A STUDY ON
THE COVERING LEMMAS

SHEN DEMIN
(B.S.(Hons), Tsinghua University)

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Chapter 1

Introduction
Fine Structure, as one of the most important tools to inner model theory, has received a lot of attention after Ronald B. Jensen’s work in the 1970’s. And the covering property plays a key role in the fine structural inner model theory as it characterizes the core models and gives good solutions to the Singular Cardinal Hypothesis in addition to Silver’s Theorem.

This survey is devoted to the investigation on the covering lemmas of the fine structural inner model theory. There are a number of publications nicely explaining the fine structure theories, however, in this survey we will concentrate merely on covering properties of different inner models to investigate the similarities and consistency among these models. The original idea of this survey is to aim some possible further development of the Covering Lemmas and the Fine Structural Inner Model Theory, although in the end this appears to be too big a goal to capture. In this paper, the author presented several proofs of covering properties for different inner models, and discussed about these analogies among the covering properties for investigation.

A large portion of this paper, including most of Sections 2 through 5, is devoted to present several analogous proofs of different covering lemmas as well as discussions on the core models. The readers are assumed to have background knowledge in Godël’s constructible universe $L$ and basic fine structure theory. Chapter 2 serves as a preliminary. In chapter 3, the author sketched a proof of the covering lemma for $L$ using fine structure tools. Chapter 3 also serves as a warm-up for later chapters where we prove the covering lemmas for larger core models. The proof is not very short and quick, however it clearly captures the idea that we will use later to prove for the Dodd-Jensen Covering Lemma for $K^{DJ}$ and $L[U]$. Chapter 4 of this survey deals with the weak covering lemma for Steel’s core model
K. The proof is sketched to be as clear and convenient to understand as possible, and sufficiently complete for the readers to capture all the important facts. This chapter is also essential for the chapter after, chapter 5, which presents a proof of the Dodd-Jensen Covering Lemma for $K^{DJ}$ and $L[U]$.

While presenting some technical lemmas in chapter 4, the intention is not so much to present the proof itself as to introduce techniques which are more important to the proofs of further chapters. Therefore for a few times, we assume stronger hypothesis which makes the proof easier as long as it still demonstrates the wanted technique. All the proofs appeared in this survey are due to original authors with citation, though there will be simplifications and modifications however not destroying the integrity of the proof and the author will point out along the way. The last part of Chapter 5 contains some discussions on the similarities of the proofs and talks about some ideas on further developments.

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Chapter 2

Preliminary

In this preliminary Chapter, we will first clarify some symbols and notations and introduce some key Lemmas. All the definitions and notations are consistent with Zeman’s book [9], therefore it is perfectly fine to immediately proceed to chapter 3 if the reader is already familiar with these. Also, this chapter only serves as a necessary and relatively simple tool box. Readers who are interested in or unfamiliar with the basic fine structural inner model theory can refer to [5] and [9] for more details.

For many of the fine structural tools developed, the motivations will only be talked about during later chapters where we actually use these tools.

2.1 Fine Structure

This section introduces basic fine structure theory which was first invented by Ronald B. Jensen in the 1970’s. Jensen presented this approach to prove the Covering Lemma for $L$, and his work was truly a brilliant breakthrough even by today’s
standards of set theoretical sophistication. The historical notes and motivations of the invention of this fine structural theory will be explained at the beginning of Chapter 3.

Jensen’s hierarchy, i.e. the $J_\alpha$-hierarchy, would be the main hierarchy throughout this paper. This hierarchy yields substantial advantages over the $L_\alpha$-hierarchy, which will be pointed out along the text. For example the $\omega$-completeness of $J_\alpha$ allows us to freely treat a finite set of ordinals as a single parameter, which otherwise would require some tedious coding.

The $\Sigma_n$-Skolem function is an important and basic concept to the fine structure of Jensen’s hierarchy. Iterated projectum(or projecta in some books), standard parameters, master codes and reducts are the other four key concepts to expand the Jensen’s hierarchy. The motivation involves preservation of condensation arguments which was essential to Gödel’s proof of relative consistency of CH. And the analogous lemma in fine structure is the so-called Downward Extensions of Embeddings Lemmata. In fact, Downward Extensions and Upward Extensions lemmas are central to the coherency and iterability of ”mice”(the essential structures to approximate core models, to be mentioned later), and Downward Extensions are also central to Jensen’s principles □.

**Definition 2.1.1** (Acceptable $J$-structure). Let $M = \langle J_\alpha^A, B \rangle$ be an amenable $J$-structure. We say $M$ is acceptable iff whenever $\xi < \alpha$ and there is a subset of $\tau$ inside $J_{\xi+1}^A - J_\xi^A$ for some $\tau < \omega \xi$, there is a surjective map $f : \tau \rightarrow \omega \xi$ in $J_{\xi+1}^A$.

First, let’s introduce the fine structure on $J_\alpha$’s, starting with the $\Sigma_1$-case:

**Definition 2.1.2.** Let $M = (J_\alpha, A)$ be an acceptable structure, then
CHAPTER 2. PRELIMINARY

1. The $\Sigma_1$-projectum $\rho_1^M$ of $M$ is the least ordinal $\rho$ such that there is a $\Sigma_1$ subset of $\rho$ which is not a member of $M$, but is $\Sigma_1$-definable in $M$ with a finite subset of $\alpha$ as parameters.

2. The $\Sigma_1$-standard parameter $p_1^M$ of $M$ is the least finite sequence $p \in [\alpha]^{<\omega}$ of ordinals such that there is some set $x \subseteq \rho_1^M$ so that $x \notin J_\alpha$, but $x$ is $\Sigma_1$-definable in $M$ from parameters in $\rho_1^M \cup p$.

3. The $\Sigma_1$-standard code is the set $A_1^M$ of pairs $(\varphi, \xi)$ such that $\xi < \rho_1^M$ and $\neg \varphi$ is the Gödel number of a $\Sigma_1$-formula $\varphi$ over $M$, with parameter $p_1^M$, such that $M \models \varphi(\xi)$.

4. The $\Sigma_1$-Skolem function $h_1^M$ of $M$ is defined as follows: Let $\langle \exists z \varphi_n : n < \omega \rangle$ be an enumeration of the $\Sigma_1$-formulas of set theory. $h_1^M(\langle n, x \rangle)$ is defined if and only if there are $y, z$ such that $M \models \varphi_n(x, y, z, p_1^M)$, and $h_1^M(\langle n, x \rangle) = y$ where $\langle \alpha', z, y \rangle$ is the lexicographically smallest triple such that $(J_{\alpha'}, A \cap \alpha') \models \varphi_n(x, y, z, p_1^M)$.

5. The $\Sigma_1$-code, $\mathcal{E}_1(M)$, of $M$ is the structure $(J_{\rho_1^M}, A_1^M)$.

**Definition 2.1.3.** Let $M = (J_\alpha, A)$ be an acceptable $J$-structure. We define the $\Sigma_n$-codes of $M$ by recursion:

$$
\rho_{n+1}^M = \rho_1^{\mathcal{E}_n(M)}, \quad p_{n+1}^M = p_1^{\mathcal{E}_n(M)}, \quad h_{n+1}^M = h_1^{\mathcal{E}_n(M)}, \quad A_{n+1}^M = A_1^{\mathcal{E}_n(M)}, \quad \mathcal{E}_{n+1}(M) = \mathcal{E}_1(\mathcal{E}_n(M)).
$$

1. We denote the $\Sigma_n$-projectum of $M$ as $\rho_n^M$, or sometimes $\omega \rho_n^M$ equivalently; and the ultimate projectum as $\rho_\infty^M$ or $\omega \rho_\infty^M$. 
2. We denote the $\Sigma_n$-standard parameter of $M$ as $p^M_n$, and the standard parameter as $p^M$;

3. We denote the $\Sigma_n$-Skolem function of $M$ as $h^M_n$;

4. We denote the $\Sigma_n$-standard code of $M$ as $A^M_n$;

5. We denote the $\Sigma_n$-code of $M$ as $\mathfrak{C}_n(M)$.

**$\Sigma^*$ – Relations**

The motivation of the $\Sigma^*$-relation was to capture the definability over the $n$-th reduct and not involving the reduct itself. We define inductively for $\Sigma^{(n+1)}_l$ to be “$\Sigma_1$ in $\Sigma^{(n)}_l$”. And this hierarchy of formulae yields stronger power than the $\Sigma_n$-hierarchy in fine structure arguments.

**Soundness**

Another important notion of fine structure theory is soundness, which enables us to reconstruct the model from its code:

**Definition 2.1.4** (Soundness). An acceptable $J$-structure $M$ is $1$-sound if it is the image of its $\Sigma_1$-projectum under the $\Sigma_1$-Skolem function. $M$ is $n$ – sound if $M$, $\mathfrak{C}_1(M)$, $\mathfrak{C}_2(M)$, ...$\mathfrak{C}_{n-1}(M)$ are all $1$-sound. And $M$ is sound if $M$ is $n$-sound for all $n \in \omega$.

(Remark : For $L$, all $J_\alpha$’s are sound.)

**Definition 2.1.5.** Let $M = < |M|, A_1, A_2, ..., A_n >$, and $X \subseteq |M|$, then we define:

$$M|X = < |M| \cap X, A_1 \cap X, A_2 \cap X, ..., A_n \cap X >$$
CHAPTER 2. PRELIMINARY

Solidity

The notion of Solidity Witness, which was first introduced by William J. Mitchell (and later rediscovered by S. Friedman), characterizes the behavior of the standard parameter along iterations, therefore together with soundness enables us to preserve fine structure information about the structures through the standard parameter:

Definition 2.1.6 (Solidity). Let $M$ be an acceptable $J$-structure. We denote the standard witness with respect to $\nu \in p^M$ as $W^\nu_M p^M$. Then, $M$ is solid iff $W^\nu_M p^M \in M$ for all $\nu \in p^M$.

Before we move on to the next section, we state the Downward Extensions of Embeddings Lemma (for L) as follows:

Lemma 2.1.7 (Downward Extensions of Embeddings Lemma). Suppose that

$$i : (J_{\rho'}, A') \prec_0 \mathcal{E}_n(J_\alpha)$$

Then there is $\alpha' \leq \alpha$ such that $(J_{\rho'}, A') = \mathcal{E}_n(J_{\alpha'})$ and $i$ extends to a $\Sigma_n$-embedding $\tilde{i} : J_{\alpha'} \to J_\alpha$. Furthermore $\tilde{i}$ preserves the first $n$ levels of fine structure, so that $\tilde{i} \circ h_{J_{\alpha'}}^J = h_{J_\alpha}^J \circ \tilde{i}$ for all $k \leq n$.

The fine structure of $J_\alpha$'s above will be sufficient for us in this paper. It also generalizes to all acceptable $J$-structures, which is very important to the theory for large core models. Readers can refer to [9] for a finer presentation.
CHAPTER 2. PRELIMINARY

2.2 Mice and Iterability

A key structure in the fine structural inner model theory is the so called ”mouse”, which we use as building stones to construct the core models. This notion was first introduced by Jensen in connection with the core model below one measurable cardinal [6][7][8], and later developed further by phases by Martin, Steel and Mitchell. A mouse is defined to be an ”iterable premouse” as follows:

**Definition 2.2.1 (Premouse).** Let $M = < J^E_\alpha, E_\omega >$ be an acceptable $J$-structure. $M$ is a premouse if the following holds:

1. $E \subset \{ < \nu, x > : \nu < \omega \alpha \& x \subset \nu \}$. Set $E_\nu = \{ x : < \nu, x > \in E \}$;

2. For each $\nu \leq \omega \alpha$, either $E_\nu = \phi$ or else $\nu$ is a limit ordinal, $J^E_\nu$ has a largest cardinal $\kappa$, $E_\nu$ is a normal measure over $J^E_\nu$ with critical point $\kappa$ and $M \parallel \nu \overset{def}{=} < J^E_\nu, E_\omega >$ is amenable;

3. (Coherency) Let $\nu \leq \omega \alpha$ and $\pi$ be the $\Sigma_0$-Ultrapower map from $J^E_\nu$ to $N$, where $N = < |N|, E' >$ for some $E'$ is the ultrapower. Then $E' \upharpoonright \nu = E \upharpoonright \nu$ and $E'_\nu = \phi$;

4. (Soundness) $M \parallel \nu$ is sound for all $\nu < \alpha$.

**Notations:** Let $M = < J^E_\alpha, E_\omega >$ be a premouse.

1. We denote the height $\alpha$ of $M$ as $ht(M)$.

2. We call the measure $E_\omega$ the top measure of $M$.

**Remark:** An important ultrapower that we use a lot in the fine structure theory is the *-ultrapower. Preservation properties of the *-ultrapower are essential to the iterations of mice. A comprehensive presentation on this can be found in chapter.
3 of [9]. We assume sufficient understanding of the fine ultrapower by the reader, and proceed to the iterations:

**Definition 2.2.2 (Iteration).** Let $M$ be a premouse. An iteration of $M$ of length $\theta$ with indices $\{\langle \nu_i, \alpha_i \rangle : i + 1 < \theta \}$ is a sequence $\{M_i : i < \theta \}$ of premice together with a sequence of commutative iteration maps $\{\pi_{ij} : i \leq j < \theta \}$ satisfying:

(a) $M_0 = M$;

(b) $\nu_i \leq \alpha_i \leq \text{ht}(M_i)$;

(c) If $E_{\nu_i, \alpha_i}^M = \phi$, then $M_{i+1} = M_i \| \alpha_i$ and $\pi_{i, i+1} = \text{id} \restriction (M_i \| \alpha_i)$;

(d) If $E_{\nu_i, \alpha_i}^M \neq \phi$, then $E_{\nu_i, \alpha_i}^M$ is a measure on $M_i \| \alpha_i$ and $\pi_{i, i+1}$ is the corresponding $\ast$-ultrapower map:

$$\pi_{i, i+1} : M_i \| \alpha_i \xrightarrow{E_{\nu_i, \alpha_i}^M}^\ast M_{i+1}$$

(e) If $\alpha_i < \text{ht}(M_i)$, we call $i$ a truncation point, and there are only finitely many truncations;

(f) For limit $\lambda$, $M_\lambda$ is the direct limit of all $\{M_i : i < \lambda \}$.

**Definition 2.2.3.** Let $M$ be a premouse.

1. We say $M$ is iterable iff any iteration of $M$ can be continued and there is no iteration of $M$ with infinitely many truncations.

2. An iteration $\tilde{s}$ of $M$ is normal iff $\nu_i < \nu_j$ whenever $i < j$ and $\alpha_i$ is always maximal such that $E_{\nu_i, \alpha_i}^M$ is a measure on $M_i \| \alpha_i$. 
3. An iteration $\tilde{s}$ of $M$ is called simple iff there are no truncations.

4. $M$ is called a mouse iff $M$ is iterable.

Next, we state a lemma related to upward extensions, which solves the problem of extending an embedding on the $\Sigma_n$-code to the whole structure. We adapt an easier version merely for $L$, because in this paper we only use it for the covering lemma for $L$. In the covering Lemmas for Steel’s $K$ and the Dodd-Jensen core model, we use finer upward extensions, such as canonical extension from fine ultrapowers by $\omega$-completeness, or Frequent Extensions of Embeddings Lemma. Therefore, we only adapt a coarse version of the Upward Extensions lemma at this moment. For the coarse version, $\Sigma_0$-ultrapower is used instead of the $*$-ultrapower to extend a given embedding $\pi: J_\kappa \to J_\kappa$ to a larger domain.

We denote, the $\Sigma_n$-ultrapower of $M$ induced by the extender $E_{\pi, \beta}$ of length $\beta$ which is associated with $\pi$, by $Ult_n(M, \pi, \beta)$, and for the $\Sigma_0$-ultrapower, we usually write $Ult(M, \pi, \beta)$ for convenience.

**Lemma 2.2.4** (Upward Extensions of Embeddings Lemma, coarse version). For a given embedding $\pi: J_\kappa \to J_\kappa$, with $\beta \leq \kappa$ and either $\omega \rho_n^\kappa \geq \min\{\nu: \pi(\nu) \geq \beta\}$ or range($\pi$) is cofinal in $\beta$ and $\pi(\omega \rho_n^\kappa) \geq \beta$, set $M_n = \mathcal{C}_n(J_\alpha)$ and $\tilde{M}_n = Ult(M_n, \pi, \beta)$. Then,

1. There is a structure $\tilde{M}_0$ such that $\tilde{M}_n$ is, formally, equal to $\mathcal{C}_n(\tilde{M}_0)$. If this structure $\tilde{M}_0$ is well-founded then there is an ordinal $\tilde{\alpha}$ such that $\tilde{M}_0 = J_{\tilde{\alpha}}$ and $\tilde{M}_n = \mathcal{C}_n(J_{\tilde{\alpha}})$. 

2. There is an embedding $\tilde{\pi} : J_\alpha \to J_\beta$ such that $\pi(\tilde{\beta}) \geq \beta$ if $\beta < \kappa$, or $\tilde{\beta} = \tilde{\kappa}$ if $\beta = \kappa$.

3. The embedding $\tilde{\pi}$ preserves the $\Sigma_k$-codes for $k \leq n$. In particular, $h^M_n \circ \tilde{\pi}(x) = \tilde{\pi}(x)$ for all $x$ of which either side is defined.

4. The embedding $\tilde{\pi}$ preserves the $\Sigma_1$-Skolem function of $M_n$ in the sense that there is a function $\tilde{h}$, which is $\Sigma_1$-definable over $\tilde{M}_n$, such that $\tilde{\pi} \circ h^{M_n}_{n+1}(x) = \tilde{h} \circ \tilde{\pi}(x)$ for all $x \in M_n$ such that either side is defined.

