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The Mañé–Conze–Guivarc'h lemma for intermittent maps of the circle

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Abstract. We study the existence of solutions g to the functional inequality $f \leq g \circ T - g + \beta$, where f is a prescribed continuous function, T is a weakly expanding transformation of the circle having an indifferent fixed point, and β is the maximum ergodic average of f. Using a method due to T. Bousch, we show that continuous solutions g always exist when the Hölder exponent of f is close to 1. In the converse direction, we construct explicit examples of continuous functions f with low Hölder exponent for which no continuous solution g exists. We give sharp estimates on the best possible Hölder regularity of a solution g given the Hölder regularity of f.

1. *Introduction*

Let $T: X \to X$ be a discrete dynamical system, and let $\mathcal{M}_T$ be the set of all Borel probability measures which are invariant under the map T. For a given continuous function $f: X \to \mathbb{R}$, we define the maximum ergodic average $\beta(f)$ by

$$\beta(f) = \sup_{\mu \in \mathcal{M}_T} \int f \, d\mu,$$

and say that $\nu \in \mathcal{M}_T$ is a *maximizing measure* for f if it satisfies $\int f \, d\nu = \beta(f)$. The study of maximizing measures has recently become the focus of significant research interest. While early articles of Bousch and Jenkinson [**2, 14**] were motivated by abstract questions concerning the geometric structure of the set of measures $\mathcal{M}_T$, questions relating to maximizing measures have also appeared in research into chaotic control [**13, 25**], Livšic-type theorems [**6**], thermodynamic formalism [**9, 15, 16**], Tetris heaps [**7**], and the Lagarias–Wang finiteness conjecture in linear algebra [**7**].

This article is concerned with a key technical tool that arises in the study of maximizing measures, which we call the *Mañé–Conze–Guivarc'h lemma*. A lemma of this type takes the following form: given a continuous function $f: X \to \mathbb{R}$ with some prescribed regularity, under suitable dynamical hypotheses there exists a continuous function

$g\colon X \to \mathbb{R}$ with the property that $f \leq g \circ T - g + \beta(f)$. This relation is equivalent to the statement that there exists a continuous g such that $\sup(f + g - g \circ T) = \beta(f)$. Conze and Guivarc'h's version of this lemma may be found in the unpublished manuscript [**10**]. It has been noted that theorems of a similar character occur in the field of optimal control, e.g. [**1, 17**]; this relationship is examined in Bousch's preprint [**5**].

We briefly describe the immediate implications of this result. First, let us rewrite the aforementioned inequality in the form $f = g \circ T - g + \beta(f) - r$, where r is continuous and satisfies $r \geq 0$. We then obtain $\int f\, d\nu = \beta(f) - \int r\, d\nu$ for every $\nu \in \mathcal{M}_T$, and so ν is maximizing for f if and only if $\int r\, d\mu = 0$. Since $r(x) \geq 0$ for all x, we conclude that the maximizing measures of f are precisely those invariant measures ν whose support lies in the compact set $r^{-1}(0)$. This leads to the *subordination principle* described by Bousch [**3**]: if invariant measures μ, ν satisfy $\operatorname{supp} \nu \subseteq \operatorname{supp} \mu$ and μ is a maximizing measure for f, then the 'subordinate' measure ν is maximizing also. It has been shown that this subordination principle can fail to hold when the regularity of f is relaxed [**6**].

A particularly interesting application of the Mañé–Conze–Guivarc'h lemma is a recent result of Bousch [**4**] which shows that for dynamical systems satisfying a Mañé–Conze–Guivarc'h lemma, measures supported on periodic orbits are the only maximizing measures that persist under Lipschitz perturbations of the observable f. A similar result was previously shown by G. Yuan and B. R. Hunt under more restrictive dynamical assumptions [**25**]. Mañé–Conze–Guivarc'h-type lemmas have also been found useful in circumstances that are not *a priori* related to maximizing measures [**20**].

When $T\colon X \to X$ is an expanding map, a subshift of finite type or an Anosov diffeomorphism, and $f\colon X \to \mathbb{R}$ is Hölder continuous, it is known that we can always find $g\colon X \to \mathbb{R}$ Hölder continuous such that $f \leq g \circ T - g + \beta(f)$ is satisfied [**3, 11, 19, 22**]. The purpose of the present article is to examine the extension of this result to a simple class of non-uniformly hyperbolic dynamical systems on the circle, namely the case in which T is uniformly expanding except in the neighbourhood of a weakly repelling fixed point.

