Math. Proc. Camb. Phil. Soc. (2005), **139**, 555 © 2005 *Cambridge Philosophical Society* 555
doi:10.1017/S0305004105008777 *Printed in the United Kingdom*

A sufficient condition for a finite-time L_2 singularity of the 3d Euler Equations

By XINYU HE
Mathematics Institute, University of Warwick, Coventry CV4 7AL.
e-mail: xinyu@maths.warwick.ac.uk

(*Received* 20 *December* 2002)

Abstract

A sufficient condition is derived for a finite-time L_2 singularity of the 3d incompressible Euler equations, making appropriate assumptions on eigenvalues of the Hessian of pressure. Under this condition $\lim_{t\uparrow T_*} \sup \|\frac{D\omega}{Dt}\|_{L_2(\Omega)} = \infty$, where $\Omega \subset \mathbb{R}^3$ moves with the fluid. In particular, $|\omega|$, $|\mathcal{S}_{ij}|$, and $|\mathcal{P}_{ij}|$ all become unbounded at one point (x_1, T_1), T_1 being the first blow-up time in L_2.

1. *Introduction*

Consider the incompressible Euler equations in $\mathbb{R}^3 \times [0, \infty)$

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p, \qquad \nabla \cdot u = 0, \tag{1}$$

where $u(x,t) = (u_1, u_2, u_3)$ denotes the unknown velocity field, p the pressure scalar. Denote the material derivative in (1) by $D/Dt = \partial/\partial t + u \cdot \nabla$, and the vorticity vector by $\omega = \nabla \wedge u$, which is governed by

$$\frac{D\omega}{Dt} = \mathcal{S}\,\omega, \qquad \nabla \cdot \omega = 0, \qquad \text{where} \quad \mathcal{S}_{ij} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{2}$$

Defining the Hessian of pressure p by

$$\mathcal{P}_{ij} := \frac{\partial^2 p}{\partial x_i \partial x_j}, \tag{3}$$

the second order derivative of ω is given by (see [**8**] and [**10**])

$$\frac{D^2\omega}{Dt^2} = -\mathcal{P}\,\omega. \tag{4}$$

Combining (2) and (4), it is shown in [**5**] that

$$\frac{D(\omega \wedge \mathcal{S}\omega)}{Dt} = -\omega \wedge \mathcal{P}\omega.$$

This means that if ω aligns with an eigenvector of $\mathcal{S}$ (call this a $\mathcal{S}-\omega$ alignment), then it must do so simultaneously with an eigenvector of $\mathcal{P}$ (call this a $\mathcal{P}-\omega$ alignment). See (15) for the converse. It is clear from (4) that only negative eigenvalues of $\mathcal{P}$ cause ω to increase in time. Intuitively, one expects that singular solutions of (1), if they exist, are related to alignments of $\mathcal{P} - \omega$ or $\mathcal{S} - \omega$. In this sense, the geometry matters.

The theorem of [**1**] states that the L_∞ norm of ω controls the smoothness of solutions of the Euler equations (1). On the other hand, the direction of vorticity plays an important role with its evolution connected to the Hessian of pressure $\mathcal{P}$ [**2**, p. 40]. It is further proved in [**3**] that if the direction of ω remains regular and the velocity is bounded, then a singularity cannot form.

There has been evidence that alignments exist in a wide classes of fluid flows. It is found in [**11**] that in the Euler singular region, the vorticity is aligned with the eigenvector of the most positive eigenvalue of the strain $\mathcal{S}$. With vortex pairs initially aligned with $\mathcal{S}$, a blow-up model is constructed [**9**]. Using a set of equations for the angle variables in terms of $\mathcal{S}$ and $\mathcal{P}$, Gibbon et al. [**5**] have recently analysed the data in [**7**], indicative of intense stretching and compression of vorticity at the singular region where the alignments occur (see [**5**, fig. 2 and 3]). See also [**6**] for the alignments associated with Navier–Stokes turbulence.

