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REGULARITY CONDITIONS AND BERNOULLI PROPERTIES OF EQUILIBRIUM STATES AND g -MEASURES

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ABSTRACT

When $T: X \rightarrow X$ is a one-sided topologically mixing subshift of finite type and $\varphi: X \rightarrow R$ is a continuous function, one can define the Ruelle operator $\mathcal{L}_\varphi: C(X) \rightarrow C(X)$ on the space $C(X)$ of real-valued continuous functions on X . The dual operator \mathcal{L}_φ^* always has a probability measure ν as an eigenvector corresponding to a positive eigenvalue ($\mathcal{L}_\varphi^* \nu = \lambda \nu$ with $\lambda > 0$). Necessary and sufficient conditions on such an eigenmeasure ν are obtained for φ to belong to two important spaces of functions, $W(X, T)$ and $\text{Bow}(X, T)$. For example, $\varphi \in \text{Bow}(X, T)$ if and only if ν is a measure with a certain approximate product structure. This is used to apply results of Bradley to show that the natural extension of the unique equilibrium state μ_φ of $\varphi \in \text{Bow}(X, T)$ has the weak Bernoulli property and hence is measure-theoretically isomorphic to a Bernoulli shift. It is also shown that the unique equilibrium state of a two-sided Bowen function has the weak Bernoulli property. The characterizations mentioned above are used in the case of g -measures to obtain results on the ‘reverse’ of a g -measure.

Introduction

We consider subshifts of finite type with a finite number of symbols. Let $k \geq 2$, and let $\Gamma = \{1, 2, \dots, k\}$ be the set of symbols. Let $A = (a_{ij})$ be a $k \times k$ matrix with each entry $a_{ij} \in \{0, 1\}$ and with no zero row and no zero column.

Let

$$X_A = \left\{ x = (x_n)_{n=0}^\infty \in \prod_0^\infty \Gamma \mid a_{x_n x_{n+1}} = 1 \ \forall n \geq 0 \right\}$$

and

$$\hat{X}_A = \left\{ x = (x_n)_{n=-\infty}^\infty \in \prod_{-\infty}^\infty \Gamma \mid a_{x_n x_{n+1}} = 1 \ \forall n \in \mathbb{Z} \right\}.$$

Both are compact sets under the product topologies on $\prod_0^\infty \Gamma$ and $\prod_{-\infty}^\infty \Gamma$ when Γ is equipped with the discrete topology. The one-sided subshift of finite type determined by A is the continuous surjection $T: X_A \rightarrow X_A$ defined by $T((x_0, x_1, \dots)) = (x_1, x_2, \dots)$. The two-sided subshift of finite type determined by A is the homeomorphism $S: \hat{X}_A \rightarrow \hat{X}_A$ defined by

$$S((\dots x_{-1}^* x_0 x_1 \dots)) = (\dots x_{-1} x_0^* x_1 x_2 \dots),$$

where the symbol $*$ is over the 0th position. Both shifts are called topologically mixing if there exists $M \geq 1$ with the product matrix $A^M > 0$, that is, every entry of A^M is non-zero. We use M for such a number throughout the paper. This is equivalent to T being topologically mixing (that is, for all non-empty open sets U, V , there exists $M \geq 1$ with $U \cap T^{-n}V \neq \emptyset$ for all $n \geq M$), and to

S being topologically mixing. If $p \leq q$ and $b_p, \dots, b_q \in \Gamma$, then ${}_p[b_p, \dots, b_q] = {}_p[b_p, \dots, b_q]_q = [b_p, \dots, b_q]_q = \{x \in X_A \mid x_i = b_i \text{ for } p \leq i \leq q\}$ when $0 \leq p$, and also ${}_p[b_p, \dots, b_q] = {}_p[b_p, \dots, b_q]_q = [b_p, \dots, b_q]_q = \{x \in \hat{X}_A \mid x_i = b_i \text{ for } p \leq i \leq q\}$ when $p \in Z$. For $n, p \geq 1$, we have $T^p(0[x_0, \dots, x_{p+n-1}]) = 0[x_p, \dots, x_{p+n-1}]$ if $0[x_0, \dots, x_{p+n-1}] \neq \emptyset$, and $T^{-p}(0[x_0, \dots, x_{n-1}]) = {}_p[x_0, \dots, x_{n-1}]$. An allowable block in X_A or \hat{X}_A is a string b_0, \dots, b_{n-1} of symbols with $a_{b_0 b_1} a_{b_1 b_2} \dots a_{b_{n-2} b_{n-1}} = 1$. If b_0, \dots, b_{n-1} is an allowable block, then we call ${}_p[b_0, \dots, b_{n-1}]$ an allowable cylinder. If $b, c \in \Gamma$ and $p, q \in N$, then $b^p c^q$ denotes the block b_0, \dots, b_{p+q-1} , where $b_i = b$ for $0 \leq i \leq p-1$ and $b_i = c$ for $p \leq i \leq p+q-1$. If x_0, \dots, x_{n-1} is an allowable block in X_A and $z = (z_0, z_1, \dots) \in X_A$ with $a_{x_{n-1} z_0} = 1$, then (x_0, \dots, x_{n-1}, z) denotes the member $y = (y_i)$ of X_A with $y_i = x_i$ for $0 \leq i \leq n-1$ and $y_{i+n} = z_i$ for $i \geq 0$.

We often write X and \hat{X} for X_A and \hat{X}_A . Consider a subshift of finite type (SFT) $T: X \rightarrow X$. We use $C(X)$ to denote the space of all real-valued continuous functions on X , equipped with the supremum norm. We let $M(X)$ denote the space of all probability measures on the Borel subsets of X , equipped with the weak*-topology, and let $M(X, T)$ denote the non-empty subset of T -invariant members of $M(X)$. We say that $\tau \in M(X)$ has support X if $\tau(U) > 0$ for every non-empty open set U . If $\varphi \in C(X)$, we let $P(T, \varphi)$ denote the pressure of T at φ [13], and let $T_n \varphi$ be the function $\sum_{i=0}^{n-1} \varphi \circ T^i$. Similar notation applies to $S: \hat{X} \rightarrow \hat{X}$. When T is a one-sided subshift of finite type, the Ruelle operator of $\varphi \in C(X)$ is denoted by $\mathcal{L}_\varphi: C(X) \rightarrow C(X)$, so that $(\mathcal{L}_\varphi f)(x) = \sum e^{\varphi(y)} f(y)$, where the sum is over all $y \in T^{-1}x$. The dual operator \mathcal{L}_φ^* always has an eigenmeasure in $M(X)$, that is, there exists $\nu \in M(X)$ and $\lambda > 0$ with $\mathcal{L}_\varphi^* \nu = \lambda \nu$. For $\varphi \in C(X)$ and T a one-sided subshift of finite type, we define $v_n(\varphi)$, for $n \geq 1$, as $v_n(\varphi) = \sup\{\varphi(x) - \varphi(x') \mid x, x' \in X \text{ and } x_i = x'_i, 0 \leq i \leq n-1\}$. We define the space $\text{Bow}(X, T)$ to be $\{\varphi \in C(X) \mid \sup_{n \geq 1} v_n(T_n \varphi) < \infty\}$ and the space $W(X, T)$ to be $\{\varphi \in C(X) \mid \sup_{n \geq 1} v_{n+p}(T_n \varphi) \rightarrow 0 \text{ as } p \rightarrow \infty\}$ [14, 15]. We have $W(X, T) \subset \text{Bow}(X, T)$, because $\varphi \in \text{Bow}(X, T)$ if and only if there is some $p \geq 0$ with $\sup_{n \geq 1} v_{n+p}(T_n \varphi) < \infty$. If $\varphi \in C(X)$ has summable variations (that is, $\sum_{n=1}^\infty v_n(\varphi) < \infty$) [11], then $\varphi \in W(X, T)$.

In [12], the author showed that for a topologically mixing subshift of finite type (TMSFT), if $\varphi \in W(X, T)$, then the Ruelle operator theorem holds (that is, there exists $\lambda > 0, \nu \in M(X)$, and $h \in C(X)$ with $h > 0$ and $\int h d\nu = 1$ such that $\mathcal{L}_\varphi h = \lambda h$, $\mathcal{L}_\varphi^* \nu = \lambda \nu$ and for all $f \in C(X)$,

$$\frac{(\mathcal{L}_\varphi^n f)(x)}{\lambda^n} \rightrightarrows h(x) \int f d\nu,$$

where \rightrightarrows denotes uniform convergence on X), φ has a unique equilibrium state μ_φ and (T, μ_φ) has a Bernoulli natural extension. Here $\mu_\varphi = h\nu$, and μ_φ is the unique g -measure for the g -function $g(x) = e^{\varphi(x)} h(x) / \lambda h(Tx)$. In [14], the author considered these questions for $\varphi \in \text{Bow}(X, T)$ and proved a weakened version of the Ruelle operator theorem. Each $\varphi \in \text{Bow}(X, T)$ has a unique equilibrium state μ_φ . We show in this paper that (T, μ_φ) has a Bernoulli natural extension. We obtain necessary and sufficient conditions on an eigenmeasure ν for $\varphi \in C(X)$ to ensure that $\varphi \in \text{Bow}(X, T)$ and to ensure that $\varphi \in W(X, T)$. These give characterizations of when $\varphi \in \text{Bow}(X, T)$ in terms of μ_φ . When this is applied to the cases $\varphi = \log g$ and g is a g -function (see Section 3), we obtain results about the ‘reverse’ of a g -measure.

