NUMERICAL MODELING AND EXPERIMENTS ON SOUND PROPAGATION THROUGH THE SONIC CRYSTAL AND DESIGN OF RADIAL SONIC CRYSTAL

ARPAN GUPTA

NATIONAL UNIVERSITY OF SINGAPORE

2012
NUMERICAL MODELING AND EXPERIMENTS ON
SOUND PROPAGATION THROUGH THE SONIC
CRYSTAL AND DESIGN OF RADIAL SONIC CRYSTAL

ARPAN GUPTA
(B-Tech Indian Institute of Technology Delhi)

A THESIS SUBMITTED
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY
DEPARTMENT OF MECHANICAL ENGINEERING
NATIONAL UNIVERSITY OF SINGAPORE

2012
Acknowledgements

Firstly, I would like to thank the Supreme Lord, for giving me the intelligence and the ability to do this work. Research work requires inspiration, knowledge, hard work, success in endeavor and many other resources. Therefore, I would like to acknowledge the mercy of the Supreme Lord to carry out this work. I hope I can use this gift for the service of mankind.

I would like to express my gratitude to my supervisor Prof. Lim Kian Meng for his very helpful suggestions and feedbacks during my PhD. He spent lot of time with me teaching me various aspects in doing research. I have benefited in various aspects, such as in computational methods, being professional in research, writing technically, etc. I am also very grateful to my supervisor Prof. Chew Chye Heng for teaching me various aspects in experimental acoustics. Both of my supervisors gave me ample opportunity to be creative and to pursue my thoughts. They also gave valuable and timely suggestions to improve my work. I am also grateful to Prof. S.P. Lim and Prof. H. P. Lee for their comments during my oral qualifying exam. Their comments helped me to be more focused in my work.

I would also like to express my deepest gratitude to my professor of numerical methods at IIT Delhi, whom I lovingly call as ‘Sir’. His contribution in my life is much more than numerical methods. He is the person who has brought some good qualities and good character in my life. He is an ideal example of a truly selfless person and a genuine
well wisher for others. I am very grateful for his wonderful teachings which have significantly transformed my life.

I would also like to thank Dr. Sujoy Roy, for being a senior friend and mentor to me during my stay here at Singapore. He gave me lot of inspiration and shared with me his valuable experiences. I would also like to thank my friends Karthik, Ruchir, Dhawal etc. for their help, friendship and happiness they shared with me during my stay here in Singapore.

I am also grateful to NUS to provide me with full research scholarship. Thanks to Dynamics lab to provide me with all the facilities to do my work. I am also thankful to Mr. Cheng for his prompt help during my experiments. Thanks to HPC (High Performance Computing) for the computational resources to carry out the numerical modeling. I am also very thankful to my lab mates Tse Kwong Ming, Guo Shieffeng, Zhu Jianghua, Liu Yang, Thein etc, for their friendship and valuable discussions.

Lastly, I would like to thank my parents, grandmother and brother for their support and patience during this work.

Arpan Gupta
# Table of Contents

Acknowledgements .............................................................................................................................................. i

Summary ........................................................................................................................................................ vi

List of figures .................................................................................................................................................. viii

List of symbols ............................................................................................................................................... xiv

Chapter 1. Introduction .................................................................................................................................... 1

  1.1 Periodic structures and band gaps ........................................................................................................ 2

  1.2 Motivation ........................................................................................................................................... 6

  1.3 Objective of the thesis ......................................................................................................................... 8

  1.4 Organization of the thesis ................................................................................................................. 9

  1.5 Original contribution of the thesis ................................................................................................. 10

  1.6 Acoustic wave propagation ............................................................................................................. 11

Chapter 2. Literature Review ........................................................................................................................ 17

  2.1 Sound insulation ................................................................................................................................. 18

  2.2 Frequency filters and acoustic waveguides ....................................................................................... 20

  2.3 Metamaterials and radial wave crystal ............................................................................................. 21

  2.4 Other applications ......................................................................................................................... 23

  2.5 Numerical Methods for calculation and optimization of the band gap ....................................... 24

  2.6 Evanescent wave ............................................................................................................................. 26
Chapter 6. Analysis of a radial sonic crystal

6.1 Problem Definition

6.1.1 Problem Definition ............................................................................................................ 83
6.1.2 Numerical Formulation .................................................................................................. 84
6.1.3 Validation of the 1-D model with the finite element simulation .................................. 88

6.2 Analysis of an intuitive radial sonic crystal ...................................................................... 93

6.3 Design of periodic structure in cylindrical coordinates .................................................. 95

6.4 Sound attenuation by the radial sonic crystal .................................................................. 99

6.5 Conclusion ........................................................................................................................ 101

Chapter 7. Experiment and finite element simulation on the radial sonic crystal

7.1 Experiment ......................................................................................................................... 104

7.2 Finite element simulation .................................................................................................. 108

7.2.1 Mesh convergence test ................................................................................................ 109

7.3 Results ............................................................................................................................... 111

7.4 Conclusion ......................................................................................................................... 117

Chapter 8. Conclusion and future direction of work

8.1 Conclusion .......................................................................................................................... 118

8.2 Future direction of work .................................................................................................... 123

References .................................................................................................................................. 126

Publications ................................................................................................................................ 140
Summary

Sound propagation through a rectangular sonic crystal with sound hard scatterers is modeled by sound propagation through a waveguide. A 1-D numerical model based on the Webster horn equation is proposed to obtain the band structure for sound propagation in the symmetry direction of the rectangular sonic crystal. The model is further modified to obtain the complex dispersion relation, which gives the additional information of decay constant of the evanescent wave. The decay constant is used to predict the sound attenuation over a finite length of the sonic crystal in the band gap region. Alternatively, sound transmission over the finite length of sonic crystal can be directly obtained using the Webster horn equation. Theoretical results from the model are compared with the finite element simulation and experiment. The model developed is used to perform a parametric study on the various geometrical parameters of the rectangular sonic crystal to find optimal design guidelines for high sound attenuation. It is found that a particular kind of rectangular structure is better suited for sound attenuation than the normal square arrangement of scatterers.

The 1-D numerical model is further extended to a quasi 2-D model for sound propagation in a waveguide. The assumption in the 1-D model was due to the Webster horn equation, which assumes a uniform pressure across the cross-section of a waveguide. Quasi 2-D model is derived from the weighted residual method and Helmholtz equation, to include a parabolic pressure profile across the cross-sectional area of the waveguide. This quasi 2-D model for sound propagation in a waveguide is used to
obtain band structure of the sonic crystal and to obtain sound attenuation over a finite length. The results match well with the 2-D finite element simulation and experimental results. The quasi 2-D model also shows significant improvement over the 1-D model based on the Webster horn equation. It is also shown that Webster horn equation is a special case of the quasi 2-D model.

Lastly, radial sonic crystal is envisioned and a numerical model is proposed to obtain its design parameters. Most of the sound sources generate pressure waves which are non-planar in nature. Instead of scatterers arranged in square lattice with a plane wave propagating through it, scatterers are arranged in radial coordinates to attenuate sound wave with circular wavefront. Sound propagation through such sonic crystal is modeled by an equation for sound propagation through radial waveguide. Although such a structure may not be physically periodic (i.e. a unit cell by simple translation can form the whole structure), but such a structure is mathematically periodic by implementing the property of invariance in translation on the governing equation. Such periodic structure in radial coordinates, are termed as radial sonic crystal. Based on the design from the numerical model, finite element simulations and experiments are performed to obtain sound attenuation for radial sonic crystal. The results are in good agreement and it shows a significant sound attenuation by radial sonic crystal in the band gap region.
List of figures

Figure 1-1 Different types of sonic crystals. (a) 1-D sonic crystal consisting of plates arranged periodically (b) 2-D sonic crystal with cylinders arranged on a square lattice (c) 3-D sonic crystal consisting of periodic arrangement of sphere in simple cubic arrangement

Figure 1-2 An example of band gaps for sonic crystal represented by the shaded region

Figure 1-3 First experimental revelation of the sonic crystal was found by an artistic structure designed by Eusebio Sempere in Madrid

Figure 3-1(a) A two dimensional periodic structure made of circular scatterers arranged on a square lattice. On the left side there is plane wave sound source. The dotted square shows a unit cell. (b) Magnified view of a unit cell with various geometric parameters

Figure 3-2 Band gap for an infinite sonic crystal corresponding to Fig. 3.1 with a = 4.25 cm and d = 3 cm, along the symmetry direction ΓX

Figure 3-3 Complex band structure for an infinite sonic crystal. (a) normal band structure. (b) Decay constant as a function of frequency. The decay constant is non-zero in the band gap regions

Figure 3-4 Sound attenuation predicted by the decay constant

Figure 3-5 (a) Sound propagating over a sonic crystal consisting of five layers of scatterers. Using symmetry of the structure, the problem is reduced to a strip model shown by rectangle ACDB. (b) A symmetric waveguide used to model sound propagation through the sonic crystal using Webster horn equation
Figure 3-6 Sound attenuation by the finite sonic crystal using the Webster horn equation and decay constant ................................................................. 44
Figure 3-7 The area function $S$ (plotted on left axis) and its derivative (plotted on right axis) for a unit cell. ............................................................. 45
Figure 3-8 Mesh convergence test for 1-D model using second order finite difference method. The results are indifferent after 1000 points. For our simulation, we have used 2000 points ................................................................. 46
Figure 3-9 Sound attenuation at 6 kHz for different mesh size. ......................... 47
Figure 4-1 Experimental setup with sound propagating over five acrylic cylinders. .... 51
Figure 4-2 Background noise of the room in which experiment was performed. ........ 53
Figure 4-3 Experimental measurements of sound attenuation from 10 experiments (each averaged 50 times). The figure shows the variation in experimental observation .......... 54
Figure 4-4 Experimental results of sound attenuation along with the results predicted by the decay constant and the Webster equation model. ........................................ 55
Figure 4-5 Boundary conditions and model for the two dimensional finite element simulation ............................................................................ 56
Figure 4-6 Finite element results along with experiment and Webster equation model. . 58
Figure 4-7 Sound attenuation for cylinder of diameter 4 cm and lattice constant 11 cm. The results compare our FE model with published work. ........................................ 60
Figure 4-8 Sound attenuation for cylinder of diameter 2 cm and lattice constant 11 cm. The results compare our FE model with published work. .............................. 61
Figure 4-9 Absolute pressure along the x axis at the highest frequency of 6 kHz. The results are indifferent from the mesh size of 2094 onwards. The simulations are performed using 5004 elements.

Figure 4-10 Absolute pressure measured at highest frequency of 6 kHz at the outlet end of the waveguide for different mesh size.

Figure 4-11 Rectangular unit cell

Figure 4-12 Parametric study of rectangular sonic crystal. (a) Sound attenuation versus frequency for rectangular sonic crystal. (b) Center frequency of band gap from numerical model and Bragg's law. (c) Maximum attenuation over a length of five unit cells, by varying $a_y$ and $d$. (d) Band gap width varying by changing $a_y$ and $d$. $a_y$ is inversely proportional to filling fraction.

Figure 5-1 Pressure plot in the strip model consisting of five circular scatterers at 3500 Hz. Pressure wave near the first and second cylinder is not uniform across the cross-section. b) Pressure amplitude at a cross-section measured 0.5 cm before the first cylinder from different methods. c) Pressure amplitude along x-axis from different methods. The pressure amplitude from the finite element model overlaps with the quasi 2-D model solution.

Figure 5-2 Sound attenuation by an array of five circular scatters. Results comparing sound attenuation from experiment, Webster horn model, quasi 2-D model and finite element model.

Figure 5-3 Complex band gap from the Webster horn model and the quasi 2-D model.
Figure 5-4 Sound attenuation predicted by the decay constant from the Webster horn model, quasi 2-D model and comparing them with experiment and finite element results. ........................................................................................................................................... 78

Figure 6-1 A conceptual/intuitive design of a radial sonic crystal with scatterers arranged periodically in the angular coordinates around a cylindrical sound source. The figure shows only a quadrant of the actual geometry. ......................................................................................................................... 80

Figure 6-2 Sound propagation through a waveguide. Sound wave is modeled with (a) planar wavefront (b) circular wavefront. ......................................................................................................................... 82

Figure 6-3 Sound propagating from a line source through a waveguide. The source and waveguide are long in the z direction so that the analysis is restricted to the 2-D $xy$ plane. ........................................................................................................................................... 84

Figure 6-4 The symmetric portion of a general waveguide. The figure shows the geometric location of an arbitrary point A in the polar coordinates. The unit normal and tangential vectors at that point are also shown. ......................................................................................................................... 85

Figure 6-5 Specific example of waveguide with perturbation of a semicircle ............... 90

Figure 6-6 Average pressure verses radial distance for wave propagating from a point source in a waveguide with circular wavefront, planar wavefront (Webster horn equation) and finite element (FE) simulations. ........................................................................................................................................... 91

Figure 6-7 (a) Radial waveguide considered for the analysis of sonic crystal in polar coordinates. (b) Sound attenuation from the intuitive radial sonic crystal. ..................... 94

Figure 6-8 (a) Unit cell of the radial sonic crystal with circular scatterers is shown by the dark line. Applying the property of invariance in translation lead to its corresponding
second periodic unit cell which was highly distorted. (b) The plot of periodic function $g(r)$ used for mapping the geometry of second unit cell. ................................................................. 96

Figure 6-9 (a) Continuous periodic function $g(r)$ used for designing RSC. (b) The symmetric part of the radial waveguide for five unit cell obtained by using the property of invariance in translation on the wave propagating equation......................................................... 97

Figure 6-10 A radial sonic crystal.............................................................................................. 99

Figure 6-11 (a) Sonic crystal made of circular scatterers based on intuitive design (b) Radial sonic crystal designed based on periodic condition. ............................................ 100

Figure 6-12 Sound attenuation as a function of frequency for radial sonic crystal and the intuitive structure made of circular scatterers of constant diameter. .......................... 101

Figure 7-1 Experimental model for testing a representative waveguide of a radial sonic crystal. The two experimental setup represents sound propagation in the waveguide with and without the elliptic scatterers. The top and bottom cover plates are not shown in this figure. ................................................................................................................................. 106

Figure 7-2 Experimental setup for testing representative waveguide of a radial sonic crystal.......................................................................................................................... 107

Figure 7-3 Finite element simulation for the symmetric part of the waveguide representing a radial sonic crystal. The surface pressure plot shows absolute pressure in the waveguide at four different frequencies................................................................. 109

Figure 7-4 Pressure profile along the radial axis for different mesh size at the highest frequency of 6 kHz......................................................................................................... 110

Figure 7-5 Absolute pressure at the outlet end of the waveguide for different mesh size. ........................................................................................................................................ 111
Figure 7-6 Sound attenuation from a representative waveguide of a radial sonic crystal based on finite element simulation, experiment, and 1-D numerical model. .................. 113

Figure 7-7 (a) Sound attenuation from the radial sonic crystal for curved edge verses straight edge design. (b) Geometry showing the difference between the straight edge (outer) verses the curved edge (inner) design of radial sonic crystal. ............................. 116
List of symbols

\( p \)   acoustic pressure or pressure fluctuation from mean pressure

\( P \)   complex amplitude of the acoustic pressure

\( P^I \)   absolute or total pressure in a fluid

\( P_I \)   amplitude of the inlet incident pressure wave

\( P_O \)   amplitude of the outgoing pressure wave

\( c \)   velocity of sound

\( \omega \)   angular frequency of the propagating wave

\( \rho \)   density

\( B \)   adiabatic bulk modulus

\( \vec{u} \)   velocity vector

\( z \)   specific acoustic impedance

\( a \)   lattice constant or a unit cell length for a square lattice arrangement

\( a_x \)   unit cell length in \( x \) direction for a rectangular lattice arrangement

\( a_y \)   unit cell length in \( y \) direction for a rectangular lattice arrangement

\( S \)   cross-sectional area of the waveguide

\( f \)   filling fraction of the scatterers in the sonic crystal

\( d \)   diameter of the cylinders

\( k \)   wavenumber

\( k_R \)   real part of the wavenumber

\( k_I \)   decay constant – imaginary part of the wavenumber

\( \phi(x) \)   periodic function used in Bloch theorem
\( A_i \) matrices formed by the finite difference discretization
\( a_i, b_i \) matrices formed by the finite difference discretization
\( f_c \) center frequency of the band gap
\( h \) step size for finite difference discretization in \( x \) direction
\( i \) complex number – square root of \((-1)\)
\( N \) total number of points used for the finite difference discretization
\( IL \) insertion loss by the sonic crystal
\( SPL \) sound pressure level measured in dB (decibels)
\( \alpha_0(x) \) constant coefficient of approximate pressure
\( \alpha_i(x) \) quadratic coefficient of approximate pressure
\( \theta(r) \) angular position of the top surface of the waveguide
\( I_i \) various integrals
\( H_0^{(1)} \) Hankel function of first kind
\( q \) periodicity of the radial sonic crystal
\( g(r) \) mapping function
Chapter 1. Introduction

Sonic crystals are artificial structures made by the periodic arrangement of scatterers in a square or triangular lattice configuration. The scatterers are sound hard (i.e., having a high acoustic impedance) with respect to the medium in which they are placed. For example, acrylic cylinders in air or steel plates in water are some examples of such sonic crystals. Sonic crystal with scatterers as cylinders arranged periodically is called a 2-D sonic crystal (Fig 1-1). When the scatterers are placed in a 1-D periodic arrangement, such as steel plates placed periodically in water, it is known as a 1-D sonic crystal. When the scatterers such as spheres are placed in a 3-D periodic arrangement (for example, simple cubic), it is known as 3-D sonic crystal. In this thesis, a 2-D sonic crystal made of acrylic cylinders in air is considered. The cylinders are arranged in a square lattice, which is later extended to the rectangular configuration.

