DESIGN OPTIMIZATION OF PERMANENT MAGNET SYNCHRONOUS MOTORS USING RESPONSE SURFACE ANALYSIS AND GENETIC ALGORITHMS

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Summary

This thesis deals with the analysis and design of buried type Interior Permanent Magnet Synchronous Motors (IPMSM). The objective is to develop a design optimization procedure for this type of motors that is more accurate than the traditional analytical methods used in AC machine analysis, and less time consuming than the usual trial and error FEM based design procedure. The main design criterion is the expansion of the power capability of the machine till very high speeds while operating as a variable speed drive with a flux weakening scheme.

It is shown that the traditional circuit modelling of the IPMSM based on motor parameters cannot provide reliable field weakening predictions. Indeed, saturation effects are important for this kind of motors and make the motor parameters variable and current dependent. A new circuit modelling based on non-linear representation of the d- and q-axis fluxes by cubic spline interpolation is proposed as an alternative. It is shown that more reliable predictions of the constant power speed range and peak torque can be obtained from this non-linear circuit modelling.

The FEM plays an important role in the method as it is used to calculate the flux linkages at the interpolation points. Two different methods to calculate these flux linkages are investigated. It is shown that since the space harmonics of
the flux can be significant, these methods are not equivalent. The method able to isolate the fundamental of the flux is preferred over the other one.

In addition, the analytical torque equation used to predict the performance of the motor is validated by FEM computation. This strengthens the confidence in the non-linear circuit modelling of the IPMSM for power capability predictions.

Response Surface Method (RSM) and Finite Element Method (FEM) are combined to relate the d- and q-axis fluxes to the design variables. The power capability of the machine can then be predicted for any set of the design variables. Different types of designs of experiments (DoE), necessary to provide the experimental data to fit the RSM models, are compared. It is shown that DoEs that require many experiments don’t yield necessarily more accurate RSM models; the Central Composite Design is shown to be the best of the DoEs investigated since it allows fitting accurate RSM models prediction from a relatively low number of experiments.

The power capability predictions obtained from the d- and q-axis fluxes RSM models are checked by FEM to validate the RSM approach. The results are globally satisfactory.

Genetic algorithm is the optimization tool chosen to optimize the IPMSM. A simple yet efficient algorithm is developed and used to show that the constant power speed range (CPSR) can be increased without limit (under the no-loss assumption) but at the expense of the peak torque available below the base speed. It is also shown that an optimal design with an infinite $CPSR$ can be achieved for a particular set of the design variables. Its main characteristic is that the magnet flux can be cancelled by the armature reaction.
List of symbols

$\lambda_m$ permanent magnet flux linkage

$L_d$ d-axis inductance

$L_q$ q-axis inductance

$\lambda_d$ d-axis flux linkage

$\lambda_q$ q-axis flux linkage

$\Omega$ mechanical speed

$\omega$ electrical speed

$p$ number of pair of poles

$V_{a,b,c}$ phase voltages

$V_{d,q}$ d-q-axis voltage

$V_{max}$ maximum output voltage of the inverter

$R$ stator resistance

$I_{a,b,c}$ phase currents

$I_{d,q}$ d-q-axis current

$I_r$ rated current

$\beta$ current angle
$CPSR$ constant power speed range
$\Omega_1$ lower limit of the $CPSR$
$\Omega_2$ upper limit of the $CPSR$
$\Omega_b$ base speed
$\Omega_{max}$ maximum speed reachable
$P$ output power
$T$ torque
$T_{max}$ maximum torque available
$\beta_{T_{max}}$ current angle producing $T_{max}$
$MMF$ magnetomotive force
$\vec{B}$ flux density
$\vec{H}$ magnetic field
$\vec{A}$ vector potential
$\vec{J}$ current density
$W$ magnetic energy
$B_r$ remanent flux density
$H_c$ coercivity
$\nu$ reluctivity
$\mu_r$ relative permeability
$\mu$ permeability
$L_{stack}$ rotor length in the $z$-direction
\begin{itemize}
\item $N_{\text{phase}}$: number of turns per phase
\item $N_{\text{coil}}$: number of turns per coil
\item $l_m$: magnet thickness
\item $\delta$: magnet position
\item $\alpha$: magnet pole angle
\item $n$: number of flux interpolation points
\item $N$: number of experimental points
\item $CCD$: central composite design
\item $\text{chromLength}$: number of bits coding the chromosome
\item $\text{popSize}$: number of individuals in the GA population
\item $\text{genMax}$: total number of generations
\item $\text{mutRate}$: mutation rate
\item $P_{\text{xover}}$: crossover probability
\item $T_{\text{inf}}$: peak torque constraint
\item $CVWM$: Coulomb virtual work method
\end{itemize}
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Chapter 1

Introduction

1.1 Background

1.1.1 The development of Permanent Magnet Machines

The first designs of permanent magnet motors were attempted as early as the 19th century by J. Henry (1831), H. Pixii (1832), W Ricthie (1833) [1]. The main idea was to replace the electromagnetic excitation with a permanent magnet, a “free” source of magnetic field, in order to increase the efficiency of the system. However the poor quality of hard magnetic material at that time (steel, tungsten steel) strongly limited the power output of the machines, and finally discouraged these attempts. The invention of Alnico in 1934 by Bell laboratories, revived the interest in permanent magnet excitation. Their high flux density and reasonable energy product (Fig. 1.1) permitted their use in power applications. However their low coercive force (resistance to demagnetization) limited their use to relatively constant current application. The advent of ceramic, or “hard ferrite” generalized the use of permanent magnets in commercial and aerospace applications [3]. With a
high coercive force they were able to withstand the conventional levels of armature reaction without risk of demagnetization and quickly many automotive motors were converted to ferrite excitation (DC commutator motor).

Figure 1.1: Magnetic characteristics of the main class of permanent magnets [2]

Finally, the development of rare earth permanent magnets in the 60’s gave a significant advantage to permanent magnet excitation. The early rare earth magnets were alloy of Samarium and Cobalt (SamCo). They provided a flux density as high as the Alnico class with a coercive force even higher than the ferrite class, resulting in energy density levels never seen before. Their relatively high cost (large quantity of Cobalt needed) was their only drawback. The second generation of rare earth permanent magnets, made of neodymium iron and boron (NdFeB), was developed by Sumitomo and General Motors in 1984. With a much lower cost than the SamCo magnet and even better magnetic characteristics, they gave permanent magnet machines the potential to compete with conventional motors in many applications. A look at the evolution of the energy density (Fig. 1.2) of
modern magnets this last century allows understanding of the recent widespread use of permanent magnet machines.

![Magnetic Energy in Kilojoules/Cubic Metre for Each Material](image)

Figure 1.2: Evolution of the magnetic material during the 20th century [2]

### 1.1.2 Features of permanent magnet motors

The use of permanent magnet material in machine design brings the following benefits, regarding economic considerations:

- high efficiency: with a proper design, the efficiency of a permanent magnet motor is higher than any other type of rotating machines. Indeed, the field ohmic losses of wound field DC or synchronous machines are eliminated when using permanent magnets. The armature current is also lower than the excitation current drawn from the energy source by induction and reluctance machines. In a modern industrialized country where more than half the electrical energy is consumed by electrical drives [4], and where energy savings
become a must, permanent magnet motors have crucial advantages.

- simplification of construction and maintenance: the simplified assembly procedure of permanent magnet machine makes them more suitable for automated assembly techniques. Indeed the wound field coil assembly is a multi step process requiring complex machinery vulnerable to breakdown and needs maintenance. In addition, insulation damage to the coils are also not uncommon during the process. The machine assembly costs for permanent magnet motors are hence lower than most other kind of motors (except switch reluctance motor). The maintenance cost are also reduced by the use of permanent magnet excitation: brushes (in the brushless DC version) or slip rings (in the AC version) are eliminated, and with them the main cause of routine maintenance. Field coil insulation failures leading to emergency repairs also disappear.

The many economic advantages mentioned above do not mean that permanent magnet motors are necessarily cheaper than their wound field equivalent. Indeed the price of the permanent magnet material can be a significant part of the machine cost, especially for mass production; the benefit of high efficiency/lower running cost must be weighted against the higher initial investments. For this reason, it has often been considered that permanent magnet motors were interesting, from an economical point of view only in low power applications (fractional horsepower), where field ohmic losses represents a high part of the overall losses. For high power application, the efficiency of electromagnetic and permanent magnet excitation become so close such that the price of permanent magnet material may not be justified.
However, price of rare earth permanent magnet keeps on decreasing because of the growing production from China. As a direct consequence, the crossover point where permanent magnet excitation becomes economically preferable over electromagnetic excitation has risen from fractional horsepower to more than one hundred hp now [5]. In many cases, it can be even higher, and perhaps in the Mega-Watt range.

Regarding technical features, the unique characteristics of permanent magnet motors are

- a very high power to weight ratio due to the very high energy densities of modern permanent magnet material. Another direct consequence of the removal of field losses is that power losses are practically all in the stator where heat can be easily removed; the cooling system requirements are then reduced. These reasons make permanent magnet motors particularly suitable for automotive applications or battery powered portable appliances, HDD spindle motors where space and weight savings are the prime considerations.

- high dynamic performances. The first reason is the high level of flux density obtainable from the magnet. The second is the low inertia of a permanent magnet motor, much lower than that for a machine with a bulky wound field rotor. Permanent magnet machines are thus the best option for servo applications like robots, machine tools where a fast response of the drive is required.

- a great flexibility of shape. The permanent magnet motor can be constructed in a variety of unconventional sizes and shapes. A magnet with high residual
flux density permits for designing machines with a larger airgap. “Ironless stator” configurations are also possible; the magnetic material in the armature is removed resulting in weight savings. This results in interesting properties like lower cogging torque and also further simplifies the assembly procedure.

Nevertheless, besides being expensive, the use of permanent magnet introduces a few limitations. The first one is the possible demagnetization of the magnet. The magnet can be operated safely at any point on the linear part of its B-H characteristic. But if the flux density is reduced beyond the knee of the characteristic, a partial yet irreversible demagnetization occurs (1.3); after being subject to a large demagnetizing field from armature conductors, the new characteristic of the magnet is a straight line parallel to but lower than the original. Temperature also

![Figure 1.3: Partial demagnetization of a permanent magnet](image)

affects the flux output of the magnet. The magnet has to be properly protected, during the design stage of the motor and inverter, against excessive armature reaction (short circuit) or high temperatures (NdFeB magnet mainly).
The second limitation is the loss of field control, which is required in variable speed application to increase the operating speed range or when the efficiency has to be optimum for different speeds. Indeed, the flux level of the magnet cannot be varied unlike that for a wound field coil and the air-gap flux is thus a constant. This point is obviously a great shortcoming in traction applications where the light weight and smaller volume of the PM machine are of interest. However, this is no more the case for the modern permanent magnet synchronous machines (PMSM). The advancements in solid-state devices and micro-controllers during the last few decades have permitted the implementation of controllers able to control the air-gap flux via a proper armature reaction, and then emulate the principle of field control. For this reason, PMSMs have received a growing interest for possible applications in traction drives.

1.2 Permanent magnet synchronous motors

1.2.1 Structures and operating principles

As mentioned earlier, the flexibility given to the designer by the use of permanent magnets has led to the development of various types of PMSM. They all have in common a stator similar to an induction machine. They differ by the configuration of the rotor, especially the position of the magnets which divide these motors in two categories.

- The exterior PMSMs: for surface permanent magnet motors, the magnets are glued on the rotor surface, requiring a relatively large air-gap to be ac-
Figure 1.4: Different rotor structures of PMSM

 commodated. Therefore they present no saliency as the relative permeability of the magnet is close to unity. In the inset type motors, the magnets are set into rotor slots. This results in saliency: an additional reluctance torque is available.

- The interior PMSMs: in the “spoked” version, the magnets are inside the rotor core. They are circumferentially magnetized and alternatively poled, which means that their flux add together to create a high air-gap flux. This machine has little reluctance torque. In the buried version, the magnets are buried in the rotor and radially magnetized. Flux barriers are necessary to prevent the magnet from being magnetically short-circuited by the iron core. Because there is a high permeance to the q-axis flux and a low-permeance to the d-axis armature reaction flux, this machine has considerable reluctance torque and flux weakening capability, giving it the ability to maintain
constant power at high speeds.

The working principle is the same for all these configurations; the magnets have to create a sinusoidal or quasi-sinusoidal flux distribution in the air-gap. The conductors are distributed sinusoidally around the stator so that when fed by three sinusoidal current waveforms, they create a rotating magnetomotive force (MMF) which interacts with the magnet field to produce torque; the magnet field will try to “catch” this rotating armature field and therefore set the rotor into motion. Both magnet and armature fields have to rotate synchronously to produce a constant torque. A position encoder is thus needed to synchronize the phase currents with the rotor position. In practice, these current waveforms can be obtained from a pulse width modulated voltage source inverter shown in Fig. 1.5. The fixed frequency single line supply voltage is rectified to a DC link and a current feedback PWM scheme controls the switching pattern of the three legs of the inverter to produce the desired current waveforms.

![Figure 1.5: PWM voltage source inverter (Bang-Bang Control)](image)
1.2.2 Inherent design issues for the buried type IPMSM

The study of any electromagnetic device is based on Maxwell’s equations. Unfortunately, the different materials and non basic geometries involved make those equations difficult if not impossible to be solved. Generally an analytical expression of the flux distribution in the machine is unattainable. Another approach usually used in electric machine analysis is the circuital representation. Using appropriate assumptions (magnetic linearity, infinite permeability of the iron) and simplifications (Carter coefficient, winding factor...) with some estimations (leakage flux, saturation factor...), it is possible to model the two axis of symmetry of the machine, direct and quadrature, by the magnetic circuits as shown in Fig. 1.6.

\[ \lambda_d = \lambda_m + L_d I_d \]  \hspace{1cm} (1.1)

\[ \lambda_q = L_q I_q \]  \hspace{1cm} (1.2)

where \( \lambda_m \), \( L_d \) and \( L_q \) are the motor parameters, namely the permanent magnet...
flux linkage, the direct axis inductance and the quadrature axis inductance. With a rotor angular velocity of $\Omega$, the flux linkage $\lambda_d$ generates a voltage $-\omega \lambda_d$ in the q axis, where $\omega$ is the electrical speed, defined as

$$\omega = p\Omega \tag{1.3}$$

$p$ is the number of pair of poles of the machine. In a similar way, the flux linkage $\lambda_q$ generates a voltage $\omega \lambda_q$ in the direct axis. Finally, the whole machine is modelled by the electrical circuit as shown in Fig. 1.7.

![Electric equivalent circuits](image)

Figure 1.7: Electric equivalent circuits

An accurate modelling of the motor parameters is then crucial to obtain reliable predictions of the motor performance during the design stage.

Analytical tools like the magnetic representation of the motor are very useful tools at the initial stages of the design. They are simple to apply and readily repeatable. They provide sizing equations to give quickly first estimations of the design variables (rotor diameter, length, magnet thickness and width...)\[31\] to meet the design specifications. The calculation of the motor parameters are remarkably accurate for surface PMSM when the air-gap is large enough (no saturation involved) [8].
However, the situation is quite different for Interior PMSMs, mainly for two reasons:

- the rotor is not homogeneous. Indeed, the magnets and flux barriers, which have a relative permeability close to unity, are buried in the iron core. From the armature flux point of view, they constitute an additional air-gap lying in the direct axis. The shape of this air-gap makes the flux path much more complex than that for exterior PMSMs. Analytical techniques have been used to tackle the problem by resolving the air-gap into d-axis and q-axis effective air-gaps with different lengths [9], similar to that used for wound field synchronous machines. However, these techniques cannot always lead to good results and are not recommended in designing high performance IPMSM for modern electric drives [10].

- saturation is inherent to this kind of motor. Indeed, the q-axis air-gap is much smaller than for surface PMSM as it doesn’t contain the magnets. As a consequence, the q-axis becomes easily saturated when all the stator current is applied along the q-axis and \( L_q \) becomes current dependant. Classical analytical tools are unable to take saturation into account in modelling these variations of \( L_q \) as non-linearities are involved. In addition, the magnetic bridges between the flux barrier and the airgap (Fig. 1.8) have to be saturated to avoid a magnetic short-circuit of the magnet. The estimation of the flux leakage through this path is quite troublesome; under certain condition, the bridge become less saturated due to the action of the demagnetizing current [11]. This also results in variation of \( L_d \) and \( \lambda_m \).

Consequentially, although being useful to give coarse estimation of the main
dimensions of an IPMSM, classical analytical tools are unreliable and too basic when accurate models of the motor parameters of an IPMSM are needed.

Numerical methods are then a must to design this kind of machine. The most popular among machine designers is the Finite Element Method due to its great flexibility in modelling awkward shapes and non linear materials [12]. Unlike analytical tools, the field distribution throughout the machine can be obtained with great accuracy. The calculation of motor parameters through this technique have shown very close agreement with measurements and this is now a well established technique to deal with IPMSM [54]. However, as a numerical tool, FEM doesn’t provide any relationship between the motor parameters and the design variables. This is the main shortcoming of the method, since this is by nature an analysis tool rather than a design tool. The designer has no choice but to follow a tedious trial
and error procedure which is time-consuming and even then without a guarantee of convergence. The method is thus ill-suited for optimization purposes.

