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A solitary-wave solution to a perturbed KdV equation

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Abstract. We derive the approximate form and speed of a solitary-wave solution to a perturbed KdV equation. Using a conventional perturbation expansion, one can derive a first-order correction to the solitary-wave speed, but at the next order, algebraically secular terms appear, which produce divergences that render the solution unphysical. These terms must be treated by a regrouping procedure developed by us previously. In this way, higher-order corrections to the speed are obtained, along with a form of solution that is bounded in space. For this particular perturbed KdV equation, it is found that there is only one possible solitary wave that has a form similar to the unperturbed soliton solution.

Dedication

The authors would like to dedicate this paper to John Dougherty on the occasion of his 65th birthday. G. R. in particular would like to thank him for his friendship over the years and for his major contribution as editor of this journal to a maintenance of standards. The present paper has its origins in a paper (Rowlands 1969) that was originally rejected by J. D. but eventually accepted after a lot of soul searching on the part of G. R. In this case, the final decision of the editor proved to be correct.

1. Introduction

The Korteweg–de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0, \tag{1}$$

in which u is the normalized ion density, governs weakly nonlinear one-dimensional waves in an ion-acoustic plasma (Infeld and Rowlands 2000). The single-soliton solution to this equation takes the form

$$u_0(x,t) = 12\eta^2 \operatorname{sech}^2[\eta(x - 4\eta^2 t)], \tag{2}$$

where η is an arbitrary real quantity. Since the discovery that the KdV equation is integrable and so can be solved by an inverse scattering transform (Gardner et al. 1967), its general solution has been studied in great detail. However, the KdV equation is of course an approximation – in its derivation, higher-order terms and dissipation have been neglected. In particular, it only applies to a uniform background plasma.

Chang et al. (1986) have shown that in the presence of a non-uniform plasma whose concentration varies linearly in the direction of propagation of the wave, if the gradient is small then the system is described by the KdV equation with a small extra linear term $\epsilon\alpha u$ on the right-hand side. A positive $\epsilon\alpha$ means a concentration increasing in the direction of the wave. In this case, the solitary-wave amplitude increases as it propagates.

In this paper, we consider a non-uniform plasma in which there is also dissipation. These effects result in the perturbed KdV equation

$$u_t + uu_x + u_{xxx} = \epsilon\alpha u + \epsilon\beta u_{xx}, \tag{3}$$

where ϵ is small. Using inverse scattering techniques, the evolution of the single-soliton solution (2) in the presence of perturbations such as the above can be analysed (Kivshar and Malomed 1989). In particular, Karpman and Maslov (1978) looked at the case where $\beta \equiv 0$, and found that the original soliton changes speed and shape, and forms a long tail at the end of which there are small oscillations in time and space. Here we shall instead look for solitary-wave solutions that propagate without changing form. We expect to find these, since if α and β are both positive then the first term on the right-hand side of (3) leads to wave steepening, while the other dissipative term will result in a decrease in amplitude. Just as a soliton results from a balance between nonlinearity and dispersion, the solitary-wave solution we are looking for would also require that the effects of the perturbing terms on the shape cancel each other out.

On performing a small-ϵ expansion about the unperturbed solution, at first order we find two types of secular terms – exponentially secular terms, which must be removed by choice of η, and what we refer to as algebraically secular terms. The latter are terms that tend to a polynomial in x for large x and therefore do not vanish at infinity. In Allen and Rowlands (1993), where a problem concerning the linear stability of solitary pulses was studied, a method was developed for dealing with such terms. The method involved regrouping certain terms at all orders to obtain a function that vanishes at infinity and has the correct asymptotic form. In the next section, we describe the application of the method to this problem and find the form and speed of the solitary-wave solution.

