J. London Math. Soc. (2) 65 (2002) 103–114
DOI: 10.1112/S002461070100268X

HARMONIC DIFFEOMORPHISMS OF NONCOMPACT SURFACES AND TEICHMÜLLER SPACES

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Introduction

Let $g : M \longrightarrow N$ be a *quasiconformal harmonic* diffeomorphism between noncompact Riemann surfaces M and N. In this paper we study the relation between the map g and the complex structures given on M and N. In the case when M and N are of finite analytic type we derive a precise estimate which relates the map g and the Teichmüller distance between complex structures given on M and N. As a corollary we derive a result that every two quasiconformally related finitely generated Kleinian groups are also related by a harmonic diffeomorphism. In addition, we study the question of whether every *quasisymmetric* selfmap of the unit circle has a quasiconformal harmonic extension to the unit disk. We give a partial answer to this problem. We show the existence of the harmonic quasiconformal extensions for a large class of quasisymmetric maps. In particular it is proved that all *symmetric* selfmaps of the unit circle have a unique quasiconformal harmonic extension to the unit disk.

Let us first mention some closely related results. In [**29**] Wolf showed that there is a unique harmonic quasiconformal map in the homotopy class of a homeomorphism between compact surfaces of finite genus. Sampson [**21**] constructed an injective map between the Teichmüller space of compact surfaces and the space of holomorphic quadratic differentials via the Hopf differentials of the harmonic maps. Wolf showed that this map is a proper and surjective parametrization of the corresponding Teichmüller space via the space of holomorphic quadratic differentials. The first study of the connections between Teichmüller theory and harmonic maps appears in the work of Earle and Eells [**7**]. Regarding this approach we also refer to the work of Tromba [**27**] (see also [**24**, **29**, **30**]).

The case of noncompact surfaces carries additional difficulties. In particular, it is not known whether the homotopy class of a quasiconformal selfmap of the unit disk necessarily contains a unique harmonic quasiconformal representative. This problem is also known as the Schoen conjecture. In [**16**] Li and Tam showed that every regular diffeomorphism of the unit circle with nonvanishing first derivative has a uniformly regular harmonic extension to the unit disk. This extension is a quasiconformal map. In [**17**] the same authors proved the uniqueness part of the Schoen conjecture in the unit disk.

Wan [**28**] showed that for every holomorphic function ϕ on the unit disk with the bounded Bers norm there exists the associated quasiconformal harmonic diffeomorphism of the unit disk such that its Hopf differential is equal to ϕ. In particular

Received 15 June 1999; revised 1 March 2001.
2000 *Mathematics Subject Classification* 32G15.

he obtained an injective map from the Bers space into the universal Teichmüller space. In this paper we show that the map introduced by Wan can be defined in the case of an arbitrary hyperbolic Riemann surface.

An interesting question related to the Schoen conjecture is to compare the Bers norm of the Hopf differential and the Teichmüller distance of the resulting point in the Teichmüller space with the base point. In the case of Riemann surfaces of finite analytic type we obtain the needed estimates by applying the main inequality of Reich and Strebel for quasiconformal maps.

Symmetric selfmaps of the unit circle are a natural generalization of diffeomorphisms of the unit circle. They form a large class among all quasisymmetric maps. In particular, there are symmetric maps which are not even absolutely continuous (see [**5**]). It was conjectured that for symmetric maps there is a solution to the Schoen conjecture (see for example [**13**]). As an application of the well-known characterization of the symmetric maps (see [**8**, **9**, **11**, **25**]) with some additional considerations we prove this conjecture.

1. *Preliminary results*

We recall a few basic definitions. Let (M,σ) and (N,ρ) be Riemann surfaces endowed with the hyperbolic metrics σ and ρ respectively. The hyperbolic metric is complete and has constant Gaussian curvature -1. If $f : M \longrightarrow N$ is a C^2 map then f is said to be *harmonic* with respect to ρ if

$$f_{z\bar{z}} + \left(\frac{\partial \log \rho}{\partial w} \circ f\right) f_z f_{\bar{z}} = 0, \tag{1}$$

where z and w are the local parameters on M and N respectively. Also f satisfies equation (1) if and only if

$$((\rho^2 \circ f) f_z \overline{(f_{\bar{z}})})(z)\, dz^2 = \varphi(z)\, dz^2 \tag{2}$$

is a holomorphic quadratic differential on M. The differential $\varphi(z)\, dz^2$ is called the *Hopf* differential and we write $\mathrm{Hopf}(f) = \varphi(z)\, dz^2$.

For $g : M \longrightarrow N$ the energy integral is given by

$$E(g,\rho) = \int_M \int (|\partial g|^2 + |\bar{\partial} g|^2)\, dV_\sigma.$$

Here

$$\partial g = \frac{(\rho \circ g)}{\sigma} gz, \quad \bar{\partial} g = \frac{(\rho \circ g)}{\sigma} g\bar{z}$$

are the partial derivatives taken with respect to the metrics ρ and σ, and dV_σ is the volume element on (M,σ). Assume that energy integral of f is bounded. Then f is harmonic if and only if f is a critical point of the corresponding functional where the homotopy class of f is the range of this functional.

