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## THE EMBEDDING OF A CYCLIC PERMUTABLE SUBGROUP IN A FINITE GROUP. II

J. COSSEY<sup>1</sup> AND S. E. STONEHEWER<sup>2</sup>

<sup>1</sup>*Department of Mathematics, Australian National University,  
Canberra, ACT 0200, Australia (john.cossey@maths.anu.edu.au)*

<sup>2</sup>*Mathematics Institute, University of Warwick,  
Coventry CV4 7AL, UK (Stewart@stonehewer.freeserve.co.uk)*

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*Abstract* In two previous papers we established the structure of the normal closure of a cyclic permutable subgroup  $A$  of a finite group, first when  $A$  has odd order and second when  $A$  has even order, but with an extra hypothesis that was unnecessary in the odd case. Here we describe the most general situation without any restrictions on  $A$ .

*Keywords:* cyclic subgroup; permutable subgroup; power automorphism

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### 1. Introduction and statement of results

Let  $A$  be a subgroup of a group  $G$ . Then  $A$  is said to be *permutable* (or *quasinormal*) if  $AX = XA$  for all subgroups  $X$  of  $G$ . Thus  $A$  is permutable precisely when the product  $AX$  is a subgroup for each subgroup  $X$  of  $G$ . Suppose that

$G$  is finite and  $A$  is a cyclic permutable subgroup.

When  $A$  has odd order, then we showed in [3] that  $[A, G]$  is abelian and  $A$  acts on it by conjugation as a group of power automorphisms. When  $A$  has even order, this result is not true in general. But we showed in [4] that  $N = [A, G]'$  has order at most 2 and lies in  $A$ , and  $A$  acts by conjugation on  $[A, G]/N$  as a group of power automorphisms, provided in each product  $AX$ , with  $X$  cyclic, the Sylow 2-subgroup has a modular subgroup lattice. (For odd primes  $p$ , the Sylow  $p$ -subgroups of  $AX$  always have a modular subgroup lattice (see [6, Hauptsatz I]).) In [4] we also gave examples in which  $[A, G]'$  has order 2. In this paper we assume no additional hypothesis about  $A$  and  $G$ . It turns out that  $[A, G]$  is always nilpotent of class at most 2. Also, if we denote by  $2'$  the set of odd primes, then the  $2'$ -component of  $[A, G]$  is abelian and  $A$  acts on it as a group of power automorphisms. If  $A_2$  is the 2-component of  $A$ , then  $[A_2, G]$  is the 2-component of  $[A, G]$ . We find that  $[A_2, G, A]$  is a normal subgroup of  $G$  lying in the centre of  $[A, G]$  and  $A$  acts on  $[A_2, G, A]$  as a group of power automorphisms.

We denote by  $A_{2'}$  the  $2'$ -component of  $A$  (always a cyclic group). Also we write  $Z(G)$  for the *centre* of a group  $G$ . Then we shall prove the following theorem.

**Theorem 1.1.** *Let  $A$  be a cyclic permutable subgroup of a finite group  $G$ . Then*

- (i)  $[A, G] = [A_2, G] \times [A_{2'}, G]$ ;
- (ii)  $[A_{2'}, G]$  is an abelian  $2'$ -group;
- (iii)  $A$  acts by conjugation on  $[A_{2'}, G]$  as a group of power automorphisms;
- (iv)  $[A_2, G]$  is a 2-group of class at most 2;
- (v)  $[A_2, G, A]$  is a normal subgroup of  $G$  and lies in  $Z([A_2, G])$ ; and
- (vi)  $A$  acts by conjugation on  $[A_2, G, A]$  as a group of power automorphisms.

All our arguments reduce quickly to the case where  $G$  is a 2-group. Here our main results are the following.

**Theorem 1.2.** *Let  $A = \langle a \rangle$  be a cyclic permutable subgroup of a finite 2-group  $G$ . Then*

- (i)  $[A, G, A] \triangleleft G$ ; and
- (ii)  $[A, G, A] = \{[u, a] \mid u \in [A, G]\}$ .