Given an embedding $\sigma : \tilde{M} \to M$ with sufficient preserving property, any iteration of $\tilde{M}$ can be turned into an iteration of $M$, this is called the ”copying process”. An important consequence is the following Dodd-Jensen Lemma:

**Lemma 2.2.5** (Dodd-Jensen Lemma). Let $M$ be a mouse, $\tilde{s}$ be an iteration of $M$ resulting in $M'$ and $\pi : M \to M'$ as the corresponding iteration map. Suppose that there is a $\Sigma^*$-preserving map $\sigma : M \to M'$. Then $\tilde{s}$ is simple and $\pi(\xi) \leq \sigma(\xi)$ for all $\xi \in M$.

Now back to the iterations of mice, a key process to characterize the class of mice is the Comparison Process, which provides us comparison between any two mice through coiteration and gives us a canonical well-ordering of the class of mice:

**Definition 2.2.6** (Coiteration). Let $M^0$, $M^1$ be premice. A pair of iterations

$$\tilde{s}^0 = (M^0_i, \pi^0_{ij} : i \leq j < \theta + 1), \quad \tilde{s}^1 = (M^1_i, \pi^1_{ij} : i \leq j < \theta + 1)$$

is a coiteration of $M^0$, $M^1$ of length $\theta + 1$ iff

(a) $M^0_0 = M^0$ and $M^1_0 = M^1$. 

(b) Both iterations satisfy that for each truncation, the $\alpha_k^i$ is chosen to be maximal as we mentioned in the definition of a normal iteration, i.e. $\alpha_k^i$ is maximal such that $E_{\nu}^{M_k^i}$ is a measure on $M_k^i \| \alpha_k^i$, for $k = 0, 1$.

(c) If $i < \theta + 1$, then $\nu_i$ is the least $\nu$ satisfying $E_{\nu}^{M_0^i} \neq E_{\nu}^{M_1^i}$, provided such a $\nu$ exists.

(d) $\nu_i$ is defined for all $i < \theta$.

Lemma 2.2.7 (Comparison Lemma). Let $M^0$, $M^1$ be premice, and suppose the coiteration of $M^0$, $M^1$ does not stop because of lack of iterability on either side. Let $\theta$ be any regular cardinal larger than the size of both of them. Then the coiteration of $M^0$, $M^1$ terminates below $\theta$.

We also point out that every mouse is solid (Solidity Theorem) and that the coiteration of two mice must satisfy that at least one side of the coiteration is simple (implied immediately by the Dodd-Jensen Lemma). Therefore together with the comparison process, these facts show that the class of mice forms a canonical well-ordering as follows:

Lemma 2.2.8 (Canonical Well-Ordering of Mice). Let $M$, $N$ be mice, and define:

1. $M \sim^* N$ iff $M$, $N$ have a common simple iterate;

2. $M <^* N$ iff there is a mouse which is a simple iterate of $M$ and not a simple iterate of $N$.

Then, $<^*$ is a well-ordering on the class of mice under the equivalent relation $\sim^*$. 
Definition 2.2.9. Let $\bar{M}$ be a premouse and $\omega \rho^M_\omega \leq \alpha \in \text{Ord} \cap |M|$. Then $\bar{M}$ is the core of $M$ above $\alpha$, denoted as $\text{core}_\alpha(M)$, iff there is a $\Sigma^*$-preserving map $\sigma: \bar{M} \to M$ such that

a) $\sigma \upharpoonright \alpha = \text{id}$;

b) $\sigma(p^{\bar{M}}) = p^M$;

c) $\bar{M}$ is the closure of $\alpha \cup p^{\bar{M}}$ under good $\Sigma^*(\bar{M})$ functions.

The map $\sigma$ is called the core map above $\alpha$. If $\alpha = \omega \rho^n_M$, we call $\bar{M}$ the nth-core of $M$. If $\alpha = \omega \rho^\omega_M$, we call $\bar{M}$ the core of $M$, denote as $\text{core}(M)$.

In analogy with the Condensation Lemma of $L$, we have a more general Condensation Lemma in the context of mice, condensing certain structures to the core or segment of an ultrapower.

Lemma 2.2.10 (Condensation Lemma). Let $\bar{M}$ be a premouse, $M$ be a mouse and

$$ \sigma: \bar{M} \xrightarrow{\Sigma^*(\bar{M})} M $$

be such that $\sigma \upharpoonright \omega \rho^{n+1}_M = \text{id}$. then $\bar{M}$ is a mouse. Suppose moreover that $\bar{M}$ is sound above $\nu$, where $\nu$ is the largest ordinal such that $\sigma \upharpoonright \nu = \text{id}$. Then one of the following holds:

a) $\bar{M} = \text{core}_\nu(M)$ and $\sigma$ is the associated core map.

b) $\bar{M}$ is a proper initial segment of $M$ above $\nu$.

c) $\bar{M}$ is a proper initial segment of $\text{Ult}^*(M \| \zeta, E^M_\nu)$ where $\zeta$ is the largest ordinal such that $E^M_\nu$ is a total measure in $M \| \zeta$. 
Finally, before we move on to the next chapter, we state the definition of the "extender models":

**Definition 2.2.11.** An extender model, or equivalently, a weasel, is a model $W$ of the form $J[E] = J^E_{\infty}$ such that $W \parallel \alpha$ is a mouse for every $\alpha \in \text{Ord}$.

**Remark:** It turns out that the same comparison process by coiteration also forms a canonical well-ordering of weasels. And moreover, a weasel can be coiterated with a mouse. A *universal weasel* is one that the coiteration with any coiterable premouse terminates. The notion of universality was first discovered by Mitchell [18].
Chapter 3

Covering Lemma for $L$

3.1 The Covering Lemma

A natural place to start with, is Gödel’s constructible universe $L$.

In 1938, Gödel came out with the constructible universe $L$ and proved the relative consistency of Continuum Hypothesis (CH). A key advantage of the $L$-hierarchy is the uniform hierarchical definition, which directly leads to the Condensation Lemma stating that any transitive elementary substructure of $L_\alpha$ is in fact some $L_\bar{\alpha}$. The argument on CH (in $L$) follows naturally: If a real is definable over $L_\alpha$, it is in fact definable over some countable transitive elementary substructure $M$ of $L_\alpha$ (Löwenheim-Skolem argument) which by Condensation Lemma is in fact some $L_{\bar{\alpha}}$, $\bar{\alpha} < \omega_1$. This allows us to enumerate every real below $L_{\omega_1}$, and hence CH in $L$ follows.

In the 1970’s, Ronald B. Jensen refined this argument in a surprisingly nice way—now known as the Fine Structure Theory. Basically Jensen worked out, uniformly, a Skolem function for $\Sigma_n$ formulae over $J_\alpha$ with a fine $\Sigma_n$ definition over $J_\alpha$. 
Jensen originally used the Lévy-hierarchy on $L_\alpha$, and later refined the theory by the invention of rudimentary functions and the $J_\alpha$-hierarchy. Jensen expanded the $J_\alpha$-hierarchy by iterated projectum, standard parameters, standard master codes and reducts. Definability was argued in $\Sigma^*$-relations. The motivation of examining the structure in such a fine way was to reduce the technical complications caused by using the Lévy-hierarchy, while preserving downward extensions in the condensation arguments which is central in the fine structure theory. We no longer have to deal with $\Sigma_{n+1}$-definability, but instead a $\Sigma_{n+1}$ formula is reduced to a $\Sigma_1$ formula over the $\Sigma_n$ code of $J_\alpha$. The $\Sigma_n$-Skolem function produces condensed substructures of $J_\alpha$’s, and while preserving the definition of the Skolem function.

Jensen proved a striking theorem, now well-known as the Covering Lemma, with this developed fine structure theory. This breakthrough in the 1970’s states the following fact:

**Theorem 3.1.1** (The Covering Lemma, Ronald B. Jensen). If $0^\sharp$ does not exist, then for every uncountable set $x$ of ordinals, there is a set $y \in L$ such that $x \subseteq y$ and $|y| = |x|$.

There are multiple ways to prove this Theorem. One very interesting proof is due to Silver, which essentially avoids fine structural argument, this proof can be found in Keith J. Devlin [8]. However, this approach doesn’t generalize to larger core models. The approach that we use, presented as below, will involve much use of the fine structure tools, and follows an analogous sketch to our later proof of the Dodd-Jensen Covering Lemma for $K^{DJ}$ and $L[U]$. Therefore it also serves as an early practice for later chapters. This proof is essentially due to William J. Mitchell, readers can refer to Hand Book of Set theory [16][17] for the original version.
CHAPTER 3. COVERING LEMMA FOR L

Proof of Theorem 3.1.1:
First we make an assumption toward a contradiction that the theorem fails, i.e. 0♯ does not exist, but there is a counter-example \( x \subseteq \kappa \) such that \( \kappa \) is the least ordinal containing such a counter-example: \( x \subseteq \kappa \) & \( \forall y \supseteq x(y \in L \to |y| > |x|) \).

A first glance at \( x \) and \( \kappa \) reveals that \( |x| < |\kappa| \), and \( x \) is cofinal in \( \kappa \). Also it is obvious that \( \kappa \) must be a cardinal in \( L \), otherwise suppose \( \lambda = |\kappa|^L \), and let \( j : \lambda \leftrightarrow \kappa, \; \bar{x} = j^{-1}x \). Since \( \bar{x} \subseteq \lambda < \kappa \), there is a set \( \bar{y} \in L \) covering \( \bar{x} \) by the minimality of the \( \kappa \). Then \( y = j''(\bar{y}) \) covers \( x \) and contradicts our earlier assumption.

Our proof essentially investigates a class of so called ”suitable sets” in \( L \), and concludes that every suitable set is in \( L \), and that every uncountable set \( x \) is contained in a suitable set of the same cardinality. Similar approach works as well for the Dodd-Jensen core model \( K \) and \( L[U] \), which will be argued later in chapter 5.

One thing to be noted is that we do not really need to cover \( x \) with a suitable set \( y \) of the same cardinality, in fact any suitable set \( y \supseteq x \) satisfying \( |y|^L < \kappa \) would be enough for our purpose. Because if we have such an \( y \), let \( \lambda = |y|^L, \; j \in L, \; J : \lambda \leftrightarrow y \). Let \( \bar{x} = j^{-1}x \). Then \( \bar{x} \subseteq \lambda < \kappa \). Then by the choice of \( \kappa \) there is a set \( \bar{Z} \in L, \; \bar{x} \subseteq \bar{Z} \subseteq \lambda, \; |\bar{Z}| = |\bar{x}| \). So \( Z = j''\bar{Z} \) gives our desired contradiction.

Now we introduce the formal notion of ”suitability”:

**Definition 3.1.2 (Suitable Sets).** Let \( X \) be a subset of \( L \), and \( \pi : N \cong X \) be the inverse collapse map. \( X \) is suitable if \( X \prec_1 J_\kappa \) for some \( \kappa \in \text{Ord} \) and \( \text{Ult}_n(J_\alpha, \pi, \beta) \) is well founded for all \((\alpha, n, \beta)\) such that the ultrapower is defined.
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Denote the class of suitable sets as $C$. Note the absoluteness of definition 3.1.2 ensures the class $C$ is definable in $L$.

Lemma 3.1.3.  1. Assume $X \prec_1 J_\kappa$ is suitable, then there is a cardinal $\rho$ of $\mathcal{L}$ and a function $h \in \mathcal{L}$, such that $\rho < \kappa$ and $X = h''(\rho \cap X)$.

2. If $X \prec_1 J_\kappa$ is suitable and $\rho < \kappa$ is a cardinal of $\mathcal{L}$, then $X \cap J_\rho$ is also suitable.

Proof of lemma: The proof of lemma 3.1.3 begins with a basic construction:

Transitive collapse of $X$ induces a non-trivial embedding (inverse collapse) $\pi : J_\kappa \rightarrow J_\kappa$. Let $(\alpha, n)$ be the lexicographically largest pair such that $\tilde{M} = \text{Ult}_n(J_\alpha, \pi, \kappa)$ is defined. Note that this largest pair always exists otherwise we can extend this embedding to a nontrivial elementary embedding from $\mathcal{L}$ to $\mathcal{L}$, which will contradict the absence of $0^\sharp$.

Now $\alpha$ is the least ordinal such that there is a bounded subset of $\bar{\kappa}$ in $J_{\alpha+\omega}$ but not in $J_\kappa$ and $n$ is the least natural number such that the set is $\Sigma_{n+1}$ in $J_\alpha$, i.e. $\omega \rho^{J_\alpha}_{n+1} < \bar{\kappa} \leq \omega \rho^{J_\alpha}_n$ and $\omega \rho^{J_{\alpha+1}}_{n_1} \geq \bar{\kappa}$ whenever $\bar{\kappa} \leq \alpha_1 < \alpha$ and $n_1 < \omega$.

By upward extensions of embeddings lemma (coarse version), $\tilde{M} = J_{\tilde{\alpha}}$ for some $\tilde{\alpha}$, and the following diagram (2.1) commutes:
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Let \( \bar{\rho} = \rho J_{n+1} + 1 \), then \( \bar{\rho} < \bar{\kappa} \) and \( J_\alpha = \tilde{\pi} J_{n+1} \bar{\rho} \), therefore \( X = \pi^\prime J_{\bar{\kappa}} = \tilde{\pi}^\prime (J_{\bar{\kappa}} \cap h_{n+1} \bar{\rho}) \cap (\tilde{\pi} \circ h_{n+1} \bar{\rho}) \cap (X \cap \rho) \), where \( \tilde{h} \in L \) is the function given by the upwards extensions of embeddings lemma (Lemma 2.0.8) and \( \rho = sup(\pi^\prime \bar{\rho}) < \kappa \).

(\( \square \)Lemma 3.1.3)

**Corollary 3.1.4.** Any suitable set \( X \prec_1 J_{\kappa} \) is in \( L \).

Proof: By induction on \( \kappa \), assume \( \kappa \) is the least such that counter example appears. Let \( X, h, \rho \) be as in lemma 3.1.3 clause 1. Then \( X \cap J_\rho \) is suitable by clause 2 and hence in \( L \) by induction hypothesis. However this gives \( X = h^\prime (X \cap \rho) \in L \).

(\( \square \)Corollary 3.1.4)

Now we fix a set \( X \) which is not suitable. Let \( \alpha, n, \beta \) be such that \( \tilde{M} = Ult_n(J_\alpha, \pi, \beta) \) is defined and not well-founded. A more careful analysis of the unsuitability of \( X \) is realized by the following:

**Definition 3.1.5** (Unsuitability Witness). Assume \( X \) is not suitable. Then the witness \( w \) to the unsuitability of \( \pi : X \prec_1 J_{\kappa} \) is a \( \omega \)-chain of \( \Sigma_0 \)-elementary embeddings \( i_k : m_k \rightarrow m_{k+1} \) such that

1. \( i_k \in X \) and \( m_k \in X \) for all \( k < \omega \);
2. The direct limit of the chain \( \pi''(w) \) equals \( \mathfrak{C}_n(J_\alpha) \) for some \( \alpha \in \text{Ord} \) and \( n \in \omega \);

3. The direct limit of the chain \( w \) is not the \( \Sigma_n \)-code of any well founded model \( J_\alpha \) for all \( n \in \omega \);

4. The critical sequence \( \beta_k : k < \omega \) where \( \beta_k \) is the critical point of \( i_k \) is nondecreasing;

5. For each \( k \) we have \( m_k \in m_{k+1} \), and exists a function \( f_k \in m_{k+1} \) such that \( f_k''(\beta_k) = i_k''(m_k) \).

We call \( \beta = \sup_k(\beta_k) \) the support of the witness \( w \), and the pair \( (\alpha, n) \) the height of \( w \) in \( X \). A witness \( w \) is said to be minimal in \( X \) if it has minimal height (lexicographic) among all witnesses with the same support.

Some modification to the definition can further make the minimal witness unique, however we need not to do so. This following technical lemma helps us further understand the role of the unsuitability witness:

**Lemma 3.1.6.** Assume \( X \prec_1 J_\kappa \). Then \( X \) is unsuitable if and only if it has a witness to its unsuitability. Furthermore, if \( w \) is such a witness, then

1. \( w \) is also a witness to the unsuitability of any \( X' \) such that \( w \subseteq X' \prec_1 X \);

2. If \( w \subseteq X' \prec_1 X \), then \( w \) is a minimal witness for \( X \) implies \( w \) is also a minimal witness for \( X' \), and other minimal witness for \( X' \) with the same support is also a minimal witness for \( X \);

3. If \( X = Y \cap J_\kappa \), where \( Y \prec_1 H(\tau) \) for some cardinal \( \tau > \kappa \), then \( w \not\in Y \).

This technical characterization lemma is adapted from Mitchell [17].
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Proof of Lemma 3.1.6:

One direction is simple: if there is a witness to $X$’s unsuitability, by definition 3.1.5 it is easy to see that $X$ is in fact unsuitable.

Now look at the other direction, assume $X$ is unsuitable, we need to find it a witness for $X$.

By assumption, there are $\alpha$, $n$ and $\beta$ such that $\text{Ult}_n(J_\alpha, \pi, \beta)$ is defined but not well-founded. Let $M_n = C_n(J_\alpha)$. Then $\tilde{M}_n = \text{Ult}_0(M_n, \pi, \beta)$ is defined but not the $\Sigma_n$-code of any well-founded structure $J_\alpha$.

If $\tilde{M}_n$ is ill-founded, then we have a sequence $\langle z_{k+1}E z_k, \ k < \omega \rangle$ where $z_k = [a_k, f_k]_\pi$, $f_k \in M_n$, $a_k \in \beta$.

If $\tilde{M}_n$ is well-founded, then since $\tilde{\pi}: M_n \rightarrow \tilde{M}_n$ is $\Sigma_1$-elementary, $\tilde{M}_n = C_n(\tilde{M})$ for some ill-founded structure $\tilde{M}$ and $\tilde{M}_n$ is mapped onto $\tilde{M}$ by the $\Sigma_n$-Skolem function. Therefore we can still obtain the sequence by the map of the $\Sigma_n$-Skolem function.