The following are Bochner's well-known formulae (see [**23**, **24**]).

$$\Delta_\sigma \log |\partial f| = |\partial f|^2 - |\bar{\partial} f|^2 - 1$$
$$\Delta_\sigma \log |\bar{\partial} f| = |\bar{\partial} f|^2 - |\partial f|^2 - 1.$$

Here Δ_σ denotes the Laplace operator taken with respect to the metric ρ.

If f is a *harmonic* diffeomorphism then the complex dilatation

$$\mathrm{Belt}(f) = \frac{f_{\bar{z}}}{f_z} \frac{d\bar{z}}{dz} = \mu \frac{d\bar{z}}{dz},$$

is of the form

$$\mu(z)\frac{d\bar{z}}{dz} = \frac{\sigma^{-2}(z)\varphi(z)}{|\partial f|^2(z)}\frac{d\bar{z}}{dz}. \tag{3}$$

If $g: M \longrightarrow N$ is a quasiconformal map then the pair (N, g) is said to be a marked Riemann surface. We identify two marked surfaces (N_1, g_1), (N_2, g_2) if and only if

$$g_2 \circ g_1^{-1}: N_1 \longrightarrow N_2$$

is homotopic to a conformal map. The corresponding quotient space of all marked surfaces M is called the Teichmüller space of M or just Teich(M). The Teichmüller metric on Teich(M) is given by

$$d(\tau_1, \tau_2) = \inf_f \frac{1}{2} \log \frac{1 + \|\mu\|_\infty}{1 - \|\mu\|_\infty},$$

where (N_1, g_1) and (N_2, g_2) are two representatives of the points $\tau_1, \tau_2 \in$ Teich(M). The infimum is taken over all quasiconformal maps f homotopic to $g_2 \circ g_1^{-1}$. Here $\mu(z)\, d\bar{z}/dz$ is the complex dilatation of the map f. Since f is quasiconformal, we have $\mu \in B_1(M)$, where $B_1(M)$ is the unit ball in the Banach space $L^\infty(M)$.

Recall that the Bers space BQD(M) is the space of all holomorphic quadratic differentials $\varphi(z)\, dz^2$ on M, which are bounded in the following sense

$$\|\varphi\| = \sup_{p \in M} |\varphi(p)|\sigma^{-2}(p) < \infty.$$

$\|\varphi\|$ denotes the Bers norm of $\varphi(z)\, dz^2$.

Wan [**28**] discovered a criteria for a harmonic diffeomorphism to be a quasiconformal map. Assume that a harmonic diffeomorphism f satisfies the condition that $(M, -|\partial f|^2 |dz|^2)$ is a complete metric space. The necessary and sufficient condition for f to be quasiconformal is that Hopf(f) $\in$ BQD(M). In the same paper Wan proved the following proposition.

PROPOSITION 1.1. *Let Δ be the unit disk and $\varphi(z)\, dz^2 \in$* BQD(Δ). *Then there is a unique harmonic diffeomorphism $f: \Delta \longrightarrow \Delta$, which fixes the points* 1, *i, and* -1 *such that* Hopf(f) $= \varphi(z)\, dz^2$ *and that $(\Delta, |\partial f|^2 |dz|^2)$ is a complete metric space. This induces an injection map $F:$* BQD(Δ) $\longrightarrow$ Teich(Δ) *which is continuous with respect to the Teichmüller metric.*

REMARK 1.1. Teich(Δ) is called the universal Teichmüller space. The map F is defined below for all hyperbolic surfaces. The case of the unit disk will be studied in Section 3.

Before we consider the case of Riemann surfaces of finite analytic type we need to generalize Proposition 1.1 to all hyperbolic Riemann surfaces. Let M be a hyperbolic Riemann surface and let $\pi: \Delta \longrightarrow M$ denote the standard universal covering map. There is a Fuchsian group Γ such that the quotient surface Δ/Γ is conformally equivalent to M. Thus we can replace M with the quotient surface Δ/Γ.

Let $\varphi(z)\, dz^2 \in$ BQD(Δ) be a quadratic differential which is equivariant with the action of the group Γ. By Proposition 1.1 there is a quasiconformal harmonic diffeomorphism $f: \Delta \longrightarrow \Delta$ such that

$$\text{Hopf}(f)(A')^2 = \varphi(z)\, dz^2$$

and $(\Delta, |\partial f|^2 |dz|^2)$ is a complete metric space. Since $\varphi(z)\, dz^2$ is equivariant with the action of the group Γ we know that

$$\mathrm{Hopf}(f \circ A) = \mathrm{Hopf}(f)$$

for all $A \in \Gamma$. By the uniqueness part of Proposition 1.1 we conclude that there is a unique Mobiüs transformation B such that

$$B \circ f = f \circ A.$$

Therefore we obtain a new Fuchsian group Γ', $B \in \Gamma'$ and

$$f^{-1}\Gamma' f = \Gamma.$$

Since f is a quasiconformal map, Δ/Γ' is a new Riemann surface.

