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Article Title: Study of two coupled three-wave interactions with a quadratic non-linearly in the presence of dissipation and frequency mismatch

Year of publication: 2003

Link to published

version: <http://dx.doi.org/10.1017/S0022377802001952>

Publisher statement: None

Study of two coupled three-wave interactions with a quadratic non-linearly in the presence of dissipation and frequency mismatch

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(Received 15 October 2001 and in revised form 2 June 2002)

Abstract. The non-linear dynamic behaviour of two three-wave systems in plasma with two waves in common has been studied, including the possibility of negative energy waves and also the effect of linear damping or growth and frequency mismatch. Depending on the various initial conditions solutions of different types have been discussed. It has also been shown that one of the triplets can be stabilized by the other one against explosive instability depending on the relative strength of the coupling factor.

1. Introduction

The coherent wave–wave interaction is an important aspect of weakly non-linear theory in plasmas and non-linear optics. For example, they can cause the filamentation and anomalous absorption of laser beams in laboratory plasmas or generate auroral radio emissions in space plasmas (Chian and Rizzato 1994; Chian et al. 1994). For non-linear optics these can produce the second harmonic, amplification, frequency up-conversion as well as phase conjugation of optical signals (Shen 1984; Yariv 1989).

The case of two three-wave interacting systems has been studied either in the form of four waves with two of them common to both triplets or in the form of five waves with only one common to both triplets in the absence of dissipation and frequency mismatch (Walters and Lewak 1977; Romeiras 1983).

Recently, Domier and Luhmann (1993) have shown that frequencies and wave numbers of finite-amplitude Alfvén waves in a magnetized plasma satisfy the criterion for four-wave resonance, which opens up the possibility of many new low-frequency non-linear phenomena in magnetized laboratory, astrophysical and upper-atmosphere plasmas where Alfvén waves abound. Chian et al. (1996) have considered the dynamical effects introduced by the presence of frequency mismatch without dissipation in two coupled three-wave interactions with two waves in common and showed numerically that its trajectory may undergo a transition to chaos as the mismatch is varied for a particular initial condition. Pakter et al. (1997) extended the problem and analysed numerically both regular and chaotic dynamics for a whole set of initial conditions. Krasnosel'skikh et al. (1998) considered the interaction of three Langmuir waves and one ion-acoustic wave at the exact

resonance condition in the dissipative region, but without any frequency mismatch and studied both decay and modified decay instabilities.

Though recently some work has been done mainly numerically in the case of four-wave interaction they either included dissipation or frequency mismatch but not both. Basu (1999) studied analytically the interaction of four electrostatic waves in a plasma medium in the presence of dissipation or growth and frequency mismatch. Depending on various initial conditions, solutions of different types were discussed.

The non-linear dynamic behaviour has been studied here for two three-wave systems with two waves in common, including the possibility of negative energy waves and also the effect of linear damping or growth and frequency mismatch.

The phenomena of explosive instability are of interest in high-power laser devices and also in the development of plasma turbulence and plasma heating (Kerst and Raether 1976). Explosive energy release during disruption of the Earth's plasma sheet can lead to magnetosphere substorms.

The solutions obtained here describe the time behaviour of the wave amplitudes. Depending on various initial conditions, various periodic and soliton-type solutions are discussed. It has also been shown that one of the triplets can be stabilized by the other one against explosive instability, depending on the relative strength of the coupling factor, whereas the converse is not possible.

2. Formulation of the system

For the non-linear interaction of two triplets with two waves in common, the resonance conditions are assumed to be of the form

$$k_{3,4} = k_1 \mp k_2, \quad \omega_{3,4} \approx \omega_1 \mp \omega_2.$$

This type of coupling is known to occur in plasmas for interactions involving:

1. electromagnetic, Langmuir and ion-acoustic waves;
2. Alfvén, fast and slow magnetohydrodynamic waves;
3. Langmuir waves and negative energy ion beam waves.

Following Pakter et al. (1997) the system of coupled mode equations in the presence of dissipation and frequency mismatch under consideration are:

$$\left. \begin{aligned} s_1 \left(\frac{de_1}{dt} + \nu_1 e_1 \right) &= \sigma e_2 e_3 e^{i\delta-t} - \tau e_2^* e_4 e^{i\delta+t} \\ s_2 \left(\frac{de_2}{dt} + \nu_2 e_2 \right) &= -\sigma e_1 e_3^* e^{i\delta-t} - \tau e_1^* e_4 e^{i\delta+t} \\ s_3 \left(\frac{de_3}{dt} + \nu_3 e_3 \right) &= -\sigma e_1 e_2^* e^{-i\delta-t} \\ s_4 \left(\frac{de_4}{dt} + \nu_4 e_4 \right) &= \tau e_1 e_2 e^{-i\delta-t}, \end{aligned} \right\} \quad (2.1)$$

where s_i is the sign of the energy of the i th wave.

σ and τ are the time-independent interaction kernels determined by the specific physical system. In the presence of dissipation the coupling coefficients σ and τ are complex.

Let them be

$$\sigma = \sigma e^{i\alpha}, \quad \tau = \tau e^{i\beta}. \quad (2.2)$$

The frequencies and wave numbers for the two sets of triplets satisfy the relations

$$\omega_{3,4} = \omega_1 \mp \omega_3 - \delta_{\mp}, \quad k_{3,4} = k_1 \mp k_2, \quad (2.3)$$

where δ_{\mp} are small linear frequency mismatches for each of the two triplets where modes 1 and 2 are common to the two interacting triplets. Following previous works (Basu 1999; De et al. 1981) the following transformations are introduced:

$$\begin{aligned} e_i &= \varepsilon A_i \exp(i\text{Re}\omega_i t), \quad i = 1, 2, 3, 4 \\ A_i &= \tilde{u}_i \exp(i\varphi_i), \quad \tilde{u}_i = |A_i|, \quad v_i = \text{Im}\omega_i. \end{aligned} \quad (2.4)$$