By definition, $M_n = C_n(J_\alpha) = < J_{\rho_n}, A_n >$. (If $n = 0$, then it is just $< J_\alpha, \phi >$.) Inductively define $\alpha_k$’s and $\beta_k$’s as following: let $\alpha_k < \rho_n$ be the least ordinal greater than $\alpha_{k-1}$ such that $\{f_1, \ldots, f_k\} \subseteq J_{\alpha_k}$, and let $\beta_k$ be the least member of $X$ such that $\{a_0, \ldots, a_k\} \subseteq \beta_k$. Finally let $\tilde{\beta}_k = \pi^{-1}(\beta_k)$ and let $\tilde{m}_k$ be the transitive collapse of the $\Sigma_1$-hull of $\tilde{\beta}_k \cap \{f_1, \ldots, f_k\}$ in $< J_{\alpha_k}, A_n \cap J_{\alpha_k} >$ with the associated collapsing map $\tilde{j}_k$. Then let $\tilde{i}_k = \tilde{j}_{k+1}^{-1} \circ \tilde{j}_k: \tilde{m}_k \rightarrow \tilde{m}_{k+1}$.

Then $\tilde{m}_k$, $\tilde{i}_k \in J_\alpha$. Set $m_k = \pi(\tilde{m}_k)$ and $i_k = \pi(\tilde{i}_k)$, $w = \langle m_k, i_k: k < \omega \rangle$, and set $\beta' = \sup(\beta_k) \leq \beta$. 

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To verify $w$ is valid for the definition of unsuitability witness, clauses 1, 3, 4 are straightforward. For clause 2, if $\bar{w} = \langle \bar{m}_k, \bar{i}_k : k < \omega \rangle = \pi^{-1}[w]$, then direct limit of $\bar{w}$ is a $\Sigma_\alpha$-elementary substructure of $M_n = C_n(J_\alpha)$ and therefore by downward extensions lemma, it is the $\Sigma_n$-code of $J_{\alpha'}$ for some $\alpha' < \alpha$. For clause 5, since $\alpha_{k+1} > \alpha_k$, the Skolem function mapping $\beta_k$ onto $j_k m_k \prec_1 J_{\alpha_k}$ with parameters $\{f_1, \ldots, f_k\}$ is a member of $J_{\alpha_{k+1}}$, this gives the desired function in clause 5. So the characterization part is proved.

For lemma 3.1.6 clause 1, by definition it is straightforward that $w \subseteq X' \prec_1 X$ to the unsuitability of $X$ is also an unsuitability witness for $X'$.

For Lemma 3.1.6 clause 2, suppose $w$ is minimal and $w'$ is a minimal witness for $X'$ with the same support $\beta$. Let $(\alpha', n')$ and $(\alpha'', n)$ be the heights of $w'$ and $w$ in $X'$. Let $\bar{\pi} = (\pi X)^{-1} \circ \pi X'$. Then $(\alpha', n') \leq (\alpha'', n)$ by minimality of $w'$, therefore

$$dirlim((\pi X)^{-1} w') = \Ult_{n'}(dirlim((\pi X')^{-1} w'), \bar{\pi}, \bar{\beta})$$

$$= \Ult_{n'}(J_{\alpha'}, \bar{\pi}, \bar{\beta})$$

$$\subseteq \Ult_n(J_{\alpha''}, \bar{\pi}, \bar{\beta})$$

$$= dirlim((\pi X)^{-1} w)$$

Therefore $dirlim((\pi X)^{-1} w')$ is well-founded, and it follows that $w'$ is a witness to the unsuitability of $X$ with support $\beta$, and by the minimality of $w$ we must have $\Ult_{n'}(J_{\alpha'}, \bar{\pi}, \bar{\beta}) = J_\alpha$ and $n' = n$. So $w'$ is also a minimal witness for $X$.

For lemma 3.1.6 clause 3, assume otherwise, $w \in Y$. Note that by the absoluteness of well-foundedness, we can find, working in $Y$, a sequence $\alpha'_k < \beta'$ of ordinals and a sequence $f'_k \in m_{k+1}$ of functions such that if $f''_k$ is the image $j_k(f'_k)$ of $f'_k$ in $dirlim(w)$, then the sets $z'_k = f''_k(d'_k)$ show that $dirlim(w)$ is not the $\Sigma_n$-code of a
well-founded structure. Then the sets $a'_k$ and $f'_k$ are members of $Y \cap J_k = X$, so the sets $\bar{z}'_k = \bar{i}_k \pi^{-1}(f'_k)(\pi^{-1}(\alpha'_k))$ show that $\text{dirlim}(\pi^{-1}[w])$ is not the $\Sigma_n$-code of a well-founded structure, contradicting 3.1.6 clause 2.

(\Box \text{Lemma 3.1.6})

Now we can finish our proof for the Covering Lemma for $L$ by the following lemma:

**Lemma 3.1.7 (Jensen).** The class $C$ of suitable sets is unbounded in $[J_\kappa]^\delta$ for any cardinal $\delta$ such that $\omega < \delta < \kappa$, i.e. any uncountable subset of $\kappa$ with cardinality $\delta$ is covered by a suitable set.

*Proof.* This proof is essentially due to Mitchell’s work in hand book of set theory, which is a bit different from Jensen’s original work, but very straight forward to understand the proving scheme. We begin with the set $Col(\delta^+, J_\kappa)$ of forcing conditions which collapses the $L$-cardinal $\kappa$ to $\delta^+$ (members of the space are functions $\sigma : \xi \to J_\kappa$ with $\xi < \delta^+$).

Now we first prove a variant version of Fodor’s Lemma:

**Proposition.** Suppose $S \subseteq Col(\delta^+, J_\kappa)$ is a stationary set such that $\text{cf}(\text{dom}(\sigma)) > \omega$ for all $\sigma \in S$ and $F$ is a regressive function on $S$ such that $F(\sigma)$ is a countable subset of $\text{ran}(\sigma)$ for all $\sigma \in S$. Then there is a stationary subset of $S$, say $S'$, and a function $\sigma_0 \in S'$ such that for all $\sigma \in S'$, $\sigma_0 \in \sigma$ and $F(\sigma) \subseteq \text{ran}(\sigma_0)$.

To show this, we first define a function $f : S \to \text{Ord}$ such that $f(\sigma) < \text{dom}(\sigma)$ be the least ordinal $\eta$ such that $F(\sigma) \subseteq \sigma^{”}\eta$. The least such ordinal $\eta$ is really necessarily less than $\text{dom}(\sigma)$ because $F(\sigma)$ is only countable but $\text{cf}(\text{dom}(\sigma)) > \omega$. We can without loss of generality assume $\text{cf}(\text{dom}(\sigma)) > \omega$ is because we keep in our mind that we are dealing with the least counter example, that is, an uncountable $x \subseteq \kappa$ cofinal in $\kappa$ with cardinality less than $\kappa$. It is always ok to extend $\sigma : \xi \to x$
to \( \sigma' : \xi + \omega_1 \to x \cup \omega_1 \), and we only need to find a cover for \( x \cup \omega_1 \) which would be fine enough to be the covering set for \( x \).

By the obvious notions of "closed" and "unbounded" (in the sense of \( P_\kappa(A) \)), readers who are not familiar with this may refer to Thomas Jech J. [14]), the ordinary Fodor’s Lemma provides us an \( S_0 \subseteq S \) which is stationary and \( f(\sigma) \) is a constant for all \( \sigma \in S_0 \) (there is a small trick here: in order to use the ordinary Fodor’s Lemma, we require the regressive function \( f \) to be \( f(\sigma) \in \text{ran}(\sigma) \), but we only have \( f(\sigma) \in \text{dom}(\sigma) \). However, noting that \( D = \{ \sigma \in \text{Col}(\delta^+, J_\kappa) \mid \kappa \subseteq \text{ran}(\sigma) \} \) is a closed and unbounded subset of the space, therefore we can take a intersection of \( D \) and \( S_0 \) first, resulting a finer stationary set in which for every \( \sigma \), \( f(\sigma) \in \text{dom}(\sigma) \subseteq \kappa \subseteq \text{ran}(\sigma) \), therefore we are free to use the ordinary Fodor’s Lemma here). Pick any \( \sigma_0 \in S_0 \) and let \( S' = \{ \sigma \in S_0 \mid \sigma_0 \subseteq \sigma \} \). Then \( S' \) and \( \sigma_0 \) are just what we want.

Now we are ready to show that \( C \) is unbounded in \( [J_\kappa]^\delta \). It suffices to show that the set \( S_0 = \{ \sigma \in \text{Col}(\delta^+, J_\kappa) \mid \text{ran}(\sigma) \notin C \& cf(\text{dom}(\sigma)) > \omega \& \text{ran}(\sigma) \prec_1 J_\kappa \} \) is non-stationary. Again we can ignore the latter two conditions because \( D_1 = \{ \sigma \in \text{Col}(\delta^+, J_\kappa) \mid cf(\text{dom}(\sigma)) > \omega \& \text{ran}(\sigma) \prec_1 J_\kappa \} \) is a club. Assume the contrary that \( S_0 \) is stationary. By lemma 3.1.5, for each \( \sigma \in S_0 \), there is a minimal witness \( w^\sigma \) with support \( \beta^w \) to the unsuitability of \( \text{ran}(\sigma) \). Apply the ordinary Fodor’s Lemma, we obtain a stationary subset \( S_1 \subseteq S_0 \) such that \( \beta = \beta^w \) is constant for all \( \sigma \in S_1 \). And by the variant of Fodor’s Lemma we just proved, there is a \( S_2 \subseteq S_1 \) and a \( \sigma_0 \in S_2 \), such that for all \( \sigma \in S_2 \), we have \( \sigma_0 \subseteq \sigma \) and \( w^\sigma \subseteq \text{ran}(\sigma_0) \). It follows that \( w^\sigma \) is a minimal witness to the unsuitability of \( \text{ran}(\sigma) \) for all \( \sigma \in S_2 \). Now consider the following set

\[
D_2 = \{ \sigma \in \text{Col}(\delta^+, J_\kappa) \mid \exists Y(Y \prec_1 H(\kappa^+) \& w^{\sigma_0} \in Y \& \text{ran}(\sigma) = Y \cap J_\kappa) \}
\]

which contains a club of \( \text{Col}(\delta^+, J_\kappa) \). However, the fact that \( S_2 \cap D_2 \neq \phi \) contradicts
clause 3 of lemma 3.1.5. Therefore we have proved lemma 3.1.6 and hence the Covering Lemma for $L$.
(Theorem 3.1.1)

3.2 Further Notes

Further to the discovery of the Covering Lemma for $L$, there have been multiple attempts to generalize the covering property to larger inner models. A direct generalization is to consider larger inner models than $L$, for example $L[U]$. However, Prikry forcing which we used to force $\neg$SCH, fails our intention to prove covering property for $L[U]$ immediately.

In 1982, Dodd and Jensen [3][4] showed the Covering Lemma for the Dodd-Jensen Core Model $K^{DJ}$ under the assumption that there is no inner model with a measurable cardinal, and an alternative covering property for $L[U]$ stating that if there is $L[U]$ but $0^\#$ does not exist, for the ”least” (in the sense of the least critical point) $L[U]$, we either have the covering property or can find a prikry sequence $C$ such that $L[U, C]$ has the covering property. Further development of fine structure theory realizes that if we do not want to reduce the strength of our definition of covering property, this is the best possible.

After that, logicians started to think about reducing the strength of conditions of the covering property. For example, changing ”cardinality” into ”order type” in the statement of covering property, however this does not work due to P. Komjáth [10].

Beginning with the idea of constructing a model with a sequence of measures (Mitchell was the first one who came out with this), a weaker version, now well-known as the weak covering lemma for the core model $K$, was developed. The
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theory of weak covering property of the core models was developed by phases by Jensen, Martin, Steel, Mitchell[7][15][18][20][21]. The weak covering lemma states that under certain anti-large cardinal assumption, even in the presence of a measurable cardinal, $K$ computes successors of all singular cardinals correctly. This week covering property still keeps enough strength to prove SCH. An essential part of the proof for the Weak Covering Theorem involves the construction of a fine e-nough and canonical inner model $K$, known as the core model. Currently the fine structural inner model theory has already reached a core model up to one Woodin cardinal, and the essential work is due to John R. Steel [7][20].

Steel’s construction of $K$ applies to all constructions of core models up to one Woodin cardinal. It is also important to know that, various definitions of the core model are all plausibly consistent. For example Steel’s $K$ coincides with Dodd-Jensen and Mitchell’s definitions if there is no inner model with a measurable cardinal. And if there is an inner model with a measurable cardinal but no $0^+$, $K = L[U]$ for some ”least” $L[U]$ with respect to the least critical point. Therefore with the help of $K$, we can take two at one time to prove both for the Covering Lemma for $K^{DJ}$ and $L[U]$.

One important objective of this survey is to present a proof of the Dodd-Jensen Covering Lemma for $K^{DJ}$ and $L[U]$ using later fine structural theories. An essential part of the proof involves ”countably complete weak covering property of $K^{CC}$”, which is also necessary to the construction of $K$. Therefore, we put Steel’s $K$ and the related weak covering theorems first, and then come back to $K^{DJ}$ and $L[U]$ in the chapter after with the developed theory.

This gives us enough motivation to proceed to the Weak Covering Lemma and
Steel’s Core Model $K$. 

Chapter 4

The Weak Covering Lemma

In this Chapter, we prove the Weak Covering Lemma for the core model $K$, asserting that in the absence of $0^\sharp$, $K$ computes the successor cardinals of sufficiently large singular cardinals correctly.

The essential construction of $K$ is due to Steel, which can in fact serve up to 1 Woodin cardinal. And Woodin’s work on stationary tower forcing shows that in the presence of one Woodin cardinal even the weak covering property would fail.

Earlier proofs by Steel([6][20]) used a technical hypothesis, that $U$ is a normal measure over $\Omega$ where $\Omega$ is strong, and constructed a core model $K$ up to $\Omega$. However, later this turned out to be unnecessary.

The proof we present in this chapter is a more modern version, which is essentially due to Steel and Jensen. We will use linear iterations to solve the weak covering problem for $K$ below $0^\sharp$, which is not as complicated as iteration trees, but still captures the same structure of proof which we are interested in. One more thing to be mentioned is that one strong cardinal is not the limit of linear iterations. Steel
later showed in fact we can reach core models containing many strong cardinals by linear iteration up to the sharp of a strong cardinal.

4.1 \( K^c \) construction

Our first step is to construct a so-called ”back ground certified core model” – \( K^c \). \( K^c \) is a universal extender model, of which the construction is necessary to the existence of the true core model \( K \). Under the presence of a measurable cardinal, we cannot even ensure the existence of \( K \) without proving the weak covering property of \( K^c \) (under certain anti-large cardinal hypothesis, of course).

The following \( K^c \)-construction is just an easier version of Steel’s \( K^c \)-construction mostly adapted from [9], we would guarantee both the iterability of each stage and the universality of \( K^c \) in our inductive construction.

In our construction, \( \omega \)-completeness is used for choice of next measure in favor of guaranteeing the iterability of each level, and this is sufficient for us at this moment. When the construction goes up to still larger models, e.g. core model up to 1 Woodin cardinal, this \( K^c \)-construction still works though we will have to in addition require the extender to be ”fully backgrounded” and to have the so called ”background certificates”, as well as \( \omega \)-completeness. The requirements about these strong partial extenders in \( V \) can be found in Steel’s chapter on Hand Book of Set Theory[6] or the original paper [20].

**Definition 4.1.1** (The \( K^c \)-construction). We inductively define a pair of sequences of mice as follows:
\[ N_0 = < J_0, \phi >, \quad M_0 = \text{core}(N_0) \]

At successor stages, setting \( N_\alpha = < J_\zeta^{E_\alpha}, E_\omega^\alpha > \) and \( M_\alpha = < J_\zeta^\bar{E}_\alpha, \bar{E}_\omega^\alpha > \),
\[
N_{\alpha+1} = \begin{cases} 
< J_\zeta^{E_\alpha}, F > & \text{if } \bar{E}_\omega^\alpha = \phi \text{ and } F \text{ is the unique } \omega\text{-complete} \\
< J_\zeta^{\bar{E}_\alpha+1}, \phi > & \text{otherwise}
\end{cases}
\]

\[ M_{\alpha+1} = \text{core}(N_{\alpha+1}) \]

At limit stages, let \( \rho_\xi \) denote \( \rho_{N_{x_\xi}} \) (which also equals \( \rho_{M_{x_\xi}} \)), let \( \alpha \) be such that \( \xi \leq \alpha \leq \infty \), set
\[
\rho_{\xi}\alpha = \min\{\rho_\eta : x_i \leq \eta < \alpha\} \\
\tau_{\xi}\alpha = (\omega \rho_{\xi}\alpha)^+\xi \\
\zeta = \sup\{\tau_{\xi}\alpha : \xi < \alpha\}
\]

Now if \( \alpha \) is a limit, we set
\[
N_\alpha = < J_\zeta^{E_\alpha}, \phi >, \text{ where } E_\alpha = \bigcup_{\xi<\alpha} E_\xi \upharpoonright \tau_{\xi}\alpha.
\]

\[ M_\alpha = \text{core}(N_\alpha) \]

This construction does not break down, stops at \( \infty \), and we obtain a hierarchy of desired premice. Finally, we define
\[
K^c = N_\infty = \bigcup_{\xi<\infty} J_\tau^{E_\xi}
\]

The particular difficulty here is that we are coring down each hierarchy to guarantee sufficient soundness condition, however this brings us to a consistency problem when reaching limit stages. The following two lemmas solve this doubt and provide us sufficient knowledge about \( K^c \) for later use.
Lemma 4.1.2. Taking the notations of the previous construction, the structure $N_\alpha$ is defined for every $\alpha < \infty$ and:

1. $N_\alpha$ is a premouse and each proper initial segment of $N_\alpha$ is a mouse;
2. every total measure in $N_\alpha$ is $\omega$-complete;
3. the extender sequence is consistent, i.e. $E^\xi \upharpoonright \tau_\xi = E^\beta \upharpoonright \tau_\xi$ and $\tau_\xi \leq (\omega p_\xi)^{+N_\beta}$ whenever $\xi \leq \beta \leq \alpha$

Particularly clause 3) holds for $\alpha = \infty$, which means, $K^c$ is well defined and the hierarchy $N_\alpha$ goes to $\infty$.