For a two-sided topologically mixing subshift of finite type, $S: \hat{X} \rightarrow \hat{X}$ and $\hat{\varphi} \in C(\hat{X})$, we let $\text{var}_n(\hat{\varphi})$, for $n \geq 1$, denote $\text{var}_n(\hat{\varphi}) = \sup\{\hat{\varphi}(x) - \hat{\varphi}(x') \mid x, x' \in \hat{X} \text{ and } x_i = x'_i \text{ for } -(n-1) \leq i \leq n-1\}$. Then let $\text{Bow}(\hat{X}_A, S) = \{\hat{\varphi} \in C(\hat{X}) \mid \sup_{n \geq 1} \text{var}_n(S_n \hat{\varphi}) < \infty\}$, where $S_n \hat{\varphi} = \sum_{i=0}^{n-1} \hat{\varphi} \circ S^i$. Bowen showed that each $\hat{\varphi} \in \text{Bow}(\hat{X}_A, S)$ has a unique equilibrium state $\hat{\mu}_{\hat{\varphi}}$ [2]. We show that $(S, \hat{\mu}_{\hat{\varphi}})$ is isomorphic to a Bernoulli shift.

1. Eigenmeasures of the Ruelle operator

LEMMA 1.1. Let $T: X \rightarrow X$ be a one-sided topologically mixing subshift of finite type, let $\varphi \in C(X)$, and let $\nu \in M(X)$ and $\lambda > 0$ satisfy $\mathcal{L}_\varphi^* \nu = \lambda \nu$. Then for all $n \geq 1, p \geq 1$ and $x \in X$, we have

$$\nu([x_0, \dots, x_{n+p-1}]) = \frac{1}{\lambda^p} \int_{z \in [x_p, \dots, x_{n+p-1}]} e^{(T_p \varphi)(x_0, \dots, x_{p-1}, z)} d\nu(z).$$

Proof.

$$\begin{aligned} \nu([x_0, \dots, x_{n+p-1}]) &= \frac{1}{\lambda^p} \int_X \mathcal{L}_\varphi^p \chi_{[x_0, \dots, x_{n+p-1}]}(z) d\nu(z) \\ &= \frac{1}{\lambda^p} \int_{[x_p, \dots, x_{n+p-1}]} e^{(T_p \varphi)(x_0, \dots, x_{p-1}, z)} d\nu(z). \quad \square \end{aligned}$$

In part (i) of the next result, the number k is the number of symbols in the topologically mixing subshift of finite type and M is a natural number for which the M th power A^M of the transition matrix A has every entry non-zero.

THEOREM 1.2. Let $T: X \rightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\varphi \in C(X)$. Let $\nu \in M(X)$ and $\lambda > 0$ satisfy $\mathcal{L}_\varphi^* \nu = \lambda \nu$. Then $\lambda = e^{P(T, \varphi)}$, ν is a supported measure, and we have the following.

(i) For all $p \geq 1$ and for all $x \in X$,

$$(e^{v_p(T_p \varphi)} \lambda^M e^{M \|\varphi\|})^{-1} \leq \frac{\nu([x_0, \dots, x_{p-1}]) \lambda^p}{e^{(T_p \varphi)(x)}} \leq k^M e^{v_p(T_p \varphi)} \frac{e^{M \|\varphi\|}}{\lambda^M}.$$

(ii) For all $n \geq 1, p \geq 1$ and for all $x \in X$,

$$e^{-v_{n+p}(T_p \varphi)} \leq \frac{\nu([x_0, \dots, x_{n+p-1}])}{\nu([x_p, \dots, x_{n+p-1}])} \frac{\lambda^p}{e^{(T_p \varphi)(x)}} \leq e^{v_{n+p}(T_p \varphi)}.$$

(iii) The measures ν and $\nu \circ T^{-p}$ are equivalent and

$$\frac{d\nu \circ T^{-p}}{d\nu} = \frac{\mathcal{L}_\varphi^p 1}{\lambda^p}.$$

(iv) For each symbol $i \in \{1, 2, \dots, k\}$, $T|_{[i]}$ is injective, so $\nu \circ T|_{[i]}$ is a measure on $[i]$ and

$$\frac{d(\nu \circ T|_{[i]})}{d(\nu|_{[i]})} = \lambda e^{-\varphi}$$

on $[i]$.

(v) For all $n \geq 1, p \geq 1$ and for all $w \in X$,

$$e^{-v_n(\log \mathcal{L}_\varphi^p 1)} \leq \frac{\nu_p([w_0, \dots, w_{n-1}])}{\nu_0([w_0, \dots, w_{n-1}])} \frac{\lambda^p}{(\mathcal{L}_\varphi^p 1)(w)} \leq e^{v_n(\log \mathcal{L}_\varphi^p 1)}.$$

Proof. (i)

$$\begin{aligned} \nu_0([x_0, \dots, x_{p-1}]) &= \int_X \chi_{0[x_0, \dots, x_{p-1}]}(z) d\nu(z) \\ &= \frac{1}{\lambda^{p+M}} \int_X \mathcal{L}_\varphi^{p+M} \chi_{0[x_0, \dots, x_{p-1}]}(z) d\nu(z). \end{aligned}$$

Now

$$\mathcal{L}_\varphi^{p+M} \chi_{0[x_0, \dots, x_{p-1}]}(z) = \sum e^{(T_{p+M} \varphi)(x_0, \dots, x_{p-1}, b_0, \dots, b_{M-1}, z)},$$

where the sum is over all b_0, \dots, b_{M-1} with the product $a_{x_{p-1}b_0} a_{b_0b_1} \dots a_{b_{M-1}z_0} > 0$. For any x_{p-1}, z_0 , such (b_0, \dots, b_{M-1}) exist by topological mixing, and there are at most k^M choices. For each such admissible (b_0, \dots, b_{M-1}) , we have $|(T_{p+M} \varphi)(x_0, \dots, x_{p-1}, b_0, \dots, b_{M-1}, z) - (T_p \varphi)(x)| \leq v_p(T_p \varphi) + M \|\varphi\|$, so we get the inequalities in (i).

The left-hand inequality in (i) shows that ν is supported. To see that $\lambda = e^{P(T, \varphi)}$, we use the fact that $(1/p) \log(\mathcal{L}_\varphi^p 1)(x) \rightrightarrows P(T, \varphi)$ [14, Theorem 1.3]. If $\varepsilon > 0$, then there exists N_ε with $e^{pP(T, \varphi) - p\varepsilon} \leq (\mathcal{L}_\varphi^p 1)(x) \leq e^{pP(T, \varphi) + p\varepsilon}$ for all $x \in X$ and for all $p \geq N_\varepsilon$. Integrating with respect to ν gives $e^{p(P(T, \varphi) - \varepsilon)} \leq \lambda^p \leq e^{p(P(T, \varphi) + \varepsilon)}$ for all $p \geq N_\varepsilon$. Hence $e^{P(T, \varphi) - \varepsilon} \leq \lambda \leq e^{P(T, \varphi) + \varepsilon}$, and since this holds for every $\varepsilon > 0$, we have $\lambda = e^{P(T, \varphi)}$.

(ii) Statement (ii) follows from Lemma 1.1 and the inequality $|(T_p \varphi)(x_0, \dots, x_{p-1}z) - (T_p \varphi)(x)| \leq v_{n+p}(T_p \varphi)$, when $z \in_0[x_p, \dots, x_{n+p-1}]$.

(iii) Fix a cylinder $0[w_0, \dots, w_{p-1}]$ in X . Then

$$T^{-p}(0[w_0, \dots, w_{n-1}]) = \bigcup_0 [y_0, \dots, y_{p-1}, w_0, \dots, w_{n-1}],$$

where the union is over all (y_0, \dots, y_{p-1}) with $a_{y_0y_1} \dots a_{y_{p-1}w_0} = 1$, and this is a disjoint union. Apply Lemma 1.1 to each such admissible $0[y_0, \dots, y_{p-1}, w_0, \dots, w_{n-1}]$ to get

$$\nu_0([y_0, \dots, y_{p-1}, w_0, \dots, w_{n-1}]) = \frac{1}{\lambda^p} \int_{z \in_0[w_0, \dots, w_{n-1}]} e^{(T_p \varphi)(y_0, \dots, y_{p-1}, z)} d\nu(z).$$

Now sum over all (y_0, \dots, y_{p-1}) to get

$$(\nu \circ T^{-p})(0[w_0, \dots, w_{n-1}]) = \int_{z \in_0[w_0, \dots, w_{n-1}]} \frac{(\mathcal{L}_\varphi^p 1)(z)}{\lambda^p} d\nu(z).$$

(iv) We have to show that for each cylinder $0[i, i_1, \dots, i_{p-1}]$, we have

$$\nu(T_0[i, i_1, \dots, i_{p-1}]) = \lambda \int_{x \in_0[i, i_1, \dots, i_{p-1}]} e^{-\varphi(x)} d\nu(x).$$

We have

$$\begin{aligned} \lambda \int_{0[i, i_1, \dots, i_{p-1}]} e^{-\varphi(x)} d\nu(x) &= \lambda \int_X e^{-\varphi} \chi_{0[i, i_1, \dots, i_{p-1}]} d\nu \\ &= \int_X \mathcal{L}_\varphi (e^{-\varphi} \chi_{0[i, i_1, \dots, i_{p-1}]}) d\nu \\ &= \int_X \chi_{0[i, i_1, \dots, i_{p-1}]}(iz) d\nu(z) \\ &= \nu(T_0[i, i_1, \dots, i_{p-1}]). \end{aligned}$$

(v) By (iii),

$$\nu_p[w_0, \dots, w_{n-1}] = \int_{z \in 0[w_0, \dots, w_{n-1}]} \frac{\mathcal{L}_\varphi^p 1}{\lambda^p}(z) d\nu(z).$$

If $w \in X$ and $z \in 0[w_0, \dots, w_{n-1}]$, then

$$e^{-v_n(\log \mathcal{L}_\varphi^p 1)} \leq \frac{(\mathcal{L}_\varphi^p 1)(z)}{(\mathcal{L}_\varphi^p 1)(w)} \leq e^{v_n(\log \mathcal{L}_\varphi^p 1)}$$

so (v) holds. \square

Recall that we use the symbol \rightrightarrows to denote uniform convergence on X .

