ON SILVER'S DICHOTOMY

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Contents

Summary

One purpose of this study is to investigate effectiveness of Silver's Dichotomy. The reverse mathematics strength of our particular version of Silver's Dichotomy is also examined in the study.

Inspired by Harrington's proof of Silver's Dichotomy, Gandy-Harrington Forcing is employed to obtain an effective version of Silver's Dichotomy in Chapter 2. We strengthen previous results by presenting a calculation of complexity of the reduction map from $\Delta(2^\omega)$ to the given Π^1_1 equivalence relation E. It turns out that the reduction map is recursive in Kleene's $\mathcal{O}$.
Moreover, with step by step construction, we could define two continuous functions a^*, z^* to witness the reduction from $\Delta(2^\omega)$ to the given Π^1_1 equivalence relation E. a^* will induce a perfect set of E-inequivalent elements and z^* will give a real to witness the inequivalence.

In Chapter 3, we examine the reverse mathematics strength of Silver's Dichotomy.

To distinguish the reverse mathematics strengths of our particular version of Silver's Dichotomy and $\Pi^1_1 - CA_0$, we use a model theoretic approach. The statement of our particular version of Silver's Dichotomy is a Σ^1_1-sentence. We construct a model M of second order arithmetic which satisfies all the true in V Σ^1_1-sentences. In the meanwhile, by Gandy's Basis Theorem, we could avoid bringing Kleene's $\mathcal{O}$ into M and make $\Pi^1_1 - CA_0$ fail in M. To check M satisfies our particular version of Silver's Dichotomy restricted to Δ^1_1 equivalence relations, upward absoluteness of Σ^1_1-sentences, and downward absoluteness of Π^1_1-sentences together with some descriptive set theoretical facts, are employed.

Furthermore, in order to compare reverse mathematics strength of our result and $\Pi^1_1 - CA_0$, a routine relativization argument is applied. By reviewing Simpson's proof, we compare the reverse mathematics strength of our result and Simpson's version of Silver's Dichotomy.

Chapter 1

Introduction

In 1980, Silver published his theorem on counting the number of equivalence classes of coanalytic equivalence relations, saying that every coanalytic equivalence relation E has either countably many equivalence classes or has a perfect set of mutually E-inequivalent elements and thus continuum many equivalence classes. This is what we call Silver's Dichotomy in this thesis.

On one hand, Silver's Dichotomy is a theorem in classical descriptive set theory, which starts since the beginning of 20 century and studies definable sets and functions in complete, separable, metric space. We call such space **Polish space**. In Polish space, an **analytic** set is the projection of some closed set and a **coanalytic** set is the complement of some analytic set. [Kechris, 1995], [Moschovakis, 2009] and [Mansfield and Weitkamp, 1985] are good textbooks of descriptive set theory. In the context of classical descriptive set theory, Silver's Dichotomy can be viewed as a generalization of Suslin's **Perfect Set Theorem** ([Lusin, 1917]) which states that every uncountable analytic set has a non-empty perfect subset. To see this, given any analytic set $A \subseteq \omega^\omega$, we can define a coanalytic equivalence relation E

as follows:

$$xEy \Leftrightarrow (x \notin A \wedge y \notin A) \vee x = y.$$

Every singleton $\{x\}$ for $x \in A$ forms an equivalence class, thus A is uncountable implies that E has uncountably many equivalence classes. By Silver's Dichotomy, the Perfect Set Theorem follows.

On the other hand, Silver's Dichotomy is a source leading to Harrington, Kechris and Louveau's result ([Harrington et al., 1990]), **Harrington-Kechris-Louveau's Dichotomy**(H-K-L's Dichotomy for short). The latter opens a new era of the theory of definable equivalence relations, which is also called invariant descriptive set theory in [Gao, 2009]. A quick glance at this subject can be found in [Kechris, 1999]. For readers who have particular interests in definable equivalence relations, [Gao, 2009] and [Kanovei, 2008] are good textbooks to read.

Recently, people started to investigate the effective theory of definable equivalence relations. In [Fokina et al., 2010], some results concerning effectiveness of previous dichotomy theorems such as Silver's Dichotomy and H-K-L's Dichotomy, were p-resented. Motivated by their results, one objective of this thesis is to investigate effectiveness of Silver's Dichotomy, expecting to reduce the complexity of required parameters. In [Fokina et al., 2010], the authors analyzed the complexity of category notion in Gandy-Harrington topology as they worked with proofs in category argument for both Silver's Dichotomy and H-K-L's Dichotomy. In this thesis, we will work with a proof in forcing argument and we will choose appropriate forcing conditions in order to restrict the complexity of induced reduction map.

Besides, Silver's Dichotomy is also a test theorem to study in reverse mathematics.

From Simpson's work ([Simpson, 2009]), some weak version of Silver's Dichotomy can be proved within ATR_0, a subsystem of second order arithmetic. Furthermore, the reverse mathematics strength of another version of Silver's Dichotomy is equivalent to $\Pi^1_1 - CA_0$, which is strictly stronger than ATR_0. This leads to another objective of this thesis: discussing reverse mathematics strength of our result on Silver's Dichotomy, especially its relationship with $\Pi^1_1 - CA_0$.

In this chapter, we briefly review Silver's Dichotomy, H-K-L's Dichotomy, previous effective results on dichotomy theorems and some materials on reverse mathematics.

1.1 Dichotomy Theorems

1.1.1 Borel Reducibility

Before we talk about dichotomy theorems, it is necessary to introduce **Borel reducibility**.

Defnition 1.1. *Given two equivalence relations E, F on Polish spaces X, Y respectively, we say that E can be reduced to F if there exists a reduction map f from X to Y such that*

$$xEy \leftrightarrow f(x)Ff(y).$$

If f is a Borel function, then we call f a **Borel reduction** from E to F. $E \leq_B F$ means E is Borel reducible to F. $E <_B F$ means $E \leq_B F$ and $F \nleq_B E$.

1.1.2 Dichotomy Theorems

Dichotomy theorems is an important topic in invariant descriptive set theory. By comparing the complexity of two given equivalence relations up to Borel reducibility, people are trying to draw a global picture of Borel reducibility hierarchy. Follow the convention of [Kanovei, 2008], given a set X, a simple equivalence relation on X is the equality relation denoted by $\Delta(X)$, i.e,

$$\forall x \in X \forall y \in X (x \ \Delta(X) \ y) \Leftrightarrow x = y.$$

A trivial linear ordering consisting of the equality relations $\Delta(n)$ for $n < \omega$ and $\Delta(\omega)$ occupy the bottom of the diagram of Borel reducibility. In this part, we have

$$\Delta(1) <_B \Delta(2) <_B \ldots <_B \Delta(\omega).$$

Then Silver's Dichotomy comes in as the first nontrivial result on Borel reducibility.
Theorem 1.2 (Silver's Dichotomy, [Silver, 1980])**.** *If E is a coanalytic equivalence relation on the space of all real numbers and has uncountably many equivalence classes, then there is a perfect set of mutually E-inequivalent reals (hence E has 2^ω many equivalence classes).*

Since a Borel equivalence relation is a coanalytic equivalence relation, Theorem 1.2 implies that up to Borel isomorphism, there is no Borel equivalence relation between $\Delta(\omega)$ and $\Delta(2^\omega)$.

The next big contribution to the diagram of Borel reducibility is the following H-K-L's Dichotomy. It gives the least element above $\Delta(2^\omega)$, E_0 defined on 2^ω by

$$xE_0y \Leftrightarrow \exists n \forall m \geq n(x(m) = y(m)).$$

Before we state H-K-L's Dichotomy, we introduce **smoothness**.

Defnition 1.3. *Given a Borel equivalence relation E on Polish space X, (i.e., E is Borel as a subset of X^2), a (countable) separating family for E is a sequence $\{A_n\}$ of subsets of X such that*

$$xEy \leftrightarrow (\forall n(x \in A_n \leftrightarrow y \in A_n)).$$

If E has a Borel separating family, then we say that E is smooth.

We present two versions of H-K-L's Dichotomy, in bold face and in light face.

Theorem 1.4 ([Harrington et al., 1990])**.** *Let X be a Polish space and E a Borel equivalence relation on X. Then exactly one of these following holds:*
(i) E is smooth or
(ii) $E_0 \sqsubseteq E$ (continuously), i.e., there is a continuous embedding of E_0 into E.

Theorem 1.5 ([Harrington et al., 1990])**.** *Let E be a Δ^1_1 equivalence relation on ω^ω. Then exactly one of the following holds:*
(i) E has a separating family $\{A_n\}$ consisting of Δ^1_1 sets (in fact uniformly, i.e., there is a separating family $\{A_n\}$ such that the set A defined by

$$(x, n) \in A \Leftrightarrow x \in A_n$$

is Δ^1_1 in $\omega^\omega \times \omega$) or
(ii) $E_0 \sqsubseteq E$ (continuously).

The former can be proved by relativizing the latter and applying the classical transfer theorem which says that given a Polish space X, B a Borel subset of X, then there is a continuous embedding from ω^ω to X and a closed set $C \subseteq \omega^\omega$ such that B is the image of C.

It is worth to note that, although Theorem 1.4 and Theorem 1.5 are both theorems in invariant descriptive set theory, they in fact originate from Glimm and Effros's earlier dichotomy theorems concerning equivalence relations induced by group actions. Basic knowledge of Polish group actions can be found in [Gao, 2009] and

[Becker and Kechris, 1996] is a book for further reading.

As we can see, up to E_0, the diagram is still linear. However, beyond E_0, the situation becomes much more complicated. In fact, it is no longer linear and there are incomparable Borel equivalence relations. For instance, it is shown in [Adams and Kechris, 2000] that there are uncountably many incomparable countable Borel equivalence relations where countable Borel equivalence relation means Borel equivalence relations such that each equivalence class is countable. A partial picture of the diagram could be found in page 68 of [Kanovei, 2008]. In this thesis, we only focus on the linear part of the Borel reducibility hierarchy.

1.2 Gandy-Harrington Topology

In both proof of Silver's Dichsotomy and H-K-L's Dichotomy, **Gandy-Harrington topology** and effective descriptive set theory playes a crucial role. Readers who are not familiar with effective descriptive set theory are referred to [C.A.Rogers, 1980], Part 4 for an introduction, as well as an elegant proof of Silver's Dichotomy. In fact, in proving our effective result on Silver's Dichotomy, we also follow Harrington's idea to execute Gandy-Harrington forcing, but in a more specific way.

The rest of this section is devoted to review some facts about Gandy-Harrington topology.

Defnition 1.6. *The Gandy-Harrington topology on Polish space X, denoted by τ, is the topology generated by all Σ^1_1 sets.*

As far as we concern, X is usually taken to be ω^ω or product spaces such as

$\omega^n \times (\omega^\omega)^m$.

One good property of τ is that it satisfies the **Baire category theorem**, i.e., the intersection of countably many dense open sets is still dense.

Gandy-Harrington forcing is the partial order $\mathbb{P}$ consisting of basic open sets of τ ordered by inclusion. Basic knowledge of forcing can be found in [Jech, 2003] and [Kunen, 1983].

The following fact of $\mathbb{P}$ implies that a $\mathbb{P}$-generic filter is equivalent to a $\mathbb{P}$-generic real.

Fact 1.1 (Lemma 30.2, [Miller, 1995])**.** *If G is $\mathbb{P}$-generic over V, then there exists $g \in \omega^\omega$ such that $G = \{p \in \mathbb{P} : \ g \in p\}$ and $\{g\} = \bigcap G$.*

We call this g $\mathbb{P}$-generic real.

There are two versions of proofs of Silver's Dichotomy, in [Miller, 1995] and [C.A.Rogers, 1980]. Although one uses forcing argument and the other uses topological argument, they are essentially the same. The crucial point in both proof is, in the Gandy-Harrington topology τ, using some effective descriptive set theory, it can be shown that either E has at most countably many equivalence classes or E is meager on some $A \times A$ in the $\tau \times \tau$ topology, where A is non-empty open in τ.

However, it is pointed out by Kechris and Martin that, in the standard topology, it is not always true that given a coanalytic equivalence relation E with uncountably many equivalence classes, E must be meager on some square $A \times A$.

1.3 Fokina-Sy.Friedman-Törnquist's Results

By replacing Borel with Hyperarithmetic, people started to study **Hyp reducibility** and obtained results in the effective theory of Borel reducibility. Here Hyperarithmetic sets are equivalent to Δ^1_1 sets.

Defnition 1.7 ([Fokina et al., 2010]). *Let E and F be equivalence relations on ω^ω. E is Hyp-reducible to F if there exists a Hyperarithmetic function*

$$f: \ \omega^\omega \to \omega^\omega$$

such that

$$xEy \leftrightarrow f(x)Ff(y)$$

which we denote by $E \leq_H F$.

$E \equiv_H F$ if and only if $E \leq_H F$ and $F \leq_H E$. If $E \equiv_H F$, then they have the same Hyp-degree.

In 2010, Fokina, Sy.Friedman and Törnquist showed in [Fokina et al., 2010] that the effective theory of Borel reducibility is quite different from the classical case. For instance, even in very low level of Hyp reducibility hierarchy, the diagram is far from linear.

In the meanwhile, they presented some effective results on Silver's Dichotomy and H-K-L's Dichotomy. Unfortunately, both effective versions of the two dichotomy theorems do not hold for Hyperarithmetic equivalence relations. Furthermore, they analyzed the parameters in both Silver's Dichotomy and H-K-L's Dichotomy and showed that instead of "Borel", the complexity of reduction map can be reduced to "Hyp in **Kleene's $\mathcal{O}$**" ($\mathcal{O}$ is the set of constructible ordinals and basic knowledge of $\mathcal{O}$ can be found in [Sacks, 1990]).