The aim of the present paper is to study geometrical configurations of $\mathcal{P}$. We shall derive a sufficient condition in Theorem 2·1 for a finite-time L_2 Euler singularity, assuming the direction of ω is parallel to an eigenvector of $\mathcal{P}$ only. Furthermore, assuming the direction of ω is parallel to both $\mathcal{P}$ and $\mathcal{S}$ in a simple way, Theorem 2·2 is obtained. Deducing from this theorem, we analyse the singular patterns in time and space by Corollary 2·3 and 2·4. Apparently, these patterns seem to be observed in [**7**] and [**11**] for the turbulent enstrophy dissipation. Finally, we discuss effectiveness of the Hessian of pressure on producing potential L_2 singularities.

Remark A. To prove the theorems, we imposed some conditions on the eigenvalues of $\mathcal{S}$ and $\mathcal{P}$. Although little is known about a relation between their eigenvalues, the conditions imposed may be justified by available numerical data. Note that the conditions already imply possible pointwise Euler singularities. However, the central point of the paper is to demonstrate that a L_2 blowup demands stronger conditions. Our condition for a pointwise singularity is not sufficient (see Remark D). Moreover, global constraints need to be satisfied, for instance only fluid elements satisfying inequality (14) become unbounded in $L_2(\Omega)$. To the author's knowledge, sufficient conditions for L_2 Euler blowup have not been precisely derived before.

2. *A sufficient condition*

Let $\mathbf{\Omega} \subset \mathbb{R}^3$ be a smooth material volume carried by the fluid. Let $\omega(x,t)$ be a sufficiently smooth solution of (1) for which we set

$$\varpi := \|\omega(t)\|^2_{L_2(\Omega)}, \quad \varpi(t) \neq 0 \;\; \forall\, t \geqslant 0 \quad \text{and} \quad \varphi_1(t) := \frac{1}{2\varpi}. \tag{5}$$

Remark B. One could also set

$$\varphi_n(t) := \frac{1}{2\,[\,\varpi\,]^{\frac{1}{n}}}, \quad n \in \mathbb{N}.$$

This would slightly improve an estimate for the constant c_0 in Theorem 2·2 below (smaller c_0 for $n > 1$). However for clarity, we take $n = 1$ as in (5).

Define a smooth function

$$v(t) := -\,\varphi_1' \tag{6}$$

so that

$$v(t) = \frac{1}{\varpi^2}\int_\Omega \omega \cdot \frac{D\omega}{Dt}\, dx \quad \text{and} \tag{7}$$

$$v'(t) = \frac{1}{\varpi^3}\left\{\left(\int_\Omega \left[\left|\frac{D\omega}{Dt}\right|^2 + \omega\cdot\frac{D^2\omega}{Dt^2}\right] dx\right)\varpi \; - 4\left(\int_\Omega \omega\cdot\frac{D\omega}{Dt}\, dx\right)^2\right\}. \tag{8}$$

Concerning the above equations, an easy estimate is

LEMMA 2·0. *Let v, v' be as in* (7) *and* (8). *Then for* $t \in [0,\infty)$

$$v(t)\,\varpi^{3/2}(t) \leqslant \left\|\frac{D\omega}{Dt}\right\|_{L_2(\Omega)} \quad \text{and} \tag{9}$$

$$v'(t)\varpi^2(t) \geqslant \int_\Omega \omega\cdot\frac{D^2\omega}{Dt^2}\, dx \; - c_1\int_\Omega \left|\frac{D\omega}{Dt}\right|^2 dx, \quad c_1 = 3. \tag{10}$$

Proof. By Cauchy–Schwarz's inequality, we get for the integral in (7):

$$\int_\Omega \omega\cdot\frac{D\omega}{Dt}\, dx \leqslant \|\omega\|_{L_2(\Omega)}\left\|\frac{D\omega}{Dt}\right\|_{L_2(\Omega)}.$$

But $\varpi = \|\omega\|^2_{L_2(\Omega)}$, giving (9). Using this relation again for the last term in (8) yields (10).