Thus we have shown that for a fixed $\varphi(z)\,dz^2 \in \mathrm{BQD}(\Delta, \Gamma)$ we have a unique quasiconformal harmonic diffeomorphism f, such that $\mathrm{Hopf}(f) = \varphi(z)\,dz^2$ and

$$f : \Delta/\Gamma \longrightarrow \Delta/\Gamma'.$$

The fact that $(\Delta/\Gamma,\ |\partial f|^2|dz|^2)$ is a complete metric space follows from the completeness of the metric space $(\Delta,\ |\partial f|^2|dz|^2)$.

We define the map $F : \mathrm{BQD}(\Delta/\Gamma) \longrightarrow \mathrm{Teich}(\Delta/\Gamma)$ as

$$F(\varphi) = (\Delta/\Gamma', f),$$

where $(\Delta/\Gamma, f)$ is a marked Riemann surface and represents a point in $\mathrm{Teich}(\Delta/\Gamma)$.

LEMMA 1.1. *Let M be a hyperbolic Riemann surface conformally equivalent to the quotient surface* (Δ/Γ). *Then the map $F : \mathrm{BQD}(M) \longrightarrow$* Teich$(M)$ *defined above is an injective and continuous map with respect to the Teichmüller metric on* Teich(M).

Proof. It is well known that the Teichmüller space $\mathrm{Teich}(\Delta/\Gamma)$ is naturally embedded in the universal Teichmüller space $\mathrm{Teich}(\Delta)$ (see [**10**, **14**]). In other words, each point in the space $\mathrm{Teich}(\Delta)$ is uniquely determined by the corresponding quasisymmetric map of the unit circle. Li and Tam (see [**17**]) showed that every quasisymmetric map of the unit circle has at most one quasiconformal harmonic extension provided that $(\Delta/\Gamma,\ |\partial f|^2|dz|^2)$ is a complete metric space. Therefore the map F is an injection. Continuity of the map F follows from the related result of Proposition 1.1. □

REMARK 1.2. It is easy to see that the map F is a diffeomorphism in the case of an arbitrary hyperbolic surface M. In particular, F is an open map.

2. *Estimates on the Teichmüller distance*

Recall that a Riemann surface M is said to be of *finite analytic type* if and only if M is obtained from a closed Riemann surface of finite genus g by deleting n points, $n \in N$. Then M is said to be of type (g, n). Since we study the *harmonic* maps with respect to the hyperbolic metric we need to make sure that our surfaces are hyperbolic. A surface of finite analytic type is hyperbolic if and only if the inequality

$$3g - 3 + 2n > 0$$

holds.

If σ denotes the hyperbolic metric on a Riemann surface M then (M, σ) is a complete, but may be a noncompact, manifold. On the other hand, if M is of finite analytic type then Teich(M) and BQD(M) are of finite dimension.

If $\varphi(z)\,dz^2 \in \mathrm{BQD}(M)$, then

$$\|\varphi\|_1 = \int_M\int |\varphi|$$

defines a Banach norm on the space BQD(M). The norms $\|\varphi\|$ and $\|\varphi\|_1$ are different, but there is a positive constant C, depending only on the type (g, n) such that

$$C^{-1}\|\varphi\| \leqslant \|\varphi\|_1 \leqslant C\|\varphi\|,$$

for all $\varphi \in \mathrm{BQD}(M)$.

Let (N, ρ) be a Riemann surface with the hyperbolic metric ρ and $f, g : M \longrightarrow N$ be two homotopic quasiconformal maps. Both maps have continuous extension to the punctures and likewise it is assumed that the homotopy extends to the punctures and keeps each puncture fixed. If $\mu(z)\,d\bar{z}/dz$ and $\tilde{\nu}(w)d\bar{w}/dw$ are the complex dilatations of f and g^{-1} respectively then for every $\varphi(z)\,dz^2 \in \mathrm{BQD}(M)$ we have

$$\|\psi\|_1 \leqslant \int_M\int |\psi| \frac{|1-\mu\frac{\psi}{|\psi|}|^2}{1-|\mu|^2}\frac{|1+(\tilde{\nu}\circ f)\theta\frac{\psi}{|\psi|}|^2}{1-|\tilde{\nu}\circ f|^2}, \tag{4}$$

where

$$\theta = \frac{1-\bar{\mu}\frac{\bar{\psi}}{|\psi|}}{1-\mu\frac{\psi}{|\psi|}}.$$

Equation (4) is known as the *main inequality* and is due to Reich and Strebel. This inequality holds for all hyperbolic Riemann surfaces if we assume that the differential $\psi(z)\,dz^2$ is integrable. For the proof of this inequality and its application to the Teichmüller theory we refer to [**4**, **10**, **20**].