2. Perturbation analysis

We first transform the equation to one in reduced variables in a reference frame in which the solitary wave with speed v is stationary by making the change of variables

$$x' = \eta(x - vt), \qquad t' = \eta^3 t, \qquad u' = \frac{u}{\eta^2}$$

and putting

$$\bar{v} = \frac{v}{\eta^2}, \qquad \bar{\alpha} = \frac{\alpha}{\eta^3}, \qquad \bar{\beta} = \frac{\beta}{\eta}.$$

As the solitary waves we are looking for are stationary in this frame, we drop the time dependence. Also dropping the primes and bars leaves us with

$$(u - v)\, u_x + u_{xxx} = \epsilon\alpha u + \epsilon\beta u_{xx}. \tag{4}$$

We look for solutions with a form given by the expansion

$$u(x) = u_0(x) + \epsilon u_1(x) + \dots \tag{5}$$

and speed

$$v = 4 + \epsilon v_1 + \epsilon^2 v_2 + \dots \quad . \tag{6}$$

We first need to consider the asymptotic behaviour of the solution. As $x \to \pm\infty$, (4) reduces to

$$\frac{d^3u}{dx^3} - \epsilon\beta\frac{d^2u}{dx^2} - (4+\epsilon v_1)\frac{du}{dx} - \epsilon\alpha u = 0 \tag{7}$$

for small ϵ. This has solutions $e^{p_i x}$, where the p_i are real and to first order in ϵ are given by

$$p_1 = 2 - \tfrac{1}{8}(\alpha + 4\beta + 2v_1)\,\epsilon, \qquad p_2 = -2 + \tfrac{1}{8}(\alpha + 4\beta - 2v_1)\,\epsilon, \qquad p_3 = -\tfrac{1}{4}\epsilon\alpha.$$

Hence physically acceptable solutions of (4) must decay as e^{2x} to lowest order as $x \to -\infty$, and, for large positive x, there will be a slowly decaying exponential in addition to the e^{-2x} terms. Note that since the p_i are real, the solution u does not have spatial oscillations as $|x| \to \infty$.

In a conventional perturbation expansion in ϵ, the slowly decaying exponential behaviour will give rise to algebraically secular terms. The present method is essentially to treat such terms separately from all others and eventually regroup to give the correct asymptotic behaviour, namely exponential decay, as $x \to \infty$.

After substituting (5) and (6) into (4), to zeroth order in ϵ, we obtain the unperturbed equation in the rest frame of the soliton,

$$u_{0xxx} + (u_0 - 4)\,u_{0x} = 0, \tag{8}$$

which has the solution

$$u_0 = 12\,\mathrm{sech}^2\,x. \tag{9}$$

To find u_1, we write the first-order equation in the form

$$\frac{d}{dx}Lu_1 = \alpha u_0 + \beta u_{0xx} + v_1\,u_{0x}, \tag{10}$$

where

$$L \equiv \frac{d^2}{dx^2} + u_0 - 4.$$

To ensure the correct asymptotic form, integrations must be performed between the limits $-\infty$ and x, and hence the inverse of L is given by

$$L^{-1}R(x) \equiv \varphi_0(x)\int^x \frac{\int_{-\infty}^{x'} R(x'')\,\varphi_0(x'')\,dx''}{\varphi_0^2(x')}\,dx', \qquad \mathrm{mod} \quad \varphi_0(x), \tag{11}$$

with $\varphi_0(x) = \mathrm{sech}^2\,x\tanh x$. (The mod φ_0 is used to remove φ_0 terms, which are unwanted since $L\varphi_0 = 0$.) Integrating (10), we obtain

$$Lu_1 = 12\alpha(\tanh x + 1) - 24\beta\,\mathrm{sech}^2\,x\tanh x + 12v_1\,\mathrm{sech}^2\,x,$$

and then applying L^{-1} gives

$$u_1 = (\tfrac{1}{2}\alpha - \tfrac{2}{5}\beta)\,e^{2x} - \tfrac{12}{5}\beta(\tanh x + 1) + (\tfrac{3}{2}\alpha + 6\beta + 3v_1)\,(\mathrm{sech}^2\,x - x\,\mathrm{sech}^2\,x\tanh x). \tag{12}$$

The first term is exponentially secular, and must be removed by choosing

$$\bar{\beta} = \tfrac{5}{4}\bar{\alpha}, \tag{13}$$

where we have momentarily reinstated the bars. (This is equivalent to the usual consistency condition applied to (10).) Referring back to the definitions of $\bar{\alpha}$ and $\bar{\beta}$ we see that this is equivalent to a condition on the soliton parameter:

$$\eta = \left(\frac{4\beta}{5\alpha}\right)^{1/2}. \tag{14}$$

Since η must be real, a solitary wave can only exist if α and β have the same sign. If α and β are of opposite signs, both perturbations will result in reduction or augmentation of the wave amplitude, and no wave of permanent form is possible. In addition, if $\beta \equiv 0$ as in Karpman and Maslov (1978), there is no solitary-wave solution.