Considering renormalization the system of equations (2.1) becomes

$$s_1 \left(\frac{du_1}{dt} + \nu_1 u_1 \right) = \varepsilon u_2 u_3 \cos(\phi_- - \theta_\alpha) - \varepsilon r u_2 u_4 \cos(\phi_+ - \theta_\beta) \quad (2.5)$$

$$s_2 \left(\frac{du_2}{dt} + \nu_2 u_2 \right) = -\varepsilon u_1 u_3 \cos(\phi_- + \theta_\alpha) - \varepsilon r u_1 u_4 \cos(\phi_+ - \theta_\beta) \quad (2.6)$$

$$s_3 \left(\frac{du_3}{dt} + \nu_3 u_3 \right) = -\varepsilon u_1 u_2 \cos(\phi_- + \theta_\alpha) \quad (2.7)$$

$$s_4 \left(\frac{du_4}{dt} + \nu_4 u_4 \right) = \varepsilon r u_1 u_2 \cos(\phi_+ + \theta_\beta) \quad (2.8)$$

$$\begin{aligned} \frac{d\phi_-}{dt} &= -\delta_- + \varepsilon \left[\left(\frac{u_1 u_2}{s_3 u_3} + \frac{u_1 u_3}{s_2 u_2} \right) \sin(\phi_- + \theta_\alpha) - \frac{u_2 u_3}{s_1 u_1} \sin(\phi_- - \theta_\alpha) \right] \\ &\quad + \varepsilon r \left[\left(\frac{u_2 u_4}{s_1 u_1} - \frac{u_1 u_4}{s_2 u_2} \right) \sin(\phi_+ - \theta_\beta) \right], \end{aligned} \quad (2.9)$$

where $\dot{\phi}_- = \dot{\phi}_1 - \dot{\phi}_2 - \dot{\phi}_3$,

$$\begin{aligned} \frac{d\phi_+}{dt} &= -\delta_+ - \varepsilon \left[\frac{u_2 u_3}{s_1 u_1} \sin(\phi_- - \theta_\alpha) + \frac{u_1 u_3}{s_2 u_2} \sin(\phi_- + \theta_\alpha) \right] \\ &\quad + \varepsilon r \left[\left(\frac{u_2 u_4}{s_1 u_1} + \frac{u_1 u_4}{s_2 u_2} \right) \sin(\phi_+ - \theta_\beta) - \frac{u_1 u_2}{s_4 u_4} \sin(\phi_- + \theta_\beta) \right], \end{aligned} \quad (2.10)$$

where $\dot{\phi}_+ = \dot{\phi}_1 + \dot{\phi}_2 - \dot{\phi}_4$.

The coupling factor r is a measure of coupling strength between the sets of two triplets and is given by $r = \tau/\sigma$.

To solve the system of equations (2.5)–(2.8), the method of perturbation (Coffey and Ford 1969) has been used to separate the secular motion from the rapidly fluctuating motion (for details see Basu 1999 and references therein).

Using the expansions

$$\begin{aligned} u_i &= y_i + \varepsilon E_i^1(\bar{y}, \phi_-, \phi_+) + \varepsilon^2 E_i^2(\bar{y}, \phi_-, \phi_+) + \dots \\ \psi_{\mp} &= -\delta_{\mp} + \varepsilon b_m^1(\bar{y}) + \varepsilon^2 b_m^2(\bar{y}) + \dots \quad m = 1, 2 \end{aligned}$$

$$\begin{aligned}\phi_- &= -\psi_- + \varepsilon G_m^1(\bar{y}, \Phi_-, \Phi_+) + \varepsilon^2 G_m^2(\bar{y}, \Phi_-, \Phi_+) + \cdots \\ \phi_+ &= -\psi_+ + \varepsilon G_m^1(\bar{y}, \Phi_-, \Phi_+) + \varepsilon^2 G_m^2(\bar{y}, \Phi_-, \Phi_+) + \cdots\end{aligned}$$

the following system of equations are obtained from different orders.

From the first-order terms one obtains

$$a_i^0 + \nu_i y_i = 0, \quad a_i^1 = 0, \quad b_i^1 = 0, \quad i = 1, 2, 3, 4 \quad (2.11)$$

$$\left. \begin{aligned} E_1^1 &= -\frac{y_2 y_3}{s_1 \delta_{1-}} \sin(\psi_- - \theta_\alpha + \eta_{1-}) + r \frac{y_2 y_4}{s_1 \delta_{1+}} \sin(\psi_+ - \theta_\beta + \eta_{1+}) \\ E_2^1 &= \frac{y_1 y_3}{s_2 \delta_{1-}} \sin(\psi_- + \theta_\alpha + \eta_{2-}) + r \frac{y_1 y_4}{s_2 \delta_{2+}} \sin(\psi_+ - \theta_\beta + \eta_{2+}) \\ E_3^1 &= \frac{y_1 y_2}{s_3 \delta_{3-}} \sin(\psi_- + \theta_\alpha + \eta_{3-}), \quad E_4^1 = -r \frac{y_1 y_2}{s_4 \delta_{4+}} \sin(\psi_+ + \theta_\beta + \eta_{4+}) \end{aligned} \right\} \quad (2.12)$$

where

$$\tan \eta_{i-} = \frac{\nu_i}{\delta_{i-}}, \quad \tan \eta_{i+} = \frac{\nu_i}{\delta_{i+}} \quad i = 1, 2, 3, 4$$

$$\begin{aligned} G_1^1 &= \frac{y_1 y_2}{s_3 y_3 \delta_-} \cos(\psi_- + \theta_\alpha) + \frac{y_1 y_3}{s_2 y_2 \delta_-} \cos(\psi_- - \theta_\alpha) \\ &+ r \left[\frac{y_2 y_4}{s_1 \delta_+ y_3} \cos(\psi_+ - \theta_\beta) - \frac{y_1 y_4}{s_2 y_2 \delta_+} \cos(\psi_+ - \theta_\beta) \right] \end{aligned} \quad (2.13)$$

$$\begin{aligned} G_2^1 &= -\frac{y_2 y_3}{s_1 y_1 \delta_-} \cos(\psi_- - \theta_\alpha) - \frac{y_1 y_3}{s_2 y_2 \delta_-} \cos(\psi_- + \theta_\alpha) \\ &+ r \left[\frac{y_2 y_4}{s_1 y_1 \delta_+} \cos(\psi_+ - \theta_\beta) + \frac{y_1 y_4}{s_2 y_2 \delta_+} \cos(\psi_+ - \theta_\beta) \right. \\ &\left. - \frac{y_1 y_2}{s_4 y_4 \delta_+} \cos(\psi_+ + \theta_\beta) \right]. \end{aligned} \quad (2.14)$$