Previously, it was shown by Souza [**23**] that for an expanding map $T\colon [0, 1] \to [0, 1]$ with a weakly repelling fixed point, a Mañé–Conze–Guivarc'h lemma can be proved when f is Hölder continuous and monotone in some neighbourhood of the indifferent fixed point z, and additionally satisfies $\int f\, d\nu_- < f(z) < \int f\, d\nu_+$ for some $\nu_-, \nu_+ \in \mathcal{M}_T$. Prior to the research described in this article, S. Branton had shown that when f is Lipschitz continuous, Souza's conditions may be removed [**8**]. In this article, using a different method from that of S. Branton, we study the case in which f is Hölder and prove a complementary result showing that solutions can fail to exist in certain situations where f is Hölder continuous with exponent close to 0.

Let $\mathbb{T} = \mathbb{R} \bmod \mathbb{Z}$, with metric d inherited from the standard metric on $\mathbb{R}$. The precise class of maps $T\colon \mathbb{T} \to \mathbb{T}$ which we study is defined as follows.

Definition 1.1. For each $\alpha > 0$, we say that a continuous function $T\colon \mathbb{T} \to \mathbb{T}$ is an *expanding map of Manneville–Pomeau type* α if it fixes 0, is differentiable with derivative greater than 1 in the interval $\mathbb{T} \setminus \{0\}$, and satisfies

$$T'(x) = 1 + \xi x^{\alpha} + o(x^{\alpha}) \quad \text{as } x \to 0^{+},$$
$$\liminf_{x \to 1^{-}} T'(x) > 1$$

for some $\xi > 0$.

The archetypal map T represented by this definition is the *Manneville–Pomeau map* defined by $x \mapsto x + x^{1+\alpha} \mod 1$. Expanding maps of Manneville–Pomeau type are studied in, for example, [**12, 18, 24**].

For each $\gamma \in (0, 1]$, let H_{γ} denote the space of all γ-Hölder continuous real-valued functions on the circle $\mathbb{T}$, and define $|f|_{\gamma} = \sup_{x \neq y} |f(x) - f(y)|/d(x, y)^{\gamma}$ for $f \in H_{\gamma}$. The set H_{γ} is a Banach space when equipped with the norm $\| \cdot \|_{\gamma}$ given by $\|f\|_{\gamma} := |f|_{\infty} + |f|_{\gamma}$. Using a method based on Young towers, S. Branton proved the following.

THEOREM. [**8**] *Let $T : \mathbb{T} \to \mathbb{T}$ be an expanding map of Manneville–Pomeau type $\alpha \in (0, 1)$. Then for every $f \in H_1$ and $\delta \in (0, 1 - \alpha)$ there exists $g \in H_{1-\alpha-\delta}$ such that $f \leq g \circ T - g + \beta(f)$.*

We are able to establish the following result.

THEOREM 1. *Let $T : \mathbb{T} \to \mathbb{T}$ be an expanding map of Manneville–Pomeau type $\alpha \in (0, 1)$, and suppose that $\alpha < \gamma \leq 1$. Then for every $f \in H_{\gamma}$ there exists $g \in H_{\gamma-\alpha}$ such that $f \leq g \circ T - g + \beta(f)$. In addition, the function g satisfies the functional equation*

$$g(x) + \beta(f) = \max_{Ty=x} [f(y) + g(y)] .$$

Furthermore, we are able to show that Theorem 1 is sharp both in the regularity of f and in the regularity of g.

THEOREM 2. *Let $T : \mathbb{T} \to \mathbb{T}$ be an expanding map of Manneville–Pomeau type $\alpha \in (0, 1)$, and suppose that $0 < \alpha < \gamma \leq 1$. Then the following hold:*

(a) *there exists $f \in H_{\gamma}$ such that if $f \leq g \circ T - g + \beta(f)$ for $g \in H_{\theta}$, then $\theta \leq \gamma - \alpha$;*

(b) *there exists $f \in H_{\alpha}$ such that $f \leq g \circ T - g + \beta(f)$ is not satisfied for any continuous function g.*

In a recent article, T. Bousch proved the following theorem, which extends a result of Yuan and Hunt [**25**].