1.3 Literature survey

As suggested before, Permanent Magnet Motors can offer many advantages in traction applications where efficiency, power to weight ratio and size are the prime considerations. For this, many researcher have investigated the field weakening capability of these motors in order to increase the range over which the rated power can be maintained (constant power speed range). The control laws for constant torque and flux weakening operation were first described in [14] and [15] by T.M. Jahns. It was shown that the current phasor has to follow an optimum trajectory in the $I_d - I_q$ plane to achieve maximum torque and power at each speed, respecting the current and voltage limitations of the motor and the inverter. This trajectory was shown to be a strong function of the motor parameters. However, the effects of varying the motor parameters on the system performances were not really shown. In [17], it was reported that the buried version of the IPMSM offered an increase in power capability over other PMSM configurations. In [18] and [11], Schiferl and Lipo studied the effects of the motor parameters on the power capability. They were the first to show the optimal field weakening design criterion $\lambda_m = L_d I_{rated}$, to obtain a theoretical infinite constant power speed range. Morimoto et al [19] extended the analysis of Jahns and Schiferl by showing that there exist two categories of “designs”; according to whether $\lambda_m$ is higher or lower than $L_d I_{rated}$, the optimum current trajectory differs and the maximum speed can be finite or infinite. Soong and Miller [20] [21] investigated the effects of the motor parameters
on high speed torque production and identified all the combinations that meet the optimal field weakening criterion. Some insights were also given about the consequences of saturation on the drive performance, as the machine parameters vary.

In the meantime, many authors proposed methods to compute the values of these motor parameters, mostly by FEM for the reasons described previously. Pavlik et al [12] estimated the motor parameters with only one axis current imposed at once, neglecting d-q-axis cross coupling effects due to iron saturation. Rahman and Zhou [22] [23] proposed a “frozen permeability” method to fully take into account the saturation. This method allowed for the computing of the motor parameters for a given operating point of the motor. Chang [24] also described a “current perturbation” method accounting for saturation but sensitive to rounding errors. Using these methods, the authors have shown that motor parameters were indeed highly dependant on the value of the stator current. These methods were thus of practical interest when coupled with the circuit modelling of the machine. Accurate predictions of the motor behavior could be obtained as current dependent motor parameters were used. However, the computational cost was relatively high as a new FEM calculation of the motor parameters had to be done at each time-step. Bianchi and Bolognani [25] proposed a simpler model of the motor parameters for design purposes. $\lambda_m$ and $L_d$ were assumed constant while $L_q$ could take two values according to the d-axis current. Chen [26] suggested a similar but slightly more complicated model. However the accuracy of both methods has not really been demonstrated to predict performances such as maximum torque or flux weakening capabilities.
Some design procedures were proposed to achieve wide operating speed range. Slemon [27] (for surface mounted PMSM) Ionel et al [8] and Liu [28] [29] (for IPMSM) proposed design methods based on analytical sizing equations to obtain the main dimensions of the machine. The actual values of the motor parameters as well as the final performances of the IPMSM were checked at the end by FEM computations. Miller developed a similar but computer aided procedure [32]. An initial dimensioning program using simple equations allowed a fast design to be refined by FEM tools. The bulk of these proposed design procedures relied on classical analytical tools, FEM being used at the end to check the performance and possibly adjust the design.

Different approaches centered on FEM were proposed for PMSM design and optimization. Bianchi [35], Chung [37] and Sim [38] combined FEM and Genetic Algorithms to optimize various motor performances like torque, efficiency, cogging torque. These methods have been used in the past to optimize, with some success, many kinds of electromagnetic devices [34]. While offering excellent accuracy, they require many FEM simulations which can lead to days of computations. Manella et al [40] and Rong et al [41] proposed to combine Response surface Method (RSM) with FEM to model electromagnetic devices. From few FEM experiments, RSM builds an empirical model relating the performances of the machine with the design variables. Traditional optimization techniques based on gradient or GA can then work on this analytical model for a faster optimization. Gillon et al applied this method extensively [42]- [47] to optimize the mean value of the back-emf of a brushless DC motor. Li et al [48] used it to minimize the cogging torque of the same motor. Finally Liu and Jabbar [39] employed this method to model the motor parameters of an IPMSM and finally optimize the operating speed range of
the machine.

1.4 Motivations

To summarize, a large part of the research on the IPMSM was focused on its analysis by numerical methods like FEM in order to calculate the motor parameters or simulate the behavior of the motor. With regard to the design target to achieve wide speed range, most methodologies were based on analytical tools and motor parameters, even though it has been reported that saturation strongly influences the values of these parameters. FEM was used only at the end of the design process to verify that the motor can meet the requirements and to analyze its performances.

This thesis has focused on the design of high field weakening capabilities IPMSMs (buried type), considering the saturation issues as an integral part of the design process. Saturation effects are indeed a challenge in high-speed design of IPMSM in the sense that the traditional linear circuital representation of the motor cannot be used for accurate prediction of the motor performances, since direct and quadrature axis inductances are not constant; the saturation level of the motor varies on the whole speed range of the motor as the current vector varies, being possibly high at low speed and likely low (even zero) at high speed because of flux weakening. As a result, using constant motor parameters in a linear model cannot lead to accurate performance predictions both at low speeds AND at high speeds. To remedy this problem, a new circuit modelling of the motor accounting for the variation of the saturation level is proposed; the traditional motor parameters are given up for a non-linear modelling of the d- and q-axis flux linkages. Analytical
models of the flux linkages are obtained as functions of the current angle from cubic spline interpolation and FEM computations. As a result, the influence of saturation is included in the circuit modelling of the machine which gives more reliable predictions of the motor power capability.

Response Surface Method is then combined with Finite Element Method to relate these flux linkages to the design variables. Empirical models of the flux interpolation points are obtained by regression from several FEM simulations. Finally, using these models with the non-linear circuit modelling allows for obtaining the power capability and field weakening performances of the IPMSM for any set of values of the design variables.

Genetic Algorithms are then chosen to work with these models to optimize the constant power speed range of the IPMSM, considering some geometric and performance constraints.

1.5 Structure of the thesis

The thesis is organized as follows:

- Chapter 2 presents the traditional linear model of the IPMSM, using constant motor parameters to relate the flux to the current. The concepts of power capability and field weakening are explained. The optimum current trajectory to obtain maximum torque from the motor at each speed is described as well as the way it is calculated for the linear model. It is then shown why this linear model of the flux is not accurate for field weakening predictions. To remedy
this problem, a non linear model of the machine is proposed. Cubic spline interpolation gives analytical models of the d-q- flux linkages at rated current, as functions of the current angle. The interpolation points are obtained from FEM computations. Finally, the way to calculate from these models the constant power speed range and other quantities (maximum torque, base speed...) is described.

- In Chapter 3, the Finite Element Method is first introduced, with its mathematical principles and the simulation procedure from the user viewpoint. Then the modelling of the IPMSM by FEM is described, especially the simplifications made to reduce the computational time, as well as the materials and geometry used. Different ways to compute the flux linkages are considered and compared. Then, the validity of the circuit modelling approach is investigated; the torque obtained by the formula \( T = \frac{3}{2}\pi (\lambda_{d}i_{q} - \lambda_{q}i_{d}) \) is compared with the torque obtained from FEM computations (Coulomb Virtual Work Method). Close agreements can be observed.

- Chapter 4 is on Response Surface Method. The background and procedure to apply RSM are first described. Then the method is applied to our problem and the choice of the design variables are explained. From FEM simulations and regression, second order models are built to express each of the flux interpolation points as a function of the three design variables of interest. The FEM method described in the previous chapter are employed to provide the experimental data. The “efficiency” of various designs of experiment for the choice of the experimental points are compared. The accuracy of the field weakening predictions obtained from the models is finally checked. The results are very satisfactory.
• The optimization of the IPMSM for wide speed operation is carried out in Chapter 5. A genetic algorithm uses the RSM models and the non linear circuit model of the IPMSM. The genetic algorithm is described and the choice of the genetic operators and the parameters, like mutation probability or crossover probability that are used, is explained. Optimizations of the constant power speed range for different rated torque constraints are performed; the results are compared with well-known results valid for the linear model of the motor.
Chapter 2

IPMSM and wide speed range operation

2.1 Introduction

The analysis of the field weakening capabilities of IPMSM, in the late 80’s and early 90’s, have always been carried out via the linear lossless model of the motor. The hypothesis of no saturation, translated by constant motor parameters, were always reported as not realistic. But, the point was to give some insight on the way to control the armature current to exploit to the maximum the potential of the motor. For the same reasons, this model and the field weakening analysis will be presented in the first two parts of the chapter. It provides the basics to understand why the alternative model, proposed in the later part of the chapter, can predict the power capability of the machine. The main point of this chapter is the saturation issue.
2.2 The linear model of the IPMSM

The IPMSM presents a large saliency (around 2 or 3 for the buried type) and is therefore traditionally analyzed in the d-q-reference frame, using Park transformation. The basic idea is to transform all the time varying quantities (voltages, currents, flux linkages...) into constant ones, by the choice of a proper reference frame (d-q), in order to simplify the analysis of the machine. This reference frame is defined by the two axes of symmetry of the rotor and rotates at the electrical angular velocity $\omega$; the polar axis is called direct (d-) axis and the interpolar axis, leading the d-axis by 90 electrical degrees, is called quadrature (q-) axis (Fig 2.1). The expression of the quantities in this new coordinate is given by the Park trans-
formation:

\[
\begin{pmatrix}
S_d \\
S_q \\
S_0
\end{pmatrix}
= \frac{2}{3}
\begin{pmatrix}
\cos \theta & \cos (\theta + \frac{2\pi}{3}) & \cos (\theta + \frac{4\pi}{3}) \\
\sin \theta & \sin (\theta + \frac{2\pi}{3}) & \cos (\theta + \frac{4\pi}{3})
\end{pmatrix}
\begin{pmatrix}
S_A \\
S_B \\
S_C
\end{pmatrix}
\] (2.1)

where \( \theta \) is the position of the d-axis with respect to the phase A axis and \( S_A, S_B \) and \( S_C \) are any three phase quantities (voltage, current, flux linkage).

Provided that

\[
\begin{align*}
S_A &= S \cos (\omega t + \gamma_0) \\
S_B &= S \cos (\omega t + \gamma_0 + \frac{2\pi}{3}) \\
S_C &= S \cos (\omega t + \gamma_0 + \frac{4\pi}{3})
\end{align*}
\] (2.2)

then \( S_d \) and \( S_q \) are independent of \( t \) (just functions of \( S \) and \( \gamma_0 \)) and \( S_0 = 0 \).

Neglecting iron losses and considering only the first harmonic of all quantities, the following steady-state equations can be written

\[
V_d = RI_d + \frac{d\lambda_d}{dt} - \omega \lambda_q
\] (2.3)

\[
V_q = RI_q + \frac{d\lambda_d}{dt} + \omega \lambda_d
\] (2.4)

where

- \( V_d \) and \( V_q \) are the d- and q-axis voltages
- \( I_d \) and \( I_q \) are the d- and q-axis currents
- \( \lambda_d \) and \( \lambda_q \) are the d- and q-axis flux linkages
- \( R \) is the armature resistance.

The magnetic field in the machine has two sources; the permanent magnet and the armature current. Assuming magnetic linearity, it is possible to express the d- and q-axis flux linkages as:

\[
\lambda_d = \lambda_m + L_d I_d
\] (2.5)
\[ \lambda_q = L_q I_q \] (2.6)

where the constants \( \lambda_m, L_d \) and \( L_q \) are the motor parameters, the permanent magnet flux linkage, direct axis and quadrature inductances, respectively. Substituting 2.5 and 2.6 into 2.3 and 2.4 we obtain the voltage equations of the linear IPMSM:

\[ V_d = R I_d + L_d \frac{dI_d}{dt} - \omega L_q I_q \] (2.7)

\[ V_q = R I_d + L_d \frac{dI_d}{dt} + \omega L_d I_d + \omega \lambda_m \] (2.8)

The machine can be represented by the electrical circuit shown on Fig 2.2.

![Electrical Circuit Diagram](image)

Figure 2.2: Definition of the direct and quadrature axis

The torque equation can be obtained from the energy conservation principle:

\[ P_{\text{input}} = \frac{3}{2} (V_d I_d + V_q I_q) \] (2.9)

Substituting \( V_d \) and \( V_q \) by (2.7) and (2.8), we finally obtain:

\[ P_{\text{input}} = \frac{3}{2} \omega (\lambda_m I_q + (L_d - L_q) I_d I_q) + \frac{3}{2} R (I_d^2 + I_q^2) + \frac{3}{4} \left( L_d \frac{dI_d^2}{dt} + L_q \frac{dI_q^2}{dt} \right) \] (2.10)
where one can identify the power developed by the machine \( (T\Omega) \), the copper losses \( (RI^2) \) and the rate of change of the magnetic energy stored in the machine \( (L\frac{dI}{dt}) \).

As the electrical speed \( \omega \) and mechanical speed \( \Omega \) differ by a factor \( p \) (the number of pairs of poles), the expression of the torque \( T \) is:

\[
T = \frac{3p}{2} \{\lambda_m I_q + (L_d - L_q)I_d I_q\}
\]

This torque consists of two components:

- the main torque, \( T_{al} = \frac{3p}{2} \lambda_m I_q \), is called the alignment torque and results from the interaction of the magnet flux and the quadrature axis current.

- the reluctance torque \( T_{rel} = \frac{3p}{2} (L_d - L_q)I_d I_q \) is the result of the saliency of the machine. As \( L_d < L_q \) for the IPMSM, \( I_d \) must be negative to benefit from this torque.

This additional torque is very helpful at high speeds as shown later. It is important to note that as \( I_d \leq 0 \), the d-axis armature flux opposes the magnet flux (2.5) and is therefore called demagnetizing current; \( I_d \) is useful for flux weakening purpose.

### 2.3 Expansion of the operating speed-range of the motors

A typical power and torque profile for traction application is shown in Fig. 2.3. The torque has to be maximum for a fast acceleration at low speed, which defines the constant torque region. At higher speeds, the torque is allowed to decrease with
the speed as only constant power is required \((T = P/\Omega)\); this is the field weakening region. It is desired that this region is as wide as possible to permit high speed operation, but is limited by the maximum voltage and current of the inverter and motor. The field weakening capabilities of a motor can be judged from its power capability, a plot of the maximum power (or torque) available at each speed. The current trajectory to achieve this will be described using a lossless linear model of the motor.

Figure 2.3: Typical torque-speed and power speed profile for traction application capability, a plot of the maximum power (or torque) available at each speed. The current trajectory to achieve this will be described using a lossless linear model of the motor.

### 2.3.1 The linear lossless model of the IPMSM

While exploring the high-speed operating characteristics of the IPMSM, we ignore the effects of the stator resistance \(R\) since the associated voltage drop is small compared to the reactive voltage and the bak-emf. The steady-state voltage equations
for the lossless linear IPSM becomes:

\[ V_d = -\omega L_q I_q \]  

(2.12)

\[ V_d = \omega L_d I_d + \omega \lambda_m \]  

(2.13)

while the torque equation remains unchanged:

\[ T = \frac{3p}{2} \{ \lambda_m I_q + (L_d - L_q) I_d I_q \} \]  

(2.14)

The optimum current trajectory that achieves the maximum power at all speeds of the motor is traditionally visualized in the \( I_d - I_q \) plane (Fig. 2.4). As mentioned before, \( I_d \) has to be negative and \( I_q \) positive to obtain positive alignment and reluctance torques. From a flux point of view, the d-axis armature flux \( L_d I_d \) is always opposing the magnet flux \( \lambda_m \) (Fig. 2.4bis). For this reason the current vector is kept in the second quadrant of the plane, but with the following limitations.
2.3.2 Voltage and current limitations

The current that can be carried by the armature in steady state is limited to a rated value $I_r$ to ensure a safe thermal loading of the motor. It follows that the $d$- and $q$-axis currents must obey the constraint:

$$I_d^2 + I_q^2 < I_r$$  \hspace{1cm} (2.15)

which defines a disk of radius $I_r$ centered on (0,0).

The voltage limit $V_{max}$ is decided by the available maximum output voltage of the inverter:

$$V_d^2 + V_q^2 < V_{max}$$  \hspace{1cm} (2.16)

and can be written as

$$(\lambda_m + L_d I_d)^2 + L_q I_q^2 \leq \left(\frac{V_{max}}{\omega}\right)^2$$  \hspace{1cm} (2.17)

which defines an ellipse centered on $\left( -\frac{\lambda_m}{L_d}, 0 \right)$ and whose size decreases with the speed.
Obviously, the vector current satisfying the current limit and voltage limit must be inside the current-limit circle and voltage-limit ellipse.

In this thesis, only motors such that $\lambda_m \geq L_d I_r$ are considered. This means that the armature flux cannot be stronger than the permanent magnet flux. This is generally true when rare earth magnets are used as they exhibit an extremely large remanent flux density ($B_r \simeq 1.15T$). The center of the voltage-limit ellipse is then outside the current-limit circle (Fig. 2.5).

![Figure 2.5: Position of the voltage-limit ellipse center](image)

The power capability of the IPMSM can be divided in two regions: the constant torque region and the constant power region. The positioning of the current vector to obtain maximum power at a given speed will be different according to which region the speed is located in.
2.3.3 Constant torque region

This is the speed range over which the motor can deliver its peak torque.

For the sake of convenience, let us express the current vector in the polar coordinate system. The current vector $\overrightarrow{I_{T_{\text{max}}}}$ producing the peak torque $T_{\text{max}}$, respecting the current constraint, can be derived \[14\] from (2.14):

$$\overrightarrow{I_{T_{\text{max}}}} = I_r \angle \beta_{T_{\text{max}}}$$  (2.18)

where

$$\beta_{T_{\text{max}}} = \arcsin\left\{ \frac{-\lambda_m + \sqrt{\lambda_m^2 + 8(L_d^2 - L_q^2)^2}I_r}{4(L_q - L_d)I_r} \right\}$$  (2.19)

This vector is represented by the point B on the current-limit circle (Fig. 2.6). In the constant torque region, this is where the current vector is positioned.