COROLLARY 1.3. *Let $T: X \rightarrow X$ be a one-sided topologically mixing subshift of finite type, and let $\varphi \in C(X)$. Let $\nu \in M(X)$ and let $\lambda > 0$ satisfy $\mathcal{L}_\varphi^* \nu = \lambda \nu$. For each fixed $p \geq 1$,*

$$\frac{\nu_{(0[x_0, \dots, x_{n+p-1}])}}{\nu_{(0[x_p, \dots, x_{n+p-1}])}} \rightrightarrows \frac{e^{(T_p \varphi)(x)}}{\lambda^p} \quad \text{as } n \rightarrow \infty,$$

and

$$\frac{\nu_{(p[w_0, \dots, w_{n-1}])}}{\nu_{(0[w_0, \dots, w_{n-1}])}} \rightrightarrows \frac{(\mathcal{L}_\varphi^p 1)(w)}{\lambda^p} \quad \text{as } n \rightarrow \infty.$$

Proof. The first statement is by Theorem 1.2(ii), since, for p fixed, $v_{n+p}(T_p \varphi) \rightarrow 0$ as $n \rightarrow \infty$, and the second statement is by Theorem 1.2(v), since, for fixed p , $v_n(\log \mathcal{L}_\varphi^p 1) \rightarrow 0$ as $n \rightarrow \infty$. \square

In fact, the case $p=1$ in the first statement of Corollary 1.3 gives

$$\frac{\nu_{(0[x_0, \dots, x_n])}}{\nu_{(0[x_1, \dots, x_n])}} \rightrightarrows \frac{e^{\varphi(x)}}{\lambda} \quad \text{as } n \rightarrow \infty,$$

and the case of general p follows from this, as does the second conclusion of the corollary.

Note that we can write the first conclusion of Corollary 1.3 as

$$\frac{\nu_{(0[x_0, \dots, x_{n+p-1}])}}{\nu_{(T^p 0[x_0, \dots, x_{n+p-1}])}} \rightrightarrows \frac{e^{(T_p \varphi)(x)}}{\lambda^p} \quad \text{as } n \rightarrow \infty,$$

and the second conclusion as

$$\frac{\nu(T^{-p}{}_0[w_0, \dots, w_{n-1}])}{\nu({}_0[w_0, \dots, w_{n-1}])} \rightrightarrows \frac{(\mathcal{L}_\varphi^p 1)(w)}{\lambda^p} \quad \text{as } n \rightarrow \infty.$$

The following result gives a condition for a probability measure to be an eigenmeasure of a Ruelle operator.

COROLLARY 1.4. *Let $T : X \rightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\nu \in M(X)$. Then $\mathcal{L}_\varphi^* \nu = \lambda \nu$ for some $\varphi \in C(X)$ and some $\lambda > 0$ if and only if ν is supported and $\nu({}_0[x_0, \dots, x_n]) / \nu({}_0[x_1, \dots, x_n])$ converges uniformly on X to a function $f : X \rightarrow (0, \infty)$.*

Proof. If $\mathcal{L}_\varphi^* \nu = \lambda \nu$ for some $\nu \in M(X)$ and some $\lambda > 0$, then

$$\frac{\nu({}_0[x_0, \dots, x_n])}{\nu({}_0[x_1, \dots, x_n])} \rightrightarrows \frac{e^{\varphi(x)}}{\lambda}$$

by Corollary 1.3.

If $\nu({}_0[x_0, \dots, x_n]) / \nu({}_0[x_1, \dots, x_n])$ converges uniformly on X to a positive function, then denote the limit by $e^{\varphi(x)}$. We show that $\mathcal{L}_\varphi^* \nu = \nu$. For any cylinder ${}_0[b_0, \dots, b_{t-1}]$,

$$\begin{aligned} & (\mathcal{L}_\varphi^* \nu)({}_0[b_0, \dots, b_{t-1}]) \\ &= \int \mathcal{L}_\varphi \chi_{{}_0[b_0, \dots, b_{t-1}]} d\nu = \int_{x \in {}_0[b_1, \dots, b_{t-1}]} e^{\varphi(b_0 x)} d\nu(x) \\ &= \lim_{n \rightarrow \infty} \int_{x \in {}_0[b_1, \dots, b_{t-1}]} \frac{\nu({}_0[b_0, b_1, \dots, b_{t-1}, x_{t-1}, x_t, \dots, x_{t+n}])}{\nu({}_0[b_1, \dots, b_{t-1}, x_{t-1}, \dots, x_{t+n}])} d\nu(x) \\ &= \lim_{n \rightarrow \infty} \sum_{x_{t-1}, \dots, x_{t+n}} \nu({}_0[b_0, b_1, \dots, b_{t-1}, x_{t-1}, \dots, x_{t+n}]) \\ &= \nu({}_0[b_0, \dots, b_{t-1}]). \quad \square \end{aligned}$$

With Corollary 1.3 in mind, we can characterize when $\varphi \in C(X)$ is a member of $W(X, T)$ in terms of an eigenmeasure ν . If \mathbb{N} is the set of natural numbers, then $BC(\mathbb{N} \times X)$ denotes the space of bounded continuous real-valued functions on $\mathbb{N} \times X$, equipped with the supremum norm.

THEOREM 1.5. *Let $T : X \rightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\varphi \in C(X)$. Let $\lambda = e^{P(T, \varphi)}$. The following statements are pairwise equivalent.*

- (i) $\varphi \in W(X, T)$.
- (ii) There exists $\tau \in M(X)$ with support X satisfying

$$\frac{\tau({}_0[x_0, \dots, x_{p+n-1}])}{\tau({}_0[x_p, \dots, x_{p+n-1}])} \frac{\lambda^p}{e^{(T_p \varphi)(x)}} \rightrightarrows 1 \quad \text{as } n \rightarrow \infty,$$

where the convergence is uniform in both $x \in X$ and $p \in \mathbb{N}$.

(iii) There exists $\nu \in M(X)$ with $\mathcal{L}_\varphi^* \nu = \lambda \nu$ such that in $\text{BC}(\mathbb{N} \times X)$, the sequence (ψ_n) , given by

$$\psi_n(p, x) = \log \left(\frac{\nu([x_0, \dots, x_{p+n-1}])}{\nu([x_p, \dots, x_{p+n-1}])} \right),$$

is convergent.

(iv) There exists $\mu \in M(X, T)$ with support X and $\ell \in C(X)$ satisfying

$$\log \left(\frac{\mu([x_0, \dots, x_{n+p-1}])}{\mu([x_p, \dots, x_{n+p-1}])} \right) \Rightarrow (T_p \varphi)(x) + \ell(x) - \ell(T^p x) - p \log \lambda \quad \text{as } n \rightarrow \infty,$$

where the convergence is uniform in both $x \in X$ and $p \geq 1$.

Proof. The implication (i) \Rightarrow (ii) follows immediately from Theorem 1.2(ii). To show that (ii) \Rightarrow (i), suppose that $\tau \in M(X)$ satisfies (ii). Then for all $\varepsilon > 0$, there exists N_ε such that $n \geq N_\varepsilon$ implies that

$$\frac{\tau([x_0, \dots, x_{n+p-1}])}{\tau([x_p, \dots, x_{n+p-1}])} \lambda^p e^{-\varepsilon} < e^{(T_p \varphi)(x)} < \frac{\tau([x_0, \dots, x_{p+n-1}])}{\tau([x_p, \dots, x_{p+n-1}])} \lambda^p e^\varepsilon \quad \forall p \geq 1, \forall x \in X.$$

If $x, z \in X$ have $x_i = z_i$ for $0 \leq i \leq p+n-1$, then when $n \geq N_\varepsilon$, $e^{-2\varepsilon} < e^{(T_p \varphi)(x) - (T_p \varphi)(z)} < e^{2\varepsilon}$. Hence $n \geq N_\varepsilon \Rightarrow \sup_{p \geq 1} \nu_{p+n}(T_p \varphi) \leq 2\varepsilon$. Hence $\varphi \in W(X, T)$.

The implication that (i) \Rightarrow (iii) follows from Theorem 1.2(ii). If (iii) holds, then the limit is $(T_p \varphi)(n) - p \log \lambda$ by Corollary 1.3, so (ii) holds.

