hypothesis of Proposition 3.1, the same remains true: the rank of $M_{\varphi,x}$

coincides with the number of non-degenerate critical points in a generic deformation of $f\colon X \longrightarrow S$.

From now on, we restrict ourselves to situations in which all the conditions of the above theorem are satisfied, namely, functions on *smoothable and unobstructed singularities* for which the condition (3.1) holds *for any one-parameter deformation*. We now show that this family of functions includes some interesting examples. First, note that it follows from the above proposition that not only does $\tau(X/S)$ coincide with the number of Morse critical points in the generic deformation, but also that for the miniversal deformation of f, the map

$$t'F\colon \mathcal{L}_{\pi,0} \longrightarrow M_{\varphi,x} \tag{3.3}$$

extends to an isomorphism of *free sheaves*. In particular, the sheaf of liftable vector fields is necessarily free.

We will now take a close look at two situations for which we can prove the extendability condition: the case described in the previous remark, and that of isolated complete intersection singularities. Let us first introduce a piece of notation. The module $M_{\varphi,x}$ is not independent of the given deformation φ. Even its length is not a well-defined invariant of the function f. To avoid such a dependence, we consider the miniversal deformation of (X,x) alone, say $\pi\colon (\mathfrak{X},0) \longrightarrow (B,0)$, and we take any extension F to the total space. We define

$$M_f = \frac{M_{\varphi,x}}{\mathfrak{m}_{B,0} M_{\varphi,x}}.$$

Note that this module is well defined, as any two extensions of f differ by an element of the maximal ideal $\mathfrak{m}_{B,0}$.

The reason for introducing this module is that if the conditions of Proposition 3.1 are fulfilled, its length will be equal to $\tau(X/S)$.

4. *Functions on space curves and complete intersections*

As remarked previously, the case of functions on smoothable and unobstructed curves falls trivially into our area of interest. The equality between the Milnor number and the Tjurina number for these is the main result of [**15**].

The authors of [**15**] define the Milnor number of a function on a space curve in terms of the *dualising module* $\omega_{X,x}$. Using the class map [**1**], or equivalently, Rosenlicht's description of $\omega_{X,x}$ as certain meromorphic forms [**4**], the module $\mathcal{O}_X df$ can be seen as a submodule of ω_X. They define the Milnor number of f as

$$\mu_f = \dim_{\mathbb{C}} \frac{\omega_{X,x}}{\mathcal{O}_{X,x} df}.$$

They also show the following interesting formula: the class map $\mathrm{cl}\colon \Omega_{X,x} \longrightarrow \omega_{X,x}$ can be dualised to obtain a submodule $\omega^*_{X,x}$ of $\Theta_{X,x}$. Then μ_f is also the length of $\Theta(f)_x / tf(\omega^*_{X,x})$.

PROPOSITION 4.1. *For a function f on a space curve,*

$$M_f = \frac{\Theta(f)_x}{tf(\omega^*_{X,x})}.$$

Proof. A space curve is a Cohen–Macaulay variety of codimension 2, and as such is defined by the maximal minors Δ_i of a $(m \times (m+1))$-matrix M with coefficients in $\mathcal{O}_{\mathbb{C}^3,x}$. Their deformations are well understood [**19**]; they are also defined by the maximal minors $\tilde{\Delta}_i$ of a perturbation $\tilde{M}$ of M.

An identical calculation to that of [**15**], but using the relative class map for the miniversal family instead of that of (X,x), shows that its dual in $\Theta_{\mathfrak{X}/B,x}$ is generated by the vector fields

$$\begin{vmatrix} \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ \dfrac{\partial \tilde{\Delta}_i}{\partial x_1} & \dfrac{\partial \tilde{\Delta}_i}{\partial x_2} & \dfrac{\partial \tilde{\Delta}_i}{\partial x_3} \\ \dfrac{\partial \tilde{\Delta}_j}{\partial x_1} & \dfrac{\partial \tilde{\Delta}_j}{\partial x_2} & \dfrac{\partial \tilde{\Delta}_j}{\partial x_3} \end{vmatrix}, \qquad 1 \leqslant i < j \leqslant l.$$