For this region:

- since the current vector is kept constant and equal to $I_r \angle \beta_{T_{\text{max}}}$, it produces a constant torque $T_{\text{max}}$ and a constant flux $\lambda_{T_{\text{max}}}$. These torque and flux values are obtained by substituting (2.18) into the torque and flux equation

$$T_{\text{max}} = \frac{3p}{2} I_r \{ \lambda_m \sin \beta_{T_{\text{max}}} + (L_d - L_q) I_r \cos(\beta_{T_{\text{max}}}) \sin(\beta_{T_{\text{max}}}) \}$$  (2.20)

$$\lambda_{T_{\text{max}}} = \sqrt{(\lambda_m + L_d I_r \cos(\beta_{T_{\text{max}}}))^2 + (L_q I_r \sin(\beta_{T_{\text{max}}}))^2}$$  (2.21)

- the voltage and power increase linearly with the speed as flux and torque are constant ($V = \omega \lambda_{T_{\text{max}}}$ and $P = T_{\text{max}} \Omega$).

The speed at which the terminal voltage reaches the inverter limit $V_{\text{max}}$ is called
Figure 2.6: Optimum current trajectory that achieves maximum power at each speed base speed $\Omega_b$.
\[
\omega_b = \frac{V_{\text{max}}}{\lambda T_{\text{max}}} \quad (2.22)
\]
In geometric terms, the voltage limit ellipse has shrunk with the speed and finally reached point B (Fig. 2.6); the voltage limitation prevent the speed to increase further unless the flux is weakened. We enter the flux weakening region.

### 2.3.4 Flux weakening region

By increasing the current angle $\beta$ above $\beta_{\text{max}}$,

- the negative d-axis current (demagnetizing current) increases and weakens the magnet flux therefore reducing the net d-axis flux $\lambda_d$ (from equation (2.5))
• the q-axis armature current also decreases and with it the q-axis flux $\lambda_q$ (from equation (2.6)).

The total flux linkage $\lambda = \sqrt{(\lambda_d)^2 + (\lambda_q)^2}$ is reduced and the speed can then increase in inverse proportion while the voltage remains equal to $V_{\text{max}}$

$$\omega = \frac{V_{\text{max}}}{\lambda}$$

(2.23)

As a counterpart, the torque has to decrease as the condition (2.18) is no more satisfied. It is kept as high as possible by maintaining the current at its maximum value $I_r$.

Regarding the trajectory in the $I_d - I_q$ plane, the current vector moves counterclockwise on the rated current circle as the speed increases (Fig. 2.6). At any speed, the current vector $\vec{I}$ corresponds to the intersection of the voltage-limit ellipse and the current-limit circle; the armature current and terminal voltage are constant and maximum. The maximum speed attainable depends on the flux weakening capability of the motor. The total flux linkage is minimum when all the armature current is applied on the d-axis ($I_d = -I_r$ and $I_q = 0$). That is for a current vector equal to $I_r \angle 180^\circ$ (point C on Fig. 2.6, called minimum flux point for this reason). The corresponding electrical speed is

$$\omega_{\text{max}} = \frac{V_{\text{max}}}{\lambda_m - L_d I_r}$$

(2.24)

The speed cannot be increased beyond $\Omega_{\text{max}}$ because the flux cannot be weakened any further. Geometrically, the voltage-limit ellipse and current-limit circle are tangent at point C. For larger speed, they are disjointed; no current vector is achievable. At $\omega_{\text{max}}$, the torque and power have dropped to zero as $I_q$ is zero (2.14).

The power, torque, voltage and current angle profile are plotted against the speed in Fig. 2.7 to illustrate all what has been said.
An interesting quantity in the evaluation of the field weakening capability of the machine is the speed range \([\Omega_1; \Omega_2]\) over which the rated power of the machine can be maintained. To eliminate the units and avoid the possible confusion of electrical speed/mechanical speed, we shall call constant power speed range \(CPSR\) the ratio of those two speeds:

\[
CPSR = \frac{\Omega_2}{\Omega_1}
\]  

(2.25)

There exists no analytical expression of the constant power speed range, so it has to be calculated numerically.

Figure 2.7: Operation on the optimum current trajectory

The power capability is determined by the values of \(\lambda_m, L_d\) and \(L_q\). From (2.24), a special case appears when

\[
\lambda_m = L_d I_r
\]  

(2.26)

The maximum speed is theoretically infinite as the magnet flux can be totally weakened by the armature flux. Point C hence becomes a zero-flux point. Geomet-
rically, the voltage-limit ellipse center lies on the current-limit circle at point C, which is thus always within the voltage and current limit constraints. In addition, the power keeps on increasing with the speed when the current vector approaches asymptotically the point C (Fig. 2.8). The constant power speed range $CPSR$ is then infinite. Designs achieving the condition (2.26) have therefore been referred to as optimal designs.

Figure 2.8: Optimal design

### 2.4 Effect of saturation

Let us recall the models of the flux linkage of section 2.2

\[ \lambda_d = \lambda_m + L_d I_d \]  \hspace{1cm} (2.27)

\[ \lambda_q = L_q I_q \]  \hspace{1cm} (2.28)
These models of the flux linkages imply that:

- the d- and q-axis armature flux $L_d I_d$ and $L_q I_q$ are proportional to their respective axis currents.
- the total d-axis flux linkage is the addition of the permanent magnet flux $\lambda_m$ and the flux $L_d I_d$ produced by the demagnetizing current $I_d$. The q-axis flux is produced exclusively by the q-axis current.

This is a consequence of the magnetic linearity assumption which permits to say that

- the magnetic field produced by a current is proportional to the value of this current
- the field resulting from two different sources A and B is equal to the addition of the field obtained from the source A acting alone and the field obtained from the source B acting alone (superposition principle).

This assumption is true when the material used in the design exhibit a linear B-H characteristic. Vacuum (air-gap and flux barrier), copper (conductors) and rare earth permanent magnets are such materials: their relative permeability $\mu_r$ is a constant and therefore their B-H curves are straight lines. Steel or iron however have more complex magnetic characteristics (Fig. 2.9) which can be divided into two parts. For low values of the magnetic field the relative permeability $\mu_r$ of the material is nearly constant and extremely high; the material opposes little resistance to the flux. This is the linear part. For higher values of H, the permeability
decreases dramatically. As a result, an increase in H doesn’t result in a significant increase in B and the B-H curve becomes flat. This is the non-linear part. A good motor is usually designed so that the iron “works” in the linear part just before the knee of its characteristic; its magnetic capabilities are fully used as it carries the maximum flux density it can, at the limit of saturation. In that case, the magnetic linearity assumption is acceptable and the models of the flux linkages (2.27) and (2.28) are valid.

The case of the IPMSM is quite different due to the reasons explained in the introduction. The relatively thin q-axis air-gap (equal to the physical air-gap) is the main reason for saturation in the motor as it offers a low reluctance path to the flux. Large variations of $I_q$ make the iron operate in the non linear portion of its characteristic. The back-iron and teeth, as flux focusing region, will then be highly saturated. Large variations of $I_d$ have less consequence as the effective d-axis air-gap (physical air-gap and magnet width in series) is large.
For this reason, the saturation level is a function of the current flowing in the winding, but also depends on how this current is distributed between $I_d$ and $I_q$. Indeed, the saturation is maximum if all the armature current is applied on the q-axis ($I_q = I_r$, $I_d = 0$); the q-axis flux is maximum as well as the net d-axis flux (no d-axis armature flux is weakening the magnet flux). On the contrary, if all the current is applied along the d-axis ($I_d = -I_r$, $I_q = 0$), the flux is minimum and no saturation occurs. The saturation level is therefore a function of the amplitude $I$ of the current vector, but also of its angle $\beta$.

To illustrate this, several FEM simulations have been carried out to measure the flux linkages at rated current, but for different values of the current angle $\beta$. This is typically how the current is varied during the flux weakening operation. The variations $\lambda_d$ (2.10) and $\lambda_q$ have been plotted against their respective axis current $I_d$ and $I_q$ to check the distortions with respect to the models (2.27) and (2.28). As expected, for high values of $I_q$, a knee in the $\lambda_q$-$I_q$ curve appears and the linearity is lost. The $\lambda_d$-$I_d$ curve shows much more linearity.

From the same simulations, the motor parameters have been computed considering the magnet flux as constant. The plot of $\lambda_m$, $L_d$ and $L_q$ against $\beta$ on Fig. 2.12 gives the confirmation that they cannot be considered as constant. Both $L_d$ and $L_q$ drops when $\beta$ approaches 90° since the flux level reaches a high value. The relative variations are around 10%.

It has been shown in section 2.3 that the power capability and flux-weakening predictions were strong functions of the motor parameters. To evaluate the consequence of using constant motor parameters in the model, the power capability of the motor has been plotted for two sets of values of the motor parameters (Fig. 2.13).
The dashed line is obtained for the values of the motor parameters at $I = I_r$ and $\beta = 180^\circ$ (“least saturated” values) and the solid line is obtained for the values of the motor parameters at $I = I_r$ and $\beta = 90^\circ$ (“most saturated” values).
The linear model of the motor based on linear flux linkage-current characteristics, though very convenient for analysis, is of little value when it comes to power capability predictions. This was reported many times in the literature [15] [20].

### 2.5 Non-linear model of the motor

As a matter of fact, a model of the motor able to take into account saturation effects requires a non-linear model of the flux linkages to be used in the equation (2.3) and (2.4).
Figure 2.13: Influence of the saturation on the power capability prediction

2.5.1 Model of the flux linkages

Sticking to the concept of motor parameters is of little interest as it leads to make them current dependent:

$$\lambda_d = \lambda_m(I_d, I_q) + L_d(I_d, I_q)I_d$$  \hspace{1cm} (2.29)

$$\lambda_q = L_q(I_d, I_q)$$  \hspace{1cm} (2.30)

The use of current dependent motor parameters makes them more complicated.

We shall instead model the flux directly from the currents

$$\lambda_d = f_d(I_d, I_q)$$  \hspace{1cm} (2.31)

$$\lambda_q = f_q(I_d, I_q)$$  \hspace{1cm} (2.32)
One option is to use FEM to build a look-up table storing the values of $\lambda_d$ and $\lambda_q$ for different values of $I_d$ and $I_q$. But this would result in many simulations being carried out as the flux has to be known on the whole $I_d - I_q$ plane. As a result, using this method to evaluate the performance of the motor in a design process would be too tedious as a new look-up table would have to be built at each iteration. The extremely high computational cost is definitely not acceptable.

However, if the main purpose of the model is to predict the power capability of the machine, then the flux linkage need not be known on the whole $I_d - I_q$ plane but only on one quarter of the rated current circle. Indeed, it has been shown in section 2.3 that the optimum current trajectory to achieve the maximum power at each speed is on the second quadrant of the rated current circle. By shifting to polar coordinates, the models of the flux linkages needed would be:

$$
\lambda_d|_{I=I_r} = f_d(\beta) \tag{2.33}
$$

$$
\lambda_q|_{I=I_r} = f_q(\beta) \tag{2.34}
$$

with $\beta$ being in the range $[90^\circ, 180^\circ]$

The method consists of computing $\lambda_d$ and $\lambda_q$ by FEM at $n$ equally spaced points on this quarter of the circle (Fig. 2.14). We obtain $n$ values of $\lambda_d$ and $\lambda_q$:

$$
\begin{cases}
\lambda_d, = \lambda_d|_{I=I_r, \beta=\beta_i} \\
\lambda_q, = \lambda_q|_{I=I_r, \beta=\beta_i}
\end{cases} \text{ for } i = 1..n \tag{2.35}
$$

with

$$
\beta_i = 90 + (i - 1) \frac{90}{n - 1} \tag{2.36}
$$

The axis-fluxes on the rated current between these “sampling points” are then interpolated using cubic spline interpolation to get the function $f_d(\beta)$ and $f_q(\beta)$. 
2.5.2 The cubic spline interpolation

The fundamental idea behind cubic spline interpolation is to use 3\textsuperscript{rd} degree polynomials to draw a smooth curve through a number of points. The coefficients of these polynomials are chosen to prevent erratic behavior and break in continuity of the curve between the points. For this, \( f_d(\beta) \) and \( f_q(\beta) \) are made into piecewise functions of the form

\[
 f(\beta) = \begin{cases} 
 f_1(\beta) & \text{if } \beta_1 \leq \beta < \beta_2 \\
 f_2(\beta) & \text{if } \beta_2 \leq \beta < \beta_3 \\
 & \vdots \\
 f_{n-1}(\beta) & \text{if } \beta_{n-1} \leq \beta < \beta_n 
\end{cases}
\] (2.37)

where \( f_i \) is a third order polynomial

\[
 f_i(\beta) = a_i(\beta - \beta_i)^3 + b_i(\beta - \beta_i)^2 + c_i(\beta - \beta_i) + d_i
\] (2.38)

for \( i = 1, 2, \ldots, n - 1 \)

The spline needs to meet the following requirements:

- The piecewise function \( f(\beta) \) must go through all the data points \( \{\lambda_d_i\}_{i=1..n} \)
for \( f_d \), \( \{\lambda_q\}_{i=1..n} \) for \( f_q \)

- \( f(\beta) \) must be continuous over the interval \([\beta_1, \beta_n] = [90^\circ, 180^\circ]\).
- \( f'(\beta) \) must be continuous over the interval \([\beta_1, \beta_n]\)
- \( f''(\beta) \) must be continuous over the interval \([\beta_1, \beta n]\)

The continuity of the function and its first and second derivatives guarantee the smoothness of the curve. These properties will also be extremely valuable later to calculate the field weakening characteristics like \( \beta_{T_{\text{max}}} \), \( T_{\text{max}} \) using the Newton-Raphson algorithm. The mathematical process to build such a spline is explained in Appendix A.

The results are shown in Fig. 2.15 for 15 interpolation points. As the flux linkage is naturally a smooth function of the current angle at constant current amplitude (nearly sinusoidal), it can be appropriately represented by the cubic splines. The choice of 15 interpolation points gives a step-size or \( \beta \) around 5 electrical degrees.

### 2.5.3 Determination of the power capability for this non-linear model

For the sake of convenience, let us rename the function \( f_d(\beta) \) as \( \lambda_d(\beta) \) and \( f_q(\beta) \) as \( \lambda_q(\beta) \), keeping in mind that these are the flux linkages at rated current.

The new steady state model of the lossless motor operating at rated current \( I_r \) is then the following:

\[
V_d = -\omega \lambda_q(\beta)
\]  \hspace{1cm} (2.39)
Figure 2.15: Flux linkages interpolations

\[ V_q = \omega \lambda_d(\beta) \]  

\[ T = \frac{3p}{2} I_r (\lambda_d(\beta) \sin(\beta) - \lambda_q(\beta) \cos(\beta)) \]  

It is now necessary to propose a new method to calculate the field weakening characteristics like \( \beta_{T_{\text{max}}}, T_{\text{max}}, \Omega_b, \Omega_{\text{max}} \) as the traditional equations described in section 2.3 cannot be used.

Regarding the constant torque region, the peak torque \( T_{\text{max}} \) of the motor as well as the current angle \( \beta_{T_{\text{max}}} \) can be found by setting the derivative of (2.41) to zero. As \( \beta \) is not expressed in radians but in electrical degrees

\[ \frac{dT}{d\beta} = \frac{3p}{2} I_r \left\{ \lambda_d'(\beta) \sin(\beta) + \lambda_d(\beta) \frac{2\pi}{360} \cos(\beta) - \lambda_q'(\beta) \cos(\beta) + \lambda_q(\beta) \frac{2\pi}{360} \sin(\beta) \right\} \]  

(2.42)
We define the quantity in the brackets as \( f(\beta) \). After rearrangement, 

\[
 f(\beta) = \sin(\beta) \left( \lambda_d'(\beta) + \frac{\pi}{180} \lambda_q'(\beta) \right) + \cos(\beta) \left( \lambda_d(\beta) - \lambda_q'(\beta) \right) 
\] 

(2.43)

The current angle \( \beta_{\text{max}} \) maximizing the torque is the solution of the equation

\[
 f(\beta) = 0 
\] 

(2.44)

This equation has obviously no analytical solution and we shall solve it numerically instead, using the Newton-Raphson algorithm. This algorithm proceeds by iteration to find the zero of a function; the idea is to approximate the function \( f \) at a given point \( \beta_{[0]} \) by its first order Taylor expansion,

\[
 f(\beta) \approx f(\beta_{[0]}) + f'(\beta_{[0]})(\beta - \beta_{[0]}) 
\] 

(2.45)

and find the zero \( \beta_{[1]} \) of this approximation:

\[
 f(\beta_{[0]}) + f'(\beta_{[0]})(\beta - \beta_{[0]}) = 0 
\] 

(2.46)

\[
 \beta_{[1]} = \beta_{[0]} - \frac{f(\beta_{[0]})}{f'(\beta_{[0]})} 
\] 

(2.47)

The process is repeated at this new point. Provided the starting point is “close enough” to the zero of this function, the algorithm will manage to find the solution of (2.44).