Remark C. Inequality (10) involves both (2) and (4), therefore it will be used to investigate various links between $\mathcal{S}$ and $\mathcal{P}$ for solutions of (1).

No rigorous estimate is known about the two terms on the right-hand side of (10), and certain assumptions will be made on geometrical arrangements of $\mathcal{S}$ and $\mathcal{P}$. First, we consider a case when there is only $\mathcal{P} - \omega$ alignment. This arrangement is shown by numerical data [**10**], which suggests the configuration to be a generic property of Euler flows. A sufficient condition can now be given.

THEOREM 2·1. *Let $\mathcal{P}\omega = -\lambda\,\omega$ in* (4) $\forall\, x \in \Omega$ *and* $t \geqslant 0$, *where* $\lambda > 0$. *Assume that at some* $t_0 > 0$, $\lambda > 3\mu_m^2$ *on* $\Omega \times [t_0,\infty)$, *where* $\mu_m = \max\{|\mu_1|, |\mu_2|, |\mu_3|\}$, μ_i *being eigenvalues of the matrix* $\mathcal{S}$. *Then there exists a finite time* $T_0 > t_0$ (*depending only on* ϖ_0 *and* v_0) *and* $T_* \in (t_0, T_0)$, *such that*

$$\lim_{t\uparrow T_*}\sup \left\|\frac{D\omega}{Dt}\right\|_{L_2(\Omega)} = \infty.$$

Proof. By Lemma 2·0, clearly

$$v'\varpi^2 \geqslant \int_\Omega \lambda(x,t)|\omega|^2\, dx - 3\int_\Omega |\mathcal{S}\omega|^2\, dx.$$

Setting $\mu_m = \max\{|\mu_i|\}$ gives

$$v'\varpi^2 \geqslant \int_\Omega \left[\lambda(x,t) - 3\mu_m^2(x,t)\right]|\omega|^2\, dx.$$

It then follows from the assumption and (9)

$$v'(t) \geqslant c\varpi(t)v^2(t), \quad t \in [t_0,\infty), \quad c \in (0,1].$$

This implies $\varphi_1' < 0$ in (6) after t_0, in turn $\varpi(t) \geqslant \varpi_0 = \varpi(t_0)$. Hence

$$v' \geqslant c\varpi_0 v^2, \quad v_0 = v(t_0) > 0.$$

One finds that for $t_0 \leqslant t < T_0$, setting $A = 1/(c\varpi_0)$,

$$v(t) \geqslant \frac{A}{T_0 - t}, \quad T_0 = t_0 + 1/(c\,\varpi_0\,v_0).$$

We see that $t_0 < T_0 < K$. According to (9), in which note $\varpi(t) \geqslant \varpi_0$,

$$\left\| \frac{D\omega(t)}{Dt} \right\|_{L_2(\Omega)} \geqslant \frac{B}{T_0 - t}, \quad B = \varpi_0^{1/2}/c.$$

This establishes the assertion.

The basic idea of Theorem 2·1 is that if λ is larger than μ_m for a certain length of time, then a L_2 singularity forms. The critical time T_0 is determined by initial ϖ_0 (the enstrophy at t_0) and v_0 (the rate change of enstrophy): higher is the initial enstrophy, shorter is the critical time.

To be precise as to how large λ needs to be, next we examine a special case of Theorem 2·1: both $\mathcal{P} - \omega$ and $\mathcal{S} - \omega$ configurations hold. Such flow geometry is often observed in numerical simulations, for example [**5**], [**10**]. Making a assumption on the eigenvalues of $\mathcal{S}$ and $\mathcal{P}$, we have