We now show that (i) \Rightarrow (iv). When $\varphi \in W(X, T)$, the unique $\nu \in M(X)$ with $\mathcal{L}_\varphi^* \nu = \lambda \nu$ satisfies (ii), as we saw above. By [11], there exists $h \in C(X)$, $h > 0$, with $\mathcal{L}_\varphi h = \lambda h$ and $\int h d\nu = 1$. The measure $\mu = h\nu \in M(X, T)$, and

$$e^{-v_t(\log h)} \leq \frac{\mu([x_0, \dots, x_{t-1}])}{\nu([x_0, \dots, x_{t-1}])h(x)} \leq e^{v_t(\log h)} \quad \forall t \geq 1, x \in X.$$

Hence

$$\frac{\mu([x_0, \dots, x_{n+p-1}])}{\mu([x_p, \dots, x_{n+p-1}])} \frac{\lambda^p}{e^{(T_p \varphi)(x)}} \frac{h(T^p x)}{h(x)} \Rightarrow 1 \quad \text{as } n \rightarrow \infty$$

uniformly in $p \geq 1$ and $x \in X$. Statement (iv) follows with $\ell = \log h$.

If (iv) holds, then, for $\varepsilon > 0$, there exists N_ε such that $n \geq N_\varepsilon$ implies that

$$e^{-\varepsilon} < \frac{\mu([x_0, \dots, x_{n+p-1}])}{\mu([x_p, \dots, x_{n+p-1}])} \frac{\lambda^p}{e^{(T_p \varphi)(x)}} \frac{h(T^p x)}{h(x)} < e^\varepsilon \quad \forall p \geq 1, \forall x \in X,$$

where $h = e^\ell$. If $(x_0, \dots, x_{n+p-1}) = (z_0, \dots, z_{n+p-1})$, then, for $n \geq N_\varepsilon$,

$$\frac{h(T^p x)}{h(T^p z)} \frac{h(z)}{h(x)} e^{-2\varepsilon} < e^{(T_p \varphi)(x) - (T_p \varphi)(z)} < \frac{h(T^p x)}{h(T^p z)} \frac{h(z)}{h(x)} e^{2\varepsilon},$$

so

$$v_{n+p}(T_p \varphi) \leq v_n(\log h) + v_{n+p}(\log h) + 2\varepsilon \leq 2v_n(\log h) + 2\varepsilon \quad \forall p \geq 1, \forall n \geq 1.$$

Hence $\varphi \in W(X, T)$. \square

Note that when $\varphi \in W(X, T)$, then the eigenmeasure ν has the property given in (ii). The property in (ii) can be written as

$$\log \left(\frac{\tau([x_0, \dots, x_{n+p-1}])}{\tau([x_p, \dots, x_{n+p-1}])} \right) \Rightarrow (T_p \varphi)(x) - p \log \lambda \quad \text{as } n \rightarrow \infty,$$

where the convergence is uniform in both $x \in X$ and $p \geq 1$.

We now characterize, in terms of eigenmeasure, those $\varphi \in C(X)$ which belong to $\text{Bow}(X, T)$.

DEFINITION 1.6. Let X be a one-sided topologically mixing subshift of finite type. Let $\tau \in M(X)$. We say that τ is *approximately multiplicative at coordinate zero* if it has support X and there exists $C > 1$ with

$$C^{-1} \leq \frac{\tau(0[x_0, \dots, x_{n+p-1}])}{\tau(0[x_0, \dots, x_{p-1}])\tau(0[x_p, \dots, x_{n+p-1}])} \leq C \quad \forall n \geq 1, p \geq 1, x \in X.$$

If we let $f^{(n)}(x) = \nu(0[x_0, \dots, x_{n-1}])$, then the condition becomes

$$C^{-1} \leq \frac{f^{(p+n)}(x)}{f^{(p)}(x)f^{(n)}(T^p x)} \leq C.$$

THEOREM 1.7. Let $T: X \rightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\varphi \in C(X)$. Let $\lambda = e^{P(T, \varphi)}$. The following statements are pairwise equivalent.

- (i) $\varphi \in \text{Bow}(X, T)$.
- (ii) There exists $\tau \in M(X)$ with support X and there exists $D > 1$ with

$$D^{-1} \leq \tau(0[x_0, \dots, x_{p-1}]) \frac{\lambda^p}{e^{(T_p \varphi)(x)}} \leq D \quad \forall p \geq 1, \forall x \in X.$$

- (iii) There exists $\nu \in M(X)$ with $\mathcal{L}_\varphi^* \nu = \lambda \nu$, and ν is approximately multiplicative at coordinate zero.

Proof. The implication (i) \Rightarrow (ii) follows from Theorem 1.2(i). If (ii) holds and $(x_0, \dots, x_{p-1}) = (z_0, \dots, z_{p-1})$, then

$$D^{-2} \leq e^{(T_p \varphi)(x) - (T_p \varphi)(z)} \leq D^2,$$

so

$$v_p(T_p \varphi) \leq 2 \log D \quad \text{and} \quad \varphi \in \text{Bow}(X, T).$$

To show (i) \Rightarrow (iii), we use Theorem 1.2(i) and (ii) to get

$$\begin{aligned} \frac{\lambda^M}{\lambda^M e^{M \|\varphi\|} e^{v_{n+p}(T_p \varphi) + v_p(T_p \varphi)}} &\leq \frac{\nu(0[x_0, \dots, x_{n+p-1}])}{\nu(0[x_0, \dots, x_{p-1}])\nu(0[x_p, \dots, x_{n+p-1}])} \\ &\leq \lambda^M e^{M \|\varphi\|} e^{v_{n+p}(T_p \varphi) + v_p(T_p \varphi)}. \end{aligned}$$

Since $v_{p+n}(T_p \varphi) \leq \text{var}_p(T_p \varphi)$, we have (i) \Rightarrow (iii). To see that (iii) \Rightarrow (ii), if C is the constant in the definition of ν being approximately multiplicative at coordinate zero, then from Theorem 1.2(ii) we have

$$C^{-1} e^{-v_{p+n}(T_p \varphi)} \leq \nu(0[x_0, \dots, x_{p-1}]) \frac{\lambda^p}{e^{(T_p \varphi)(x)}} \leq C e^{v_{p+n}(T_p \varphi)} \quad \text{for all } n \geq 1.$$

Let $n \rightarrow \infty$ to give (ii). □

The equivalence of (i) and (ii) is well known. Other equivalent conditions for $\varphi \in \text{Bow}(X, T)$ can be found in [14].

As we shall see in Section 3, Theorem 1.7 gives a nice characterization of which g -measures correspond to a g with $\log g \in \text{Bow}(X, T)$.

COROLLARY 1.8. Let $T: X \rightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\varphi \in \text{Bow}(X, T)$. The unique equilibrium state μ_φ of φ is approximately multiplicative at coordinate zero.

Proof. When $\varphi \in \text{Bow}(X, T)$, there is a unique $\nu \in M(X)$ with $\mathcal{L}_\varphi^* \nu = \lambda \nu$ [14], and ν is approximately multiplicative at coordinate zero by Theorem 1.7. Also, $\mu_\varphi = h\nu$ for some measurable $h: X \rightarrow [a, b]$ with $0 < a < b$, and $\mathcal{L}_\varphi h = \lambda h$ and $\int h d\nu = 1$ [14]. Hence

$$a\nu({}_p[x_0, \dots, x_{n-1}]) \leq \mu_\varphi({}_p[x_0, \dots, x_{n-1}]) \leq b\nu({}_p[x_0, \dots, x_{n-1}]) \\ \forall p \geq 1, n \geq 1, x \in X,$$

so that μ_φ is approximately multiplicative at coordinate zero. \square

Since μ_φ is T -invariant, we can write the approximately multiplicative at coordinate zero condition for μ_φ as follows. There exists $D > 1$ with

$$D^{-1} \leq \frac{\mu_\varphi[{}_0(x_0, \dots, x_{n+p-1})]}{\mu_\varphi[{}_0(x_0, \dots, x_{p-1})]\mu_\varphi[{}_p(x_p, \dots, x_{n+p-1})]} \leq D \quad \forall p \geq 1, n \geq 1, x \in X.$$

We use this condition in the next definition.

DEFINITION 1.9. Let $T: X \rightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\tau \in M(X)$. We say that τ has approximate product structure if it has support X and if there exists $C > 1$ with

$$C^{-1} \leq \frac{\tau[{}_0(x_0, \dots, x_{n+p-1})]}{\tau[{}_0(x_0, \dots, x_{p-1})]\tau[{}_p(x_p, \dots, x_{n+p-1})]} \leq C \quad \forall n \geq 1, p \geq 1, x \in X.$$

If $\tau \in M(X, T)$, then clearly τ is approximately multiplicative at coordinate zero if and only if τ has approximate product structure. To investigate the relationship between the two conditions when τ is an eigenmeasure ν for \mathcal{L}_φ , we can use the following deduction from Theorem 1.2(iii).