Lemma 4.1.3. $K^c$ is a weasel, i.e. $ht(K^c) = \infty$. Let $E = E^{K^c}$

1. If $E_\tau$ is a total measure in $K^c$ then $E_\tau$ is $\omega$-complete.
2. If $F$ is an $\omega$-complete total measure on $K^c$ and $< J^F, F >$ is a premouse, then $F = E_\tau$.

By the construction above, we have successfully constructed $K^c$, which is in fact a universal weasel. The proof of the universality of $K^c$ can be found in Jensen’s paper([13]), using an additional assumption that $On$ is inaccessible. A ZFC version of the proof is due to Zeman and Schindler ([12]).

Now before we proceed to the proofs of the weak covering lemmas, we will have a quick glance at $0^\dagger$, which can be considered as the first mouse with a measure of order 1, and is necessary to be ruled out with the anti-large cardinal hypothesis $"-0^\dagger"$ when proving the weak covering lemmas.
Recall that $0^+$ is the first mouse with respect to $L$. $0^+$ can be understood as the first mouse containing a measure of order 1. Its relationship to weasels is just similar to that of $0^1$ to the constructible universe. It can be proven that $0^+$ exists and is unique iff there is a mouse containing a measure of order 1. Moreover, $0^+$ is $\Pi_1$-definable over $H_{\omega_1}$.

Here we sketch a lemma stating that $0^+$ cannot be added by generic set-size forcing, for later usage in proving the weak covering lemma for Steel’s core model $K$.

**Lemma 4.1.4.** Assume $0^+$ does not exist, $\mathbb{P} \in V$ is a forcing notion and $G$ a $\mathbb{P}$-generic filter over $V$, then $0^+ \notin V[G]$ either.

**Proof of Lemma:**

Suppose otherwise, $0^+ \in V[G]$, then there is a forcing notion $p \in \mathbb{P}$ such that $p \forces \exists x \varphi(x)$, where $\varphi$ is the defining formula for $0^+$. Let $H$ be $\text{Col}(\omega, P(\mathbb{P}))$-generic over $V$. Then in $V[H]$ we only have countably many subsets of $\mathbb{P}$. Let $\bar{G}$ be $\mathbb{P}$-generic over $V[H]$ such that $p \in \bar{G}$. Then $V[\bar{G}] \models \exists x \varphi(x)$. Let $y \in V[\bar{G}]$ be the unique $y$ such that $V[\bar{G}] \models \varphi(y)$. $y$ is the unique real such that $V[H] \models \varphi(y)$ by the genuineness of $0^+$(Since iterability can be realized below $H_{\omega_1}$, $0^+$ in any $\text{ZF}^-$ model $M \subset V$ containing $\omega_1$ is the true $0^+$ in any larger $\text{ZF}^-$ models $M \subset M' \subset V$).

For every $n \in \omega$

$$n \in y \iff V[H] \models \exists x \varphi(x) \& n \in x \iff \vdash_{\text{Col}(\omega, P(\mathbb{P}))} \exists x \varphi(x) \& \bar{n} \in x;$$

It follows that $y \in V$. The above argument runs independently of the choice of $H$.

We then have $H_{\omega_1}^V \models \varphi(y)$ and therefore $V \models \varphi(y)$. 

(\Box\text{Lemma 4.1.4})
4.2 Countably Closed Weak Covering Theorem for $K^c$

The first difficulty to prove weak covering in a core model in the presence of large cardinals is to isolate a nice definable inner model, rich enough, and canonical. In Steel’s construction, the weak covering property of $K^c$ is necessary for the existence of the true core model $K$. In this section, we adapt a slightly weaker version (countably complete case) of the weak covering, whose proof is technically easier and still suffices to guarantee the existence of $K$. And furthermore, this Countable Complete Weak Covering property is essentially important to the proofs of the next chapter, where we use it to prove for the Dodd-Jensen Covering Lemma.

Theorem 4.2.1 (Countably Closed Weak Covering Theorem for $K^c$). Assume $0^+$ does not exist, then for every $\omega$-closed cardinal $\kappa$

$$cf(\kappa^{+K^c}) \geq \kappa$$

Particularly if $\kappa$ is singular in $V$,

$$\kappa^{+K^c} = \kappa^+$$

Proof: Assume otherwise toward a contradiction, there is a $\omega$-closed singular cardinal $\kappa$. Throughout this proof, let $E$ denote $E^{K^c}$ for convenience.

Let $\tau = \kappa^{+K^c}$ Suppose that $\gamma = cf(\tau) < \kappa$. We aim to show that $\tau$ is collapsed inside $K^c$. Let $b$ be a sequence of ordinals of order type $\gamma$ which is cofinal in $\tau$, and $\theta$ be regular above $\kappa^+$. By taking $\omega$-closed hulls of $b \cup \{ J^E_\tau \}$ in $H_\theta$, we can easily obtain a structure $X$ in $\omega_1$ steps satisfying:

1. $b \subset X$ & $J^E_\tau \in X$;
2. $X$ is $\omega$-closed;
3. $\text{card}(X) < \kappa$;

Now we fix a few notations. Let $\bar{H}$ be the transitive collapse of $X$, and $\sigma$ be the uncollapsing map, $X \cong \bar{H} \prec H_\theta$. $\sigma$ is a nontrivial elementary embedding from $\bar{H}$ to $H_\theta$. Let $\alpha = \text{cr}(\sigma)$, $\bar{\tau}$, $\bar{\kappa}$, $\bar{E}$ be such that

$$\sigma(\bar{\tau}, \bar{\kappa}, J^\bar{E}_\tau) = \tau, \kappa, J^E_\tau$$

Notice that $\bar{H}$ is closed under $\omega$-sequences by construction, and further $\sigma \upharpoonright J^\bar{E}_\tau$ is $\Sigma_0$-elementary and cofinal. We denote $J^E_\tau$ by $\bar{K}$.

Notice that we are collapsing $\tau$ which is greater than $\kappa$ to $\bar{\tau}$ less than $\kappa$, so the critical point $\alpha$ is obviously not larger than $\bar{\kappa}$ and therefore less than $\bar{\tau}$. Consider $\bar{\vartheta} = \alpha^{+\bar{K}}$, and $\vartheta = \alpha^{+K^c}$. Now we have $\bar{K}$ carrying information of $K^c$ from our counter assumption, and the coiteration with $K^c$ will carry on these until the stage that both branches reach a common end, and comparison yields disagreement which leads to a contradiction.

The first important fact is that we will show $\bar{\vartheta} \neq \vartheta$. This will lead to the fact that the power set of $\alpha$ in $\bar{K}$ and $K^c$ are not equal, because if the two power sets are same, then the cardinal successors are also same by encoding each ordinal below the cardinal successor of one structure into a subset and then mapping to the other structure and decoding. This ensures us non-triviality of comparison between $K$ and $K^c$. And actually this is the only place we use the assumption that $0^\sharp$ does not exist.

**Lemma 4.2.2.** $\bar{\vartheta} < \vartheta$

Proof of Lemma: Assume otherwise, i.e. $\bar{\vartheta} = \vartheta$, then we denote it as $\vartheta$. The following fine structural argument shows that $\bar{E} \upharpoonright \vartheta = E \upharpoonright \vartheta$: 


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Let $\xi < \vartheta$ be such that $\alpha$ is the largest cardinal in $\bar{K}\|\xi$, let $\zeta_\xi < \vartheta$ be such that $\alpha$ is the largest cardinal in $\bar{K}\|\xi$ but not in $\bar{K}\|((\zeta_\xi + 1))$. $\xi$ is a cardinal successor of $\alpha$ in $\bar{N} = \bar{K}\|\xi$ and is collapsed in $\bar{K}\|((\zeta_\xi + 1))$. Let $a_0$ be a subset of $\alpha$ coding the ordinal $\xi$ inside $\bar{K}\|((\zeta_\xi + 1))$.

Consider the mouse $N = \sigma(\bar{N})$ (note $N$ is fully iterable) and $\bar{\sigma} = \sigma \upharpoonright \bar{N} : \bar{N} \to N$, apply the Condensation Lemma (note that $\sigma$ is fully elementary and captures all the requisite properties), clause a) and c) of the Condensation Lemma are easily excluded. c) is immediately excluded since $\alpha$ is not a $\bar{N}$-successor. For a), Since $\bar{N}$ is sound, we can use a subset $b \subset \alpha$ which is in $\bar{K}$ to encode $\bar{N}$. Preservation properties of $\sigma$ and acceptability of $K^e$ enable us to reconstruct $\bar{N}$ inside $K^e\|\vartheta$, which is an initial segment of $N$. Therefore $\bar{N} \in N$, $\bar{N}$ cannot be the core of $N$ and clause c) fails.

Now we have proved $\bar{K}\|\xi$ is an initial segment of $K^e$. Ordinals of the form $\zeta_\xi$ are cofinal in $\vartheta$, therefore $E^K \upharpoonright \vartheta = E \upharpoonright \vartheta$.

Now the structures $J^E_\vartheta$ and $J^E_\tau$ agree up to $\vartheta$.

Let $U = \{x \in J^E_\vartheta; \alpha \in \sigma(x)\}$ be a measure on $J^E_\vartheta$ derived from $\sigma$. $U$ is easily seen to be an $\alpha$-complete ultrafilter over $J^E_\vartheta$; we want to show $U$ is normal.

Pick a regressive function $f : a \to \alpha$ from $J^E_\vartheta$, where $a \in U$. Then $\alpha \in \sigma(a)$; set $\delta = \sigma(f)(\alpha)$. $\bar{a} = \{\xi < \alpha; f(\xi) = \delta\}$, we have $\alpha \in \sigma(\bar{a})$. We show that the measure $U$ is weakly amenable with respect to $J^E_\vartheta$.

Pick $g : \alpha \to P(\alpha)$ from $J^E_\vartheta$. We show that $c = \{\xi < \alpha; g(\xi) \in U\}$ is an element
of $J^E_\varphi$. But for $\xi < \alpha$ we have
\[ g(\xi) \in U \iff \alpha \in \sigma(f(\xi)) = \sigma(f)(\xi), \]
so $c \in J^E_\varphi$ and hence $c \in J^E_\varphi$ by acceptability.

In order to apply this measure $U$ to our iteration, we have to check that it is $\omega$-complete. Pick a countable sequence $< x_i; i \in \omega >$ of sets from $U$, then $< \sigma(x_i); i \in \omega >$ is in $X$. Therefore $\sigma(< x_i; i \in \omega >) = < \sigma(x_i); i \in \omega >$. The existence of $\alpha$ shows that the intersection of all $\sigma(x_i)$ must be nonempty, and thus the intersection of all $x_i$(by elementarity of $\sigma$).

Next we apply this measure $U$ to form an ultrapower $Ult(J^E_\varphi, U)$, and want to show that $Ult(J^E_\varphi, U) \parallel \varphi = < J^E_\varphi, E_\varphi >$. This is by a standard solidity argument used in the proof of Solidity Theorem. This tedious fine structural argument takes up pages and is better not embedded in our proof here. Readers can assume this as preliminary knowledge, and refer to chapter 5 of [9] for details.

Once we obtain this, we can finally reach our desired contradiction. If $E_\varphi = \phi$, then $< J^E_\varphi, U >$ is a premouse(all clauses of the definition can be either checked trivially or guaranteed by the previous paragraphs). Since $U$ is $\omega$-complete, by lemma 4.1.3, we know that $U = E_\varphi$, so $E_\varphi$ cannot indeed be empty. Then $M = < J^E_\varphi, E_\varphi, U >$ is an $s$-premouse in which all total measures are $\omega$-complete, this gives us the existence of $0^\dagger$. Contradiction.

(\square) Lemma 4.2.2

Now we coiterate $\bar{K}$ with $K^c$, and conclude that the coiteration between $\bar{K}$ and $K^c$ is identity on the $\bar{K}$ side.

**Lemma 4.2.3.** In the coiteration between $\bar{K}$ and $K^c$, no measure(extender) is
applied on the $\bar{K}$ side.

Proof of Lemma: Assume otherwise, let $l \in \text{Ord}$ be the first stage that $\bar{K}$ is moved.

By lemma 4.2.2, $\bar{\vartheta} < \vartheta$. Hence there is a $\zeta < \vartheta$ such that $\bar{\vartheta}$ is a cardinal in $K^c\|\zeta$ but not in $K^c\|\zeta + 1$. Let $M_0$ denote $K^c\|\zeta$.

Then $\bar{\vartheta} = \alpha + M_0$ and $\bar{E} \upharpoonright \bar{\vartheta} = E^{M_0} \upharpoonright \bar{\vartheta}$; $M_0$ is sound and projects to $\alpha$. It follows that the coiteration of $\bar{K}$ with $K^c$ is above $\alpha$, and universality of $K^c$ ensures that the $\bar{K}$ side of coiteration is simple. Therefore on the $\bar{K}$ side, all structures have the same subsets of $\alpha$ as $\bar{K}$. On the $K^c$ side, the first iteration index used must be at most $\zeta$ since $\omega^\alpha \omega^\zeta \subseteq \alpha$. It follows that this coiteration can be identified with the coiteration of $\bar{K}$ with $M_0$.

Let $M_i$, $\tilde{K}_i$ be the corresponding structures in the coiteration, $\nu_i$ be the iteration indices, $\kappa_i$ be the associated critical points. Let $\zeta_i$ be the maximal such that $\nu_i$ is the cardinal successor of $\kappa_i$ in $M_i\|\zeta_i$, i.e. $\bar{E}_{\nu_i}$ is a total measure over $M_i\|\zeta_i$ (note that $E^{\zeta_i}_{\nu_i} = \bar{E}_{\nu_i}$, where $\bar{E}$ corresponds to $\bar{\kappa}$). If $\zeta_i < ht(M_i)$, then $M_i\|\zeta_i$ projects to $\kappa_i$ and is sound above $\kappa_i$. Otherwise, we show that $M_i$ projects to $\kappa_i$ and is sound above $\kappa_i$.

This follows from the following claim:

Each mouse $M_i$ projects to $\kappa_i$ and is sound above $\kappa_i$ \hspace{1cm} (4.1)

Take an induction on the length of the iteration.

The $i = 0$ case is trivial.

For Successor stage $M_{i+1}$, assume we have proved for $M_i$, consider $M_{i+1} = Ult^*(M_i^*, E^{M_i}_{\nu_i})$ where $M_i^* = M_i\|\zeta_i$. It follows that $M_i^*$ projects to $\kappa_i$ and is sound above $\kappa_i$. 
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Let \( x \in M_{i+1} \), then \( x = \pi_{i,i+1}(f)(\kappa_i) \) where \( \pi_{i,i+1} \) is the ultrapower map. Let \( g \) be the good \( \Sigma_1^{(n-1)}(M_i^*) \) function and \( p \in M_i^* \) such that \( f(\xi) \simeq g(\xi,p) \). Since \( M_i^* \) is sound above \( \kappa_i \), \( p = \tilde{h}_{M_i}^{n+1}(\xi_0, p_{M_i}) \) for some \( \xi_0 < \kappa_i \). Then there is a good \( \Sigma_1^{(n-1)}(M) \) function \( g' \) with the same functionally absolute definition as \( g \) such that \( x = g'(\kappa_i, \pi_{i,i+1}(p)) = g'(\kappa_i, \tilde{h}_{M_i}^{n+1}(\xi_0, p_{M_i})). \) It follows that \( x \) is \( \Sigma_1^{(n)}(M_{i+1}) \)-definable from \( p_{M_i+1} = \pi_{i,i+1}(p_{M_i}) \) and parameters below \( \kappa_i+1 \), therefore \( M_{i+1} = \tilde{h}_{M_i}^{n+1}((\kappa_i+1) \cup \{ p_{M_i+1} \}). \) The successor case holds.

For limit stage \( M_\lambda \), take an \( i \) large enough such that there are no truncations between \( i \) and \( \lambda \). Apply similar computation as above, we obtain that \( M_\lambda = \tilde{h}_{M_\lambda}^{n+1}((\kappa_\lambda) \cup \{ p_{M_\lambda} \}). \) Thus the limit case also holds.

Now we look at the coiteration, clearly \( \bar{K} \) side is simple and therefore \( \bar{E}_\nu \) is a total measure in \( \bar{K} \). It follows that \( \sigma(\bar{E}_\nu) = E_{\sigma(\nu)} \) is a total measure in \( \bar{K}^c \). Apply lemma 4.1.2, we have that \( \sigma(\bar{E}_\nu) \) is \( \omega \)-complete. By elementarity of \( \sigma \), \( \bar{E}_\nu \) is \( \omega \)-complete in the sense of \( \bar{H} \), and therefore \( \omega \)-complete in the sense of \( \bar{V} \) provided that \( \bar{H} \) is closed under \( \omega \)-sequences. Now we apply \( \bar{E}_\nu \) on \( M_i^* \), \( \pi : M_i^* \rightarrow \bar{E}_\nu \). Then the premouse \( \bar{M} \) is normally iterable above \( \kappa_i \). Now we coiterate \( M_i^* \) with \( \bar{M} \). Again by the standard solidity argument we mentioned earlier in the proof of Lemma 4.2.2, we infer that \( E_{\bar{M}_i} = \bar{E}_\nu \). However, this leads to a contradiction since coiteration of \( M_i^* \) with \( \bar{K} \) at stage \( l \) also uses these two measures and they must disagree with each other.