The following two theorems are their effective results on Silver's Dichotomy and H-K-L's Dichotomy.

In convenience to state the results, we introduce some notations.

Defnition 1.8 ([Fokina et al., 2010]). *For every $n \in \omega$, $n \geq 1$, $=_n$ is the Hyp-degree of the following equivalence relation on ω^ω defined by*

$$x \equiv y \Leftrightarrow x(0) = y(0) \text{ or both } x(0), y(0) \geq n - 1.$$

$=_\omega$ is the Hyp-degree of the equivalence relation on ω^ω defined by

$$x \equiv y \Leftrightarrow x(0) = y(0)$$

$=_{\mathcal{P}(\omega)}$[1] is the Hyp-degree of the equality relation $=$ on $\mathcal{P}(\omega)$, the power set of ω.

Theorem 1.9 ([Fokina et al., 2010]). *Let E be a Hyp equivalence relation on ω^ω. Then either*

(1)

$$E \leq_H =_\omega$$

or

(2)

$$=_{\mathcal{P}(\omega)} \leq_{\Delta^1_1(\mathcal{O})} E.$$

Theorem 1.10 ([Fokina et al., 2010]). *Let E be a Hyp equivalence relation on ω^ω. Then either*

(1)

$$E \leq_H =_{\mathcal{P}(\omega)}$$

or

(2)

$$E_0 \leq_{\Delta^1_1(\mathcal{O})} E.$$

Theorem 1.9 and Theorem 1.10 say that with regard to Hyp reducibility, in the second case of Silver's Dichotomy and H-K-L's Dichotomy, there are reduction

[1] In this thesis, we interchange between $=_{\mathcal{P}(\omega)}$ and $\Delta(2^\omega)$ when we cite results in [Fokina et al., 2010].

maps which are "Hyp in Kleene's $\mathcal{O}$". In fact, Theorem 1.9 is one of the work from which our result in Chapter 2 is motivated since we would like to know whether "Hyp in Kleene's $\mathcal{O}$" is the best possible parameter.

1.4 Reverse Mathematics

The main question of reverse mathematics is: what is the foundation of mathematics and what is the appropriate axiom system of mathematics? In other words, the major subject of reverse mathematics is to study under what axiom system, a given theorem of ordinary mathematics can be proved?

There are some results on Silver's Dichotomy with regard to reverse mathematics strength in [Simpson, 2009]. Out of curiosity about reverse mathematics strength of our result on Silver's Dichotomy, we include the discussion on Silver's Dichotomy as a test theorem in reverse mathematics. Contents in Chapter 3 can be viewed as discussion in adjoint part between descriptive set theory and reverse mathematics. Purpose of this section is not to present deep facts in reverse mathematics but only to let the readers get a quick glance at some necessary terminologies used in this thesis. For readers who are particularly interested in foundation of mathematics, it is suggested to read Simpson's Book, [Simpson, 2009], for a better understanding of this subject. All the definitions and theorems presented in this section follow [Simpson, 2009]'s convention. In addition, to understand the rest of this section, basic knowledge of model theory is needed. [Marker, 2002] or [Shoenfield, 1967] is referred to readers for a first acquaintance of model theory.

1.4.1 Second Order Arithmetic

Being different from first order arithmetic whose language has only number variables, the language of second order arithmetic has two kinds of variables. One is number variables ranging over ω and the other is set variables ranging over all subsets of ω. There are two constant symbols, 0 and 1, two binary operation symbols, $+$ and $\cdot$, which are intended to represent addition and multiplication of natural numbers respectively. Besides propositional connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$ and number quantifiers $\forall n$, $\exists n$, there are also set quantifiers $\forall X$, $\exists X$. Terms, atomic formulas and formulas are formed conventionally. We denote the language of second order arithmetic by L_2.

Next, the following is the formal system of second order arithmetic, denoted by Z_2.

Defnition 1.11 (second order arithmetic)**.** *The axioms of second order arithmetic consist of the universal closures of the following L_2-formulas:*

(i) basic axioms:

$$\begin{aligned} m+1 &\neq 0 \\ (m+1=n+1) &\rightarrow m=n \\ m+0 &= m \\ m+(n+1) &= (m+n)+1 \\ m\cdot 0 &= 0 \\ m\cdot(n+1) &= (m\cdot n)+m \\ &\neg(m<0) \\ (m<n+1) &\leftrightarrow (m<n \vee m=n) \end{aligned}$$

(ii) induction axiom:

$$(0\in X \wedge \forall n(n\in X \rightarrow n+1\in X)) \rightarrow \forall n(n\in X)$$

(iii) comprehension scheme:

$$\exists X \forall n(n \in X \Leftrightarrow \varphi(n))$$

where $\varphi(n)$ is any formula of L_2 in which X does not occur freely.

Defnition 1.12 (L_2-structure)**.** *A structure for L_2 is an ordered 7-tuple*

$$M = (|M|, S_M, +_M, \cdot_M, 0_M, 1_M, <_M),$$

where $|M|$ is a set which serves as the range of the number variables, S_M is a set of subsets of $|M|$ serving as the range of the set variables, $+_M$ and $\cdot_M$ are binary operations on $|M|$, 0_M and 1_M are distinguished elements of $|M|$, $<_M$ is a binary relation on $|M|$.

Lastly, we introduce some L_2-structures which will appear later.

Example 1.13 (intended model)**.** *The intended model for L_2 is*

$$(\omega, \mathcal{P}(\omega), +, \cdot, 0, 1, <).$$

Example 1.14 (ω-model)**.** *An ω-model of L_2-structure is of the form*

$$(\omega, S, +, \cdot, 0, 1, <)$$

where S is a non-empty collection of subsets of ω.

Example 1.15 (β-model)**.** *A β-model is an ω-model $(\omega, S, +, \cdot, 0, 1, <)$ with the following property:*

If φ is any Π^1_1 or Σ^1_1-sentence with parameters from S, then $(\omega, S, +, \cdot, 0, 1, <)$ satisfies φ if and only if the intended model satisfies φ.

1.4.2 RCA_0, ACA_0, $\Pi^1_1 - CA_0$ and ATR_0

In this part, we introduce some subsystems of Z_2.

The first subsystem of Z_2 to introduce is RCA_0. Before we define RCA_0, it is necessary to define Σ^0_1 **induction** and Δ^0_1 **comprehension**.

Defnition 1.16 (Σ^0_1 induction)**.** *The Σ^0_1 induction scheme is the restriction of the second order induction scheme (as in Definition 1.11 (ii)) to L_2-formulas $\varphi(n)$ where φ is Σ^0_1.*

Defnition 1.17 (Δ^0_1 comprehension)**.** *The Δ^0_1 comprehension scheme consists of (the universal closures of) all formulas of the form*

$$\forall n(\varphi(n) \leftrightarrow \xi(n)) \rightarrow \exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$

where $\varphi(n)$ is any Σ^0_1-formula, $\xi(n)$ is any Π^0_1-formula, n is any number variable, and X is a set variable which does not occur freely in $\varphi(n)$.

Similarly, we can define **arithmetic comprehension**, Π^1_1 **comprehension** by replacing Δ^0_1 with arithmetic, Π^1_1 in **Definition 1.17** respectively.

Defnition 1.18 (RCA_0)**.** *RCA_0 is the subsystem of Z_2 consisting of the basic axioms in Definition 1.11 (i), the Σ^0_1 induction scheme and the Δ^0_1 comprehension scheme.*

Similarly, we define ACA_0 and $\Pi^1_1 - CA_0$.

Defnition 1.19 (ACA_0)**.** *ACA_0 is the subsystem of Z_2 consisting of the basic axioms in Definition 1.11 (i), the induction axiom in Definition 1.11, and the arithmetic comprehension scheme.*

Defnition 1.20 ($\Pi^1_1 - CA_0$)**.** *$\Pi^1_1 - CA_0$ is the subsystem of Z_2 by replacing arithmetic comprehension with Π^1_1 comprehension in ACA_0 .*

Obviously, RCA_0 is the weakest and $\Pi^1_1 - CA_0$ is the strongest among the above three subsystems of Z_2.

Next, we define another subsystem of Z_2, ATR_0, consisting of ACA_0 plus the scheme of **arithmetical transfinite recursion**.

Defnition 1.21 (arithmetical transfinite recursion)**.** *$\theta(n, X)$ is an arithmetical formula with a free number variable n and a free set variable X. Note that $\theta(n, X)$*

may also contain parameters, i.e., additional free number and set variables. Define an "arithmetical operator" $\Theta : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ *by*

$$\Theta(X) = \{n \in \omega : \theta(n, X)\}.$$

Let $A, <_A$ *be any countable well ordering and consider the set* $Y \subseteq \omega \times A$ *obtained by transfinitely iterating the operator* Θ *along* $A, <_A$ *defined by the following conditions:*
(i) $Y \subseteq \omega \times A$*;*
(ii) For each $a \in A$, $Y_a = \Theta(Y^a)$ *where* $Y^a = \{(n, b) : \ n \in Y_b \wedge b <_A a\}$*. Thus,* Y^a *is the result of iterating* Θ *along the initial segment of* $A, <_A$ *up to but not including* a *and* Y_a *is the* a*-section of* Y *if such* Y *exists, i.e.,* $Y_a = \{m : \ (m, a) \in Y\}$*.*

Arithmetical transfinite recursion is the axiom scheme asserting that for every arithmetical operator Θ and every countable well ordering $A, <_A$, such a set Y exists.

A fact that is not so obvious is the reverse mathematics strength of ATR_0 is weaker than $\Pi^1_1 - CA_0$, thus is between ACA_0 and $\Pi^1_1 - CA_0$. In discussion of Chapter 3, Silver's Dichotomy is closely related to ATR_0 and $\Pi^1_1 - CA_0$.

1.4.3 Additional Words

In Chapter 3, the approach we used to judge the reverse mathematics strength of our specific version of Silver's Dichotomy is model theoretical.
In order to prove that version of Silver's Dichotomy is weaker than $\Pi^1_1 - CA_0$, we constructed a model M which is capable to "recognize and satisfy" the specific Silver's Dichotomy within itself but can not be too strong such that model $\Pi^1_1 - CA_0$. After constructing M, we have to make sure that all the argument can be captured by M. To achieve this, we need upward (downward) absoluteness of

of Σ^1_1 (Π^1_1) sentences, together with some coding. Having the above, by Gödel's completeness theorem, the result follows.

Chapter 2

Effectiveness of Silver's Dichotomy

As mentioned before, this chapter is devoted to study the effectiveness of Silver's Dichotomy.

We prove the following effective version of Silver's Dichotomy.

Theorem 2.1. *Let E be a Π^1_1 equivalence relation on ω^ω. Then either*

(1) E has countably many equivalence classes or

(2)

$$\Delta(2^\omega) \leq_{Rec(\mathcal{O})} E$$

where $\leq_{Rec(\mathcal{O})}$ means the reduction map is recursive in $\mathcal{O}$.

2.1 Preliminaries

In this section, we briefly review Harrington's proof of Silver's Dichotomy and indicate the key lemma which we would strengthen to imply Theorem 2.1. Readers can refer to [Miller, 1995] or [C.A.Rogers, 1980] for more details about Harrington's proof. Here we follow [Miller, 1995]'s convention.

Harrington's proof of Silver's Dichotomy is completed by a sequence of lemmas. Given a coanalytic equivalence relation E on ω^ω, let $\mathbb{P}$ denote the Gandy-Harrington forcing mentioned in Chapter 1.

We consider the set B, which is the union of all Δ^1_1 sets which is contained in a single equivalence class, i.e.,

$$B = \bigcup\{D \subseteq \omega^\omega : \ D \text{ is } \Delta^1_1 \wedge \forall x \forall y (x, y \in D \to xEy)\}.$$

To calculate the complexity of B, we use the following Δ^1_1 coding theorem (Theorem1.7.4, [Gao, 2009]).

Theorem 2.2 (Δ^1_1 coding)**.** *Given a Polish space X, there are Π^1_1 subsets P^+, $P^- \subseteq \omega \times X$ and $C \subseteq \omega$ such that*

(i) for any $n \in C$, P^+_n, P^-_n are complements of each other, and

(ii) for any Δ^1_1 set D, there is $n \in C$ such that $D = P^+_n$.

By Δ^1_1 coding theorem,

$$z \in B \Leftrightarrow \exists n (n \in C \wedge z \in P^+_n \wedge \forall x \forall y (x, y \notin P^-_n \to xEy)).$$

B is Π^1_1.

If $B = \omega^\omega$, then E has only countably many equivalence classes since there are only countably many Δ^1_1 sets.

Otherwise, $A = \omega^\omega \setminus B$ is a nonempty Σ^1_1 set and thus a condition in $\mathbb{P}$. The next lemma indicates that A forces the $\mathbb{P}$-generic reals should appear in a new equivalence class:

Lemma 2.3 (Lemma 30.5, [Miller, 1995])**.** *Suppose $c \in \omega^\omega \cap V$. Then*

$$A \Vdash_{\mathbb{P}} \neg(\check{c}\breve{E}\dot{g})$$

where $\dot{g}$ is a name for the $\mathbb{P}$-generic real.

It can be derived from Lemma 2.3 that two mutually $\mathbb{P}$-generic reals are E-inequivalent.