**Theorem 1.3.** *Let  $A$  be a cyclic permutable subgroup of a finite 2-group  $G$  and put  $B = [A, G, A]$ . Then*

- (i)  $B \leq Z([A, G])$ ;
- (ii)  $A$  centralizes  $[A, G]/B$ ;
- (iii)  $A$  acts by conjugation on  $B$  as a group of power automorphisms; and
- (iv)  $[A, G]$  has nilpotency class at most 2.

**Remark 1.4.** It was shown by Cooper in [2] that a power automorphism of a finite abelian group is always *universal*, i.e. each element maps to the *same* power.

In §2 we deduce Theorems 1.1 and 1.3 from Theorem 1.2. Then we establish several preliminary lemmas for the proof of Theorem 1.2 in §§3 and 4. We denote by  $H^G$  and  $H_G$ , respectively, the *normal closure* and the *core* of a subgroup  $H$  in a group  $G$ . When  $G$  is a  $p$ -group, for some prime  $p$ , then  $\Omega(G)$  is the subgroup generated by the elements of order  $p$ . All other notation is standard.

## 2. Preliminary lemmas and proofs of Theorems 1.1 and 1.3

The proof of Theorem 1.3 will follow easily from Theorem 1.2 and one of the most useful results in [4], which we record again here for convenience.

**Lemma 2.1** (see **Lemma 2.3** in [4]). *Let  $G = AX$  be a finite 2-group, where  $A = \langle a \rangle$  and  $X = \langle x \rangle$  are cyclic subgroups and  $A$  is permutable in  $G$ . Then*

- (i)  $G$  is metacyclic;
- (ii)  $G' = \langle [a, x] \rangle$ ;
- (iii) for each integer  $i$ ,  $\langle [a^i, x] \rangle = \langle [a, x]^i \rangle$ ;
- (iv)  $a$  conjugates  $[a, x]$  to a power congruent to 1 modulo 4; and
- (v) each element of  $G'$  has the form  $[a^i, x]$ , for some integer  $i$ .

From part (iii) we immediately obtain the following corollary.

**Corollary 2.2.** *Let  $A = \langle a \rangle$  be a cyclic permutable subgroup of a finite 2-group  $G$ . Then  $[A, G] = \langle [a, g] \mid g \in G \rangle$ .*

**Proof of Theorem 1.3.** We have a cyclic permutable subgroup  $A = \langle a \rangle$  of a finite 2-group  $G$  and  $B = [A, G, A]$ . By Theorem 1.2 (ii), each cyclic subgroup of  $B$  has the form  $\langle [u, a] \rangle = K$ , say, for some element  $u$  in  $[A, G]$ . But by Lemma 2.1 (i),  $A$  normalizes  $K$  and so  $A$  normalizes every subgroup of  $B$ . By Theorem 1.2 (i),  $B \triangleleft G$  and therefore it follows that  $A^G$  normalizes every subgroup of  $B$ . Thus  $B (\leq A^G)$  has all its subgroups normal and by [5] and [1] (see also [9, Theorem 2.3.12]),  $B$  is either abelian or the direct product of a quaternion group of order 8 and an elementary abelian 2-group. In the latter case,  $B$  has exponent 4 and so  $A$  centralizes  $B$ , by Lemma 2.1 (iv). But then  $A^G$  also centralizes  $B$  and  $B$  is abelian, a contradiction. Thus  $B$  must be abelian and then (iii) follows from [2].

Suppose that  $a$  conjugates each element of  $B$  to its  $r$ th power. Let  $x \in B$  and  $g \in G$ . Then

$$x^{[a, g]} = x^{a^{-1}g^{-1}ag} = ((x^{a^{-1}g^{-1}})^r)^g = (x^r)^{a^{-1}} = x, \quad (2.1)$$

and so from Corollary 2.2 we see that  $[A, G]$  centralizes  $B$ . Therefore, (i) is true. Part (ii) is trivial. Finally, because  $B \triangleleft G$ , we deduce that  $A^G$  centralizes  $[A, G]/B$  and thus  $[A, G]/B$  is abelian. Hence (iv) is true.  $\square$

In order to prove Theorem 1.1, all we need now is a well-known basic result.