The second-order terms of a_i^2 ($i = 1, 2, 3, 4$) can be obtained from the ψ -independent part of the following system of equations:

$$\begin{aligned} a_1^2 + \frac{\partial E_1^1}{\partial \psi_-} G_1^1 + \frac{\partial E_1^1}{\partial \psi_+} G_2^1 &= \frac{1}{s_1} (E_2^1 y_3 + E_3^1 y_2) \cos(\psi - \theta_\alpha) \\ &- \frac{y_2 y_3}{s_1} G_1^1 \sin(\psi_- - \theta_\alpha) + \frac{r}{s_1} y_2 y_4 G_2^1 \sin(\psi_+ - \theta_\beta) \\ &- \frac{r}{s_1} (E_2^1 y_4 + E_4^1 y_2) \cos(\psi_+ - \theta_\beta) \end{aligned}$$

and the similar equations for other a_i ($i = 1, 2, 3, 4$).

Using the relation

$$\dot{y}_i = a_i^0 + \varepsilon a_i^1 + \varepsilon^2 a_i^2 + \cdots$$

the following system of equations are obtained for the wave amplitudes:

$$\begin{aligned}
 \dot{y}_1 + \nu_1 y_1 = & y_1 y_2^2 \left\{ \frac{\cos(\eta_{1-}/2 - \theta_\alpha)}{2s_1 s_3 \delta_- \delta_{1-}} - \frac{\sin(\eta_{3-}/2 + \theta_\alpha)}{2s_1 s_3 \delta_{3-}} + \frac{\sin \theta_\alpha}{2s_1 s_3 \delta_-} \right. \\
 & \left. + r^2 \left[\frac{\sin(\theta_\beta + \eta_{4+}/2)}{2s_1 s_4 \delta_{4+}} + \frac{\cos(\eta_{1+}/2)}{2s_1 s_4 \delta_+ \delta_{1+}} \right] \right\} \\
 & + y_1 y_3^2 \left[\frac{\sin(\eta_{2-}/2 + \theta_\alpha)}{2s_1 s_2 \delta_{2-}} + \frac{\cos(\eta_{1-}/2 - \theta_\alpha)}{2s_1 s_2 \delta_- \delta_{1-}} + \frac{\sin \theta_\alpha}{2s_1 s_2 \delta_-} \right] \\
 & - r^2 y_1 y_4^2 \left[\frac{\cos(\eta_{1+}/2)}{2s_1 s_2 \delta_+ \delta_{1+}} + \frac{\eta_{2+}/2}{2s_1 s_2 \delta_{2+}} \right] - \frac{y_2^2 y_3^2 \cos(\eta_{1-}/2)}{2y_1 s_1^2 \delta_- \delta_{1-}} \\
 & - r^2 \frac{y_2^2 y_4^2 \cos(\eta_{1+}/2)}{2y_1 \delta_+ \delta_{1+}} \tag{2.15}
 \end{aligned}$$

$$\begin{aligned}
 \dot{y}_2 + \nu_2 y_2 = & y_2 y_1^2 \left\{ \frac{\sin(\eta_{3-}/2)}{2s_2 s_3 \delta_{3-}} - \frac{\cos(\eta_{2-}/2)}{2s_2 s_3 \delta_- \delta_{2-}} \right. \\
 & \left. + r^2 \left[\frac{\sin \theta_\beta}{2s_2 s_4 \delta_+} + \frac{\sin(\eta_{4+}/2)}{2s_2 s_4 \delta_{4+}} - \frac{\sin(\eta_{2+}/2 - \theta_\beta)}{2s_2 s_4 \delta_{4+}} \right] \right\} \\
 & + y_2 y_3^2 \left[\frac{\sin(\eta_{2-}/2 - \theta_\alpha)}{2s_1 s_2 \delta_{1-}} + \frac{\cos(\eta_{2-}/2 + \theta_\alpha)}{2s_1 s_2 \delta_- \delta_{2-}} - \frac{\sin \theta_\alpha}{2s_1 s_2 \delta_-} \right] \\
 & - r^2 y_2 y_4^2 \left[\frac{\sin(\eta_{1+}/2)}{2s_1 s_2 \delta_{1+}} + \frac{\sin(\eta_{2+}/2)}{2s_1 s_2 \delta_{2+}} \right] - \frac{y_2^2 y_3^2 \cos(\eta_{2-}/2)}{2y_2 \delta_- \delta_{2-}} \\
 & - r^2 \frac{y_1^2 y_4^2 \sin(\eta_{2+}/2)}{2y_2 \delta_+ \delta_{2+}} \tag{2.16}
 \end{aligned}$$

$$\begin{aligned}
 \dot{y}_3 + \nu_3 y_3 = & y_3 y_1^2 \left[\frac{\cos(\eta_{3-}/2)}{2s_2 s_3 \delta_- \delta_{3-}} - \frac{\sin(\eta_{2-}/2)}{2s_2 s_4 \delta_+ \delta_{2+}} \right] + \frac{y_1^2 y_2^2 \cos(\eta_{3-}/2)}{2y_3 \delta_- \delta_{2-}} \\
 & + y_3 y_2^2 \left[\frac{\sin(\eta_{1-}/2 - \theta_\alpha)}{2s_1 s_3 \delta_{1-}} - \frac{\sin \theta_\alpha}{2s_1 s_3 \delta_-} - \frac{\cos(\eta_{3-}/2 + \theta_\alpha)}{2s_1 s_3 \delta_- \delta_{3-}} \right] \tag{2.17}
 \end{aligned}$$

$$\begin{aligned}
 \dot{y}_4 + \nu_4 y_4 = & y_4 y_1^2 r^2 \left[\frac{\cos(\eta_{4+}/2 + \theta_\beta)}{2s_2 s_4 \delta_+ \delta_{4+}} + \frac{\sin(\eta_{2+}/2 - \theta_\beta)}{2s_2 s_4 \delta_{2+}} - \frac{\sin \theta_\beta}{2s_1 s_3 \delta_+} \right] \\
 & - r^2 \frac{y_1^2 y_2^2 \cos(\eta_{4+}/2)}{2y_2 \delta_+ \delta_{4+}} \\
 & + y_4 y_2^2 r^2 \left[\frac{\sin(\eta_{1+}/2 - \theta_\beta)}{2s_1 s_4 \delta_{1+}} - \frac{\sin \theta_\beta}{2s_1 s_4 \delta_+} - \frac{\cos(\eta_{1+}/2 + \theta_\beta)}{2s_1 s_3 \delta_- \delta_{3-}} \right]. \tag{2.18}
 \end{aligned}$$