![Sonic crystals](image)

Figure 1-1 Different types of sonic crystals. (a) 1-D sonic crystal consisting of plates arranged periodically (b) 2-D sonic crystal with cylinders arranged on a square lattice (c) 3-D sonic crystal consisting of periodic arrangement of sphere in simple cubic arrangement.
1.1 Periodic structures and band gaps

Due to the periodic arrangement of scatterers, sonic crystals have a unique property of selective sound attenuation in specific range of frequencies. This range of frequencies is known as the band gap, and it is found that sound propagation is significantly reduced in this band gap region [1]. The reason for such sound attenuation is due to the destructive interference of wave in the band of frequencies. It is also shown numerically in this thesis that the propagating wave has an evanescent behavior (decaying amplitude) which causes the sound attenuation to take place in the band gap region.

Periodic structures, in general, can significantly alter the propagation of wave through them. The earliest realization of this principle was at the level of atomic structure in metals and semiconductors. According to quantum physics [2], atoms are arranged in a periodic lattice in a solid. When electron (wave) passes though the crystal structure, it experiences a periodic variation in potential energy caused by the positive core of metal ions. The solution of Schrödinger equation over such a periodic arrangement is obtained by the Bloch theorem [2], or the Floquet theorem for 1-D case [3]. The wave propagating in such periodic structure is given as,

$$\psi(r) = u(r)e^{ikr} \tag{1-1}$$

where $\psi(r)$ is the Bloch function representing the electron wave function and $u(r)$ is a periodic function with periodicity of the lattice.
The solution of the Bloch wave for periodic potential leads to the formation of bands of allowed and forbidden energy regions. The allowed energy region is known as conduction and valence band, whereas, the forbidden band of energies where there is no solution for the Bloch wave is known as the band gap. These band gaps are quite common in semiconductor materials and they form the basis of all modern electronic devices.

Another application of the same principle of wave interacting with periodic structures is in the field of photonic crystals [4, 5]. When electromagnetic wave (light wave) passes through a periodic arrangement of dielectric material with different dielectric constants, photonic band gaps are formed. Therefore, there are certain frequencies of light that are allowed to pass through the structure and certain frequencies are restricted. The formation of band gap allows the design of optical materials to control and manipulate the flow of light. One such practical application is the design of photonic crystal fiber [6], which uses microscale photonic crystal to confine and guide light.

The same principle is being extended and applied to the acoustic wave passing through the periodic structures. When an acoustic wave interacts with a periodic structure it forms bands of frequencies (Fig 1-2), where certain frequencies are allowed to pass through the structure without much attenuation, while certain frequencies are attenuated. This leads to significant sound attenuation in the frequency band. The band gap is represented by the shaded region in Fig. 1-2, where there is no solution of the frequency for a given wavenumber \( k \). The band gap that extends for all directions of wave
propagation is known as a complete band gap. However, in the present work, wave propagation is considered along one of the symmetry directions. The details of the band gap are discussed in chapter 3.

One major difference between the periodic structure in the photonic crystal and in the sonic crystal is the size of the scatterers. For periodic structures to interact with waves, the scatterer dimension and the spacing between them should be of the order of wavelength of propagating wave [2]. In a photonic crystal, the size of scatterers is of the order of microns [7] which is also the order of magnitude of the wavelength of electromagnetic wave. So a photonic crystal of the order of few millimeters has thousands of periodic units arranged in a periodic manner. An ideal or infinite periodic structure should have repeating units which extends till infinity. The band gap is actually

![Graph showing band gaps for a sonic crystal]

**Figure 1-2 An example of band gaps for sonic crystal represented by the shaded region.**
obtained for an infinite structure. Photonic crystal having thousands of periodic units resembles an infinite periodic structure, and therefore in the band gap region, there is no propagation of electromagnetic wave. For sound wave in audible region (20 Hz – 20 kHz), the wavelength is of the order of few centimeters (1700 cm – 1.7 cm). Therefore, sonic crystal due to practical consideration consists of few (3 – 10) scatterers arranged periodically and there is a significant sound attenuation in the band gap region. The thesis will present some numerical methods to obtain band gap and also to obtain sound attenuation through a finite size of sonic crystal.

The first experimental measurement of sound attenuation by the sonic crystal was reported by Martinez-Sala et al. [1] in 1995 and published in Nature. The sonic crystal was an artistic creation by Eusebio Sempere in Madrid consisting of a periodic array of steel cylinders as shown in Fig. 1-3. Experimental tests on this sculpture showed that there was a significant sound attenuation (~15 dB) at 1.67 kHz. This seminal work led to further investigation of acoustic wave passing over periodic structures. Such structures are called ‘Sonic Crystals’ (SC) or ‘Phononic crystals’. Phononic crystals generally refer to structures made of similar host and scatterer material, such as nickel cylinders embedded in copper matrix etc, while sonic crystal refer to structure made of dissimilar materials, such as, steel cylinders in water etc. Phononic crystal made of solid materials are for elastic wave propagation having both longitudinal and transverse wave components; while in the sonic crystal, only longitudinal wave component is considered.
Figure 1-3 First experimental revelation of the sonic crystal was found by an artistic structure designed by Eusebio Sempere in Madrid.

1.2 Motivation

Sound attenuation is very important and is required in many situations. The benefit of sonic crystal is that it can attenuate sound significantly (~ 30 dB) in a particular frequency band. Also the property of selective sound attenuation by the sonic crystal can be useful in designing frequency filters. The conventional method uses a partition or solid barrier. But in sonic crystal, the sound attenuation is due to interaction of wave with the periodic structure. The periodic structure is an open structure and therefore allows for the passage of wind/fluid, which may be required for ventilation, for example to dissipate heat in some situations. Hence this structure can be used where acoustic insulation and heat transfer are simultaneously required. Therefore, developing numerical models for
sound propagation in sonic crystals may help in solving more complicated problems which may arise in the future.

As mentioned, sonic crystals have a finite number of scatterers and it is important to evaluate the performance of such finite structures. There are different standard numerical methods (discussed in the next chapter) to obtain the band gap of the periodic structure. The band gap just predicts the frequency range for which no wave propagation exists for an infinite structure. However, these methods do not predict anything about the sound attenuation from a finite size sonic crystal. There is only one recent method known as extended plane wave expansion method [8], which discusses about complex band gap and it can predict sound attenuation through a finite sonic crystal. This motivated us to develop a numerical method, based on Webster horn equation which can predict the sound attenuation from a finite sonic crystal.

Sonic crystals are one of the growing interests because sound behaves differently in them than in the ordinary material or structures. The periodic property of these structures causes them to exhibit such unique properties. This motivated us to explore the “periodic” nature in polar coordinates. The numerical models developed for rectangular sonic crystals are extended to the polar coordinates to design radial sonic crystal. To our knowledge, this is a new concept, and such kind of sonic crystals can help in sound attenuation from a point or a line source.
1.3 **Objective of the thesis**

The main objective of this thesis is to develop numerical models for obtaining sound attenuation through the sonic crystal and validate them with the experiment and finite element simulations. A one dimensional numerical model based on the Webster horn equation is presented and is compared with the experiments and finite element simulations. The 1-D model is used to perform a geometric parametric study on rectangular sonic crystal to determine the optimal design guidelines for high sound attenuation. The 1-D model is later extended to a quasi 2-D model. The quasi 2-D model is a general model and an improvement to the Webster horn equation for sound propagation in a waveguide. Unlike Webster horn equation which assumes a uniform pressure across the cross-section of the waveguide, a quasi 2-D model includes a quadratic pressure profile, and therefore its predictions are more accurate.

The numerical method developed is further extended to the polar coordinates to design novel structures known as the radial sonic crystal. These structures are periodic in nature in the polar coordinate. The unique property of these structures is that, such structures are aperiodic from a physical point of view, but they have a mathematical basis of periodicity in their design. These novel structures are shown to provide significant sound attenuation in the band gap region. Results from the numerical model are verified by experiments and finite element simulations.
1.4 Organization of the thesis

This chapter gives a brief introduction and motivation of studying the wave propagation through the periodic structures. It also discusses about the basic equation for acoustic wave propagation in free space. Chapter 2 describes some of the recent and past developments in the field of sonic crystals and some of its applications.

Chapter 3 presents a one dimensional model based on the Webster horn equation to predict the band gap and sound attenuation by the sonic crystal. The results from this model are validated in Chapter 4 by experiment and finite element simulation. Chapter 5 presents an improved numerical model, the quasi 2-D model, for modeling sound propagation in a waveguide. This model is used to predict sound attenuation from a sonic crystal and also to obtain the band structure. The result from the quasi 2-D model matches well with the experiment and the finite element simulation and shows a significant improvement over the 1-D model.

Chapter 6 presents a novel concept of radial sonic crystal where scatterers, instead of being placed in a square periodic arrangement, are placed along an arc around a line/point source. Numerical model is developed to study wave propagation in a waveguide with circular wavefront. The model is first used to test an intuitive design of radial sonic crystal. The mathematical background for the design of radial sonic crystal is presented. Chapter 7 presents the experiments and simulations performed to test the design of radial sonic crystal.
1.5 Original contribution of the thesis

In this thesis, firstly a numerical model based on the Webster horn equation is presented for sound propagating through the symmetry direction in a sonic crystal. For an infinite periodic structure, band gaps are obtained. The method is further modified to obtain the complex dispersion relation, which gives additional information of the decay constant of the evanescent wave in the band gap region. The decay constant can be used to predict sound attenuation over a finite length of the sonic crystal. The sound transmission by the sonic crystal can also be directly obtained using the Webster horn equation. These results are compared with those obtained from the finite element simulations. An experiment was also performed to validate these results. A parametric study is performed on the rectangular sonic crystal to obtain the optimal design parameters for high sound attenuation.

We also developed another improvement over the previous model, referred as the quasi 2-D model. The model can be used to obtain both, the complex band gap and sound attenuation by the sonic crystal. The results predicted by the quasi 2-D model are in good agreement with the finite element simulations and experiments.

Lastly, a radial sonic crystal is envisioned and designed based on the mathematical principle of invariance in translation of a unit cell. A governing equation for sound propagation in waveguide with circular wavefront, similar to the Webster horn equation, is obtained. This equation is used to design periodic structure in radial coordinates, known as the radial sonic crystal. Based on the design from the numerical
model, finite element simulations and experiments were performed to evaluate sound attenuation by the radial sonic crystal. The results are in good agreement and such novel structure promise a new design paradigm for sound attenuation and frequency filter especially from a divergent sound source.

1.6 Acoustic wave propagation

Acoustic wave propagating in a medium [9] is given by the wave equation.

\[
\nabla^2 p - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0 \tag{1-2}
\]

The above equation is the linearized, lossless wave equation in which \( p \) is the pressure fluctuation, or disturbance in mean pressure, and \( c \) is velocity of sound which is 343 m/s at 20 °C. Acoustic wave in fluids are longitudinal in nature, i.e. the fluid molecules move back and forth in the direction of propagation of wave, producing adjacent regions of compressions and rarefactions.

In the above Eq. 1-2, fluid medium is assumed to be inviscid and there is no energy loss or dissipation in the medium. Also the medium is assumed to be homogenous and isotropic. The analysis of wave is limited to waves with relatively small amplitude. These assumptions hold well for the medium as air, and for the conditions in which the experiments were conducted in this work.
Acoustic process is adiabatic in nature. Therefore, the adiabatic equation of state is used to model the process. For perfect gas the equation for adiabatic process is given by Eq. 1-2.

\[ \frac{P^A}{P_o^A} = \left( \frac{\rho}{\rho_o} \right)^\gamma \]  

(1-3)

where \( P^A \) is the instantaneous absolute or total pressure and \( \rho \) is the density of the fluid. (The convenient symbol of \( P \) is avoided here, as it will be used to represent some other variable used throughout the thesis). The subscript \( o \) represents the constant equilibrium values for pressure and density. For fluids other than perfect gas, the adiabatic equation of state is more complex, and is determined experimentally by using Taylor series expansion as shown below.

\[ P^A = P_o^A + \left( \frac{\partial P^A}{\partial \rho} \right)_\rho_o (\rho - \rho_o) + \frac{1}{2} \left( \frac{\partial^2 P^A}{\partial \rho^2} \right)_\rho_o (\rho - \rho_o)^2 + \ldots \]  

(1-4)

The above expression is restricted to the linear term.

\[ P^A - P_o^A = B \left( \frac{\rho - \rho_o}{\rho_o} \right) \]  

(1-5)

which can be further written as

\[ \rho = B \left( \frac{\Delta \rho}{\rho_o} \right) \]  

(1-6)
where \( B \) is the adiabatic bulk modulus given by \( B = \rho_0 \left( \frac{\partial P^A}{\partial \rho} \right) \rho_0 \) and \( p \) is the change in mean pressure, or pressure fluctuation given by \( p = P^A - P_o^A \).

The equation of continuity is basically the conservation of mass. It simply means that the net rate of mass flowing through the surface of a control volume must equal the rate at which mass is increasing in the volume. The equation of continuity is given by Eq. 1-7 which is a non linear equation, because both, the density and the velocity are variable.

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0
\]  

(1-7)

By applying the Newton’s second law on the control volume, we obtain the Euler’s equation. The force due to viscosity of the fluid is neglected. The net force on a control volume due to pressure fluctuations is given by Eq. 1-8.

\[
\vec{j} = -\nabla P^A dV
\]  

(1-8)

The net acceleration of a fluid element is given by Eq. 1-8 which can be rewritten in a more compact way given by Eq. 1-10

\[
a = \frac{\partial \vec{u}}{\partial t} + u_x \frac{\partial \vec{u}}{\partial x} + u_y \frac{\partial \vec{u}}{\partial y} + u_z \frac{\partial \vec{u}}{\partial z}
\]  

(1-9)

\[
a = \frac{\partial \vec{u}}{\partial t} + (\vec{u}, \nabla) \vec{u}
\]  

(1-10)
Applying Newton’s second law on the control volume by using force and acceleration expression in Eqs. 1-8, and 1-10 we obtain,

\[- \nabla p^A = \rho \left[ \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} \right] \quad (1-11)\]

This is a nonlinear, and inviscid force equation known as Euler’s equation. It can be further simplified by assuming \( \frac{\partial \vec{u}}{\partial t} >> (\vec{u} \cdot \nabla)\vec{u} \) for small oscillations, and variation of density is small for acoustic process. This results in a linear inviscid force equation, valid for acoustic processes of small amplitude given by Eq. 1-12.

\[ \rho_0 \frac{\partial \vec{u}}{\partial t} = -\nabla p \quad (1-12) \]

Combining the three equations – adiabatic process equation, continuity equation and the force balance equation results in the wave equation given by Eq. 1-2.

When such a propagating wave encounters a solid surface, depending on surface properties, there is reflection, absorption and transmission of sound wave. The reflective property of a surface is mainly governed by the impedance ratio of the two surfaces. The specific acoustic impedance is defined as

\[ z = \frac{p}{u} \quad (1-13) \]
where $p$ is the acoustic pressure in a medium and $u$ is the associated particle speed. For plane wave, the specific acoustic impedance simplifies to

$$z = \rho_0 c$$  \hspace{1cm} (1-14)

where $c$ is the velocity of sound in that material given by, $c = \sqrt{\frac{B}{\rho_0}}$, $B$ is the bulk modulus of material and $\rho_0$ is the density of the material. The product $\rho_0 c$ has much greater significance from the acoustic point of view, and is known as the characteristic impedance of the medium. The value of characteristic impedance of some of the materials is summarized in Table 1-1 below [9].

**Table 1-1 Characteristic acoustic impedance for some materials at 20°C**

<table>
<thead>
<tr>
<th>Material</th>
<th>$z$ (Pa.s/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
<td>415</td>
</tr>
<tr>
<td>Water</td>
<td>$1.48 \times 10^6$</td>
</tr>
<tr>
<td>Steel</td>
<td>$47 \times 10^6$</td>
</tr>
<tr>
<td>Acrylic</td>
<td>$3.26 \times 10^6$</td>
</tr>
</tbody>
</table>

When we consider the acoustic wave propagating through the air and interacting with solid surface such as steel or acrylic, the ratio of characteristic acoustic impedance determines the reflection of the acoustic wave from the surface. The ratio of characteristic acoustic impedance for different materials to air are $z_{\text{steel}}/z_{\text{air}} = 1.1 \times 10^5$ and $z_{\text{acrylic}}/z_{\text{air}} =$
$7.8 \times 10^3$. This implies that steel and acrylic are good reflectors of acoustic wave in air, and therefore, can be considered as sound hard with respect to sound propagating in air.
Chapter 2. Literature Review

Sound propagation over a periodic arrangement of scatterers has been of interest over the past few decades. The first experimental observation of sound attenuation by a 2-D sonic crystal was made by Martinez et al. [1] in 1995, when it was found that an artistic creation has a possible engineering application. The structure was based on minimalistic design (an art movement in 1950’s based on simplistic forms and designs), and consist of hollow steel rods, 3 cm in outer diameter, arranged on a square lattice with a lattice constant of 10 cm. When sound propagates through this structure, it was found that certain bands of frequencies (centered around 1670 Hz) were significantly attenuated compared to other frequencies. The frequency corresponds to the destructive interference due to Bragg’s reflection and thus the experimental measurements were explained by the opening up of first band gap in the periodic structure. This finding led to increased interest among researchers to explore sound propagation through periodic structures. Sigalas and Economou [10] obtained the band gap for the same experiment on the sculpture using the plane wave expansion method.

Dowling [11] has initially drawn correspondence from the electronic band gap in semiconductors and photonic band gap in photonic crystal and applied to sound wave. He has showed theoretically in 1992 that a one dimensional structure made by periodic variation in density of fluid will exhibit band gaps. James et al. [12] has demonstrated one dimensional sonic crystal made of perspex plates in water, theoretically and
experimentally. They have also demonstrated that a narrow pass band can be obtained within the band gap by introducing a defect in the sonic crystal. Thus such structure can be used as a frequency filter.