Corollary 2.4 (Harrington, [Miller, 1995])**.** *If* (g_0, g_1) *is* $\mathbb{P} \times \mathbb{P}$*-generic over* V*. then*

$$(A, A) \Vdash_{\mathbb{P}\times\mathbb{P}} \neg \dot{g}_0 \breve{E} \dot{g}_1.$$

Proof. Let G_0 be the corresponding $\mathbb{P}$-generic filter for g_0. $\breve{A}$ is the name for A.

$$V[g_0] \models g_0 \in \breve{A}/G_0$$

where $\breve{A}/G_0$ is the interpretation of $\breve{A}$ by G_0.
Since g_0, g_1 are mutually $\mathbb{P}$-generic, g_1 is $\mathbb{P}$-generic over $V[g_0]$, and therefore by Lemma 2.3,

$$(A, A) \Vdash_{\mathbb{P}\times\mathbb{P}} \neg(\dot{g}_0 \breve{E} \dot{g}_1).$$

□

To complete the proof, we take V_κ containing enough information.
In particular, V_κ knows

$$(A, A) \Vdash_{\mathbb{P}\times\mathbb{P}} \neg(\dot{g}_0 \breve{E} \dot{g}_1).$$

Let M be the transitive collapse of a countable elementary substructure of $(V_\kappa, \in)$.
Note that we do not have to assume there are $\mathbb{P}$-generic reals over V.
A typical splitting construction provides a perfect set of reals mutually $\mathbb{P}$-generic over M.
Note that "E is an equivalence relation" is a Π^1_1 statement. Using absoluteness of Π^1_1-sentences, a perfect set of mutually $\mathbb{P}$-generic over M reals produces a perfect set of E-inequivalent reals.

Lemma 2.5 (Lemma 30.6, [Miller, 1995])**.** *Suppose* M *is a countable transitive model of a sufficiently large fragment of* ZFC *and* $\mathbb{P}$ *is a partially ordered set in* M*. Then there exists a "perfect" set of* $\mathbb{P}$*-filters* $\{G_\alpha : \alpha \in 2^\omega\}$ *such that for every* $\alpha \neq \beta$*,* (G_α, G_β) *is* $\mathbb{P} \times \mathbb{P}$*-generic over* M*.*

Take $\{G_\alpha :\ \alpha \in 2^\omega\}$ as in Lemma 2.5 with $A \in G_\alpha$ for all α and let

$$P = \{g_\alpha :\ \alpha \in 2^\omega\}$$

be the set of corresponding $\mathbb{P}$-generic reals. By Lemma 2.3, for every α, $\beta \in 2^\omega$,

$$\alpha \neq \beta \to \neg(g_\alpha E g_\beta).$$

Moreover, from the construction, we can require the map $\alpha \mapsto g_\alpha$ to be continuous. Thus P is perfect.
This finishes proof of Silver's Dichotomy.

Note that in Lemma 2.5, the complexity of the map $\alpha \mapsto g_\alpha$ is not estimated. In the next section, we will give an analysis of the complexity of such map.

2.2 Proof of Theorem 2.1

In Harrington's proof, the construction of the reduction map involves the following two steps:
(1) prove two mutually $\mathbb{P}$-generic reals are E-inequivalent,
(2) construct a perfect set of mutually $\mathbb{P}$-generic reals over a sufficiently large countable transitive model M.
Step 2 is completed by a typical splitting construction and the induced map is continuous without imposing extra requirement on those forcing conditions during the construction. In this section, we will take care of the complexity of the reduction map making sure that it is recursive in Kleene's $\mathcal{O}$.

Proof. (Proof of Theorem 2.1)
We follow Harrington's proof of Silver's Dichotomy assuming that the set A defined

in Section 2.1 is nonempty and do Gandy-Harrington forcing.

Along with the forcing, we construct a map $\mu : 2^{<\omega} \to \omega^{<\omega}$ inductively. Before the construction, we carry out some necessary calculation.
The following facts will be used in the calculations.
Firstly, note that $\mathcal{O}$ is Π^1_1 complete (Theorem 5.4, Chapter 1, [Sacks, 1990]), hence all the Π^1_1 sets are many-one reducible to $\mathcal{O}$, i.e., given a Π^1_1 set $P \subseteq \omega$, there is a recursive function h witnessing that for all e,

$$e \in P \leftrightarrow h(e) \in \mathcal{O}.$$

Using the above fact, we can show that determining whether a Σ^1_1 subset of ω^ω is nonempty is recursive in $\mathcal{O}$.
To see this, take any Σ^1_1 set $S \subseteq \omega^\omega$ and take T_S to be a recursive tree on $\omega \times \omega$ representing S.
WFG is the collection of Gödel numbers (a definition of Gödel number can be found in [Shoenfield, 1967]) of all well-founded recursive trees. By Theorem 4.9, [Mansfield and Weitkamp, 1985], WFG is a Π^1_1 but not Σ^1_1 set of integers. Thus there is a recursive function h witnessing for all e, $e \in WFG \leftrightarrow h(e) \in \mathcal{O}$. Fix this h.
Let e_S be the Gödel number of recursive tree T_S.

$$S = \emptyset \leftrightarrow T_S \text{ is well-founded} \leftrightarrow e_S \in WFG \leftrightarrow h(e_S) \in \mathcal{O}.$$

Secondly, by Harrington's result, 3.2, [Harrington et al., 1990], there is a "**good**" universal system $U^{\omega^\omega} \subseteq \omega \times \omega^\omega$ for Σ^1_1 subsets of ω^ω which is defined by the two properties below.
(1) For any Σ^1_1 $S \subseteq \omega^\omega$, there is an $n \in \omega$ such that $S = U^{\omega^\omega}_n$ where $U^{\omega^\omega}_n$ is the n-section of U^{ω^ω}. Hence we can view any $n \in \omega$ as a code of some Σ^1_1 subset of ω^ω.

(2) For any $m \in \omega$, there is a recursive function $S^{m,\omega^\omega} : \omega^{m+1} \to \omega$ such that

$$(e, k_1, \ldots, k_m, x) \in U^{\omega^m \times \omega^\omega} \leftrightarrow (S^{m,\omega^\omega}(e, k_1, \ldots, k_m), x) \in U^{\omega^\omega}.$$

Fix such a "good" universal system U^{ω^ω}.

Fix some notations.

Since U^{ω^ω} itself is a Σ^1_1 set, let $T_{U^{\omega^\omega}}$ be the recursive tree representation of U^{ω^ω}.

Given a recursive tree T, let O_T denote the Gödel number of T.

Using this good universal system, we can calculate codes of some objects which will be used in construction of μ.

(a) Calculating codes of Σ^1_1 subsets of A.

Fix a code of the Σ^1_1 set A, denoted by n_A. Consider the intersection of A and some Σ^1_1 set $U^{\omega^\omega}_k \subseteq \omega^\omega$ for some $k \in \omega$. Its code can be calculated as follows: consider H_k such that

$$(n_A, k, x) \in H_k \Leftrightarrow (n_A, x) \in U^{\omega^\omega} \wedge (k, x) \in U^{\omega^\omega},$$

then H_k is Σ^1_1, hence there is an $e_k \in \omega$ such that

$$(n_A, k, x) \in H_k \leftrightarrow (e_k, n_A, k, x) \in U^{\omega^2 \times \omega^\omega} \leftrightarrow (S^{2,\omega^\omega}(e_k, n_A, k), x) \in U^{\omega^\omega}.$$

Hence, $S^{2,\omega^\omega}(e_k, n_A, k)$ gives a code of $A \cap U^{\omega^\omega}_k$.

Denote $k \mapsto e_k$ by e_1. An appropriate good universal system guarantees this map is recursive. $S^{2,\omega^\omega}(e_1(\cdot), n_A, \cdot)$ with domain ω is a recursive function which outputs codes of Σ^1_1 subset of A. Abbreviate $S^{2,\omega^\omega}(e_1(\cdot), n_A, \cdot)$ by S_A.

(b) Calculating codes of $N_\varsigma \cap (U^{\omega^\omega}_m \cap A)$ where $N_\varsigma = \{x \in \omega^\omega : \varsigma \subseteq x\}$.

View $\omega^{<\omega}$ as ω. Given $m \in \omega$ and a finite sequence $\varsigma \in \omega^{<\omega}$, to find a code for $N_\varsigma \cap (U^{\omega^\omega}_m \cap A)$, we consider the set $Q_{m,\varsigma}$ such that

$$(\varsigma, m, x) \in Q_{m,\varsigma} \Leftrightarrow (S_A(m), x) \in U^{\omega^\omega} \wedge \varsigma \subseteq x.$$

Since $Q_{m,\varsigma}$ is Σ_1^1, there is an $e_{m,\varsigma} \in \omega$ such that

$(\varsigma, m, x) \in Q_{m,\varsigma} \leftrightarrow (e_{m,\varsigma}, \varsigma, S_A(m), x) \in U^{\omega^2 \times \omega^\omega} \leftrightarrow (S^{2,\omega^\omega}(e_{m,\varsigma}, \varsigma, S_A(m)), x) \in U^{\omega^\omega}$.

Hence, $S^{2,\omega^\omega}(e_{m,\varsigma}, \varsigma, S_A(m))$ gives a code of $N_\varsigma \cap (U_m^{\omega^\omega} \cap A)$.

Denote $(m, \varsigma) \mapsto e_{m,\varsigma}$ by e_2. An appropriate good universal system guarantees this map is recursive. $S^{2,\omega^\omega}(e_2(\cdot,\cdot), \cdot, S_A(\cdot))$ with domain $\omega^{<\omega} \times \omega$ is a recursive function which outputs codes of $N_\varsigma \cap (U_m^{\omega^\omega} \cap A)$ for $\varsigma \in \omega^{<\omega}$ and $m \in \omega$. Abbreviate $S^{2,\omega^\omega}(e_2(\cdot,\cdot), \cdot, S_A(\cdot))$ by S'_A.

$S'_A(\varsigma, m)$ is a code of $N_\varsigma \cap (U_m^{\omega^\omega} \cap A)$ and thus $N_\varsigma \cap (U_m^{\omega^\omega} \cap A) = U^{\omega^\omega}_{S'_A(\varsigma,m)}$. Let $T_{U^{\omega^\omega}_{S'_A(\varsigma,m)}}$ the representing recursive tree.

Moreover,

$$\begin{aligned} N_\varsigma \cap (U_m^{\omega^\omega} \cap A) \neq \emptyset &\leftrightarrow T_{U^{\omega^\omega}_{S'_A(\varsigma,m)}} \text{ is ill-founded} \\ &\leftrightarrow O_{T_{U^{\omega^\omega}_{S'_A(\varsigma,m)}}} \notin WFG \leftrightarrow h(O_{T_{U^{\omega^\omega}_{S'_A(\varsigma,m)}}}) \notin \mathcal{O}. \end{aligned}$$

h and S'_A are both recursive functions. Hence, whether $N_\varsigma \cap (U_m^{\omega^\omega} \cap A)$ is nonempty is recursive in $\mathcal{O}$.

(c) Finding codes of two Σ_1^1 subsets of A which splits the finite sequence $\varsigma \in \omega^{<\omega}$ determined by $\mathbb{P}$-condition $U_m^{\omega^\omega} \cap A$.

Given $m \in \omega$, $\varsigma \in \omega^{<\omega}$, let $L_{\varsigma,m}$ be the collection of (ζ_0, ζ_1) satisfying the following:

(i) $(\varsigma \subseteq \zeta_0) \wedge (\varsigma \subseteq \zeta_1)$;

(ii) $\zeta_0 \restriction (n-1) = \zeta_1 \restriction (n-1)$ where n is the length of ζ_0, ζ_1;

(iii) $(N_{\zeta_0} \cap (U_m^{\omega^\omega} \cap A) \neq \emptyset) \wedge (N_{\zeta_1} \cap (U_m^{\omega^\omega} \cap A) \neq \emptyset)$.

For each ς, m, $L_{\varsigma,m}$ is intended to contain all the pairs of sequences that split ς where ς is determined by the $\mathbb{P}$-condition $U_m^{\omega^\omega} \cap A$.

(i) and (ii) are obviously recursive. A similar calculation as in (b) shows that (iii) is recursive in $\mathcal{O}$. Therefore, the set $L_{\varsigma,m}$ is recursive in $\mathcal{O}$.

Furthermore, to pick up a representative from $L_{\varsigma,m}$ is recursive in $\mathcal{O}$.

To see this, we introduce two well orderings, $<^*$ on $\omega^{<\omega}$ and $<_*$ on $\omega^{<\omega} \times \omega^{<\omega}$ for convenience.

Defnition 2.6. *Given* $s,\, t \in \omega^{<\omega}$, *if* $s = (s_0, \ldots, s_{m-1})$, $t = (t_0, \ldots, t_{n-1})$, *then*

$$s <^* t \Leftrightarrow (s \subsetneqq t) \vee (\exists i < \min\{m, n\}(\forall j < i(s_j = t_j) \wedge s_i < t_i)).$$

Defnition 2.7. *Given* (s,t), (s', t') *in* $\omega^{<\omega} \times \omega^{<\omega}$,

$$(s,t) <_* (s', t') \Leftrightarrow (s <^* s' \vee (s = s' \wedge t <^* t')).$$

Let $(\varsigma_0, \varsigma_1)$ be the $<_*$-least element in $L_{\varsigma,m}$. Since $<_*$ is a recursive well ordering, computing $(\varsigma_0, \varsigma_1)$ from $L_{\varsigma,m}$ is also recursive in $\mathcal{O}$.
Therefore, $N_{\varsigma_0} \cap (U_m^{\omega^\omega} \cap A)$ and $N_{\varsigma_1} \cap (U_m^{\omega^\omega} \cap A)$ are two Σ_1^1 subsets of A which split $\varsigma \in \omega^{<\omega}$. We call them splitting subsets of $N\varsigma \cap (U_m^{\omega^\omega} \cap A)$. $S'_A(\varsigma_0, m)$ and $S'_A(\varsigma_1, m)$ are codes of $N_{\varsigma_0} \cap (U_m^{\omega^\omega} \cap A)$ and $N_{\varsigma_1} \cap (U_m^{\omega^\omega} \cap A)$ respectively.