The derivative of \( f(\beta) \) is obtain from (2.43)

\[
 f'(\beta) = \sin(\beta) \left( \lambda_d''(\beta) + \frac{\pi}{180} \lambda_q''(\beta) \right) + \frac{\pi}{180} \cos(\beta) \left( \lambda_d'(\beta) + \frac{\pi}{180} \lambda_q'(\beta) \right) 
\]

\[ + \cos(\beta) \left( \frac{\pi}{180} \lambda_d'(\beta) - \lambda_q'(\beta) \right) - \frac{\pi}{180} \sin(\beta) \left( \frac{\pi}{180} \lambda_d(\beta) - \lambda_q'(\beta) \right) 
\]

\[
 = \sin(\beta) \left( \lambda_d''(\beta) - \frac{\pi^2}{180^2} \lambda_d(\beta) + 2 \frac{\pi}{180} \lambda_q'(\beta) \right) 
\]

\[ + \cos(\beta) \left( \frac{\pi^2}{180^2} \lambda_q(\beta) - \lambda_q''(\beta) + 2 \frac{\pi}{180} \lambda_d'(\beta) \right) 
\] 

(2.48)
as the use of cubic spline interpolation guarantee the existence and continuity of \( \lambda_d', \lambda_q', \lambda_d'' \), and \( \lambda_q'' \).

A good starting point is very important for the convergence of this algorithm. If it is too far from the root and \( f \) not monotonous, the algorithm will never converge. If the starting point is near a local optimum of the torque function (2.41), then the algorithm will remain stuck in it and miss \( T_{\text{max}} \). For this reason, the torque is first calculated at the \( n \) interpolation points \( \beta_i \) from (2.41), which gives us the \( n \) values \( \{T(\beta_i)\}_{i=1..n} \). We choose the starting point \( \beta_{[0]} \) as the interpolation point that gives the highest torque among those \( \{T(\beta_i)\}_{i=1..n} \):

\[
\beta_{[0]} = \beta_{\text{best}} \text{ with } i_{\text{best}} \text{ such that } T(\beta_{\text{best}}) = \max \{T(\beta_i)\}_{i=1..n} \tag{2.49}
\]

In addition, to avoid oscillation problems (2.47) is modified as

\[
\beta_{[k+1]} = \beta_{[k]} - a \frac{f(\beta_{[k]})}{f'(\beta_{[k]})} \tag{2.50}
\]

with the value

\[
a = 0.2 \tag{2.51}
\]
determined empirically.

Once \( \beta_{T_{\text{max}}} \) is obtained, the peak torque \( T_{\text{max}} \) is easily calculated by substituting \( \beta_{T_{\text{max}}} \) in (2.41)

\[
T_{\text{max}} = \frac{3p}{2} I_r \left\{ \lambda_d(\beta_{T_{\text{max}}}) \sin(\beta_{T_{\text{max}}}) - \lambda_q(\beta_{T_{\text{max}}}) \cos(\beta_{T_{\text{max}}}) \right\} \tag{2.52}
\]

The power capability in the constant torque region is plotted using

\[
P(\Omega) = T_{\text{max}} \Omega \tag{2.53}
\]

The base speed \( \Omega_b \) at which the field-weakening region begins is calculated from (2.39) and (2.40):

\[
\Omega_b = \frac{V_{\text{max}}}{p \sqrt{\lambda(\beta_{T_{\text{max}}})^2 + \lambda(\beta_{T_{\text{max}}})^2}} \tag{2.54}
\]
as it is by definition the speed at which the voltage reaches its maximum value when the motor is accelerated with its peak torque $T_{max}$ and corresponding flux $\lambda_{T_{max}}$.

Finally the maximum speed attainable $\Omega_{max}$ is obtained at $\beta = 180^\circ$ where both d- and q-axis flux are minimum.

$$\Omega_{max} = \frac{V_{max}}{p\sqrt{\lambda_d(180^\circ)^2 + \lambda_q(180^\circ)^2}}$$

It is noted that $\lambda_q(180^\circ)$ is equal to zero as $I_q = 0$ for $\beta = 180^\circ$.

The power capability of the motor in this field weakening region is obtained by plotting

$$P(\beta) = T(\beta)\Omega(\beta) = \frac{3}{2} I_r V_{max} \frac{\lambda_d(\beta) \cos(\beta) + \lambda_q(\beta) \sin(\beta)}{\sqrt{\lambda_d(\beta)^2 + \lambda_q(\beta)^2}}$$

against

$$\Omega(\beta) = \frac{V_{max}}{p\sqrt{\lambda_d(\beta)^2 + \lambda_q(\beta)^2}}$$

for $\beta$ varying between $\beta_{T_{max}}$ and $180^\circ$.

### 2.5.4 Comparison

To see the usefulness of this new model, and have some insight on how the saturation influences the motor performances, we shall compare the power capability predictions of an IPMSM obtained from

- this non-linear model
- the linear model of section 2.3.1 with the motor parameters computed at $\vec{I} = I_r \angle 180^\circ$, point at which they are the least saturated. This model will be called “linear least saturated model” in the followings.
bullet the linear model of section 2.3.1 with the motor parameters computed at
\( \overrightarrow{I} = I_r \angle 90^\circ \), point at which they are the most saturated. This model will be
called “linear most saturated model” in the followings.

These comparisons are made for three different values of the rated current \( I_{r_1} = 1.92 \) \( A \), \( I_{r_2} = 2.85 \) \( A \) and \( I_{r_3} = 3.8 \) \( A \) as saturation effects are expected to be predominant for high value of the current. The motor parameters to be used in the linear models are therefore computed (by Finite Element Method) for \( I_{r_1} \), \( I_{r_2} \) and \( I_{r_3} \) at the corresponding current angle (Table 2.1). As expected because of saturation, the values of the inductances are larger for \( \beta = 180^\circ \) than for \( \beta = 90^\circ \) for a given value of the rated current. The prediction of the power capability by these three models

<table>
<thead>
<tr>
<th>( I_{r_1} = 1.9A )</th>
<th>Motor parameters at 180°</th>
<th>530</th>
<th>47.7</th>
<th>165.9</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Motor parameters at 90°</td>
<td>530</td>
<td>40.9</td>
<td>136.7</td>
</tr>
<tr>
<td>( I_{r_2} = 2.85A )</td>
<td>Motor parameters at 180°</td>
<td>510</td>
<td>44.4</td>
<td>159.3</td>
</tr>
<tr>
<td></td>
<td>Motor parameters at 90°</td>
<td>510</td>
<td>36.1</td>
<td>115.6</td>
</tr>
<tr>
<td>( I_{r_3} = 3.8A )</td>
<td>Motor parameters at 180°</td>
<td>487</td>
<td>42.2</td>
<td>152.0</td>
</tr>
<tr>
<td></td>
<td>Motor parameters at 90°</td>
<td>487</td>
<td>32.9</td>
<td>99.3</td>
</tr>
</tbody>
</table>

are presented in Figs. 2.16, 2.17 and 2.18. The prediction by the non-linear model (dashed line) is compared with the prediction by the linear “least saturated” model (solid line) on the left side, and with the prediction of the linear “most saturated” model (solid line also) on the right side. The corresponding peak torque, CPSR, and speed predictions are shown in Tables 2.2, 2.3 and 2.4.

Let us first compare the non-linear model with the linear “least saturated” model; it can be seen that the power capability curves of the non-linear model and linear
Figure 2.16: Power capability predictions from the two models for $I_r = 1.9\text{Amps}$

Figure 2.17: Power capability predictions from the two models for $I_r = 2.85\text{Amps}$

Figure 2.18: Power capability predictions from the two models for $I_r = 3.8\text{Amps}$
“least saturated” model are very close at high speeds. The reason is that the flux level becomes very low at such high speeds thus the linear “least saturated” model can be considered as a good model of this low saturated IPMSM. At low speeds however, in the constant torque region where saturation occurs, the torque prediction of this linear “least saturated” model are too optimistic compared with the torque prediction of the non-linear model. Obviously, the prediction error of the linear “least saturated” model increases with the value of the rated current. In quantitative terms, a glance at Table 2.2, 2.3 and 2.4 tells that the maximum speed $\Omega_{\text{max}}$ and the $CPSR$ upper limit speed $\Omega_2$ are exactly predicted by this linear model. The error on $T_{\text{max}}$, on the contrary, can reach +10% and as a consequence, the predictions of $\Omega_b$, $\Omega_1$ and the $CPSR$ are quite erroneous.

Table 2.2: Comparison of the performance predictions obtained by the three models for $I_{r_1} = 1.92A$

<table>
<thead>
<tr>
<th>$I_{r_1} = 1.92A$</th>
<th>Non linear model</th>
<th>Linear model with parameters computed at $\beta = 180^\circ$</th>
<th>Linear model with parameters computed at $\beta = 90^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{T_{\text{max}}} (^\circ)$</td>
<td>114</td>
<td>114 (+0%)</td>
<td>110 (-3%)</td>
</tr>
<tr>
<td>$T_{\text{max}}$ (Nm)</td>
<td>4.8</td>
<td>4.92 (+2%)</td>
<td>4.74 (-1%)</td>
</tr>
<tr>
<td>$CPSR$</td>
<td>2.27</td>
<td>2.33 (+3%)</td>
<td>2.18 (-4%)</td>
</tr>
<tr>
<td>$\Omega_b$ (rpm)</td>
<td>2581</td>
<td>2474 (-4%)</td>
<td>2587 (+0%)</td>
</tr>
<tr>
<td>$\Omega_1$ (rpm)</td>
<td>1591</td>
<td>1554 (-2%)</td>
<td>1612 (+1%)</td>
</tr>
<tr>
<td>$\Omega_2$ (rpm)</td>
<td>3616</td>
<td>3625 (+0%)</td>
<td>3507 (-3%)</td>
</tr>
<tr>
<td>$\Omega_{\text{max}}$ (rpm)</td>
<td>3895</td>
<td>3893 (+0%)</td>
<td>3723 (-4%)</td>
</tr>
</tbody>
</table>

The situation is reversed when one compares the linear “most saturated” model predictions with the non-linear model predictions; if both power predictions in the constant torque region seem to match, it is clearly not the case in the field
Table 2.3: Comparison of the performance predictions obtained by the three models for $I_{r2} = 2.85A$

<table>
<thead>
<tr>
<th>$I_{r2} = 2.85A$</th>
<th>Non linear model</th>
<th>Linear model with parameters computed at $\beta = 180^\circ$</th>
<th>Linear model with parameters computed at $\beta = 90^\circ$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{T_{max}}$ (°)</td>
<td>120</td>
<td>119 (-1%)</td>
<td>114 (-5%)</td>
</tr>
<tr>
<td>$T_{max}$ (Nm)</td>
<td>7.4</td>
<td>7.9 (+6%)</td>
<td>7.2 (-4%)</td>
</tr>
<tr>
<td>$CPSR$</td>
<td>2.95</td>
<td>3.12 (+6%)</td>
<td>2.6 (-12%)</td>
</tr>
<tr>
<td>$\Omega_b$ (rpm)</td>
<td>2453</td>
<td>2208 (-10%)</td>
<td>2508 (+2%)</td>
</tr>
<tr>
<td>$\Omega_1$ (rpm)</td>
<td>1540</td>
<td>1456 (-5%)</td>
<td>1601 (+4%)</td>
</tr>
<tr>
<td>$\Omega_2$ (rpm)</td>
<td>4536</td>
<td>4547 (+0%)</td>
<td>4160 (-8%)</td>
</tr>
<tr>
<td>$\Omega_{max}$ (rpm)</td>
<td>4747</td>
<td>4744 (+0%)</td>
<td>4301 (-9%)</td>
</tr>
</tbody>
</table>

weakening region. At very high speeds, the power capability predictions from the linear “most saturated” model are under-estimated and the curves of these linear “most saturated” model and non-linear models are clearly distinct; indeed the linear “most saturated” linear uses highly saturated motor parameters values although little saturation occurs in this low flux region. The trend is more obvious for larger values of rated current. The prediction errors of $\Omega_2$, $\Omega_{max}$ and of the $CPSR$ can be as high as 20% for the highest value of the rated current.

This comparison clearly brings to light the shortcomings of the linear model of the IPMSM. The variations of the saturation level makes predictions by this constant motor parameters model unsatisfactory throughout the entire speed range. The predicted power capability is clearly distorted, either in the constant torque region or in the field weakening region.
Table 2.4: Comparison of the performance predictions obtained by the three models for \( I_{r3} = 3.8A \)

<table>
<thead>
<tr>
<th>( I_{r3} = 3.8A )</th>
<th>Non linear model</th>
<th>Linear model with parameters computed at ( \beta = 180^\circ )</th>
<th>Linear model with parameters computed at ( \beta = 90^\circ )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_{T_{max}} ) (°)</td>
<td>125</td>
<td>119 (-4%)</td>
<td>117 (-6%)</td>
</tr>
<tr>
<td>( T_{max} ) (Nm)</td>
<td>10.1</td>
<td>11.1 (+10%)</td>
<td>9.5 (-6%)</td>
</tr>
<tr>
<td>( CPSR )</td>
<td>3.86</td>
<td>4.26 (+10%)</td>
<td>3.05 (-21%)</td>
</tr>
<tr>
<td>( \Omega_b ) (rpm)</td>
<td>2378</td>
<td>1990 (-16%)</td>
<td>2484 (+4%)</td>
</tr>
<tr>
<td>( \Omega_1 ) (rpm)</td>
<td>1518</td>
<td>1377 (-9%)</td>
<td>1616 (+6%)</td>
</tr>
<tr>
<td>( \Omega_2 ) (rpm)</td>
<td>5856</td>
<td>5863 (+0%)</td>
<td>4935 (-16%)</td>
</tr>
<tr>
<td>( \Omega_{max} ) (rpm)</td>
<td>6045</td>
<td>6039 (+0%)</td>
<td>5047 (-17%)</td>
</tr>
</tbody>
</table>

### 2.6 Conclusion

In this chapter, a new steady state circuit modelling of the buried type IPMSM to be used as a design criterion has been described to predict the power capability of the motor. This is only valid for the motor operating at rated current. Its predictions take into account the saturation effects by modelling the d- and q-axis flux linkages as non linear functions of the current angle \( \beta \). These functions are obtained by using a combination of Finite Element Method and Cubic Spline Interpolation. The algorithms to compute the field-weakening characteristics \( \beta_{T_{max}}, T_{max}, \Omega_{base} \) and \( \Omega_{max} \) of the motor using this non-linear model have been described. A comparison of this non-linear model with the traditional linear model of the motor (based on constant motor parameters) has shown a significantly improved accuracy in the power capability predictions.
Chapter 3

Finite Element Method and computations of the motor characteristics

3.1 Introduction

The Finite Element Method is used to find the solution of partial differential equations in a given domain including its boundary conditions. It was first introduced in structural mechanics for stress field analysis and then adapted in many other fields like heat transfer, fluid mechanics and electro-magnetics. Its flexibility in representing very complex geometry and dealing with non-linear equations gave it a definitive advantage over other numerical tools like Boundary Elements and Finite Difference method for the analysis of electromagnetic devices. The idea behind it is to divide the continuum by a finite number of sub-domains or elements, where
the potential to be found is represented by interpolation functions that contain, as unknowns, the value of this potential at the respective nodes of the element. An energy functional is built from the differential equations and expressed with these interpolation functions. Its minimization generates a system of equations and the potential values at the nodes can be determined using direct or iterative methods [49]. The accuracy of the method increases with the number of elements used, which is also the size of the system of equation to be solved, and is therefore only limited by the computational power available. The tremendous development of computers performances over the last three decades has naturally resulted in the wide-spread use of Finite Element as a design tool. Many FEM commercial packages like the one used in this thesis (Flux2D) are now available so the user doesn’t need to write his own program anymore. These packages have paid much attention to the user interface and post-processing of solution to make the FEM transparent and allow the user to focus on the modelling of the machine and interpretation of the results [55]. As a result the user doesn’t need to know much about the FEM theory, but the principles have to be understood to use this tool properly to obtain meaningful results. For this reason, the mathematical principle of the method is briefly explained at the beginning of this chapter.

The way to model the IPMSM with the software Flux2D is then described, with all the simplifications and assumptions made. The non-linear relation flux-current in the motor are obviously implicitly accounted via the input of the non-linear BH curve of the stator laminations and rotor core in the FEM software. The saturation inherent to the IPMSM will therefore be completely accounted for when deriving quantities from the vector potential distribution returned by a FEM computation.
The very purpose of the FEM here in this thesis is to compute the values of the fundamental of the flux linkages, $\lambda_d$ and $\lambda_q$, at rated current for a given value of the current angle, to be used after in the non-linear circuit model of the IPMSM presented in the previous chapter (or to fit the RSM polynomials introduced in the next chapter). Two methods to compute these fluxes from the vector potential distribution obtained by a FEM simulation will therefore be described: one of the methods provides the net flux linkage, which means it unfortunately includes all its spatial harmonics with the fundamental. The other is able to isolate the first spatial harmonic of the airgap flux, but unfortunately cannot account for the stator leakage flux. The spatial harmonics of the airgap fluxes will therefore be computed to evaluate their significance and estimate the magnitude of the stator flux leakage. This will allow us to decide which of the two methods should be used to compute the $\lambda_d$ and $\lambda_q$.

### 3.2 Principle of the Method

The magnetic field in an electric machine obeys the Maxwell’s equations. As the field frequencies are generally low, the displacement current is neglected and the fundamental equations can be written as:

$$\begin{align*}
\text{curl}\vec{H} &= \vec{J} \\
\text{div}\vec{B} &= 0
\end{align*}$$

where

- $\vec{H}$ is the magnetic field intensity
- $\vec{B}$ is the magnetic flux density and
- $\vec{J}$ represent the source current density.
The magnetic properties of the material gives a third equation relating $\vec{H}$ with $\vec{B}$.

$$\vec{H} = \nu \vec{B}$$  \hspace{0.5cm} (3.2)

where $\nu$ is the reluctivity of the material, usually a function of $\vec{B}$.