**Lemma 2.3** (see **Lemma 5.2.11** in [9]). *Let  $A$  be a cyclic permutable subgroup of a group  $G$ . Then every subgroup of  $A$  is also permutable in  $G$ .*

**Proof of Theorem 1.1.** We have a cyclic permutable subgroup  $A$  of a finite group  $G$ . By [7],  $A$  is subnormal in  $G$  and so  $A^G = A[A, G]$  is nilpotent. Thus (i) follows immediately from  $A = A_2 \times A_{2'}$ . Also  $[A_2, G]$  and  $[A_{2'}, G]$  are the 2- and 2'-components, respectively, of  $[A, G]$ .

By Lemma 2.3,  $A_2$  and  $A_{2'}$  are both permutable subgroups of  $G$ . Then parts (ii) and (iii) are proved in Theorem 1.1 of [3], bearing in mind that  $A_2$  centralizes  $[A_{2'}, G]$ .

Elements in  $G$  of odd order normalize and therefore centralize  $A_2$ . Thus for any Sylow 2-subgroup  $G_2$  of  $G$ , we have  $A_2 \leq G_2$  and  $[A_2, G] = [A_2, G_2]$ . Then (iv) follows from Theorem 1.3 (iv). Similarly,  $[A_2, G, A] = [A_2, G_2, A_2] = B$ , say. By Theorem 1.2,  $B \triangleleft G_2$ ; and since the elements of odd order in  $G$  actually centralize  $[A_2, G]$ , we even have  $B \triangleleft G$ . By Theorem 1.3 (i),  $B \leq Z([A_2, G])$ . This proves (v).

Finally, (vi) follows from Theorem 1.3 (iii), observing that  $A_{2'}$  centralizes  $B$ .  $\square$

In the remainder of this section we establish some new and record some old results that will be required for the proof of Theorem 1.2 in §§ 3 and 4. The first comes from [4] and is both applicable to, and important in, this more general situation.

**Lemma 2.4.** *Let  $A$  be a cyclic permutable subgroup of a finite 2-group  $G$ . Then  $[A^2, G] = [A, G]^2$ .*

This allows us to prove our next result.

**Lemma 2.5.** *Let  $A = \langle a \rangle$  be a cyclic permutable subgroup of a finite 2-group  $G = \langle a, g_1, \dots, g_r \rangle$ . Then there is a commutator  $[a, g_i]$  of order equal to the exponent of  $[A, G]$ .*

**Proof.** We argue by induction on  $|A|$ . Let  $[A, G]$  have exponent  $2^n$  and let  $2^m$  be the maximal order of a commutator  $[a, g_i]$ ,  $1 \leq i \leq r$ . So  $m \leq n$  and we may assume that  $m \geq 1$ , since  $[A, G] = \langle [a, g_i] \mid i = 1, \dots, r \rangle^G$ . By Lemma 2.3,  $A^2$  is permutable in  $G$ . Put  $H = \langle a^2, g_1, \dots, g_r \rangle$ . Then  $G = AH$ . Also by induction there is a commutator  $[a^2, g_i]$  of order equal to the exponent of  $[A^2, H]$ . But, by Lemma 2.4,  $[A^2, H] = [A^2, G] = [A, G]^2$ . Therefore, by Lemma 2.1 (iii) this exponent is  $2^{m-1}$ . But clearly it is either  $2^n$  or  $2^{n-1}$ . Thus it must be  $2^{n-1}$  and  $m = n$  as required.  $\square$

We recall that permutable subgroups of order 2 are well understood.