Next we define a function $\lambda:\ \omega^{<\omega} \times \omega \to \omega \times \omega$ which can compute the codes of splitting subsets of $N_\varsigma \cap (U_m^{\omega^\omega} \cap A)$ for any given $\varsigma \in \omega^{<\omega}$ and $m \in \omega$ as follows:
Fix l, $l' \in \omega$ such that l, l' are not in range of S_A.
$\lambda(\varsigma, m) = (j_0, j_1)$ if
(1) $h(O_{T_{U^{\omega^\omega}_{S'_A(\varsigma,m)}}}) \notin \mathcal{O}$
(2)

$$j_0 = S'_A(\varsigma_0, m) \wedge j_1 = S'_A(\varsigma_1, m)$$

where $(\varsigma_0, \varsigma_1)$ is the $<_*$-least element in $L_{\varsigma,m}$.
$\lambda(\varsigma, m) = (l, l')$ if $h(O_{T_{U^{\omega^\omega}_{S'_A(\varsigma,m)}}}) \in \mathcal{O}$.

By our calculation (a), (b), (c), λ is an $\mathcal{O}$-recursive function.

Now we start forcing and construct $\mu:\ 2^{<\omega} \to \omega^{<\omega}$ using λ defined above.
Let $\langle\cdot\rangle$ denote the empty sequence. At the beginning, simply let $\mu(\langle\cdot\rangle) = \langle\cdot\rangle$.

Let G_0, G_1 denote two mutually $\mathbb{P}$-generic filters and $\dot{g}_0$, $\dot{g}_1$ be the names of corresponding $\mathbb{P}$-generic reals.

Input m_0 where m_0 is a code of ω^ω and $r_0 = \langle\cdot\rangle$. We can find $\lambda(r_0, m_0) = (m_0^0, m_0^1)$ and $(s_0,\, t_0)$ which is the least $<_*$-least element in L_{r_0,m_0}. Let n_0 be the length of s_0, t_0.

Let $p_0 = A \cap N_{s_0}$ and $p_1 = A \cap N_{t_0}$, then

$$(p_0, p_1) \Vdash_{\mathbb{P}\times\mathbb{P}} (\dot{g}_0 \restriction (n_0-1) = \dot{g}_1 \restriction (n_0-1) = \check{s}_0 \restriction (n_0-1)) \wedge (\dot{g}_0 \restriction n_0 = \check{s}_0) \wedge (\dot{g}_1 \restriction n_0 = \check{t}_0).$$

Define

$$\mu(\langle 0\rangle) = s_0, \mu(\langle 1\rangle) = t_0.$$

Since λ is $\mathcal{O}$-recursive, subsequently, $\mathcal{O}$ can recursively compute s_0, t_0.

Suppose we have constructed μ for $\rho \in 2^{<(k+1)}$ and obtained all the intermediate information.

The next step is to define $\mu(\rho)$ where $\rho \in 2^{k+1}$.

Find $\lambda(r_{k+1}, m_{k+1}) = (m_{k+1}^0, m_{k+1}^1)$ such that r_{k+1} is $\mu(\varrho)$ for some $\varrho \in 2^k$ and m_{k+1} is a code of the forcing condition p_ϱ forcing that $\mu(\varrho)$ is an initial segment of the $\mathbb{P}$-generic real where $\varrho \in 2^k$.

$(s_{k+1},\, t_{k+1})$ are the $<_*$-least element in $L_{r_{k+1},m_{k+1}}$. Let n_{k+1} be the length of s_{k+1}, t_{k+1}.

Let $p_{\varrho^\frown 0} = p_\varrho \cap N_{s_{k+1}} \neq \emptyset$ and $p_{\varrho^\frown 1} = p_\varrho \cap N_{t_{k+1}} \neq \emptyset$.

$$(p_{\varrho^\frown 0}, p_{\varrho^\frown 1}) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{g}_0 \restriction (n_{k+1} - 1) = \dot{g}_1 \restriction (n_{k+1} - 1) = \check{s}_{k+1} \restriction (n_{k+1} - 1)$$

and

$$(p_{\varrho^\frown 0}, p_{\varrho^\frown 1}) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{g}_0 \restriction n_{k+1} = \check{s}_{k+1} \wedge \dot{g}_1 \restriction n_{k+1} = \check{t}_{k+1}.$$

Define

$$\mu(\varrho^\frown 0) = s_{k+1}, \mu(\varrho^\frown 1) = t_{k+1}.$$

In this way, μ is constructed in countably many steps and it is $\mathcal{O}$-recursive.

Finally, let $\mu^* : 2^\omega \to \omega^\omega$ be defined by

$$\mu^*(\alpha) = \bigcup_{n\in\omega} \mu(\alpha \upharpoonright n)$$

μ^* is a reduction map of $\Delta(2^\omega)$ to E. This is because, by our construction, if $\alpha \neq \beta$, then $\mu^*(\alpha)$, $\mu^*(\beta)$ are two mutually $\mathbb{P}$-generic reals and they are E-inequivalent. Moreover, μ^* is continuous since for any N_s with s of length n, there is some $m \leq n$ such that a $\mathbb{P}$-condition p_γ with $\gamma \in 2^m$ determines s.
Lastly, recall definition of code of continuous function.

Defnition 2.8 ([Mansfield and Weitkamp, 1985]). *Let f be a continuous function from a set of reals into reals. A real δ is a code for f iff for every $k \in \omega$, $\delta(k) = 0$ exactly when k codes a pair $\langle s,t\rangle$ such that $f(N_S) \subseteq N_t$.*

By definition, μ can be viewed as a code for μ^*. Since μ is $\mathcal{O}$-recursive, μ^* is also $\mathcal{O}$-recursive . □

Using a different approach, it is proved in Theorem 1.9 that a reduction map can be Δ^1_1 in Kleene's $\mathcal{O}$. We can get a corollary from the following theorem.

Theorem 2.9 ([Fokina et al., 2010]). *Let z be a real in which Kleene's $\mathcal{O}$ is not hyperarithmetic. Then there is a Hyp equivalence relation E such that $=_{\mathcal{P}(\omega)} \leq_{\Delta^1_1(\mathcal{O})} E$, but $=_{\mathcal{P}(\omega)} \not\leq_{\Delta^1_1(z)} E$.*

Note that by Theorem 17, [Fokina et al., 2010], any such Hyp equivalence relation E actually has uncountably many equivalence classes. Thus by Theorem 2.1, E satisfies that $=_{\mathcal{P}(\omega)} \leq_{\mathrm{Rec}(\mathcal{O})} E$. Thus we have the following corollary.

Corollary 2.10. *Let z be a real in which Kleene's $\mathcal{O}$ is not hyperarithmetic. Then there is a Hyp equivalence relation E such that $=_{\mathcal{P}(\omega)} \leq_{Rec(\mathcal{O})} E$, but $=_{\mathcal{P}(\omega)} \not\leq_{\Delta^1_1(z)} E$.*

2.3 Witness

In this section, we prove a stronger form of Silver's Dichotomy concerning the witness of two E-inequivalent reals. This result will be used in next chapter.

Theorem 2.11. *If E is a Π^1_1 equivalence relation on ω^ω, T is a recursive tree on $\omega \times \omega \times \omega$ such that $\forall x, y \in \omega^\omega$,*

$$\neg(xEy) \text{ iff } \exists w \forall n (T(w \upharpoonright n, x \upharpoonright n, y \upharpoonright n)),$$

then either

(1) E has countably many equivalence classes or

(2)

$$\exists a \exists z \forall \sigma, \tau \in 2^{<\omega} (\sigma \neq \tau \rightarrow T(z(\sigma, \tau), a(\sigma), a(\tau))).$$

Moreover, $a : 2^{<\omega} \rightarrow \omega^{<\omega}$ and $z : 2^{<\omega} \times 2^{<\omega} \rightarrow \omega^{<\omega}$ induce continuous functions $a^* : 2^\omega \rightarrow \omega^\omega$ and $z^* : 2^\omega \times 2^\omega \rightarrow \omega^\omega$ defined by

$$a^*(\alpha) = \bigcup_{n \in \omega} a(\alpha \upharpoonright n) \text{ and } z^*(\alpha, \beta) = \bigcup_{n \in \omega} z(\alpha \upharpoonright n, \beta \upharpoonright n)$$

where α, $\beta \in 2^\omega$.

Proof. We still use Gandy-Harrington forcing $\mathbb{P}$ and start with the Σ^1_1 set A assuming that E has uncountably many equivalence classes. As well, the work is carried out in a countable transitive set M which knows sufficient information. We will focus on handling the problem of keeping track of the witnesses where the function z arises. Functions a and z are constructed along with the forcing process.

Let $\langle \cdot \rangle$ denote the empty sequence. At the beginning, simply let $a(\langle \cdot \rangle) = \langle \cdot \rangle$ and $z(\langle \cdot \rangle, \langle \cdot \rangle) = \langle \cdot \rangle$.

Let G_0, G_1 denote two mutually $\mathbb{P}$-generic filters and $\dot{g}_0$, $\dot{g}_1$ be the names of corresponding $\mathbb{P}$-generic reals. By Lemma 2.3 and Corollary 2.4, we have

$$(A, A) \Vdash_{\mathbb{P}\times\mathbb{P}} \exists w \forall n (T(w \restriction n, \dot{g}_0 \restriction n, \dot{g}_1 \restriction n)).$$

Let $\dot{w}$ be a $\mathbb{P} \times \mathbb{P}$-name with

$$(A, A) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{w} \in \omega^\omega \wedge (\forall n (T(\dot{w} \restriction n, \dot{g}_0 \restriction n, \dot{g}_1 \restriction n))).$$

Since g_0, g_1 are mutually $\mathbb{P}$-generic, there exist $n_0 \in \omega$, $r_0 \in \omega^{n_0-1}$, s_0, $s_1 \in \omega^{n_0}$, $r_0 \subseteq s_0$, $r_0 \subseteq s_1$, $s_0(n_0-1) \neq s_1(n_0-1)$ such that

$$(p_0, p_1) \Vdash_{\mathbb{P}\times\mathbb{P}} (\dot{g}_0 \restriction (n_0-1) = \dot{g}_1 \restriction (n_0-1) = \breve{r}_0) \wedge \dot{g}_0 \restriction n_0 = \breve{s}_0 \wedge \dot{g}_1 \restriction n_0 = \breve{s}_1,$$

where $p_0 = A \cap N_{s_0} \neq \emptyset$, $p_1 = A \cap N_{s_1} \neq \emptyset$.
Moreover,

$$(p_0, p_1) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{w} \in \omega^\omega \wedge (\forall n (T(\dot{w} \restriction n, \dot{g}_0 \restriction n, \dot{g}_1 \restriction n))).$$

Let $\sigma_0 \in \omega^{n_0}$ and $(p'_0, p'_1) \leq (p_0, p_1)$ be such that

$$(p'_0, p'_1) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{w} \restriction n_0 = \breve{\sigma}_0.$$

Now we can define

$$a(\langle 0 \rangle) = s_0, a(\langle 1 \rangle) = s_1$$

and

$$z(\langle 0 \rangle, \langle 1 \rangle) = \sigma_0.$$

Also, we define $z(\langle 1 \rangle, \langle 0 \rangle) = \sigma_0$ to make z be symmetric.