(\( \square \)Lemma 4.2.3)

Finally, we are ready to finish proof of Theorem 4.3.1:
Consider the coiteration between \( M_0 \) and \( \bar{K} \). The last one on the \( M_0 \) side is a lengthening of \( \bar{K} = J^E_\pi \). Either all \( \kappa_i \)'s and \( \nu_i \)'s are below \( \bar{\kappa} \), or there is a last
ultrapower on the $M_0$ side such that the iteration index equals \( \bar{\tau} \), that is

\[
\pi^M_{\lambda, \lambda+1} : M^*_\lambda \xrightarrow{E_\psi} M_{\lambda+1}
\]

Denote $M_\lambda$ as either the last mouse for the former case, or the mouse of last ultrapower for the second case (on the $M_0$ side). Recall that

\[
\sigma \upharpoonright J^E_\tau : J^E_\tau \rightarrow J^E_\tau \text{ is } \Sigma_0 \text{ and cofinal}
\]

We want to canonically extend this to $\tilde{\sigma} : M \rightarrow N$ where we define $M$ to be $M_\lambda \| \eta$ where $\eta$ is the maximal $\eta'$ such that $\bar{\tau} = \bar{\kappa}^{+M_\lambda \| \eta'}$.

By definition of $M$, $M$ projects to $\bar{\kappa}$ and is sound above $\bar{\kappa}$ (if $M$ is shorter than $M_\lambda$ then this is obvious, otherwise this follows from (4.1)); $\bar{\tau} = \bar{\kappa}^{+M}$; and $J^E_{\tau}^{M} = J^E_{\tau}$.

Let $n \in \omega$ be such that $\omega \rho^n_{M} \leq \bar{\kappa} < \omega \rho^n_{N}$.

Since $\sigma$ is $\omega$-complete, we canonically extend $J^E_\tau \rightarrow J^E_\tau$ to $\tilde{\sigma} : M \rightarrow N$. And such canonical extension satisfies that:

1. $\omega \rho^n_{N} \leq \kappa < \omega \rho^n_{N}$;

2. $N$ is sound above $\kappa$

3. $\tilde{\sigma}(\bar{\kappa}, \bar{\tau}) = \kappa, \tau$

4. $\tau = \kappa^{+N}$

Also, the $\omega$-completeness of $\sigma$ guarantees that the canonical extension $N$ is normally iterable above $\kappa$. $N$ is coiterable with $K^c$ above $\kappa$ since $N$ agrees with $K^c$ above $\kappa$ and in both models $\kappa$ is a cardinal. Coiterate $N$ with $K^c$, since $K^c$ is universal, then the coiteration terminates say after $\theta^*$ many steps, and the $N$ side of coiteration is simple. Denoting the last model on the corresponding side to be
Now look at the above fine structure properties 1, 2 about $N$. This implies that $N$ is in fact the $\Sigma_1(N)$-hull of $\kappa \cup \{p_N - (\kappa + 1)\}$, i.e. $N = \tilde{h}_N^{n+1}(\kappa \cup \{p_N - (\kappa + 1)\})$, therefore $N$ can be encoded into a $\Sigma_1(N)$ subset $b \subset \kappa$. Since the coiteration is simple on the $N$ side, $b \in \Sigma_1(N_{\theta^*})$. If $W_{\theta^*} \neq N_{\theta^*}$, then $b \in W_{\theta^*}$, and $b \in K^c$ since the iteration is above $\kappa$. If $W_{\theta^*} = N_{\theta^*}$, then we take the last truncation point $W_k$ on the $K^c$ side, and consider the mouse after truncation $W_k^*$, the iteration from $W_k^*$ to $W_{\theta^*} = N_{\theta^*}$ is simple above $\kappa$, therefore $b$ is $\Sigma_1(W_k^*)$, and therefore in $W_k$ and therefore $b \in K^c$.

Then we can decode or reconstruct $N$ from $b$ inside $K^c$. Since $N = \tilde{h}_N^{n+1}(\kappa \cup \{p_N - (\kappa + 1)\})$ and $\tilde{h}_N^{n+1}$ is definable over $N$, $\tilde{h}_N^{n+1}$ is a partial function in $K^c$: $\xi \mapsto \tilde{h}_N^{n+1}(\xi, p_N)$ which maps $\kappa$ on to $\tau = \kappa + K^c$. Contradiction.

(\square Theorem 4.2.1)

### 4.3 Weak Covering Theorem for $K$

In analogy with Jensen’s Covering Lemma for $L$, the core model was built up to be ”canonical” and yet has the similar ”close to V” property, which is the covering property. Jensen first showed that if $0^\sharp$ does not exist, then $L = K$ is the canonical model, and has the covering property. Jensen also showed that if $0^\sharp$ exists but $0^{2\sharp}$ does not, then $K = L[0^\sharp]$ is the canonical core model.

The first order and second order definitions of $K$ are well presented in [5][6][9], we assume the existence and definitions of $K$ in this article, as otherwise it can take

$N_{\theta^*}$ and $W_{\theta^*}$ and the former is therefore an initial segment of the latter.
up a whole series of chapters to talk about the core models. $K$ can be similarly observed as $K^c$ as it is also a universal weasel, but more than that $K$ is in a sense "maximal" (containing all measures that belong to some universal weasel). Also, $K$ has the rigidity property asserting that there is no nontrivial elementary embedding from $K$ to $K$, and the generic forcing absoluteness asserting that $K$ is invariant under generic set-size forcing.

It is important to know that certain anti-large cardinal hypothesis is necessary since without this we cannot even ensure that $K$ exists. For the remainder of this chapter, we assume $\neg 0^+$ if without specific explanation.

The next theorem states the weak covering property of $K$:

**Theorem 4.3.1** (Weak Covering Theorem for $K$). Assume $\kappa \geq \omega_2$ and $\kappa$ is a cardinal in $K$, then

$$\text{cf}(\kappa^{+K}) \geq |\kappa|$$

**Proof of Theorem 4.3.1**: The proof closely follows the proof of theorem 4.2.1. Novelties arise as we have to use maximality of $E^K$ to absorb measures rather than $\omega$-completeness, and we need a better upward extension technique which will make the earlier proof applicable for all $\kappa$’s, which is the following Frequent Extensions of Embeddings Lemma:

**Lemma 4.3.2** (Frequent Extensions of Embeddings Lemma). Consider a (linear) system of structures and maps with the following properties:

1. $< \alpha_\xi : \xi \leq \alpha >$ is a normal sequence of ordinals such that $\alpha_\alpha = \alpha$ where $\alpha \geq \omega_2$ is a regular cardinal;

2. $< Q_\xi = J_{E^r_\xi} : \xi \leq \alpha >$ is a system of premice such that $Q_\alpha = Q = J^E_r$
is a mouse with the largest cardinal \( \kappa \geq \alpha \) and \( \text{card}(Q_\xi) < \alpha \) for \( \xi < \alpha \); moreover, \( \kappa_\xi \) is the largest cardinal in \( Q_\xi \);

3. \( \sigma_{\xi\xi'} : Q_\xi \rightarrow Q_{\xi'} \) is a commutative system of \( \Sigma_0 \)-preserving maps such that \( \sigma_{\xi\xi'}(\kappa_\xi) = \kappa_{\xi'} \), and for limit ordinal \( \lambda \leq \alpha \), \( < Q_\lambda, \sigma_{\xi\lambda} > \) is the direct limit of the diagram \( < Q_\xi, \sigma_{\xi\xi'} >_{\xi \leq \xi' < \lambda} \), i.e. \( Q_\lambda = \bigcup\{\text{range}(\sigma_{\xi\lambda}) : \xi < \lambda\} \).

Setting \( \tilde{\tau}_\xi = \sup(\sigma_{\xi\alpha}, \tau_\xi) \) and \( \tilde{Q}_\xi = J_{E_{\tilde{\tau}_\xi}} \), we have

\[
\tau_\xi : Q_\xi \xrightarrow{\Sigma_0} \tilde{Q}_\xi \text{ cofinally.}
\]

Let \( S \) be a set of indices such that \( S \subset \alpha \) is a set of ordinals of uncountable cofinality, and for each \( M_\xi \) that \( \xi \in S \):

(a). \( M_\xi \) is a lengthening of \( Q_\xi \)

(b). \( \tau_\xi \) is the cardinal successor of \( \kappa_\xi \) in \( M_\xi \);

(c). Either \( \omega_{\rho^*_M_\xi} > \kappa_\xi \), or \( \omega_{\rho^*_M_\xi} \leq \kappa_\xi \) and \( M_\xi \) is sound above \( \kappa_\xi \).

Then, under all the above assumptions, for all but non-stationarily many \( \xi \), we have

1. The canonical extension \( \tilde{\sigma}_\xi : M_\xi \rightarrow \tilde{M}_\xi \) exists;

2. \( \tilde{M}_\xi \) is iterable if \( \omega_{\rho^*_M_\xi} > \kappa_\xi \), and otherwise normally iterable above \( \kappa_\xi \).

For the \( \omega \)-complete case, chapter 3 of [2] has a very detailed analysis of extendability of fine ultrapowers and the canonical extension immediately follows from the \( \omega \)-completeness. Theorem 4.3.2 is a somehow finer technique, asserting that, for a certain kind of systems, even if there is no way to ensure canonical extensions for all structures, however we could ensure a stationarily many.
Proof of Lemma 4.3.2:
Fine ultrapowers and extendability are key to the proof of Lemma 4.3.2, and are well presented in chapter 3 of Zeman’s book([9]). We will not bother to develop and present these theories and arguments in this paper. We only state the Interpolation Lemma as follows:

**Definition 4.3.3.** Let $Q$, $Q'$, $M$ be acceptable and

1. $Q = J^A_\tau$, $Q' = J^{A'}_{\tau'}$;
2. $\sigma : Q \xrightarrow{\Sigma_0} Q'$ cofinally;
3. $M = \langle J^A_\alpha, B \rangle$ and $\tau$ is a regular cardinal in $M$.

$k(M)$ is defined to be the least number $k < \omega$ such that there is a good $\Sigma_1^{(k)}(M)$ function singularizing $\tau$. In such case we have that if $Q$ has a largest cardinal then $\omega p_M^{k+1} < \tau \leq \omega p_M^k$.

The canonical extension $\tilde{\sigma}$ of $\sigma$ to $M$ is defined to be the canonical $k$-extension $\tilde{\sigma} : M \rightarrow N$ where $N$ is the $k$-pseudo ultrapower of $M$ by $\sigma$, and $\tilde{\sigma}$ is the associated pseudo ultrapower map.

**Lemma 4.3.4** (Interpolation Lemma). Let all the notations be consistent with above. Let $\omega p_M^n \geq \tau$ and either $n = k$ or else $n \leq k$ and $\omega p_M^{n+1} < \tau$. Suppose

$$\sigma : M \xrightarrow{\Sigma_0^{(n)}} M^*$$

where $M^* = \langle J^A_\alpha^*, B^* \rangle$. Let $\tau^* = \sigma(\tau)$, $\tau' = \sup(\sigma^\ast \tau)$ and $A' = A^* \cap J^A_{\tau^*}$. Let $Q = J^A_\tau$, $Q' = J^{A'}_{\tau'}$. Then

1. The canonical extension $\tilde{\sigma} : M \rightarrow N$ of $\sigma \upharpoonright Q : Q \rightarrow Q'$ exists.

2. There is a unique $\Sigma_0^{(n-1)}$-preserving map $\sigma' : N \rightarrow M^*$ such that $\sigma = \sigma' \circ \tilde{\sigma}$ and $\sigma' \upharpoonright \tau' = \text{id.}$
3. If $\tilde{\sigma}$ is cofinal, then $\sigma'$ is $\Sigma_0^{(n)}$-preserving.

The following commutative diagram illustrates the Interpolation Lemma:

Now we sketch a proof of Lemma 4.3.2. Note that clause (c) of the lemma, which is phrased by cases, will result in argument by cases for the proof. However, the former and latter cases are not much different in complexity and fine structure arguments. We take the latter case i.e. the case that $\omega_\rho M \leq \kappa_\xi$ and $M_\xi$ is sound above $\kappa_\xi$ to prove, and leave the former case where $\omega_\rho M_\xi > \kappa_\xi$ for the reader to complete.

The following diagram helps to understand the structure of embeddings of the proof:
We first prove clause 1 of Lemma 4.3.2. Assume toward a contradiction that the
canonical extensions do not exist for stationarily many \( \xi \in S \), then without loss of
generality we can assume that the canonical extensions do not exist for all \( \xi \in S \).
Given \( \xi \in S \) we can further assume that \( \tau_\xi \) is the minimal in the sense that canoni-
cal extensions exist for corresponding structures below \( \tau_\xi \), again by stationarity.

Remembering that we are arguing for the latter case of clause (c), \( \omega \rho_{\nu+1}^\omega M \leq \kappa_\xi \) and
\( M_\xi \) is sound above \( \kappa_\xi \), the stationarity allows us to further restrict \( S \) so that there
is a fixed number \( n \in \omega \), such that \( \omega \rho_{\nu+1}^n \leq \kappa_\xi < \omega \rho_{\nu+1}^n M \) for all \( \xi \in S \).

To each \( \xi \in S \), fix an \( \omega \)-sequence \( f_\xi \in \Gamma(\sigma_\xi) = \Gamma(\kappa_\xi, M_\xi) \), and \( \beta_\xi^i < \kappa_\xi \) to
witness the ill-foundedness of the canonical extension, that is

\[ (\beta_\xi^{i+1}, \beta_\xi^i) \in \sigma_\xi a_{\xi}^{i+1,i}, \]
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Where \( u_{i+1,i} = \{ \langle \eta_1, \eta_0 \rangle; \ f_{i+1}^\xi(\eta_1) \inf_{i}^\xi(\eta_0) \} \). Fix parameters \( q_i^\xi \in M_\xi \) such that \( f_i^\xi \) is a good \( \Sigma_1^{n-1}(M_\xi) \)-function in \( q_i^\xi \).

Given \( \xi \in S \), let \( X_\xi \) be the smallest elementary substructure of \( M_\xi \) containing all \( q_i^\xi \)'s, then \( X_\xi \) is countable.

Let \( M_\xi^* \) be the transitive collapse of \( M_\xi \upharpoonright X_\xi \) and \( \sigma_\xi^* : M_\xi^* \to M_\xi \) be the corresponding inverse collapsing map. Then \( \sigma_\xi^* \) is elementary, \( p_{M_\xi^*} \in X_\xi \), \( M_\xi^* \) is a premouse and \( \sigma_\xi^*(p_{M_\xi}) = p_{M_\xi} \). Set \( \kappa_\xi^* = \sigma_\xi^*-1(\kappa_\xi) \) and we have that

\[
\omega^\rho_{M_\xi^*} \leq \kappa_\xi^* < \omega^\rho_{M_\xi^*}
\]

and \( M_\xi^* \) is sound above \( \kappa_\xi^* \). Let \( q_i^\xi* = \sigma_\xi^*-1(q_i^\xi) \), and \( f_i^\xi* \) corresponds to the same functionally absolute definition with \( f_i^\xi \) and therefore \( \sigma_\xi^*(f_i^\xi*) = f_i^\xi \).

We argue that

\( \sigma_\xi^* \) maps \( \tau_\xi^* = (\kappa_\xi^*)^{M_\xi^*} \) cofinally into \( \tau_\xi \),

(4.2)

thus showing \( \tau_\xi \) is \( \omega \)-cofinal.

Suppose \( \tilde{\tau}_\xi^* = \sup(\sigma_\xi^* \tau_\xi) < \tau_\xi \), then we apply the Interpolation Lemma and infer the existence of the canonical extension \( \tilde{\sigma}_\xi^* : M_\xi^* \to \tilde{M}_\xi^* \) as indicated in the picture (4.3.2), where \( Q_\xi^* = J_{\tilde{E}^\xi_\xi} \). And there is a \( \Sigma_0^0 \)-preserving map \( \sigma_\xi^* : \tilde{M}_\xi^* \to M_\xi \) such that the diagram commutes.

It follows that, \( \sigma_\xi^* \mid \tilde{\tau}_\xi^* = id \), and \( \sigma_\xi^* \tilde{\tau}_\xi = \tau_\xi \) if \( \tilde{\tau}_\xi^* \in \tilde{M}_\xi^* \). We aim to show that \( \tilde{M}_\xi^* \) is a counter example to the minimality of \( \tau_\xi \). Fine ultrapower arguments infer that \( \omega^\rho_{\tilde{M}_\xi^*} \leq \kappa_\xi < \omega^\rho_{\tilde{M}_\xi^*} \), \( \tilde{M}_\xi^* \) is sound above \( \kappa_\xi \) and that \( \sigma_\xi^* \) is \( \Sigma_0^{(n)} \)-preserving.

Apply the Condensation Lemma and we have iterability of \( \tilde{M}_\xi^* \) guaranteed by the preservation properties of \( \sigma_\xi^* \).

Now let \( \tilde{f}_i^\xi = \tilde{\sigma}_\xi^* (f_i^\xi) \), then \( \sigma_\xi^* \tilde{f}_i^\xi = f_i^\xi \). Computation shows that \( [\beta_i^\xi, \tilde{f}_i^\xi] \) represents a descending sequence w.r.t. the membership relation in \( D^{\tau_\xi}(\tilde{M}_\xi^*, \sigma_{\xi \alpha}) \) by preservation properties of \( \sigma_\xi^* \). And therefore we have showed (4.2).
Note $X_\xi$ is countable and $c.r.(\xi) > \omega$, there is a $g(\xi) < \xi$ such that $X_\xi \cap Q_\xi \subseteq \text{ran}(\sigma_{g(\xi)}(\xi))$. Then by stationarity we can assume without loss of generality that $g(\xi) = \bar{\xi}$ is a constant in $S$ and $\bar{\check{\xi}} < \min\{\xi\}$.

Given $\xi \in S$, we define a $\Sigma_0$-preserving map $\sigma_\xi : Q_\xi^* \to Q_\xi$ by $\sigma_\xi = \sigma_{\bar{\xi}}^{-1} \circ \sigma_\xi^*$ as illustrated in picture 4.3.2.