Substituting (13) into (12), we obtain

$$u_1 = -3\alpha(\tanh x + 1) + 3(3\alpha + v_1)(\operatorname{sech}^2 x - x \operatorname{sech}^2 x \tanh x). \tag{15}$$

The first term is an algebraically secular term. However, owing to the correct choice of the limits of integration in (11) and when integrating (10), it has the correct underlying asymptotic form, vanishing as $x \to -\infty$ and tending to a constant as $x \to +\infty$, since $p_3 = 0$ to lowest order.

To next order, we obtain

$$\frac{d}{dx} L u_2 = \alpha u_1 + \beta u_{1xx} - \frac{1}{2}\frac{d}{dx} u_1^2 + v_1 u_{1x} + v_2 u_{0x}. \tag{16}$$

At this point, it is helpful to introduce the functions $A_n(x)$, defined by

$$A_n(x) = \int_{-\infty}^{x} A_{n-1}(x')\, dx', \qquad A_0(x) = \tanh x + 1, \tag{17}$$

from which we have

$$A_1(x) = \ln(1 + e^{-2x}) + 2x,$$
$$A_2(x) = \tfrac{1}{2}\operatorname{Li}_2(-e^{-2x}) + x^2 + \tfrac{1}{12}\pi^2,$$

in which $\operatorname{Li}_n(z)$ is the polylogarithmic function (Wolfram 1991). On considering the asymptotic behaviour of $A_0(x)$ for large $|x|$, it follows from the definition that the asymptotic behaviour of the $A_n(x)$ to leading order is given by

$$[A_n(x)]_{x \to -\infty} \simeq 2^{1-n} e^{2x}, \tag{18}$$

$$[A_n(x)]_{x \to +\infty} \simeq \frac{2}{n!} x^n. \tag{19}$$

From the above, we see that any term proportional to $A_n(x)$ is an algebraically secular term with the correct underlying asymptotic form for this problem.

After integrating (16) and then using *Mathematica* (Wolfram 1991) to apply (11), we obtain an expression for u_2 with an exponentially secular term

$$-\tfrac{1}{16}\alpha(9\alpha + 2v_1)\, e^{2x}.$$

We must therefore choose

$$v_1 = -\tfrac{9}{2}\alpha \tag{20}$$

to remove it. This leaves us with

$$\begin{aligned} u_2 = \alpha^2[&\tfrac{3}{4}A_1(x) - \tfrac{9}{16}A_0(x) + \tfrac{9}{8}\operatorname{sech}^2 x + \tfrac{9}{16}x\operatorname{sech}^2 x + \tfrac{27}{32}x^2\operatorname{sech}^2 x \\ &- \tfrac{9}{4}A_1(x)\operatorname{sech}^2 x - \tfrac{81}{64}x^2\operatorname{sech}^4 x - \tfrac{9}{64}x\operatorname{sech}^2 x\tanh x \\ &+ \tfrac{9}{4}A_2(x)\operatorname{sech}^2 x\tanh x] + 3v_2(\operatorname{sech}^2 x - x\operatorname{sech}^2 x\tanh x). \end{aligned} \tag{21}$$

To demonstrate the regrouping of algebraic secular terms and obtain the second-order correction to the speed of the solitary wave, we proceed to third order:

$$\frac{d}{dx}Lu_3 = \alpha u_2 + \beta u_{2xx} - \frac{d}{dx}(u_1 u_2) + v_1 u_{2x} + v_2 u_{1x} + v_3 u_{0x}. \tag{22}$$