Since g_0, g_1 are mutually $\mathbb{P}$-generic, there exist $n_1 \in \omega, n_1 > n_0, r \in \omega^{n_1-1}$, s^*_{00}, $s^*_{01} \in \omega^{n_1}$, $r \subseteq s^*_{00}$, $r \subseteq s^*_{01}$ and $s^*_{00}(n_1-1) \neq s^*_{01}(n_1-1)$ such that

$$(p_{00}, p_{01}) \Vdash_{\mathbb{P}\times\mathbb{P}} (\dot{g}_0 \restriction (n_1-1) = \dot{g}_1 \restriction (n_1-1) = \breve{r}) \wedge \dot{g}_0 \restriction n_1 = \breve{s}^*_{00} \wedge \dot{g}_1 \restriction n_1 = \breve{s}^*_{01},$$

where $p_{00} = p'_0 \cap N_{s^*_{00}} \neq \emptyset$ and $p_{01} = p'_0 \cap N_{s^*_{01}} \neq \emptyset$.
Let $s^*_1 \in \omega^{n_1}$, $s_1 \subseteq s^*_1$ and $p''_1 \leq p'_1$ be such that

$$(p_{00}, p''_1) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{g}_0 \restriction n_1 = \breve{s}^*_{00} \wedge \dot{g}_1 \restriction n_1 = \breve{s}^*_1.$$

Since $p_{00} \leq p'_0$,

$$(p_{00}, p''_1) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{w} \restriction n_0 = \breve{\sigma}_0.$$

Let $\sigma_{00,1} \in \omega^{n_1}$, $\sigma_0 \subseteq \sigma_{00,1}$, and $(p'_{00}, p^{(3)}_1) \leq (p_{00}, p''_1)$ be such that

$$(p'_{00}, p^{(3)}_1) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{w} \restriction n_1 = \breve{\sigma}_{00,1}.$$

In the meanwhile,

$$(p_{01}, p^{(3)}_1) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{g}_0 \restriction n_1 = \breve{s}^*_{01} \wedge \dot{g}_1 \restriction n_1 = \breve{s}^*_1.$$

Since $p_{01} \leq p'_0$,

$$(p_{01}, p^{(3)}_1) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{w} \restriction n_0 = \breve{\sigma}_0.$$

Let $\sigma_{01,1} \in \omega^{n_1}$, $\sigma_0 \subseteq \sigma_{01,1}$, and $(p'_{01}, p^{(4)}_1) \leq (p_{01}, p^{(3)}_1)$ be such that

$$(p'_{01}, p^{(4)}_1) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{w} \restriction n_1 = \breve{\sigma}_{01,1}.$$

Since g_0, g_1 are mutually $\mathbb{P}$-generic, let $n'_1 \in \omega$, $n'_1 > n_1$ satisfy
$\exists l \leq n'_1$, $\exists t \in \omega^{l-1}$, $\exists s_{10},\ s_{11} \in \omega^{n'_1}$, $t \subseteq s_{10}$, $t \subseteq s_{11}$ and $s_{10}(l-1) \neq s_{11}(l-1)$ such that

$$(p_{10}, p_{11}) \Vdash_{\mathbb{P}\times\mathbb{P}} (\dot{g}_0 \restriction (l-1) = \dot{g}_1 \restriction (l-1) = \breve{t}) \wedge \dot{g}_0 \restriction n'_1 = \breve{s}_{10} \wedge \dot{g}_1 \restriction n'_1 = \breve{s}_{11},$$

where $p_{10} = p^{(4)}_1 \cap N_{s_{10}} \neq \emptyset$, $p_{11} = p^{(4)}_1 \cap N_{s_{11}} \neq \emptyset$.
Let $s_{00}, s_{01} \in \omega^{n'_1}$ be such that $s^*_{00} \subseteq s_{00}$, $s^*_{01} \subseteq s_{01}$, and $(p''_{00}, p''_{01}) \leq (p'_{00}, p'_{01})$,

$$(p''_{00}, p''_{01}) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{g}_0 \restriction n'_1 = \breve{s}_{00} \wedge \dot{g}_1 \restriction n'_1 = \breve{s}_{01}.$$

This finishes searching for s_{00}, s_{01}, s_{10} and s_{11}.

Now we consider choices of the corresponding witnesses.

Following lexicographic order, the first witness to consider is $\sigma_{00,01}$. By above, n'_1 is the length of s_{ij} where $i,\ j \in \{0,1\}$. So let $\sigma_{00,01} \in \omega^{n'_1}$, $(p^{(3)}_{00}, p^{(3)}_{01}) \leq (p''_{00}, p''_{01})$ be such that

$$(p^{(3)}_{00}, p^{(3)}_{01}) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{w} \restriction n'_1 = \check{\sigma}_{00,01}.$$

Now we finish searching for $\sigma_{00,01}$.

Similarly, we find $\sigma_{00,10}$, $\sigma_{00,11}$, $\sigma_{01,10}$, $\sigma_{01,11}$, $\sigma_{10,11}$.

For $\sigma_{00,10}$, since $p^{(3)}_{00} \leq p'_{00}$, $p_{10} \leq p^{(3)}_1$,

$$(p^{(3)}_{00}, p_{10}) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{w} \restriction n_1 = \check{\sigma}_{00,1}.$$

Let $\sigma_{00,10} \in \omega^{n'_1}$, $\sigma_{00,1} \subseteq \sigma_{00,10}$, $(p^{(4)}_{00}, p'_{10}) \leq (p^{(3)}_{00}, p_{10})$ be such that

$$(p^{(4)}_{00}, p'_{10}) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{w} \restriction n'_1 = \check{\sigma}_{00,10}.$$

For $\sigma_{00,11}$, since $p^{(4)}_{00} \leq p'_{00}$, $p_{11} \leq p^{(3)}_1$,

$$(p^{(4)}_{00}, p_{11}) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{w} \restriction n_1 = \check{\sigma}_{00,1}.$$

Let $\sigma_{00,11} \in \omega^{n'_1}$, $\sigma_{00,1} \subseteq \sigma_{00,11}$, $(p^{(5)}_{00}, p'_{11}) \leq (p^{(4)}_{00}, p_{11})$ be such that

$$(p^{(5)}_{00}, p'_{11}) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{w} \restriction n'_1 = \check{\sigma}_{00,11}.$$

For $\sigma_{01,10}$, since $p^{(3)}_{01} \leq p'_{01}$, $p'_{10} \leq p^{(4)}_1$,

$$(p^{(3)}_{01}, p'_{10}) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{w} \restriction n_1 = \check{\sigma}_{01,1}.$$

Let $\sigma_{01,10} \in \omega^{n'_1}$, $\sigma_{01,1} \subseteq \sigma_{01,10}$, $(p^{(4)}_{01}, p''_{10}) \leq (p^{(3)}_{01}, p'_{10})$ be such that

$$(p^{(4)}_{01}, p''_{10}) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{w} \restriction n'_1 = \check{\sigma}_{01,10}.$$

For $\sigma_{01,11}$, since $p^{(4)}_{01} \leq p'_{01}$ $p'_{11} \leq p^{(4)}_1$,

$$(p^{(4)}_{01}, p'_{11}) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{w} \restriction n_1 = \check{\sigma}_{01,1}.$$

Let $\sigma_{01,11} \in \omega^{n'_1}$, $\sigma_{01,1} \subseteq \sigma_{01,11}$, $(p^{(5)}_{01}, p''_{11}) \leq (p^{(4)}_{01}, p'_{11})$ be such that

$$(p^{(5)}_{01}, p''_{11}) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{w} \restriction n'_1 = \breve{\sigma}_{01,11}.$$

Lastly, for $\sigma_{10,11}$, since $(p''_{10}, p''_{11}) \leq (p_{10}, p_{11})$,

$$(p''_{10}, p''_{11}) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{w} \in \omega^\omega \wedge (\forall n (T(\dot{w} \restriction n, \dot{g}_0 \restriction n, \dot{g}_1 \restriction n))).$$

Just let $\sigma_{10,11} \in \omega^{n'_1}$, $(p^{(3)}_{10}, p^{(3)}_{11}) \leq (p''_{10}, p''_{11})$ be such that

$$(p^{(3)}_{10}, p^{(3)}_{11}) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{w} \restriction n'_1 = \breve{\sigma}_{10,11}.$$

Now we can define a, z for σ, τ of length 2,

$$a(\langle i, j\rangle) = s_{ij} \text{ where } i, j \in \{0, 1\},$$

$$z(\langle i_1, j_1\rangle, \langle i_2, j_2\rangle) = \sigma_{i_1 j_1, i_2 j_2} \text{ where } i_1, i_2, j_1, j_2 \in \{0, 1\}.$$

Moreover, by symmetry, we define

$$z(\langle i_1, j_1\rangle, \langle i_2, j_2\rangle) = z(\langle i_2, j_2\rangle, \langle i_1, j_1\rangle).$$

Continue in this way, we can carry out the rest of the construction and get the induced functions a^*, z^*.
Note that a^* is continuous since for any N_s with s of length n, there is some $m \leq n$ such that a $\mathbb{P}$ condition $p^{(k)}_\gamma$ for some $k \in \omega$ with $\gamma \in 2^m$ determining s. Similarly, z^* is also continuous.
This finishes the proof. □

Note that in [Fokina et al., 2010], it is proved that if E is a Δ^1_1 equivalence relation with only countably many equivalence classes, then

$$E \leq_H =_\omega .$$

Remark 2.1. *The proof of the above result in [Fokina et al., 2010] involves effective descriptive set theory. The fact that the code set of all* Δ^1_1 *sets is* Π^1_1 *is used in the proof. The argument can not be applied when* E *is a* Π^1_1 *equivalence relation.*

Therefore, if E is a Δ^1_1 equivalence relation, "E has countably many equivalence classes" in Theorem 2.11 can be strengthened to "$E \leq_H =_\omega$". The following corollary follows.

Corollary 2.12 (Silver's Dichotomy for Δ^1_1 equivalence relations)**.** *If* E *is a* Δ^1_1 *equivalence relation on* ω^ω*, and* T *is a recursive tree on* $\omega \times \omega \times \omega$ *such that* $\forall x, y \in \omega^\omega$*,*

$$\neg(xEy) \text{ iff } \exists w \forall n (T(w \restriction n, x \restriction n, y \restriction n)),$$

then either (1)

$$E \leq_H =_\omega$$

or (2)

$$\exists a \exists z \forall \sigma, \tau \in 2^{<\omega}(T(z(\sigma, \tau), a(\sigma), a(\tau)))$$

Moreover, $a : 2^{<\omega} \to \omega^{<\omega}$ *and* $z : 2^{<\omega} \times 2^{<\omega} \to \omega^{<\omega}$ *induce continuous functions* $a^* : 2^\omega \to \omega^\omega$ *and* $z^* : 2^\omega \times 2^\omega \to \omega^\omega$ *defined by*

$$a^*(\alpha) = \bigcup_{n\in\omega} a(\alpha \restriction n) \text{ and } z^*(\alpha, \beta) = \bigcup_{n\in\omega} z(\alpha \restriction n, \beta \restriction n)$$

where α*,* $\beta \in 2^\omega$*.*

In Chapter 3, we will construct an ω-model of second order arithmetic in which Corollary 2.12 "holds".

Regarding to complexity of a and z, we have the following corollary.

View two reals x, y as subsets of ω. $x \oplus y$ is called the **join** of x and y, defined by

$$x \oplus y = \{2n : \ n \in x\} \cup \{2m + 1 : \ m \in y\}.$$

Corollary 2.13. *There are a, z satisfying Theorem 2.11 and*

$$\omega_1^{CK,a\oplus z} = \omega_1^{CK}.$$

Proof. Observing that the sentence

$$\exists a \exists z \forall \sigma, \tau \in 2^{<\omega}(T(z(\sigma,\tau), a(\sigma), a(\tau)))$$

is Σ^1_1, by Gandy's Basis Theorem (Theorem A.1.4, [Gao, 2009]), the result follows. □

From now on, we call a real x is "**low**" if $\omega_1^{CK,x} = \omega_1^{CK}$.

Chapter 3

$\Pi^1_1 - CA_0$ and Silver's Dichotomy

Main result of this chapter is to construct an ω-model of second order arithmetic M such that Corollary 2.12 "holds" in M but M does not satisfy $\Pi^1_1 - CA_0$. By doing this, we establish that our particular version of Silver's Dichotomy does not require Π^1_1-comprehension. In addition, we draw comparison of our result with some other version of Silver's Dichotomy in [Simpson, 2009]. It turns out that they have different reverse mathematics strengths.

3.1 Preparation

In Theorem 2.1 and Theorem 2.11, equivalence relation E is on Baire space ω^ω. However, set variables range over $\mathcal{P}(\omega)$ in ω-models of second order arithmetic. It is necessary to interpret objects in ω^ω into objects in 2^ω.

In this section, we introduce some notations in order to present our main result of this chapter.

Given M as an ω-model of second order arithmetic, we consider interpretation of

$x \in \omega^\omega$ in M.

In order to carry out the interpretation, we introduce a pairing function.

Defnition 3.1 ([Shoenfield, 1967]). *Given two natural numbers $n, k \in \omega$, ordered pair $\lceil n, k \rceil$ is calculated by the pairing function $\lceil \cdot, \cdot \rceil$:*

$$\lceil n, k \rceil = (n + k) \cdot (n + k) + n + 1.$$

Define a map $\pi : \ \omega^\omega \to 2^\omega$ by

$$(x(n) = k) \leftrightarrow \pi(x)(\lceil n, k \rceil) = 1.$$

This map is recursive and one-to-one.

Hence, given a real $x \subseteq \omega^\omega$, $x^M = \pi(x)$ is an interpretation of x in M.

Next we consider the interpretation of a and z in M where a and z are as in Theorem 2.11.

Define $a^M : \ 2^{<\omega} \to 2^{<\omega}$ using a as follows:

Given $\sigma \in 2^{<\omega}$, and $a(\sigma) = s^*$, let n be the length of s^* and $s^*(i) = k_i$ where $i = 0, 1, \ldots, n-1$. Then $a^M(\sigma)$ is of length $\lceil n-1, k_{n-1} \rceil + 1$ and

$$a^M(\sigma)(\lceil i, k_i \rceil) = 1 \leftrightarrow \pi(s^*)(\lceil i, k_i \rceil) = 1 \leftrightarrow a(\sigma)(i) = k_i.$$

Similarly, we define $z^M : \ 2^{<\omega} \times 2^{<\omega} \to 2^{<\omega}$ using z as follows:

Given $\sigma, \tau \in 2^{<\omega}$, and $z(\sigma, \tau) = s^*$, let n be the length of s^* and $s^*(i) = k_i$ where $i = 0, 1, \ldots, n-1$. Then $z^M(\sigma)$ is of length $\lceil n-1, k_{n-1} \rceil + 1$ and

$$z^M(\sigma, \tau)(\lceil i, k_i \rceil) = 1 \leftrightarrow \pi(s^*)(\lceil i, k_i \rceil) = 1 \leftrightarrow z(\sigma, \tau)(i) = k_i.$$

Furthermore, if $a^* : 2^\omega \to \omega^\omega$ and $z^* : 2^\omega \times 2^\omega \to \omega^\omega$ are as in Theorem 2.11, and given $\alpha, \beta \in 2^\omega$,

$$(a^*)^M(\alpha) = \pi \circ a^*(\alpha)$$

and

$$(z^*)^M(\alpha, \beta) = \pi \circ z^*(\alpha, \beta).$$

Finally, we consider interpretation of Δ^1_1 equivalence relation E in M.
Let E be a Δ^1_1 equivalence relation in V.
Fix Σ^1_1-formulas $\varphi(x, y)$, $\psi(x, y) \in L_2$ with all free variables shown such that

$$V \models \forall x \forall y(\varphi(x, y) \leftrightarrow (\neg\psi(x, y))) \tag{3.1}$$

and

$$V \models \forall x \forall y(xEy \leftrightarrow \varphi(x, y)). \tag{3.2}$$

Then we have

$$V \models \forall x \forall y \forall z(\varphi(x, x) \wedge (\varphi(x, y) \rightarrow \varphi(y, x)) \wedge ((\varphi(x, y) \wedge \varphi(y, z)) \rightarrow \varphi(x, z))).$$

E itself is not an element of M, but in M, we can describe E using φ and ψ.