**Lemma 2.6 (see Theorem 5.2.9 in [9]).** *Let  $G$  be a finite 2-group and let  $A$  be a permutable subgroup of order 2. Then  $[A, G] \leq Z(G)$ .*

In the proof of Theorem 1.2 (i) we shall reduce to the case where  $G$  is generated by  $A$  and at most two other elements. The remaining results in this section examine that situation. They all extend trivially to the case where  $G$  has any number of generators. The first has already appeared in [4].

**Lemma 2.7 (see Lemma 3.3 in [4]).** *Let  $A = \langle a \rangle$  be a cyclic permutable subgroup of a finite 2-group  $G = \langle a, x, y \rangle$ . Then*

$$[A, G] = (A \cap [A, G])\langle [a, x] \rangle \langle [a, y] \rangle.$$

**Corollary 2.8.** *With the same hypotheses as Lemma 2.7 and for any integer  $i \geq 0$ , we have*

$$[A^{2^i}, G] = (A^{2^i} \cap [A^{2^i}, G])\langle [a^{2^i}, x] \rangle \langle [a^{2^i}, y] \rangle. \quad (2.2)$$

The proof is straightforward and may be omitted. A complementary result is the following.

**Lemma 2.9.** *With the same hypotheses as Lemma 2.7 and for any integer  $i \geq 0$ , we have*

$$[A^{2^i}, G] = \langle [a^{2^i}, x] \rangle \langle [a^{2^i}, y] \rangle [A^{2^i}, G, G]. \quad (2.3)$$

**Proof.** We keep the same notation as in Corollary 2.8 and denote the right-hand side of (2.3) by  $K$ . Clearly,  $[B, G] \geq K$ . Conversely,  $K \triangleleft G$  and, modulo  $K$ ,  $b$  commutes with  $a$ ,  $x$  and  $y$ , i.e. with  $G$ . Therefore,  $[B, G] \leq K$  and so we have equality.  $\square$

We need one more preliminary result for the proof of Theorem 1.2.

**Lemma 2.10.** *With the same hypotheses as Lemma 2.7 and for any integer  $i \geq 0$ ,*

$$[A^{2^i}, G, G] \leq \langle [a^{2^i}, x, y] \rangle \langle [a^{2^i}, y, x] \rangle [A^{2^{i+1}}, G]. \quad (2.4)$$

**Proof.** Again we keep the notation of Corollary 2.8 and denote the right-hand side of (2.4) by  $L$ . By Lemmas 2.3 and 2.6 we have  $[B, G, G, G] \leq [B^2[B^2, G], G] = [B^2, G]$ . Therefore,  $[B, G, G]$  is central in  $G$  modulo  $[B^2, G]$ . Thus  $L \triangleleft G$ . But, by Lemma 2.1,

$$[b, x, x] \in \langle [b, x]^2 \rangle = \langle [b^2, x] \rangle \leq [B^2, G].$$

Similarly,  $[b, y, y] \in [B^2, G]$ . Since  $A \cap [B, G] \leq B^2$ , it follows that, modulo  $L$ , each of the three factors on the right-hand side of (2.2) is centralized by  $G$ . Therefore,  $[B, G, G] \leq L$  by Corollary 2.8, as required.  $\square$

### 3. Proof of Theorem 1.2 (i)

We have a finite 2-group  $G$  with a permutable cyclic subgroup  $A = \langle a \rangle$  and we must show that  $[A, G, A] = K$ , say, is normal in  $G$ . We begin by reducing to the case where

$$G \text{ is generated by } A \text{ and at most two other elements.} \quad (3.1)$$