By (4.2) we have that both $\sigma_\xi$ and $\sigma_{\bar{\xi}} \xi$ are cofinal. Similar to the Interpolation, fine ultrapower arguments yield that:

- The canonical extension $\bar{\sigma}_\xi : M^*_\xi \to \bar{M}_\xi$ of $\sigma_\xi$ exists and there is a $\Sigma_0$-preserving map $\sigma'_\xi : \bar{M}_\xi \to M_\xi$ extending $\sigma_{\bar{\xi}} \xi$ defined by
  $$\sigma'_\xi(\bar{\sigma}_\xi(f)(\zeta)) = \sigma^*_\xi(f)(\sigma_{\bar{\xi}} \xi(\zeta))$$
  for $f \in \Gamma(\kappa^*_\xi, \ M^*_\xi)$ and $\zeta < \kappa_\xi$. Moreover, $\bar{\sigma}_\xi$ is $\Sigma^{(k)}_1$-preserving if $\omega^{\rho^k_{\bar{\xi} \xi} + 1} \bar{M}_\xi > \kappa^*_\xi$.

- $\omega^{\rho^n_{\bar{\xi} \xi}} \leq \kappa_\xi < \omega^{\rho^n_{\bar{\xi} \xi}} \bar{M}_\xi$ is sound and solid above $\kappa_\xi$ and that $\bar{\sigma}_\xi$ is $\Sigma^{(n)}_0$-preserving.

Standard fine ultrapower computation yields the preservation properties of $\sigma'_\xi$, and it immediately follows that $\bar{M}_\xi$ is normally iterable above $\kappa_\xi$.

Now we are ready to derive a contradiction. For any $\xi, \xi_1 \in S$, coiterability of $\bar{M}_\xi$ and $\bar{M}_{\xi_1}$ above $\kappa_\xi$ are ensured as above. And the coiteration computes that in fact $\bar{M}_\xi$ and $\bar{M}_{\xi_1}$ are always equal.

We choose $\xi, \xi_1$ such that $\beta_1^\xi \in \text{ran}(\sigma_{\xi_1 \alpha})$ for all $i < \omega$. Set $\bar{f}^\xi_i = \bar{\sigma}_\xi(f^*_i)$. This definition makes sense since $\bar{\sigma}_\xi$ is sufficiently preserving, and each $\bar{f}^\xi_i$ is an element of $\Gamma(\kappa_\xi, \bar{M}_\xi)$. 
Preservation properties of $\sigma'_\xi$’s allows us to compute the following infinite descending sequence:

$$\sigma'_\xi(f_0^\xi)(\sigma^{-1}_{\xi\alpha}(\beta_0^\xi)) \ni \sigma'_\xi(f_1^\xi)(\sigma^{-1}_{\xi\alpha}(\beta_1^\xi)) \ni \cdots$$

and thus resulting in a contradiction with the existence of the canonical extension $\sigma'_\xi$.

**Remark:** The advantageous property that $\bar{M}_\xi = \bar{M}_{\check{\xi}}$ we verified in the above proof only holds as we assumed the latter case of clause (c) in the statement of Lemma 4.3.2, however the former case of (c) implies full iterability and coiteration between $\bar{M}_\xi$ and $\bar{M}_{\check{\xi}}$ will still do the work. The argument for the former case of (c) is basically repetitive work of the above proof for the latter case.

To prove for clause 2 of Lemma 4.3.2, we follow an analogous argument to the above one for clause 1. Again we take the latter case of (c), and we assume toward a contradiction that the conclusion fails on a stationary set $S$. By stationarity we can assume w.o.l.g. it holds for all $\xi \in S$, and $\tau_\xi$ is in the sense minimal as previously, and similar for the constant number $n < \omega$ as before.

Now we have the canonical extension $\bar{\sigma}_\xi$ exists since we proved clause 1. For each $\xi \in S$, fix a countable premouse $M^\square_\xi$ together with a $\Sigma^*$-preserving map $\sigma^\square_\xi : M^\square_\xi \to \bar{M}_\xi$ to witness the failure of iterability of $\bar{M}_\xi$ such that

$$M^\square_\xi \text{ is not normally iterable above } \kappa^\square_\xi = (\sigma^\square_\xi)^{-1}(\kappa_\xi)$$

Let $\langle x_\xi^i : i \in \omega \rangle$ be an enumeration of $M^\square_\xi$, and $f_\xi^i \in \Gamma(\sigma_\xi, M_\xi)$ and $\beta_\xi^i < \kappa_\xi$ such that $\sigma_\xi(x_\xi^i) = \bar{\sigma}_{\xi\alpha}(f_\xi^i)(\beta_\xi^i)$. Let $M^*_\xi$, $\sigma^*_\xi$, $\kappa^*_\xi$, $\tau^*_\xi$ be as before, then there is no difficulty in generating the following statement in analogy to (4.2).

$$\sigma^*_\xi \text{ maps } \tau^*_\xi = (\kappa^*_\xi)^{M^*_\xi} \text{ cofinally into } \tau_\xi.$$  

(4.3)
With (4.3) we can again find a constant \( \bar{\xi} < \min\{S\} \) with the premise \( \bar{M}_\xi \) constructed in the same way extending \( Q_\xi \) and embeddings \( \bar{\sigma}_\xi : \bar{M}_\xi \to \bar{M}_\xi, \sigma'_\xi : \bar{M}_\xi \to M_\xi \) as before. Again \( \bar{M}_\xi = \bar{M}_{\xi_1} \) for all \( \xi, \xi_1 \in S \). We are then ready to derive a contradiction by embedding \( M_\xi^{\Box} \) into some \( M_{\xi_1} \) with sufficiently preserving properties.

We fix \( \xi, \xi_1 \in S \) such that \( \xi < \xi_1 \) and \( \beta_i^{\xi} \in \text{ran}(\sigma_{\xi_1}) \) for all \( i < \omega \). Set \( \bar{f}_i^{\xi} = \bar{\sigma}_\xi(f_i^{\xi}) \). This definition makes sense since \( \bar{\sigma}_\xi \) is sufficiently preserving, and each \( \bar{f}_i^{\xi} \) is an element of \( \Gamma(\kappa_\xi, M_\xi) \). Then the assignment

\[
 x_i \mapsto \sigma'_{\xi_1}(\bar{f}_i^{\xi})(\sigma_{\xi_1}^{-1}(\beta_i^{\xi})),
\]

which is a \( \Sigma_0^{(n)} \)-preserving embedding of \( M_\xi^{\Box} \) into \( M_{\xi_1} \) mapping \( \kappa_\xi^{\Box} \) to \( \kappa_{\xi_1} \) will do the work: \( M_\xi^{\Box} \) is not normally iterable above \( \kappa_\xi^{\Box} \), however \( M_{\xi_1} \) is fully iterable, contradiction.

(\( \square \) Lemma 4.3.2)

Now we have the Frequent Extensions of Embeddings Lemma, and we continue with the proof of Theorem 4.3.1. The proof exactly follows the same structure as that of Theorem 4.2.1, though there are a few novelties.

Again we assume otherwise, there is a counter-example \( \kappa. \kappa \geq \omega_1 \) is a cardinal in \( K \), \( \tau = \kappa^{+K}, \gamma = cf(\tau), \) and \( \gamma < \kappa \).

The aim is to collapse \( \tau \) inside \( K \). Notice that this time \( \kappa \) need not be a real cardinal, instead of directly constructing an \( \omega \)-closed structure to contain all the bad information given by the counter assumption, we apply the Frequent Extensions of Embeddings Lemma to get a stationarily many structures to secure a desired
Let $\alpha$ be a regular cardinal such that $\max(\gamma, \omega_1) < \alpha \leq \kappa$. We collapse $\tau$ to $\alpha$ by working in $V[G]$ where $G$ is $Col(\alpha, \tau)$-generic over $V$. $V[G]$ does not add any partial functions of size less than $\alpha$. And by Lemma 4.1.4, $0^{\sharp}$ also does not exist in $V[G]$, and $K$ is the core model in the sense of $V[G]$ since $K$ is generic forcing invariant.

In $V[G]$, fix a surjective mapping $f : \alpha \rightarrow J^E_{\tau}$. Since $c(\tau) < \alpha$ and $\alpha$ remains regular in $V[G]$, let $\alpha^* < \alpha$ be such that $f^* \alpha^*$ is cofinal in $\tau$. Consider the following closed and unbounded set $C \subseteq \alpha$:

$$C = \{\bar{\alpha} | \alpha^* < \bar{\alpha} < \alpha, f^* \alpha^* \cap \alpha = \bar{\alpha} \& J^E_{\tau}|(f^* \bar{\alpha}) \text{ is an elementary substructure of } J^E_{\tau}\}$$

Then let

1. $< \alpha_\xi : \xi < \alpha >$ be a normal enumeration of $C$;

2. $Q_\xi = J^E_{\tau \xi}$ be the transitive collapse of $J^E_{\tau}|(f^* \bar{\alpha})$ and let $\sigma_{\xi \alpha}$ be the inverse collapsing map. Let $\sigma_{\xi \xi'} = \sigma_{\xi \alpha}^{-1} \circ \sigma_{\xi \alpha}$ for $\xi' > \xi$;

3. $\kappa_\xi$ be the largest cardinal in $Q_\xi$, $\alpha_\xi = crit(\sigma_{\xi \alpha}) = crit(\sigma_{\xi \xi'})$ and $\eta_\xi = \alpha_\xi^{+Q_\xi}$.

We thus obtain a commutative diagram of structures satisfying the requirements 1-3 of Frequent Extensions of Embedding Lemma. Let $S$ be the set of all ordinals less than $\alpha$ with uncountable cofinality.

**Lemma 4.3.5.** For all but non-stationarily many $\xi \in S$,

$$P(\alpha_\xi) \cap Q_\xi \subseteq P(\alpha_\xi) \cap K \quad (4.4)$$
This lemma is an analogue to Lemma 4.2.2. Recall in Lemma 4.2.2, we obtained the non-triviality of the power sets of \( \alpha \) in \( Q \) and \( K \) by showing that the successor cardinal \( \vartheta \) in respective models are not same. Now, we assume otherwise of lemma 4.3.5, then \( \alpha_\xi \) is an inaccessible cardinal both in \( Q_\xi \) and \( K \), and \( \vartheta_\xi = \alpha_\xi^+K \) for a stationarily many \( \xi \)'s in some \( S' \subset S \). Also \( \sigma_{\xi \alpha}(\alpha_\xi) = \alpha \), and \( K \) agrees with \( Q_\xi \) below \( \vartheta_\xi \) by the Condensation Lemma.

Consider the following system:

\[
< \bar{Q}_\xi, \sigma_{\xi \xi'} \upharpoonright \bar{Q}_\xi : \xi < \xi' \leq \alpha >, \text{ where } \bar{Q}_\xi = J_{\vartheta_\xi}^{E_{\xi}}
\]

This system of structures satisfies requirements 1-3 of Frequent Extensions of Embeddings Lemma. Then we let \( M_\xi = K \) be the proper lengthening of \( \bar{Q}_\xi \) for \( \xi \in S' \) then we can obtain from a simpler variant of the Frequent Extensions of Embeddings Lemma that for all but non-stationarily \( \xi \in S' \), the canonical 0-extension

\[
\tilde{\sigma}_{\xi \alpha}^* : K \to \tilde{K}_\xi
\]

of \( \sigma_{\xi \alpha} \upharpoonright \bar{Q}_\xi \) exists. And \( \tilde{K}_\xi \) is a weasel, therefore a simple iterate of \( K \), and \( \tilde{\sigma}_{\xi \alpha}^* \) is the associated iteration map with critical point \( \alpha \). Then the first iteration index used must be \( \vartheta_\xi \). It follows that \( E^K_{\vartheta_\xi} \neq \phi \) and \( E^K_{\vartheta_\xi} = \phi \). But \( \vartheta_\xi = \alpha_\xi^+K \leq \alpha \), \( \tilde{K}_\xi \) is a lengthening of \( J_{\sup(\sigma_{\xi \alpha} \upharpoonright \vartheta_\xi)}^{E_{\xi}} \), and \( E^K_{\vartheta_\xi} = E^{\tilde{K}_\xi}_{\vartheta_\xi} \), contradiction. (\( \square \) Lemma 4.3.5)

Therefore, without loss of generality, we can assume that Lemma 4.3.5 holds for all \( \xi \in S \). Similar to Lemma 4.2.3, we have the following lemma:

**Lemma 4.3.6.** For all but non-stationarily many \( \xi \in S \), \( Q_\xi \) is not moved in the coiteration of \( Q_\xi \) with \( K \)

Proof of Lemma 4.3.6:

Suppose otherwise, the coiteration of \( Q_\xi \) with \( K \) is nontrivial on the \( Q_\xi \) side for
stationarily many \( \xi \)'s. Following exactly the same argument as the proof of \( K^c \), in the coiteration of \( Q_\xi \) with \( K \), we truncate on the \( K \)-side from the first step, where the coiteration is realized by coiteration of \( Q_\xi \) with \( M_{\xi,0} \). We denote the corresponding notations to the coiteration by \( \nu_{\xi,i}, \kappa_{\xi,i} \) and \( M_{\xi,i} \) where \( i \) is the index of the stage. Then \( \vartheta_\xi \) is still the successor cardinal of \( \alpha_\xi \) in \( M_{\xi,0} \), and \( E^\xi \) agrees with \( E^{M_{\xi,0}} \) on \( \vartheta_\xi \), and \( M_{\xi,0} \) is sound and projects to \( \kappa_{\xi,0} \).

In order to carry out the argument in Lemma 4.2.3, the only novelty is that we need to verify the canonical extension \( \tilde{M}_\xi = \text{Ult}(M^*_{\xi}, E^\xi_{\nu(\xi)}) \) exists and is normally iterable above \( \kappa(\xi) \) for stationarily many \( \xi \)'s.

As \( Q_\xi \) has no top measure, we have \( \nu(\xi) \in \xi \), and an easy pressing down argument gives us a \( \bar{\xi} < \alpha \) and a stationary \( S' \subset S \) such that \( \nu(\xi) \in \text{range}(\sigma_{\xi\alpha}) \). And since the size of \( Q_\xi \) is strictly less than \( \alpha \), there is a \( \bar{\nu} \in Q_\xi \) such that \( \bar{\nu} = \sigma_{\xi\xi}^{-1}(\nu(\xi)) \) for stationarily many \( \xi \)'s in \( S' \). Without loss of generality, we can assume this holds for all \( \xi \)'s. Now \( \bar{\nu} \) is the cardinal successor of some \( \bar{\kappa} \) in \( Q_\xi \) by the pull back. So \( \kappa(\xi) = \sigma_{\xi\xi}(\bar{\kappa}) \). Let \( \kappa' = \sigma_{\xi\alpha}(\bar{\kappa}) \) and \( \nu' = \sigma_{\xi\alpha}(\bar{\nu}) \). Now we have the following system:

\[
< Q^*_\xi, \sigma_{\xi\xi}; \xi < \xi' \leq \alpha >
\]

where \( Q^*_\xi = J^E_{\nu(\xi)} \) and \( \sigma^\ast_{\xi\alpha} = \sigma_{\xi\alpha} \upharpoonright Q^*_\xi \), satisfying the requirements 1-3 of Frequent Extensions of Embeddings Lemma, and \( M^*_{\xi} \) is a lengthening of \( Q^*_\xi \) for each \( \xi \in S' \) satisfying requirements a-c of Frequent Extensions of Embeddings Lemma. Moreover, we can choose a stationary subset such that all \( n_\xi \)'s have the same value, therefore without loss of generality, we assume all \( n_\xi = n \) for some natural number \( n \).
CHAPTER 4. THE WEAK COVERING LEMMA

Apply the Frequent Extensions of Embeddings Lemma, we have for all but nonstationarily many \( \xi \in S' \):

- The canonical extension \( \tilde{\sigma}^*_\xi : M'_\xi \rightarrow M'_\xi \) of \( \sigma^*_\xi \) exists, and is \( \Sigma_0^{(n)} \) cofinal;

- \( \omega \rho_{M'_\xi}^{(n+1)} \leq \kappa' < \omega \rho_{M'_\xi}^n \), and \( M'_\xi \) is sound above \( \kappa' \)

- \( M'_\xi \) is normally iterable above \( \kappa' \)

\( \kappa' \) is a cardinal in \( K \), so \( M'_\xi \) are \( K \) are coiterable above \( \kappa' \). Next we show that \( M'_\xi \) is an initial segment of \( K \).

Suppose not, then one of the structures is moved in the coiteration.

If \( M'_\xi \) is not moved, then let \( K' \) be the last model on the \( K \) side, \( M'_\xi \) would be a proper initial segment of \( K' \). If there is no truncation then this case is trivial. Otherwise, we let \( \nu'_0 \) be the first coiteration index and \( \tilde{\nu}(\xi) = sup((\sigma^*_\xi)''(\nu(\xi))) \). Then \( M'_\xi \) agrees with \( K \) below \( \tilde{\nu}(\xi) \) and therefore \( \nu'_0 \geq \tilde{\nu}(\xi) \). Since \( \nu'_0 \) is a cardinal in \( K' \) and \( M'_\xi \) projects to \( \kappa' < \nu'_0 \), \( M'_\xi \) is a proper initial segment of \( K' || \nu'_0 \), therefore a proper initial segment of \( K \).

If \( M'_\xi \) is moved in the coiteration. As the last model on the \( M'_\xi \) side is not sound but an initial segment of \( K' \), there must be a truncation on the \( K \) side and therefore nontrivial. Let \( K'' \) be the result of the last truncation on the \( K \) side, then \( M'_\xi \) agrees with \( K'' \) and therefore apply the same argument as the above case, we know that \( M'_\xi \) must be an initial segment of \( K \).