Regarding to tree representation of Σ^1_1-formulas, we consider interpretations of recursive trees on $\omega \times \omega \times \omega$ in M.
Let T_1, T_2 be recursive trees on $\omega \times \omega \times \omega$ such that for $x, y \in \omega^\omega$,

$$xEy \leftrightarrow \exists w \forall n(T_1(w \restriction n, x \restriction n, y \restriction n))$$

and

$$xEy \leftrightarrow \neg(\exists w \forall n(T_2(w \restriction n, x \restriction n, y \restriction n))).$$

Given a finite sequence of natural numbers σ, let $ln(\sigma)$ denote the length of σ.
Using T_i, we define T^M_i on $2 \times 2 \times 2$.
$(\eta^*, \sigma^*, \tau^*) \in T^M_i$ if the following holds:

(1) $\exists(\eta,\sigma,\tau)\in T_i(ln(\sigma)<ln(\sigma^*))$;

(2)

$$\forall m<ln(\sigma)\forall k<ln(\sigma^*)\forall k'<ln(\sigma^*)(\lceil m,k\rceil<ln(\sigma^*)\wedge\lceil m,k'\rceil<ln(\sigma^*))\rightarrow$$

$$(\sigma^*(\lceil m,k\rceil)=1\leftrightarrow(\sigma(m)=k))\wedge(\tau^*(\lceil m,k'\rceil)=1\leftrightarrow(\tau(m)=k'));$$

(3) if $\exists m<ln(\sigma)\exists k<ln(\sigma^*)\exists k'<ln(\sigma^*)$, $\sigma^*(\lceil m,k\rceil)=1$ and $\tau^*(\lceil m,k'\rceil)=1$, and $\lceil m,\eta(m)\rceil<ln(\sigma^*)$, then

$$\eta^*(\lceil m,\eta(m)\rceil)=1.$$

Otherwise,

$$\eta^*(\lceil m,\eta(m)\rceil)=0.$$

Then, if $T_{i,(x,y)}$ has a path, then $T^M_{i,(\pi(x),\pi(y))}$ has a path. Conversely, if $T^M_{i,(\alpha,\beta)}$ has a path, we can find a path in $T_{i,(\pi^{-1}(\alpha),\pi^{-1}(\beta))}$.

Thus, we can define E^* on 2^ω by

$$\alpha E^*\beta\Leftrightarrow\exists\gamma\forall n(T^M_1(\gamma\restriction n,\alpha\restriction n,\beta\restriction n))$$

and

$$\alpha E^*\beta\Leftrightarrow\neg(\exists\gamma\forall n(T^M_2(\gamma\restriction n,\alpha\restriction n,\beta\restriction n))).$$

In this way, if E is a Δ^1_1 equivalence relation in V, then E^* is an interpretation of E in M. We denote it by E^M.

3.2 A Model M

The main purpose of this section is to prove the following theorem.

Theorem 3.2. *If E is a Δ^1_1 equivalence relation in V, then there is an ω-model of second order arithmetic M such that*

$$M \models (E^M \leq_H =^M_\omega)^1 \vee (\exists a^M \exists z^M \forall \sigma, \tau \in 2^{<\omega}(T^M(z^M(\sigma,\tau), a^M(\sigma), a^M(\tau))))$$

where a^M, z^M, T^M are as in Section 3.1, but without satisfying $\Pi^1_1 - CA_0$.

Proof. We will construct an $M = (\omega, S, +, \cdot, 0, 1, <)$ and it will satisfy the following requirements:

(1) M is a β-model.

(2) If E is a Δ^1_1 equivalence relation in V, then

$$M \models (E^M \leq_H =^M_\omega) \vee (\exists a^M \exists z^M \forall \sigma, \tau \in 2^{<\omega}(T^M(z^M(\sigma,\tau), a^M(\sigma), a^M(\tau)))).$$

(3)

$$M \not\models \Pi^1_1 - CA_0$$

We start with the standard model of first order arithmetic $\mathcal{N} = (\omega, +, \cdot, 0, 1, <)$.

Before we execute the construction, we prove the following two claims.

Claim 3.1. *Given a real x, if y is a $\Delta^1_1(x)$ real, then $\{y\}$ is a $\Sigma^1_1(x)$-singleton.*

Claim 3.2. *If y is a $\Delta^1_1(x)$ real and $\exists z \phi(z,y)$ is a Σ^1_1-sentence with y as the only parameter, then $\exists z \phi(z,y)$ can be written as a Σ^1_1-sentence with x as the only parameter.*

Claim 3.1 says that if x is a witness of some Σ^1_1-sentence, then every $\Delta^1_1(x)$ reals is a witness of some Σ^1_1 sentence and will be added to $\mathcal{N}$ eventually when we complete the construction.

Claim 3.2 guarantees that a Σ^1_1-sentence with a $\Delta^1_1(x)$ real as the only parameter where x has already been added to $\mathcal{N}$ is still a $\Sigma^1_1(x)$-sentence.

[1] $=^M_\omega$ is defined by using the approach introduced in Section 3.1.

Proof. (Proof of Claim 3.1)
If y is a Δ^1_1 real, then there are two Σ^1_1-formulas $\varphi(n)$ and $\psi(n)$ with

$$\forall n(\varphi(n) \leftrightarrow \neg\psi(n))$$

which defines y by

$$\forall n((n \in y \rightarrow \varphi(n)) \wedge (\neg\psi(n) \rightarrow n \in y)). \tag{3.3}$$

We define A by

$$y \in A \Leftrightarrow \forall n((n \in y \rightarrow \varphi(n)) \wedge (\neg\psi(n) \rightarrow n \in y)).$$

Since (3.3) is a Σ^1_1-formula and defines the Δ^1_1 real y, $A = \{y\}$ is a Σ^1_1-singleton. By relativizing to x, we conclude that if y is a $\Delta^1_1(x)$ real, then $\{y\}$ is a $\Sigma^1_1(x)$-singleton. □

Proof. (Proof of Claim 3.2)
By Claim 3.1, we can replace the appearance of y by a $\Sigma^1_1(x)$-formula. Thus $\exists z\phi(z, y)$ can be written as

$$\exists z\exists y\phi(z, y) \wedge \text{ (specification of } y \text{ as a } \Sigma^1_1\text{-singleton).}$$

□

Now we start our construction.
Fix an enumeration of Σ^1_1-formulas $\{\varphi_{j,i}\}_{j,i\in\omega}$ where $\varphi_{j,i}$ denotes the j-th Σ^1_1-formula with an i-tuple parameter. In particular, $\varphi_{j,0}$ denotes the j-th Σ^1_1-formula with the empty set as its parameter set. Each $\varphi_{j,i}$ is in form of $\exists x\psi_{j,i}(x, \vec{X}_i)$ where $\vec{X}_i$ is an i-tuple and $\psi_{j,i}(x, \vec{X}_i)$ is a Π^0_1-formula.
We will find a sequence of reals $\{x_l\}_{l\in\omega}$ such that

$$\forall i\forall j\forall\langle x_{j_0}, x_{j_1}, \ldots, x_{j_{i-1}}\rangle\exists l \geq \max(j_0, j_1, \ldots, j_{i-1})\psi_{j,i}(x_l, \langle x_{j_0}, x_{j_1}, \ldots, x_{j_{i-1}}\rangle) \tag{3.4}$$

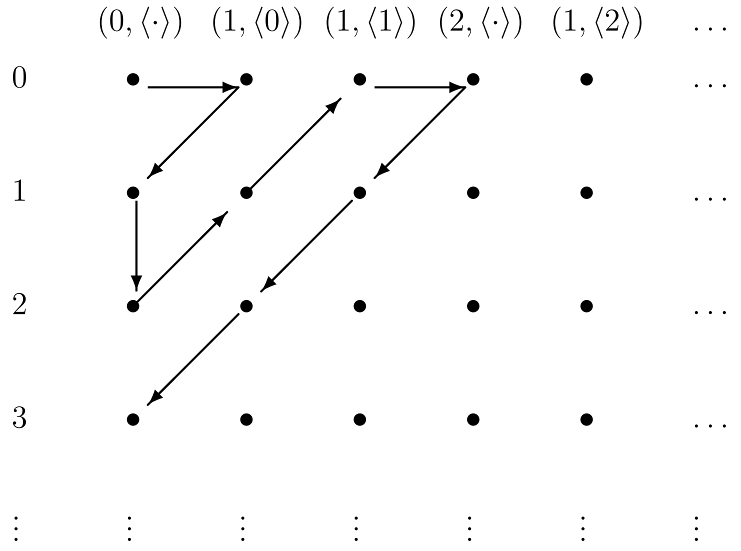
and

$$\forall i(x_0 \oplus x_1 \ldots \oplus x_i \not\geq_T \mathcal{O}). \tag{3.5}$$

Define an order $<_\circ$ on $\bigcup_{i\in\omega}(\{i\} \times \omega^i)$ as follows:
For each $(i, \langle a_0, \ldots, a_{i-1}\rangle) \in \bigcup_{i\in\omega}(\{i\} \times \omega^i)$, we abbreviate $i + \Sigma_{0\leq j\leq i-1} a_j$ by I.
$<_{lex}$ denotes the lexicographic order on $\bigcup_{i\in\omega}(\{i\} \times \omega^i)$.

$$(i, \langle a_0, \ldots, a_{i-1}\rangle) <_\circ (i', \langle a_0, \ldots, a_{i'-1}\rangle)$$
$$\Leftrightarrow I < I' \vee ((I = I') \wedge ((i, \langle a_0, \ldots, a_{i-1}\rangle) <_{lex} (i', \langle a_0, \ldots, a_{i'-1}\rangle)).$$

$<_\circ$ on $\bigcup_{i\in\omega}(\{i\} \times \omega^i)$ is a well ordering of order type ω.
$(0, \langle\cdot\rangle)$, $(1, \langle 0\rangle)$, $(1, \langle 1\rangle)$, $(2, \langle\cdot\rangle)$, $\ldots$ is an initial segment of $(\bigcup_{i\in\omega}(\{i\} \times \omega^i), <_\circ)$.
From now on, we view $\omega \times \bigcup_{i\in\omega}(\{i\} \times \omega^i)$ as $\omega \times \omega$ and well order $\omega \times \bigcup_{i\in\omega}(\{i\} \times \omega^i)$ by canonical well-ordering on $\omega \times \omega$. We denote this well ordering by $<_\bullet$.
The following picture presents an initial segment of $(\omega \times \bigcup_{i\in\omega}(\{i\} \times \omega^i), <_\bullet)$



In our construction, we go along with this well ordering to find the sequence $\{x_l\}_{l\in\omega}$.

Actually, the construction defines a partial function inductively

$$\nu:\ \omega \times (\omega \times \omega^{<\omega}) \to \omega$$

such that if $\nu(j, i, \langle a_0, a_1, \dots, a_{i-1}\rangle) = l$, then

$$V \models \psi_{j,i}(x_l, \langle x_{a_0}, x_{a_1}, \dots, x_{a_{i-1}}\rangle).$$

For each $\varphi_{j,i}$, there are two cases.

Case 1:

$$V \models \exists x \psi_{j,i}(x, \vec{X}_i).$$

Case 2:

$$V \models \forall x(\neg\psi_{j,i}(x, \vec{X}_i)).$$

At Stage 0, we consider $\varphi_{0,0}$.
If Case 1 holds, then by Gandy's Basis Theorem, there is an x_0 with $\omega_1^{CK,x_0} = \omega_1^{CK}$ and

$$V \models \psi_{0,0}(x_0).$$

We add x_0 to $\mathcal{N}$ and define $\nu(0, 0, \langle\cdot\rangle) = 0$.
If Case 2 holds, then we add nothing to $\mathcal{N}$ and ν is undefined at $(0, 0, \langle\cdot\rangle)$.
Trivially, $x_0 \not\geq_T \mathcal{O}$.

Suppose ν has been constructed for k stages.
At Stage $k+1$, let $(j^k, i^k, \langle a_0^k, \dots, a_{i^k-1}^k\rangle)$ be a tuple such that

$$\nu(j^k, i^k, \langle a_0^k, \dots, a_{i^k-1}^k\rangle) = k.$$

In other words, $(j^k, i^k, \langle a_0^k, \dots, a_{i^k-1}^k\rangle)$ is the k-th input at which ν halts. Note that $a_{i^k-1}^k < k$.