Suppose now that f is a harmonic diffeomorphism. The complex dilatation of f is given by

$$\mu(z)\frac{d\bar{z}}{dz} = \left(|\mu|\frac{|\varphi|}{\varphi}\right)(z)\frac{d\bar{z}}{dz},$$

where $\varphi(z)\,dz^2 = \mathrm{Hopf}(f) \in \mathrm{BQD}(M)$. We apply the main inequality (4). Take $\varphi(z)\,dz^2 = \psi(z)\,dz^2$. Since $|\theta|\varphi|/\varphi| = 1$ we estimate the expression under the integral in (4) as

$$\frac{|1-\mu\frac{\psi}{|\psi|}|^2}{1-|\mu|^2}\frac{|1+(\tilde{\nu}\circ f)\theta\frac{\psi}{|\psi|}|^2}{1-|\tilde{\nu}\circ f|^2} \leqslant \frac{1-|\mu|}{1+|\mu|}\frac{1+|\tilde{\nu}\circ f|}{1-|\tilde{\nu}\circ f|}.$$

Therefore the right side in (4) becomes

$$\|\varphi\|_1 \leqslant \int_M\int |\varphi|\frac{1-|\mu|}{1+|\mu|}\frac{1+|\tilde{\nu}\circ f|}{1-|\tilde{\nu}\circ f|}.$$

We obtain

$$\|\varphi\|_1 \leqslant \frac{1+\|\tilde{\nu}\|_\infty}{1-\|\tilde{\nu}\|_\infty}\int_M\int |\varphi|\frac{1-|\mu|}{1+|\mu|}. \tag{5}$$

Let τ denote the point in Teich(M) represented by the pair (N, f). If τ_0 is the origin in Teich(M) (τ_0 is represented by (M, id)) then the Teichmüller distance $d(\tau_0, \tau)$ is given by

$$d(\tau_0, \tau) = \inf \frac{1}{2}\log\frac{1+\|\nu\|_\infty}{1-\|\nu\|_\infty}, \tag{6}$$

where $\nu(z)\,d\bar{z}/dz$ is the complex dilatation of g and the infimum is taken over all quasiconformal maps g homotopic to f. It is well known that $\|\nu\|_\infty = \|\tilde{\nu}\|_\infty$ (note that $\nu(z)\,d\bar{z}/dz$ is defined on M and $\tilde{\nu}(w)\,d\bar{w}/dw$ is defined on N). By varying the map g in (5) over the whole homotopy class of f and by applying (6) we have

$$\|\varphi\|_1 \leqslant \left(\int_M\!\!\int |\varphi|\frac{1-|\mu|}{1+|\mu|}\right) e^{2d(\tau_0,\tau)}, \tag{7}$$

where $d(\tau_0,\tau)$ is the Teichmüller distance between τ_0 and τ. If ∂f and $\bar{\partial} f$ are the partial derivatives of f taken with respect to the metrics σ and ρ we have

$$|\varphi|\frac{1-|\mu|}{1+|\mu|} = |\varphi|\frac{1-|\mu|^2}{(1+|\mu|)^2}.$$

From $|\varphi| = \sigma^2|\partial f||\bar{\partial} f|$ we obtain

$$|\varphi|\frac{1-|\mu|}{1+|\mu|} = \frac{|\partial f|^2-|\bar{\partial} f|^2}{(1+|\mu|)^2}\sigma^2. \tag{8}$$

Also

$$\int_M\!\!\int \frac{(|\partial f|^2-|\bar{\partial} f|^2)}{(1+|\mu|)^2}dV_\sigma \leqslant \int_N\!\!\int dV_\rho = \mathrm{Area}(N,\rho). \tag{9}$$

By replacing the equality (8) in (7), we obtain

$$\|\varphi\|_1 \leqslant \left(\int_M\!\!\int \frac{(|\partial f|^2-|\bar{\partial} f|^2)}{(1+|\mu|)^2}dV_\sigma\right) \leqslant e^{2d(\tau_0,\tau)}.$$

By using (9) we estimate the right side of the inequality (7) and we obtain

$$\|\varphi\|_1 \leqslant \mathrm{Area}(N,\rho)e^{2d(\tau_0,\tau)}. \tag{10}$$

Let $\mathrm{Area}(N,\rho) = A(g,n)$. $A(g,n)$ is a constant which depends only on the type (g,n). Thus, we have proved the following theorem.

THEOREM 2.1. *Suppose that M and N are hyperbolic Riemann surfaces of finite analytic type (g,n) and let $f: M \longrightarrow N$ be a quasiconformal harmonic diffeomorphism. If $\varphi(z)\,dz^2$ denotes the Hopf differential of f then*

$$\int_M\!\!\int |\varphi| \leqslant A(g,n)e^{2d(\tau_0,\tau)}. \tag{11}$$

Here $\tau \in \mathrm{Teich}(M)$ is the point in the Teichmüller space represented by the pair (N,f) and τ_0 is the origin in $\mathrm{Teich}(M)$.

REMARK 2.1. The proof above can be used to prove a more general form of Theorem 2.1. In fact, this theorem holds for all harmonic diffeomorphisms (not necessarily quasiconformal) between any two hyperbolic Riemann surfaces if we assume that $\mathrm{Hopf}(f)$ is an integrable (2, 0) holomorphic form (see **[18]** for the extension of the main inequality to the case of nonquasiconformal diffeomorphisms).