THEOREM. [**4**] *Let $T : X \to X$ be a continuous surjection of a compact metric space. Suppose that for all $f \in H_1$, there exists $g \in H_1$ such that $f \leq g \circ T - g + \beta(f)$ and $|g|_1 \leq C|f|_1$ for some $C > 0$ independent of f. Suppose also that $\mu \in \mathcal{M}_T$ is a maximizing measure for every element of some non-empty open set $U \subset H_1$. Then μ is supported on a periodic orbit of T.*

We remark that while uniformly expanding dynamical systems have been shown to satisfy the hypotheses of this theorem (see [**3, 11, 22**]), Theorem 2(a) demonstrates that the required hypotheses do not hold for maps of Manneville–Pomeau type.

2. *Proof of Theorem 1*

We use a fixed-point method that was employed in the work of Bousch [**2, 4**]. We begin with the following lemma.

LEMMA 2.1. *Let T be of Manneville–Pomeau type α, and take $z_1, z_2 \in \mathbb{T}$ with $d(z_1, z_2)$ sufficiently small. Then*

$$d(Tz_1, Tz_2) \geq d(z_1, z_2)(1 + C_0\, d(z_1, z_2)^\alpha)$$

for some constant C_0 that depends only on T.

Proof. We consider separately two cases depending on whether the shortest arc connecting z_1 and z_2 does or does not contain 0.

We begin with the latter case. Choose representatives $a_1, a_2 \in [0, 1)$ of $z_1, z_2 \in \mathbb{T}$, respectively, assuming without loss of generality that $0 \leq a_1 \leq a_2 < 1$. If $d(z_1, z_2)$ is small enough, then

$$\begin{aligned} d(Tz_1, Tz_2) = \int_{z_1}^{z_2} |T'(s)|\, ds &\geq \int_{a_1}^{a_2} 1 + \rho_0 s^\alpha\, ds \\ &\geq (a_2 - a_1) + \rho_1(a_2 - a_1)^{1+\alpha} = d(z_1, z_2) + \rho_1\, d(z_1, z_2)^{1+\alpha} \end{aligned}$$

for some small $\rho_0, \rho_1 > 0$ not depending on z_1 and z_2. This completes the proof in this case.

Now suppose that 0 lies in the arc connecting z_1 and z_2, with the triple $(z_1, 0, z_2)$ being positively oriented. Arguing as previously, we have $d(Tz_2, 0) \geq d(z_2, 0) + \rho_1 d(z_2, 0)^{1+\alpha}$. Since T has derivative bounded away from 1 in any small interval of the form $(-\delta, 0)$, there is a $\rho_2 > 0$ such that $d(Tz_1, 0) \geq (1 + \rho_2)d(z_1, 0)$ when $d(z_1, 0)$ is small enough. Combining these estimates yields

$$d(Tz_1, Tz_2) = d(Tz_1, 0) + d(0, Tz_2) \geq d(z_1, z_2) + \rho_1\, d(z_2, 0)^{1+\alpha} + \rho_2\, d(z_1, 0).$$

If we take $C_0 = \min\{\rho_1/2^{1+\alpha}, \rho_2/2\}$, then by separating the cases $d(z_1, 0) \geq d(z_2, 0)$ and $d(z_1, 0) \leq d(z_2, 0)$ we obtain

$$\rho_1\, d(z_2, 0)^{1+\alpha} + \rho_2\, d(z_1, 0) \geq C_0\, d(z_1, z_2)^{1+\alpha}$$

for every sufficiently close choice of z_1 and z_2 separated by 0. Combining the above two inequalities completes the proof. □

LEMMA 2.2. *Let T be of Manneville–Pomeau type α, and let $\gamma \in (\alpha, 1]$. Then there exists $C_\gamma > 0$ with the following property: for every $x_1, x_2, y_1 \in \mathbb{T}$ with $Ty_1 = x_1$, we may choose $y_2 \in T^{-1}\{x_2\}$ such that*

$$d(y_1, y_2)^{\gamma-\alpha} + C_\gamma\, d(y_1, y_2)^\gamma \leq d(x_1, x_2)^{\gamma-\alpha}. \tag{1}$$