3. Solutions of the coupled mode equations

To make the non-linear equations tractable analytically it is assumed that all the damping or growth terms are equal. The system of equations from the second-order

terms become:

$$\begin{aligned}
\dot{X}_1 = & X_1 X_2 \left\{ \frac{\cos(\eta_{k-}/2 - \theta_\alpha)}{s_1 s_3 \delta_- \delta_{k-}} - \frac{\sin(\eta_{k-}/2 + \theta_\alpha)}{s_1 s_3 \delta_{k-}} + \frac{\sin \theta_\alpha}{s_1 s_3 \delta_-} \right. \\
& \left. + r^2 \left[\frac{\sin(\theta_\beta + \eta_{k+}/2)}{s_1 s_4 \delta_{k+}} + \frac{\cos(\eta_{k+}/2)}{s_1 s_4 \delta_+ \delta_{k+}} \right] \right\} \\
& + X_1 X_3 \left[\frac{\sin(\eta_{k-}/2 + \theta_\alpha)}{s_1 s_2 \delta_{k-}} + \frac{\cos(\eta_{k-}/2 - \theta_\alpha)}{s_1 s_2 \delta_- \delta_{k-}} + \frac{\sin \theta_\alpha}{s_1 s_2 \delta_-} \right] - X_2 X_3 \frac{\cos(\eta_{k-}/2)}{\delta_- \delta_{k-}} \\
& - r^2 X_1 X_4 \left[\frac{\cos(\eta_{k+}/2)}{s_1 s_2 \delta_+ \delta_{k+}} + \frac{\sin(\eta_{k+}/2)}{s_1 s_2 \delta_{2+}} \right] - r^2 X_2 X_4 \frac{\cos(\eta_{k+}/2)}{\delta_+ \delta_{k+}} \quad (3.1)
\end{aligned}$$

$$\begin{aligned}
\dot{X}_2 = & X_1 X_2 \left\{ \frac{\sin(\eta_{k-}/2)}{s_2 s_3 \delta_{k-}} - \frac{\cos(\eta_{k-}/2)}{s_2 s_3 \delta_- \delta_{k-}} \right. \\
& \left. + r^2 \left[\frac{\sin(\eta_{k+}/2)}{s_2 s_4 \delta_{k+}} + \frac{\sin \theta_\beta}{s_2 s_4 \delta_+} + \frac{\sin(\eta_{k+}/2 - \theta_\beta)}{s_2 s_3 \delta_{k-}} \right] \right\} \\
& + X_2 X_3 \left[\frac{\sin(\eta_{k-}/2 - \theta_\alpha)}{s_1 s_2 \delta_{k-}} + \frac{\cos(\eta_{k-}/2 + \theta_\alpha)}{s_1 s_2 \delta_- \delta_{k-}} - \frac{\sin \theta_\alpha}{s_1 s_2 \delta_-} \right] - X_1 X_3 \frac{\cos(\eta_{k-}/2)}{\delta_- \delta_{k-}} \\
& - r^2 X_2 X_4 \left[\frac{\sin(\eta_{k+}/2)}{s_1 s_2 \delta_+ \delta_{k+}} + \frac{\sin(\eta_{k+}/2)}{s_1 s_2 \delta_{k+}} \right] - r^2 X_1 X_4 \frac{\sin(\eta_{k+}/2)}{\delta_+ \delta_{k+}} \quad (3.2)
\end{aligned}$$

$$\begin{aligned}
\dot{X}_3 = & X_3 X_1 \left[\frac{\cos(\eta_{k-}/2)}{2 s_2 s_3 \delta_- \delta_{k-}} - \frac{\sin(\eta_{k-}/2)}{s_2 s_4 \delta_+ \delta_{k+}} \right] + X_1 X_2 \frac{\cos(\eta_{k-}/2)}{\delta_- \delta_{k-}} \\
& + X_3 X_2 \left[\frac{\sin(\eta_{k-}/2 - \theta_\alpha)}{s_1 s_3 \delta_{k-}} - \frac{\sin \theta_\alpha}{s_1 s_3 \delta_-} - \frac{\cos(\eta_{k-}/2 + \theta_\alpha)}{s_1 s_3 \delta_- \delta_{k-}} \right] \quad (3.3)
\end{aligned}$$

$$\begin{aligned}
\dot{X}_4 = & X_4 X_1 r^2 \left[\frac{\cos(\eta_{k+}/2 + \theta_\beta)}{s_2 s_4 \delta_+ \delta_{k+}} + \frac{\sin(\eta_{k+}/2 - \theta_\beta)}{s_2 s_4 \delta_{k+}} - \frac{\sin \theta_\beta}{s_1 s_3 \delta_+} \right] \\
& - r^2 X_1 X_2 \frac{\cos(\eta_{k+}/2)}{\delta_+ \delta_{k+}} \\
& + X_4 X_2 r^2 \left[\frac{\sin(\eta_{k+}/2 - \theta_\beta)}{s_1 s_4 \delta_{k+}} - \frac{\sin \theta_\beta}{s_1 s_4 \delta_+} - \frac{\cos(\eta_{k+}/2 + \theta_\beta)}{s_1 s_3 \delta_- \delta_{k-}} \right] \quad (3.4)
\end{aligned}$$

where

$$\left. \begin{aligned}
x_i &= X_i \exp(-2\nu t), & \tau &= \frac{1}{4\nu} (1 - e^{4\nu t}), & y_i^2 &= x_i \\
\tan \eta_{k-} &= \frac{\nu}{\delta_{k-}}, & \tan \eta_{k+} &= \frac{\nu}{\delta_{k+}}, & \nu_i (i = 1, 2, 3, 4) &= \nu \\
\eta_{i-} (i = 1, 2, 3, 4) &= \eta_{k-}, & \eta_{i+} (i = 1, 2, 3, 4) &= \eta_{k+} \\
\frac{1}{\delta_{k-}} &= \frac{1}{\sqrt{1 + \nu^2/\delta_-^2}} & \text{and} & & \frac{1}{\delta_{k+}} &= \frac{1}{\sqrt{1 + \nu^2/\delta_+^2}}.
\end{aligned} \right\} \quad (3.5)$$