Suppose $(j^{k+1}, i^{k+1}, \langle a_0^{k+1}, \ldots, a_{i^{k+1}-1}^{k+1}\rangle)$ is the successor of $(j^k, i^k, \langle a_0^k, \ldots, a_{i^k-1}^k\rangle)$ in $(\omega \times \bigcup_{i\in\omega}(\{i\} \times \omega^i), <_\bullet)$. Hence the next Σ_1^1-formula to be considered is $\varphi_{j^{k+1},i^{k+1}}(\langle x_{a_0^{k+1}}, \ldots, x_{a_{i^{k+1}-1}^{k+1}}\rangle)$. Again, $a_{i^{k+1}-1}^{k+1} < k+1$.
View $\varphi_{j^{k+1},i^{k+1}}(\langle x_{a_0^{k+1}}, \ldots, x_{a_{i^{k+1}-1}^{k+1}}\rangle)$ as a $\Sigma_1^1(x_0, x_1, \ldots, x_k)$-sentence.
If Case 1 holds, then by Gandy's Basis Theorem relativized to $x_0 \oplus x_1 \ldots \oplus x_k$, there is an x_{k+1} with $\omega_1^{CK,x_{k+1}} = \omega_1^{CK,x_0\oplus x_1\ldots\oplus x_k}$ and

$$V \models \psi_{j^{k+1},i^{k+1}}(x_{k+1}, \langle x_{a_0^{k+1}}, \ldots, x_{a_{i^{k+1}-1}^{k+1}}\rangle).$$

Since x_{k+1} is "low" in $x_0 \oplus x_1 \ldots \oplus x_k$ and by induction, $x_0 \oplus x_1 \ldots \oplus x_k$ is also "low". Furthermore, $x_0 \oplus x_1 \ldots \oplus x_{k+1}$ is "low" and thus $x_0 \oplus x_1 \ldots \oplus x_{k+1} \not\geq_T \mathcal{O}$.
We add a new real x_{k+1} to $\mathcal{N}$ and define $\nu(j^{k+1}, i^{k+1}, \langle a_0^{k+1}, \ldots, a_{i^{k+1}-1}^{k+1}\rangle) = k+1$.
If Case 2 holds, then we add nothing to $\mathcal{N}$ and ν is undefined at
$(j^{k+1}, i^{k+1}, \langle a_0^{k+1}, \ldots, a_{i^{k+1}-1}^{k+1}\rangle)$.

Continue in this way, after countably many steps, we add a sequence $\{x_l\}_{l\in\omega}$ satisfying (3.4) and (3.5) to $\mathcal{N}$ and get a new model $M = (\omega, S, +, \cdot, 0, 1, <)$ where $S = \{x_l : l \in \omega\}$. M is a β-model since for every $x \in S$, the witnesses of all the Σ_1^1-formulas with x appearing as a parameter are added to S at some stage later. Note that for all $x \in S$, x is "low". But Kleene's $\mathcal{O}$ is not in M since by our construction all the reals added satisfy (3.5). Since the definition of Kleene's $\mathcal{O}$ is a Π_1^1-formula, M is not a model of $\Pi_1^1 - CA_0$.
By above, we see that M satisfies requirement (1) and (3).

Before we verify M satisfies requirement (2), we do some preparation.
The next claim shows that if V thinks E is a Δ_1^1 equivalence relation, then M also thinks E is a Δ_1^1 equivalence relation.

Claim 3.3. *If*

$$V \models \forall x \forall y(\varphi(x, y) \leftrightarrow (\neg\psi(x, y)))$$

and

$$V \models \forall x \forall y \forall z((\neg\psi(x,x)) \wedge (\varphi(x,y) \to (\neg\psi(y,x))) \wedge ((\varphi(x,y) \wedge \varphi(y,z)) \to (\neg\psi(x,z)))),$$

then

$$M \models \forall x \forall y(\varphi(x,y) \leftrightarrow (\neg\psi(x,y)))$$

and

$$M \models \forall x \forall y \forall z((\neg\psi(x,x)) \wedge (\varphi(x,y) \to (\neg\psi(y,x))) \wedge ((\varphi(x,y) \wedge \varphi(y,z)) \to (\neg\psi(x,z)))).$$

Proof. Note that $(\neg\psi(x,y) \to (\varphi(x,y)))$ is a Σ^1_1-formula. Since M is a Σ^1_1 elementary submodel of V,

$$V \models \forall x \forall y(\neg\psi(x,y) \to (\varphi(x,y))) \Rightarrow M \models \forall x \forall y(\neg\psi(x,y) \to (\varphi(x,y))).$$

Since $\forall x \forall y(\varphi(x,y) \to (\neg\psi(x,y)))$ is a Π^1_1-sentence, by downward absoluteness of Π^1_1-sentences,

$$V \models \forall x \forall y(\varphi(x,y) \to (\neg\psi(x,y))) \Rightarrow M \models \forall x \forall y(\varphi(x,y) \to (\neg\psi(x,y))).$$

Similarly, since

$$\forall x \forall y \forall z((\neg\psi(x,x)) \wedge (\varphi(x,y) \to (\neg\psi(y,x))) \wedge ((\varphi(x,y) \wedge \varphi(y,z)) \to (\neg\psi(x,z))))$$

is a Π^1_1-sentence, by downward absoluteness of Π^1_1-sentences,

$$V \models \forall x \forall y \forall z((\neg\psi(x,x)) \wedge (\varphi(x,y) \to (\neg\psi(y,x))) \wedge ((\varphi(x,y) \wedge \varphi(y,z)) \to (\neg\psi(x,z))))$$

implies

$$M \models \forall x \forall y \forall z((\neg\psi(x,x)) \wedge (\varphi(x,y) \to (\neg\psi(y,x))) \wedge ((\varphi(x,y) \wedge \varphi(y,z)) \to (\neg\psi(x,z)))).$$

□

Remark 3.1. *Note that being a Δ^1_1 equivalence relation is a Π^1_2-sentence. It is not absolute between V and ω-model of second order arithmetic M in general. It may happen that " M thinks 'E' is a Δ^1_1 equivalence relation, but in V, E is not a Δ^1_1 equivalence relation".*

Next we define code of recursive trees on $\omega \times \omega \times \omega$ in M.
Note that M is a β-model and thus $M \models ACA_0$. By Lemma V.1.4, [Simpson, 2009], ACA_0 proves the normal form theorem for Σ^1_1-formulas,

$$M \models \forall x(\varphi(x) \leftrightarrow (\exists f \forall m \theta(x \upharpoonright m, f \upharpoonright m)))$$

where φ is a Σ^1_1-formula and θ is a Σ_0-formula.

Thus, for the two Σ^1_1-formulas φ, ψ defining E, there exist two Σ_0-formulas θ, ρ such that

$$M \models \forall x \forall y(\varphi(x, y) \leftrightarrow (\exists w \forall n \theta(w \upharpoonright n, x \upharpoonright n, y \upharpoonright n))),$$

and

$$M \models \forall x \forall y(\psi(x, y) \leftrightarrow (\exists w \forall n \rho(w \upharpoonright n, x \upharpoonright n, y \upharpoonright n))).$$

Define two recursive trees T_1 and T_2 on $\omega \times \omega \times \omega$ using θ and ρ by

$$(\eta, \sigma, \tau) \in T_1 \Leftrightarrow \theta(\eta, \sigma, \tau),$$

and

$$(\eta, \sigma, \tau) \in T_2 \Leftrightarrow \rho(\eta, \sigma, \tau).$$

Following the convention of [Simpson, 2009], in RCA_0, we code finite sequences of natural numbers by natural numbers. Hence for any given finite sequence of natural numbers σ, we denote its code by c_σ which is a natural number.
Define a real C_i by

$$n \in C_i \Leftrightarrow \exists n_1 < n \exists n_2 < n \exists n_3 < n \exists \sigma_1 \in \omega^{<\omega} \exists \sigma_2 \in \omega^{<\omega} \exists \sigma_3 \in \omega^{<\omega}$$

$$((\sigma_1, \sigma_2, \sigma_3) \in T_i \wedge n_1 = c_{\sigma_1} \wedge n_2 = c_{\sigma_2} \wedge n_3 = c_{\sigma_3} \wedge n = c_{\langle n_1, n_2, n_3 \rangle}).$$

This is an arithmetical definition. Thus, we can regard T_1 and T_2 as two reals C_1 and C_2 respectively. Moreover, C_1 and C_2 are in M by arithmetical comprehension. From now on, we use C_1 and C_2 to represent T_1 and T_2 in M.

M satisfies ACA_0, thus we have

$$M \models \varphi(x, y) \leftrightarrow (\exists w \forall n C_1(w \upharpoonright n, x \upharpoonright n, y \upharpoonright n)),$$

and

$$M \models \psi(x, y) \leftrightarrow (\exists w \forall n C_2(w \upharpoonright n, x \upharpoonright n, y \upharpoonright n)).$$

Here $C_i(w \upharpoonright n, x \upharpoonright n, y \upharpoonright n)$ means $c_{\langle c_{w\upharpoonright n}, c_{x\upharpoonright n}, c_{y\upharpoonright n}\rangle} \in C_i$.

The next claim states that some facts is absolute between V and M.

Claim 3.4. *By absoluteness of Δ^1_1-formulas to M, we can show that*

$$\forall x, y \in M, M \models \exists w \forall n C_1(w \upharpoonright n, x \upharpoonright n, y \upharpoonright n) \text{ iff } V \models \exists w \forall n T_1(w \upharpoonright n, x \upharpoonright n, y \upharpoonright n) \tag{3.6}$$

and

$$\forall x, y \in M, M \models \neg(\exists w \forall n C_2(w \upharpoonright n, x \upharpoonright n, y \upharpoonright n)) \text{ iff } V \models \neg(\exists w \forall n T_2(w \upharpoonright n, x \upharpoonright n, y \upharpoonright n)). \tag{3.7}$$

Proof. By upward absoluteness of Σ^1_1-formulas, we have

$$\forall x, y \in M, M \models \exists w \forall n C_1(w \upharpoonright n, x \upharpoonright n, y \upharpoonright n) \Rightarrow V \models \exists w \forall n T_1(w \upharpoonright n, x \upharpoonright n, y \upharpoonright n).$$

This shows one direction for (3.6).

For the other direction for (3.6), consider the following.

By (3.1), we have

$$\forall x, y \in M,\ V \models \exists w \forall n T_1(w \upharpoonright n, x \upharpoonright n, y \upharpoonright n) \Rightarrow V \models \neg(\exists w \forall n T_2(w \upharpoonright n, x \upharpoonright n, y \upharpoonright n)).$$

By downward absoluteness of Π^1_1-formulas,

$$V \models \neg(\exists w \forall n T_2(w \restriction n, x \restriction n, y \restriction n)) \Rightarrow M \models \neg(\exists w \forall n C_2(w \restriction n, x \restriction n, y \restriction n)).$$

By (3.1) and Claim 3.3, we have

$$M \models \neg(\exists w \forall n C_2(w \restriction n, x \restriction n, y \restriction n)) \Rightarrow M \models \exists w \forall n C_1(w \restriction n, x \restriction n, y \restriction n).$$

(3.7) can be proved in the same way. □

Now we verify that M satisfies requirement (2):
if E is a Δ^1_1 equivalence relation in V, then

$$M \models (E^M \leq_H =^M_\omega) \vee (\exists a^M \exists z^M \forall \sigma, \tau \in 2^{<\omega}(T^M(z^M(\sigma,\tau), a^M(\sigma), a^M(\tau))))$$

where a^M, z^M and T^M are as in Section 3.1.

In V, by Corollary 2.12, there are two cases.
Case 1: There is a Hyp function $f:\ \omega^\omega \to \omega^\omega$ and

$$\forall x \forall y (xEy \leftrightarrow (f(x)(0) = f(y)(0))).$$

Now fix $x_0, y_0 \in M$ such that $x_0 E y_0$ in V, i.e., $V \models \varphi(x_0, y_0)$.
By Claim 3.3, in M, "$x_0 E y_0$" as well.
If Case 1 holds in V, let f be the Hyp reduction from E to $=_\omega$, then we have

$$V \models (\exists w \forall n T_1(w \restriction n, x_0 \restriction n, y_0 \restriction n)) \to (f(x_0)(0) = f(y_0)(0))$$

and

$$V \models (f(x_0)(0) = f(y_0)(0)) \to \neg(\exists w \forall n T_2(w \restriction n, x_0 \restriction n, y_0 \restriction n)).$$

$(\exists w \forall n T_1(w \upharpoonright n, x_0 \upharpoonright n, y_0 \upharpoonright n)) \to (f(x_0)(0) = f(y_0)(0))$ and $(f(x_0)(0) = f(y_0)(0)) \to (\neg(\exists w \forall n T_2(w \upharpoonright n, x_0 \upharpoonright n, y_0 \upharpoonright n)))$ are both Π^1_1-sentences with parameters x_0, y_0 from M, thus by downward absoluteness of Π^1_1-sentences,

$$M \models (\exists w \forall n C_1(w \upharpoonright n, x_0 \upharpoonright n, y_0 \upharpoonright n)) \to (f(x_0)(0) = f(y_0)(0))$$

and

$$M \models (f(x_0)(0) = f(y_0)(0)) \to (\neg(\exists w \forall n C_2(w \upharpoonright n, x_0 \upharpoonright n, y_0 \upharpoonright n))).$$

So if Case 1 holds in V, then

$$M \models \forall x \forall y((\exists w \forall n C_1(w \upharpoonright n, x \upharpoonright n, y \upharpoonright n)) \to (f(x)(0) = f(y)(0)))$$

and

$$M \models \forall x \forall y((f(x)(0) = f(y)(0)) \to (\neg(\exists w \forall n C_2(w \upharpoonright n, x \upharpoonright n, y \upharpoonright n))),$$

i.e., Case 1 holds in M.