THEOREM 2·2. *Let* $\mathcal{P}\omega = -\lambda\omega$ *in* (4) *and* $\mathcal{S}\omega = \mu\omega$ *in* (2) $\forall x \in \Omega$ *and* $t \geqslant 0$, *where* $\lambda, \mu > 0$. *Assume that at some* $t_0 > 0$, $\lambda = c_0\mu^2$ *on* $\Omega \times [t_0, \infty)$ *with some constant* $c_0 > 3$. *Then there exists a finite time* $T_0 > t_0$ *and* $T* \in (t_0, T_0)$, *such that*

$$\lim_{t \uparrow T_*} \sup \left\| \frac{D\omega}{Dt} \right\|_{L_2(\Omega)} = \infty.$$

Proof. The proof is similar to that of Theorem 2·1. Here for T_0, we have

$$T_0 = t_0 + 1/(c\varpi_0 v_0), \quad c = c_0 - 3 > 0. \tag{11}$$

Remark D. When both $\mathcal{P} - \omega$ and $\mathcal{S} - \omega$ alignments hold, there may exist many functional relations between their eigenvalues, $\lambda = f(\mu)$. The hypothesis in the theorem, $\lambda = c_0\mu^2$ with $c_0 \in (3, 3 + \epsilon)$, is a requirement for the L_2 blowup (but note not every fluid element satisfying the relation can blowup, see (14) below). This requirement already implies pointwise singular solutions. For such singularities, a similar relation is $\lambda = c_p\,\mu^2$ with $c_p \in (1, 1 + \epsilon)$ (see the proof of Corollary 2·3). Notice that $c_p < c_0$ for $\epsilon \in (0, 1)$.

This case is the simplest to analyse structures of the L_2 blowup. To do so we will further assume that μ is the only positive eigenvalue of $\mathcal{S}$, as suggested by an analysis [**11**, p. 309]. Thus the very first blow-up time in L_2 is identified by:

COROLLARY 2·3 (Temporal interval). *Suppose in Theorem* 2·2 *that* μ *is the only positive eigenvalue of* $\mathcal{S}$. *Then there exists a smallest time* $T_1 \in (t_0, T_0)$ *such that*

$$\lim_{t \uparrow T_1} \sup |\omega|_{L_\infty} = \infty, \quad \lim_{t \uparrow T_1} \sup |\mathcal{S}_{ij}|_{L_\infty} = \infty \quad \text{and} \quad \lim_{t \uparrow T_1} \sup |\mathcal{P}_{ij}|_{L_\infty} = \infty.$$

In fact, $[T_1, T_0) = \{t | T_1 \leqslant t < T_0\}$ *is the interval of blow-up.*

Proof. Let $\Omega_0 = \overline{\Omega}(t_0)$ and $\mu^0(x) = \mu(x, t_0)$ for $x \in \Omega_0$. Consider a fluid element located at $\alpha \in \Omega_0$. Differentiating $D\omega/Dt = \mu\omega$ and using (4), one obtains by following the element: $\mu'(t) = \lambda - \mu^2$. Inserting $\lambda = c_0\mu^2$ gives $\mu' = (c_0 - 1)\mu^2$. This equation admits a solution which ceases to be regular at a finite-time

$$\mu(t;\alpha) = \frac{(c_0-1)^{-1}}{T_* - t}, \quad T_* = t_0 + 1/[(c_0-1)\mu^0(\alpha)]. \tag{12}$$

Note $\inf \mu^0(\Omega_0) \leqslant \mu^0(\alpha) \leqslant \sup \mu^0(\Omega_0) \ \forall \alpha \in \Omega_0$. Define

$$T_1 := \inf_{\alpha\in\Omega_0} T_*(\alpha) = t_0 + 1/\big[(c_0-1)\mu_1^0\big], \quad \mu_1^0 = \sup \ \mu^0(\Omega_0). \tag{13}$$