2.1 Sound insulation

Since the sound wave is inhibited to propagate through the sonic crystal in the band gap region, one of the direct applications of sonic crystal is in selective sound reduction. Multilayer partition based on periodic arrangement of layers showed the properties of one dimensional sonic crystal [13]. Such a one dimensional sonic crystal was able to overcome the increase in transmission at the critical frequency of the panel. Trees arranged in a two dimensional periodic arrangement [14] gave better sound attenuation compared to a green belt or forest. The frequencies attenuated corresponded to the periodicity of the lattice, and the array of trees works like a sonic crystal. Hence it was proposed that these periodic arrays of trees can be used as green acoustic screens. Similarly, in another study [15], periodic structures of size 1.11 m x 7.2 m with cylinders of diameter 16 cm were used as acoustic barriers in outdoors. The results showed good agreement with those predicted by Maekawa [16] for barriers.

Goffaux et al. [17] has also proposed using sonic crystals as an insulation partition. A comparison of sound attenuation by the sonic crystal inside the band gap is made with the mass law. It is shown that beyond nine periods of repeating unit, sonic crystal performs better than the mass law. There are similarly many experimental
demonstration of sound attenuation in the band gap region [18-20]. Kushwaha [21] has proposed a multiperiodic tandem structure for obtaining a sound attenuation over a wide frequency range.

Batra et al. [22] has experimentally demonstrated three dimensional sonic crystal made of lead spheres and brass beads in unsaturated polyester resin. The sound attenuation is explained on the basis of band gaps and it is proposed that such structures can be used for selective noise reduction.

Recently Krynkin et al. [23] has also studied the effect of a nearby surface on the acoustic performance of sonic crystal. They have validated their work with semi-analytical predictions based on multiple scattering theory and numerical simulations based on a boundary element formulations. It is concluded that the destructive interference of sound reflection by ground surface can significantly affect the transmission spectra.

The same author has also explained scattering by coupled resonating elements [24]. They have considered different types of resonators – empty N slip pipes and latex cylinder covered by a concentric PVC cylinder with four slits. It is shown that increase in slits causes an increase in frequency of Helmholtz resonator. Using such coupled resonators in sonic crystal can lead to low frequency sound attenuation.
Recently elastic shells with different material properties have also been used as the scattering element [25]. The resonance of such shells result is can improve in sound attenuation in the low frequency below the first band gap.

### 2.2 Frequency filters and acoustic waveguides

Sonic crystal can also be used as a frequency filter which does not allow sound wave to propagate in the band gap region. Another way of using sonic crystal as frequency filter is by introducing defect in the periodic structure. The defect mode corresponds to a narrow frequency pass band within the band gap. A one dimensional model of defect mode was presented by Munday et al [26]. Khelif et al. demonstrated tunable narrow pass band in sonic crystal consisting of steel cylinders in water theoretically and experimentally [27, 28]. They [29, 30] have also shown that waveguides can be obtained by removing a single row of the scatterers. These waveguides are also demonstrated for tight bending of acoustic waves in the sonic crystal. Li et al. [31] has shown bending and branching of acoustic waves in V shape waveguides made from two dimensional sonic crystal.

Pennec et al. [32] has theoretically investigated propagation of acoustic waves through the waveguides of steel hollow cylinders arranged periodically in water. They have demonstrated presence of narrow pass band inside a broad stop band. The pass band can be adjusted by appropriately selecting the inner radius of the hollow cylinders or by filling the cylinders with a different density fluid. They have also extended this to a waveguide with hollow steel cylinders with two different inner radii varying
alternatively. Such a novel waveguide has been shown to have two narrow pass bands corresponding to individual radii of hollow cylinder. An active guiding device is proposed by changing the fluid in these two different cylinders. Also, a Y-shaped waveguide is shown to act as a multiplexer and demultiplexer for separating and merging signals with different frequencies.

Miyashita et al. [33, 34] has demonstrated experimentally sonic crystal waveguide made by acrylic cylinders in air. Straight waveguide and sharp bending waveguide composed of line of single defect are shown to have a good transmission in a narrow pass band. They have proposed waveguides based on defect in sonic crystal as potential application in acoustic circuits made on sonic crystal slab.

2.3 Metamaterials and radial wave crystal

Recently, there has been immense growth in the area of metamaterials, due to their ability to manipulate light and sound waves, which are not available in nature. Metamaterials are artificial structures made by periodic arrangement of scatterers, similar to sonic crystal, but in this case, the periodic units are much smaller than the wavelength propagating over the structure. As a result, the wave sees an ‘effective material properties’. It is something like, replacing natural atoms by larger man-made atoms. The result of such design leads to unique material properties such as negative density, negative bulk modulus, negative refractive index etc.
The first concept of metamaterial was proposed by Vesalago for electromagnetic wave in 1960 [35]. Later on Pendry et al. [36, 37] proposed artificial structure materials having effective negative permeability and permittivity. The negative refractive index material was first demonstrated at GHz frequency [38, 39].

The first experimental evidence of acoustic metamaterial was observed by Liu et al. [40], where locally resonant sonic materials demonstrated negative effective dynamic density. Recently, Fang et al. [41] has also proposed acoustic metamaterial based on Helmholtz resonators which exhibit negative effective modulus. The feasibility of such negative effective material properties has led to very interesting applications. One such application is in cloaking, or making an object invisible to electromagnetic [42, 43] or acoustic wave [44-47].

Torrent et al. [48, 49], has also recently proposed a new kind of metamaterial known as radial wave crystal which are metamaterials in polar or radial coordinate. They have demonstrated that such metamaterials possesses anisotropic material properties. The material properties of density tensor and bulk modulus were obtained from the property of invariable in translation on the governing wave equation from one unit cell to another. We have used similar concept to design radial sonic crystal to attenuate sound propagating with circular wavefront.

Arc shared phononic crystal have also been studied recently using transfer matrix method in cylindrical coordinates [50].
2.4 Other applications

Another interesting application of sonic crystal is in sound diffusers [51] in room. Such diffusers can help in improving the acoustic performance of a room by reducing the echo and increasing the sound field diffusiveness, especially at low frequencies.

Sonic crystals have also been shown for the application of acoustic diode for unidirectional sound propagation [52]. Previously it has been demonstrated that it requires strongly nonlinear materials to break the time reversal symmetry in a structure [53, 54]. However, in this recent work Li et al. [52] have experimentally realized unidirectional sound transmission through the sonic crystal. The nonreciprocal sound transmission is controlled simply by mechanically rotating the square cylinders of the sonic crystal. Li et al. [52] also claims that the new model of sonic crystal based acoustic diode being a linear system is more energy efficient and operates at a broader bandwidth than the acoustic rectification based on nonlinear materials.

Sonic crystal is also used for an application of liquid sensor [55]. The liquid sensor is designed based on the transmission spectra of the sonic crystal. The shift in the band gap is used to predict the material properties of the liquid. Sonic crystal is also used to exhibit the phenomenon of resonant tunneling [56, 57].
2.5 Numerical Methods for calculation and optimization of the band gap

All the applications of the sonic crystal are primarily based on the prediction of the band gap. The band gap is characterized by the position (center frequency) and the width of the band gap. There are methods such as plane wave expansion method \([10, 58-60]\) and multiple scattering method \([61, 62]\) for calculating the band structure. These methods were mainly developed for photonic and phononic crystals which were also extended to sonic crystal.

The first complete band gap calculation for periodic elastic composites was presented by Kushwaha et al. in 1993 \([63]\). The computation was performed for nickel alloy cylinders in aluminum alloy background and vice versa. Phononic band gaps have also been obtained using a 1-D and 2-D periodic spring mass system by Jensen \([64]\). Acoustic band gaps have also been shown for silica cylinders in viscous liquid \([65]\). Plane wave expansion method is used to investigate the effect of viscosity and it has been shown that viscosity can lead to larger band gaps. Band gaps are also shown for time varying materials \([66]\). Thus by varying the material properties of the phononic crystal, band gaps can be modified dynamically. Band gaps have also been optimized using structure topology optimization \([67]\). Band gaps can also be tuned by placing an additional rods in the unit cell \([68]\).
Miyashita [69] has demonstrated experimentally sound attenuation by an array of 10 x 10 acrylic cylinders. He has explained the sound reduction by obtaining sound transmission over a finite periodic structure using the finite difference time domain method. However, this method is computationally expensive, as one solves the wave propagation over two dimensional periodic structures in real time. The space discretization is restricted by the wavelength of propagating wave and the time step is restricted by the Courant’s condition, which makes this method computationally very expensive. In another similar work [58], they have used 10 x 10 copper cylinders in air. Sound transmission through the sonic crystal was obtained experimentally, which was compared with the numerical prediction from the finite difference time domain method and band gap calculation from the plane wave expansion method. It was further shown that since the impedance mismatch between copper and air is very high, sound transmission through the hollow and filled cylinders is the same. Therefore, the cylinders boundary can be effectively modeled as sound hard boundaries.

An improved numerical method based on finite difference time domain method was proposed by Cao et al. [70] to obtain the band structure. The method is shown to overcome the convergence problem of the plane wave expansion method for liquid cylinder in solid matrix. The computational time is also reduced compared to the conventional finite difference time domain method.

Chen and Ye [71] have shown that as the cylinders are randomized, the sound attenuation is improved over a wide range of frequencies. Band gap have also been
shown to increase by reducing the symmetry of the structure[72]. Goffaux [73] has also shown that the band gaps can be tuned or controlled by rotating the sonic crystal made of rectangular scatterers. Tunable acoustic band gap are also shown for sonic crystal made of dielectric elastomer cylindrical actuator [74]. Applying voltage to the dielectric elastomer cylindrical actuator causes a radial strain in the cylinders which affects the band gap.

Kushwaha [75] has also studied a three dimensional sonic crystal made of air bubbles in water in different configurations such as, face-centered cubic, body-centered cubic and simple-cubic. Such structure with huge difference in acoustic impedance has been reported to have widest band gap [76]. Kushwaha et al. [77] has performed band gap calculations for three dimensional sonic crystal made of rigid spheres and cubes in air. They have also proposed a tandem structure that allows for an ultrawideband filter for environmental or industrial noise in the desired frequency range. Band gap calculated for cubical array of hydrogen spheres in air [78].

2.6 Evanescent wave

Recently, an extended plane wave expansion (EPWE) method has been proposed [8, 79] to compute the complex band structure [80] of the sonic crystal. The complex band structure gives the additional information of the decay constant (imaginary part of wavenumber) of the sound wave in the band gap region. Also, it has been shown that for a finite structure or for structures having defects, the propagating wave is evanescent in
nature [81]. Therefore, using the decay constant, sound attenuation through the finite arrangement of scatterers can be obtained.

The complex band gap is important because we rarely find an ‘infinite’ sonic crystal, especially for applications of sound wave in audible frequency range. The dimension of sonic crystal (based on bragg’s reflection) depends on the wavelength of the sound wave, which is quite large (~ few cm) for the audible frequency range. Therefore, the sonic crystal has a finite number of scatterers and it is important to evaluate the performance of a finite sonic crystal. There are methods such as finite difference time domain method to obtain transmission coefficient through the finite sonic crystal, but such methods are computationally very expensive. However, obtaining the attenuation through the decay constant gives a quick check on the sound attenuation level expected by the structure in the band gap region. In the present work, we have proposed a novel method based on Webster horn equation to obtain the decay constant of the evanescent wave in the band gap region, which can predict the sound attenuation through the finite sonic crystal.

2.7 Webster horn equation

The Webster horn equation [82, 83] is an approximate one dimensional equation for the acoustic wave propagation in a horn or a waveguide. Webster assumed the wavefront as an isophase surface. For Cartesian coordinates system it is generally assumed that P(x) is a function of x only and therefore it is a plane wave. However, in
some another coordinate system, Webster horn equation can also model non-planar wavefront.

A detailed review on the Webster horn equation and its solution was given by Eisner [84]. There have been some improvements over the Webster equation. Martin [85] had obtained a hierarchy of one dimensional ordinary differential equations for an axis-symmetric waveguide. The equations were obtained by solving the Helmholtz equation using power-series expansion method in a stretched radial coordinate. The lowest approximation leads to the Webster equation and second approximation leads to a fourth order differential equation. Rienstra [86] also discusses about sound propagation in waveguide with a mean flow, duct with acoustic lining etc. Webster horn equation has many applications in predicting sound propagation through waveguides, horns, musical instruments etc. Recently we have used Webster horn equation for predicting frequency band structure in sonic crystal [87].
Chapter 3. One dimensional model for sound propagation through the sonic crystal.

In this chapter, a one dimensional model is presented to obtain the band structure of an ‘infinite’ sonic crystal. The model is modified to obtain complex band structure which gives additional information of the decay constant for the evanescent wave in the band gap region. Using this decay constant, sound attenuation over a finite length of sonic crystal can be predicted. The last section presents a direct way to obtain sound attenuation through a finite sonic crystal by modeling the sonic crystal as a waveguide. The results from the finite sonic crystal model match well with the prediction of band gap and with the sound attenuation by the decay constant.

3.1 Computation of band structure

Interaction of sound wave with a periodic arrangement of scatterers leads to formation of frequency bands where sound cannot pass through the structure. Such bands of frequencies are known as band gaps.

In the present work, sound propagation over a two dimensional sonic crystal is considered. The scatterers are sound hard cylinders with diameter 3 cm, arranged on a square lattice with lattice constant of 4.25 cm. The sonic crystal is shown in Fig. 3-1(a). The sound wave with a planar wavefront propagates along the symmetry direction (ΓX)
of the periodic structure. The triangle $\Gamma X M$ represents the irreducible part of the first Brillion zone[2]. Because of the symmetry, wave propagation in any direction can be represented by wave propagation along the irreducible part of the Brillion zone. However, in the present study, sound propagation is considered only along the symmetric direction $\Gamma X$.

The periodic structure is represented by a unit cell as shown by the dotted square in Fig 3-1(a). Simple translation of this unit cell can form the whole periodic structure. Figure 3-1(b) shows the magnified view of the unit cell with some geometric parameters. The lattice constant is given by $a$ which is the center to center distance between the adjacent scatterers. Since this is a square lattice therefore, the lattice constant is same in both $x$ and $y$ directions.

![Figure 3-1(a) A two dimensional periodic structure made of circular scatterers arranged on a square lattice. On the left side there is plane wave sound source. The dotted square shows a unit cell. (b) Magnified view of a unit cell with various geometric parameters.](image)
Sound propagation through the unit cell takes place through the air (shaded region in Fig. 3-1(b)) as the scatterers are sound hard with respect to air. Sound propagation can thus be modeled by the Webster horn equation [83, 88].

\[
\frac{d^2 p(x,t)}{dt^2} = \frac{c^2}{S} \frac{d}{dx} \left( S \frac{dp(x,t)}{dx} \right)
\]  

(3-1)

where \( p \) is the pressure, \( c \) is the velocity of sound in air and \( S \) is the variable cross sectional area. The cross sectional area \( S \) is perpendicular to the direction of wave propagation and is represented by the vertical bold line in Fig. 3-1(b). Furthermore, we consider the case when pressure is harmonic, and the signal can be represented as a sum of harmonic functions.

\[
p(x,t) = \text{Re} \left( P(x) e^{i\omega t} \right)
\]  

(3-2)

where \( P(x) \) is a complex-valued amplitude.

For harmonic response, the Webster horn equation further reduces to Eq. (3-3).

\[
\frac{d^2 P}{dx^2} + \frac{dP}{dx} \left( \frac{S'}{S} \right) + \frac{\omega^2}{c^2} P = 0
\]  

(3-3)

where \( S' \) represents the first derivative of \( S \) with respect to \( x \) and \( \omega \) is the angular frequency of the propagating wave.

Wave propagation in a periodic structure, as shown by Fig 3-1(a), can be restricted to a unit cell (Fig 3-1(b)) by using the Bloch theorem [2] or the Floquet theorem for 1-D case [3]. The wave propagating in such a periodic structure is given as,

\[
p(x) = e^{ikx} \phi(x)
\]  

(3-4)
where \( \phi(x) \) is a periodic function with periodicity \( a \), i.e. \( \phi(a + x) = \phi(x) \). Bloch waves are plane wave (\( e^{ikx} \)) modulated with a periodic function \( \phi(x) \), which has the same periodicity as that of the lattice structure.

Using the Floquet theorem with the Webster horn equation in a unit cell, the governing equation reduces to the following equation.

\[
\left\{ \frac{d^2 \phi}{dx^2} + \frac{d\phi}{dx} \left( 2ik + \frac{S'}{S} \right) + \phi \left( ik \frac{S'}{S} - k^2 \right) \right\} + \frac{\omega^2}{c^2} \phi = 0
\]  
(3-5)

The above equation is solved using the finite difference method. The unit cell is divided into \( N \) segments of length \( h \) with \( (N+1) \) points in the \( x \) direction. Second order central difference scheme is used for the finite difference discretization.

\[
\frac{d^2 \phi}{dx^2} = \left( \phi_{j+1} - 2\phi_j + \phi_{j-1} \right) \frac{1}{h^2} + O(h^2)
\]  
(3-6)

\[
\frac{d\phi}{dx} = \left( \frac{\phi_{j+1} - \phi_{j-1}}{2h} \right) + O(h^2)
\]  
(3-7)

As the structure is periodic, periodic boundary conditions are applied at the first and last node of the discretized geometry.

This transforms Eq. (3-5) to an eigenvalue problem given by Eq. (3-8).

\[
[A(k)]\{\phi\} + \frac{\omega^2}{c^2}\{\phi\} = 0
\]  
(3-8)

where \( A \) is a matrix whose coefficients are function of the wavenumber \( k \) which is varied in the first Brillouin zone\( \left( -\frac{\pi}{a} \leq k \leq \frac{\pi}{a} \right) \). The solution to the eigenvalue problem (Eq. (3-8)
results in the feasible values of frequencies that can propagate through the structure. This results in the band structure or the dispersion relation for periodic structure as shown in Fig. 3-2. The band structure is a plot between the frequency and wavenumber. The band gaps are represented by the shaded region where there is no real wave that can propagate through the periodic structure.