On the other hand, the relative class map

$$\mathrm{cl}_{\mathfrak{X}/B,x} \colon \Omega_{\mathfrak{X}/B,x} \longrightarrow \omega_{\mathfrak{X}/B,x}$$

(or rather, a representative of it) is an isomorphism whenever the fibre is smooth. As the generic fibre is indeed smooth, the set where it fails to be bijective is of codimension at least 2. Hence its dual is an isomorphism everywhere, and if F is any extension of f to $(\mathfrak{X},x)$ then we have

$$M_f = \frac{\Theta(F)_x}{tF(\Theta_{\mathfrak{X}/B,x})} + \mathfrak{m}_{B,0}\Theta(F)_x = \frac{\Theta(f)_x}{tf(\omega^*_{X,x})}. \qquad \square$$

Example 4.2. We can use the above calculation to compute versal deformations of functions on space curves. For example, the union of the three coordinate axes in $(\mathbb{C}^3,0)$ is defined by the (2×3)-minors of $M = \left(\begin{smallmatrix} x & y & 0 \\ 0 & y & z \end{smallmatrix}\right)$. The miniversal deformation of a function $f = x^p + y^q + z^r$ is therefore obtained by considering the miniversal deformation of the curve together with the unfolding $F = x^p + \sum_{i=1}^{p-1} a_i x^{p-i} + y^p + \sum_{i=1}^{q-1} b_i y^{q-i} + z^r + \sum_{i=1}^{r-1} c_i z^{r-i} + d$. In [**9**], where simple functions on curves are classified, this singularity is referred to as $C_{p,q,r}$.

We now go on to study the case of functions on complete intersections. Let $f\colon (X,x) \longrightarrow (S,0)$ be a germ with an isolated singularity on a n-dimensional complete intersection. Let $g_1,\dots,g_k$ be elements defining the ideal of (X,x) in $(\mathbb{C}^{n+k},x)$.

If $n \geqslant 2$, a submodule of $\Theta_{X,x}$ whose members are clearly tangent to all the fibres of f is generated by the maximal minors of the following matrix.

$$\begin{pmatrix} \dfrac{\partial}{\partial x_1} & \cdots & \dfrac{\partial}{\partial x_{n+k}} \\ \dfrac{\partial f}{\partial x_1} & \cdots & \dfrac{\partial f}{\partial x_{n+k}} \\ \dfrac{\partial g_1}{\partial x_1} & \cdots & \dfrac{\partial g_1}{\partial x_{n+k}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial g_k}{\partial x_1} & \cdots & \dfrac{\partial g_k}{\partial x_{n+k}} \end{pmatrix} \tag{4.1}$$

LEMMA 4.3. *For $n \geqslant 2$, the vector fields in* (4.1) *generate* $\Theta_{X/S}$.

Proof. Let $\varphi = (f, g_1, \ldots, g_k)$. The module $\Theta_{X/S,x}$ is the kernel of

$$\Theta_{\mathbb{C}^{n+k},x} \otimes \mathcal{O}_{X,x} \xrightarrow{t\varphi\otimes\mathbb{I}} \Theta_{\mathbb{C}^{1+k},0} \otimes \mathcal{O}_{X,x}. \tag{4.2}$$

As f has an isolated singularity and (X, x) is Cohen–Macaulay, the depth of the ideal in $\mathcal{O}_{X,x}$ generated by the maximal minors of (4.2) is n; that is, it is the greatest possible. It follows that the Eagon–Northcott complex is exact [**6**], and the kernel is generated by the above vector fields. □

COROLLARY 4.4. *For a germ $f\colon (X, x) \longrightarrow (S, 0)$ with an isolated singularity on a complete intersection of dimension $n \geqslant 1$, $\tau(X/S)$ coincides with the number of non-degenerate critical points of a generic deformation. If n denotes the dimension of (X, x), then*

$$M_f \simeq \frac{\omega_{X,x}}{df \wedge \Omega^{n-1}_{X,x}}. \tag{4.3}$$

Proof. If $n = 1$, then X is a curve, the fibres of f are just points, and there are no tangent vector fields to the fibres of f. For $n \geqslant 2$, we see from Lemma 4.3 that the extendability condition holds. In both cases, Lemma 4.3 is also telling us which vector fields are tangent to all the fibres of a deformation of a complete intersection. We simply take (X, x) to be the ambient space $(\mathbb{C}^{n+k}, x)$ and change $(S, 0)$ by $(\mathbb{C}^k, 0)$. We see that $\Theta_{\mathfrak{X}/B,x}$ is generated by the maximal minors of (4.1) with the row involving f deleted. The equality (4.3) is now made evident by differentiating f with respect to this set of generators of $\Theta_{\mathfrak{X}/B,x}$. □

REMARK 4.5. The equality between Tjurina and Milnor numbers for functions on complete intersections is proved, using unrelated methods, in [**7**].