4. Constants of motion and the wave solution (special cases)

By suitable choices of ν , η_{k-} , η_{k+} , θ_α and θ_β from (3.1)–(3.4), the following constants of motion are obtained:

$$\left. \begin{aligned} X_4 &= r^2 X_3, & X_1 X_2 X_3^2 &= Q^2 \\ \frac{X_2(1 - \delta_{k-})}{s_1 s_3 \delta_- \delta_{k-}} + \frac{X_1}{s_2 s_4 \delta_+ \delta_{k+}} &= \sqrt{(1 - r^4)} [P X_3 - R], \end{aligned} \right\} \quad (4.1)$$

where P , Q and R are arbitrary constants and could be determined from the initial conditions. Equation (4.1) can be considered to be the Manley–Rowe relations that imply conservation of wave energy of the interacting waves.

The solution of the amplitude X_3 is given by (for $r < 1$)

$$\dot{X}_3 = P \sqrt{(1 - r^4)} \left[X_3^4 - 2 \frac{R}{P} X_3^3 + \frac{R^2}{P^2} X_3^2 - \frac{Q^2}{P^2 s_1 s_2 s_3 s_4} \right]^{1/2}, \quad (4.2)$$

where $Q'^2 = 4Q^2(1 - \delta_{k-})/\delta_- \delta_+ \delta_{k-} \delta_{k+}$, whereas for ($r > 1$) X_3 is given by

$$\dot{X}_3 = P \sqrt{(r^4 - 1)} \left[-X_3^4 + 2 \frac{R}{P} X_3^3 - \frac{R^2}{P^2} X_3^2 + \frac{Q^2}{P^2 s_1 s_2 s_3 s_4} \right]^{1/2}, \quad (4.3)$$

where $Q'^2 = 4Q^2(1 - \delta_{k-})/\delta_- \delta_+ \delta_{k-} \delta_{k+}$.

The solutions of (4.2) and (4.3) can be found in terms of Jacobian elliptic functions; however, the character of the solutions would mainly depend on the nature of the roots, the ordering of the roots in magnitude and the relative strength of the coupling factor.

5. Analysis of the solution

Case 1: $r < 1$

When all s'_j are of the same sign, (4.2) can be written as

$$\left. \begin{aligned} \dot{X}_3 &= P \sqrt{(1 - r^4)} \left[X_3^4 - 2 \frac{R}{P} X_3^3 + \frac{R^2}{P^2} X_3^2 - \frac{Q^2}{P^2} \right]^{1/2} \\ \dot{X}_3 &= P \sqrt{(1 - r^4)} [(X_3 - \alpha_1)(X_3 - \alpha_2)(X_3 - \alpha_3)(X_3 + \alpha_4)] \end{aligned} \right\} \quad (5.1)$$

and is thus formally equivalent to the equation for a non-linear oscillator subject to the potential $\pi(X_3)$, where $\pi(X_3)$ is given by

$$\left(\frac{dX_3}{d\tau} \right)^2 + \pi(X_3) = 0$$

and the solution for $X_3(\tau)$ only exists in the region where $\pi(X_3) > 0$.

In Fig. 1 the bounded solution of the coupled mode equation corresponds to X_3 oscillating between A and D and the explosive instability corresponds to X_3 lying on the portion D ∞ of the curve. Also, from Fig. 1 for the instability to occur one requires $X_3(0) > \alpha_3$, so that a threshold exists for the onset of the instability.

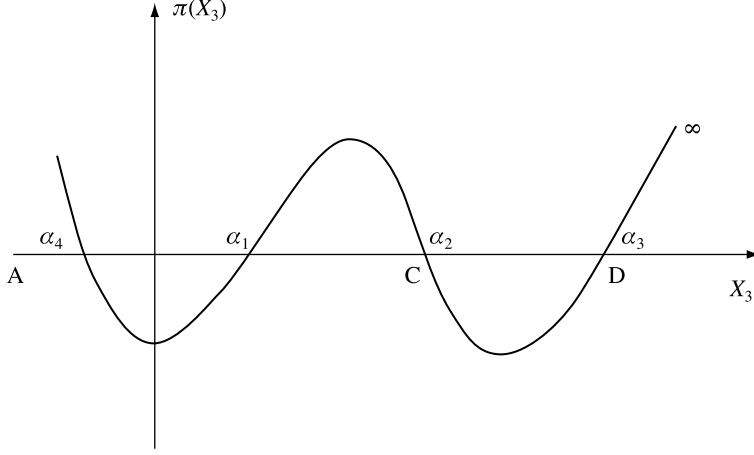


Figure 1. Schematic picture of $\pi(X_3)$ corresponding to (5.1) with four real roots.