We claim $T_1 < T_0$ as defined in (11). Computing $c\ w_0\ v_0$ in T_0 by use of the Second Mean-Value Theorem for Integrals in (7), we get $c\ w_0\ v_0 = (c_0-3)\mu^0(\beta)$ for some $\beta \in \Omega_0$. The fact $(c_0-1)\mu_1^0 > (c_0-3)\mu^0(\beta)\ \forall\ \beta \in \Omega_0$ suffices for the claim. Consequently, T_1 is the first time in the blow-up interval $[T_1,\ T_0)$, in which corresponding $\mu^0(\alpha)$ necessarily satisfy

$$\mu^0(\alpha) \geqslant \mu^0(\beta_*)\ (c_0-3)/(c_0-1), \ \ \beta_* \in \Omega_0. \tag{14}$$

We now ask what functions are singular at T_1? Since both matrices $\mathcal{S}$ and $\mathcal{P}$ are symmetric, we have only to consider their eigenvalues. Let μ_a and μ_b be the two other eigenvalues of $\mathcal{S}$ whose eigenvectors are not aligned with the vorticity vector. By the incompressibility condition, $\mu > \max\{|\mu_a|, |\mu_b|\}$ as it is the only positive eigenvalue. Thus it is obvious from (12) and (13) that $|\mathcal{S}_{ij}|_{L_\infty}$ is unbounded at T_1. This means, by the theorems of [**1**] and [**12**], that $|\omega|_{L_\infty}$ also fails to be smooth at the same time. Finally we turn to the Hessian of pressure. Let λ_ζ and λ_η be the two other eigenvalues of $\mathcal{P}$ while $-\lambda$ is the negative eigenvalue associated with the eigenvector aligned to ω. Note that λ_ζ or λ_η cannot blow up at any time earlier than T_1, because if this happened, it can be shown by (2) and (4) that $|\omega|_{L_\infty}$ would have blown up at a time earlier than T_1, contradicting (13). Now given $\delta > 0 \forall t \in (T_1-\delta, T_1)$, either (a) $\sup_{x\in\Omega}\lambda \geqslant \max\{|\lambda_\zeta|, |\lambda_\eta|\}$, or (b) $\sup_{x\in\Omega}\lambda < \max\{|\lambda_\zeta|, |\lambda_\eta|\}$. We know that $\lim_{t\uparrow T_1}\sup|\mathcal{S}_{ij}|_{L_\infty} = \infty$, which is equivalent to $\lim_{t\uparrow T_1}\sup_{x\in\Omega}\lambda = \infty$ by the alignment relation $\lambda = c_0\mu^2$. Thus inequality (a) is left as the only choice. Evidently $\lim_{t\uparrow T_1}\sup|\mathcal{P}_{ij}|_{L_\infty} = \infty$. The proof is complete.

It is natural to wonder what would be the singular set in space. In this direction we can show.

Corollary 2·4 (Spatial set). *Let $x_1 \in \Omega$ be the space point where $|\mathcal{S}_{ij}|_{L_\infty} = \infty$ as $t \to T_1$. Then $|\omega|_{L_\infty}$ and $|\mathcal{P}_{ij}|_{L_\infty}$ also blow up at $(x_1,\ T_1)$.*

Proof. Without loss of generality, let us assume that at time t_0, there is only one fluid element having $\mu_1^0 = \sup\ \mu^0(\Omega_0)$. Suppose $|\omega|_{L_\infty}$ blows up at (y, T_1), $y \neq x_1$, however this is impossible. At the time T_1, y is a position reached by a fluid element with initial point $\mu^0(y) \neq \mu_1^0$, which is not singular at that time. We then conclude $y = x_1$. To find the singular location of $|\mathcal{P}_{ij}|_{L_\infty}$ we recall from Corollary 2·3 that $\sup_{x\in\Omega}\lambda \geqslant \max\{|\lambda_\zeta|, |\lambda_\eta|\}$ for $t \in (T_1-\delta, T_1)$. If $\sup_{x\in\Omega}\lambda > \max\{|\lambda_\zeta|, |\lambda_\eta|\}$, then it is unbounded at (x_1, T_1) by the alignment relation. If $\sup_{x\in\Omega}\lambda = \max\{|\lambda_\zeta|, |\lambda_\eta|\}$, this means both $\sup_{x\in\Omega}\lambda$ and $\max\{|\lambda_\zeta|, |\lambda_\eta|\}$ blow up at T_1. Having stated $\sup_{x\in\Omega}\lambda$ is singular at (x_1, T_1), let us suppose $\max\{|\lambda_\zeta|, |\lambda_\eta|\}$ is singular at $(z, T_1), z \neq x_1$. A similar argument to the one above for $|\omega|_{L_\infty}$ shows we must have $z = x_1$.