PROPOSITION 1.10. Let $T: X \rightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\varphi \in C(X)$. Let $\nu \in M(X)$ satisfy $\mathcal{L}_\varphi^* \nu = \lambda \nu$ with $\lambda = e^{P(T, \varphi)}$. Let $D \geq 1$, and let $p \in \mathbb{N}$. Then

$$D^{-1} \leq \frac{(\mathcal{L}_\varphi^p 1)(x)}{\lambda^p} \leq D \quad \forall x \in X$$

if and only if

$$D^{-1} \leq \frac{\nu[{}_p[z_0, \dots, z_{n-1}]]}{\nu[{}_0[z_0, \dots, z_{n-1}]]} \leq D \quad \forall n \geq 1, z \in X.$$

Proof. By Theorem 1.2(iii),

$$\nu[{}_p[z_0, \dots, z_{n-1}]] = \int_{w \in {}_0[z_0, \dots, z_{n-1}]} \frac{(\mathcal{L}_\varphi^p 1)(w)}{\lambda^p} d\nu(w). \quad (*)$$

Clearly the first statement of the proposition implies the second statement. Now suppose that the second statement holds and let

$$U = \left\{ x \in X \mid \frac{(\mathcal{L}_\varphi^p 1)(x)}{\lambda^p} > D \right\}.$$

Then U is open. Suppose that $U \neq \emptyset$. For $z \in U$, choose some n with ${}_0[z_0, \dots, z_{n-1}] \subset U$. Then, by (*), $\nu_p[{}_0[z_0, \dots, z_{n-1}]] > D\nu({}_0[{}_0[z_0, \dots, z_{n-1}]])$, which contradicts the assumption. Hence $U = \emptyset$, so $(\mathcal{L}_\varphi^p 1)(x)/\lambda^p \leq D$ for all $x \in X$. Similarly, we get $D^{-1} \leq (\mathcal{L}_\varphi^p 1)(x)/\lambda^p$ for all $x \in X$. \square

COROLLARY 1.11. *Let $T : X \rightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\varphi \in C(X)$. Let $\nu \in M(X)$ satisfy $\mathcal{L}_\varphi^* \nu = \lambda \nu$ with $\lambda = e^{P(T, \varphi)}$. Suppose that there exists $D \geq 1$ with*

$$D^{-1} \leq \frac{(\mathcal{L}_\varphi^p 1)(x)}{\lambda^p} \leq D \quad \forall p \geq 1, \forall x \in X.$$

Then ν is approximately multiplicative at coordinate zero if and only if ν has approximate product structure.

COROLLARY 1.12. *Let $T : X \rightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\varphi \in C(X)$. Let $\lambda = e^{P(T, \varphi)}$. Then $\varphi \in \text{Bow}(X, T)$ if and only if both of the following statements hold.*

- (i) *There exists $B > 1$ with $B^{-1} \leq (\mathcal{L}_\varphi^p 1)(x)/\lambda^p \leq B$ for all $p \geq 1, x \in X$.*
- (ii) *There exists $\nu \in M(X)$ with $\mathcal{L}_\varphi^* \nu = \lambda \nu$ and ν has approximate product structure.*

Proof. If $\varphi \in \text{Bow}(X, T)$, then (i) holds by [14, p. 337], and (ii) holds by Theorem 1.7 and Corollary 1.11. If (i) and (ii) hold, then, by Corollary 1.11, ν is approximately multiplicative at coordinate zero and hence $\varphi \in \text{Bow}(X, T)$ by Theorem 1.7. \square

The following lemma is well known.

LEMMA 1.13. *Let $T : X \rightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\varphi \in C(X)$ and $\lambda = e^{P(T, \varphi)}$. Suppose that $\nu \in M(X)$ satisfies $\mathcal{L}_\varphi^* \nu = \lambda \nu$, and suppose that there is a measurable $h : X \rightarrow [a, b]$ with $0 < a \leq b$ and $\mathcal{L}_\varphi h = \lambda h$. Suppose that h is normalized so that $\int h d\nu = 1$. Then $\mu = h\nu$ is an equilibrium state for φ .*

Proof. We have $\mu \in M(X, T)$ since, for $f \in C(X)$,

$$\int f \circ T d\mu = \int f \circ Th d\nu = \lambda^{-1} \int \mathcal{L}_\varphi(f \circ Th) d\nu = \lambda^{-1} \int f \mathcal{L}_\varphi(h) d\nu = \int fh d\nu = \int f d\mu.$$

To see that μ is an equilibrium state, we use Theorem 1.2(i) to get

$$\begin{aligned} -\frac{1}{p} \log(\lambda^M e^{M\|\varphi\|}) - \frac{v_p(T_p \varphi)}{p} &\leq -\frac{1}{p} \log \nu({}_0[x_0, \dots, x_{p-1}]) + \frac{1}{p} (T_p \varphi)(x) - \log \lambda \\ &\leq \frac{1}{p} \log \left(\frac{k^M e^{M\|\varphi\|}}{\lambda^M} \right) + \frac{v_p(T_p \varphi)}{p}. \end{aligned}$$

Since $a\nu({}_0[x_0, \dots, x_{p-1}]) \leq \mu({}_0[x_0, \dots, x_{p-1}]) \leq b\nu({}_0[x_0, \dots, x_{p-1}])$ and

$$v_p(T_p \varphi) \leq \sum_{i=1}^p v_i(\varphi),$$

we have

$$-\frac{1}{p} \log \mu_0[x_0, \dots, x_{p-1}] + \frac{1}{p} (T_p \varphi)(x) \Rightarrow \log \lambda = P(T, \varphi).$$

Integrating via μ gives

$$\frac{1}{p} H \left(\bigvee_{i=0}^{p-1} T^{-i} \xi \right) + \int \varphi d\mu \longrightarrow P(T, \varphi),$$

where ξ is the partition into states at coordinate zero. Since ξ is a one-sided generator, we have $h_\mu(T) + \int \varphi d\mu = P(T, \varphi)$. □

There is another characterization of $\varphi \in \text{Bow}(X, T)$.

THEOREM 1.14. *Let $T : X \rightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\varphi \in C(X)$ and $\lambda = e^{P(T, \varphi)}$. Then $\varphi \in \text{Bow}(X, T)$ if and only if all of the following three statements hold.*

- (i) φ has a unique equilibrium state μ_φ .
- (ii) μ_φ has approximate product structure.
- (iii) There exists $B > 1$ with $B^{-1} \leq (\mathcal{L}_\varphi^p 1)(x) / \lambda^p \leq B$ for all $p \geq 1, x \in X$.

Proof. If $\varphi \in \text{Bow}(X, T)$, (i) and (iii) hold by [14], and (ii) holds by Corollary 1.8. Now assume that the three conditions hold. Let $\nu \in M(X)$ satisfy $\mathcal{L}_\varphi^* \nu = \lambda \nu$. By (iii), there is a measurable $h : X \rightarrow [a, b]$ with $0 < a < b, \int h d\nu = 1$, and $\mathcal{L}_\varphi h = \lambda h$ [14, p. 341]. Then, by Lemma 1.13, $\mu = h\nu$ is T -invariant and is an equilibrium state for φ . By (i), $\mu_\varphi = h\nu$. By (ii), μ_φ is approximately multiplicative at coordinate zero, so ν is also. Hence $\varphi \in \text{Bow}(X, T)$ by Theorem 1.7. □

2. The weak Bernoulli property

In this section, we show that if $\varphi \in \text{Bow}(X, T)$, then the natural extension of T with respect to the unique equilibrium state μ_φ of φ is a Bernoulli shift. We shall also show that when $S : \hat{X} \rightarrow \hat{X}$ is a two-sided topologically mixing subshift of finite type and $\hat{\varphi} \in \text{Bow}(\hat{X}, S)$, then $(S, \hat{\mu}_{\hat{\varphi}})$ is isomorphic to a Bernoulli shift where $\hat{\mu}_{\hat{\varphi}}$ is the unique equilibrium state of $\hat{\varphi}$.

When $\hat{\varphi} \in \text{Bow}(\hat{X}, S)$, then Bowen [2] showed that there exists $C > 1$ with

$$C^{-1} \leq \frac{\hat{\mu}_{\hat{\varphi}}([x_0, \dots, x_{n-1}])}{e^{(S_n \hat{\varphi})(x) - nP(S, \hat{\varphi})}} \leq C \quad \forall x \in \hat{X}, \forall n \geq 1.$$

It readily follows that $\hat{\mu}_{\hat{\varphi}}$ has approximate product structure (or rather that $\hat{\mu} \circ \pi^{-1}$ does, where $\pi : \hat{X} \rightarrow X$ is the natural projection). The definition of approximate product structure also makes sense for a measure on the two-sided shift space \hat{X} .

Whereas every $\hat{\varphi} \in W(\hat{X}, S)$ is cohomologous in $C(\hat{X})$ to some $\varphi \circ \pi$ with $\varphi \in W(X, T)$ [1], there are examples of $\hat{\varphi} \in \text{Bow}(\hat{X}, S)$ which are not cohomologous in $C(\hat{X})$ to a one-sided function [10].

For every $\mu \in M(X, T)$, there is a unique $\hat{\mu} \in M(\hat{X}, S)$ with $\hat{\mu} \circ \pi^{-1} = \mu$. We have $\hat{\mu}_p([b_0, \dots, b_{n-1}]) = \mu_q([b_0, \dots, b_{n-1}])$, for all $p \in \mathbb{Z}, q \geq 0$. This gives a bijection between $M(X, T)$ and $M(\hat{X}, S)$.