The band gap calculation shows that the first band gap opens up at a frequency band of 2795 Hz – 5470 Hz. Band gap is represented by two parameters: center frequency and band width. The center frequency of the band gap is the mean frequency of the band gap, which in this case is 4132 Hz. The center frequency of the periodic structure can also be predicted by the Bragg’s law [89] given by Eq. (3-9).

\[ f_c = \frac{c}{2a} \]  

(3-9)

The center frequency from the Bragg’s law prediction is 4035 Hz, which is quite close to the center frequency predicted by the numerical model (4132 Hz).
Figure 3-2 Band gap for an infinite sonic crystal corresponding to Fig. 3.1 with $a = 4.25$ cm and $d = 3$ cm, along the symmetry direction $\Gamma X$.

Another representative parameter of a band gap is the band width which is the difference between the upper and lower frequencies of a band gap. The band width for first band gap is 2675 Hz.

In the band gap region of 2795 Hz – 5470 Hz, no propagating wave exist for an ‘infinite sonic crystal’. There are other band gaps at higher frequencies, three of which are shown by the shaded region in Fig. 3-2. However, the band widths of the three band gaps are quite small.
3.2 Complex frequency band structure and decay constant

The previous section gives a method to obtain band gaps where there is no solution to the wave frequency for a given wavenumber varying in the first Brillion zone. As mentioned in chapter 2, there are many other methods to obtain band gap. However, band gap does not give any information about the sound attenuation over the finite sonic crystal.

In this section, a method is presented to obtain decay constant and complex band gap which can predict sound attenuation by the sonic crystal in the band gap region. Decay constant is the imaginary part of the wavenumber, which predicts evanescent wave in the band gap region. The attenuation in the evanescent wave over a finite length can be obtained numerically, which gives the sound attenuation by the finite sonic crystal. The combined plot of band gap along with the decay constant is referred as the complex band gap or complex dispersion relation.

In the previous section, it was shown that Webster horn equation along with Floquet theorem leads to a differential equation given by Eq. (3-5), which reduces to an eigenvalue problem (Eq. (3-8)) using the finite difference discretization and periodic boundary condition. For this case, the independent variable was wavenumber $k$ and the problem was solved for feasible values of angular frequency $\omega$. Equation (3-5) can be reformulated in the reverse way with the independent variable being angular frequency, and solving for the dependent variable wavenumber.
The above equation when discretized using finite difference equations (Eq. 3-6, Eq. 3-7) along with periodic boundary condition leads to the following quadratic eigenvalue problem in wavenumber.

\[
\begin{bmatrix}
[A_2] \{\phi\}
\end{bmatrix} k^2 + \begin{bmatrix}
[A_1] \{\phi\}
\end{bmatrix} k + \begin{bmatrix}
[A_0] \{\phi\}
\end{bmatrix} = 0
\] (3-11)

where, \(A_i\) are all matrices of size N x N, with known coefficients which are function of frequency and \(\{\phi\}\) is the eigenvector of size N x 1. The matrices \(A_0\) and \(A_1\) are periodic tridiagonal matrices, whose coefficients are given as,

\[
(A_0)_{m,m-1} = \frac{2}{h^2} - \frac{S'}{2Sh} , (A_0)_{m,m} = \left(\frac{2}{h^2}\right) - \frac{2}{h^2}, (A_0)_{m,m+1} = \frac{1}{h^2} + \frac{S'}{2Sh}
\]

for \(2 \leq m \leq N-1\)

Due to the periodicity of the unit cell, the first and last rows are given by,

\[
(A_0)_{1,1} = \left(\frac{2}{h^2}\right) - \frac{2}{h^2}, (A_0)_{1,2} = \frac{1}{h^2} + \frac{S'}{2Sh} , (A_0)_{1,N} = \frac{1}{h^2} - \frac{S'}{2Sh}
\]

\[
(A_0)_{N,1} = \frac{1}{h^2} + \frac{S'}{2Sh} , (A_0)_{N,N-1} = \frac{1}{h^2} - \frac{S'}{2Sh} , (A_0)_{N,N} = \left(\frac{2}{h^2}\right) - \frac{2}{h^2}
\]

Similarly coefficients of periodic tridiagonal matrix \(A_1\) are

\[
(A_1)_{m,m-1} = -\frac{i}{h} , (A_1)_{m,m} = i\frac{S'}{S} , (A_1)_{m,m+1} = \frac{i}{h} \quad \text{for} \quad 2 \leq m \leq N-1
\]

\[
(A_1)_{1,1} = i\frac{S'}{S} , (A_1)_{1,2} = \frac{i}{h} , (A_1)_{1,N} = -\frac{i}{h}
\]

\[
(A_1)_{N,1} = \frac{i}{h} , (A_1)_{N,N-1} = -\frac{i}{h} , (A_1)_{N,N} = i\frac{S'}{S}
\]

Lastly, \(A_2\) is a diagonal matrix with diagonal element as
\((A_2)_{m,m} = -1\) for \(1 \leq m \leq N\).

The above quadratic eigenvalue problem (Eq. (3-11)) is solved by rearranging the equation in a linear form as shown below,

\[
\begin{pmatrix}
-A_0 & 0 \\
0 & I
\end{pmatrix}
\begin{pmatrix}
\phi \\
k\phi
\end{pmatrix}
= k
\begin{pmatrix}
A_1 & A_2 \\
I & 0
\end{pmatrix}
\begin{pmatrix}
\phi \\
k\phi
\end{pmatrix}
\]

(3-12)

which is solved using a standard eigenvalue solver in Matlab to obtain \(k\). In general, the wavenumber \(k\) for a given frequency is found to be a complex number.

\[
k = k_R + ik_I,
\]

(3-13)

where \(k_R\) and \(k_I\) are the real and imaginary parts, respectively.

In a unit cell, the pressure variation is given by Eq. (3-4). The complex wavenumber causes an exponential decay in the amplitude of wave due to the imaginary part of the wavenumber \(k_I\).

\[
P(x) = \phi(x) \exp(-k_I x) \exp(ik_R x)
\]

(3-14)

If \(k\) is real valued \((k_I = 0)\), wave propagates with constant amplitude. For a positive imaginary part \((k_I > 0)\), wave has an evanescent behavior, leading to attenuation in the amplitude of pressure wave in the propagating direction. The value of \(k_I\) gives us the decay in the amplitude of the wave, and hence it is referred as the decay constant. The decay constant and complex band structure is plotted in Fig 3-3. The first band gap predicted is from 2795 Hz – 5470 Hz with center frequency of 4132 Hz. The results match with the band gap calculation in section 3.1. The maximum value of decay constant at the center frequency is found to be 24.75.
Figure 3-3 Complex band structure for an infinite sonic crystal. (a) normal band structure. (b) Decay constant as a function of frequency. The decay constant is non-zero in the band gap regions.

As it can be seen from the figure, band gap corresponds to the range of frequencies where a nonzero value of $k_I$ exists. Thus, for a finite sonic crystal, the propagating wave undergoes sound attenuation for the frequencies corresponding to the band gap. Based on Eq. (3-14), sound attenuation (in dB) over a length $x$ is calculated as

$$\text{Attenuation} = 20 \times \log(1/\exp(-k_Ix))$$

(3-15)

The sound attenuation over a length of five unit cells (0.21m) is predicted by the decay constant and is plotted in Fig. 3-4 for a frequency range of 500 Hz – 6000 Hz. The sound attenuation in the band gap region is not uniform; rather it is elliptic in nature. The maximum sound attenuation is found to be 45 dB at the center frequency of the band gap.
Figure 3-4 Sound attenuation predicted by the decay constant.

### 3.3 Sound attenuation by the sonic crystal using the Webster horn equation

Sound attenuation by a finite sonic crystal can be directly obtained by modeling the sonic crystal as a waveguide and obtaining sound transmission through the waveguide using the Webster horn equation. The difference between previous sections and this section is that, no periodic condition such as Bloch theorem or Floquet theorem is implemented in the formulation. The periodicity is in the geometry of the waveguide.

The problem considered is shown in Fig. 3-5(a) with sonic crystal consisting of five layers of scatterers. A planar sound wave propagates along the symmetry direction
(ΓΧ) of the sonic crystal. Since the problem is symmetric about AB and CD, the model can be reduced to a strip model as shown by the rectangle in Fig. 3-5(a).

Figure 3-5 (a) Sound propagating over a sonic crystal consisting of five layers of scatterers. Using symmetry of the structure, the problem is reduced to a strip model shown by rectangle ACDB. (b) A symmetric waveguide used to model sound propagation through the sonic crystal using Webster horn equation.

The model is further reduced by taking the symmetry about the center line to give a waveguide as shown in Fig. 3-5(b). The top and bottom surfaces including the cylinders are modeled as sound hard boundaries. There is a sound source at the inlet end, and radiation boundary condition is applied at the outlet end. The problem is effectively reduced to one of sound propagation through a symmetric waveguide as shown in Fig. 3-5(b) which is solved using the Webster horn equation.
Sound propagating through a waveguide with a variable cross-sectional area is modeled by the Webster horn equation. The Webster horn equation considers pressure to be a function of the direction of wave propagation, and constant over the cross-section of the waveguide. This reduces the problem to a 1-D model represented by an ordinary differential equation. For harmonic excitation, the Webster horn equation is given by the Eq. (3-3), where \( S \) is represented by the cross-sectional area of the waveguide as shown in Fig. 3-5(b).

The Webster horn equation is discretized along the \( x \)-axis using the second-order finite difference method to obtain a system of linear equations. The radius of cylinders \( r \) is 1.5 cm and lattice spacing \( a \) is 4.25 cm. The cross-sectional area function \( S(x) \) is shown in Fig. 3-5(b) by the dashed line. The numerical results for pressure were found to converge for 2000 mesh points, for frequency up to 6000Hz.

Sound attenuation by the sonic crystal is given by the insertion loss.

\[
IL = SPL_{\text{without SC}} - SPL_{\text{with SC}}
\]

(3-16)

where \( SPL_{\text{without SC}} \) and \( SPL_{\text{with SC}} \) are the sound pressure levels at the same position without and with the sonic crystal, respectively. Sound pressure is obtained 10 cm (~2.5\( a \)) away from the last cylinder. The pressure amplitude obtained at this position corresponds to sound pressure level with the sonic crystal \( (SPL_{\text{with SC}}) \). When the cylinders are removed, the waveguide becomes a straight channel with uniform pressure amplitude across the cross-section, as the other end has the radiation boundary condition.
Therefore, the sound pressure level without cylinders at the same position is the same as that corresponding to the incident wave amplitude when the cylinders are present.

Using the standard definition of sound pressure level, the above expression for insertion loss reduces to

$$IL = 20 \times \log_{10} \left( \frac{P_I}{P_O} \right)$$

(3-17)

where $P_I$ is the amplitude of the inlet pressure wave that is incident on the sonic crystal and $P_O$ is the amplitude of the outgoing pressure wave measured 10 cm after the last cylinder.

At the inlet boundary, a sound source with pressure of 1 Pa was prescribed. However, the prescribed pressure at the inlet boundary is not the forward traveling incident wave. The pressure at the inlet boundary is the resultant of forward traveling incident wave and backward traveling reflected wave. Pressure and velocity condition at the inlet boundary are used to extract the incident pressure wave.

The net pressure at the inlet can be written as a combination of forward and backward travelling wave.

$$p(x, t) = \text{Re}\{P_I e^{i(\omega t - kx)} + P_R e^{i(\omega t + kx)}\} = \text{Re}\{P(x)e^{i\omega t}\}$$

(3-18)

therefore,

$$P(x) = P_I e^{-ikx} + P_R e^{ikx}$$

(3-19)

where $P_I$ and $P_R$ are the amplitudes of the incident and reflected wave, respectively.
Similarly the velocity is given as [83],

\[ u(x,t) = \text{Re} \left\{ \frac{1}{\rho c} \left( P_i e^{i(\sigma t-kx)} - P_r e^{i(\sigma t+kx)} \right) \right\} = \text{Re} \left\{ U(x) e^{i\omega t} \right\} \quad (3-20) \]

where,

\[ U(x) = \frac{1}{\rho c} \left( P_i e^{-ikx} - P_r e^{ikx} \right) \quad (3-21) \]

At the inlet \((x = 0)\), Eqs. 3.19 and 3.21 reduces to

\[ P_{i\mid x=0} = P_i + P_r \quad (3-22) \]

\[ U_{\mid x=0} = \frac{P_i}{\rho c} - \frac{P_r}{\rho c} \quad (3-23) \]

The above set of simultaneous equations is solved to get the amplitude of incident forward traveling wave as a function of inlet boundary pressure \(P\) and velocity \(U\) at the boundary.

\[ P_i = \left( \frac{P + \rho c U}{2} \right)_{\mid x=0} \quad (3-24) \]

The outgoing wave amplitude \((P_O)\) can be directly obtained from the pressure field in the output region as there is no reflected wave in the output region.

In this way, sound attenuation at a particular frequency is obtained. The procedure is repeated for a range of frequencies from 500 Hz – 6000 Hz, with a frequency step of 10 Hz to obtain the sound attenuation by five cylindrical scatterers arranged periodically. Sound attenuation by the finite sonic crystal using the Webster horn equation and from the decay constant is plotted in Fig 3-6.
Figure 3-6 Sound attenuation by the finite sonic crystal using the Webster horn equation and decay constant.

It is also important note is that the derivative $S'$ in the Webster horn equation is not defined at the points where cylinders/circle intersect the horizontal line of the waveguide. However, this derivative is numerically evaluated using second order finite difference scheme, which approximates this discontinuity with a finite value. The area function $S$ (in meters) and its derivative $S'$ is shown for one unit cell in Fig 3-7 below. The area function is plotted on the left axis while the derivative is plotted on the right side axis. There is a jump in the magnitude of $S'$ at the discontinuity points. In the region where the cylinder is not present, the area derivative is zero.
Figure 3-7 The area function $S$ (plotted on left axis) and its derivative (plotted on right axis) for a unit cell.

However, the discontinuity is handled by the second order finite difference method, as the results are validated by mesh convergence test. The mesh convergence test for sound attenuation is shown for different mesh sizes of 100, 400, 1000, 2000 and 4000 grid points. It can be seen that sound attenuation result for 100 and 400 mesh points are not so accurate. However, for the mesh size of 1000 points onwards, the results are practically indifferent. The sound attenuation from 1000 points, 2000 points and 4000 points have overlapped each other. To get a clearer picture, sound attenuation at highest frequency (6 kHz) is compared for different mesh points in Fig 3-9. The result shows that
the sound attenuation at 6 kHz has converged for 1000 points onwards. For our present work we have used a mesh size consisting of 2000 points.

Figure 3-8 Mesh convergence test for 1-D model using second order finite difference method. The results are indifferent after 1000 points. For our simulation, we have used 2000 points.
Figure 3-9 Sound attenuation at 6 kHz for different mesh size.

The sound attenuation by the Webster horn equation for a finite sonic crystal shows significant sound attenuation (> 20 dB) in the band gap region – 2795 Hz – 5470 Hz, as predicted by the model in section 3.1. Further the sound attenuation magnitude is quite close to the predictions made by the decay constant in section 3.2. Decay constant is based on an infinite sonic crystal, therefore, for the frequencies not in the band gap (such as below 2500 Hz), sound attenuation is zero. However, for a finite sonic crystal, there is a finite sound attenuation, which increases significantly in the band gap region.
3.4 Conclusion

In section 3.1, a one dimensional numerical model based on the Webster horn equation and Floquet theorem is proposed to obtain the band gap for an infinite sonic crystal. The model solves for the allowable frequency for a given wavenumber. For a given geometry of circular scatterers and their lattice constant, first band gap was obtained as 2795 Hz – 5470 Hz. The band gap gives the information that for frequencies within the band gap, there exists no propagating wave for an infinite sonic crystal.

In section 3.2, the same problem was reformulated to obtain the wavenumber for a given frequency. The reformulated problem leads to complex band gap with additional information of decay constant. Decay constant is the imaginary part of the wavenumber which gives the information of the band gap and also of the decay in amplitude of the evanescent wave for a finite sonic crystal. It was shown that band gap exist in the region with non-zero decay constant. Thus sound attenuation by the sonic crystal over a finite length could be predicted by the help of decay constant. Sound attenuation over the band gap was found to be elliptic in nature. A maximum sound attenuation of 45 dB was observed at a central frequency of 4130 Hz for a finite sonic crystal consisting of five scatterers arranged periodically.

Finally in section 3.3, sound attenuation by the same finite sonic crystal was obtained directly from the Webster horn equation and verified by the predictions of decay constant. Using symmetry in the model, sonic crystal was modeled as a waveguide. Sound propagation through the waveguide was modeled by the Webster horn equation.
Sound attenuation by the sonic crystal measured by the insertion loss was obtained. The sound attenuation predicted by the 1-D model matches well with the band gap predictions and with the sound attenuation obtained from the decay constant.
Chapter 4. Validation of 1-D model by experiment and finite element simulation

In this chapter, experiment and finite element model are discussed which are used to validate the one dimensional model developed in the previous chapter. In the last section, a parametric study is performed with the validated numerical model on rectangular sonic crystal to find geometric parameters for high sound attenuation.

4.1 Experiment

As explained in section 3.3, sound propagation through a sonic crystal consisting of five layers of scatterers can be reduced to a strip model with sound propagating through a rectangular channel over an array of five scatterers. The problem is explained in Fig. 3-5(a) with rectangle ABCD being the representative strip model because of the symmetry in the problem. The experiment is designed based on this representative model.

The experiment (Fig. 4-1) consists of sound wave propagating through an array of five acrylic cylinders. The outer diameter of cylinder is 3 cm and distance between the cylinders is 4.25 cm. The height of the cylinders is 25 cm. The cylinders were mounted on a periodic grid, while care was taken to ensure that the side walls are parallel. A speaker (Philips FWB-MX970RS) was used as the sound source. The distance between the speaker and cylinder was 80 cm so that the wavefront that interacts with cylinder is close to a plane wavefront. Acoustic foam was used in regions around the source and receiver (as shown by thick lines in Fig. 4-1) to reduce reflections. The acoustic foam on
the source serves the purpose of absorbing the diverging source wave and ensuring that
the incident wave is mostly planar. However, some inclined incident wave may be
present, which may further get reflected by the acrylic walls on sides and at the top,
leading to some minor errors in the experimental measurements. The microphone (1/4
inch) used on the receiver end was kept at a distance of 10 cm from the last cylinder of
the sonic crystal. The microphone was connected to an analyzer (HP 35670A). A
harmonic analysis was performed on the system by sending a fixed frequency signal to
the speaker and the microphone measurement at the same frequency was noted. The
sound pressure level obtained from the microphone was averaged 50 times to obtain a
consistent experimental reading. This process is performed over a range of frequencies
from 500 Hz to 6 KHz at a frequency step of 3.5 Hz.