By Corollary 2.2,  $[A, G] = \langle [a, g] \mid g \in G \rangle$ . So a typical element of  $[A, G]$  has the form  $u = u_1 u_2 \dots u_m$ , where  $u_i = [a, g_i]$ ,  $g_i \in G$ . Then  $K$  is generated by elements  $v = [u, a]$ , by Lemma 2.1 (iii). It is sufficient to show that  $v^y \in K$  for all such  $v$  and  $y \in G$ . But

$$v^y = [u_1 \dots u_m, a]^y = [u_1, a]^{y_1} [u_2, a]^{y_2} \dots [u_m, a]^{y_m}, \quad (3.2)$$

where  $y_i = u_{i+1} u_{i+2} \dots u_m y$ . Thus with  $G_i = \langle a, g_i, y_i \rangle$ , we have

$$[u_i, a^{y_i}] \in [A, G_i, A]^{y_i} = [A, G_i, A]$$

if the theorem is true for  $A$  as a permutable subgroup of  $G_i$ . Then each factor on the right-hand side of (3.2) lies in  $K$  and the theorem follows. Therefore, we may assume that (3.1) holds and so

$$G = \langle a, x, y \rangle. \quad (3.3)$$

Suppose that the theorem is false when (3.1) holds and let  $G$  (given by (3.3)) be a counterexample of minimal order. Then  $K_G = 1$  and  $K \neq 1$ . We distinguish two cases.

**Case 1.** Suppose that  $\langle [a, x] \rangle \cap \langle [a, y] \rangle \neq 1$ . By Lemma 2.7 we may assume without loss of generality that  $[a, x, a] \neq 1$ . Then  $1 \neq \langle [a, x, a] \rangle \leq \langle [a, x] \rangle$  by Lemma 2.1 (ii). So  $N = \Omega(\langle [a, x] \rangle) = \Omega(\langle [a, y] \rangle) \leq K \cap Z(G)$ , contradicting  $K_G = 1$ .

**Case 2.** Suppose that  $\langle [a, x] \rangle \cap \langle [a, y] \rangle = 1$ . Without loss of generality we may assume that  $|[a, x]| \geq |[a, y]|$ . So  $|[a, x]| = 2^n$  (say) is the exponent of  $[A, G]$ , by Lemma 2.5. If  $|[a, y]| < 2^n$ , then  $|[a, xy]| = 2^n$ , since  $G = \langle a, xy, y \rangle$  (using Lemma 2.5 again). Therefore, replacing  $y$  by  $xy$  if necessary, we may assume that

$$|[a, x]| = |[a, y]| = 2^n \text{ is the exponent of } [A, G].$$

To simplify notation, we write  $A_i = A^{2^i}$  and  $a_i = a^{2^i}$  for each integer  $i \geq 0$ . Choose  $i$  such that  $a$  acts non-trivially on  $[A_i, G]$  and trivially on  $[A_{i+1}, G]$ . So  $0 \leq i < n - 2$ , by Lemma 2.1 (iv). (We could have chosen our counterexample  $G$  with  $|A|$  minimal. Then  $[A^2, G, A^2] \triangleleft G$  and so  $[A^2, G, A^2] = 1$ . This implies that  $a$  centralizes  $[A^4, G]$  and hence  $i = 0$  or  $1$ . But this additional information does not appear to shorten our argument.) By Corollary 2.8,  $a$  cannot centralize both  $[a_i, x]$  and  $[a_i, y]$ . Therefore, suppose without loss of generality that  $a$  acts non-trivially on  $\langle [a_i, x] \rangle$ . Let  $|[a_i, x]| = 2^t$ . So  $t = n - i$ , by Lemma 2.1 (iii). Since  $a$  centralizes  $[a_{i+1}, x]$ ,  $a$  must conjugate  $[a_i, x]$  to  $[a_i, x]^\rho$ , where  $\rho = 2^{t-1} + 1$ , by Lemma 2.1 (iv). In the same way we see that  $a$  either centralizes  $[a_i, y]$  or conjugates it to  $[a_i, y]^\rho$ .