Proof. Given $x_1, x_2, y_1 \in \mathbb{T}$ with $Ty_1 = x_1$, we claim that there exists $y_2 \in T^{-1}\{x_2\}$ such that

$$d(y_1, y_2)(1 + \rho_3\, d(y_1, y_2)^\alpha) \leq d(x_1, x_2) \tag{2}$$

for some $\rho_3 > 0$ independent of x_1, x_2, y_1. Taking $\rho_4 = (1 + \rho_3)^{\gamma-\alpha} - 1 > 0$, we have $(1 + \rho_3 t)^{\gamma-\alpha} \geq 1 + \rho_4 t$ for all $t \in [0, 1]$. Applying this to (2) yields (1) with $C_\gamma = \rho_4$.

We now prove the claim. We begin by noting that T expands sufficiently long intervals by a uniform factor: for every $\delta > 0$, there exists $K_\delta > 0$ such that if $d(x_1, x_2) \geq \delta$, then y_2 may be chosen with

$$(1 + K_\delta)\, d(y_1, y_2) \leq d(x_1, x_2).$$

Thus, given some fixed $\delta > 0$, (2) holds for every case in which $d(x_1, x_2) \geq \delta$ by taking $\rho_3 \leq K_\delta$. On the other hand, if $d(x_1, x_2) < \delta$ for some sufficiently small fixed $\delta > 0$, then we may choose $y_2 \in T^{-1}\{x_2\}$ with $d(y_1, y_2) \leq d(x_1, x_2) < \delta$ and apply Lemma 2.1 to obtain

$$d(y_1, y_2)(1 + C_0\, d(y_1, y_2)^\alpha) \leq d(x_1, x_2),$$

so that taking $\rho_3 = \min\{K_\delta, C_0\}$ completes the proof. □

We now prove Theorem 1. Let $\gamma \in (\alpha, 1]$ and define a subset of $C(\mathbb{T})$ by

$$K = \{g \in H_{\gamma-\alpha} : |g|_{\gamma-\alpha} \leq C_\gamma^{-1} |f|_\gamma\},$$

where $C_\gamma > 0$ is as in Lemma 2.2. Let $K_0 = K/\mathbb{R}$, the set of equivalence classes of elements of K modulo addition of a constant. Clearly, K_0 is compact with respect to uniform distance. For each $g \in K$, define $L_f g \in C(\mathbb{T})$ by $(L_f g)(x) = \max_{Ty=x} (f + g)(y)$. We assert that L_f is a continuous transformation of K with respect to uniform distance.

Given $x_1, x_2 \in \mathbb{T}$ and $g \in K$, choose $y_1 \in T^{-1}x_1$ such that $(L_f g)(x_1) = (f + g)(y_1)$. Invoking Lemma 2.2, we may choose $y_2 \in T^{-1}x_2$ such that (1) holds and therefore

$$\begin{aligned}(L_f g)(x_1) - (L_f g)(x_2) &\leq (f + g)(y_1) - (f + g)(y_2)\\ &\leq |f|_\gamma\, d(y_1, y_2)^\gamma + |g|_{\gamma-\alpha}\, d(y_1, y_2)^{\gamma-\alpha}\\ &\leq C_\gamma^{-1} |f|_\gamma\, d(x_1, x_2)^{\gamma-\alpha}.\end{aligned}$$

We conclude that $|L_f g|_{\gamma-\alpha} \leq C_\gamma^{-1}|f|_\gamma$ for all $g \in K$ and therefore $L_f K \subseteq K$. A simple argument shows that $|L_f g_1 - L_f g_2|_\infty \leq |g_1 - g_2|_\infty$ for $g_1, g_2 \in K$ so that L_f is a continuous transformation of K. It follows that L_f induces a continuous transformation of K_0. Hence, by the Schauder–Tychonoff theorem, there exists $h \in K$ such that $L_f h = h$ mod $\mathbb{R}$. Let $b \in \mathbb{R}$ be chosen such that $h(x) = b + \max_{Ty=x}(f + h)(y)$ for all $x \in \mathbb{T}$; a simple argument as in [**2**] shows that $b = \beta(f)$. The proof of Theorem 1 is complete.