![Experimental setup with sound propagating over five acrylic cylinders.](image)

Figure 4-1 Experimental setup with sound propagating over five acrylic cylinders.
The side walls, representing sound hard boundary condition were made of acrylic which is 1 cm thick. The impedance of acrylic is $3 \times 10^6$ Pa.s/m, while impedance of air at 20 °C is 415 Pa.s/m. Therefore, acrylic wall and cylinder both act as sound hard surfaces with respect to air.

The background noise measurements were carried out for the frequency range of interest (500 – 6 kHz) and are shown below in Fig. 4-2. The maximum sound pressure level is 29 dB at 500 Hz. The experimental measurements of sound pressure level with speaker were around 70 dB. So, even after a maximum attenuation of 30 dB by the sonic crystal, the SPL is around 40 dB which is 10 dB more than the maximum ambient noise. Therefore the experimental error due to background noise can be considered negligible.
The sound attenuation by the sonic crystal was measured by insertion loss ($IL$) as explained by Eq. 3-16. $SPL_{\text{without SC}}$ and $SPL_{\text{with SC}}$ is sound pressure level measured at the same position (10 cm from the center of last cylinder) without and with the sonic crystal, respectively. The sound attenuation is shown in Fig. 4-3 below. It shows the variation in sound attenuation from 10 different experiments, each experiment being averaged 50 times. It can be seen that for some frequencies, the variation in experimental measurements is high. This can be due to those frequencies corresponding to some resonant mode in the experimental setup, which is sensitive to some perturbations.
Figure 4-3 Experimental measurements of sound attenuation from 10 experiments (each averaged 50 times). The figure shows the variation in experimental observation.

The experimental results along with attenuation predicted by the decay constant and Webster horn equation over five cylinders is shown in Fig. 4-4. The result from experiment verifies the prediction of the band gap, with a significant attenuation of more than 20 dB in the region of 2900 Hz – 5000 Hz. For frequencies below and above this band gap sound attenuation is quite low (< 10 dB). However, comparing with the band gap prediction from the 1-D model of 2795 Hz – 5470 Hz, the experimental results show
a frequency shift in the sound attenuation curve. This can be due to the one dimensional nature of the numerical model.

Figure 4-4 Experimental results of sound attenuation along with the results predicted by the decay constant and the Webster equation model.

4.2 Finite element simulation

To validate the results from the Webster equation model and experiment, finite element simulations were performed on the strip model in Fig 3-5(a) using the software COMSOL Multiphysics 3.4. A two dimensional schematic model along with boundary conditions is shown in Fig. 4-5, which consists of an array of five circular scatterers having the same geometric parameters as mentioned before.
The pressure is assumed to be harmonic in time, and the wave propagation is modeled by the Helmholtz equation given by Eq. 4-1.

\[ \nabla^2 P(x,y) + \frac{\omega^2}{c^2} P(x,y) = 0 \]  \hspace{1cm} (4-1)

where \( p(x, y, t) = \text{Re}(P(x, y)e^{i\omega t}) \) and \( P(x,y) \) is the complex amplitude.

The radiation boundary condition for a plane wave in COMSOL is given as

\[ -n \cdot \nabla P + ikP = 0 \]  \hspace{1cm} (4-2)

where \( n \) is the outward pointing normal vector.

Figure 4-5 Boundary conditions and model for the two dimensional finite element simulation.

For the finite element analysis, triangular quadratic elements (T6) were used for meshing the domain. The model consists of 5004 elements, and the simulation was performed over a range of frequency from 500 Hz to 6000 Hz in steps of 10 Hz. The simulation result was checked for convergence by using a finer mesh for the pressure field at the highest frequency of 6000 Hz. For the current model, the maximum element
size is 5.6 mm, which was less than 1/10 of the wavelength at the highest frequency simulated (6000 Hz corresponds to a wavelength of 57 mm in air).

The sound attenuation by the finite element model was computed based on the insertion loss as given by Eq. 3-17. In the case finite element simulation, there are two ways to obtain the sound attenuation. One is the same as presented in section 3-3 by applying constant pressure boundary condition at the inlet boundary and then decoupling the forward travelling wave to obtain $P_I$. The other simpler method is to use the inbuilt radiation boundary condition in COMSOL with a radiating sound pressure of 1 Pa at the inlet boundary which is the forward travelling wave. The sound attenuation is calculated by the insertion loss give by Eq. 3-17.

The sound attenuation by the two dimensional finite element model along with experiment results and 1-D Webster horn model is shown in Fig. 4-6. Finite element simulation predicts a significant sound attenuation in frequency range of 2720 Hz – 4820 Hz with a maximum sound attenuation of 38 dB. The sound attenuation results from the 1-D Webster horn model shows a significant sound attenuation (more than 20 dB) in the frequency range of 2900 Hz – 5500 Hz, with a maximum sound attenuation of 39 dB. The two results also agree with the experiment results. Below 2500 Hz, there is no significant sound attenuation.
Although the results are in reasonable agreement, some observations can be made. The finite element results are closer to the experimental results compared to the sound attenuation from the 1-D model. Above 2500 Hz, the sound attenuation curve from the 1-D model is shifted to the right by about 200 Hz to 500 Hz at 2500 Hz and 5000 Hz, respectively, compared to the experiment and the finite element results.

As it will be explained in chapter 5, the reason for this frequency shift in sound attenuation curve at high frequencies is due to the one dimensional assumption of uniform pressure across the cross-section in the Webster equation model. This assumption may hold true at low frequencies or when the cross sectional area is varying slowly. However, near the cylinders or for high frequencies, the pressure varies...
significantly in both directions, and hence we need to consider 2-D effect in the numerical modeling.

Also, we can consider simulating the 3-D problem, similar to the experiment. However, due to lack of computational resources, we limit ourselves to 2-D simulations, which serve the purpose of validating our numerical models. 3-D modeling can be taken up as a future work, which can in better comparing with the experimental observations.

The finite element method is an approximate numerical method. Therefore it is important to validate the finite element method against some published work and also by mesh convergence.

4.2.1 Validation of finite element simulation with published work

The finite element model developed above is validated by performing simulation for a published work. We try to simulate the experimental work presented by Sanchez-Perez, J.V., et al., in *Sound attenuation by a two-dimensional array of rigid cylinders*. Physical Review Letters, 1998 [20]. They have considered cylinders of different diameter placed in square array with lattice constant of 11 cm. The sound attenuation is measured from 500 Hz – 2500 Hz for 10 cylinders. We have simulated their work, using the same geometry. The simulation results for two cases with diameter of cylinders being 4 cm and 2 cm is shown in Fig. below. The results are in good agreement and show a high sound attenuation in the band gap region as mentioned in their work.
Figure 4-7 Sound attenuation for cylinder of diameter 4 cm and lattice constant 11 cm. The results compare our FE model with published work.
4.2.2 Mesh convergence study

The mesh convergence study for finite element simulation is performed for mesh consisting of 1787, 2094, 3306, 5004 and 7036 triangular T6 elements. As it can be seen in Fig. 4-9 and 4-10, the absolute pressure results have converged for a mesh size of 2094. The simulations performed in this work are done using a mesh size of 5004 elements.
Figure 4-9 Absolute pressure along the x axis at the highest frequency of 6 kHz. The results are indifferent from the mesh size of 2094 onwards. The simulations are performed using 5004 elements.

The results for mesh size of 2094 onwards have overlapped each other. To get a clearer picture, absolute pressure at the outlet end for different mesh size is shown in Fig. 4-10. The result shows that the pressure field has converged for mesh size of 2094 element onwards. For our simulation, we have used mesh size of 5004 elements.
Figure 4-10 Absolute pressure measured at highest frequency of 6 kHz at the outlet end of the waveguide for different mesh size.

4.3 Parametric study on rectangular sonic crystal

The numerical model of decay constant was validated by the help of experiment and finite element simulation in the previous sections. This model can predict the band gap and the decay constant of the sonic crystal. Band gap is characterized by center frequency and band width. In this section, based on the 1-D model of decay constant, rectangular sonic crystal will be studied. Also the effects of varying the geometric parameters on the band gap and sound attenuation by the sonic crystal will be studied.
In a sonic crystal, there are mainly three geometric parameters that can be varied - lattice constant $a_x$ (distance between cylinders in $x$ direction), $a_y$ (distance between cylinders in $y$ direction) (Fig. 4-5) and diameter of the cylinders $d$. These dimensions, when varied, modify the area function $S(x)$.

The dimension $a_x$ was varied first to study its effect on the band gap. An experiment was conducted to verify the numerical prediction for a rectangular sonic crystal with the lattice constant $a_x$ being double the value of $a_y$. The experimental setup is same as described in section 4.1, but with an exception of $a_x$ being 8.5 cm.

Sound attenuation from the sonic crystal with $a_x = 2a_y = 8.5$ cm is shown in Fig. 4-6(a) for a frequency range of 500 Hz to 6000 Hz. The numerical model predicts two band gaps in this frequency range, 1420 – 2500 Hz and 3430 Hz – 4720 Hz, with maximum sound attenuation of 30 dB and 36 dB respectively. The experimental results
validate the numerical model. Further, comparing the band gap of square arrangement of scatterers [87] \((a_x = a_y = 4.25 \text{ cm})\), with the present rectangular arrangement, the band gap has shifted from 2795 Hz – 5470 Hz to 1420 – 2500 Hz. In the literature the same phenomenon is explained by the Bragg’s law and an analytical expression for the center frequency of band gap is given in literature [89] as \(c/2a\). The comparison between this analytical expression and the numerical model for different values of \(a_x\) is shown in Fig. 4-12(b). Thus it was found that the parameter \(a_x\) directly affects the center frequency.

The parameters \(a_y\) and \(d\) affect the maximum attenuation, and not the center frequency. Both parameters were varied individually, keeping the other as constant, and the maximum attenuation by five cylinders was calculated. The parameters \(a_y\) and \(d\), can be scaled to a single parameter called the filling fraction so that it is easy to compare them on a same scale. The filling fraction is defined as the fraction of space occupied by scatterers or cylinders.

\[
f = \frac{\pi d^2}{4(a_x a_y)}
\]  

(4-3)

The results are shown in Fig. 4-6(c) which shows the maximum attenuation by five layers of cylinders versus different values of filling fraction by varying \(a_y\) and \(d\). It can be seen that maximum attenuation increases with increasing filling fraction \(f\). As \(f\) is proportional to \(d\) and inversely proportional to \(a_y\) (Eq. 4-3), sound attenuation by the structure increases with increasing diameter \(d\) or decreasing \(a_y\). The point of intersection of the two curves refers to the square arrangement of scatterers where \(a_y = a_x\). As we
move right of this point, it is interesting to note that the increase in maximum attenuation is more effective by decreasing $a_y$, than by increasing $d$. Therefore, for practical design purpose, a rectangular arrangement of scatterers with $a_y < a_x$ will give much better sound attenuation than by increasing the diameter. Conversely, going left of the point of square arrangement, it is not desirable to increase $a_y$; rather, decreasing the cylinder diameter will give better attenuation than increasing $a_y$.

The above results can be explained by considering the effects of parameters $a_y$ and $d$ on the number of scatterers per unit area. Although the filling fraction can be same
for either decreasing $a_y$ or increasing $d$; by decreasing $a_y$ the number of scatterers per unit area increases. And the structure becomes more dense and hence offers more scattering to the incoming wave. However, in increasing the diameter, while keeping other parameters same, the number of scatterers per unit area decreases. Therefore, the attenuation by increasing diameter is not as effective as by decreasing $a_y$.

Changing the geometrical parameters $a_y$ and $d$ also affects the band width of the band gap. The band width represents the frequency range over which attenuation takes place. A parametric study on the band width is shown in Fig. 4-12(d). The band width follows the same trend as that of maximum attenuation because the both of them are related to the decay constant. The band width for the sonic crystal increases more by decreasing $a_y$ than by increasing the diameter of the cylinders.

Although decreasing the lattice parameter $a_y$ will enhance the sound attenuation, there should be a limit to sound attenuation due to sonic crystal. For a case with very small $a_y$ we also need to consider sound propagation through the vibration of the structure, which is not considered in this study. When $a_y$ is very small, there will be a significant sound propagation through the vibration of such structure. Hence, sonic crystal will act more like a barrier with sound attenuation governed mainly by the mass law. This can be studied in future to include the effect of vibration in the sonic crystal.
4.4 Conclusion

In this chapter, experiment and finite element simulation on the sound attenuation by the sonic crystal was described. From the experiments and simulations significant sound attenuation (~ 20 dB) was observed in the band gap region of 2900 Hz – 5000 Hz, as predicted by the 1-D model. Also the maximum attenuation of ~ 40 dB was observed which matches well with the prediction of 1-D model using decay constant.

Based on the 1-D model of decay constant, a parametric study on rectangular sonic crystal is performed. The three geometric parameters considered are: lattice constants $a_x$, $a_y$ and scatterer diameter $d$. An experiment was also performed to validate the numerical model for a rectangular sonic crystal. The parametric study shows that parameter $a_x$ influences the position or the center frequency of the band gap. This was in agreement with the Bragg’s law prediction. Parameters $a_y$ and $d$ affect the decay constant and hence they affect the maximum attenuation and the band width. It was found that decreasing $a_y$ is more effective in sound attenuation than increasing $d$ for a same filling fraction. Rectangular structures with $a_y < a_x$ are more effective in sound attenuation than the normal square arrangement of scatterers.
Chapter 5. Quasi 2-D model for sound attenuation through the sonic crystal

Sound propagation along the symmetry direction of a sonic crystal is modeled in chapter 3 by a one dimensional numerical model based on Webster horn equation. This 1-D model can predict the band gap, decay constant and sound attenuation through the sonic crystal; the results being validated in the last chapter by the help of experiment and finite element simulation. Results in Fig. 4-6 shows that the 1-D model is able to capture the behavior of the sonic crystal and predict its sound attenuation properties. However, there is significant scope of improvement in the 1-D model which can help to make the numerical model more accurate compared to the 2-D finite element simulation. Particularly sound attenuation curve from the 1-D model (Fig 4-6) is shifted to right compared to the finite element and experiment results by ~ 500 Hz. The quasi 2-D model is developed with a purpose to rectify this frequency shift in sound attenuation by the 1-D model.

In this chapter, a quasi 2-D model for sound propagation through a waveguide with variable cross-section area is developed. The quasi 2-D model assumes a parabolic pressure profile across the cross-section of the waveguide. The governing equation for this model is derived from the weighted residual method. The predictions from the quasi 2-D model for sound attenuation through the sonic crystal are much closer to the finite element simulation and experiment, than the 1-D model.
5.1 **Quasi 2-D model for sound propagation through the sonic crystal**

The 2-D pressure field in the sonic crystal can be studied through the finite element simulation and is shown in Fig. 5-1(a). The figure shows the sound pressure in the strip model at 3500 Hz where the sound attenuation is quite significant (~37 dB). The pressure plot shows that there is a very feeble outgoing wave, and most of the incoming wave is attenuated by the sonic crystal. The frequency of high sound attenuation is chosen to observe the interaction of sound wave with the sonic crystal.

As it can be seen in Fig. 5-1(a), sound pressure at a distance of even one unit cell $a$ before and after the sonic crystal is uniform across the vertical cross-section. However, the pressure field near the cylinders is not uniform across the cross-sectional area. Therefore, the assumption in the Webster horn equation is certainly questionable. To improve upon the Webster horn equation, we have developed a method to include a non-uniform (parabolic) pressure profile across the cross-section.

The method is based on implementing the weighted residual method [90] on the two dimensional Helmholtz equation (Eq. 4-1) for harmonic wave propagation. The residue of the Helmholtz equation is integrated over the cross-section, with $y$ varying from 0 to $S(x)$. The pressure across the cross-section is assumed to be a linear combination of constant and parabolic pressure profile in $y$ direction as given by Eq. 5-1.
The parabolic pressure profile is chosen so that the pressure is symmetric in the waveguide about the x axis.

\[ P(x, y) = \alpha_0(x) + \alpha_1(x)y^2 \]  
(5-1)

The Galerkin method [90] is used so that the weighting functions are the same as the pressure profiles (1 and \(y^2\)). This leads to the weighted residual equations as,

\[ \int_0^{S(x)} \left\{ \frac{\partial^2 P(x, y)}{\partial x^2} + \frac{\partial^2 P(x, y)}{\partial y^2} + \frac{\omega^2}{c^2} P(x, y) \right\} dy = 0 \]  
(5-2)

\[ \int_0^{S(x)} \left\{ \frac{\partial^2 P(x, y)}{\partial x^2} + \frac{\partial^2 P(x, y)}{\partial y^2} + \frac{\omega^2}{c^2} P(x, y) \right\} y^2 dy = 0 \]  
(5-3)

The above formulation results in two ordinary differential equations

\[ \left( \alpha''_0 + S' \alpha' + \frac{\omega^2}{c^2} \alpha_0 \right) + \frac{1}{S} \frac{d}{dx} \left( (\alpha_0 - P_0)S' \right) = 0 \]  
(5-4)

\[ \left( \alpha''_1 + 3S' \alpha_1' + \frac{\omega^2}{c^2} \alpha_1 \right) + \frac{3}{S} \frac{d}{dx} \left( (\alpha_1 - P_1)S^2 S' \right) + \frac{6}{S^2} (\alpha_0 - \alpha_1) = 0 \]  
(5-5)

where \( \alpha_0 \) and \( \alpha_1(x) \) are pressure related quantities, defined in terms of coefficients of Eq. 5-1.

\[ \frac{\int_0^{S(x)} P(x, y) dy}{\int_0^{S(x)} dy} = \frac{\alpha_0(x) + \alpha_1(x) S^2}{3} \]  
(5-6)
\[
\alpha_i(x) = \frac{\int_0^{s(x)} P(x, y) y^2 dy}{\int_0^{s(x)} y^2 dy} = \alpha_0(x) + \alpha_i(x) \frac{3S^2}{5}
\] (5-7)

Also, \( P|_y \) is the pressure at \( y = S \)

\[
P|_y = \alpha_0(x) + \alpha_i(x)S^2
\] (5-8)

Substituting Eq. 5.6-8, in Eq. 5.4-5, leads to the following set of equation in \( \alpha_0(x) \) and \( \alpha_i(x) \).

\[
\left( \alpha''_0 + \alpha'_0 \frac{S'}{S} + \alpha_0 \frac{\omega^2}{c^2} \right) + \left( \alpha''_i S^2 + \alpha'_i SS' + \alpha_i \frac{S^2 \omega^2}{3 c^2} \right) = 0
\] (5-9)

\[
\left( \alpha''_0 + 3\alpha'_0 \frac{S'}{S} + \alpha_0 \frac{\omega^2}{c^2} \right) + \left( \alpha''_i \frac{3}{5} S^2 + 3\alpha'_i SS' + \alpha_i \left( \frac{3}{5} \frac{\omega^2}{c^2} S^2 - 4 \right) \right) = 0
\] (5-10)

The above equations were solved simultaneously using the finite difference method to obtain the values of \( \alpha_0(x) \) and \( \alpha_i(x) \). The pressure \( P(x,y) \) is readily obtained using the quasi 2-D model, for a given frequency.