Case 2: If Case 1 fails in V, then by Theorem 2.11,

$$V \models \exists a \exists z \forall \sigma, \tau \in 2^{<\omega}(T_2(z(\sigma, \tau), a(\sigma), a(\tau))),$$

since M is a β-model,

$$M \models \exists a^M \exists z^M \forall \sigma, \tau \in 2^{<\omega}(C_2(z^M(\sigma, \tau), a^M(\sigma), a^M(\tau))),$$

i.e., Case 2 holds in M.

Note that a^*, z^* are also in M since a^*, z^* can be defined by the following arithmetical formulas with parameters a, z from M

$$(\alpha, x) \in a^* \Leftrightarrow \forall n((\alpha \upharpoonright n, x \upharpoonright n) \in a)$$

and

$$(\alpha, \beta, x) \in z^* \Leftrightarrow \forall n((\alpha \upharpoonright n, \beta \upharpoonright n, x \upharpoonright n) \in z).$$

This finishes proof of Theorem 3.2. □

Now we consider arithmetic equivalence relations.
Given an arithmetic equivalence relation F defined by $\phi(x,y)$ in V. Then we have

$$V \models \forall x \forall y (xFy \leftrightarrow \phi(x,y))$$

and

$$V \models \forall x \forall y \forall z (\phi(x,x) \wedge (\phi(x,y) \rightarrow \phi(y,x)) \wedge ((\phi(x,y) \wedge \phi(y,z)) \rightarrow \phi(x,z))).$$

Both

$$\forall x \forall y (xFy \leftrightarrow \phi(x,y))$$

and

$$\forall x \forall y \forall z (\phi(x,x) \wedge (\phi(x,y) \rightarrow \phi(y,x)) \wedge ((\phi(x,y) \wedge \phi(y,z)) \rightarrow \phi(x,z)))$$

are Π^1_1-sentences. Since M is a β-model, being an arithmetic equivalence relation is absolute between V and M.

Similar argument as in proof of Theorem 3.2 shows that

Corollary 3.3. *There is an ω-model of second order arithmetic M so that if E is an arithmetic equivalence relation in V, then*

$$M \models (E^M \leq_H =^M_\omega) \vee (\exists a^M \exists z^M \forall \sigma, \tau \in 2^{<\omega} (T^M(z^M(\sigma,\tau), a^M(\sigma), a^M(\tau))))$$

where a^M, z^M and T^M are as in Section 3.1. But M does not satisfy $\Pi^1_1 - CA_0$.

3.3 Relativization

By examining the proof of Corollary 2.12, we can relativize Corollary 2.12 as follows:

Corollary 3.4. *Given $p \in \omega^\omega$, if E is a $\Delta^1_1(p)$ equivalence relation on ω^ω, and T is a tree recursive in p on $\omega \times \omega \times \omega$ such that $\forall x, y \in \omega^\omega$,*

$$\neg(xEy) \leftrightarrow \exists w \forall n (T(w \upharpoonright n, x \upharpoonright n, y \upharpoonright n)),$$

then either (1)

$$E \leq_{Hyp(p)} =_\omega$$

or (2)

$$\exists a \exists z \forall \sigma, \tau \in 2^{<\omega}(T(z(\sigma, \tau), a(\sigma), a(\tau)))$$

Moreover, $a : 2^{<\omega} \to \omega^{<\omega}$ and $z : 2^{<\omega} \times 2^{<\omega} \to \omega^{<\omega}$ induce continuous functions $a^* : 2^\omega \to \omega^\omega$ and $z^* : 2^\omega \times 2^\omega \to \omega^\omega$ defined by

$$a^*(\alpha) = \bigcup_{n\in\omega} a(\alpha \upharpoonright n) \text{ and } z^*(\alpha, \beta) = \bigcup_{n\in\omega} z(\alpha \upharpoonright n, \beta \upharpoonright n)$$

where α, $\beta \in 2^\omega$.

We relativize proof of Proposition 2, [Fokina et al., 2010] to prove Corollary 3.4. Suppose E has countably many equivalence classes. To show case (1) holds, recall

$$B = \bigcup \{D \subseteq \omega^\omega : \ D \text{ is } \Delta^1_1 \wedge \forall x, y \in D, \ xEy\}.$$

Since E is a $\Delta^1_1(p)$ equivalence relation, B is $\Pi^1_1(p)$. Let C be the set of codes of $\Delta^1_1(p)$ set contained in a single equivalence class as above. It is a classical result of effective descriptive set theory that C is $\Pi^1_1(p)$.

Consider the relation

$$R = \{(x, c) : \ c \in C \wedge x \in H(c), \text{the } \Delta^1_1(p) \text{ set coded by } C\}.$$

R is $\Pi^1_1(p)$, and can be uniformized by a $\Pi^1_1(p)$ function F. Since the value of F are all natural numbers, F is $\Delta^1_1(p)$ and by separation theorem, there is a $\Delta^1_1(p)$ set D such that $range(F) \subseteq D \subseteq C$.

Define an equivalence relation E^* on D by

$$d_0 E^* d_1 \Leftrightarrow (\forall x_0, x_1)((x_0 \in H(d_0) \wedge x_1 \in H(d_1)) \to x_0 E x_1)$$

$$\Leftrightarrow (\exists x_0, x_1)((x_0 \in H(d_0) \wedge x_1 \in H(d_1)) \wedge x_0 E x_1).$$

Thus $d_0 E^* d_1$ if and only if $H(d_0)$ and $H(d_1)$ are subsets of the same E-equivalence class. Since E is $\Delta^1_1(p)$, E^* is $\Delta^1_1(p)$. Furthermore, F witnesses that E is $\Delta^1_1(p)$ reducible to E^*.

Lastly, E^* is $\Delta^1_1(p)$ reducible to $=_\omega$.

To see this, view $=_\omega$ as equality relation on ω and define $f: \omega \to \omega$ by

$$f(c) = c^* \leftrightarrow cE^*c^* \wedge (\forall c'(c'E^*c^* \to c^* \leq c')).$$

Therefore, by transitivity, E is $\Delta^1_1(p)$ reducible to $=_\omega$.

If E has uncountably many equivalence classes, then recall $A = \omega^\omega \setminus B$. A is $\Sigma^1_1(p)$. The rest of the proof of Theorem 2.11 follows.

Relativization of Theorem 3.2 also holds. Explicitly, we have

Corollary 3.5. *Given M as constructed in Theorem 3.2 and $p \in M$. If E is a $Hyp(p)$ equivalence relation in V, then*

$$M \models (E^M \leq_{Hyp(p)} =^M_\omega) \vee (\exists a^M \exists z^M \forall \sigma, \tau \in 2^{<\omega}(T^M(z^M(\sigma,\tau), a^M(\sigma), a^M(\tau)))).$$

where a^M, z^M and T^M are as in Section 3.1. But M does not satisfy $\Pi^1_1 - CA_0$.

Corollary 3.6. *Given M as constructed in Corollary 3.3 and $p \in M$. If E is a $\Sigma^0_n(p)$ equivalence relation for some n in V, then*

$$M \models (E^M \leq_{Hyp(p)} =^M_\omega) \vee (\exists a^M \exists z^M \forall \sigma, \tau \in 2^{<\omega}(T^M(z^M(\sigma,\tau), a^M(\sigma), a^M(\tau)))).$$

where a^M, z^M and T^M are as in Section 3.1. But M does not satisfy $\Pi^1_1 - CA_0$.

To see this, just note that

Observation 1. *Since $p \in M$, therefore, if V thinks E is a $\Delta^1_1(p)$ or $\Sigma^0_n(p)$ (for some n) equivalence relation, then M also recognizes E as a $\Delta^1_1(p)$ or $\Sigma^0_n(p)$ (for some n) equivalence relation. The rest of the proof of Theorem 3.2 can be easily relativized to p.*

Remark 3.2. *We need $p \in M$ to ensure that when we complete the relativization, M can still recognize E as a $\Delta_1^1(p)$ or $\Sigma_n^0(p)$ (for some n) equivalence relation.*

3.4 Comparison with Simpson's Theorem

In this section, we compare the reverse mathematics strengths of our version of Silver's Dichotomy with some other version of Silver's Dichotomy.

It is shown by Simpson that over RCA_0, some version of Silver's Dichotomy is equivalent to Π_1^1-comprehension.

Defnition 3.7 (Silver's Theorem, [Simpson, 2009])**.** *If E is a coanalytic equivalence relation, then either*

(1) there exists a sequence of points $\langle y_n : n \in \omega \rangle$ such that

$$\forall x \exists n (x E y_n)$$

or

(2) there exists a perfect set P such that

$$\forall x \forall y ((x, y \in P \wedge x \neq y) \rightarrow (\neg (x E y))).$$

Theorem 3.8 ([Simpson, 2009])**.** *The following statements are pairwise equivalent over RCA_0.*

(i) Π_1^1-comprehension.

(ii) Silver's theorem.

(iii) Silver's theorem restricted to equivalence relations on ω^ω which are Δ_2^0 definable (with parameters).

Since our constructed model M satisfies Corollary 3.4 restricted to arithmetical (in particular Δ_2^0) in p equivalence relations for $p \in M$ but not Π_1^1-comprehension, a simple observation will lead us to the following question:

Question 3.1. *Is there any contradiction between Theorem 3.8 and our result?*

The following discussion answers Question 3.1.

In [Simpson, 2009], Simpson firstly proves the following version of Silver's theorem is provable in ATR_0:

Theorem 3.9 (an ATR_0 version of Silver's Theorem, [Simpson, 2009])**.** *The following is provable in ATR_0. If E is a coanalytic equivalence relation, then either (1) there exists a sequence of Borel codes (Definition V.3.1,[Simpson, 2009]) $\langle B_n : n \in \omega \rangle$ such that*

$$\forall x \exists n (x \in B_n)$$

and

$$\forall n \forall x \forall y (((x, y \in B_n) \rightarrow (xEy))$$

or

(2) there exists a perfect set P such that

$$\forall x \forall y ((x, y \in P \wedge x \neq y) \rightarrow (\neg (xEy))).$$

Consider the reverse mathematics strength of Definition 3.7, Theorem 3.9 and Corollary 3.4. The strength of Definition 3.7 is different from Theorem 3.9, and is also different from Corollary 3.4.

Our model M is a β-model. From Chapter VII, [Simpson, 2009], it is a model of ATR_0, and hence it models Theorem 3.9.

Case (1) of Definition 3.7 claims that if there are only countably many equivalence classes, then we can pick up for each equivalence class a representative. This is stronger than case (1) in Corollary 3.4 which only claims there exists a $Hyp(p)$ reduction from E to $=_\omega$. The construction of M gives no clue that we should believe that M satisfies Definition 3.7 or (iii) in Theorem 3.8. In fact, if M satisfies either of them, we will have a contradiction.

Claim 3.5. *M does not satisfy (iii) in Theorem 3.8.*

Proof. Firstly, note that (iii) implies arithmetic comprehension.
As mentioned in page 105, [Simpson, 2009], to see this, we only have to show that (iii) implies that every function $g: \omega \to \omega$ has a range.
Given $g: \omega \to \omega$, we can define an equivalence relation E_g as follows:

$$\forall x \forall y (x E_g y \Leftrightarrow \forall n \forall n' (n \in x \to (\exists m (g(m) = n))) \wedge (n' \in y \to (\exists m' (g(m') = n'))).$$

E_g is a Π^0_2 equivalence relation with parameter g. Obviously, there are only two equivalence classes for E_g. One is the range of g and the other is the complement of the range of g. By (iii), there are y_1, y_2 representing the two equivalence classes respectively and thus g has a range y_1.
Therefore, (iii) implies arithmetic comprehension.
Let ϕ_0 be a Π^1_1-formula which defines Kleene's $\mathcal{O}$, i.e., $\mathcal{O} = \{m : \phi_0(m)\}$. Thus the complement of $\mathcal{O}$ is defined by a Σ^1_1-formula $\varphi_0 = \neg\phi_0$. By Kleene's normal form theorem, we can write $\varphi_0(m)$ as $\exists f \theta(m, f)$ where θ is Π^0_1. Define a Δ^0_2 equivalence relation E_θ on $\omega \times \omega^\omega$ by

$$(m, f) E_\theta (n, g) \Leftrightarrow (m = n \wedge (\theta(m, f) \leftrightarrow \theta(n, g))).$$

By definition of E_θ, E_θ has only countably many equivalence classes. By (iii), there are a sequence of representatives $\langle (m_k, f_k) : k \in \omega \rangle$ such that

$$\forall m \forall f \exists k (m, f) E_\theta (m_k, f_k).$$

Then

$$\forall m (\exists f \theta(m, f) \leftrightarrow \exists k (m = m_k \wedge \theta(m_k, f_k)).$$

$\exists f \theta(m, f)$ is equivalent to an arithmetic formula with a sequence of parameters $\langle (m_k, f_k) : k \in \omega \rangle$.

Hence $\{m : \varphi_0(m)\} = \{m : \exists f \theta(m, f)\}$ exists by arithmetic comprehension. Therefore, if such parameters exist in M, then we can define $\mathcal{O}$ in M which is impossible. □

From the above discussion, we can see that there is no contradiction between Theorem 3.8 and our results.

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ON SILVER'S DICHOTOMY

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