We make a few observations about the above results. (i) Geometrical arrangements can limit the set of singularities. In the case of the double alignments, we have shown that $|\omega|$, $|\mathcal{S}_{ij}|$ and $|\mathcal{P}_{ij}|$ all blowup at one point (x_1, T_1). (ii) The L_2 singularity condition is stronger, namely the integral relation (10) has to be satisfied as a constraint. In this instance, although in (12) any fluid element could locally blow up at T_*, only those satisfying the inequality (14) can actually make up the L_2 singularity. (iii) Taking the divergence of (1) results in $|\omega|^2/2 - \mathcal{S}^2 = \mathcal{P}_{ii} = \lambda_\zeta + \lambda_\eta - \lambda$. From Corollary 2·4, we see that in any neighborhood of (x_1, T_1), the above equation has an indefinite sign of $\infty - \infty$.

3. *Necessity for L_2 blow-up*

On the right-hand side of (10), if the first integral is persistently greater than the second, then a singularity could result. In our above theorems, we only used the geometric conditions on the integrands, which is more restrictive than the integral requirement. However in general cases when there is not any coherent configuration, it seems hard to proceed. In what follows, we shall discuss solutions of (1) having some coherence in the Hessian of pressure.

To simplify the discussion, let $\mathcal{S}$ and $\mathcal{P}$ be diagonalised on $\Omega \times [0, \infty)$ with respect to the principal axes. Since (10) is invariant under the coordinate transformations, we can write referring to these axes

$$v'\varpi^2 \geqslant -\int_\Omega \left[\lambda_\zeta \omega_\zeta^2 + \lambda_\eta \omega_\eta^2 + \lambda_\xi \omega_\xi^2\right] dx \; - 3\int_\Omega \left[\mu_a^2 \omega_a^2 + \mu_b^2 \omega_b^2 + \mu_c^2 \omega_c^2\right] dx,$$

where ζ, η, and ξ denote the principal axes of $\mathcal{P}$, a, b and c the principal axes of $\mathcal{S}$, respectively. It appears that a $\mathcal{P} - \omega$ alignment with a negative eigenvalue would be an effective way for attaining the requirement, for the following reason.

As shown in the Introduction, when a $\mathcal{P} - \omega$ alignment occurs, we have

$$-\omega \wedge \mathcal{P}\omega \equiv 0 \Longrightarrow \omega \wedge \mathcal{S}\omega = \text{constant}. \tag{15}$$

Let us write out three components of the invariant $(\omega \wedge \mathcal{S}\omega)$:

$$\omega_c \omega_b (\mu_c - \mu_b) = c_1; \quad \omega_a \omega_c (\mu_a - \mu_c) = c_2; \quad \omega_b \omega_a (\mu_b - \mu_a) = c_3. \tag{16}$$