Since the Banach norms $\|\varphi\|_1$ and $\|\varphi\|$ are equivalent, inequality (11) can be written as

$$\|\varphi\| \leqslant C(g,n)e^{2d(\tau_0,\tau)}, \tag{12}$$

where $C(g,n)$ is a constant depending only on the type (g,n).

Let $F : \mathrm{BQD}(M) \longrightarrow \mathrm{Teich}(M)$ be the map defined by Lemma 1.1. Then inequality (12) yields an estimate on the behaviour of the function F. In particular F is a proper map. This means that if $\varphi_n(z)\,dz^2$ is a sequence in $\mathrm{BQD}(M)$ and $\|\varphi_n\| \to \infty$, then $d(\tau_0, \tau_n) \to \infty$ where $\tau_n = F(\varphi_n)$. Next we prove the following theorem.

THEOREM 2.2. *If M is a hyperbolic Riemann surface of finite analytic type (g, n), then the map $F : \mathrm{BQD}(M) \longrightarrow \mathrm{Teich}(M)$ satisfies the growth condition*

$$\|\mathrm{Hopf}(f)\| \leqslant C(g,n)e^{2d(\tau_0,\tau)}, \tag{13}$$

where $F(\mathrm{Hopf}(f)) = \tau$ and $C(g,n)$ is a constant depending only on the type (g,n). In particular F is a proper homeomorphism of BQD(M) *onto* Teich(M).

REMARK 2.2. It is possible to show that the above estimate is the best possible in the sense that the majorant function in inequality (13) is the best possible. This can be verified when M is the square punctured torus.

Proof of Theorem 2.2. Inequality (13) follows from inequality (12). The fact that the map f is proper follows directly from (13). We already know by Lemma 1.1 that F is a continuous injection. The surjectivity of F is a consequence of the fact that Teich(M) is a finite dimensional metric space. □

REMARK 2.3. Actually F is a real analytic map. This follows from the corresponding result for the universal Teichmüller space.

COROLLARY 2.1. *Let M and N be hyperbolic Riemann surfaces of type (g,n), and let $p_1, \ldots, p_n$ and $\tilde{p}_1, \ldots, p_n$ denote the punctures on M and N respectively. Let $g : M \longrightarrow N$ be a homeomorphism which extends continuously to the punctures such that $g(\tilde{p}_i) = \tilde{p}_i$, $1 \leqslant i \leqslant n$. Then there is a unique quasiconformal harmonic diffeomorphism $f : M \longrightarrow N$ such that f is homotopic to g and in particular $f(p_i) = \tilde{p}_i$.*

Proof. It is well known that there is a quasiconformal h which is homotopic to g (see [**1**, **10**]). Then the theorem follows from the surjectivity of the map F which is proved in Theorem 2.3 □

REMARK 2.4. In [**12**], Jost asked whether the Teichmüller space of a closed Riemann surface with finitely many disks removed, can be parametrized by the harmonic diffeomorphisms. In this case the homotopy is not required to fix the boundary of deleted disks and therefore the dimensions of the corresponding Teichmüller space and the space of quadratic differentials are finite. One can apply a similar procedure as in the proof of Theorem 2.2 and prove that there is the desired parametrization by harmonic diffeomorphisms.

Theorem 2.2 can be applied to the deformation theory of Kleinian groups.

COROLLARY 2.2. *Assume that Γ and Γ' are finitely generated Kleinian groups without elliptic transformations acting on the Riemann sphere $\bar{\mathbf{C}}$. Let $g : \bar{\mathbf{C}} \longrightarrow \bar{\mathbf{C}}$ be a quasiconformal conjugation map between Γ and Γ', $g^{-1}\Gamma g = \Gamma'$. Then there is a unique quasiconformal map f such that*
(1) $f^{-1}\Gamma f = \Gamma'$;

(2) $f^{-1} \circ g$ *is homotopic to the identity map relative the limit set of the group* Γ*;*

(3) f *is harmonic with respect to the hyperbolic metric which is given on each component of the set of discontinuity.*

Proof. Let Ω, Ω' and Λ, Λ' be respectively the domains of discontinuity and the limit sets of the groups Γ and Γ'. Since Γ and Γ' act freely and properly discontinuously on Ω, Ω' respectively, the quotients Ω/Γ, Ω'/Γ' are well defined. By the Ahlfors finiteness theorem, Ω/Γ and Ω/Γ' are finite unions of Riemann surfaces of finite analytic type, which are denoted by $M_1, \ldots, M_n$ and $N_1, \ldots, N_n$ respectively.