We have

$$[a_i, yx] = [a_i, x][a_i, y][a_i, y, x]. \quad (3.4)$$

Since  $A$  centralizes  $A_{i+1}[A_{i+1}, G] = A_{i+1}^G$ , so also does  $A^G$ . Therefore, by Lemmas 2.3 and 2.6 we have

$$[A_i, G, G] \leq A_{i+1}[A_{i+1}, G] \leq Z(A^G). \quad (3.5)$$

Therefore,

$$a \text{ centralizes } [a_i, y, x]. \quad (3.6)$$

Suppose first that  $a$  centralizes  $[a_i, y]$ . Then it follows from (3.4) that

$$a \text{ centralizes } [a_i, x]^{-1}[a_i, yx]. \quad (3.7)$$

Let  $[a_i, yx]^a = [a_i, yx]^\beta$ . From (3.7) we get

$$[a_i, x]^{-\rho}[a_i, yx]^\beta = [a_i, x]^{-1}[a_i, yx].$$

But we can assume that  $\langle [a, x] \rangle \cap \langle [a, yx] \rangle = 1$ , since otherwise Case 1 applies to  $G = \langle a, x, yx \rangle$ . Therefore,  $\rho \equiv 1 \pmod{2^t}$ , which is a contradiction.

Thus we may suppose that  $[a_i, y]^a = [a_i, y]^\rho$ . By Lemma 2.1,

$$[A_i, G]/[A_{i+1}, G]$$

is generated by elements of order 2, all centralized by  $A$  and therefore by  $A^G$ . So this quotient is elementary abelian. It follows from (3.5) that

$$[A_i, G] \text{ has class at most 2 and } [A_i, G]' \text{ is elementary abelian.} \quad (3.8)$$

Since  $\langle [a, x] \rangle \cap \langle [a, y] \rangle = 1$ ,  $a$  does not centralize  $[a_i, x][a_i, y]$ . Therefore, by (3.4) and (3.6),

$$a \text{ does not centralize } [a_i, yx]. \quad (3.9)$$

Clearly,  $[A_i, G]$  has exponent  $2^t$ , by Lemmas 2.3 and 2.5. To complete the proof we distinguish two cases.

Suppose that  $|[a_i, yx]| < 2^t$ . By Lemma 2.1 (iv),  $t \geq 3$  and so  $2^{t-1} = \rho - 1 \geq 4$ . Therefore, raising both sides of (3.4) to the power  $\rho - 1$ , we get

$$1 = [a_{n-1}, x][a_{n-1}, y][a_i, y, x]^{\rho-1},$$

using Lemma 2.1 (iii) and (3.8). Thus

$$1 \neq [a_{n-1}, x][a_{n-1}, y] = [a_i, y, x]^{\rho-1} = [[a_i, y]^{\rho-1}, x] = w,$$

say, by (3.5). Hence  $w = [a_{n-1}, y, x]$ , again by Lemma 2.1 (iii). But by Lemmas 2.3 and 2.6,

$$[A_{n-1}, G, G, G] \leq [A_n[A_n, G], G] = [A_n, G] = 1.$$

Thus  $w$  is a non-trivial element in  $Z(G)$ . Also  $[a_{n-1}, x] \in [A_i, \langle x \rangle, A] \leq K$  and similarly  $[a_{n-1}, y] \in K$ . Therefore,  $w \in K$ , contradicting  $K_G = 1$ .

Finally, suppose that  $|[a_i, yx]| = 2^t$ . Then, by (3.9),  $[a_i, yx]^a = [a_i, yx]^\rho$  and hence, by (3.4) and (3.8),  $[a_i, y, x]^a = [a_i, y, x]^\rho$ . Thus from (3.6) we get  $|[a_i, y, x]| \leq 2^{t-1}$ . In the same way, interchanging  $x$  and  $y$ , we may also assume that  $|[a_i, x, y]| \leq 2^{t-1}$ . Since  $[A_i, G, G]$  is abelian (by (3.5)), it follows from Lemma 2.10 that  $[A_i, G, G]$  has exponent  $2^{t-1}$ . Therefore, in centralizing  $[A_i, G, G]$ ,  $a$  is in fact raising each element to its  $\rho$ th power. Thus by Lemma 2.9 and (3.9),  $a$  conjugates each element of  $[A_i, G]$  to its  $\rho$ th power.