3. *Proof of Theorem 2*

In this section we shall take the liberty of using the fundamental domain $[0, 1)$ as a model for $\mathbb{T}$ and treating T as a $[0, 1) \to [0, 1)$ map in the obvious fashion. Let $u_1 = \min\{u \in (0, 1) : Tu = 0\}$ and define a sequence $(u_n)_{n\geq 1}$ in $[0, 1)$ by $u_n := \min\{u \in (0, 1) : Tu = u_{n-1}\}$. We require two simple lemmas.

LEMMA 3.1. *There is $C_1 > 1$ such that for all $n \geq 1$,*

$$C_1^{-1} n^{-1-1/\alpha} \leq u_n - u_{n+1} \leq C_1 n^{-1-1/\alpha}$$

and

$$C_1^{-1} n^{-1/\alpha} \leq u_n \leq C_1 n^{-1/\alpha}.$$

Proof. This follows from the relation $Tu_n - u_n = \xi u_n^{1+\alpha} + o(u_n)^{1+\alpha}$ in a fairly straightforward fashion; see, for instance, [**24**]. □

LEMMA 3.2. *Let* $f\colon [0,1) \to \mathbb{R}$. *Assume* $f(0)=0$, *and suppose that there is* $C>0$ *such that for all* $\kappa \in (0,1)$,

$$|f(\kappa)| \le C\kappa^{\gamma_1}$$

and

$$\sup_{\substack{x,y\in[\kappa,1]\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|} \le C\kappa^{-\gamma_2},$$

where $\gamma_1, \gamma_2 > 0$ *and* $\gamma_1+\gamma_2 \ge 1$. *Then* f *is* $\gamma_1/(\gamma_1+\gamma_2)$*-Hölder continuous throughout* $[0,1)$.

Proof. Let $0 \le x < y < 1$, and let $\lambda = y^{-\gamma_1-\gamma_2}(y-x)$ and $\gamma = \gamma_1/(\gamma_1+\gamma_2)$. If $\lambda > 1/2$, then $y^{\gamma_1+\gamma_2} < 2(y-x)$ and hence

$$|f(x)-f(y)| \le |f(x)| + |f(y)| \le 2Cy^{\gamma_1} < 2^{1+\gamma}C|y-x|^{\gamma}.$$

Otherwise, $y - x = \lambda y^{\gamma_1+\gamma_2} \le \lambda y \le y/2$; so $0 < y \le 2x$ and hence

$$\begin{aligned}|f(x)-f(y)| &\le Cx^{-\gamma_2}(y-x)^{1-\gamma}(y-x)^{\gamma}\\ &= C\lambda^{1-\gamma}\left(\frac{y}{x}\right)^{\gamma_2}(y-x)^{\gamma} \le 2^{\gamma-1+\gamma_2}C(y-x)^{\gamma},\end{aligned}$$

as required. □

3.1. *Proof of part (a).* Given $0<\alpha<\gamma\le 1$, let $K_\gamma = C_1\sum_{n=2}^{\infty} n^{-\gamma/\alpha} < \infty$. Define f by $f(x) = x^\gamma$ for all $x\in[0,u_3]$, by $f(x) = -K$ for all $x \in [u_2,u_1]$, and by linear interpolation in the intervals $[u_3,u_2]$ and $[u_1,1)$ subject to the constraint $\lim_{x\to 1} f(x) = 0$ which ensures that f yields a continuous function $\mathbb{T}\to\mathbb{R}$. Note that $f(x) \le u_k^\gamma$ when $u_{k+1} \le x \le u_k$ and that $f \in H_\gamma$.

We claim that $\beta(f)=0$. Since the Dirac measure δ_0 is invariant and $f(0)=0$, it is clear that $\beta(f)\ge 0$. By a lemma of Peres [**21**], there exists $x\in\mathbb{T}$ such that $\sum_{j=0}^{n-1} f(T^jx) \ge n\beta(f)$ for all $n \ge 0$; so to prove that $\beta(f)\le 0$, it is sufficient to show that for each $x\in[0,1]$ we may find $v(x)>0$ such that $\sum_{j=0}^{v(x)-1} f(T^jx) \le 0$.