The quasiconformal map g induces the maps $\tilde{g}_i : M_i \longrightarrow N_i$, $1 \leqslant i \leqslant n$. Each map $\tilde{g}_i$ is a quasiconformal map. By Corollary 2.1 we know that there are *unique* quasiconformal harmonic maps $\tilde{f}_i : M_i \longrightarrow N_i$, satisfying the condition that $\tilde{f}_i$ is homotopic to $\tilde{g}_i$ (*rel* punctures if any). Let $\tilde{\mu}_i$ be the Beltrami dilatation of $\tilde{f}_i$ ($\tilde{\mu}_i$ is given on M_i). Denote by μ the lift of $\tilde{\mu}_i$ given on Ω. The differential μ is defined on the set of the discontinuity. We extend the μ onto the whole complex plane by letting μ to be zero on the limit set. Let h be the solution of the Beltrami equation $h_{\bar{z}} = \mu h_z$. The map h is a quasiconformal map and it is conformal if and only if the differential μ is zero almost everywhere. Therefore, h is equivariant with the action of the group Γ, so we may define

$$h\Gamma h^{-1} = \Gamma_1.$$

Γ_1 is a finitely generated Kleinian group. Denote by Ω_1, Λ_1 respectively the corresponding domain of the discontinuity and the limit set of the group Γ_1. The quotient Ω_1/Γ_1 is a finite union of Riemann surfaces $R_1, \ldots, R_n$. The map h induces the quasiconformal maps

$$\tilde{h}_i : M_i \longrightarrow R_i,$$

and we have

$$\mathrm{Belt}(\tilde{h}_i) = \mathrm{Belt}(\tilde{f}_i).$$

We conclude that the maps $k_i = \tilde{h}_i \circ (\tilde{f}_i)^{-1}$ are conformal maps. Since k_i is conformal we find that the map $\tilde{h}_i$ is a harmonic quasiconformal map. h is the lift of harmonic maps $\tilde{h}_i$ and we see that the map h is a harmonic quasiconformal map with respect to the hyperbolic metric on Ω_1.

Let

$$g \circ h^{-1} = \alpha.$$

Once again there are induced quasiconformal maps $\tilde{\alpha}_i R_i \longrightarrow N_i$, satisfying the equality $\tilde{\alpha}_i = \tilde{g}_i \circ (\tilde{h}_i)^{-1}$. Since $\tilde{g}_i$ is homotopic to $\tilde{f}_i$ and $f_i \circ (\tilde{h}_i)^{-1}$ is a conformal map, we conclude that the maps $\tilde{\alpha}_i$ are homotopic to some conformal maps. Therefore the quasiconformal map α is a trivial deformation of Ω_1/Γ_1. The classical theorem (see [**19**]) asserts that α is a trivial deformation of $(\mathbf{C}, \Gamma_1)$ as well. Recall that the deformation space $\mathrm{Def}(\mathbf{C})$ is the space of all quasiconformal maps defined on $\mathbf{C}$, where we identify two maps q_1 and q_2 if $q_1 \circ (q_2)^{-1}$ is a Möbius transformation. Also $\mathrm{QC}_0(\mathbf{C}, \Gamma_1)$ denotes the spaces of all trivial deformations, that is all quasiconformal maps on $\mathbf{C}$ which are isotopic to the identity map $\mathrm{rel}\,\Lambda_1$. By the definition we have

$$\mathrm{Teich}(\mathbf{C}, \Gamma_1) = \mathrm{Def}(\mathbf{C})/\mathrm{QC}_0(\mathbf{C}, \Gamma_1).$$

Since α is a trivial deformation of $(\mathbf{C}, \Gamma_1)$ we conclude that there is a Möbius transformation A, which agrees with the map α on the limit set Λ_1. Also $A(\Omega_1) = \Omega'$.

Now we construct the map f which satisfies the conditions of this corollary. Define the map f by

$$f = A \circ h.$$

The quasiconformal map f is also a harmonic map with respect to the hyperbolic metric on the open set Ω'. Since $g = \alpha \circ h$ and also A agrees with α on Λ_1, we conclude that f and g agree on the limit set Λ. In addition we have

$$f \circ \Gamma \circ f^{-1} = \Gamma'.$$

Therefore the map f satisfies conditions (1), (2) and (3), and the corollary is proved. □

3. *Symmetric maps of the unit circle*

The aim of this section is to give a partial answer to the Schoen conjecture about existence of harmonic quasiconformal maps with prescribed boundary values. We study the map $F : \mathrm{BQD}(\Delta) \longrightarrow \mathrm{Teich}(\Delta)$ introduced in Section 1. Let $\mathrm{BQD}_0(\Delta)$ be a closed linear subspace of $\mathrm{BQD}(\Delta)$ given by

$$\mathrm{BQD}_0(\Delta) = \{\varphi(z)\,dz^2 \in \mathrm{BQD}(\Delta) : \lim_{r\to 1}(\sup_{1-\rho<|z|<1} \rho^{-2}(z)|\varphi(z)|) = 0\}.$$

$\mathrm{BQD}_0(\Delta)$ is a Banach space. Our aim is to describe the range of the restriction $F|_{\mathrm{BQD}_0(\Delta)}$. First we recall the following alternative definition of the universal Teichmüller space. Let $B_1(\Delta)$ be the unit ball in the Banach space $L^\infty(\Delta)$ of all essentially bounded measurable functions defined on the unit disk Δ. We say that two elements μ, $\nu \in M(\Delta)$ are equivalent $\mu \sim \nu$ if the normalized quasiconformal maps f^μ and f^ν of the unit disk coincide on the unit circle. We say that a map is normalized if fixes the points 1, i and -1. Here we use the standard notation $\mathrm{Belt}(f^\mu) = \mu$ and $\mathrm{Belt}(f^\nu) = \nu$. The quotient space $M/\sim$ is the universal Teichmüller space. If $\mu \in M(\Delta)$, the corresponding point in $\mathrm{Teich}(\Delta)$ is denoted by $[\mu]$.