Putting  $L = [A_i, G]$ , we now have  $L^{\rho-1} \leq [L, A] \leq L^{\rho-1}$ . So  $[L, A] = L^{\rho-1} \triangleleft G$ . But  $1 \neq [L, A] \leq K$ , contradicting  $K_G = 1$ . This completes the proof of Theorem 1.2 (i).  $\square$

#### 4. Proof of Theorem 1.2 (ii)

Here  $A = \langle a \rangle$  is a cyclic permutable subgroup of a finite 2-group  $G$  and we must show that  $[A, G, A] = \{[u, a] \mid u \in [A, G]\}$ . We argue by induction on  $|A|$ . If  $|A| \leq 4$ , then  $|[a, g]| \leq 4$  for all  $g \in G$ , by Lemma 2.1 (iii). Thus  $a$  centralizes all  $[a, g]$ , by Lemma 2.1 (iv), so  $[A, G, A] = 1$  and the theorem is true. Therefore, suppose that  $|A| \geq 8$  and assume the usual induction hypothesis. Since  $A^2$  is permutable in  $G$  by Lemma 2.3, we deduce that

$$[A^2, G, A^2] = \{[v, a^2] \mid v \in [A^2, G]\} = K,$$

say. By Theorem 1.2 (i),  $K \triangleleft G$ . By Lemma 2.1 (ii),  $a$  normalizes each cyclic subgroup  $\langle [v, a^2] \rangle$  and so  $A$  normalizes every subgroup of  $K$ . Then  $A^G (\geq [A, G])$  does the same and hence  $K$  has all its subgroups normal. Suppose that  $K$  is not abelian, so it is the direct product of a quaternion group of order 8 and an elementary abelian 2-group (as we saw in the proof of Theorem 1.3). But then the commutators  $[v, a^2]$  all have order at



most 4 and are centralized by  $a$ , by Lemma 2.1 (iv). Thus  $A$  centralizes  $K$  and therefore  $K \leq Z(A^G)$ , contradicting the fact that  $K$  is not abelian. So

$K$  is abelian.

By analogy with (2.1) in the proof of Theorem 1.3, we see that

$$[K, [A, G]] = 1. \quad (4.1)$$

Let  $v \in [A^2, G]$ . By Lemma 2.1 (ii), there are elements  $g_1, \dots, g_n$  in  $G$  such that  $v = [a^2, g_1] \dots [a^2, g_n]$ . Therefore, by (4.1),

$$[v, a^2] = [[a^2, g_1] \dots [a^2, g_n], a^2] = [a^2, g_1, a^2] \dots [a^2, g_n, a^2].$$

But  $\langle [a^2, g_i, a^2] \rangle = \langle [[a^2, g_i]^2, a] \rangle$ , by Lemma 2.1 applied to the product  $\langle [a^2, g_i] \rangle \langle a \rangle$ , in which both factors are permutable subgroups. Therefore, with  $M = \langle [a^2, g_i]^2 \rangle \langle a \rangle$ , we have  $M' = \langle [a^2, g_i, a^2] \rangle$ . Again by Lemma 2.1, each element of  $M'$  has the form  $[x, a]$ , with  $x \in \langle [a^2, g_i]^2 \rangle \leq [A^4, G]$ . Thus  $[a^2, g_i, a^2] = [x_i, a]$  for some element  $x_i \in [A^4, G]$ . Therefore,

$$[v, a^2] = [x_1, a] \dots [x_n, a] = [x_1 \dots x_n, a]$$

by (4.1). It follows that every element of  $K$ , as an element of  $[A, G, A]$ , has the form that we are trying to establish. Since  $K \triangleleft G$ , there is a central series of  $G$  passing through  $K$ . Then using the identity  $[u_1 u_2, a] = [u_1, a]^{u_2} [u_2, a]$ , a simple induction allows us to assume that