It can be shown that when \( \alpha_i(x) = 0 \),

\[
P(x, y) = \alpha_0(x)
\] (5-11)

The above equation (Eq. 5-9) reduces to the Webster horn equation (Eq. 3-3). This shows that the Webster horn equation is a special case of the quasi 2-D model.
The pressure obtained from the quasi 2-D model and the Webster equation model is compared with the finite element simulation. The results are compared at a frequency of 3500 Hz. Pressure amplitude across a vertical cross-section, 0.5 cm before the first cylinder is plotted in Fig. 5-1(b). The result shows that there is a significant pressure variation across the cross-section. It clearly shows that the quasi 2-D model is able to represent the 2-D pressure variation better than the Webster horn equation, as it is closer to the finite element results. Sound pressure along the $x$ axis is also compared in Fig. 5-1(c). Finite element results overlap with the quasi 2-D model results. The comparison shows that quasi 2-D model is a better representation than the original Webster equation.

Figure 5-1 Pressure plot in the strip model consisting of five circular scatterers at 3500 Hz. Pressure wave near the first and second cylinder is not uniform across the cross-section. b) Pressure amplitude at a cross-section measured 0.5 cm before the first cylinder from different methods. c) Pressure
amplitude along x-axis from different methods. The pressure amplitude from the finite element model overlaps with the quasi 2-D model solution.

Using the above quasi 2-D model, sound attenuation is calculated for the range of frequencies from 500 Hz – 6000 Hz. The sound attenuation curve for this model matches well with the finite element simulation and the experimental results. The results also show a significant improvement over the Webster horn equation model (Fig. 5-2). The quasi 2-D model does not exhibit the shift in frequency of the attenuation band as seen for the Webster horn equation model. This confirms the hypothesis that the frequency shift in the attenuation band in the Webster model is due to the assumption of uniform pressure across the cross-section.

It should also be noted that using a quadratic pressure profile does not satisfies the boundary condition at the sound hard surface exactly. The boundary condition of $\frac{dp}{dy} = 0$ can be satisfied by using a higher order pressure profile.
5.2 Decay constant for the sonic crystal using the quasi 2-D model

The quasi 2-D model for sound propagation through a waveguide can be applied along with the Floquet theorem in a unit cell to obtain complex dispersion relation as it was implemented in section 3.2. However, unlike section 3.2, the wave propagation considered in the quasi 2-D model has a parabolic pressure profile across the cross-section. This leads to improved band gap calculations, which help in improving in the prediction of sound attenuation by the decay constant.

The Floquet theorem for a one dimensional case is stated in Eq. 3-4. For a two dimensional or for quasi-2D case with sound propagating in x direction,
\[ p(x, y) = \phi(x, y)e^{ikx} \quad (5-12) \]

The periodic function \( \phi(x, y) \) can also be expanded along \( y \) direction using quadratic expansion as

\[ \phi(x, y) = \phi_0(x) + \phi_1(x)y^2 \quad (5-13) \]

Using Eqs. 5-12,13 and 5-1 we can individually write Floquet theorem for individual pressure components.

\[ \alpha_0(x) = \phi_0(x)e^{ikx} \quad (5-14) \]

\[ \alpha_1(x) = \phi_1(x)e^{ikx} \quad (5-15) \]

Substituting Eqs. 5-14,15 in Eqs. 5-9,10 leads to the following equations.

\[ \begin{align*}
\left\{ \phi_0'' + \phi_0 \left( 2ik + \frac{S'}{S} \right) + \phi_0 \left( ik \frac{S'}{S} - k^2 + \frac{\omega^2}{c^2} \right) \right\} + \\
+ \left\{ \phi_0'' \frac{S'^2}{3} + \phi_0 \left( \frac{2}{3} ikS'^2 + SS' \right) + \phi_0 \left( ikSS' - k^2 \frac{S'^2}{3} + \frac{\omega^2 S'^2}{c^2} \right) \right\} &= 0 \\
\left\{ \phi_0'' + \phi_0 \left( 2ik + \frac{3S'}{S} \right) + \phi_0 \left( 3ik \frac{S'}{S} - k^2 + \frac{\omega^2}{c^2} \right) \right\} + \\
+ \left\{ \phi_0'' \frac{3S'^2}{5} + \phi_0 \left( \frac{6}{5} ikS'^2 + 3SS' \right) + \phi_0 \left( 3ikSS' - k^2 \frac{3S'^2}{5} + \frac{\omega^2 3S'^2}{c^2} - 4 \right) \right\} &= 0
\end{align*} \quad (5-16) \]

The above equations can be rearranged to form a quadratic eigenvalue problem in wavenumber \( k \).

\[ \begin{bmatrix}
    a_0 & a_1 \\
    b_0 & b_1 
\end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} k^2 + \begin{bmatrix}
    a_2 & a_3 \\
    b_2 & b_3 
\end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} k + \begin{bmatrix}
    a_4 & a_5 \\
    b_4 & b_5 
\end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = 0 \quad (5-18) \]
where $a_i$ and $b_i$ are matrices of size $N \times N$ formed by finite difference discretization. The quadratic eigenvalue problem is solved by the same method as described in section 3.2, which results in improved complex band gap as shown in Fig. 5-3.

![Graph showing complex band gap from the Webster horn model and the quasi 2-D model.](image)

**Figure 5-3** Complex band gap from the Webster horn model and the quasi 2-D model.

The decay constant (imaginary part of wavenumber $k$), as mentioned in section 3.2, can predict the sound attenuation by a finite sonic crystal in the band gap region. Thus sound attenuation obtained from the Webster horn model and the quasi 2-D models are compared against the experiment and finite element results in Fig. 5-4. The quasi-2D model prediction of sound attenuation by the decay constant is more accurate than the Webster horn model.
Figure 5-4 Sound attenuation predicted by the decay constant from the Webster horn model, quasi 2-D model and comparing them with experiment and finite element results.

5.3 Conclusion

This chapter presents an improvement over the Webster horn model developed in chapter 3. As discussed in chapter 4 from the finite element simulation, sound wave propagating through the sonic crystal is not planar, especially near the scatterers. Due to this the assumption involved in the Webster horn equation, it causes some deviations in results from the experimental and finite element simulation.

To overcome this assumption, quasi 2-D model was developed for sound propagation with parabolic pressure profile. The model was developed from the Helmholtz equation and the weighted residual method. Sound attenuation by the finite
sonic crystal was obtained by modeling sonic crystal as a waveguide. The results match well with the experiment and finite element simulation.

The quasi 2-D model was used along with the Floquet theorem to obtain complex band gap. The band gap from the quasi 2-D model was compared with the band gap using the Webster horn model. The results show significant improvement. Decay constant from the quasi 2-D model was used to predict the sound attenuation over five unit cell, which are in good agreement with experiment and finite element results. As a result the quasi 2-D model shows significant improvement over the Webster horn model and predicts the sound attenuation through the sonic crystal more accurately.

It should also be noted that the assumption of plane wave involved in the Webster horn model is due to our choice of Cartesian coordinate system and assuming pressure as a function of $x$ only. However, Webster horn equation can also take to non-planar wavefront depending on the coordinate system/transformation used. Even for the present quasi 2-D model, it is quite possible that in some other transformed coordinate system, say , where the pressure function can take a quadratic profile corresponding to the Cartesian coordinates. However, to our knowledge, such a study of transformation of coordinates is not there in literature and it can be recommended as a future work.
Chapter 6. Radial sonic crystal

Taking inspiration from the normal sonic crystal with scatterers arranged in square lattice, it was conceived that scatterers could be placed in some other arrangement for the purpose of sound attenuation. Up to the last chapter, sound sources considered were planar sources. However, most of the sound sources encountered are closer to line/point sources with pressure wave propagating as cylindrical/spherical wave. It was envisioned that instead of placing scatterers in square lattice, if the scatterers are arranged in ‘some periodic arrangement’ around the line/point source, it could help in better sound attenuation. One such intuitive arrangement of scatterers around a cylindrical line source is shown in Fig. 6-1. The structure show a quadrant of the geometry where sound hard cylinders are placed periodically in the angular coordinate. The cylinders were placed equidistant along the radial coordinate.

Figure 6-1 A conceptual/intuitive design of a radial sonic crystal with scatterers arranged periodically in the angular coordinates around a cylindrical sound source. The figure shows only a quadrant of the actual geometry.
Although the intuitive design is periodic in the angular coordinate, the position and size of scatterers in radial coordinate is questionable. As the sound wave propagates away from the source, there is an inbuilt divergence in the space due to polar or cylindrical coordinates. Therefore, design of periodic structure in polar coordinate is not obvious.

This chapter presents a systematic method to obtain periodic structure in polar coordinates based on a 1-D numerical model. Such structures are referred as radial sonic crystal (RSC). Section 6.1 presents the derivation of a 1-D equation for sound propagation in a two dimensional waveguide with circular wavefront. Section 6.2 uses this equation to analyze the intuitive radial sonic crystal proposed for sound attenuation properties. Section 6.3 presents a design procedure based on the 1-D model for designing periodic structure in radial coordinates. In section 6.4, sound attenuation by the RSC is compared with the intuitive structure proposed in section 6.2. The comparison shows significant improvement in sound attenuation due to the periodic property of RSC. Lastly, section 6.5 presents the conclusion to this chapter.

6.1 Sound propagation in two dimensional waveguide with circular wavefront

Similar to the previous approach in chapter 3, sound propagation in the radial sonic crystal is modeled by sound propagation through a waveguide. To analyze a sonic
crystal such as in Fig. 6-1, an equation for sound propagation through a waveguide with circular wavefront is required. As mentioned in the last chapter, Webster horn equation in Cartesian coordinate is an approximate equation with planar wavefront. The derivation in this section presents a differential equation (1-D model) which is the Webster horn equation in polar coordinates, for sound propagating with circular wavefront. The difference in these two concepts is shown in Fig. 6-2. Webster horn equation in Cartesian coordinate models sound propagation with planar wavefront (Fig. 6-2(a)) while the equation derived in this section is used to model sound propagation from a point source O propagating with circular wavefront (Fig. 6-2(b)).

![Diagram of sound propagation through a waveguide](image)

Figure 6-2 Sound propagation through a waveguide. Sound wave is modeled with (a) planar wavefront (b) circular wavefront.

The Webster horn equation can be obtained by implementing weighted residual method on Helmholtz equation. The same concept is used in this section, but for Helmholtz equation in polar coordinates with appropriate weighting function. This leads to the equation for wave propagation in a general waveguide with circular wavefront. The
same equation can also be obtained from the general form of Webster equation along with suitable transformations. The equation obtained in this section will be used to design and analyze radial sonic crystal.

The equation obtained is used to evaluate pressure field for different geometries of the waveguide and the results are validated with 2-D finite element simulations. The results from the differential equation match exactly with the finite element simulation for a uniform waveguide. It is shown that for such waveguide in form of sector of circle (with no curvature) the differential equation reduces to the Bessel’s equation of zero order. Hence it gives exact solution. However, as the waveguide is made non-uniform by introducing perturbation (semicircle) in the waveguide, the accuracy of the 1-D model depends upon the frequency of the propagating wave and the degree of perturbation. At low frequencies, the results match exactly with the finite element simulations, but at high frequencies, the 1-D model predictions deviate slightly from the finite element results.

6.1.1 Problem Definition

The problem considered for the derivation of mathematical model is shown in Fig. 6-3, where sound from a line source propagates through a waveguide. Considering that the line source and the waveguide is long enough in the z direction, the pressure can be assumed to be uniform in the z direction, and hence the analysis is restricted to the 2-D \( xy \) plane.
6.1.2 Numerical Formulation

Sound propagation with cylindrical wavefront (3-D) is given by the Helmholtz equation in cylindrical coordinates.

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 p}{\partial \phi^2} + \frac{\partial^2 p}{\partial z^2} + \left( \frac{\omega}{c} \right)^2 p = 0
\]  

(6-1)

As explained in Fig. 6-3, the pressure variation in the \( z \) direction is not significant, and hence the equation reduces to Eq. 6-2 for the 2-D geometry.

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) + \frac{\partial^2 p}{\partial \phi^2} + \left( \frac{\omega}{c} \right)^2 p = 0
\]  

(6-2)
The problem (Fig. 6-2) is symmetric, about the x-axis, therefore only the top half of the geometry is considered and is shown in Fig. 6-4. The waveguide considered is made of sound hard material with respect to air. The top surface of the waveguide is represented by the polar coordinates \((r, \theta(r))\) as shown in Fig. 6-4.

Sound propagation in such 2-D geometry is given by Eq. 6-2 which is a partial differential equation in pressure with respect to radial and angular coordinates. The aim in this section is to reduce the partial differential equation to an ordinary differential equation in the radial coordinate. For this, the numerical method of weighted residual method [90] is used, which is an integral method to obtain an approximate solution to any differential equation.

![Figure 6-4 The symmetric portion of a general waveguide. The figure shows the geometric location of an arbitrary point A in the polar coordinates. The unit normal and tangential vectors at that point are also shown.](image)
The weighted residual method can find an approximate solution to any differential equation by equating the integral of the equation with respect to a weighting function over a domain to zero. In the present case with Eq. 6-2, we chose to integrate the equation over a circular arc with center at the origin O and with an integrating variable $\phi$. The variable angle $\phi$ varies across the arc from zero to $\theta(r)$. It will be shown later that the circular arc corresponds to the wavefront of the propagating wave with source placed at origin O.

The implementation of weighted residual method on Eq. 6-2 is shown in Eq. 6-3 which uses weighting function as unity.

$$\int_0^\theta \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 p}{\partial \phi^2} + \left( \frac{\omega}{c} \right)^2 p \right] d\phi = 0$$  \hspace{1cm} (6-3)

The integral equation can be split into three integrals for the ease of analysis.

$$I_1 = \int_0^\theta \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) \right) d\phi = 0$$  \hspace{1cm} (6-4)

$$I_2 = \int_0^\theta \left( \frac{1}{r^2} \frac{\partial^2 p}{\partial \phi^2} \right) d\phi = 0$$  \hspace{1cm} (6-5)

$$I_3 = \int_0^\theta \left( \left( \frac{\omega}{c} \right)^2 p \right) d\phi = 0$$  \hspace{1cm} (6-6)

As the waveguide is symmetric about the axis $\phi = 0$, the pressure is also symmetric and hence,
\[ \frac{\partial p}{\partial \phi} \bigg|_{\phi=0} = 0 \]  
\[ I_1 = \frac{1}{r} \left[ \frac{d}{dr} \left( r \left( \frac{d}{dr} p \phi - p \theta' \right) \right) \right] - \left( r \frac{\partial p}{\partial r} \right) \theta' \]  
\[ I_2 = \frac{1}{r^2} \frac{\partial p}{\partial \phi} \bigg|_{\theta} \]  
\[ I_3 = \left( \frac{\omega}{c} \right)^2 \int_0^\theta p d\phi = 0 \]  

where prime denotes derivative with respect to \( r \).

The upper surface (curve) of the waveguide is sound hard boundary which is given as by the boundary condition as,
\[ \hat{n} \cdot \nabla p = 0 \]  

where \( \hat{n} \) is the surface normal at a point A as shown in Fig. 6-4. The expression when simplified leads to the following equation.
\[ \left[ \frac{\partial p}{\partial r} \theta' - \frac{\partial p}{\partial \phi} \frac{1}{r^2} \right] \bigg|_{\theta} = 0 \]

As it can be seen by adding the three integrals, the sound hard boundary condition at the top surface appears in the expression leading to the following equation.
The sound source is placed at the origin O, and assuming that the wave propagates with a circular wavefront, the pressure will be constant over any circular arc. With this assumption, pressure is now only a function of the radial distance. Therefore, the above integral expression can be simplified to an ordinary differential equation in radial coordinate.

\[
\frac{1}{r} \frac{d}{dr} \left[ r \left( \frac{d}{dr} \int_0^\theta p d\phi \right) \right] + \left( \frac{\omega}{c} \right)^2 \int_0^\theta p d\phi = 0
\]  

(6-13)

The above equation is similar to the Webster horn equation (Eq. 3-3), however, the above equation is for sound propagating with circular wavefront in a two dimensional waveguide.