REMARK 4.6. Using (4.3) and a well-known result [**20**], we can interpret the rank of M_f, and hence $\tau(X/S)$, as the rank of a certain vanishing homology, namely $H_n(X_b, Y_s)$ for Milnor fibres of (X, x) and f.

5. *Multiplication on the sheaf of liftable vector fields*

Whenever the map $t'F$ of (3.3) extends to an isomorphism of sheaves, we can use it to define a multiplication on $\mathcal{L}_\pi$ by pulling back the algebra structure on M_φ. If (X, x) is also smoothable, then this defines a multiplication on the tangent bundle of the complement $B\backslash\Delta$ of the discriminant of the fibration. We begin recalling the definition of an F-manifold from [**10**, Chapter 1].

DEFINITION 5.1. A complex manifold with an associative and commutative multiplication $\star$ on the tangent bundle is called an *F-manifold* if:

(i) (*unity*) there exists a *global* vector field e such that $e \star u = u$ for any $u \in \Theta_M$, and

(ii) (*integrability*) $\mathrm{Lie}_{u\star v}(\star) = u \star \mathrm{Lie}_v(\star) + \mathrm{Lie}_u(\star) \star v$ for any $u, v \in \Theta_M$.

An *Euler vector field* E (of weight 1) for M is defined by the condition

$$\mathrm{Lie}_E(\star) = \star.$$

The main consequence of this definition is the integrability of multiplicative subbundles of TM; in other words, if in a neighbourhood U of a point $p \in M$ we can decompose TU as a sum of unitary subalgebras $A \oplus B$ such that $A \star B = 0$, then A and B are integrable.

By choosing good representatives in the sense of [**14**] for all the germs involved, we have the following statement.

Proposition 5.2. *The map $t'F$ endows the sheaf of liftable vector fields $\mathcal{L}_\pi$ with the structure of a commutative and associative $\mathcal{O}_B$-algebra $\star$ such that, for any $u, v \in \mathcal{L}_\pi$:*

$$\mathrm{Lie}_{u\star v}(\star) = \mathrm{Lie}_u(\star) \star v + u \star \mathrm{Lie}_v(\star). \tag{5.1}$$

The class of F in $\pi_* M_\varphi$ corresponds to an Euler vector field of weight 1.

Proof. It is enough to show that (5.1) holds off Δ. Let $\mu = \operatorname{rk} \pi_* M_\varphi$. For a generic point $b \in B \backslash \Delta$, the function F has μ quadratic singularities on the smooth fibre $\pi^{-1}(b)$. Hence $\pi_* M_\varphi$ decomposes into μ one-dimensional unitary subalgebras. In a neighbourhood $U \subset S \backslash \Delta$ of such a point, the integrability condition is equivalent to the image L of the map

$$\text{supp } M_\varphi \ni x \longmapsto d_x F \in T^*_{\pi(x)} B \tag{5.2}$$

being a Lagrangian subvariety of T^*B (see [**10**, Theorem 3.2]). If α denotes the canonical 1-form on T^*B and $p\colon T^*B \longrightarrow B$ the projection, it is easy to check that the diagram

$$\begin{array}{ccc} \text{supp } M_\varphi & \xrightarrow{\qquad\qquad} & p_*\mathcal{O}_L \\ & \nwarrow \quad \nearrow & \\ & \Theta_B & \end{array}$$

is commutative. The homomorphism on the right-hand side is given by evaluation, so that it can also be expressed as $\alpha(\tilde{u})$ where $\tilde{u}$ is a lift of $u \in \Theta_B$ to Θ_{T^*B}. Hence α_L is the relative differential of F when thought of as a map on L via the identification (5.2). It follows that α_L is exact and hence closed, so that L is Lagrangian.