From (3.1), it is possible to define a magnetic vector potential such that:

$$\text{curl} \vec{A} = \vec{B}$$  \hspace{0.5cm} (3.3)

Substituting (3.3) into (3.1), we obtain the equation:

$$\text{curl}(\nu \text{curl} \vec{A}) = \vec{J}$$  \hspace{0.5cm} (3.4)

In electric machines, it is common to consider the field as 2-dimensional (in the x-y plane). This approximation is valid provided the machine length along the z-axis is large enough. The 3 dimensional effects like skewing or end winding field can be taken into account later by correction factors applied to the 2 dimensional solution, or by using a 3-D FEM package to model the end winding [54]. With $\vec{H}$ and $\vec{B}$ restricted to the x-y plane ($B_z=H_z=0$) and $\vec{J}$ restricted to the z-axis ($\vec{J} = J\vec{z}$), it follows that $\vec{A}$ is along the z-axis, that is:

$$\vec{A} = Az$$  \hspace{0.5cm} (3.5)

$A$ is therefore the variable in the FEM and $\vec{B}$ can then be easily obtained later from $A$ by:

$$\vec{B} = \frac{\partial A}{\partial y} \vec{x} - \frac{\partial A}{\partial x} \vec{y}$$  \hspace{0.5cm} (3.6)

Under the 2-dimensional assumption, equation (3.4) can then be simplified to:

$$\frac{\partial}{\partial x} \left( \nu \frac{\partial A}{\partial x} \right) + \frac{\partial}{\partial y} \left( \nu \frac{\partial A}{\partial y} \right) = -J$$  \hspace{0.5cm} (3.7)
This differential equation governs the magnetic field inside the domain and is accompanied by the boundary conditions, which describe the behavior of the field at the boundaries. In a way, the boundary conditions summarize the influence of everything lying outside the domain.

\[
\begin{align*}
\begin{cases}
A(x, y) = A_{s_1}(x, y) & \text{on the boundary } S_1 (\text{Dirichlet condition}) \\
\frac{\partial A(x, y)}{\partial n} = 0 & \text{on the boundary } S_2 \ (\text{Neuman condition})
\end{cases}
\end{align*}
\]  (3.8)

are the most frequent boundary conditions.

These boundary conditions cause a lot of trouble when trying to solve the partial differential equation. For this reason, a functional approach is preferred in FEM. Indeed, according to Euler’s principle, it can be shown that solving (3.7) with (3.8) is equivalent to minimizing the energy functional

\[
W = \iint_{\text{Domain}} \left[ \int_0^B \nu B \, dB - JA \right] \, dxdy \quad \text{with} \quad A(x, y)\big|_{(x,y)\in S_1} = A_{S_1}(x, y) \quad (3.9)
\]

where

\[
B = \|\vec{B}\| = \sqrt{\frac{\partial A^2}{\partial x} + \frac{\partial A^2}{\partial y}}
\]  (3.10)

In other words, the vector potential distribution \(A(x, y)\) that minimizes \(W\) is the solution of the problem.

To illustrate the Finite Element Method, we shall consider \(\nu\) constant in the following (keeping in mind that FEM can perfectly handle the case \(\nu = f(B)\)). The energy density can be evaluated as,

\[
\int_0^B \nu B \, dB = \frac{\nu}{2} B^2
\]  (3.11)

and equation (3.11) becomes:

\[
\iint_{\text{Domain}} \left\{ \frac{\nu}{2} \left[ \left( \frac{\partial A}{\partial x} \right)^2 + \left( \frac{\partial A}{\partial y} \right)^2 \right] - JA \right\} \, dxdy \quad \text{with} \quad A(x, y)\big|_{(x,y)\in S_1} = A_{S_1}(x, y)
\]  (3.12)
As mentioned earlier, the principle is to subdivide the domain into elements, usually using triangles or quadrangles, whose vertex are called nodes. Then the field over each element is approximated by an interpolation function, in most cases using first or second order polynomials.

Let us consider the case of a triangle $ijk$ (Fig. 3.1) for which the values of $A$ at the nodes are $A_i = A(x_i, y_i)$, $A_j = A(x_j, y_j)$ and $A_k = A(x_k, y_k)$. Using the first order polynomial approximation, the vector potential representation is given by

$$A(x, y) = a + bx + cy$$

(3.13)

over the element $ijk$. Writing this equation at the vertex of this triangle gives us

$$A_i = a + bx_i + cy_i$$

$$A_j = a + bx_j + cy_j$$

$$A_k = a + bx_k + cy_k$$

(3.14)

which may be rewritten using matrix notation

$$\begin{bmatrix}
    A_i \\
    A_j \\
    A_k 
\end{bmatrix} =
\begin{bmatrix}
    1 & x_i & y_i \\
    1 & x_j & y_j \\
    1 & x_k & y_k 
\end{bmatrix}
\begin{bmatrix}
    a \\
    b \\
    c 
\end{bmatrix}$$

(3.15)

The coefficient $a$, $b$, $c$ are obtained by

$$\begin{bmatrix}
    a \\
    b \\
    c 
\end{bmatrix} =
\begin{bmatrix}
    1 & x_i & y_i \\
    1 & x_j & y_j \\
    1 & x_k & y_k 
\end{bmatrix}^{-1}
\begin{bmatrix}
    A_i \\
    A_j \\
    A_k 
\end{bmatrix}$$

(3.16)
Finally, substituting (3.16) into (3.13), we obtain

\[
A(x, y) = \begin{bmatrix} 1 & x & y \\ 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}^{-1} \begin{bmatrix} A_i \\ A_j \\ A_k \end{bmatrix} \quad (3.17)
\]
or

\[
A(x, y) = \begin{bmatrix} N_i(x, y), N_j(x, y), N_k(x, y) \end{bmatrix} \begin{bmatrix} A_i \\ A_j \\ A_k \end{bmatrix} \quad (3.18)
\]

where \(N_i(x, y), N_j(x, y)\) and \(N_k(x, y)\) are called shape functions of the triangular element as they only depend on the position of the nodes of the element, not on the nodal values \(A_i, A_j\) and \(A_k\). Over each element, the vector potential distribution is therefore expressed in terms of the nodal values. As a result, the knowledge of the global field in the continuum is reduced to the knowledge at the nodes. The nodal values become the unknowns of the problem.

Equation (3.12) can be rewritten as a sum of the functionals of all the elements

\[
W = \sum_{e \in \text{Domain}} W_e \quad (3.19)
\]

where

\[
W_e = \int_e \left\{ \frac{\nu_e}{2} \left[ \left( \frac{\partial A}{\partial x} \right)^2 + \left( \frac{\partial A}{\partial y} \right)^2 \right] - JA \right\} dxdy \quad (3.20)
\]

Substituting (3.18) into (3.20) gives

\[
W_e = \int_e \left\{ \frac{\nu_e}{2} \left[ \left( \frac{\partial[N_e][A_e]}{\partial x} \right)^2 + \left( \frac{\partial[N_e][A_e]}{\partial y} \right)^2 \right] - J[N_e][A_e] \right\} dxdy \quad (3.21)
\]
The solution of the field distribution being the one that minimizes the energy functional, we can differentiate (3.21) with respect to each of the three nodal values of the element \(A_i, A_j\) and \(A_k\):

\[
\frac{\partial W_e}{\partial A_n} = \int_e \left\{ \nu_e \left[ \frac{\partial[N_e][A_e]}{\partial x} \frac{\partial[N_e][A_e]}{\partial x} + \frac{\partial[N_e][A_e]}{\partial y} \frac{\partial[N_e][A_e]}{\partial y} \right] - J[N_e][A_e] \right\} dxdy \quad (3.22)
\]

\[
\frac{\partial W_e}{\partial A_n} = \int_e \nu_e \left[ \frac{\partial[N_e][A_e]}{\partial x} \frac{\partial[N_e][A_e]}{\partial x} + \frac{\partial[N_e][A_e]}{\partial y} \frac{\partial[N_e][A_e]}{\partial y} \right] dxdy [A_e] - \int_e J[N_e][A_e] dxdy
\]
where \( n = i, j, k \)

We obtain the matrix equation for the element \( e \)

\[
\begin{bmatrix}
\frac{\partial W}{\partial A_n}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial W}{\partial A_i} & \frac{\partial W}{\partial A_j} & \frac{\partial W}{\partial A_k}
\end{bmatrix}
= \begin{bmatrix}
h_{ii} & h_{ij} & h_{ik} \\
h_{ji} & h_{jj} & h_{jk} \\
h_{ki} & h_{kj} & h_{kk}
\end{bmatrix}
\begin{bmatrix}
A_i \\
A_j \\
A_k
\end{bmatrix}
- \begin{bmatrix}
p_i \\
p_j \\
p_k
\end{bmatrix}
= [H_e][A_e] - [P_e] \quad (3.23)
\]

where

\[
h_{ij} = \int_e \int \nu_e \left( \frac{\partial [N_e]^i}{\partial x} \frac{\partial [N_e]^j}{\partial x} + \frac{\partial [N_e]^i}{\partial y} \frac{\partial [N_e]^j}{\partial y} \right) dx dy \quad (3.24)
\]

and

\[
p_i = \int_e \int J[N_e] dx dy \quad (3.25)
\]

the current density and reluctivity being considered constant within the element for the integration.

As this is not the energy of a single element but the energy on the whole domain that has to be minimized, the matrix \([H_e], [A_e]\) and \([P_e]\) of all the elements are assembled in domain matrix \([H], [A]\) and \([P]\) so that:

\[
\frac{\partial W}{\partial A_n} = \sum_{e \in \text{Domain}} \frac{\partial W_e}{\partial A_n} = 0 \quad \text{for all nodes} \quad (3.26)
\]

is equivalent to

\[
[H][A] = [P] \quad (3.27)
\]

This equation where \([A]\) is the unknown is usually solved using algorithms such as the Choleski matrix decomposition or conjugate gradient methods, as \([H]\) is far too large to be inverted. The number of iterations needed is around \(N^2\) if \(N\) is the number of nodes. The Dirichlet boundary conditions are applied before solving by setting the values of \(A\) on the boundary to their specified values.

When the domain contains non linear material like steel or iron, (3.28) becomes non linear:

\[
[H(A)][A] = [P] \quad (3.28)
\]
as the reluctivity $\nu$ is a function of the flux density and hence also a function of $A$. This equation has to be solved by iteration using the Newton-Raphson algorithm.

Finally, from the matrix of the nodal values $[A]$, the vector potential $A(x, y)$ can be known throughout the domain using (3.18).

In practice, the whole calculation procedure to obtain the vector potential distribution doesn’t need any intervention of the FEM software user. The user is involved before (pre-processing) and after (post-processing). In addition to inputting all the geometric and material data, the user plays a crucial role in the meshing of the domain, the definition of the elements to be used for computation; the user has to define the node density throughout the domain in an intelligent way to optimize the accuracy of the computation and to reduce the computation time. From the mathematical part that has been described above, it is possible to derive several rules regarding the mesh definition for this purpose:

- The variable of interest for the user, $\vec{B}$, is interpolated from the nodal values. If we take the first order interpolation function, from (3.13) and (3.6), $\vec{B}$ is actually a constant all over an element. If we take a second order interpolation function, $\vec{B}$ will vary linearly over the element. As a result, to obtain a better accuracy, the node density should be higher in the regions where the field is expected to vary quickly, for example, at the corner of the slot. Similarly, to decrease the computation time, the mesh can be made coarse in regions where the field is nearly constant.

- As $A$ is interpolated between the nodes, the mesh density should be higher in the region where it varies a lot to obtain a better accuracy. Since $\vec{B}$ is in
some way the “derivative” of $A$, high values of $B$ correspond to fast spatial variations of $A$; the regions where the mesh has to be refined are where $B$ is expected to be high, like in the saturated regions (magnetic bridge for example) or flux focusing regions. Similarly, the mesh can be coarse where the field is weak, like in the slots or shaft; indeed their reluctivity is very high (air) and the flux prefer to ”avoid” these regions.

- As the solution is found by minimizing an energy functional, which actually corresponds to the energy of the domain, the regions believed to contain high energy density must have a fine mesh. For a motor, this corresponds to the air-gap where the bulk of the magnetic energy is present.

- the triangular elements should be ideally equilateral. If the angle of an element is too sharp, the error in the local solution will be increased. This will also affect the accuracy of the global solution. In practice, this is achieved by making variations of the mesh density smooth enough.

In addition, the mesh should be defined according to what the user is investigating. Should it be the computation of a very sensitive quantity, like the cogging torque, or the investigation of a local quantity, like the value of the flux density at the tooth-tip, the mesh in the region of interest must be very refined to obtain reliable results. On the contrary, for the the computation of a global quantity like flux linkage, such a fine mesh would increase unnecessarily the computation time, without improving significantly the accuracy of the results. The user has to be aware that the computation time increases approximately with the square of the number of nodes; finding a trade-off between accuracy and computational cost is necessary.
Finally, one should also keep in mind, when analyzing the results, that the vector potential distribution obtained is optimum in a global sense; indeed it optimizes the total energy of the domain. For this reason, there is no guarantee that local errors in the field do not occur. As a result, when one has to calculate quantities like torque or inductances that are obtainable from different methods, it is preferable to rely on the methods that use the magnetic energy rather than on the ones that need local values of the flux density.

The FEM has been extensively applied in this thesis to analyze the IPMSM and to calculate the d- and q-axis flux linkages needed in its non linear circuit modelling of the previous chapter.

### 3.3 Modelling of the IPMSM using FEM

As mentioned before, most of the FEM analysis of electric machine is done by considering the problem as 2-dimensional. This is also the case here. For this reason, the skewing effect and end winding flux leakages are neglected. The FEM software chosen is Flux2D. This section describes the pre-processing part, that is the definition of the geometry, materials, sources and meshing. All these data are the input of the processing part described above.

#### 3.3.1 Material and Geometry

The IPMSM to be modelled is shown in Fig. 3.2 and has the following characteristics:
• the 4 poles rotor carries 4 NeFeB magnets with a remanent flux density $B_r = 1.15T$ and a relative permeability $\mu_r = 1.05$. The magnets are actually modelled by a region with relative permeability $\mu_r = 1.05$ and a constant current density at its boundary that represents the magnetization; this is the way FEM handles this special material to make it "compliant" with the equation (3.11). The flux barriers are made of non magnetic material $\mu_r = 1$. The core is made of non-linear steel whose characteristics are presented in the Appendix B.

• the stator is an induction machine type stator with 24 slots. This gives 4 slots per pole per phase. The winding is thus a double layer one short-pitched by one slot; one phase pole is then made of two coils of 39 turns each spanning 75° (mechanical) and shifted by 15°. The winding for one phase is represented
in Fig. 3.3. In practice, each coil is modelled using a current source in a half-slot equal to the current times the number of conductors. The relative permeability of the slot is chosen to be equal to 1 as the conductors are made of copper, a non-magnetic material. The stator lamination is made of steel, the same material as the rotor core.

The main dimensions and characteristics of the motors are given in Appendix B.

![Figure 3.3: Coil arrangement of one phase winding](image)

### 3.3.2 Static analysis

The steady state analysis of a 400 Watt synchronous machine can be regarded as a static type problem. Indeed, there is no relative motion between the stator and the rotor MMFs as both rotate at the synchronous speed; the d- and q-axis quantities
are independent of the rotor position. As a result, the IPMSM can be analyzed from a single FEM computation, a snapshot of the motor state at a given instant $t_0$. This means that we neglect

- the eddy current and hysteresis effects, due to local variations of the magnetic field
- the air-gap permeance harmonics (the teeth modify the air-gap seen by the flux when it rotates) as well as MMF harmonics (the conductors distribution is not exactly sinusoidal).

This static approach is very attractive as it allows the computational time to be a minimum, while keeping a very good accuracy. This low computation time is crucial as it will be shown in the next chapter when FEM is combined with Response Surface Method. Finally, the limitations (no iron losses or harmonics) are not a problem since we use FEM to get data for the non-linear circuit modelling that already neglected these "second order" effects. The essential point is that the complex geometry and non linearity of the iron are fully taken into account.

### 3.3.3 Positioning of the MMF

The magnet (rotor) and the armature (stator) MMFs have to be properly positioned to model the steady state operation of the motor corresponding to a given current vector.

At the instant $t_0$ the phase currents are chosen such that:

$$I_A = I \cos(\alpha)$$
$$I_B = I \cos(\alpha - 120^\circ)$$
$$I_C = I \cos(\alpha - 240^\circ)$$

(3.29)
which means that the stator MMF lead the phase A axis by an angle $\alpha$ (Fig. 3.4). The rotor position is chosen such that the symmetry axis of a north pole magnet, the $d$-axis by definition, is aligned with the phase A axis. As a result, the stator MMF leads the rotor MMF by the angle $\alpha$ and we have the relations

$$
I_d = I \cos(\alpha) \\
I_q = I \sin(\alpha)
$$

Therefore, to simulate the steady-state operation of the IPMSM at the current vector $I_r \angle \beta$, we choose $I = I_r$ and $\alpha = \beta$.