THEOREM 2.1. *Let $T: X \rightarrow T$ be a one-sided topologically mixing subshift of finite type with transition matrix A . Let $\mu \in M(X, T)$ have approximate product structure. If $A^M > 0$, then there exists $D > 1$ such that for all $n \geq 2M$, all $p, q \geq 1$, and all allowable cylinders ${}_0[x_0, \dots, x_{p-1}]$, ${}_0[y_0, \dots, y_{q-1}]$, we have*

$$D^{-1} \leq \frac{\mu({}_0[x_0, \dots, x_{p-1}] \cap T^{-(p+n)}{}_0[y_0, \dots, y_{q-1}])}{\mu([x_0, \dots, x_{p-1}])\mu([y_0, \dots, y_{q-1}])} \leq D.$$

Proof. Let C be the constant in the approximate product structure condition for μ . Let $[x_0, \dots, x_{p-1}]$ and $[y_0, \dots, y_{q-1}]$ be given. Let $n \geq 2M$. Since $A^M > 0$, there is a point $w \in X$ of the form $w = (x_0, \dots, x_{p-1}, w_p, \dots, w_{p+n-1}, y_0, y_1, \dots)$. Moreover, for any allowable choice of the cylinder $[w_{p+M-1}, \dots, w_{p+n-M}]$, there is such a point. Since

$$C^{-4} \leq \frac{\mu[x_0, \dots, x_{p-1}, w_p, \dots, w_{p+n-1}, y_0, \dots, y_{q-1}]}{\mu[x_0, \dots, x_{p-1}]\mu[w_p, \dots, w_{p+M-2}]\mu[w_{p+M-1}, \dots, w_{p+n-M}]\mu[w_{p+n-M+1}, \dots, w_{p+n-1}]\mu[y_0, \dots, y_{q-1}]} \leq C^4,$$

we have

$$C^{-4}d^2 \leq \frac{\mu({}_0[x_0, \dots, x_{p-1}] \cap T^{-(p+n)}{}_0[y_0, \dots, y_{q-1}])}{\mu({}_0[x_0, \dots, x_{p-1}])\mu({}_0[y_0, \dots, y_{q-1}])} \leq C^4,$$

where d is the minimal μ -measure of a cylinder of length $M - 1$.

Therefore, put $D = C^4d^{-2}$. □

If we consider the corresponding two-sided measure $\hat{\mu}$, then the conclusion of Theorem 2.1 can be written as

$$D^{-1} \leq \frac{\hat{\mu}({}_{-p}[x_{-p}, \dots, x_{-1}] \cap S^{-n}{}_0[y_0, \dots, y_{q-1}])}{\hat{\mu}([x_{-p}, \dots, x_{-1}])\hat{\mu}([y_0, \dots, y_{q-1}])} \leq D$$

for all $n \geq 2M$ and all cylinders ${}_{-p}[x_{-p}, \dots, x_{-1}]$, ${}_0[y_0, \dots, y_{q-1}]$ with $p, q \geq 1$.

By the usual approximation arguments, one can readily get

$$D^{-1} \leq \frac{\hat{\mu}(B_1 \cap S^{-n}B_2)}{\hat{\mu}(B_1)\hat{\mu}(B_2)} \leq D \tag{†}$$

whenever $n \geq 2M$, $B_1 \in \mathcal{B}_{-\infty}^{-1}$, $B_2 \in \mathcal{B}_0^\infty$ and $\hat{\mu}(B_1)\hat{\mu}(B_2) > 0$. Here $\mathcal{B}_{-\infty}^{-1}$ is the σ -algebra generated by all cylinders ${}_{-p}[x_{-p}, \dots, x_{-1}]_{-1}$ with $p \geq 1$, and \mathcal{B}_0^∞ is the σ -algebra generated by all cylinders ${}_0[y_0, \dots, y_{q-1}]_{q-1}$ with $q \geq 1$.

We shall use the following result.

COROLLARY 2.2. *Let $S: \hat{X} \rightarrow \hat{X}$ be a one-sided topologically mixing subshift of finite type and let $\hat{\mu} \in M(\hat{X}, S)$ be of approximate product type. Then S is strongly mixing with respect to $\hat{\mu}$.*

Proof. This follows from a result of Ornstein [9] if we can show that S^i is ergodic for each $i \geq 1$ and there is a constant d with

$$\limsup_{n \rightarrow \infty} \hat{\mu}(B_1 \cap S^{-n}B_2) \leq d\hat{\mu}(B_1)\hat{\mu}(B_2) \quad \forall B_1, B_2 \in \mathcal{B}.$$

From (†), we get

$$\limsup_{n \rightarrow \infty} \hat{\mu}(B_1 \cap S^{-n}B_2) \leq D\hat{\mu}(B_1)\hat{\mu}(B_2)$$

whenever $B_1 \in \mathcal{B}_{-\infty}^{\ell}$ and $B_2 \in \mathcal{B}_{-j}^{\infty}$, $\ell, j > 0$, so an approximation argument gives the same inequality whenever $B_1, B_2 \in \mathcal{B}$.

Similarly,

$$\liminf_{n \rightarrow \infty} \hat{\mu}(B_1 \cap S^{-n}B_2) \geq D^{-1} \hat{\mu}(B_1) \hat{\mu}(B_2)$$

whenever $B_1, B_2 \in \mathcal{B}$. This latter inequality gives the total ergodicity required for Ornstein's result, since, if $S^i B_2 = B_2$, then

$$0 = \hat{\mu}((\hat{X} \setminus B_2) \cap B_2) \geq D^{-1} \hat{\mu}(\hat{X} \setminus B_2) \hat{\mu}(B_2),$$

so either $\hat{\mu}(B_2)$ or $\hat{\mu}(\hat{X} \setminus B_2)$ is 0.

We now use results of Bradley to show that having approximate product structure \square implies the weak Bernoulli property.

THEOREM 2.3. *Let $S: \hat{X} \rightarrow \hat{X}$ be a two-sided topologically mixing subshift of finite type, and let $\hat{\mu} \in M(\hat{X}, S)$ have approximate product structure. Then for all $\varepsilon > 0$, there exists N so that $n \geq N$ implies that*

$$e^{-\varepsilon} \hat{\mu}(B_1) \hat{\mu}(B_2) \leq \hat{\mu}(B_1 \cap S^{-n}B_2) \leq e^{\varepsilon} \hat{\mu}(B_1) \hat{\mu}(B_2)$$

whenever $B_1 \in \mathcal{B}_{-\infty}^{-1}$ and $B_2 \in \mathcal{B}_0^{\infty}$. Hence the natural partition into states at coordinate zero is a weak Bernoulli partition for S , so $(S, \hat{\mu})$ is isomorphic to a Bernoulli shift.

Proof. From Theorem 2.1, the inequalities (\dagger) hold. Since, by Corollary 2.2, $(S, \hat{\mu})$ is strongly mixing, then, by a result of Bradley [3], if

$$\psi_n^* = \sup \left\{ \frac{\hat{\mu}(B_1 \cap S^{-n}B_2)}{\hat{\mu}(B_1) \hat{\mu}(B_2)} \mid B_1 \in \mathcal{B}_{-\infty}^{-1}, B_2 \in \mathcal{B}_0^{\infty}, \hat{\mu}(B_1) \hat{\mu}(B_2) > 0 \right\}$$

$$\psi_n' = \inf \left\{ \frac{\hat{\mu}(B_1 \cap S^{-n}B_2)}{\hat{\mu}(B_1) \hat{\mu}(B_2)} \mid B_1 \in \mathcal{B}_{-\infty}^{-1}, B_2 \in \mathcal{B}_0^{\infty}, \hat{\mu}(B_1) \hat{\mu}(B_2) > 0 \right\},$$

then (i) either $\psi_n^* \rightarrow 1$ as $n \rightarrow \infty$ or $\psi_n^* = 0$ for all n , (ii) either $\psi_n' \rightarrow 1$ as $n \rightarrow \infty$ or $\psi_n' = 0$ for all n . Since inequalities (\dagger) hold, we have $\psi_n^* \leq D$ and $D^{-1} \leq \psi_n'$, so $\psi_n^* \rightarrow 1$ and $\psi_n' \rightarrow 1$. This gives the condition in the statement of the theorem. The isomorphism result is due to Friedman and Ornstein [5]. \square

We state the following results.

THEOREM 2.4. *Let $T: X \rightarrow X$ be a one-sided, and let $S: \hat{X} \rightarrow \hat{X}$ be the corresponding two-sided topologically mixing subshift of finite type.*

(i) *If $\varphi \in \text{Bow}(X, T)$ and μ_{φ} is its unique equilibrium state, then the natural extension of (T, μ_{φ}) is isomorphic to a Bernoulli shift.*

(ii) *If $\hat{\varphi} \in \text{Bow}(\hat{X}, S)$ and $\hat{\mu}_{\hat{\varphi}}$ is its unique equilibrium state, then $(S, \hat{\mu}_{\hat{\varphi}})$ is isomorphic to a Bernoulli shift.*

Proof. (i) By Corollary 1.8, μ_{φ} has approximate product structure, so $\hat{\mu}_{\hat{\varphi}}$ has approximate product structure. Now Theorem 2.3 gives the result.