The statement about the Euler vector is an easy calculation, which we leave to the reader (see [**10**, Theorem 3.3]). $\square$

The above proposition establishes the structure of the F-manifold, at least off Δ. In the case where $\mathcal{L}_\pi$ coincides with the sheaf of tangent vector fields to Δ, denoted by $\Theta(-\log \Delta)$, we can in fact define the F-manifold structure on each of the strata of the logarithmic stratification induced by $\Theta(-\log \Delta)$ (see [**18**]). First we need a lemma.

Lemma 5.3. *For any ideal sheaf $I \subset \mathcal{O}_B$, the kernel of the map*

$$\mathcal{L}_\pi / I\mathcal{L}_\pi \longrightarrow \Theta_B / I\Theta_B$$

is identified by $t'F$ with an ideal of $\pi_* M_\varphi / I\pi_* M_\varphi$.

Proof. Choosing a Stein representative of π, we can interpret the diagram in Lemma 2.1 as an isomorphism of free resolutions of $\pi_* T^1_{\mathfrak{X}/B}$. The vertical arrows remain isomorphisms after tensorising the diagram with $\mathcal{O}_B/I$. Hence the kernel of

$$\mathcal{L}_\pi/I\mathcal{L}_\pi \longrightarrow \Theta_B/I\Theta_B$$

is isomorphic via $t'F$ to the kernel of

$$\pi_*M_\varphi/I\pi_*M_\varphi \longrightarrow \pi_*T^1_{\mathfrak{X}/S\times B}/I\pi_*T^1_{\mathfrak{X}/S\times B}.$$

Note now that this latter map is $\pi_*\mathcal{O}_\mathfrak{X}$-linear, and hence its kernel is a $\pi_*\mathcal{O}_\mathfrak{X}$-module; that is, an ideal for M_φ is (isomorphic to) a quotient of $\mathcal{O}_\mathfrak{X}$. □

THEOREM 5.4. *If $\mathcal{L}_\pi = \Theta(-\log\Delta)$, then each stratum of the logarithmic stratification has the structure of an F-manifold with an Euler vector field of weight* 1.

Proof. Let $b \in B$, and let S_b be the stratum in which b lies. Let V be an open neighbourhood of b in which $S_b \cap V$ is an analytic subset of V defined by the ideal I_{S_b}. The sheaf $\Theta_{S_b\cap V}$ can be identified with

$$\frac{\operatorname{im}(\mathcal{L}_\pi|_V \longrightarrow \Theta_B|_V)}{I_{S_b}\operatorname{im}(\mathcal{L}_\pi|_V \longrightarrow \Theta_B|_V)}.$$

Let $\mathcal{K}$ denote the sheaf $\operatorname{Tor}_1^{\mathcal{O}_B}(\pi_*T^1_{\mathfrak{X}/B}, \mathcal{O}_B/I_{S_b})$. The map $t'F$ descends to the above quotient, and it yields an isomorphism of $\mathcal{O}_{S_b\cap V}$-modules

$$\Theta_{S_b\cap V} \xrightarrow{\simeq} \frac{\pi_*M_\varphi|_V}{I_{S_b}\pi_*M_\varphi|_V + t'F(\mathcal{K}|_V)}.$$

According to the previous lemma, the right-hand side is an $\mathcal{O}_B$-algebra. The above isomorphism defines the multiplication on the tangent bundle of the stratum S_b. From Proposition 5.2 it follows that it is an F-manifold with Euler vector field of weight 1 given by the class of F in the corresponding algebra. □

REMARK 5.5. If the stratum S_b is a massive F-manifold, that is, if there exist coordinates $u_1, \dots, u_l$ such that

$$\frac{\partial}{\partial u_i} \star \frac{\partial}{\partial u_j} = \delta_{ij}\frac{\partial}{\partial u_i} \qquad \text{for all } i, j,$$

then the critical values of F are generically local coordinates on S_b. In particular, this always holds on the stratum $B - \Delta_\varphi$. In the case of space curves, that the critical values of F off the bifurcation diagram are local coordinates is shown for simple functions by V. Goryunov in [**8**]. He also conjectured the analogous result for non-simple functions.

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