Consider the following ultrapower map:

\[ \pi' : K \xrightarrow{E_{\nu'}} \hat{K} \]

\( E_{\nu'} \) is a total measure in \( K \) with critical point \( \kappa' \) because \( E^\xi_{\nu(\xi)} \) is a total measure on \( Q_\xi \) and the coiteration between \( Q_\xi \) and \( K \) is simple on the \( Q_\xi \) side.
Then \( \tilde{M}_\xi' = \pi'(M'_\xi) \) is an initial segment of \( \tilde{K} \) and therefore a mouse. \( \tilde{\sigma}_{\xi\alpha}^* \) is extension of \( \sigma_{\xi\alpha}^* \). We have

\[
\forall x \subset \kappa(\xi) (x \in E^\xi_{\nu(\xi)} \iff \tilde{\sigma}_{\xi\alpha}^*(x) \in E_{\nu'}).
\]

Through direct computation we know that \([f] \mapsto \pi' \circ \tilde{\sigma}_{\xi\alpha}^*(f)(\kappa')\) gives us the \(^*\)-ultrapower \( \tilde{M}_\xi' \) through collapsing of \( \mathbb{D}^*(M'_\xi, E^\xi_{\nu(\xi)}) \). Therefore \( \tilde{M}_\xi \) exists. And we obtain the following diagram:

\[
\begin{array}{c}
M'_\xi \\
\downarrow_{\kappa} \\
M''_\xi \\
\end{array}
\quad \xymatrix{
M'_\xi 
\ar[r]^= \ar[d]_{\kappa} & \\ \\
\tilde{M}'_\xi 
\ar[u] \ar[r]^= & \\
\tilde{M}_\xi 
\end{array}
\]

Now we have showed that \( \tilde{M}_\xi \) exists and is normally iterable above \( \kappa(\xi) \), repeat the argument of the proof of Theorem 4.2.3 and we obtain the desired claim.
(\( \square \) Lemma 4.3.6)

Now we can finish the proof of the Weak Covering Lemma for \( K \) following a similar argument to that of Theorem 4.2.1. From the above lemma, we assume without loss of generality that \( Q_\xi \) is not moved in the coiteration with \( K \) for all \( \xi \in S \). Then following exactly the same argument as Theorem 4.2.1, we find an iterate \( M_\xi, \theta(\xi) \) of \( M_{\xi,0} \) which lengthens \( Q_\xi \), and an initial segment \( M_\xi \) of \( M_\xi, \theta(\xi) \) such that \( M_\xi \) projects to \( \kappa_\xi \) and \( \tau_\xi \) is the successor cardinal of \( \kappa_\xi \) in \( M_\xi \). Apply Frequent Extensions of Embeddings Lemma, we have that for all but non-stationarily many \( \xi \in S \):

- The canonical extension \( \tilde{\sigma}_{\xi\alpha} : M_\xi \rightarrow N_\xi \) of \( \sigma_{\xi\alpha} : Q_\xi \rightarrow J^K_\tau \) exists and is \( \Sigma_0^{(\alpha)} \) and cofinal;
- $\omega^{(n+1)}_{\rho_{N_{\xi}}} \leq \kappa < \omega^{(n)}_{\rho_{N_{\xi}}}$, and $N_{\xi}$ is sound above and normally iterable above $\kappa$.

Then we observe that $N_{\xi}$ is coiterable with $K$ as the coiteration is above $\kappa$. Coiterate $N_{\xi}$ with $K$, then the coiteration terminates say after $\theta^*_\xi$ many steps, and the $N_{\xi}$ side of coiteration is simple. Denoting the last model on the corresponding side to be $N_{\eta^*_\xi}$ and $W_{\eta^*_\xi}$ and the former is therefore an initial segment of the latter.

Now the fine structural properties of $N_{\xi}$ we obtained above implies that $N_{\xi}$ is in fact the $\Sigma^1_1(N_{\xi})$-hull of $\kappa \cup \{p_{N_{\xi}} - (\kappa + 1)\}$, i.e. $N_{\xi} = h^{n+1}_{N_{\xi}}(\kappa \cup \{p_{N_{\xi}} - (\kappa + 1)\})$, therefore $N_{\xi}$ can be encoded into a $\Sigma^1_1(N_{\xi})$ subset $b \subset \kappa$. Since the coiteration is simple on the $N_{\xi}$ side, $b \in \Sigma^1_1(N_{\eta^*_\xi})$. If $W_{\eta^*_\xi} \neq N_{\eta^*_\xi}$, then $b \in W_{\eta^*_\xi}$, and $b \in K$ since the iteration is above $\kappa$. If $W_{\eta^*_\xi} = N_{\eta^*_\xi}$, then we take the last truncation point $W_k$ on the $K$ side, and consider the mouse after truncation $W^*_k$, the iteration from $W^*_k$ to $W_{\eta^*_\xi} = N_{\eta^*_\xi}$ is simple above $\kappa$, therefore $b$ is $\Sigma^1_1(W^*_k)$, and therefore in $W_k$ and therefore $b \in K$.

Then we can decode or reconstruct $N_{\xi}$ from $b$ inside $K^c$. In fact $N_{\xi} \in K$. Then since $N_{\xi}$ projects to $\kappa$ and lengthens $J^E_{\tau}$, $K$ must contain a subset of $\kappa$ which lies outside $J^E_{\tau}$, therefore $\tau$ is collapsed to $\kappa$ inside $K$. Contradiction.

($\square$ Theorem 4.3.1)
Chapter 5

The Dodd-Jensen Covering Lemma for $K^{DJ}$ and $L[U]$}

Finally we have had enough preparation for the desired proof of the Dodd-Jensen Covering Lemma for $K^{DJ}$ and $L[U]$.

The statement of the theorems are as follows:

**Theorem 5.0.1.** If there is no inner model with a measurable cardinal, then for any set $x$ of ordinals there is a $y \in K^{DJ}$ such that $y \supseteq x$ and $|y| = |x| + \omega_1$, i.e. $K^{DJ}$ has the covering property.

**Theorem 5.0.2.** If there is an inner model with a measurable cardinal, but no $0^\dagger$, and the model $L[U]$ are chosen such that $\kappa = \text{crit}(U)$ is the least, then one of the following holds:

1. $L[U]$ satisfies the covering property

2. $L[U]$ does not satisfy the covering property, but there is a Prikry generic sequence $C \subseteq \kappa$ over $L[U]$, and $L[U,C]$ has the covering property.
5.1 The Proof

The original idea of $K^{DJ}$ was to find some model between $L$ and $L[U]$, which is as large as possible, and yet still satisfies the Covering Lemma. In the absence of $L[U]$, $K^{DJ}$ was defined by $L[M]$ where $M$ is the class of mice, which are approximations to the measure $U$. Though the Covering Lemma for $K^{DJ}$ only holds below one measurable cardinal, the definition of $K^{DJ}$ could be as simple as just iterating the measure $U$ out of the universe when $L[U]$ does exist.

The definitions of mice are all consistent, however, when we talk about the Dodd-Jensen core model below $0^+$, since there are no full measures, the definitions of mice and other fine structural concepts are somehow simpler than that we used to talk about sequence of measures with extenders, for example when there is no inner model with a measurable cardinal, all mice are comparable.

**Definition 5.1.1** (Dodd-Jensen Core Model). We define the Dodd-Jensen core model $K^{DJ} = J_\infty[E]$ by recursion:

Suppose $K^{DJ}_\kappa = J_\kappa[E \upharpoonright \kappa]$ is defined, let $\mathcal{M}$ be the set of mice $M = J_\alpha[M^M]$ such that $M$ has no full measures, $M$ projects to $\kappa$ and $E^M$ agrees with $E$ below $\kappa$, then $K^{DJ}_\kappa = \bigcup \mathcal{M}$ where $\tilde{\kappa} = \sup\{\alpha^M : M \in \mathcal{M}\}$.

As we mentioned earlier at the end of Chapter 3, plausible consistency appears between definitions of the core models. If there is no $0^+$, then $K = L$; if there is $0^-$ but no $L[U]$(inner model with a measurable cardinal), then $K = K^{DJ}$; if there is an $L[U]$ but no $0^+$, then $K = L[U]$ for some $L[U]$ where the measure $U$ has the least critical point $\text{crit}(U)$ [17].

With the already developed theory of $K$, especially the Countable Complete Weak
CHAPTER 5. THE DODD-JENSEN COVERING LEMMA FOR $K^{DJ}$ AND $L[U]$

Covering property, we are now able to prove Theorem 5.0.1 and 5.0.2 by the following Lemma in a very close analogy to that proof of Theorem 3.1.1 presented in Chapter 3:

**Lemma 5.1.2.** If $0^+$ does not exist, then The core model $K$ satisfies the Dodd-Jensen Covering Lemma, Theorem 5.0.1 and 5.0.2.

Under the strong anti-large cardinal hypothesis that $0^+$ does not exist, we make free use of countably complete weak covering property of $K$. We are using $K$ to prove the Dodd-Jensen Covering Lemma for $K^{DJ}$ and $L[U]$ at one time, however, to claim the relationship between $K$, $K^{DJ}$ and $L[U]$ requires the countable complete weak covering property of $K$. In fact, we use a result about the relationship between existence of $0^+$ and nontrivial elementary embedding of $K^{DJ}$, which is analogous to Kunen’s result about $0^+$ and elementary embedding of $L$, but the proof of the result itself, involves partial use of countable complete weak covering property of $K$. Therefore, although we are using such results without proofs in this article, for the integrity of entire knowledge, we must avoid circular proof and put the weak covering theorems in the previous chapter before we can use it to prove for the Covering Lemmas for $K^{DJ}$ and $L[U]$.

Although many parts of the proof are reusing the previous techniques, the assumption that $0^+$ does not exist, which is equivalent to assuming that every premouse could only have at most one full ultrafilter, simplifies the proof substantially.

Since for this theorem we only need to involve sequence of measures without extenders, the theory in Chapter 4 still applies but somehow becomes simpler. For convenience of reading, from here on the notation $E$ represents a sequence of measures only.
Before we start proving for Lemma 5.1.2, we list a few technical Lemmas that we use as preliminary knowledge in our proof as follows:

**Lemma 5.1.3.** Assume that $\neg 0^*$ and $K$ satisfies the Countably Complete Weak Covering property. If there is a nontrivial elementary embedding $i : K \rightarrow M$, then $K = L[U]$ where $U$ is a measure in $L[U]$ with $\text{crit}(U) = \text{crit}(i)$.

The following proof was originally presented by William J. Mitchell in [17].

**Proof of Lemma 5.1.2:**
Recall that in order to prove the Covering Lemma for $L$, we used a basic construction for suitable sets $X \prec J_\kappa$ and showed that suitable sets are contained in $L$ and the class of suitable sets is unbounded. The proof we are going to present here is quite similar, we have a slightly more complicated basic construction for suitable sets $X \prec K_\kappa$, and show that the class of suitable sets is unbounded. There also involves an new argument to prove for Covering Lemma for $K = L[U]$, in which we analyze the indiscernibles generated during the basic construction, and yield the Prikry generic sequence $C$ over $K = L[U]$. 

First, we refine the definition of "suitable sets":

**Definition 5.1.4.** Assume $X \prec_1 K_\kappa$, and the transitive collapse of $X$ is $\bar{K} = J_\kappa[\bar{E}]$. Denote the inverse collapse map as $\pi : \bar{K} \rightarrow X$.

Then we say $X$ is suitable iff $\text{Ult}_n(M, \pi, \beta)$ is always defined whenever $n \in \omega$, $\beta \leq \kappa$ and $M = J_\alpha[E']$ is an iterable premouse ($\alpha$ could be Ord) such that $\text{Ult}_n(M, \pi, \beta)$ is defined and $E'$ and $\bar{E}$ agree below $\bar{\beta}$ where $\bar{\beta}$ is defined to be least ordinal such that $\pi(\bar{\beta}) \geq \beta$.
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Similar to the proof in Chapter 3, we have:

**Definition 5.1.5.** $X \prec_1 K_\kappa$ is countably closed if there is a $Y$ such that $\text{``}Y \subseteq Y, Y \prec H(\tau)\text{'' for some } \tau > \kappa$ and $X = Y \cap K_\kappa$.

**Definition 5.1.6** (Unsuitability Witness). Assume $X$ is not suitable. Then the witness $w$ to the unsuitability of $\pi : X \prec_1 K_\kappa$ is a $\omega$-chain of $\Sigma_0$ elementary embeddings $i_k : m_k \to m_{k+1}$ such that

1. $i_k \in X$ and $m_k \in X$ for all $k < \omega$;

2. The direct limit of the chain $\pi^{-1}(w)$ equals the $\Sigma_n$-code of some mouse $\tilde{M}_0$;

3. Either $\tilde{M}_0$ is ill-founded or else it has an ill-founded iteration;

4. The critical sequence $< \beta_k : k < \omega >$ where $\beta_k$ is the critical point of $i_k$ is nondecreasing;

5. For each $k$ we have $m_k \in m_{k+1}$, and exists a function $f_k \in m_{k+1}$ such that $f_k^{-1}(\beta_k) = i_k^{-1}(m_k)$.

The minor novelty appears in clause 2 and 3 in comparison with Definition 3.1.5. In clause 2, we changed the requirement of $\text{dir lim}(\pi^{-1}(w)) = C_n(J_\alpha)$ for some $\alpha$ into $\text{dir lim}(\pi^{-1}(w)) = \Sigma_n$ code of some mouse $\tilde{M}_0$; in clause 3, we require this mouse to have no well-founded iteration.

We are now ready to begin with our basic construction for suitable sets. Similar to the proof in chapter 3, we have a suitable set $X \prec_1 K_\kappa$, and not transitive. Let $\pi : \tilde{K} = J_\kappa[E] \to K_\kappa$ to be the inverse collapse of $X$. Then $\pi$ is not identity.

Recall diagram (2.1) of the basic construction in Chapter 3. We similarly compare the models $\tilde{K}$ and $K$, however, it is not as simple as in the constructible universe,
and we need the help of iterated ultrapowers. On the other hand, the same argument as Theorem 4.2.1 yields that, the coiteration between $\bar{K}$ and $K$ is nontrivial, and never moved on the $\bar{K}$ side. By universality of $K$, the coiteration ends at $\theta$ and $M_\theta = J_{\alpha_\theta}[E_\theta]$ such that $\bar{E} = E_\theta \upharpoonright \bar{\kappa}$.

Therefore we have the following diagram:

\[
\begin{array}{c}
M_\theta \supseteq M \\
\uparrow \quad \uparrow \\
\bar{K} = J_{\kappa}[\bar{E}] \quad \xrightarrow{\bar{\pi}} \quad X \prec K_\kappa
\end{array}
\]

Let $M = J_{\alpha}[E^M]$ and $\tilde{M} = J_{\bar{\alpha}}[\tilde{E}]$. $\pi : \bar{K} \to X$ is the inverse collapse map. $M_\theta$ is the end-structure on the $K$ side of the coiteration.

Once the model $M_\theta = J_{\alpha_\theta}[E^{M_\theta}]$ is constructed, we can complete the above diagram: Let $M = J_{\alpha}[E^M]$, where $(\alpha, n)$ is the largest pair below $(\alpha_\theta, n_\theta)$ such that $Ult_n(J_{\alpha}[E^M], \pi, \kappa)$ is defined. Set $\tilde{M} = Ult_n(M, \pi, \kappa)$.

Recall the proof of Theorem 4.2.1, we have the following facts:

**Proposition 5.1.7.** Assume $\neg 0^+$, and $X \prec_1 K_\kappa$ is a suitable set which is not transitive. Let $\pi : \bar{K} = J_{\kappa}[\bar{E}] \to K_\kappa$ to be the inverse collapse of $X$. Then
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1. Let $\eta = \text{crit}(\pi)$, and $\bar{\vartheta} = \eta+K$, $\vartheta = \eta^+K$, then $\bar{\vartheta} < \vartheta$ therefore $P(\eta)^K \subsetneq P(\eta)^K$.

2. In the coiteration of $\bar{K}$ and $K$, the $\bar{K}$ side is trivial.

3. Either $\bar{K}$ is an initial segment of $K$, or else, we take truncation on the $K$ side from the first step.

4. The coiteration ends after $\theta$ steps, and $\bar{K}$ is an initial segment of $M_\theta$ which is the last model on the $K$ side of coiteration.

5. $\bar{M} \in K$.

Clauses 1,2,4 are directly from proof of theorem 4.2.1. For clause 3, and 5, we make the following additional assumption:

If $K = L[U]$, where $U$ is a measure on a cardinal $\mu$ of $K$, then either $\mu^{+K} \subseteq X$ or else $\kappa \leq \mu^{+K}$.

Assuming (5.0) does not involve in any loss of generality, because for the former case is just a relativization of the proof for $L$ and shows that any set $x_0$ of size at most $\mu^{+K}$ is contained in a set $y$ in $K$ of size $\mu^{+K}$, and the latter case shows that $x_0$ can be covered by a subset of $y$, say $y'$, which satisfies the Covering Theorem for $L[U]$.

For clause 3 of the proposition, we only care about the nontrivial case, which is $\bar{K}$ is not an initial segment of $K$. Then using (5.0), we know that any full ultrafilter in $K$ with critical point less that $\eta$ would also be in $\bar{K}$, and therefore cannot be used in the coiteration, therefore the coiteration on the $K$ side truncate immediately.