When (5.1) has two real and two complex roots it can be written as (all s_j are of the same sign)

$$\int_b^z \frac{dx}{[(t^2 + a^2)(t^2 - b^2)]^{1/2}} = \frac{1}{\sqrt{a^2 + b^2}} nc^{-1} \left(\frac{z}{b}, \frac{a^2}{a^2 + b^2} \right) \quad 0 < b < z.$$

The solution for X_3 is given by

$$X_3 = \frac{q + p a n c \{ [aP \sqrt{(1-r^4)} / (p-q)] (\tau - \tau_0) |k\}}{1 + a n c \{ [aP \sqrt{(1-r^4)} / (p-q)] (\tau - \tau_0) |k\}}, \quad k^2 = \frac{a^2}{a^2 + b^2}. \quad (5.2)$$

In the case of one of the waves common to both triplets being negative (either s_1 or s_2) (5.1) becomes

$$\dot{X}_3 = P \sqrt{(1-r^4)} \left[X_3^4 - 2 \frac{R}{P} X_3^3 + \frac{R^2}{P^2} X_3^2 + \frac{Q^2}{P^2} \right]^{1/2}. \quad (5.3)$$

Equation (5.3) can be written in the form

$$\int_b^z \frac{dx}{[(t^2 + a^2)(t^2 + b^2)]^{1/2}} t = \frac{1}{a} sc^{-1} \left(\frac{z}{b}, \frac{a^2 - b^2}{a^2} \right) \quad b < a.$$

Following Milne-Thomson (1950, p. 27, the details are given in the appendix), the solution of (5.3) is given by

$$X_3 = \frac{q + p a s c \{ [aP \sqrt{(1-r^4)} / (p-q)] (\tau - \tau_0) |k\}}{1 + a s c \{ [aP \sqrt{(1-r^4)} / (p-q)] (\tau - \tau_0) |k\}}, \quad k^2 = \frac{a^2 - b^2}{a^2}, \quad (5.4)$$

where τ_0 is given by

$$\tau_0 = \frac{p-q}{aP \sqrt{(1-r^4)}} sc^{-1} \left[\frac{q - X_3(0)}{a(X_3(0) - p)} \middle| k \right]. \quad (5.5)$$

In the present case

$$p+q = \frac{\sigma - \mu}{\rho - \lambda}, \quad pq = \frac{\lambda\sigma - \mu\rho}{\rho - \lambda}, \quad \mu = ab, \quad \sigma = cd, \quad a+b = -2\lambda, \quad c+d = -2\rho$$

$$\begin{aligned}
 a &= -\lambda \pm \sqrt{(\lambda^2 - \mu)}, & b &= -\lambda \mp \sqrt{(\lambda^2 - \mu)} \\
 \sigma &= \frac{R^2}{2P^2} + \frac{1}{2}\sqrt{\frac{R^4}{P^4} + 4Q^2}, & \mu &= \frac{R^2}{2P^2} - \frac{1}{2}\sqrt{\frac{R^4}{P^4} + 4Q^2} \\
 \rho &= -\frac{R}{2P} \frac{R^2 + \sqrt{R^4/P^4 + 4Q^2}}{\sqrt{R^4/P^4 + 4Q^2}}, & \lambda &= \frac{R}{P} \frac{R^2 + \sqrt{R^4/P^4 + 4Q^2}}{\sqrt{R^4/P^4 + 4Q^2}}.
 \end{aligned}$$

The solution (5.4) is generally periodic with a period of oscillation

$$\frac{4k}{\omega} \quad \text{where} \quad \omega = \frac{aP\sqrt{(1-r^4)}}{p-q}.$$

The solution for the other wave amplitudes can be obtained from (5.4) and the conservation laws (4.1). They are also periodic with the same period. The amplitudes of all four waves become infinite when

$$sc \left[\frac{aP\sqrt{(1-r^4)}}{p-q} (\tau - \tau_0) |k \right] = -\frac{1}{a} \quad (5.6)$$

and the explosion time τ_∞ is given by

$$\tau_\infty = \tau_0 + \frac{1}{\omega} sc^{-1} \left(-\frac{1}{a}, k \right).$$

The growth rate is the reciprocal of the explosion time

$$\Gamma_{\text{growth}} = \frac{1}{\tau_\infty}.$$

The effects of the frequency mismatch and the dissipation ν_i are to introduce threshold values to the initial amplitudes for the explosion to occur and to increase explosion time.

Case 2: $r > 1$

When the s_i (s_1 or s_2 are of different signs) (4.3) becomes

$$\dot{X}_3 = P\sqrt{(r^4 - 1)} \left[-X_3^4 + 2\frac{R}{P}X_3^2 - \frac{R^2}{P^2}X_3^2 + \frac{Q^2}{P^2} \right]^{1/2}. \quad (5.7)$$

In the case where (5.7) has two real α_1, α_2 ($\alpha_1 > \alpha_2$) and two complex roots, it can be written as

$$\dot{X}_3 = P\sqrt{(r^4 - 1)} [(X_3 + \alpha_2)(\alpha_1 - X_3)(X_3^2 - 2\alpha_3X_3 + \alpha_4)]^{1/2}. \quad (5.8)$$

The bounded solutions of the coupled mode equations correspond to X_3 oscillating between A and B on the curve shown in Fig. 2. No explosive solution occurs.

Depending on the assignment of the roots to the values $\alpha_1, \alpha_2, \alpha, \bar{\alpha}$ (complex), where $X_3 > \alpha_2, \alpha, \bar{\alpha}$, from (5.7) the solution for X_3 is given by

$$X_3(\tau) = \frac{q + pb \operatorname{cn}[P\sqrt{(r^4 - 1)}(a^2 + b^2)^{1/2}(\tau - \tau_0), k]}{1 + b \operatorname{cn}[P\sqrt{(r^4 - 1)}(a^2 + b^2)^{1/2}(\tau - \tau_0), k]}, \quad (5.9)$$

where the constant τ_0 is defined as

$$\tau_0 = \frac{1}{P\sqrt{r^4 - 1}} \operatorname{cn}^{-1} \left[\frac{q - X_3(0)}{b(X_3(0) - p)}, k \right], \quad k^2 = \frac{b^2}{a^2 + b^2}.$$

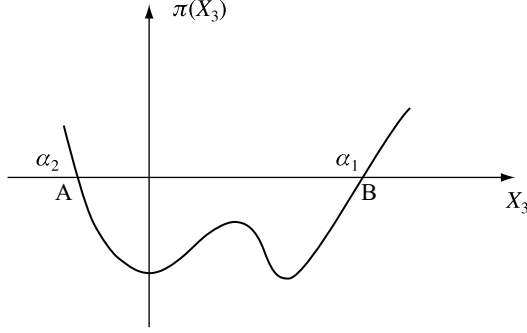


Figure 2. Schematic picture of the potential $\pi(X_3)$ of (5.8) with two real and two complex roots.