We introduce the following subspace $T_0(\Delta)$ of $\mathrm{Teich}(\Delta)$. For $\tau \in \mathrm{Teich}(\Delta)$, the boundary dilatation is given by

$$b(\tau) = \lim_{r\to 0}\left(\{\sup_{\nu\in[\mu]=\tau} \|\nu(z)\|_\infty : 1 - r < |z| < 1\}\right).$$

If $b(\tau) = 0$ then $\tau \in T_0(\Delta)$. $T_0(\Delta)$ is a closed subspace of the universal Teichmüller space. Moreover $T_0(\Delta)$ is the maximal topological group contained in $\mathrm{Teich}(\Delta)$ (see [**11**]). It was conjectured (see [**13**]) that

$$F(\mathrm{BQD}_0(\Delta)) = T_0(\Delta). \tag{14}$$

In order to prove equality (14) first we show that $F(\mathrm{BQD}_0(\Delta)) \subset T_0(\Delta)$. This fact is already known (see [**13**]). For the sake of completeness we present another proof.

Lemma 3.1. $F(\mathrm{BQD}_0(\Delta)) \subset T_0(\Delta)$.

Proof. Given $\varphi(z)\,dz^2 \in \mathrm{BQD}_0(\Delta)$ set $\tau = F(\varphi(z)\,dz^2)$ and let $f : \Delta \longrightarrow \Delta$ be the corresponding harmonic diffeomorphism $\mathrm{Hopf}(f) = \varphi(z)\,dz^2$. Then $\mathrm{Belt}(f) = \mu(z)\,d\bar{z}/dz$ is an element of the equivalence class τ. Equation (3) asserts that

$$\mu(z)\frac{d(\bar{z})}{dz} = \frac{\sigma^{-2}(z)\varphi(z)}{|\partial f|^2(z)}\frac{d\bar{z}}{dz}.$$

Since $(\Delta, |\partial f|^2|dz|^2)$ is a complete metric space, we have $|\partial f|(z) \geqslant 1$ for all $z \in \Delta$. Also $\varphi(z)\,dz^2 \in \mathrm{BQD}_0(\Delta)$ and we obtain

$$\lim_{r\to 1}(\sup\{|\mu(z)| : 1 - r < |z| < 1\}) = 0.$$

Therefore the boundary dilatation of τ is equal to zero, $b(\tau) = 0$, and $\tau \in T_0(\Delta)$. □

Let $q : K \longrightarrow K$ be a quasisymmetric map of the unit circle. The map q is said to be symmetric (see [**11**, **25**]) if

$$\lim_{t\to 0} \frac{q(e^{i(t+t_0)}) - q(e^{it_0})}{q(e^{it_0}) - q(e^{(it_0-t)})} = 1, \tag{15}$$

uniformly in t for all points $z_0 \in K$. Here K denotes the unit circle. The well-known characterization of Strebel and Fehlmann (see [**9**, **10**, **25**]) states that $\tau \in T_0(\Delta)$ if and only if the corresponding quasisymmetric map is symmetric. Symmetric maps form a very large class. For an example all maps of the class C^1 with nonvanishing first derivate are symmetric. This follows from the work of Strebel [**25**]. In general, a symmetric map q may not be even absolutely continuous (see [**5**]). In particular if q is a real analytic map with nonvanishing first derivate then the corresponding point τ in the universal Teichmüller space belongs to the subspace $T_0(\Delta)$, $\tau \in T_0(\Delta)$.

Denote by $N(\Delta)$ the subset of $T_0(\Delta)$ such that $\tau \in N(\Delta)$ if the corresponding quasisymmetric map $q : K \longrightarrow K$ is a real analytic map with nonvanishing first derivate. In [**11**] Gardiner and Sullivan showed that the set $N(\Delta)$ is dense in $T_0(\Delta)$ with respect to the Teichmüller metric.

Lemma 3.2. *$N(\Delta)$ is dense in $T_0(\Delta)$, with respect to the Teichmüller metric.*

On the other hand Li and Tam [**16**] proved that every C^1 quasisymmetric map $q : K \longrightarrow K$ with a nonvanishing first derivate has a quasiconformal harmonic extension (see also [**2**]). By using the estimates of Yau and Chang, Tam and Wan [**26**] proved the following proposition.

Proposition 3.1. *There is a universal constant $P > 0$, such that the ball of radius P, centred at an arbitrary element of $N(\Delta)$, is contained in the open set $F(\mathrm{BQD}(\Delta)) \subset$ Teich(Δ).*

Now we prove the main results of this section.