![Figure 3.4: Positioning of the MMFs](image)

### 3.3.4 Boundary conditions

The boundary conditions are extremely important, they have to be chosen with care to represent adequately the behavior of the flux over the boundaries of the
domain. They can also be used to reduce the size of the domain to be modelled, by using symmetry considerations, and therefore to reduce the computation time. It can be first noticed that the geometry is periodic over one pole pitch (90 mechanical °). In addition, the sources (magnets and currents) are such that

\[ J(r, \theta + 90^\circ) = -J(r, \theta) \]  

(3.31)

In consequence,

\[ A(r, \theta + 90^\circ) = -A(r, \theta) \]  

(3.32)

It follows that the study of the machine can be reduced to only one pole pitch, by applying anti-cyclic boundary conditions to the lines \( L_1 \) and \( L_2 \) (Fig. 3.5). These two boundaries are linked so that the value of \( A \) is unknown but of opposite sign on homologous nodes. It should be noticed that there is no need to choose the quarter machine bounded by radial lines; any radial path, straight or curved will do equally well as long as it is matched by a similar boundary exactly 90° away [52]. The number of nodes saved by modelling one quarter of the motor instead of the whole motor will allow to use a higher node density and obtain more accurate results, for the same amount of computation time.

The Dirichlet boundary condition \( A = 0 \) is applied on the outer surface of the stator \( L_3 \). This means that the magnetic flux cannot cross \( L_3 \) as the value of \( A \) is constant along this line:

\[ B_r = \frac{1}{r} \frac{\partial A}{\partial \theta} = 0 \quad \text{on} \quad L_3 \]  

(3.33)

where \( B_r \) is the radial component of \( \vec{B} \).

This boundary condition is justified by the fact that the motor is surrounded by air, whose permeability is several hundred times less than the permeability of iron. As a result most of the flux is confined within the motor and \( L_3 \) is a flux line.
3.3.5 Meshing of the geometry

The machine is modelled with a total of 3500 nodes as shown in Fig. 3.6. As the vector potential in the air-gap will be needed for the computation of the flux linkage, three layers of nearly equilateral triangular elements are used to mesh this region (around 0.2 mm long edges). Half of the total elements are actually used to mesh the air-gap.

The magnetic bridges, where the value of the flux density will be very high, are also carefully meshed with very small elements.

The magnet is meshed with three layers of elements since, being surrounded by iron with a relative permeability 1000 times higher, it constitutes a high variation of permeability seen by the flux.

The rotor iron is meshed with bigger elements as the flux density will be low and relatively constant in magnitude and direction. The iron in the stator is meshed
with smaller element than in the rotor as some part, like the teeth and the corner of the slots, will be surely saturated. Due to the presence of non-linear iron, around 10 iterations are needed to obtain the matrix of the nodal values \([A]\), which corresponds to 3 minutes of computation time on a Sun-Blade-1000 Work Station.

### 3.4 Computation of the flux

The non-linear equivalent circuit of the IPMSM described in the previous chapter requires the values of the fundamentals of the d-q-flux linkages at the interpolation points \(I_r\angle\beta_1, I_r\angle\beta_2, \ldots I_r\angle\beta_n\) to build the models of the flux \(\lambda_d(\beta)|_{I=I_r}\) and \(\lambda_q(\beta)|_{I=I_r}\) at rated current. FEM are perfectly suited to fulfill this function since we have seen in the previous section it is able to gives us with the utmost degree of accuracy the vector potential distribution (and hence flux density distribution) throughout the machine, accounting for the non-linear B-H characteristic of the iron. These fluxes

![Figure 3.6: Magnetic bridge and air-gap mesh](image)
are calculated in the post-processing phase, from the vector potential distribution returned by the FEM computation for a specific set of phase currents initially input by the user (to match the desired current vector $I_r \angle \beta_i$) in the pre-processing phase.

Two methods to compute these values are described in this section. The first one is based on the computation of the vector potential in the slots to obtain directly the total value of the flux linkages by Stokes theorem. The second method is based on a Fourier analysis of the vector potential in the air-gap to obtain the fundamental of the flux per pole and then the flux linkages.

### 3.4.1 Method 1

This method is based on the Stokes theorem that can be used to show that the line integral of the vector potential along a closed path is equal to the flux linking that path.

\[
\psi_S = \oint_S \vec{B} \cdot d\vec{S} = \int_S (\text{curl } \vec{A}) \cdot d\vec{S} = \oint_L \vec{A} \cdot d\vec{l} \tag{3.34}
\]

Let us consider the two conductors on Fig. 3.7, being the go and return conductor of a coil. The vector potential is everywhere parallel to these conductors (2D problem). Applying the theorem on the path indicated by the dashed line, we get $A_1 \cdot L$ for path 1, 0 for path 2 as the dot product is zero, $-A_2 \cdot L$ for path 3 and 0 for path 4. The total flux per unit depth linking the circuit is then

\[
\Psi = A_1 - A_2 \quad \text{(Wb/m)} \tag{3.35}
\]
Since a coil is made of a bundle of conductors, its total flux linkage is expressed as

$$\lambda = \sum_{k=1}^{N} (A_{k+} - A_{k-}) \text{ (Wb/m)}$$  \hspace{1cm} (3.36)$$

where

- $k_+$ is the go conductor and $A_{k+}$ its associated potential value
- $k_-$ is the return conductor and $A_{k-}$ its associated potential value
- $N$ is the number of turns in the coil.

In our finite element problem, the individual conductors are not represented, but as they are much smaller in diameter than the element size, they will occupy all possible positions in each element; we can thus use the average value of the vector potential in the slot instead:

$$\lambda_{coil} = N \left( \langle A_{S+} \rangle - \langle A_{S-} \rangle \right) L_{\text{stack}}$$

$$\langle A_{S+} \rangle = \frac{\iint_{S_+} A \, dS}{S_+}$$  \hspace{1cm} (3.37)$$

where $S_+$ and $S_-$ are the area of the half-slot housing the go and return conductors of the coil, $L_{\text{stack}}$ is the length of the stator in the $z$-direction.

As only one fourth of the machine is modelled, the slot housing the conductors of a coil may be outside the domain, for example coil 2 of phase A (Fig. 3.8). In that case, the property (3.32) is used to find the average value of $A$ in that slot from
Figure 3.8: Distribution of the different phase conductors in the slots

the slot 90° ahead.

\[ \langle A_{S_{14}} \rangle = - \langle A_{S_{2}} \rangle \]  \hspace{1cm} (3.38)

Finally, the flux linking the two coils in series constituting one pole of phase A is given by

\[ \lambda_{\text{poleA}} = N_{\text{coil}} (\langle A_{S_{12}} \rangle - \langle A_{S_{1}} \rangle + \langle A_{S_{14}} \rangle - \langle A_{S_{3}} \rangle) L_{\text{stack}} \]  \hspace{1cm} (3.39)

which gives the total flux linkage for phase A (4 poles)

\[ \lambda_{A} = 4 N_{\text{coil}} (\langle A_{S_{12}} \rangle - \langle A_{S_{1}} \rangle - \langle A_{S_{2}} \rangle - \langle A_{S_{3}} \rangle) L_{\text{stack}} \]  \hspace{1cm} (3.40)

where \( N_{\text{coil}} = 39 \) turns/coil and \( L_{\text{stack}} = 52 \)mm.

Similarly, the flux linkage for phase B and C are given by

\[ \lambda_{B} = 4 N_{\text{coil}} (-\langle A_{S_{10}} \rangle - \langle A_{S_{11}} \rangle - \langle A_{S_{9}} \rangle - \langle A_{S_{8}} \rangle) L_{\text{stack}} \]  \hspace{1cm} (3.41)

\[ \lambda_{C} = 4 N_{\text{coil}} (\langle A_{S_{6}} \rangle + \langle A_{S_{7}} \rangle + \langle A_{S_{4}} \rangle + \langle A_{S_{5}} \rangle) L_{\text{stack}} \]  \hspace{1cm} (3.42)
The d- and q-axis flux linkage are finally obtained from the A, B and C flux linkages using Park transformation:

\[
\begin{pmatrix}
\lambda_d \\
\lambda_q \\
\lambda_o
\end{pmatrix} = \frac{2}{3} \begin{pmatrix}
\cos 0 & \cos \left(0 + \frac{2\pi}{3}\right) & \cos \left(0 + \frac{4\pi}{3}\right) \\
\sin 0 & \sin \left(0 + \frac{2\pi}{3}\right) & \sin \left(0 + \frac{4\pi}{3}\right) \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{pmatrix} \begin{pmatrix}
\lambda_A \\
\lambda_B \\
\lambda_C
\end{pmatrix}
\] (3.43)
as the d-axis and the phase A axis are aligned.

It should be noted that \(\lambda_A\), \(\lambda_B\) and \(\lambda_C\) contain not only the fundamental of the flux, but also all its harmonics. As a result, these harmonics are unfortunately also included in \(\lambda_d\) and \(\lambda_q\).

### 3.4.2 Method 2

This method is based on the computation of the fundamental of the flux per pole from a Fourier analysis of the vector potential in the air-gap. \(\lambda_d\) and \(\lambda_q\) are then obtained by a projection of this flux per pole on the d- and q-axis.

Let us denote \(A(\theta)\) as the value of the vector potential in the middle of the air-gap at the angle \(\theta\) (electrical). To perform a discrete Fourier analysis of this quantity, \(A\) has to be sampled along a circular path in the middle of the air-gap. The spatial period of \(A(\theta)\) is two poles (2\(\pi\) radians) but as \(A(\theta + \pi) = -A(\theta)\), this path only needs to cover one pole pitch (90 mechanical degrees). The vector potential is sampled at \(N\) points over this path, with a sampling period (electrical) of \(\Delta\theta\). The first harmonic of \(A(\theta)\) is given by:

\[
A_1(\theta) = R_1 \cos(\theta) + I_1 \sin(\theta)
\]

with

\[
\begin{align*}
R_1 &= \frac{2 \Delta\theta}{2\pi} 2 \sum_{k=0}^{N-1} A(k \Delta\theta) \cos(k \Delta\theta) \\
I_1 &= \frac{2 \Delta\theta}{2\pi} 2 \sum_{k=0}^{N-1} A(k \Delta\theta) \sin(k \Delta\theta)
\end{align*}
\] (3.44)
with \( N\Delta\theta = \pi \).

This equation can be rewritten in a simpler way as

\[
A_1(\theta) = A_1 \cos(\theta - \theta_1)
\]

with

\[
\begin{aligned}
A_1 &= \sqrt{R_1^2 + I_1^2} \\
\theta_1 &= \cos^{-1}\left( \frac{R_1}{\sqrt{R_1^2 + I_1^2}} \right)
\end{aligned}
\] (3.45)

The vector potential in the air-gap is maximum at \( \theta = \theta_1 \), therefore the radial component of the flux density \( \vec{B} \) is maximum at \( \theta = \theta_1 - \frac{\pi}{2} \) as

\[
B_{1r}(\theta) = \frac{1}{R} \frac{\partial A_1(\theta)}{\partial \theta} = \frac{A_1}{R} \cos(\theta - \theta_1 + \frac{\pi}{2})
\] (3.46)

The fundamental of the flux per pole \( \Phi_p \) is thus obtained by integrating \( B_{1r} \) between \( \theta = \theta_1 - \pi \) and \( \theta = \theta_1 \) (Fig. 3.9), or by simply using Stokes theorem:

\[
\Phi_p = [A_1(\theta_1) - A_1(\theta_1 - \pi)] L_{stack}
\]

\[
\Phi_p = 2A_1 L_{stack}
\] (3.47)

This flux can be resolved in two components \( \Phi_{pd-axis} \) and \( \Phi_{pq-axis} \).
\[
\begin{align*}
\Phi_{pd-axis} &= \Phi_p \cos(\theta_1 - \frac{\pi}{2} - \alpha) \\
\Phi_{pq-axis} &= \Phi_p \sin(\theta_1 - \frac{\pi}{2} - \alpha)
\end{align*}
\]

(3.48)

where \( \alpha \) is the angle between the d-axis and the origin of \( \theta \).

Finally, using the traditional analysis of AC machines [53], the fluxes \( \lambda_d \) and \( \lambda_q \) linking the imaginary windings carried respectively by the d-axis and q-axis are given by:

\[
\begin{align*}
\lambda_d &= N_{\text{phase}} K_s \Phi_{pd-axis} \\
\lambda_q &= N_{\text{phase}} K_s \Phi_{pq-axis}
\end{align*}
\]

(3.49)

where \( K_s \) is the winding factor for the first harmonic, and \( N_{\text{phase}} \) the total number of turns per phase (39 turns/coil * 2 coils/poles * 4 poles = 272 turns.) Indeed, this imaginary winding has the same characteristics (arrangement and number of turns) as each of the three phase windings.

The winding factor accounts for the fact that the winding is made of coils that do not span a full pole pitch, and that are space shifted. As a result, the flux linking the winding is less than the flux per pole. For our winding, \( K_s = 0.933 \). The same reasoning can be applied to \( \lambda_q \).

Finally, for the rotor position adopted in our geometry, where \( \alpha = \frac{7\pi}{12} \), the d- and q- axis flux linkages are given by

\[
\begin{align*}
\lambda_d &= -2 N_{\text{phase}} K_s L_{\text{stack}} A_1 \cos(\theta_1 - \frac{\pi}{12}) \\
\lambda_q &= -2 N_{\text{phase}} K_s L_{\text{stack}} A_1 \sin(\theta_1 - \frac{\pi}{12})
\end{align*}
\]

(3.50)

with

\begin{align*}
N_{\text{phase}} &= 272 \text{ turns} \\
K_s &= 0.933 \\
L_{\text{stack}} &= 52 \text{ mm}
\end{align*}
It should be noted that, unlike method 1, the stator leakage flux is not taken into account; indeed, a small part of the stator flux doesn’t cross the air-gap, but still links the winding by taking a path through the slots.

To sum up,

- method 1 gives the total d-q axis flux linkages,
- method 2 gives the first harmonic of the d-q axis airgap flux
- whereas these are the fundamentals of the d-q axis flux linkages that are required in the non-linear circuit model of the IPMSM described in the previous chapter.

In other words, method 1 makes an ”error” in the estimation of the needed quantities that is equal to the harmonic content in the d-q flux linkages, while method 2 makes an ”error” that is equal to the stator flux leakage. Both ”errors” has to be estimated to choose which method is the most suitable to predict the flux linkage to be used in the circuit model.

This comparison will be carried out based on the analysis of the harmonic content of the vector potential distribution in the airgap.

3.4.3 Comparison of the two methods

The comparison is made at the \( n \) interpolation points \( I_r \angle \beta_1, I_r \angle \beta_2 \ldots I_r \angle \beta_n \), required to build the non-linear analytical models of the flux linkages of Chapter 2, in the following way:
For each current angle $\beta_i$,

- a FEM computation is carried out with the set of phase currents producing the corresponding current vector $I_r \angle \beta_i$. This computation returns the vector potential distribution throughout the domain.

- from this vector potential distribution,
  
  - the $\lambda_{d_i}$ and $\lambda_{q_i}$ are calculated using both methods 1 and 2
  
  - a Fourier analysis of the vector potential in the airgap is carried out to obtained its significant harmonics ($3^{rd}$, $5^{th}$, $7^{th}$ and $9^{th}$). The principle is the same as in Section 3.4.2; from the $k^{th}$ harmonic of the vector potential in the airgap,

\[
A_k(\theta) = A_k \cos(k\theta - \theta_k)
\]

the $k^{th}$ harmonics $\lambda_{d_i}^k$ and $\lambda_{q_i}^k$ of the flux linkages are obtained using

\[
\lambda_{d_i}^k = 2A_k N_{phase} K_{s_k} L_{stack} \cos\left(\frac{\theta_k}{k} - \frac{\pi}{2k} - \alpha\right)
\]

\[
\lambda_{q_i}^k = 2A_k N_{phase} K_{s_k} L_{stack} \sin\left(\frac{\theta_k}{k} - \frac{\pi}{2k} - \alpha\right)
\]

(3.52)

where $K_{s_k}$ is the winding factor associated to the $k^{th}$ harmonic.

The flux linkages obtained by methods 1 and 2 as well as the harmonics are plotted against the current angle on Fig. 3.10. As expected, there is a difference between results given by method 1 and 2 (square and circle “points” on Fig. 3.10). From these figures, we note that the harmonics of the q-axis flux are negligible; method 1 provides therefore exactly the fundamental of the q-axis flux linkage $\lambda_{q_i}$, whereas method 2 provides it with an error corresponding to the stator leakage.
Figure 3.10: Flux linkages obtained by method 1 and 2 and harmonics

Flux linkages obtained by method 1
Flux linkages obtained by method 2 (fundamental)

This relative error is small and nearly constant equal to 4.5%: for calculation of $\lambda_q$, we might therefore prefer method 1, keeping in mind method 2 could also do the job.

Regarding the d-axis flux, the relative difference between the results of the two methods increases with the current angle from 1% at 90° to nearly 50% at 180°. This trend can be explained by the 3rd harmonic, the only significant one; its amplitude increases with the current angle and becomes comparable to the fundamental, as the amplitude of the fundamental is low for high values of $\beta$ (the flux weakening is maximum). As a result, if method 1 is used to calculate the fundamental of the flux linkages $\lambda_d$, it will give erroneous results; this method is therefore ruled out.

To estimate the d-axis stator leakage flux, the d-axis flux obtained by method 1 is compared to the fundamental of the d-axis flux obtained by method 2 completed with its harmonics (the addition of all the components previously obtained by the
Fourier analysis of the vector potential in the airgap). We can see in Fig. 3.11 that
the difference is negligible, which means that the d-axis flux stator flux leakage is
extremely low; method 2 is therefore very suitable to calculate the fundamental of
the flux linkage $\lambda_d$.

![Figure 3.11: Total flux linkages](image)

The conclusion of the comparison is that method 2 is the most suitable of the
2 methods to calculate $\lambda_d$ and $\lambda_q$, the fundamental of the d-q-axis flux linkages,
since the the d-q-stator leakage fluxes are low enough.