A lengthy calculation using *Mathematica* yields an expression for u_3 with over 40 terms. The parts of u_3 that do not vanish for large x are

$$\begin{aligned} &\tfrac{1}{256}\alpha(27\alpha^2 - 32v_2)\,e^{2x} - \alpha^3[\tfrac{3}{16}A_2(x) + \frac{3v_2}{8\alpha^2}A_1(x) \\ &\qquad + \tfrac{9}{64}xA_0(x) + \tfrac{9}{64}(A_1(x)\,e^{-2x} - 1) - \tfrac{207}{512}A_0(x)]. \end{aligned}$$

The first term in this expression is the only exponentially secular one. It must be removed by choosing

$$v_2 = \tfrac{27}{32}\alpha^2. \tag{23}$$

The remaining secular terms are all algebraic.

We are now ready to treat the algebraically secular terms. The leading algebraically secular terms in the expansion up to third order can be written as

$$-3\epsilon\alpha[A_0(x) - \tfrac{1}{4}\epsilon\alpha\,A_1(x) + (\tfrac{1}{4}\epsilon\alpha)^2 A_2(x)]. \tag{24}$$

Using (19), it can be seen that for large positive x, the above expansion is proportional to the expansion of the slowly decaying asymptotic form $e^{p_3 x}$. The leading terms (24) also appear to be part of an expansion of a function $-3\epsilon\alpha\mathscr{A}(x)$, where

$$\begin{aligned} \mathscr{A}(x) = A_0(x) - \tfrac{1}{4}\epsilon\alpha\int_{-\infty}^{x} A_0(x')\,dx' + (\tfrac{1}{4}\epsilon\alpha)^2\int_{-\infty}^{x}\int_{-\infty}^{x'} A_0(x'')\,dx'\,dx'' \\ - \ldots = \left(1 + \tfrac{1}{4}\epsilon\alpha\int_{-\infty}^{x} dx\right)^{-1} A_0(x). \end{aligned}$$

This is equivalent to the differential equation

$$\frac{d\mathscr{A}(x)}{dx} + \tfrac{1}{4}\epsilon\alpha\,\mathscr{A}(x) = A_0(x),$$

whose solution is

$$\mathscr{A}(x) = \int_{-\infty}^{x} A_0(x')\,e^{(\epsilon\alpha/4)(x'-x)}\,dx'. \tag{25}$$

It can be seen that $\mathscr{A}(x)$ vanishes for large x. Hence the algebraic secular terms (up to second order at least) have been regrouped to form a physically acceptable function. There are other algebraically secular terms at third order, but it seems reasonable to assume that these can be regrouped with higher-order terms in a similar manner.

Replacing the $\tanh x+1$ in u_1 by the function $\mathscr{A}(x)$, combining the results (9), (15), (20) and (23), and then rewriting in terms of the original variables, we find that the solitary wave has the form

$$u(x) = \left(12\eta^2 - \frac{9}{2}\frac{\epsilon\alpha}{\eta}\right)\operatorname{sech}^2 \eta x - \frac{3\epsilon\alpha}{\eta}\mathscr{A}(\eta x) + \frac{9}{2}\epsilon\alpha x \operatorname{sech}^2 \eta x \tanh \eta x + O(\epsilon^2) \quad (26)$$

and travels at a speed

$$v = 4\eta^2 - \frac{9}{2}\frac{\epsilon\alpha}{\eta} + \frac{27}{32}\frac{\epsilon^2\alpha^2}{\eta^4} + O(\epsilon^3), \quad (27)$$

where η is given by (14).

3. Conclusions

We have found the form and speed of a new solitary-wave solution to a perturbed KdV equation. Whereas the unperturbed KdV equation has a one-parameter family of one-soliton solutions (2), the perturbed equation admits only one solution that is similar to the soliton solution for a given value of α and β.

To arrive at a bounded solution, we have had to regroup algebraically secular terms following an approach first developed in Allen and Rowlands (1993). The method justifies using the simple consistency condition to give the lowest-order value for v. It is applicable to any third- or higher-order soliton-bearing equation (which is not necessarily integrable) with a perturbation of similar form.

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