A key point here is that from the instant t_0 at which $\mathcal{P} - \omega$ occurs for some fluid elements, the constants in (16) are fixed in time following the same elements. The configuration of a vortex tube would give an interesting example of (16). Suppose at t_0, the fluid elements have $\mu_a > 0$, and $\mu_b, \mu_c < 0$ with $\mu_b = \mu_c$. This leads to initially, $c_1 = 0$, $c_2 > 0$ and $c_3 < 0$. We obtain in (16) $\omega_a = c_2/\omega_c(\mu_a + |\mu_c|)$. In this formula: (i) $c_2 > 0$ is fixed; (ii) it is not clear how $(\mu_a + |\mu_c|)$ changes in time (Theorem 2·2 is not applicable); (iii) ω_c decreases according to (2), since μ_c remains negative to keep $c_1 = 0$, due to the incompressibility. So there is a tendency for ω_a to increase in time, keeping the vortex-tube state alive, and such a state will be strengthened if there are some symmetries existing in the flow at t_0. This (extreme) example illustrates that a $\mathcal{P} - \omega$ alignment "freezes" the initial straining states by (15), and if the initial configuration favours vortex stretching, then these vortex lines would have to be stretched indefinitely. This suggests that the Hessian of pressure alone could possibly produce a L_2 singularity.

The Euler equation is rich in its geometrical structures (see [**4**]). One further speculates whether the geometry of $\mathcal{P} - \omega$ or $\mathcal{S} - \omega$ is a necessary condition for solutions of (1) to develop finite-time singularities. Note a $\mathcal{S} - \omega$ alignment automatically implies a $\mathcal{P} - \omega$ alignment, but the converse is not true. Reflecting that the alignment enforces growth of ω ([**8**, p. 192]), and in view of analytical and numerical works on the subject, we may loosely make a:

Conjecture. Let $\Omega \subset \mathbb{R}^3$. Suppose (1) has a $L_2(\Omega)$ singularity at $T_* < +\infty$. Then ω, $\mathcal{S}$, and $\mathcal{P}$ blow up at the same space point $x_* \in \Omega \iff$ there exists a $\mathcal{S} - \omega$ alignment.

REFERENCES

[**1**] J. T. Beale, T. Kato and A. J. Majda. Remarks on the breakdown of smooth solutions for the 3d Euler equations. *Comm. Math. Phys.* **94** (1984), 61–66.

[**2**] P. Constantin. Geometric and analytical studies in turbulence. *Appl. Math. Sci.* **100** (1994).

[**3**] P. Constantin, C. Fefferman and A. J. Majda. Geometric constraints on potentially singular solutions for the 3d Euler equations. *Comm. Partial Differential Equations* **21** (1996), 559–571.

[**4**] J. D. Gibbon. A quaternionic structure in the 3d Euler and ideal MHD equations. *Physica D* **166** (2002), 17–28.

[**5**] J. D. Gibbon, B. Galanti and R. M. Kerr. Stretching and compression of vorticity in the 3d Euler equations. In *Turbulence Structure and Vortex Dynamics* (ed. J. C. R. Hunt & J. C. Vassilicos), (Cambridge University Press, 2000).

[**6**] X. He. An invariant for the 3d Euler equations. *Applied Mathematics Letters* **12** (1999), 55–58.

[**7**] R. M. Kerr. Evidence for a singularity of the 3d, incompressible Euler equations. *Phys. Fluids A* **5** (1993), 1725–1746.

[**8**] A. J. Majda. Vorticity and the mathematical theory of incompressible fluid flow. *Comm. Pure Appl. Math.* **39** (1986), S187–S220.

[**9**] H. K. Moffatt. The interaction of skewed vortex pairs: a model for blow-up of the Navier–Stokes equations. *J. Fluid Mech.* **409** (2000), 51–68.

[**10**] K. Ohkitani. Eigenvalue problems in 3d Euler flows. *Phys. Fluids* **5** (1993), 2570–2572.

[**11**] R. B. Pelz. Symmetry and the hydrodynamic blow-up problem. *J. Fluid Mech.* **444** (2001), 299–320 (see also: Discrete groups, symmetric flows and hydrodynamic blowup, in *Tubes, Sheets and Singularities in Fluid Dynamics* (Eds. K. Bajer and H. K. Moffatt), IUTAM Symposium Series, (Kluwer, 2002).

[**12**] G. Ponce. Remarks on a paper by J. T. Beale, T. Kato and A. J. Majda. *Comm. Math. Phys.* **98** (1985), 349–353.