Theorem 3.1. *The map $F : \mathrm{BQD}(\Delta) \longrightarrow$ Teich(Δ) satisfies $F(\mathrm{BQD}_0(\Delta)) = T_0(\Delta)$. Moreover, let P be the constant given by Proposition 3.1. Then the ball of radius P centred at τ, $\tau \in T_0(\Delta)$, is contained in the open set $F(\mathrm{BQD}(\Delta)) \subset$ Teich(Δ). In particular, every symmetric map of the unit circle can be extended to a unique harmonic quasiconformal map of the unit disk.*

Proof. Let $\tau \in T_0(\Delta)$. By Lemma 3.2 there is a sequence $\{\tau_n\}$, $\tau_n \in N(\Delta)$ such that $d(\tau, \tau_n) \longrightarrow 0$, where d is the Teichmüller distance. There exists an integer $n_0 \in N$, such that for $n > n_0$ we have $d(\tau, \tau_n) < P$. This shows that every point $\tau \in T_0(\Delta)$ is contained in the open set $F(\mathrm{BQD}(\Delta))$. Also, there is a quadratic differential $\varphi(z)\,dz^2 \in \mathrm{BQD}(\Delta)$, such that $F(\varphi) = \tau$. We want to show that $\varphi(z)\,dz^2$ belongs to $\mathrm{BQD}_0(\Delta)$.

Let $\varphi_n(z)\,dz^2$ denote a sequence in $\mathrm{BQD}(\Delta)$ such that $F(\varphi_n) = \tau_n$. Since the quasisymmetric map associated to τ_n, is real analytic, by the result of Tam and Wan (see [**26**]) we find that $\varphi_n(z)\,dz^2 \in \mathrm{BQD}_0(\Delta)$. F is a diffeomorphism of $\mathrm{BQD}(\Delta)$ onto the open set $F(\mathrm{BQD}(\Delta)) \subset T(\Delta)$. Therefore from the fact that $\tau_n \to \tau$ in the Teichmüller metric, we have $\varphi_n \to \varphi$ in the Bers norm. $\mathrm{BQD}_0(\Delta)$ is a closed subspace of $\mathrm{BQD}(\Delta)$. Therefore $\varphi(z)\,dz^2 \in \mathrm{BQD}_0(\Delta)$. We have proved that $T_0(\Delta) \subset F(\mathrm{BQD}_0(\Delta))$. In Lemma 3.1 we proved that the converse inclusion holds. Thus, the equality

$$F(\mathrm{BQD}_0(\Delta)) = T_0(\Delta)$$

is proved.

Finally if $d(\tau_n, \tau) = \epsilon_n$, then the ball of radius $P - \epsilon_n$ centred at τ is contained in the open set $F(\mathrm{BQD}(\Delta))$. Since $\epsilon_n \to 0$, the second part of the theorem follows. □

THEOREM 3.2. *Let τ be an element of* Teich(Δ), *such that the boundary dilatation $b(\tau)$ satisfies the inequality $b(\tau) < \tilde{P}$. The constant $\tilde{P}$ is defined by*

$$\tilde{P} = \frac{e^{2P} - 1}{e^{2P} + 1},$$

where P is the constant introduced in Proposition 3.1. Then $\tau \in F(\mathrm{BQD}(\Delta))$. In particular, every quasisymmetric map whose boundary dilatation is less than P can be extended to a unique harmonic quasiconformal map of the unit disk.

Proof. To prove this result it is sufficient to show that there is $\tau_0 \in T_0(\Delta)$ such that $d(\tau_0, \tau) < P$. Since $b(\tau) < \tilde{P}$, there is a quasiconformal map $f : \Delta \longrightarrow \Delta$, with $\mathrm{Belt}(f) = \mu$, such that $[\mu] = \tau$. In fact, f can be chosen such that there exists $0 < r_0 < 1$ and $\|\mu(z)\| < \tilde{P}$ for $r_0 < |z| < 1$. Let $\alpha = \mathrm{Belt}(f^{-1})$. Since $|\alpha \circ f| = |\mu|$, one can find $0 < r_1 < 1$ such that $\|\alpha(z)\|_\infty < \tilde{P}$ for $r < |z| < 1$. Further, let $\beta(z) = \alpha(z)$ for $|z| < r_0$, and $\beta(z) = 0$ for $r_0 < |z| < 1$. Also let $g : \Delta \longrightarrow \Delta$ be the quasiconformal map with $\mathrm{Belt}(g) = \beta$. Set $\mathrm{Belt}(g^{-1}) = \nu$, and $[\nu] = \tau_0$. We show that $d(\tau, \tau_0) < P$. Indeed, by the definition of the distance d we have

$$d(\tau, \tau_0) \leqslant \frac{1}{2} \log \frac{1 + \|\mathrm{Belt}(g \circ f)\|_\infty}{1 - \|\mathrm{Belt}(g \circ f)\|_\infty}.$$

From the construction it follows that $\|\mathrm{Belt}(g \circ f)\|_\infty < \tilde{P}$. Since $\tilde{P} = (e^{2P} - 1)/(e^{2P} + 1)$ we establish the inequality $d(\tau, \tau_0) < P$, and the theorem is proved. □

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