The solutions for the other wave amplitudes can be obtained from (5.9) and the conservation laws (4.1).

The solution (5.9) is generally periodic with a period of oscillation $4K/\gamma$, where K is the complete elliptic integral of the first kind:

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{(1 - k^2 \sin^2 \theta)^{1/2}}.$$

6. Soliton solution

For a different choice of ν , η_{k-} , η_{k+} , θ_α and θ_β from (3.1)–(3.4), the following constants of motion are derived:

$$\left. \begin{aligned} X_4 &= r^2 X_3, & X_1 X_2 X_3 &= Q_1^2 \\ \frac{X_1}{s_2 s_4 \delta_+ \delta_{k+}} + \frac{X_2 (1 - \delta_k)}{s_1 s_3 \delta_- \delta_{k-}} &= R_1, \end{aligned} \right\} \quad (6.1)$$

where Q_1 and R_1 are arbitrary constants.

Using (6.1) in (3.3), one obtains

$$\dot{X}_3 = R_1 X_3 \left(X_3^2 - \frac{4Q_1^2 P_1}{R_1^2} X_3 \right)^{1/2}. \quad (6.2)$$

The solution to (6.2) is given by

$$X_3 = \frac{4P_1 Q_1^2 / R_1^2}{\{1 - [c_1 - (2P_1 Q_1^2 / R_1) \tau]\}^2}, \quad (6.3)$$

where

$$P_1 = \frac{(1 - \delta_{k-})}{s_1 s_2 s_3 s_4 \delta_- \delta_+ \delta_{k-} \delta_{k+}}, \quad c_1^2 = 1 - \frac{4P_1 Q_1^2}{R_1^2 X_3(0)}.$$

The solution (6.3) represents a soliton, X_3 , which is limited to a maximum value $4Q_1^2 P_1 / R_1^2 (1 - c^2)$ and tends to zero for large times even for large negative values of time τ . Similarly for the other wave amplitudes X_1 , X_2 and X_4 . The decrease of the amplitudes after the maximum corresponds to the collapse of the waves. This type of soliton solution was obtained in the case of three-wave interactions (Weiland and Wilhelmsson 1977; De et al. 1981).

7. Constants of motion and wave solution (general case)

For the general case, when all v_i 's are different using numerical calculation, individual wave amplitudes have been plotted against time for the system of equations (3.1)–(3.4), where all the variables are in normalized form. It has been shown that though in the absence of dissipation or growth the solutions of the wave amplitudes are unstable (Fig. 3), the inclusion of dissipation can make the amplitudes stable (Fig. 4). Also, it has been shown that only the increment of the value of the frequency mismatch helps to stabilize the wave amplitudes (Fig. 5).

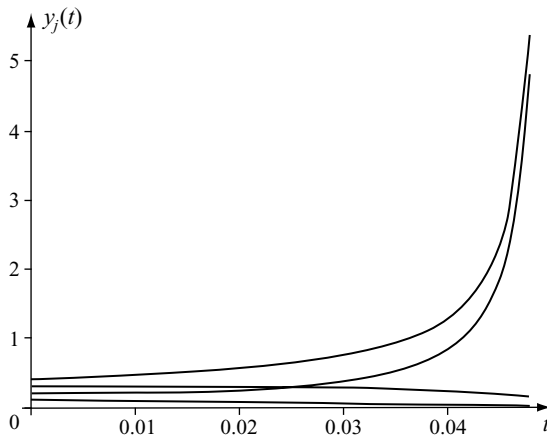


Figure 3. All v_s are zero, $\Delta\omega_- = 0.01$, $\Delta\omega = 0.03$ and $r < 1$.

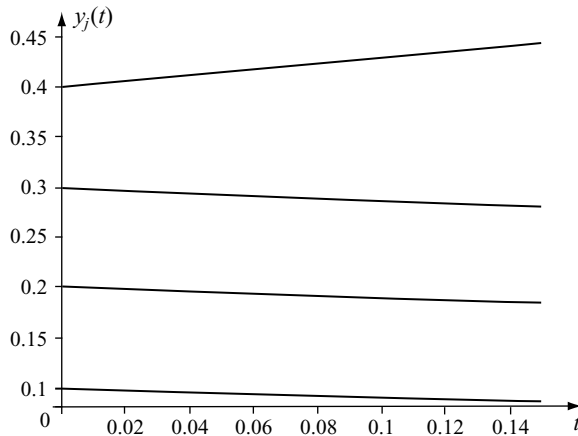


Figure 4. All v_s are non-zero (0.15, 0.25, 0.3, 0.35), $\Delta\omega = 0.03$, $\Delta\omega_- = 0.01$, $r < 1$.

8. Solutions for two coupled three-wave interactions with one triplet unstable

Next, the case has been considered where one triplet is unstable (first) when viewed as an isolated three-wave interaction but the other triplet (second) is not. From (4.2) if $r < 1$ the solution is divergent, but from (4.3) if $r > 1$, which could be interpreted

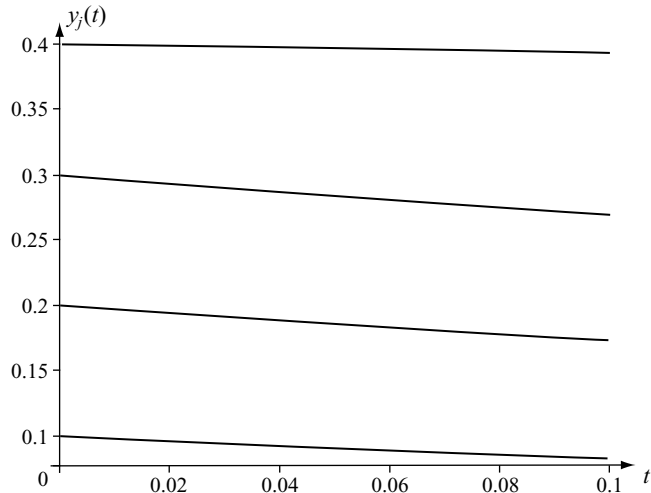


Figure 5. All ν s are zero, $\Delta\omega = 0.25$, $\Delta\omega_- = 0.2$ and $r < 1$.