If $x=0$ or $x\in[u_2,1)$, then clearly we may take $v(x)=1$. Otherwise, we have $x\in[u_{r+1},u_r]$ for some $r\ge 2$. Applying Lemma 3.1, we obtain

$$\sum_{j=0}^{r} f(T^jx) \le \sum_{j=0}^{r-2}(T^jx)^\gamma - K \le \sum_{k=2}^{r} u_k^\gamma - K \le C_1\sum_{k=2}^{\infty} k^{-\gamma/\alpha} - K = 0,$$

so that taking $v(x) = r+1$ proves the claim.

Now suppose that $f \le g\circ T - g + \beta(f)$, where $g\in H_\theta$. For every $n>0$ and $r\ge 3$, we have

$$g(u_{n+r}) + \sum_{j=0}^{n-1} f(T^ju_{n+r}) \le g(T^nu_{n+r})$$

and hence

$$g(u_r) \geq \sum_{k=r+1}^{r+n} f(u_k) + g(u_{n+r}) \geq C_1^{-1} \sum_{k=r+1}^{r+n} k^{-\gamma/\alpha} + g(u_{n+r}).$$

Taking the limit as $n \to \infty$ gives

$$g(u_r) \geq C_1^{-1} \sum_{k=r+1}^{\infty} k^{-\gamma/\alpha} + g(0) \geq \tilde{C} r^{1-\gamma/\alpha} + g(0),$$

and therefore

$$\tilde{C} r^{-1-\gamma/\alpha} \leq |g(0) - g(u_r)| \leq |g|_\theta u_r^\theta \leq |g|_\theta C_1^\theta r^{-\theta/\alpha}$$

for every $r \geq 3$. We deduce that $\theta \leq \gamma - \alpha$. □

3.2. *Proof of part (b).* Define $f(0) = 0$, $f(x) = 0$ for all $x \in [u_1, 1)$ and, for each $n \geq 0$,

$$f(u_{2^{4n}}) = f(u_{2^{4n+2}}) = 0,$$
$$f(u_{2^{4n+1}}) = -2^{-4n},$$
$$f(u_{2^{4n+3}}) = \tau 2^{-4n},$$

where $\tau \in (0, 1)$ is a real number to be fixed later. Extend f to the whole of $[0, 1)$ by interpolating linearly in each interval $[u_{2^{4n+k+1}}, u_{2^{4n+k}}]$.

We will show that f is α-Hölder. Suppose that $u_{2^{4n+4}} \leq \kappa \leq u_{2^{4n}}$ for some $n \geq 0$; then

$$|f(\kappa)| < 2^{-4n} \leq C_1^\alpha u_{2^{4n}}^\alpha \leq C_1^\alpha \kappa^\alpha. \tag{3}$$

We must estimate the Lipschitz norm of f in the interval $[\kappa, 1)$. To do this, we require the simple lower bound

$$\begin{aligned} u_{2^{r+1}} - u_{2^r} &= \sum_{\ell=0}^{2^r-1} u_{2^r+\ell+1} - u_{2^r+\ell} \geq \sum_{k=2^r}^{2^{r+1}-1} C_1^{-1} k^{-1-1/\alpha} \\ &\geq \tilde{C}(2^{-r/\alpha} - 2^{-(r+1)/\alpha}) \geq \tilde{C} 2^{-r/\alpha} \end{aligned}$$

for all $r > 0$, where we have used Lemma 3.1. It follows that when $u_{2^{4n+4}} \leq \kappa \leq u_{2^{4n}}$, the gradient of f in $[\kappa, 1)$ is bounded by

$$\sup_{\substack{0\leq k\leq n \\ 0\leq \ell<4}} \frac{2^{-4k}}{|u_{2^{4k+\ell+1}} - u_{2^{4k+\ell}}|} \leq \sup_{\substack{0\leq k\leq n \\ 0\leq \ell<4}} \frac{2^{-4k}}{\tilde{C} 2^{-(4k+\ell)/\alpha}} = \tilde{C} 2^{-4k+4k/\alpha} \leq \tilde{C} \kappa^{\alpha-1}. \tag{4}$$

Combining estimates (3) and (4) with Lemma 3.2, we deduce that $f \in H_\alpha$.

We next compute $\beta(f)$. Since $f(0) = 0$ and the Dirac measure δ_0 is T-invariant, we have $\beta(f) \geq 0$. To prove that $\beta(f) = 0$, we proceed as in part (a) by showing that for each $x \in [0, 1)$, there is $v(x) > 0$ such that $\sum_{j=0}^{v(x)-1} f(T^j x) \leq 0$.