6.1.3 Validation of the 1-D model with the finite element simulation

Sound propagation through the 2-D waveguide can be modeled by the 1-D ordinary differential equation derived above. The purpose of this sub-section is to validate the 1-D model by the 2-D finite element simulation and to observe the conditions for which the 1-D model predictions are comparable with the 2-D simulations.

To accomplish this goal, a specific example of a waveguide in the form of a sector is considered. A perturbation of a semicircle is introduced in the sector so that it perturbs the pressure field and forces it to be two dimensional. The symmetric part of the
waveguide is shown in Fig. 6-5. A point source is located at the center O, which is also the center of the sector. The geometry is also a representative unit cell of intuitive RSC, which will be discussed later. The waveguide for the present case extends over a radius varying from \( r_i \) (0.1 m) to \( r_o \) (0.45 m). The perturbation of a semicircle is introduced in the waveguide at a radial distance of \( r_c \) (0.28 m). This perturbation makes the waveguide non-uniform in the angular direction. Different waveguides were considered by varying the radius of semicircle \( r_p \) as zero, 2 cm and 5 cm.

For the same geometries of the waveguide, 2-D models were constructed in finite element software COMSOL Multiphysics. The solution to the 1-D differential equation was obtained by finite difference method by using second order central difference discretization. A pressure boundary condition of one Pascal was applied at the inlet \((r_i)\), while radiation boundary condition was applied at the outlet \((r_o)\) for both finite element and the 1-D model. For the 1-D model, radiation boundary condition was implemented by the radiation boundary condition for cylindrical wave which is give by following equation[91].

\[
\frac{dp}{dr} - \frac{H''_0(kr)}{H'_0(kr)} k p = 0
\]

(6-15)

where \( H''_0 \) is the Hankel function of first kind.

The average pressure \( p(r) \) from the 1-D model is plotted along with the finite element results for different geometries at 500 Hz and 5 kHz in Fig. 6-6. A comparison is also made with the Webster horn equation for the same geometries and frequencies.
Figure 6-5 Specific example of waveguide with perturbation of a semicircle

The results (Fig. 6-6(a) and 6-6(b)) show that 1-D model predicts the average pressure exactly when the waveguide is uniform ($r_p = 0$). For this case, when there is no perturbation, the pressure is uniform in the angular direction and hence our assumption for the 1-D model holds exactly true. It can also be seen from the Eq. 6-14, when the term $\theta' = 0$, the equation reduces to the Bessel equation of zero order. And hence for this case, the 1-D model turns out to be an exact equation.

For the subsequent cases when the perturbation parameter, $r_p$ is non zero, the waveguide is non-uniform in the angular direction, and so is the pressure. Still the prediction by the 1-D model matches well with 2-D finite element simulation. At low frequency (500 Hz) or when the perturbation ($r_p$) is small, the pressure from 1-D model is quite close to the 2-D finite element results (Fig. 6-6(a)-(e)). However, the results vary slightly, especially for highly non-uniform waveguide at high frequency (Fig. 6-6(f)).
such a case, we need to include higher modes of pressure in the angular direction to obtain a better solution.

Figure 6-6 Average pressure verses radial distance for wave propagating from a point source in a waveguide with circular wavefront, planar wavefront (Webster horn equation) and finite element (FE) simulations.
A one dimensional model for sound propagation from a point source through a two dimensional waveguide has been proposed. The model assumes uniform pressure across the circular wavefront and is obtained from integrating the governing Helmholtz equation in polar coordinates using weighted residual method. The results obtained are compared with 2-D finite element simulation for different waveguide geometries and frequencies. The results are in good agreement. However, at high frequencies and for non-uniform waveguide, the one dimensional model prediction differs slightly from the finite element results. The 1-D model can be improved in the future by including a non-uniform pressure in the angular coordinate to obtain higher order variants of this 1-D model.

It also interesting to note that the same equation can be obtained from the general form of the Webster horn equation derived in the Pierce’s textbook[92]. The general form of Webster equation, using the present convention of variables is

\[
\frac{1}{S(x)} \frac{\partial}{\partial x} \left( S(x) \frac{\partial p}{\partial r} \right) = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} \tag{6-16}
\]

Considering the area \(S(x)\), as a function of \(r\), in polar coordinates it can be written as \(S(r) = r\theta(r)\). Substituting this expression in the general equation and for a harmonic analysis, the equation leads to the same Eq. 6-14 derived in this section.

The equation for wave propagation with circular wavefront provides with a simple yet accurate model for sound propagating from the line source, enclosed by some
sound hard surface acting as a waveguide. The predictions by this 1-D model are accurate at low frequency and when the waveguide is not so distorted. The model derived in this section can be used to analyze and design radial sonic crystal.

6.2 Analysis of an intuitive radial sonic crystal

As mentioned in the introduction to this chapter, Fig. 6-1 shows an intuitive design of a radial sonic crystal with sound hard cylinders arranged periodically in the angular coordinate. To analyze such a structure a radial waveguide was considered and is shown in Fig. 6-7(a). The structure considered consists of five cylinders placed equidistant in the radial direction and the cylinders are placed periodically at an angular interval of thirty degrees.
Figure 6-7 (a) Radial waveguide considered for the analysis of sonic crystal in polar coordinates. (b) Sound attenuation from the intuitive radial sonic crystal.

The radial waveguide shown in Fig. 6-7(a) was analyzed based on sound propagation with circular wavefront emanating from the origin O, using Eq. 6-14. The geometry of waveguide, shown by the dark line in Fig. 6-7(a) starts at a radial position of 10 cm. The cylinders or circular scatterers were 3 cm in diameter and they were placed at a radial distance of 5 cm center to center. Eq. 6-14 was solved for this geometry using the same boundary conditions as discussed in previous section. Sound attenuation through the radial waveguide was obtained as a function of frequency using the method described in section 3-3. Thus sound attenuation through such an intuitive sonic crystal was obtained and is plotted in Fig. 6-7(b). The result shows that such an intuitive radial sonic
crystal is not effective in providing sound attenuation as the maximum sound attenuation is around 10 dB.

One reason for such poor sound attenuation is due to lack of periodicity in such structure. The reason will be analyzed in more depth in the next section. But for now, as one goes away from the origin, the free space for wave propagation increases in the waveguide. There is an inbuilt divergence in space in polar coordinates, which makes the arrangement with constant diameter cylinder as aperiodic.

6.3 Design of periodic structure in cylindrical coordinates

In a recent article on acoustic metamaterials [49], Torrent et al. proposed ‘Radial wave crystal’ based on anisotropic material properties. They have shown that sound propagation along the radial direction in such a periodic structure follows Bloch states. The anisotropic material properties of bulk modulus and density tensor are obtained from the property of governing wave equation being invariance in translation from one unit cell to another. In this section, the same property is used for the sound propagation equation in radial waveguide to obtain the cross-section area of the radial waveguide. This forms the basis of designing a radial sonic crystal.

Sound propagation through a radial waveguide is given by Eq. 6-14. For a periodic structure along the radial coordinate, the governing equation should be invariant from one unit cell to another. Therefore, the coefficients of the equations should also be
invariant from one unit cell to another which implies that mapping function \( g(r) \) is invariant in translation \( r \to r + nq \)

\[
g(r) = \left( \frac{\theta'(r)}{\theta(r)} + \frac{1}{r} \right) = \left( \frac{\theta'(r + nq)}{\theta(r + nq)} + \frac{1}{r + nq} \right)
\]  \hspace{1cm} (6-17)

where \( q \) represents the periodicity of the RSC and \( n \) is any positive integer. Therefore, for a given shape of waveguide in first unit cell, the shape of waveguide or RSC can be obtained by mapping the first unit cell to the corresponding position in other unit cells.

For the intuitive sonic crystal shown in Fig 6-7(a), consider one unit cell consisting of a half cylinder as shown in Fig. 6-8(a) by the dark line. Mapping the first unit cell to the second unit cell using Eq. 6-17 leads to a highly distorted and diverged structure as shown by the dotted line. Such a structure when extended to more number of unit cell leads to highly distorted structure which are not circular therefore the intuitive structure with circular scatters of constant diameter cannot be radial sonic crystal.

![Figure 6-8](image)

**Figure 6-8** (a) Unit cell of the radial sonic crystal with circular scatterers is shown by the dark line. Applying the property of invariance in translation lead to its corresponding second periodic unit cell.
which was highly distorted. (b) The plot of periodic function g(r) used for mapping the geometry of second unit cell.

The reason for this distorted structure is because of discontinuity in the function g(r), as shown in Fig. 6-8(b). This discontinuity leads to the distorted structure of the second unit cell. To redesign the structure, a new mapping function g(r) similar to the original function but which was continuous from one unit cell to another unit cell was chosen. The new g(r) was in terms of cosine function (Fig. 6-9(a)) and it resulted in a continuous geometry rather than a distorted geometry.

Thus, using the property of invariance on the propagating wave equation, the coefficients were mapped for a chosen function g(r). Using the Eq. 6-17, the periodic structure in polar coordinates can be obtained and is shown in Fig. 6-9(b). The structure in this case does not turn out to be circular scatterers. The size of such scatterers increases as one goes away from the center O.

![Figure 6-9](image)

Figure 6-9 (a) Continuous periodic function g(r) used for designing RSC. (b) The symmetric part of the radial waveguide for five unit cell obtained by using the property of invariance in translation on the wave propagating equation.
The waveguide obtained can be rotated by using symmetry along the angular coordinate to obtain radial sonic crystal as shown in Fig. 6-10. The figure shows that the radial sonic crystal turn out to be complicated structure which resembles like a fish-bone structure. The radial sonic crystal grows in size as it goes away from the source, which was expected due to inbuilt divergence in space in polar coordinates. The scatterers appear like elliptic objects, however, these elliptic scatterers are connected to each other. This makes the radial sonic crystal as distinct from the normal sonic crystal. The reason for this is due to the polar coordinates, the designed waveguide (Fig. 6-9 (b)) has a curvature in its top surface. This point will be addressed and dealt in the end of chapter 7.
6.4 Sound attenuation by the radial sonic crystal

In the last section, design of radial sonic crystal was outlined by using the property of invariance on the wave propagating equation. In this section, sound attenuation by radial sonic crystal is obtained by solving the circular wave propagating equation (Eq. 6-14). The results are also compared with section 6.2 where scatterers were cylinders of constant diameter. The waveguide for the two cases are shown in Fig. 6-11.
Figure 6-11 (a) Sonic crystal made of circular scatterers based on intuitive design (b) Radial sonic crystal designed based on periodic condition.

The solution to the wave propagating equation with circular wavefront (Eq. 6-14) was obtained using the second order finite difference method. A pressure boundary of 1 Pa was applied at the inlet radius $r_i = 0.1$ m while at outlet a radiation boundary condition for cylindrical wave was applied (Eq. 6-15). The sound pressure was solved in the domain. The incident pressure wave was computed based on the pressure and velocity conditions at the inlet as explained in section 3.3. Thus the sound attenuation through the radial sonic crystal is obtained and plotted in Fig. 6-12.
The sound attenuation from the radial sonic crystal clearly shows a significant sound attenuation (~ 35 dB) due to the periodic structure. The sonic crystal made of circular scatterers also offers a very small sound attenuation (~ 10 dB). It shows that introducing the periodicity in structure can highly amplify the sound attenuation by such structure.

### 6.5 Conclusion

In this chapter, a 1-D model of wave propagation equation with circular wavefront was obtained from the weighted residual method and Helmholtz equation in
polar coordinates. The equation is same as the Webster horn equation in the polar coordinates. This equation is used to model sound propagation through radial sonic crystal. The results from the 1-D model are compared with finite element simulation and they are in good agreement except for high frequencies.

In section 6.2, an intuitive sonic crystal in radial or polar coordinates is proposed. The scatterers are cylinders with constant diameter and are placed at a periodic angular interval of thirty degrees. The structure consists of five scatterers placed equidistantly in the radial direction. The waveguide formed by such sound hard scatterers is analyzed based on the wave propagation equation and it was found that such structure perform very poorly in providing sound attenuation (~ 10 dB). A proposed reason was that in polar coordinates, there is an inbuilt divergence in space, therefore using scatterers with constant diameter makes the structure aperiodic. It requires nonuniform structure in polar coordinates to make a periodic structure.

Periodic structures were designed in section 6.3 by using the property of invariance in translation over the governing wave propagation equation. Initially the first unit cell consisting of circular scatterer was mapped to the second unit cell to obtain the geometry of second unit cell. This led to a non-circular scatterer in the second unit cell. Finally a new mapping function was chosen and using the periodic property of invariance in translation, it led to the design of radial sonic crystal. The sound attenuation through such RSC was obtained and was compared with intuitive structure designed in section
6.2. The result shows significant improvement in sound attenuation (~ 35 dB) which is due to the periodic property of RSC.

The design of radial sonic crystal in this section was based on the 1-D wave equation which assumes circular wavefront. In the next chapter, a 2-D analysis based on the finite element simulation will be presented and compared with experimental results. As it was observed in chapter 5, the 1-D model based on Webster horn equation was good enough to predict the sound attenuation properties. However, there was some shift in the sound attenuation curve, especially at high frequencies. Similarly, as we discussed in section 6.1, the 1-D model developed in this chapter, works well at low frequencies. So the results predicted in this chapter (Fig. 6.12) may be expected to match well with the 2-D simulation at low frequencies. However, 1-D model may not be able to predict the sound attenuation phenomenon at high frequencies.

The design based on the 1-D model gives a simple way to design radial sonic crystal which is a novel structure for sound attenuation from a line source. The model can be improved by including higher order mode in the wave propagation equation similar to the quasi 2-D model proposed in chapter 5. In the present case, pressure was assumed to be a function of radial distance and constant over the angular variation. The higher modes can include pressure variation over the angular position and appropriate weighting function can be used to obtain higher order wave propagation equation.
Chapter 7. Experiment and finite element simulation on the radial sonic crystal

This chapter validates the concept of radial sonic crystal by experiment and finite element simulation performed on a representative waveguide. The results of sound attenuation measured from the experiment and finite element simulation are compared with the sound attenuation based on the 1-D model presented in the last chapter. The experimental results are in good agreement with the finite element simulation and predict a high sound attenuation. The 1-D model being an approximate model predicts some trends of the actual results, but is not able to capture some of the finer details. The high sound attenuation obtained in the waveguide is due to the periodic property of the designed radial sonic crystal. The design of radial sonic crystal is quite effective in offering high sound attenuation from a point source and it can be further improved in the future.

7.1 Experiment

There were quite a few challenges in designing an experiment to validate the concept of radial sonic crystal. The most important challenge was to obtain a sound source which is close to a point source and also to perform the experiment for validating a two dimensional model. The smallest sound source that we could find was a speaker, 1.4 cm in diameter, which could provide sufficient sound pressure level (~ 90 dB) for frequency range of 1000 – 6000 Hz. The speaker was chosen to be loud enough so that even after attenuation, the sound pressure level is sufficiently higher than the ambient
noise (~ 35 dB). The maximum attenuation expected from the 1-D model (Fig. 6-12) was around 35 dB. The speaker was tested for its repeatability so that while measuring sound pressure level with and without the scatterers, the speaker’s performance remains the same.

To work with circular wavefront, we chose to perform an experiment which is close to a 2-D model. The experimental model is shown in Fig. 7-1. The figure shows a waveguide which forms the representative unit of a radial sonic crystal. Fig. 7-1(b) shows the waveguide without the elliptic scatterers. The scatterers considered in this geometry are only the elliptic geometry objects. The structure was made from 1.5 cm thick acrylic sheet. The speaker was located at the inlet while at the outlet acoustic foam was used along the side walls to prevent reflections.

The vertical thickness of the structure was 1.5 cm, which is around quarter wavelength at the highest frequency (5.7 cm at 6000 Hz). The small thickness was chosen to avoid standing wave pattern in the z direction. This ensures that the pressure variation in the z direction is mostly uniform and sound propagation can be considered as 2-D in the xy plane. The top and bottom surface of the waveguide was covered with 1 cm thick acrylic sheet which acted as a rigid wall. The experimental setup is shown in Fig. 7-2.

The waveguide was redesigned to start from the origin so that the scatterers of radial sonic crystal encounter circular wavefront and not planar wavefront. If the scatterers are placed far away from the source then the wavefront will be mostly planar.
Also, it helps to keep the experiment compact. Some initial radial gap was provided at the origin to accommodate the speaker. The unit cell of the periodic structure was 5 cm thick and the scatterer spread over a distance of 3 cm within the unit cell. The waveguide consists of periodic structure with five scatterers which added to 25 cm in length. For sound wave to radiate in the free space, an additional flare of 10 cm was provided after the periodic structure, which was followed by lining of black acoustic foam (Fig. 7-2). Sound pressure level measurements were made at the end of the periodic structure (~ 22 cm from the speaker) in line with the speaker.

Figure 7-1 Experimental model for testing a representative waveguide of a radial sonic crystal. The two experimental setup represents sound propagation in the waveguide with and without the elliptic scatterers. The top and bottom cover plates are not shown in this figure.
The speaker was excited by a harmonic sound source which was varied from 1000 Hz to 6000 Hz in frequency steps of 6 Hz. A ½ inch microphone was used to measure the pressure at the outlet end of the waveguide. The measurements were averaged 50 times to obtain a consistent pressure reading. For the two geometries (as shown in Fig. 7-1) with and without scatterers, sound pressure level was measured at the same position and the insertion loss was obtained (Eq. 3-16). The insertion loss represents the sound attenuation by the radial sonic crystal.

![Figure 7-2 Experimental setup for testing representative waveguide of a radial sonic crystal.](image)

The speaker being so close to the base wall and the sonic crystal could lead to vibration of structure which can be further transmitted to the microphone. This can lead to some errors or fluctuations in measurements. Speaker isolation can be included in future work by using some vibration dampening material to support the speaker.
7.2 Finite element simulation

The finite element simulation is based on solving the Helmholtz equation in the 2-D domain. The domain consists of symmetric part of the waveguide which is shown along with the absolute pressure plots in Fig. 7-3 obtained from the simulation. The simulation was implemented by the commercial software COMSOL Multiphysics 3.4. The domain was meshed with 1952 quadrilateral Q8 elements and the model was checked for mesh convergence at the highest frequency of 6000 Hz. The element size was smaller than 1/10 of the smallest wavelength. The boundary condition at the inlet boundary was a radiation pressure of 1 Pa from a point source placed at the bottom-left corner of the waveguide. The top and bottom boundary of the waveguide was modeled as sound hard surface. The outlet boundary was modeled by radiation boundary condition to simulate the realistic experimental situation.