### 3.5 Validation of the circuit modelling

This section investigates the accuracy of the non-linear model of the IPMSM that
predicts the torque from the flux linkages, using the relation:

$$T = \frac{3p}{2} I_r [\lambda_d(\beta) \sin(\beta) - \lambda_q(\beta) \cos(\beta)] \quad (3.53)$$
For this purpose, the results given by this formula can be compared to the value of the torque computed numerically by the FEM.

Several methods are available to compute the torque from an FEM solution:

- the Maxwell stress tensor is based on a numerical integration of $\vec{B}$ along a path lying in the middle of the air-gap. However, this method is well known to be little reliable as it uses a local quantity; the torque computed depends greatly on the path chosen [56].

- the virtual work method is based on the formula

$$T = \frac{\partial W}{\partial \theta}$$

This differentiation is made numerically; the magnetic energy stored in the motor is calculated for two different motor positions spaced by a small angle $\Delta \theta$, keeping the same amount of current in the slot. Even if based on an energy calculation (the most reliable), the method is sensitive to the choice of $\Delta \theta$ that can lead to rounding errors if too small, or to inaccurate results if too large.

- The Coulomb Virtual Work Method (CVWM) is based on the same formula, but the differentiation is made analytically [51]. As a result, this method is known to provide better results than the two others and is chosen here.

The motor torque is not expected to be constant when the rotor rotates since the harmonics of the MMFs and airgap permeance interact with each other, creating a pulsating torque. For this reason, the torque has to be computed for several rotor positions and averaged if one wants to compare it with the values
given by (3.53). Indeed this formula gives the torque resulting from the interaction
between the fundamentals of the stator and rotor MMFs, and should therefore be
constant.

For this, a pseudo-dynamic FEM simulation of the motor in rotation is performed
in the following way: at each step, the rotor is rotated from \( \theta_k \) to \( \theta_{k+1} \) by an angle
\( \Delta \theta \) (electrical degrees) and the current sources in the slot become

\[
I_A = I \cos(\theta_{k+1} + \beta) \\
I_B = I \cos(\theta_{k+1} + \beta - 120^\circ) \\
I_C = I \cos(\theta_{k+1} + \beta - 240^\circ)
\]

(3.55)
to simulate a steady state operation at a current angle \( \beta \).

At each angle \( \theta_k \),

- the instantaneous torque \( T_{CVWM}(\theta_k) \) is computed by CVWM,
- the flux linkages \( \lambda_d(\theta_k) \) and \( \lambda_q(\theta_k) \) are computed by method 2 and the cor-
  responding torque \( T_{method2}(\theta_k) \) is obtained from (3.53).

This simulation is made for two different sets of magnet dimensions (Fig. 3.12,
top row), one producing a high magnet flux, and another one with a much weaker
magnet flux to account for different saturation levels in the motor. The current
angle is chosen as \( \beta = 120^\circ \).

From the plot of the instantaneous torque \( T_{CVWM} \) against the rotor position,
a periodicity of 30° appears (Fig. 3.12, middle row), which corresponds to the slot
pitch (15 mechanical °).

\( T_{CVWM} \) and \( T_{method2} \) are compared over this range in Fig. 3.12 (bottom row). The
following observations can be made:

- the torque \( T_{method2}(\theta_k) \) is nearly constant as expected,
• it seems to correspond to the average value of the instantaneous torque \( T_{CVWM} \).

The average values and maximum variations (about this average value) of these two torques are shown in Table 3.1 and confirm these observations; the variations of \( T_{\text{method}_2}(\theta_k) \) is below 1% for both sets of magnet dimensions. The average value of the instantaneous torque is very close to the value of \( T_{\text{method}_2} \) (−0.2% for the small magnet, −4% for the large magnet).

<table>
<thead>
<tr>
<th></th>
<th>Small magnet</th>
<th></th>
<th>Large magnet</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_{CVWM} )</td>
<td>2.51</td>
<td>-20.9%</td>
<td>3.99</td>
<td>-27.0%</td>
</tr>
<tr>
<td>( T_{\text{method}_2} )</td>
<td>2.52</td>
<td>0.3%</td>
<td>4.15</td>
<td>0.8%</td>
</tr>
</tbody>
</table>

This comparison tends to validate the circuit modelling approach to calculate the average torque from the d- and q-axis flux linkages. The accuracy is more than satisfactory.

3.6 Conclusion

This chapter has shown how to model the IPMSM using Finite Element Method. The key point of the FEM is that it can return the vector potential overall the machine produced by a given current vector, handling easily the non-linear magnetic characteristic of the iron. Therefore, the flux linkages calculated from this vector potential fully takes into account the saturation. It has been shown that the popular method using the value of the vector potential in the slots, called method
1 in the chapter, is not recommended in our case as it could induce large errors if used in the non-linear circuit of chapter 2 to predict torque and voltage over a wide speed range; indeed, it is unable to isolate the fundamental of the d-q-axis flux linkages. A different method based on a Fourier analysis of the vector potential is therefore preferred as it can isolate the first harmonics of $\lambda_d$ and $\lambda_q$, the only one considered in the non-linear circuit.

Finally, the torque equation associated to the circuit modelling has been validated by a dynamic FEM simulation; it has been shown that this equation gives the average value of the torque, “filtering” the cogging torque and torque harmonic effects.

To conclude, the torque and voltage when the motor is operated at a given current angle $\beta$ (and at rated current) can be accurately calculated from $\lambda_d$ and $\lambda_q$ obtained by a single FEM computation.
Figure 3.12: Comparison of the two torque computation methods
Chapter 4

Response Surface Method

4.1 Introduction

Response surface method is a part of the field of design and analysis of experiments, initially developed for the study of biological and agricultural processes [57] and based on experimentation and statistical analysis. Indeed, a mechanistic model, written down from physical laws is not always available to describe a process; in mathematical terms, a mathematical relation between the quantity of interest (response) and the level of the factors assumed to affect it is out of reach. This is the case when the underlying phenomena are too complex, or simply not known. An alternative approach is to consider the process as a black box and to observe the response for different levels of the factors. An estimation of the relation between the response and the factors can be obtained by a regression analysis of the data collected. Response Surface Method is basically a collection of statistical tools to
• design the experimental plan, that decides how many observations need to be made, and at which levels of the factors

• build an empirical model relating the response and the factors from regression

• check the ability of this model to effectively represent the response

• find the values of the factors that yield the highest response.

This method was originally developed by Box and Wilson in the early 50’s and has been successfully applied in many diverse fields such as chemical engineering, industrial development and process improvement, agricultural and biological research [58]. Its application in motor design is relatively recent and not very common, maybe because the machine theory and its equations have been well established for more than a century. However, for the reasons stated in the introduction, its application for unconventional motors are obvious. In this chapter, it is shown how simple second order polynomials can accurately relate the model of the machine to the design variables. The model of the machine can be either the classic linear model, in that case each of the three motor parameters is expressed as a polynomial of the design variables, or it can be the non-linear model described throughout this thesis in which case the polynomials are used to represent each flux interpolation point. The “experimental data” used to fit the polynomials are naturally provided by FEM.

The whole procedure to build these empirical models is described in the first section and then applied to the motor parameters \( \lambda_m, L_d \) and \( L_q \). Different designs of experiments are investigated and compared in terms of accuracy and experimental costs. In the third section the same method is applied directly to the flux
interpolation points of the non-linear model of the IPMSM. As a result, our non-linear model of the motors can provide the field weakening performance of the IPMSM for any set of values of the design variables. The validity of the method is checked by investigating the reliability of those predictions.

4.2 RSM procedure

Let us consider the process shown in Fig. 4.1. The output, called true response,

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{process.png}
\caption{Process}
\end{figure}

is denoted by \( \eta \); the inputs, called factors believed to have some effect on \( \eta \) are denoted by \( \xi_1, \xi_2...\xi_n \) and define a n-dimensional space. \( \eta \) is actually inaccessible as the measurement process introduces an experimental error denoted by \( \varepsilon_{\text{exp}} \), usually assumed to be a random variable with zero mean and with a variance \( \sigma^2 \).

The observation of the response \( y \) can thus be written as:

\[
y = \eta + \varepsilon_{\text{exp}} \quad \text{with} \quad E(y) = \eta \quad \text{and} \quad Var(y) = \sigma^2
\]  

(4.1)

If we suppose that there exists a deterministic relationship \( f \) between \( \eta \) and \( \xi_1, \xi_2...\xi_n \), we can write:

\[
y = f(\xi_1,\xi_2...\xi_n) + \varepsilon_{\text{exp}}
\]  

(4.2)

As \( f \) is unknown, the idea is to approximate it by a low order polynomial that can be considered as its Taylor series expansion [60]. This approximation is of
course not expected to represent $f$ over the whole space of the input variables. However this could be a good local approximation within a given region of interest that has to be studied. Obviously, the size of this region and the accuracy of the approximation are strongly determined by the natural smoothness of $f$. The coefficients of these polynomials are unknown but can be estimated from several observations of the response.

The first step of the RSM procedure is then to choose the region where the behavior of the response is of interest. Let us denote by $\xi_{i_{\text{min}}}$ and $\xi_{i_{\text{max}}}$ the minimum and maximum values that can be taken by the $i^{th}$ input variable $\xi_i$. A normalization of the factors is then carried out to remove the units and prevent rounding errors during the later regression analysis [58]. Each factor $\xi_i$ is coded by a variable $x_i$ in the following way:

$$x_i = \frac{\xi_i - \bar{\xi}_i}{\Delta \xi_i} \quad (4.3)$$

where

$$\bar{\xi}_i = \frac{\xi_{i_{\text{max}}} + \xi_{i_{\text{min}}}}{2} \quad \text{and} \quad \Delta \xi_i = \frac{\xi_{i_{\text{max}}} - \xi_{i_{\text{min}}}}{2} \quad (4.4)$$

This way, the coded variable takes the simple value $1 \ 0 \ -1$ when the factor is at its maximum\middle\minimum values.

The model of the observed response has then to be chosen. Usually first or second order polynomials are preferred because of their simplicity. First order models are not meant to model the response. They are used when RSM is applied for optimization purposes; indeed they can provide an estimation of the gradient of $f$, which is pointing to the maximum of the response. These simple models give the direction to follow and allows the region of optimum of the response to be located after few iterations.
Second order models are generally used when one wants to model the response with accuracy. Indeed, they is usually a good compromise between accuracy and complexity; higher order models require many more experiments to estimate their coefficients and are therefore seldom used. This second order model has therefore been chosen. In the case of three factors ($n = 3$), the model of the observed response (4.2) is

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \beta_{12} x_1 x_2 + \beta_{13} x_1 x_3 + \beta_{23} x_2 x_3 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \beta_{33} x_3^2 + \epsilon$$  

(4.5)

where

$\beta_i$, $\beta_{ii}$ and $\beta_{ij}$ are constant coefficients to be estimated

$\epsilon$ is the prediction error, composed of the experimental error $\epsilon_{exp}$ and the bias error $\epsilon_{bias}$ induced by the polynomial approximation of $f$

The matrix notation is often preferred as it simplifies the equations:

$$y = [x]^T [\beta] + \epsilon$$  

(4.6)

where $[x]$ represents the point at which the response is predicted

$$[x]^T = [1 \ x_1 \ x_2 \ x_3 \ x_1 x_2 \ x_1 x_3 \ x_2 x_3 \ x_1^2 \ x_2^2 \ x_3^2]$$  

(4.7)

and $[\beta]$ the vector of the coefficients

$$[\beta]^T = [\beta_0 \ \beta_1 \ \beta_2 \ \beta_3 \ \beta_{12} \ \beta_{13} \ \beta_{23} \ \beta_{11} \ \beta_{22} \ \beta_{33}]$$  

(4.8)

The next step consists of choosing the design of experiments, that is the points in the 3D space of the coded variables where the response will be observed. Such a design is not chosen arbitrarily; it is required to provide a satisfactory distribution of information throughout the region of interest to ensure that the predictions made
by the RSM model, fitted from these observations, is as close as possible to the true values of the response\cite{59}. Several standard designs with different interesting properties are at the disposal of the experimenter. They will be described and investigated in the next section. However, the number of observations made is often much larger than the number of coefficients of the polynomial (10 for the second order model of three factors used here)

Let us denote the \( N \) observation points chosen as follow (\( N > 10 \)):

<table>
<thead>
<tr>
<th>Points</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( x_{11} )</td>
<td>( x_{21} )</td>
<td>( x_{31} )</td>
</tr>
<tr>
<td>2</td>
<td>( x_{12} )</td>
<td>( x_{22} )</td>
<td>( x_{32} )</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>( N )</td>
<td>( x_{1N} )</td>
<td>( x_{2N} )</td>
<td>( x_{3N} )</td>
</tr>
</tbody>
</table>

A vector \([x]_i\) is associated to each point \( i \) according to (4.7) and the design matrix \([X]\) is built as

\[
[X] = \begin{pmatrix}
[x]_1^T \\
[x]_2^T \\
\vdots \\
[x]_N^T
\end{pmatrix}
\] (4.9)

The vector of the observations at those \( N \) points is written as

\[
[Y] = \begin{pmatrix}
y_1 \\
y_2 \\
\vdots \\
y_N
\end{pmatrix}
\] (4.10)

and the vector of the errors

\[
[E] = \begin{pmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\vdots \\
\varepsilon_N
\end{pmatrix}
\] (4.11)

such that when (4.6) is applied to all the points, we obtain the matrix equation

\[
[Y] = [X][\beta] + [E]
\] (4.12)
[Y] and [X] being fixed, the error [E] is only a function of the coefficients in [β] which is thus the unknown in this equation. It should be noted that [E] cannot be made equal to zero as the system [Y] = [X][β] has more equations than unknowns (N > 10); so [E] has to be minimized. Different methods can be used to estimate [β], according to which component (bias or experimental) of the error vector [E] one wants to minimize. In practice [58], those methods give nearly the same results so we choose the most straightforward called least square estimator; the estimate of [β] is chosen such that it minimizes the sum of the square of the errors ε, i.e. the norm of [E].

This estimate [β̂] is obtained from the pseudo-inverse of [X] as

\[ \betâ = ([X]^T [X])^{-1} [X]^T [Y] \] (4.13)

[β̂] gives us finally the RSM fitted model that can now be used to predict the response at any point [x] of the space of the factors:

\[ \hat{y} = [x]^T [β̂] \] (4.14)

where \(\hat{y}\) is the prediction of y.

An analysis of variance is then performed to see how well this fitted model is able to predict the response at the observation points. For this, three important quantities related to the variations of the response are defined.

The total sum of squares (SST) measures the total variations of the N observations of the response. If we define \(\bar{y}\) as the average of those observations

\[ \bar{y} = \frac{y_1 + y_2 + \cdots + y_N}{N} \] (4.15)

then it can be written as

\[ SST = \sum_{i=1}^{N} (y_i - \bar{y})^2 \] (4.16)
and has \(N - 1\) degrees of freedom.

This total sum of squares is made of two components; the sum of squares due to regression and the sum of squares of the residuals.

The first component measures the variations accounted for by the fitted model and is given by

\[
SSR = \sum_{i=1}^{N} (\hat{y}_i - \bar{y})^2
\]

(4.17)

where \(\hat{y}_i\) is the prediction of the fitted model (4.14) at the point \(i\). \(p - 1\) degrees of freedom are associated to this SSR, where \(p\) is the number of coefficients (10 here).

The second component measures the variations of the response not accounted by the fitted model; it consists of the sum of squares of the residuals at the observation points.

\[
SSR = \sum_{i=1}^{N} (y_i - \hat{y})^2
\]

(4.18)

and has \(N-p\) degrees of freedom associated.

An important statistic that is often calculated in the analysis of a fitted model is

\[
R^2 = \frac{SSR}{SST}
\]

(4.19)

The value of \(R^2\) is a measure of the proportion of total variation of the \(y_i\)'s explained by the fitted model. However, this statistic tends to approach 100% when the number of observations \(N\) is close to the number of coefficient of the model \(p\) (few degree of freedom for the SSE), even if the model is not appropriate.

For this reason, the adjusted statistic \(R_A^2\) is often preferred

\[
R_A^2 = 1 - \frac{SSE/(N - 1)}{SST/(N - p)}
\]

(4.20)

as the number of degrees of freedom are involved in its calculation.
4.3 Application of RSM to the linear-model of the IPMSM

It has been said previously that it is extremely difficult to obtain a reliable analytical expression of the motor parameters of an IPMSM, even if one considers them as constant (independent of the saturation level). RSM thus seems an obvious alternative to express $\lambda_m$, $L_d$ and $L_q$ in terms of the design variables; these motor parameters can be considered as three responses of factors that are the design variables. The experimental observations of these responses can be done using FEM computations. At this point, it should be mentioned that the number of experiments required to build an RSM model increases exponentially with the number of variables. The number of design variables has to be limited to the minimum necessary to keep the computational time within acceptable limits. For this reason, three of the design variables that are known to have the greatest influence on these motor parameters [30] are chosen (4.2):

- the magnet thickness $l_m$ expressed in millimeters
- the magnet radial position $\delta$ expressed as the ratio of the distance between the shaft and magnet and the airgap radius.
- the magnet pole angle $\alpha$ expressed in mechanical degrees.

These three variables obviously determine the shape of the magnet and therefore $\lambda_m$. In addition, they (mainly $l_m$) also determine the reluctance of the d-axis path as the magnet is seen as an additional airgap by the d-axis armature flux. Consequently, they strongly affect $L_d$. Finally, the reluctance of the q-axis path is
also indirectly affected by $\alpha$ that decides its width, and then the saturation level of this path. The validity of these considerations will be checked with the value of the coefficients of the RSM model that will be fitted.