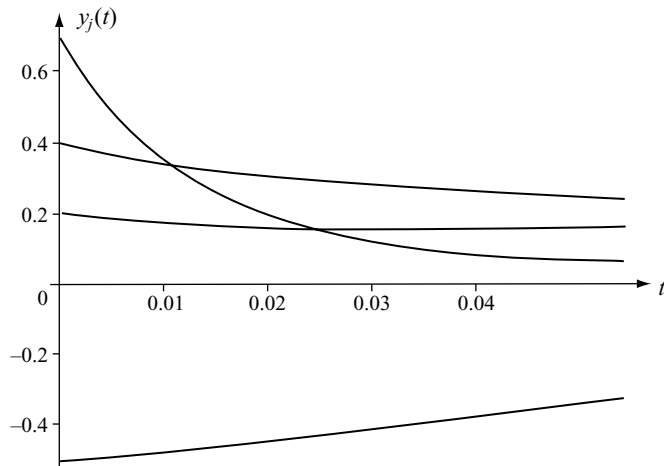


Figure 6. Amplitude against time, where the first triplet contains one negative energy wave.

as the second triplet being stronger than the first, no divergent solution occurs. The significant conclusion is that even though one triplet is explosively unstable by itself, the presence of the second (stronger) triplet can stabilize the solutions, whereas the converse does not happen. In the weakly turbulent case it has been shown that the wave system will be unstable if any explosive triplet exits (Coppi et al. 1969). This is not true for the coherent waves.

The different wave amplitudes are plotted against time for the general case with $r > 1$ where the first triplet contains a negative energy wave. The wave amplitudes are found to be stabilized due to the presence of the second triplet (Fig. 6).

Next, different wave amplitudes are plotted against time when the second wave triplet contains a negative energy wave; the unstable solutions were obtained (Fig. 7).

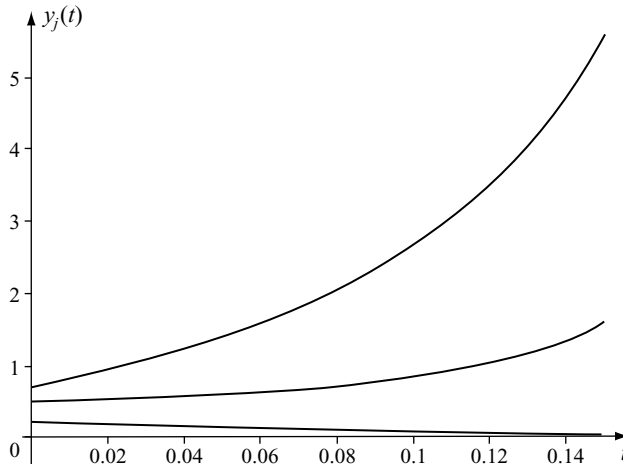


Figure 7. Amplitude against time, where the second triplet contains one negative energy wave.

9. Conclusion

Depending on certain initial conditions, a periodic solution in terms of the Jacobian elliptic function are obtained. The period of oscillation, growth rate and the explosion time have been calculated. It has been shown that the effect of the dissipation and frequency mismatch were to introduce a threshold value and to increase the time of explosion. Depending on different initial conditions periodic, soliton-type and shock-like solutions were obtained. It was found that in the coherent case for two waves triplets, if one is explosively unstable by itself the presence of the second one can stabilize the solutions depending on the relative strength of the coupling factor, while in the incoherent case this does not happen. The coupling constants contain all the information necessary for studying how the efficiency of the wave coupling depends on the plasma parameters. The growth rate, explosion time and threshold value of the excited waves for such a system can be obtained by direct application of my theory. Also, the effects of dissipation and frequency mismatch on these parameters can be calculated.

Appendix

Considering the integral of the type (Akhiezer 1990)

$$\int \frac{dz}{\sqrt{f(z)}} \quad (\text{A } 1)$$

where $f(z) = a_0(z^2 + 2\lambda z + \mu)(z^2 + 2\rho z + \sigma)$ and the coefficients a_0, λ, μ, ρ and σ are real.

If $\lambda = \rho$ then let $z + \lambda = t$ and $f(z) = a_0(t^2 + \alpha)(t^2 + \beta)$, where α and β are real. But if $\lambda \neq \rho$ let $z = (pt + q)/(t + 1)$. Considering there should not be any term with a square power of t , the following two relations would derive p and q :

$$pq + \lambda(p + q) + \mu = 0, \quad pq + \rho(p + q) + \sigma = 0,$$

where p and q are the roots of the quadratic equation

$$X^2 + \frac{\mu - \sigma}{\lambda - \rho} X - \frac{\lambda\sigma - \mu\rho}{\lambda - \rho} = 0.$$

Denoting the roots of the polynomial $z^2 + 2\lambda z + \mu$ by a and b , and the roots of $z^2 + 2\rho z + \sigma$ by c and d , where $\mu = ab$, $2\lambda = -a - b$, $\sigma = cd$, $2\rho = -c - d$.

Equation (A1) turns into integrals of the following types:

$$\begin{aligned} \text{I. } z &= \sqrt{(a^2 - t^2)(b^2 - t^2)} & \text{II. } z &= \sqrt{(a^2 - t^2)(t^2 - b^2)} \\ \text{III. } z &= \sqrt{(a^2 - t^2)(b^2 + t^2)} & \text{IV. } z &= \sqrt{(t^2 - a^2)(b^2 + t^2)} \\ \text{V. } z &= \sqrt{(t^2 + a^2)(t^2 + b^2)} & & (a \text{ and } b \text{ are positive numbers}). \end{aligned}$$

If $t^2 < b^2$, let $t = bx$ and $k^2 = b^2/a^2$, which gives

$$z = ab\sqrt{(1 - x^2)(1 - k^2x^2)},$$

where as if $t^2 > a^2$, let $t = a/x$, which gives

$$z = \frac{a^2}{x^2}\sqrt{(1 - x^2)(1 - k^2x^2)}.$$

As z varies in $[-1, 1]$ in both cases setting $z = \text{sn}(u; k)$, where u is in the interval $[-K, K]$ and

$$K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}.$$

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