If $x \geq u_2$ or $x = 0$ or $u_{2^{4n+2}} \leq x \leq u_{2^{4n}}$ for some $n > 0$, then $f(x) \leq 0$ and we may take $v(x) = 1$. We therefore restrict our attention to the case in which $u_{2^{4n+4}} < x < u_{2^{4n+2}}$ for some $n \geq 0$. Assuming this, suppose that

$$u_{2^{4n+2}+k+1} \leq x \leq u_{2^{4n+2}+k},$$

where $0 \le k < 2^{4n+4} - 2^{4n+2}$. We choose $v(x) = k + 2^{4n+1} + 2$. First we note that

$$\sum_{j=0}^{k} f(T^j x) \le \tau k 2^{-4n} \le 12\tau. \tag{5}$$

Using the monotonicity of f in $[u_{2^{4n+1}}, u_{2^{4n}}]$, we obtain

$$\begin{aligned}
\sum_{j=k+1}^{k+2^{4n+1}+1} f(T^j x) &\le \sum_{\ell=0}^{2^{4n+1}} f(u_{2^{4n+1}+\ell}) = -\sum_{\ell=1}^{2^{4n+1}} 2^{-4n} \frac{|u_{2^{4n+1}} - u_{2^{4n+1}+\ell}|}{|u_{2^{4n+1}} - u_{2^{4n+2}}|} \\
&\le -\sum_{\ell=1}^{2^{4n+1}} 2^{-4n} u_{2^{4n+1}}^{-1} (u_{2^{4n+1}} - u_{2^{4n+1}+\ell}) \\
&\le -C_1^{-1} 2^{-1/\alpha-4n+4n/\alpha} \sum_{\ell=1}^{2^{4n+1}} \sum_{j=0}^{\ell-1} (u_{2^{4n+1}+j} - u_{2^{4n+1}+j+1}) \\
&\le -C_1^{-2} 2^{-1/\alpha-4n+4n/\alpha} \sum_{\ell=1}^{2^{4n+1}} \ell 2^{-(4n+2)(1+1/\alpha)} \\
&\le -\frac{1}{C_1^2 2^{2+3/\alpha}} 2^{-8n} \sum_{\ell=1}^{2^{4n+1}} \ell \le -\frac{1}{C_1^2 2^{5+2/\alpha}} = -\varepsilon < 0,
\end{aligned}$$

say, where we have twice used Lemma 3.1. Combining this estimate with (5), we deduce that $\sum_{j=0}^{v(x)-1} f(T^j x) \le \max\{0, 12\tau - \varepsilon\}$ for each $x \in [0, 1)$; thus, if τ is taken smaller than $\varepsilon/12$, then $\beta(f) = 0$.

Our final task is to show that the relation $f \le g \circ T - g + \beta(f)$ is impossible for continuous g. Following the method of the preceding estimate, for each $n > 0$ we have

$$\begin{aligned}
\sum_{\ell=2^{4n+2}}^{2^{4n+3}} f(u_\ell) &\ge \tau \sum_{\ell=1}^{2^{4n+2}} 2^{-4n} \frac{|u_{2^{4n+2}} - u_{2^{4n+2}+\ell}|}{|u_{2^{4n+2}} - u_{2^{4n+3}}|} \\
&\ge \tau \tilde{C} 2^{-4n+4n/\alpha} \sum_{\ell=1}^{2^{4n+2}} \sum_{j=0}^{\ell-1} (u_{2^{4n+2}+j} - u_{2^{4n+2}+j+1}) \\
&\ge \tau \tilde{C} 2^{-8n} \sum_{\ell=1}^{2^{4n+2}} \ell \ge \delta_\tau > 0,
\end{aligned}$$

say. Suppose now that $f \le g \circ T - g + \beta(f)$ is satisfied; then for each $n > 0$ we have

$$g(u_{2^{4n+2}}) \ge g(u_{2^{4n+3}}) + \sum_{j=0}^{2^{4n+3}-2^{4n+2}} f(T^j u_{2^{4n+3}}) \ge g(u_{2^{4n+3}}) + \delta_\tau.$$

If g is continuous at 0, letting $n \to \infty$ then yields

$$g(0) \ge g(0) + \delta_\tau > g(0),$$

which is a contradiction. □

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