For clause 5 of the proposition, assume otherwise, $\bar{M} \notin K$. We show that $\bar{M}$ has no full measure $U$ with critical point less than $\kappa$. Let $\mu = \text{crit}(U)$, then
we have $\mu^+K = \kappa$ since otherwise $\mu^+K < \kappa$ and $U = E^M_\gamma$ for some $\gamma < \kappa$. Then $\tilde{\pi}^{-1}(U) = E^M_{\pi^{-1}(\gamma)}$, and $\pi^{-1}(\gamma) < \bar{\kappa}$. By the construction it follows that $\tilde{\pi}^{-1}(U) \in \bar{K}$ which implies $U \in K$, contradict with (5.0). We borrow the next lemma stating that $\kappa$ cannot be a successor cardinal of $K$, and the proof of it will come later. Then we have shown that $\tilde{M}$ has no full measure $U$ with critical point less than $\kappa$. Then the suitability of $X$ implies $\tilde{M}$ is iterable. And $\tilde{M}$ is sound above $\kappa$ by definition. On the other hand, the ultimate projectum $\rho$ of $\tilde{M}$ is no smaller than $\kappa$, because otherwise $\tilde{M}$ is an iterated ultrapower of some mouse $M'$ sized not larger than $\rho$, but then $M' \in K_\kappa \subseteq M$ which is impossible. Therefore $\tilde{M}$ is sound, and thus in $K$.

The following lemma will complete the proof of the above:

**Lemma 5.1.8.** Keeping all the previous notations, then $\kappa$ is not a successor cardinal of $K$.

Proof of 5.1.8: First we assume that $\tilde{M}$ has no full measure $U$ with critical point less than $\kappa$. The idea is to show that if $\kappa$ is a successor cardinal in $K$, then there is an $\eta < \kappa$ that $X = \tilde{h}^\kappa(X \cap \eta)$, which shows that $cf(\kappa) < \kappa$, contradiction. To argue this, we will need to consider the indiscernibles generated by the iteration $i$.

If $M$ is a proper initial segment of $M_\theta$, then $M$ is a mouse. Exactly following the construction for $L$ in the proof of theorem 3.1.1, we know that $X = \tilde{h}^\kappa(X \cap \rho)$, where $\tilde{h}$ comes from Lemma 2.2.4(Upward Extension) and $\rho = \pi(\rho_{n+1}^M)$ where $\rho_{n+1}^M < \bar{\kappa} \leq \rho_{n}^M$. And we are done.

If $M = M_\theta$, then we know from the previous proposition that the coiteration truncate immediately on the $K$ side. Let $l$ be the last truncate on the $K$ side, then the iteration from $M_l^*$ to $M_\theta$ is simple. And $M_l^*$ is $n$-sound, all the remaining
iterated ultrapower has degree $n$. Let

\[ \bar{C} = \{ i_{l,\nu}(\kappa_l) : \text{for all } l, \nu \text{ such that } l < \nu < \theta \} \]

Then $\bar{C}$ is a sequence of indiscernibles for $M_\theta$.

Consider the $\Sigma_n$-projectum of $M_{\theta}$, denoted as $\bar{\rho}$, $\bar{\rho} = \rho_{n+1}^* = \rho_{n+1}^{M_\theta}$. Denote the $\Sigma_n$-Skolem function of $M_\theta$ as $\bar{h}$, then $M_\theta = \bar{h}^*(\bar{\rho} \cup \bar{C})$ by soundness of $M_i^\ast$.

Now let $\rho = \sup(\pi^\ast \bar{\rho})$ and $C = \pi^\ast \bar{C}$. Let $\tilde{h} = \tilde{h}^X$ be the function given by Lemma 2.2.4 (Upward Extension), then we have

\[ X = K_\kappa \cap \pi^\ast M_\theta = K_\kappa \cap \pi^\ast h^*(\bar{\rho} \cup \bar{C}) \subseteq \tilde{h}^*(\rho \cup C). \]

$\kappa$ is a successor cardinal in $K$, therefore $\bar{\kappa}$ is a successor cardinal in $\bar{K}$, therefore $\bar{C}$ cannot be unbounded in $\bar{\kappa}$, therefore $\eta = \sup(\rho \cup C) < K$ satisfies the desired claim. So we have finished proof for the case that $\tilde{M}$ has no full measure $U$ with critical point less than $\kappa$.

If $\tilde{M}$ has a full measure $U$ with critical point less than $\kappa$. Then as we argued in clause 5 of the previous proposition, $\kappa = \mu^K$ where $\mu = \text{crit}(U)$. In this case, the same argument as above shows that $\tilde{M}' = \text{Ult}(M', \pi, \kappa) \in K$, where $M'$ is the result of carrying out one more step of the iteration $i$ on the $K$ side. Exactly the same argument using $M'$ and $\tilde{M}'$ in place of $M$ and $\tilde{M}$ finishes the proof.

(\Box Lemma 5.1.8.)

Now we have completed our basic construction for suitable sets.

Till now, our argument is almost same as the one we used to prove the Covering Lemma for $L$. Next, in order to finish the proof of the Covering Lemma for $K^{DJ}$
and $L[U]$, we need to investigate in more details and analyze the indiscernibles sequence $C$, which is the end critical point sequence, introduced in the above Lemma 5.1.8. The use of indiscernibles from an iterated ultrapower as a Prikry sequence is well discussed in section 2.2 of [16].

One novelty that differs from the proof for $L$ is that we need to analyze the indiscernibles which are generated by the iterated ultrapower that we used in the construction. In the case when the iterated ultrapower is infinite, these indiscernibles would yield a generic Prikry sequence $C$ over $K = L[U]$.

Now, for an arbitrary suitable set $X$, we aim to find an $f \in K$ and $\eta < \kappa$ such that $X = f''(\eta \cap X)$ or else $C$ is a Prikry sequence and $X = f''(C \cup (\eta \cap X))$. (In fact we can further show that $C$ is unique modulo finite differences.)

In the proof of Lemma 5.1.8, we have the fine structure property of $M$ implies $M = h''(\rho \cup \tilde{C})$. In fact we have

$$\forall \xi \in (\bar{\kappa} - \bar{C}) (\xi \in \tilde{h}''(\bar{\rho} \cup (\bar{C} \cap \xi)))$$ (5.1)

Keep the notations of $\rho$, $C$, and $\tilde{h}$ as in Lemma 5.1.8. Then it follows that $X = \tilde{h}''((X \cap \rho) \cup C)$, and if $\xi \in X \cap \kappa$ then $\xi \in \tilde{h}''((X \cap \rho) \cup (C \cap \kappa + 1))$.

If $\bar{C}$ is finite then we can define $f(x) = \tilde{h}(x, C)$, so that $f \in K$ and the first alternative of our aim holds.

If $C$ is infinite, then we analyze the indiscernibles as follows:

For convenience, we put a superscript $X$ for each $C$ to represent that $C$ is built up by the construction with respect to the arbitrary suitable set $X$.

\textbf{Definition 5.1.9.} Let $C$ be the class of suitable sets $X$ such that $C^X$ is either
finite or else \( K = L[U] \), the set \( C^X \) is a Prikry sequence for \( U \), and \( C^X \) is maximal in the sense that \( C - C^X \) is finite whenever \( C \) is any other Prikry sequence for \( L[U] \).

We will finish proof of Lemma 5.1.2 and therefore the Dodd-Jensen Covering Lemma by the following lemma:

**Lemma 5.1.10.** If \( 0^\dagger \) does not exist then the class \( C \) is unbounded in \( [K_\kappa]^\delta \) whenever \( \kappa \) is a cardinal of \( K \) and \( \delta \) is an uncountable regular cardinal below \( \kappa \).

Proof of Lemma 5.1.10: This proof is in analogy with Lemma 3.1.7. Again we work in the space \( \text{Col}(\delta, K_\kappa) \). The elements are partial functions \( \sigma : \xi \to K_\kappa \) with \( \xi < \delta \). Let

\[
S = \{ \sigma \in \text{Col}(\delta, K_\kappa) : \text{cf}(\text{dom}(\sigma)) > \omega \& \text{range}(\sigma) \text{ is suitable but not a member of } C \}
\]

Then toward a contradiction, we assume that \( S \) is stationary in the space. (In the proof we usually use superscripts to represent corresponding notations for convenience, for example \( C^\sigma \) actually represents \( C^{\text{ran}(\sigma)} \)).

By the variant Fodor’s Lemma in the proof of Lemma 3.1.7, we have that there is a \( \sigma_0 \in S \) and a stationary set \( S_0 \subseteq S \) such that \( \sigma_0 \subseteq \sigma \) and \( C^\sigma \subseteq \text{ran}(\sigma_0) \) for all \( \sigma \in S_0 \).

**Definition 5.1.11.** We say \( a \subseteq^* b \) iff \( a - b \) is a finite set. And \( a =^* b \) iff \( a \subseteq^* b \& b \subseteq^* a \).

To prove Lemma 5.1.10, we first need the following observation:
Claim 1. If $X_0, X_1$ are two suitable sets, and $X_0 \subseteq X_1$, then $C^{X_1} \cap X_0 \subseteq^* C^{X_0}$.

Proof of Claim 1: Use $M^{X_0}, \pi^{X_0}, \bar{K}^{X_0}$ to denote the mouse $M$, ultrapower map $\pi$ and the transitive collapse $\bar{K}$ in our basic construction for $X_0$, and similarly $M^{X_1}, \pi^{X_1}, \bar{K}^{X_1}$ for $X_1$. Let $\nu$ be any member of $X_0 \cap (C^{X_1} - C^{X_0})$. And let $\nu_0$ be such that $\pi^{X_0}(\nu_0) = \bar{\nu}$. Then $\nu_0 \notin (\pi^{X_0})^{-1}(C^{X_0})$, and so $\nu_0 \in h_0 \nu_0$ where $h_0$ is the Skolem function of $M^{X_0}$.

Now let $\tau : \bar{K}^{X_0} \to \bar{K}^{X_1}$ be such that $\tau = (\pi^{X_1})^{-1} \circ \pi^{X_0}$, and $\tilde{\tau}$ be the ultrapower map such that $\tilde{\tau} : M^{X_0} \to M^* = \text{Ult}(M^{X_0}, \tau, \bar{\kappa}^{X_1})$.

Then $\nu_1 = \tau(\nu_0) \in h^* \nu_1$ where $h^*$ is given by the Upwards Extensions of Embeddings Lemma. Then $M^*$ is sound above $\bar{K}^{X_1}$ and agree with $\bar{K}^{X_1}$ up to $\bar{\kappa}^{X_1}$. The coiteration between $M^*$ and $M^{X_1}$ shows that one must be an initial segment of the other. As every bounded subset of $\bar{K}^{X_1}$ in $M^*$ is a member of $\bar{K}^{X_1}$, so $M^*$ must be an initial segment of $M^{X_1}$. Then $h^*$ is definable in $M^{X_1}$ from parameter.

Now $h^*$ is a function definable in $M^{X_1}$ such that $\xi \in h^* \xi$ for all but boundedly many $\xi \in (\pi^{X_1})^{-1}((C^{X_1} \cap X_0) - C^{X_0})$, however, this can only hold for finitely many $\xi \in (\pi^{X_1})^{-1}(C^{X_1})$. Since we chose $\nu \in X_0 \cap (C^{X_1} - C^{X_0})$ arbitrarily, there are only finitely many $\nu$’s, and therefore $C^{X_1} \cap X_0 \subseteq^* C^{X_0}$.

(\square Claim 1.)

Now we consider the previous $\sigma_0$ and $S_0$ given by the variant Fodor’s Lemma. Since all $\sigma \in S_0$ is extends $\sigma_0$, by claim 1 we know that $C^{\sigma} \subseteq^* C^{\sigma_0}$ for all $\sigma \in S_0$.

We assign, for each $\sigma \in S_0$, a unsuitability witness $w(\sigma)$ as $\text{ran}(\sigma) \notin C$. 


Claim 2. There is a function $w$ defined on $S_0$ such that $w(\sigma) \subseteq \text{ran}(\sigma)$ for all \( \sigma \in S_0 \), and that for any \( \sigma_1, \sigma_2 \in S_0 \) with \( \sigma_1 \subseteq \sigma_2 \) & \( w(\sigma_2) \subseteq \text{ran}(\sigma_1) \), we have \( C^{\sigma_1} \subseteq^* C^{\sigma_2} \).

Proof of Claim 2: We already know that \( C^{\sigma_1} \subseteq^* C^{\sigma_0} \) and we will show that \( C^{\sigma_0} \subseteq^* C^{\sigma_2} \). Let \( D = C^{\sigma_0} - C^{\sigma_2} \). If \( D \) is finite then we are done. Now assume \( D \) is infinite and we enumerate \( D \) as \( < d_m : m < \omega > \) in increasing order. Let \( \bar{d}_m \) be the inverse of \( d_m \) under the map \( \pi^{\sigma_1} \), then \( \bar{d}_m \in h^{M^{\sigma_1}} d_m \), where \( h^{M^{\sigma_1}} \) is the Skolem function for the mouse \( M^{\sigma_1} \) in the basic construction.

We define the function \( w \) by a slight modification of unsuitability witness: Replace clause 3 of the definition of Unsuitability Witness into the following clause:

There is a function \( h \) which is \( \Sigma_n \) over \( \text{dirlim}(w) \) such that \( \forall d \in D \ (d \in h'' d) \).

To verify that this definition of \( w \) satisfies Claim 2, we let \( \sigma_1 \subseteq \sigma_2 \) be members of \( S_0 \) and such that \( w(\sigma_2) \subseteq \text{range}(\sigma_1) \). We follow a similar argument like claim 1:

Consider \( \tau = (\pi^{\sigma_1})^{-1} \circ \pi^{\sigma_2} : \hat{K}^{\sigma_1} \rightarrow \hat{K}^{\sigma_2} \) that if we denote direct limit of \( (\pi^{\sigma_1})^{-1}(w(\sigma_2)) \) as \( \bar{m} \), \( \tau \) extends to an elementary embedding \( \bar{\tau} : \bar{m} \rightarrow m \). As \( C^{\sigma_2} \subseteq^* C^{\sigma_1} \), it follows that the measure on \( \bar{k}^{\sigma_1} \) in \( \bar{m} \) agrees with that in \( M^{\sigma_1} \). Therefore the two structures \( \bar{m} \) and \( M^{\sigma_1} \) agree up to \( \bar{k}^{\sigma_1} \). By soundness one must be an initial segment of the other. \( \bar{m} \) has to be an initial segment of \( M^{\sigma_1} \) because otherwise there would be a bounded subset of \( \bar{k}^{\sigma_1} \) in \( \bar{m} - \hat{K}^{\sigma_1} \), which is impossible. Hence the Skolem function of \( \bar{m} \) is definable in \( M^{\sigma_1} \), and every sufficiently large member \( d \) of \( (\pi^{\sigma_1})^{-1}(D) \) is in \( h^{M^{\sigma_1}} d \).

As there are only finitely many such members \( d \) of \( (\pi^{\sigma_1})^{-1}(D) \) is in \( h^{M^{\sigma_1}} d \), \( D \cap C^{\sigma_1} \) has to be finite, and Claim 2 follows.

(\( \square \) Claim 2.)
Now we are ready to finish the proof of Lemma 5.1.10. Apply a second time of the variant of Fodor’s Lemma, we can find a $\sigma_1 \in S_0$ and a stationary $S_1 \subseteq S_0$ such that $\sigma_1 \subseteq \sigma$ and $w(\sigma) \subseteq \text{ran}(\sigma_1)$ for all $\sigma \in S_1$. By Claim 1 $C^\sigma \subseteq^* C^{\sigma_1}$ and by Claim 2 $C^{\sigma_1} \subseteq^* C^\sigma$. It follows that $C^\sigma =^* C^{\sigma_1}$ for all $\sigma \in S_1$.

As $S_1$ is stationary, there is a $\sigma \in S_1$ such that $\text{ran}(\sigma) = Y \cap K_\kappa$ for some $Y \prec H(\kappa^+)$ with $C^{\sigma_1} \in Y$. Therefore $C^\sigma \in Y$. $C^\sigma$ is a Prikry sequence for the measure $U$ of $K = L[U]$, and to witness $\text{ran}(\sigma) \notin C$ this cannot be a maximal Prikry sequence, i.e. there is another Prikry sequence $C_1$ such that $C_1 - C^\sigma$ is infinite.

Then by elementarity there is such a sequence in $Y$, say $C'$. Then $C' \subseteq \text{ran}(\sigma)$, so any other member $\alpha$ of $C' - C^\sigma$ is in $\check{h}^{\text{ran}(\sigma)}''\alpha$, and since $\check{h}^{\text{ran}(\sigma)} \in K$, we have that $C' - C^\sigma$ is finite as $C'$ is a Prikry sequence. Contradiction.

This completes the proof of Lemma 5.1.10 and therefore the Dodd-Jensen Covering Lemma 5.0.1 and 5.0.2.

### 5.2 Some Discussion

Basically saying, the covering lemmas assert that, under certain anti-large cardinal hypothesis, the core model is "close" to the universe $V$. These lemmas, together with the construction of the core models, become a major part of the fine structural inner model theory.

Fine structure yields substantial advantages in the power of our arguments, in both condensation and extendability. In the proofs of covering lemmas, we usually assume toward a contradiction, that we have a cofinal sequence in the least
counter-example $\tau = \kappa^+K$ to witness the counter assumption. Then we take out this sequence and collapse it in a smaller structure. Note that this small structure now contains some “bad” information from the counter assumption.

The rest is just to compare the smaller structure with some good ones, for example $K$. The coiteration always terminates until that one of the end-structure is an initial segment of the other. Soundness allows us to code the bad information of the small structure below $\kappa$, and Solidity preserves the standard parameter $p_M$ and all the fine structure notions. What’s happening here is that the bad information is coded into a small package and passed on to the end of one side of the coiteration, then carried by the other side, and all the preservation properties guarantee us that we could correctly decode the bad information that is passed back into the good structure. And this will result in a contradiction as desired.

We have seen that iterability plays a key role in such theory, and in fact the development of iterability is key to the development of constructions of larger core models. When the construction goes far beyond a strong cardinal, where linear iterations don’t apply, iteration tree can serve instead for a core model up to 1 Woodin cardinal. The corresponding construction and proof of weak covering lemma would follow almost the same structure. The advantage of using iteration tree is that sometimes when the extender cannot immediately apply to continue the iteration, we wait until it becomes applicable. By showing there is always a well-founded branch, we obtain stronger iterability and the construction goes further thereafter.
Bibliography