$$K = [A^2, G, A^2] = 1. \quad (4.2)$$

Using Lemma 2.1 again, we see that  $[A, G, A^2]$  is generated by elements of the form

$$[[a, h_1] \dots [a, h_r], a^2] = \prod_{i=1}^r [a, h_i, a^2]^{c_i}, \quad (4.3)$$

for suitable elements  $h_i$  in  $G$  and  $c_i = [a, h_{i+1}] \dots [a, h_r]$ . By (4.2),  $a$  induces an automorphism of order at most 2 in any subgroup of the form  $\langle [a^2, g] \rangle$ . So by Lemma 2.1 (iv),  $a$  centralizes  $\langle [a^2, g]^2 \rangle = \langle [a^4, g] \rangle$ . Therefore,  $A$  centralizes  $[A^4, G]$  and hence so also does  $A^G = A[A, G]$ . But again by Lemma 2.1 (iv), we have

$$[a, g, a^2] \in \langle [a, g]^8 \rangle \leq [A^4, G]$$

and therefore each  $[a, h_i, a^2]$  commutes with  $c_i$  in (4.3). Thus  $[A, G, A^2]$  is generated by the elements  $[a, g, a^2]$ ,  $g \in G$ .

Choose  $g \in G$  and let  $L = \langle [a, g]^2 \rangle \langle a \rangle$ , so that  $L' = \langle [[a, g]^2, a] \rangle = \langle [a, g, a^2] \rangle$ . Each element of  $L'$  has the form  $[y, a]$ , where  $y \in \langle [a, g]^2 \rangle = \langle [a^2, g] \rangle \leq [A^2, G]$ . Therefore,  $[a, g, a^2] = [y, a]$ , for such an element  $y$  in  $[A^2, G]$ . It follows that  $[A, G, A^2]$  is generated by elements of the form  $[y, a]$ . But with  $y_1, y_2 \in [A^2, G]$ , we have  $[y_1, a][y_2, a] = [y_1 y_2, a]$ , by (4.2). Thus  $[A, G, A^2] \subseteq \{[y, a] \mid y \in [A^2, G]\}$ . In particular,  $[A, G, A^2] \leq [A^2, G, A]$ . Conversely,  $[A, G, A^2] = [A, G, A]^2$ , by Lemma 2.4 (replacing  $G$  there by  $A^G = A[A, G]$ ).

Therefore,  $[A, G, A^2] \triangleleft G$ , by Theorem 1.2 (i), and it follows from the Three Subgroup Lemma (see [8, 5.1.10]) that  $[A^2, G, A] \leq [A, G, A^2]$ . Thus

$$[A, G, A^2] = [A^2, G, A] = \{[y, a] \mid y \in [A^2, G]\} \triangleleft G. \quad (4.4)$$

Just as we were able to assume above that  $K = 1$ , so now from (4.4) we may suppose that

$$[A, G, A^2] = [A^2, G, A] = 1. \quad (4.5)$$

By Lemma 2.1 we have  $[A, G, A] \leq [A^4, G]$ , and  $[A^2, G, [A, G]] = 1$ , by (4.5). Therefore,

$$[A, G, A, [A, G]] = 1. \quad (4.6)$$

Each element of  $[A, G]$  has the form  $[a, y_1] \dots [a, y_m]$ ,  $y_i \in G$ , by Corollary 2.2. Thus  $[A, G, A]$  is generated by elements of the form

$$[[a, y_1] \dots [a, y_m], a] = [a, y_1, a] \dots [a, y_m, a], \quad (4.7)$$

by (4.6), i.e.  $[A, G, A]$  is generated by elements  $[a, g, a]$ ,  $g \in G$ . But, by (4.7), products of such elements have the form  $[u, a]$ ,  $u \in [A, G]$ . This completes the proof of Theorem 1.2 (ii).  $\square$

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