(ii) Bowen [2] showed that $\hat{\mu}_{\hat{\varphi}}$ has the property that there exists $C > 1$ with

$$C^{-1} \leq \frac{\hat{\mu}_{\hat{\varphi}}[x_0, \dots, x_{n-1}]\varphi}{e^{(S_n \hat{\varphi})(x) - nP(S, \hat{\varphi})}} \leq C \quad \forall n \geq 1, x \in \hat{X}.$$

Hence $\hat{\mu}_\varphi$ has approximate product structure, so the result follows by Theorem 2.3. □

3. g -measures

We interpret the results in Sections 1 and 2 for the case of g -measures. If $T: X \rightarrow X$ is a one-sided topologically mixing subshift of finite type, let $\mathcal{G}(X, T)$, or \mathcal{G} , denote the set

$$\left\{ g \in C(X) \mid g(x) > 0 \ \forall x \in X \text{ and } \sum_{y \in T^{-1}x} g(y) = 1 \ \forall x \in X \right\}.$$

If $g \in \mathcal{G}$, we can consider $\mathcal{L}_{\log g}$, and $\mu \in M(X)$ is called a g -measure if $\mathcal{L}_{\log g}^* \mu = \mu$ [7]. Such a measure always belongs to $M(X, T)$. The condition can be formulated in several ways. For example, one can show that $\mu \in M(X, T)$ is a g -measure if and only if μ is an equilibrium state of $\log g$ ([8], see also [11]). Since $P(T, \log g) = 0$ for $g \in \mathcal{G}$, this condition becomes $h_\mu(T) + \int \log g \, d\mu = 0$. Let $\mathcal{M}(X, T) = \{ \mu \in M(X, T) \mid \mu \text{ is a } g\text{-measure for some } g \in \mathcal{G}(X, T) \}$.

The following results are special cases of the results in Sections 1 and 2, obtained by considering φ of the form $\log g$ with $g \in \mathcal{G}(X, T)$. Again, k is the number of symbols used for the subshift of finite type, and M is a natural number with $k^M > 0$.

THEOREM 3.1. *Let $T: X \rightarrow X$ be a one-sided topologically mixing subshift of finite type and let $g \in \mathcal{G}$. Let μ be a g -measure. Then μ has support X and each of the following holds.*

(i) *For all $p \geq 1, x \in X$,*

$$(\inf g)^M e^{-v_p(T_p \log g)} \leq \frac{\mu([x_0, \dots, x_{p-1}])}{g(x)g(Tx) \dots g(T^{p-1}x)} \leq k^M e^{v_p(T_p \log g)}.$$

(ii) *For all $n, p \leq 1, x \in X$,*

$$e^{-v_{n+p}(T_p \log g)} \leq \frac{\mu([x_0, \dots, x_{p+n-1}])}{\mu([x_p, \dots, x_{p+n}])g(x)g(Tx) \dots g(T^{p-1}x)} \leq e^{v_{n+p}(T_p \log g)}.$$

COROLLARY 3.2. *For $T: X \rightarrow X, g \in \mathcal{G}$ and $\mu \in M(X, T)$ as in Theorem 3.1, we have for each fixed $p \geq 1$,*

$$\frac{\mu([x_0, \dots, x_{p+n-1}])}{\mu([x_p, \dots, x_{p+n-1}])} \rightarrow g(x)g(Tx) \dots g(T^{p-1}x) \quad \text{as } n \rightarrow \infty.$$

COROLLARY 3.3. *Let $T: X \rightarrow X$ be a one-sided topologically mixing subshift of finite type and let $\mu \in M(X, T)$. Then μ is a g -measure for some $g \in \mathcal{G}$ if and only if μ has support X and $\mu([x_0, \dots, x_{n-1}]) / \mu([x_1, \dots, x_{n-1}])$ converges uniformly on X as $n \rightarrow \infty$ to a function $f: X \rightarrow (0, \infty)$.*

With Corollary 3.2 in mind, we can characterize those g with $\log g \in W(X, T)$ as follows.

THEOREM 3.4. *Let $T: X \rightarrow X$ be a one-sided topologically mixing subshift of finite type and let $g \in \mathcal{G}$. The following statements are pairwise equivalent.*

(i) $\log g \in W(X, T)$.

(ii) There exists $\mu \in M(X, T)$, with support X , satisfying

$$\log \left(\frac{\mu([x_0, \dots, x_{p+n-1}])}{\mu([x_p, \dots, x_{p+n-1}])} \right) \Rightarrow (T_p(\log g))(x) \quad \text{as } n \rightarrow \infty,$$

where the convergence is uniform in both $x \in X$ and $p \geq 1$.

(iii) There exists a g -measure μ such that in $\text{BC}(\mathbb{N} \times X)$, the sequence (ψ_n) , given by

$$\psi_n(p, x) = \log \left(\frac{\mu([x_0, \dots, x_{p+n-1}])}{\mu([x_p, \dots, x_{p+n-1}])} \right),$$

is convergent.

Note that the unique g -measure μ , when $\log g \in W(X, T)$, satisfies the condition in (ii).

We shall use Theorem 3.4 later in an application. The following result characterizes those g with $\log g \in \text{Bow}(X, T)$.

THEOREM 3.5. *Let $T: X \rightarrow X$ be a one-sided topologically mixing subshift of finite type and let $g \in \mathcal{G}(X, T)$. Then $\log g \in \text{Bow}(X, T)$ if and only if there is a g -measure which has approximate product structure. When $\log g \in B(X, T)$, there is a unique g -measure μ and the coordinate zero partition is weak Bernoulli for μ so that the natural extension of (T, μ) is isomorphic to a Bernoulli shift.*

Proof. Since a g -measure is exactly an eigenmeasure for $\mathcal{L}_{\log g}^*$ and is T -invariant, the first statement follows from Theorem 1.7. Let $\log g \in \text{Bow}(X, T)$. By [14, Theorem 3.2], there is a unique g -measure, and the Bernoulli properties follow from Theorem 2.4. \square

Notice that when μ is g -measure, then μ has approximate product structure if and only if $\log g \in \text{Bow}(X, T)$.

We now consider the question of whether the ‘reverse’ of a g -measure is also a g -measure. Let $T: X \rightarrow X$, where $X = X_A$, be a one-sided topologically mixing subshift of finite type, and let $S: \hat{X} \rightarrow \hat{X}$ be the corresponding two-sided topologically mixing subshift of finite type. Let $\pi: \hat{X} \rightarrow X$ be the natural projection given by $\pi\{x_n\}_{-\infty}^{\infty} = \{x_n\}_0^{\infty}$. Then $\pi S = T\pi$, and there is a natural bijection $M(\hat{X}, S) \rightarrow M(X, T)$ given by $\hat{\mu} \rightarrow \hat{\mu} \circ \pi^{-1}$. We denote $\hat{\mu} \circ \pi^{-1}$ by $\hat{\mu}_+$.

The other one-sided space $X_- = \{\{x_n\}_{-\infty}^0 \mid \exists x_i \text{ for } i \geq 1 \text{ with } \{x_n\}_{-\infty}^{\infty} \in \hat{X}\}$ together with the shift $T_-: X_- \rightarrow X_-$, given by $T_-(\dots, x_{-2}, x_{-1}, x_0) = (\dots, x_{-2}, x_{-1})$, can be considered as the one-sided shift on the space X_{A^t} , where A^t is the transpose of the matrix A . Let $\pi: \hat{X} \rightarrow X_-$ be given by $\pi\{x_n\}_{-\infty}^{\infty} = \{x_n\}_{-\infty}^0$ and then $\pi S^{-1} = T_- \pi$. Since $M(\hat{X}, S^{-1}) = M(\hat{X}, S)$, we have a natural bijection $M(\hat{X}, S) \rightarrow M(X_-, T_-)$ given by $\hat{\mu} \rightarrow \hat{\mu} \circ \pi^{-1} \equiv \hat{\mu}_-$, so that $\hat{\mu}_+ \rightarrow \hat{\mu}_-$ gives a natural bijection $M(X, T) \rightarrow M(X_-, T_-)$. For an allowed cylinder $[i_1, \dots, i_r]$ in \hat{X} , we have $\hat{\mu}_+(s[i_1, \dots, i_r]) = \hat{\mu}_-([i_1, \dots, i_r]_t)$ for all $s \geq 0, t \leq 0$. Clearly $\hat{\mu}_+$ has support X if and only if $\hat{\mu}_-$ has support X_- . We can define what it means for $\hat{\mu}_-$ to have approximate product structure by considering the natural conjugacy $X_- \rightarrow X_{A^t}$, given by $(\dots, x_{-2}, x_{-1}, x_0) \rightarrow (x_0, x_{-1}, x_{-2}, \dots)$, of T_- to the topologically mixing subshift of finite type on X_{A^t} . Then $\hat{\mu}_-$ has approximate product structure if and only if $\hat{\mu}_+$ has approximate product structure if and only if $\hat{\mu}$ has approximate product structure.

Let \mathcal{G}_+ denote $\mathcal{G}(X, T)$, the space of all positive g -functions for T , and let \mathcal{G}_- denote $\mathcal{G}(X_-, T_-)$, the space of all positive g -functions for T_- . Hence

$$\mathcal{G}_- = \left\{ g \in C(X_-) \mid g(z) > 0 \ \forall z \in X_- \text{ and } \sum_{w \in T_-^{-1}z} g(w) = 1 \ \forall z \in X_- \right\}$$

If \mathcal{M}_+ denotes $\mathcal{M}(X, T)$ and \mathcal{M}_- denotes $\mathcal{M}(X_-, T_-)$, then the map $\hat{\mu}_+ \rightarrow \hat{\mu}_-$ need not map \mathcal{M}_+ into \mathcal{M}_- . Kalikow constructed examples to show this when X is the full shift space on two symbols [6]. One can construct a family of such examples, inspired by Kalikow, as follows.