Sound attenuation was measured by the insertion loss given by Eq. 3-17. The forward travelling wave amplitude $P_I$ was obtained from the radiation pressure at the inlet boundary. The outgoing wave amplitude $P_O$ was obtained by taking the average pressure across the outlet boundary.

The sound source was excited by a harmonic signal which was varied from 1000 Hz to 6000 Hz in steps of 10 Hz. Sound attenuation was measured for this frequency range. Figure 7-2 shows pressure contour at four different frequencies which will be used to analyze the results in the next section. The pressure plot is shown for absolute pressure, so that it gives an idea of actual sound pressure in the domain.
7.2.1 Mesh convergence test

To validate the finite element model, mesh convergence studies have been performed for the radial sonic crystal. The results are shown in Fig. 7-4 and 7-5.

Different mesh size consisting of 728, 1238, 1638, 1979 and 4217 elements were considered. Pressure profile along the radial axis for different mesh size is plotted at the highest frequency of 6 kHz in Fig 7-4. It can the results are indifferent for mesh size of 1238 elements onwards. For a better understanding, we plot pressure only at the outlet end for different mesh sizes in Fig. 7-5. The figure shows that the results have converged.
for mesh size of 1638 elements onwards. For our simulation, we have used 1979 elements.

Figure 7-4 Pressure profile along the radial axis for different mesh size at the highest frequency of 6 kHz.
7.3 Results

The sound attenuation from the waveguide representing the radial sonic crystal is obtained from the experiment and finite element simulation and is shown in Fig. 7-6. As the experiment was performed close to a 2-D model, the finite element simulation is able to model the experiment quite accurately and the two results compare well. Some deviations in the two results is due to the sound source in the experiment not being a
perfectly point source with circular wavefront. This could lead to the deviations in sound attenuation results around 2500 Hz.

Fig. 7-3 shows the finite element simulation plot for absolute pressure field at some of the critical frequencies in Fig. 7-6. The frequency of 2750 Hz is the point of maximum sound attenuation based on the finite element simulation and experiment. (The other two points of high sound attenuation predicted by the finite element simulation are due to antiresonance.) The maximum sound attenuation obtained is around 30 dB (Fig. 7-6) which is quite significant from a 25 cm radial sonic crystal compared to the 10 dB from the intuitive structure proposed in chapter 6. The pressure plot at this frequency (Fig. 7-3(a)) shows that most of sound wave is reflected by the first scatterer (similar to Fig. 5-1(a)) and there is a constructive interference at the inlet. At the outlet, there is a very feeble outgoing wave, and this result in high sound attenuation.

Fig. 7-3 (b) and (d) shows pressure plot at two antiresonance peaks (Fig. 7-6) at frequency of 3200 Hz and 4600 Hz. The antiresonance can be seen in the figures 7-3 (b) and (d) which causes a very feeble outgoing wave. The minimum absolute pressure for these two cases is lowest among all four cases. This causes the sound attenuation to jump to a very high value. It can also be see that the antiresonance peaks are also observed for the experimental results at 3200 Hz and 4600 Hz.

Fig. 7-3(c) also shows another interesting frequency of 4300 Hz where there is hardly any sound attenuation. Consistent results are obtained from the finite element
simulation and experiment. The reason can be analyzed from Fig. 7-3(c). There is some resonance in the model but it happens such the pressure wave is able to propagate through the waveguide (the minimum pressure in Fig. 7-3(c) is much higher than the other cases).

![Image of sound attenuation from a representative waveguide of a radial sonic crystal based on finite element simulation, experiment, and 1-D numerical model.](image)

*Figure 7-6 Sound attenuation from a representative waveguide of a radial sonic crystal based on finite element simulation, experiment, and 1-D numerical model.*

The sound attenuation based on the 1-D model presented in the last chapter for the present waveguide is also shown in Fig. 7-6 by the dash-dotted line. The 1-D model predictions are based on the assumption that sound pressure profile inside the waveguide has a circular wavefront. However, it can be seen from Fig. 7-3 that sound pressure profile in the waveguide does not have an exact circular wavefront. For the case shown in
Fig. 7-3 (a) and (b), the pressure profile is closer to circular wavefront than for the cases in Fig. 7-3 (c) and (d). Therefore, it can be seen that 1-D model predictions of sound attenuation in Fig. 7-6 match well with the finite element simulation for the frequencies 2750 Hz and 3200 Hz, compared to 4300 Hz and 4600 Hz.

It can also be observed from Fig. 7-6 that there is a frequency shift between the 1-D model and 2-D finite element simulation of ~300 Hz around 4300 Hz. A similar shift in sound attenuation was also observed and discussed in chapter 4 and 5. The reason for this shift is due to the assumptions involved in the 1-D model and it can be improved in future by a 2-D or a quasi 2-D model as presented in chapter 5.

The 1-D model prediction of the sound attenuation for radial sonic crystal is similar to the sound attenuation due to the band gap for the normal rectangular sonic crystal. The reason is because for a 1-D model, the geometry is a periodic structure. The sound attenuation curve therefore follows an elliptic trend from 2500 Hz – 4500 Hz. The analysis of such structure by 2-D finite element simulation and experiment reveals that sound attenuation partly follows the trend predicted by the 1-D model. The sound attenuation increases from frequency of 2500 Hz, but it starts dropping from 3200 Hz onwards. For sound attenuation from the periodic structure, it was expected that sound attenuation would follow the elliptic trend throughout the band of frequency 2500 Hz – 4500 Hz. The drop in sound attenuation from 3200 Hz may be due to the design of radial sonic crystal which is based on 1-D model as presented in the last chapter. A quasi 2-D model can be developed for sound propagation from a point source and it can be used to
redesign the radial sonic crystal in future. Such a radial sonic crystal would provide high sound attenuation throughout the band gap frequency and follow the elliptic trend as it was observed in the case of rectangular sonic crystal. The quasi 2-D model will also help in overcoming the problem of frequency shift for sound attenuation curve by the 1-D model. The radial sonic crystal can be designed to make a true 2-D periodic structure, as there is a periodicity along the radial and angular directions.

Another thing that can be noted about the previous model (as mentioned in chapter 6) is that the top edge of the waveguide has some curvature and is not a straight line. Making a full sonic crystal from such structure would lead to intermediate shapes which connect the scatterers as it was shown in Fig. 6-10. The structure was referred as fish-bone structure. From design point of view, it is expected that sonic crystal will have some scatterers (which are not connected). To remove the in between structure, a modification was made to the geometry by connecting the inlet end to the outlet end by a straight line. The elliptic scatterers were made by extending the present shape using cubic spline. This eliminates the curved intermediate potion and helps us to obtain separate elliptic scatterers. The modification in design is shown in Fig. 7-7(b). The figure shows the design of radial sonic crystal with the curved and straight top edge. Finite element simulations were performed on both designs and the results are shown in Fig. 7-7(a).
The results show that modifying the top edge to a straight leads to some shift in sound attenuation (~300 Hz). The overall sound attenuation is same or even slightly better than the original design. The modified geometry with straight top edge also shows a high sound attenuation in the band gap region which is due to the periodic scatterers. So even by removing the intermediate connecting structure and having only the elliptic scatterers it would lead to similar sound attenuation.
7.4 Conclusion

In this chapter, the concept of radial sonic crystal was validated by performing an experiment and finite element simulation on a representative waveguide. The experiment was designed to represent a two dimensional model and therefore, the finite element predictions were quite close to the experimental results. The sound attenuation results were also compared with the 1-D numerical model based on circular wavefront. The 1-D model predicts an elliptic sound attenuation curve, as in the case of a rectangular sonic crystal, which was partly observed for the present case with the experiment and finite element simulation. The reason for this is because the radial sonic crystal designed in chapter 6 is based on the 1-D model. Including a two dimensional or quasi 2-D model could help in better design of radial sonic crystal which would perform better in offering high sound attenuation in the band gap region. However, the sound attenuation results for the present design shows a significant improvement in sound attenuation (~ 30 dB) compared to the intuitive design proposed in chapter 6 (~10 dB) made of circular scatterers of constant diameter. The reason for increased sound attenuation is due to the periodicity in the structure which is based on the 1-D model.

The design of radial sonic crystal was also modified to changing its top edge as a straight line so that the scatterers are disjointed from each other, unlike as shown in Fig. 6-10. To validate this, finite element simulations were performed on modified geometry with straight top edge. The finite element simulations gave good results and it still predicts a high sound attenuation, but with some shift in sound frequency.
Chapter 8. Conclusion and future direction of work

8.1 Conclusion

In this thesis, numerical models for sound propagation through the sonic crystal are developed. Sound propagation through the sonic crystal in symmetry direction is modeled as a waveguide phenomenon, which is mathematically modeled by the Webster horn equation. Floquet-Bloch theorem for periodic structure is used to reduce the analysis of an ‘infinite’ sonic crystal to a unit cell. The analysis of such a unit cell gives an eigenvalue problem which leads to the formation of band gaps. Band gaps are the frequency ranges for which acoustic waves cannot propagate through the infinite periodic structure.

The eigenvalue problem is reformulated to form a quadratic eigenvalue problem. The problem is solved for the wavenumber for a given range of frequencies. Although the order of problem has increased from a linear eigenvalue problem to a quadratic eigenvalue problem, this gives very important and additional information known as the decay constant. The decay constant is the imaginary part of the wavenumber which predicts sound attenuation or decay in sound over a finite length of the sonic crystal. Sound attenuation through a finite sonic crystal is important from a practical point of view. The distance between the scatterers of the sonic crystal is of the order of magnitude of the wavelength for sound attenuation. For acoustic waves in audible frequency, this distance turns out to be of the order of few centimeters. Therefore, sonic crystal consists
of a finite number of scatterers arranged periodically. Sound propagation through the 
‘finite’ sonic crystal leads to a high sound attenuation in the frequencies corresponding to 
the band gap region. The decay constant represents the exponential attenuation in sound 
over the finite length of the sonic crystal.

Sound attenuation by the finite sonic crystal can also be directly obtained by 
solving the Webster horn equation over the geometry of the waveguide with appropriate 
boundary conditions. The waveguide in this case consists of five repetitive unit cells. The 
sound attenuation from this model predicts a significant sound attenuation in the band 
gap region which compares well with the sound attenuation predicted by the decay 
constant.

The models based on the Webster horn equation (both for the ‘infinite SC’ and for 
the ‘finite SC’) are 1-D model because Webster horn equation assumes pressure to be a 
function of the direction of wave propagation only. Webster horn equation solves for the 
average pressure across the cross-section of the waveguide. To validate the 1-D model, 2-
D finite element simulation and experiments are performed.

The results from experiment and finite element simulation show that the 
predictions by the 1-D model are good enough for the sound attenuation in the band gap 
region. Based on the 1-D model, a parametric study is performed on the geometric 
parameters of the rectangular sonic crystal. It is found that geometric spacing between the 
scatterers in the direction of sound propagation affects the center frequency of the band
gap. Reducing the geometric spacing between the scatterers in the direction perpendicular to the sound propagation helps in better sound attenuation. Such rectangular arrangement of scatterers gives better sound attenuation than the regular square arrangement of scatterers. The model for parametric study is also supported by experimental results.

The results of sound attenuation from the 1-D model, finite element simulation and experiment also show that sound attenuation by the 1-D model is shifted in frequency (~ 500 Hz) compared to the experiment and finite element simulation. The reason for this shift is due to the one dimensional nature of the Webster horn equation.

To improve upon the Webster horn equation, numerical method of the weighted residual method along with the governing Helmholtz equation is used. It is found that implementing the weighted residual method with uniform pressure across the cross-section of the waveguide leads to the standard Webster horn equation. From the finite element simulation, we observe that the 2-D pressure variation can be approximated by including a parabolic pressure profile. Therefore, we chose the pressure as a linear combination of constant and parabolic pressure profile over the cross-section of the waveguide. This leads to the quasi 2-D model developed in chapter 5. Webster horn equation is found to be a specific case of this quasi 2-D model. The quasi 2-D model is then used to obtain sound attenuation through the finite sonic crystal and also to obtain the band gap for an infinite sonic crystal. It was found that the sound attenuation predicted by the quasi 2-D model is as good as the finite element simulation. The pressure profile from the quasi 2-D model compares well with the pressure profile
predicted by the finite element simulation. The quasi 2-D model explains the shift in the sound attenuation in the band gap region and it shows a significant improvement over the 1-D model based on the Webster horn equation. The quasi 2-D model developed can also be used in other applications of a waveguide as an improved model of the Webster horn equation.

Chapter 6 and 7 describe a novel type of sonic crystal called the radial sonic crystal. The radial sonic crystal is designed to attenuate sound from a line source with scatterers placed in a periodic arrangement in the angular coordinate. The design for such a structure was not obvious. Initially, like the sonic crystal based on square lattice, we intuitively tried to build these structures using array of five cylinders of constant diameter placed periodically in angular coordinate around a line source. However, the numerical simulation of such structure resulted in very poor sound attenuation (~10 dB). The reason for this poor performance was due to the divergence in sound wave as it moves away from the source, while the scatterers were of constant diameter. Therefore, it was expected that as the radial distance from the source increases, the scatterers should also increase in size so that the sound wave experiences a periodic variation.

The concept of periodicity in polar coordinate was implemented mathematically by using the property of invariance in translation on the wave propagation equation with circular wavefront. The wave propagation equation with circular wavefront was obtained from the weighted residual method implemented on the Helmholtz equation in polar coordinates. The same equation can also be obtained by transforming the Webster horn
equation to polar coordinates and choosing the appropriate cross-sectional area. Implementing the property of invariance in translation on this equation leads to the design of novel structures called as the radial sonic crystal which is periodic structure in radial coordinate. The scatterers turn out to be non-circular and their size increases with the increase in radial coordinate. The overall structure resembles a fish-bone like structure. The numerical simulation based on the 1-D numerical model gives a very good result compared to the structure based on circular scatterers. The sound attenuation is found to increases from 10 dB to 30 dB.

The concept of radial sonic crystal is validated by designing a representative waveguide, which forms the symmetry part of the radial sonic crystal. The waveguide is tested by performing experiment and finite element simulation. The insertion loss by the radial sonic crystal was measured experimentally and it matched well with the finite element simulation. The sound attenuation from the 1-D model was also obtained but it cannot capture the finer aspects of the 2-D problem. However, the predictions were close and the maximum sound attenuation of 30 dB was achieved in both experiment and finite element simulation. The 1-D model predicts an elliptic sound attenuation curve, as in the case of a rectangular sonic crystal. However, such observations were partly observed using the experiment and finite element simulation for the radial sonic crystal. The reason for this is because the radial sonic crystal design is based on a 1-D model which assumes a circular wavefront. The finite element simulation reveals that sound propagation in the waveguide does not happen as circular wavefront at all the frequencies. Therefore, including a two dimensional or quasi 2-D model could help in better design of radial
sonic crystal which would perform better in offering high sound attenuation in the band gap region. However, the present design of radial sonic crystal, which is a periodic structure based on a 1-D model performs well in attenuating sound from a line sound source.

8.2 Future direction of work

The numerical model such as the 1-D model and quasi 2-D model for rectangular sonic crystal is restricted to sound propagation in symmetry direction, that is, sound propagating along the direction of periodicity of the structure. In future, the model can be extended to include sound propagation in any general direction, although the analysis may be difficult using the present method.

The quasi 2-D model includes only the parabolic pressure variation. The model can be extended to a generalized problem, which can include any arbitrary pressure variation. A series expansion (either polynomial or Fourier series) can be used to expand the sound pressure across the waveguide, and this may lead to more accurate models for sound propagation in waveguide. A related study can be to compare the computational time from the different models and the finite element simulation. For the present case, we could not perform this study because the quasi 2-D model was developed in Matlab which is quite slow for bigger problems, while the commercial software COMSOL is optimized for fast solution.
The parametric study performed in section 4.3 can also be improved by including sound transmission through the structure vibration along with the sound attenuation by the sonic crystal. From the results of the present parametric study it was concluded that as the distance between scatterers perpendicular to the direction of wave propagation is reduced, the sound attenuation increases. It was shown numerically the maximum sound attenuation can be as high as 80 dB. It can also be theoretically argued that if this distance is reduced to a very small value, such that scatterers are almost touching each other vertically, such a structure will have a very high sound attenuation based on this model. However, sound attenuation by the sonic crystal will also be restricted by the sound transmission through the structure vibration. As the distance between the scatterers perpendicular to the direction of wave propagation is reduced, the layer of scatterers effectively behaves like a barrier and the sound transmission will be governed by the mass law, which is due to structure vibration.

The radial sonic crystal designed in the present work is based on the 1-D model which considers sound propagation with circular wavefront. However, the finite element simulation reveals that the sound propagation in the geometry is more complicated. Therefore, the 1-D model can be improved by including the angular variation in pressure which can help in better design of the radial sonic crystal. This can be accomplished by model similar to the quasi 2-D model developed for the rectangular sonic crystal. Further, the analysis of radial sonic crystal by the improved model will lead to more accurate predictions of sound attenuation.
The numerical models developed in this work were implemented using Matlab, which was quite slow for bigger problems. Finite element simulations were performed using commercial software COMSOL which is optimized for better performance. An interesting work is to compare the computational efficiency of the different numerical models with the finite element solver in COMSOL. For this, the numerical models have to be implemented using C or FORTRAN. Standard linear equation solver such as LAPACK, LINPACK, BLAS etc can be used to accelerate the computational efficiency of the methods developed. Similarly the eigenvalue solvers from standard libraries in C and FORTRAN can be used for the numerical models for band gap calculations.
References


Publications