Let $X = \prod_0^\infty \{0, 1\}$ be the space of all sequences (x_0, x_1, \dots) with each $x_n \in \{0, 1\}$. Let $\{d_n\}_{n=0}^\infty$ be such that $d_n \in [0, 1)$ for all $n \geq 0$, $d_n \rightarrow 0$ as $n \rightarrow \infty$, and $\sum_{n=0}^\infty (d_n / (1 + d_n)) = \infty$. Such a sequence is given by $d_n = 1 / (n + 1)$. Define $g : X \rightarrow (0, 1)$ as follows. For $k \geq 0, \ell \geq 0$, put

$$\begin{aligned} g(000^k 1^\ell 101 \dots) &= \frac{1}{2}(1 - d_{k+\ell}), \\ g(100^k 1^\ell 101 \dots) &= \frac{1}{2}(1 + d_{k+\ell}), \\ g(000^k 1^\ell 100 \dots) &= \frac{1}{2}(1 + d_{k+\ell}), \\ g(100^k 1^\ell 100 \dots) &= \frac{1}{2}(1 - d_{k+\ell}), \end{aligned}$$

and $g(x) = \frac{1}{2}$ at all other points. The value of g depends on the first occurrence of cylinder [10] in (x_2, x_3, \dots) and on whether this occurrence of [10] is followed by a 0 or a 1. Then $g \in \mathcal{G}_+$. Suppose that $\hat{\mu}_+$ is a g -measure and let $\hat{\mu}_-$ correspond to it under the natural bijection $M(X, T) \rightarrow M(X_-, T_-)$. Here X_- is the space $\prod_{-\infty}^0 \{0, 1\}$. One can easily show that $\hat{\mu}_+([1^m 00]) / \hat{\mu}_+([1^m 0])$ does not depend on m for $m \geq 2$ so has a constant value $c \in (0, 1)$. Hence

$$\frac{\hat{\mu}_-([1^m 00])}{\hat{\mu}_-([1^m 0])} = \frac{\hat{\mu}_+([1^m 00])}{\hat{\mu}_+([1^m 0])} = c.$$

Suppose that $\hat{\mu}_-$ is a g_- -measure for some $g_- \in \mathcal{G}_-$. Then $g_-(1^\infty 00) = \hat{\mu}_-[1^m 00] / \hat{\mu}_-[1^m 0]$ for all $m \geq 2$, so $g_-(1^\infty 00) = c$. However, one can use the properties of $\{d_n\}$ to show that, for each fixed $m \geq 1$, $\hat{\mu}_+([0^n 1^m 0]) / \hat{\mu}_+([0^n 1^m 00]) \rightarrow 0$ as $n \rightarrow \infty$. Hence $\hat{\mu}_-([0^n 1^m 00]) / \hat{\mu}_-([0^n 1^m 0]) \rightarrow 1$ as $n \rightarrow \infty$, so $g_-(0^\infty 1^m 00) = 1$ for all $m \geq 1$. Therefore g_- cannot be continuous, because $\lim_{m \rightarrow \infty} g_-(0^\infty 1^m 00) = 1 \neq c = g_m(1^\infty 00)$. □

We now show that if $g_+ \in \mathcal{G}_+$ and $\log g_+ \in W(X, T)$ and $\hat{\mu}_+$ is the unique g_+ -measure, then $\hat{\mu}_-$ is the unique g_- -measure for some $g_- \in \mathcal{G}_-$ with $\log g_- \in W(X_-, T_-)$.

THEOREM 3.6. *Let $S : \hat{X} \rightarrow \hat{X}$ be a two-sided topologically mixing subshift of finite type and let $T : X \rightarrow X, T_- : X_- \rightarrow X_-$ be the corresponding one-sided topologically mixing subshifts of finite type. Let $\hat{\mu}_+ \rightarrow \hat{\mu}_-$ be the natural bijection from $M(X, T)$ to $M(X_-, T_-)$ described above. Let $g_+ \in \mathcal{G}_+$ and let $\hat{\mu}_+$ be a g_+ -measure. If $\log g_+ \in W(X, T)$, then $\hat{\mu}_-$ is a g_- -measure for some $g_- \in \mathcal{G}_-$ and $\log g_- \in W(X_-, T_-)$. We have*

$$\left| \log \left(\frac{\hat{\mu}_-([x_{-n}, \dots, x_0])}{\hat{\mu}_-([x_{-n}, \dots, x_{-1}])} \right) - \log g_-(x) \right| \leq 2 \liminf_{j \rightarrow \infty} v_{n+j}(T_j \log g_+) \quad \forall n \geq 1, x \in X_-.$$

The functions $\log g_+ \circ \pi$ and $\log g_- \circ \pi_-$ are cohomologous in $C(\hat{X})$.

Proof. For $n \geq 1$, define $b_n : X_- \rightarrow (0, 1)$ by

$$b_n(x) = \frac{\hat{\mu}_-([x_{-n}, \dots, x_0])}{\hat{\mu}_-([x_{-n}, \dots, x_{-1}])} = \frac{\hat{\mu}_+([x_{-n}, \dots, x_0])}{\hat{\mu}_+([x_{-n}, \dots, x_{-1}])}.$$

We show that $(\log b_n)$ is a Cauchy sequence in (C, X_-) . Since

$$\frac{b_n(x)}{b_{n+j}(x)} = \frac{\hat{\mu}_+([x_{-n}, \dots, x_0])}{\hat{\mu}_+([x_{-n-j}, \dots, x_0])} \frac{\hat{\mu}_+([x_{-n-j}, \dots, x_{-1}])}{\hat{\mu}_+([x_{-n}, \dots, x_{-1}])},$$

we can use Theorem 3.1 to get

$$e^{-v_{n+j+1}(T_j \log g_+) - v_{n+j}(T_j \log g_+)} \leq \frac{b_n(x)}{b_{n+j}(x)} \leq e^{v_{n+j+1}(T_j \log g_+) + v_{n+j}(T_j \log g_+)}.$$

Therefore $|\log b_n(x) - \log b_{n+j}(x)| \leq 2v_{n+j}(T_j \log g_+)$, and since $\log g_+ \in W(X, T)$, we have $(\log b_n)$ is a Cauchy sequence in $C(X_-)$. Hence $\log b_n(x) \rightrightarrows \psi(x)$ for some $\psi \in C(X_-)$. Since $\sum_{z \in T^{-1}(x)} b_n(z) = 1$ for all $n \geq 1$, we have $\sum_{z \in T^{-1}x} e^{\psi(z)} = 1$. Let $g_- = e^\psi$. Then g_- is a g -function for $T_- : X_- \rightarrow X_-$ and $\hat{\mu}_-$ is a g_- -measure by Corollaries 3.2 and 3.3.

We get

$$|\log b_n(x) - \log g_-(x)| \leq 2 \liminf_{j \rightarrow \infty} v_{n+j}(T_j \log g_+).$$

To see that $\log g_- \in W(X_-, T_-)$, we use Theorem 3.4. Since $\log g_+ \in W(X, T)$, we have that

$$(p, x) \rightarrow \log \left(\frac{\hat{\mu}_+([x_0, \dots, x_{p+n-1}])}{\hat{\mu}_+([x_p, \dots, x_{p+n-1}])} \right)$$

is a Cauchy sequence in $\text{BC}(\mathbb{N} \times X)$. This is the equivalent to

$$(p, z) \rightarrow \log \left(\frac{\hat{\mu}_-([z_{-(n+p-1)}, \dots, z_0])}{\hat{\mu}_-([z_{-(n+p-1)}, \dots, z_{-p}])} \right)$$

being a Cauchy sequence in $\text{BC}(\mathbb{N} \times X_-)$, and hence $\log g_- \in W(X_-, T_-)$.

We use [15, Theorem 1.4] to see that $\log g_+ \circ \pi$ is cohomologous to $\log g_- \circ \pi_-$ in $C(\hat{X})$. Since $\log g_+ \in W(X, T)$, $\log g_+ \circ \pi$ is cohomologous in $C(\hat{X})$ to $\varphi_- \circ \pi_-$ for some $\varphi_- \in C(X_-)$. By of [15, Lemma 13], $\varphi_- \in W(X_-, T_-)$. Hence φ_- is cohomologous in $C(X_-)$ to $\log g_1$ for some g -function $g_1 : X_- \rightarrow (0, 1)$ [12]. Since μ_- is a g -measure for g_1 and g_- , we have $g_1 = g_-$. Hence $\log g_+ \circ \pi$ is cohomologous to $\log g_- \circ \pi_-$ in $C(\hat{X})$. \square

We do not know if the corresponding result holds when $W(X, T)$ is replaced by $\text{Bow}(X, T)$, but we do have the following.

THEOREM 3.7. *Let $S : \hat{X} \rightarrow \hat{X}$ be a two-sided topologically mixing subshift of finite type, and let $T : X \rightarrow X, T : X_- \rightarrow X_-$ be the corresponding one-sided topologically mixing subshifts of finite type. Let $\hat{\mu}_+ \rightarrow \hat{\mu}_-$ be the natural bijection from $M(X, T)$ to $M(X_-, T_-)$ described above. Let $\hat{\mu}_+$ be a g_+ -measure and $\hat{\mu}_-$ be a g_- -measure for some $g_+ \in \mathcal{G}_+$ and some $g_- \in \mathcal{G}_-$. Then $\log g_+ \in \text{Bow}(X, T)$ if and only if $\log g_- \in \text{Bow}(X_-, T_-)$.*

Proof. From Theorem 3.5, we know that $\log g_+ \in \text{Bow}(X, T)$ if and only if $\hat{\mu}_+$ has approximate product structure, and $\log g_- \in \text{Bow}(X_-, T_-)$ if and only if $\hat{\mu}_-$ has approximate product structure. \square

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