Finally, it should be mentioned that the variable having the greatest influence on these motor parameters, i.e the air-gap length $l_g$, is not chosen as a design variable: this choice is based on the fact that we want to focus on the rotor geometry whose effects on the motor parameters are difficult to predict quantitatively. All the other design variables (number of turns, rated current, stator geometry and airgap length...) are considered fixed in the following and are the same as in chapter 3. The following ranges are decided for the design variables

$$
\begin{align*}
1 \text{ mm} & \leq l_m \leq 2 \text{ mm} \\
60\% & \leq \delta \leq 75\% \\
60^\circ & \leq \alpha \leq 75^\circ
\end{align*}
$$

(4.21)
which are thus coded by the formula

\[
\begin{align*}
  x_1 &= \frac{l_m - 1.5}{0.5} \\
  x_2 &= \frac{\delta - 67.5\%}{67.5\%} \\
  x_3 &= \frac{\alpha - 67.5\%}{67.5}\% 
\end{align*}
\]

(4.22)

The second order RSM models for the three motor parameters are written as,

\[
\begin{align*}
  \lambda_m &= [x]^T[\beta_{\lambda_m}] + \varepsilon_{\lambda_m} \\
  L_d &= [x]^T[\beta_{L_d}] + \varepsilon_{L_d} \\
  L_q &= [x]^T[\beta_{L_q}] + \varepsilon_{L_q}
\end{align*}
\]

(4.23)

using the notation of the previous section.

The error $\varepsilon$ doesn’t have any experimental component here; replicate observations at the same point will give the same results since the experiments are numerical computations.

Three different designs of experiments will be tried to fit these models: the full factorial design, the central composite design, and the Box-Benhken design. The three of them are very different in terms of the number of experiments to be performed and choice of the experimental points.

- The Full Factorial Design is the most costly, as it requires to observe the response at three different levels (-1, 0 and 1) for each of the factors; this means in our case , $3^3 = 27$ runs. In geometrical terms, the observations have to be done on the vertex of a cube and in the middle of its faces and edges as well as at the origin (Fig. 4.3). The observations collected at these points are shown in Tab. C.2 in Appendix C.

- The Central Composite Design (CCD) is the most popular of all the designs. Indeed, it possesses the property of rotatability which makes the precision of
the RSM model fitted on it independent of the direction, but depends only on the distance from the origin. The CCD is made of 15 points: the eight vertices of a cube \((\pm1, \pm1, \pm1)\) (cube points), 6 star points at \((\pm1.682, 0, 0)\), \((0, \pm1.682, 0)\) and \((0, 0, \pm1.682)\) and the origin \((0,0,0)\) (Fig. 4.4). The observations collected at these points are shown in Tab. C.1 in Appendix C.

- The Box-Behnken Design is a subset of the Full Factorial Design (Fig. 4.5). It is the most economical of these three designs as it requires only 13 runs. In addition, it doesn’t require any experiments to be performed so far from the origin unlike the CCD and its star-points. This can be convenient when these experiments cannot be realized for practical reasons (due to physical constraints for example). The observations collected at these points are shown in Tab. C.3 in Appendix C.
The models of $\lambda_m$, $L_d$ and $L_q$ fitted on each design are presented in Tab. 4.1, 4.2 and 4.3 with their analysis of variance. It can be seen that the coefficients of the polynomials obtained from the different designs show close agreement, for each of
the motor parameters. The $R^2_A$ statistic, always above 95%, tends to show that the second order polynomials are very suited to represent the motor parameters for this range of the design variables. By looking closer, this statistic is always slightly less for the models fitted on the CCD. This can be explained by the larger size of this design; the polynomial approximation loses its accuracy at the remote star points.

<table>
<thead>
<tr>
<th></th>
<th>$\lambda_{m_{full}}$</th>
<th>$\lambda_{m_{CCD}}$</th>
<th>$\lambda_{m_{BOX}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_0$</td>
<td>365.6</td>
<td>366.0</td>
<td>365.5</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>36.1</td>
<td>37.8</td>
<td>36.4</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>73.8</td>
<td>73.9</td>
<td>74.0</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>71.7</td>
<td>71.0</td>
<td>72.1</td>
</tr>
<tr>
<td>$\beta_{12}$</td>
<td>6.7</td>
<td>6.4</td>
<td>7.4</td>
</tr>
<tr>
<td>$\beta_{13}$</td>
<td>8.3</td>
<td>7.8</td>
<td>9.1</td>
</tr>
<tr>
<td>$\beta_{23}$</td>
<td>-2.9</td>
<td>-2.9</td>
<td>-2.8</td>
</tr>
<tr>
<td>$\beta_{11}$</td>
<td>-8.8</td>
<td>-10.1</td>
<td>-8.4</td>
</tr>
<tr>
<td>$\beta_{22}$</td>
<td>0.2</td>
<td>-0.7</td>
<td>0.2</td>
</tr>
<tr>
<td>$\beta_{33}$</td>
<td>-8.7</td>
<td>-5.8</td>
<td>-8.7</td>
</tr>
<tr>
<td>SST</td>
<td>216340</td>
<td>165030</td>
<td>96921</td>
</tr>
<tr>
<td>SSR</td>
<td>216240</td>
<td>164840</td>
<td>96902</td>
</tr>
<tr>
<td>SSE</td>
<td>100</td>
<td>190</td>
<td>19</td>
</tr>
<tr>
<td>$R^2$</td>
<td>99.95%</td>
<td>99.88%</td>
<td>99.98%</td>
</tr>
<tr>
<td>$R^2_A$</td>
<td><strong>99.93%</strong></td>
<td><strong>99.67%</strong></td>
<td><strong>99.92%</strong></td>
</tr>
</tbody>
</table>
Table 4.2: RSM models of $L_d$ fitted on the three designs

<table>
<thead>
<tr>
<th></th>
<th>$L_{d_{full}}$</th>
<th>$L_{d_{CCD}}$</th>
<th>$L_{d_{BOX}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_0$</td>
<td>64.5</td>
<td>64.7</td>
<td>64.3</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>-11.6</td>
<td>-12.4</td>
<td>-11.6</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>-3.5</td>
<td>-3.6</td>
<td>-3.5</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>-2.5</td>
<td>-2.8</td>
<td>-2.3</td>
</tr>
<tr>
<td>$\beta_{12}$</td>
<td>-0.1</td>
<td>-0.4</td>
<td>0.3</td>
</tr>
<tr>
<td>$\beta_{13}$</td>
<td>0.1</td>
<td>-0.2</td>
<td>0.6</td>
</tr>
<tr>
<td>$\beta_{23}$</td>
<td>-0.1</td>
<td>-0.1</td>
<td>-0.1</td>
</tr>
<tr>
<td>$\beta_{11}$</td>
<td>2.7</td>
<td>2.8</td>
<td>2.9</td>
</tr>
<tr>
<td>$\beta_{22}$</td>
<td>-0.5</td>
<td>-1.3</td>
<td>-0.4</td>
</tr>
<tr>
<td>$\beta_{33}$</td>
<td>-0.3</td>
<td>0.9</td>
<td>-0.3</td>
</tr>
</tbody>
</table>

SST  | 2810 | 2251 | 1254 |
SSR  | 2794 | 2521 | 1253 |
SSE  | 16   | 30   | 1    |
$R_2$ | 99.40% | 98.84% | 99.95% |
$R_A^2$ | **99.09%** | **96.75%** | **99.78%** |

However, for a fair comparison, the RSM models obtained from these three designs have to be tested on all the experimental points (the union of all those designs, that is the 33 points shown in Fig. 4.6) rather than only on the points where they have been fitted. The performance criteria will be the average relative error

$$Error_{\text{average}} = \frac{1}{33} \sum_{i=1}^{33} \left| \frac{y_i - \hat{y}_i}{y_i} \right|$$  \hspace{1cm} (4.24)
Table 4.3: RSM models of $L_q$ fitted on the three designs

<table>
<thead>
<tr>
<th></th>
<th>$L_{qFull}$</th>
<th>$L_{qCCD}$</th>
<th>$L_{qBOX}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_0$</td>
<td>174.1</td>
<td>173.5</td>
<td>173.8</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>-4.9</td>
<td>-4.9</td>
<td>-4.9</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>-5.2</td>
<td>-5.2</td>
<td>-5.2</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>-8.2</td>
<td>-8.1</td>
<td>-8.3</td>
</tr>
<tr>
<td>$\beta_{12}$</td>
<td>-1.2</td>
<td>-1.3</td>
<td>-1.1</td>
</tr>
<tr>
<td>$\beta_{13}$</td>
<td>-2.5</td>
<td>-2.6</td>
<td>-2.4</td>
</tr>
<tr>
<td>$\beta_{23}$</td>
<td>-2.3</td>
<td>-2.3</td>
<td>-2.2</td>
</tr>
<tr>
<td>$\beta_{11}$</td>
<td>0.5</td>
<td>0.9</td>
<td>0.7</td>
</tr>
<tr>
<td>$\beta_{22}$</td>
<td>-0.8</td>
<td>-0.3</td>
<td>-0.8</td>
</tr>
<tr>
<td>$\beta_{33}$</td>
<td>-0.6</td>
<td>-1.5</td>
<td>-0.4</td>
</tr>
<tr>
<td>SST</td>
<td>2297</td>
<td>1743</td>
<td>1007</td>
</tr>
<tr>
<td>SSR</td>
<td>2295</td>
<td>1737</td>
<td>1006</td>
</tr>
<tr>
<td>SSE</td>
<td>2</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>$R_2$</td>
<td>99.91%</td>
<td>99.66%</td>
<td>99.91%</td>
</tr>
<tr>
<td>$R_A^2$</td>
<td><strong>99.87%</strong></td>
<td><strong>99.06%</strong></td>
<td><strong>99.63%</strong></td>
</tr>
</tbody>
</table>

and the maximum relative error

$$Error_{max} = \max\left\{ \left| \frac{y_i - \hat{y}_i}{y_i} \right| \right\}_{i=1..33} \tag{4.25}$$

The results of this comparison are shown in table Tab. 4.4, 4.5 and 4.6. For each response, the trend is the same: the average error of the models fitted on the three designs are nearly the same (maybe slightly larger for the model fitted on the CCD). However, there is clearly a difference regarding the maximum error; this error is always significantly smaller for the model fitted on the CCD. Furthermore,
Figure 4.6: Union of the three design: the benchmark

The maximum error made by the two other models is located on one of the star points. This is not a surprise as these points are far outside the Full Factorial and Box-Behnken designs. To sum up, we can say that the CCD is the most versatile design of experiment among those tested. Indeed, it explores a larger domain of the design variables space than the Box-Behnken and Full-Factorial designs during

<table>
<thead>
<tr>
<th>$\lambda_m$</th>
<th>Full Factorial</th>
<th>CCD</th>
<th>Box-Beinken</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Error_{average}$</td>
<td>0.81%</td>
<td>0.81%</td>
<td>0.78%</td>
</tr>
<tr>
<td>$Error_{max}$</td>
<td>4.39%</td>
<td>2.63%</td>
<td>4.79%</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$L_d$</th>
<th>Full Factorial</th>
<th>CCD</th>
<th>Box-Beinken</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Error_{average}$</td>
<td>1.55%</td>
<td>1.75%</td>
<td>1.53%</td>
</tr>
<tr>
<td>$Error_{max}$</td>
<td>6.88%</td>
<td>3.76%</td>
<td>7.46%</td>
</tr>
</tbody>
</table>
the observation stage; as a result the model fitted on it can predict the response near the origin and at more remote points accurately. It is also really interesting to note that the accuracy of a model doesn’t necessarily increase with the number of experiments used to fit it; with about half the number of runs required by the Full Factorial Design, the CCD and Box-Behnken designs allow for fitting models that have similar accuracy. Their “efficiency” is therefore far better and these designs should be recommended.

Finally, it is interesting to note that an empirical method using simple second order polynomials succeeds where well established analytical tools have failed. The motor parameters are modelled with excellent accuracy in terms of the design variables.

### 4.4 Application of RSM on the non-linear model of the IPMSM

This empirical model building procedure can also be applied to the non-linear model of the IPMSM, but at a higher experimental cost. Indeed, the non-linear model is totally described by $2n$ flux interpolation points ($\{\lambda_{di}\}_{i=1..n}$ and ($\{\lambda_{qi}\}_{i=1..n}$ using the same notation as in section 2.5), and therefore needs $2n$ response surfaces to

<table>
<thead>
<tr>
<th>$L_q$</th>
<th>Full Factorial</th>
<th>CCD</th>
<th>Box-Behnken</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Error_{average}$</td>
<td>0.26%</td>
<td>0.38%</td>
<td>0.30%</td>
</tr>
<tr>
<td>$Error_{max}$</td>
<td>2.42%</td>
<td>1.11%</td>
<td>2.54%</td>
</tr>
</tbody>
</table>
be fitted. For each interpolation point, the value of the d- and q-axis flux will be related to the design variables.

If the design variables are the same as in the previous section and the RSM models $2^{nd}$ degree polynomials, these $2n$ responses are:

$$\lambda_{di} = [x]^T[b_{di}] + \varepsilon_{di}$$

$$\lambda_{qi} = [x]^T[b_{qi}] + \varepsilon_{qi}$$

(4.26)

for $i = 1, 2, \ldots n$. The $[b_{di}]$ and $[b_{qi}]$ are the coefficients of the models to be fitted. As

![Figure 4.7: Responses for the non-linear IPMSM](image)

one FEM experiment can provide simultaneously one observation of $\lambda_{di}$ and one observation of $\lambda_{qi}$, using a design of experiments consisting of $N$ runs will require $N \times n$ FEM experiments to be carried out. This problem is a typical case where the number of experimental runs has to be kept low by the choice of an appropriate design of experiments. The Full factorial design is therefore ruled out. The CCD is chosen here as it has proved to combine accuracy and efficiency in the last section. The number of interpolation points, $n$ is chosen to be 16; 240 FEM computations are therefore performed to collect data to fit the models.

The $R^2_A$ statistics for the 30 fitted models are shown on Table 4.7. These statistics are always higher than 99.5% for any of the $\{\lambda_{di}\}_{i=1..n}$, which means
Table 4.7: $R^2_A$ statistic for the $2n$ responses

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\lambda_d$</th>
<th>$\lambda_q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ($\beta = 90^\circ$)</td>
<td>99.79%</td>
<td>99.54%</td>
</tr>
<tr>
<td>2 ($\beta = 96^\circ$)</td>
<td>99.78%</td>
<td>99.57%</td>
</tr>
<tr>
<td>3 ($\beta = 102^\circ$)</td>
<td>99.76%</td>
<td>99.63%</td>
</tr>
<tr>
<td>4 ($\beta = 108^\circ$)</td>
<td>99.77%</td>
<td>99.74%</td>
</tr>
<tr>
<td>5 ($\beta = 114^\circ$)</td>
<td>99.77%</td>
<td>99.77%</td>
</tr>
<tr>
<td>6 ($\beta = 120^\circ$)</td>
<td>99.77%</td>
<td>99.50%</td>
</tr>
<tr>
<td>7 ($\beta = 126^\circ$)</td>
<td>99.75%</td>
<td>98.60%</td>
</tr>
<tr>
<td>8 ($\beta = 132^\circ$)</td>
<td>99.73%</td>
<td>97.03%</td>
</tr>
<tr>
<td>9 ($\beta = 138^\circ$)</td>
<td>99.72%</td>
<td>95.33%</td>
</tr>
<tr>
<td>10 ($\beta = 144^\circ$)</td>
<td>99.70%</td>
<td>92.22%</td>
</tr>
<tr>
<td>11 ($\beta = 150^\circ$)</td>
<td>99.69%</td>
<td>88.47%</td>
</tr>
<tr>
<td>12 ($\beta = 156^\circ$)</td>
<td>99.69%</td>
<td>86.11%</td>
</tr>
<tr>
<td>13 ($\beta = 162^\circ$)</td>
<td>99.68%</td>
<td>83.36%</td>
</tr>
<tr>
<td>14 ($\beta = 168^\circ$)</td>
<td>99.68%</td>
<td>79.66%</td>
</tr>
<tr>
<td>15 ($\beta = 174^\circ$)</td>
<td>99.68%</td>
<td>73.62%</td>
</tr>
<tr>
<td>16 ($\beta = 180^\circ$)</td>
<td>99.68%</td>
<td>83.05%</td>
</tr>
</tbody>
</table>

that second order models are suitable to model these responses. Regarding the $\{\lambda_q\}_{i=1..n}$, the statistics is good for low values of $\beta$; however above 150$^\circ$, it drops below 90% to a minimum of 73.62% at 174$^\circ$ which is less satisfactory.

To check the quality of these RSM models, we will compare the $\lambda_d(\beta)$ and $\lambda_q(\beta)$ obtained by:

- the cubic spline interpolation using the $\{\lambda_d\}_{i=1..n}$ and $\{\lambda_q\}_{i=1..n}$ observed by FEM
the cubic spline interpolation using the \( \lbrace \lambda_{di} \rbrace_{i=1:n} \) and \( \lbrace \lambda_{qi} \rbrace_{i=1:n} \) predicted by the RSM models on each of the 15 points of the CCD.

The results are shown on Fig. 4.9, 4.10 and 4.11; the flux linkages (in Wb) are plotted against the current angle (in electrical degree). It is obvious that the predictions on these points are satisfactory, as the curves are often joined. The predictions of \( \lambda_{di} \) are a little bit different from the measurements at the points (1.68,0,0) and (-1.68,0,0), corresponding to very extreme values of the magnet thickness.

By combining these 2n RSM polynomials with the non-linear modelling of the IPMSM from chapter 2, it is now possible to obtain a prediction of \( \lambda_{d}(\beta) \) and \( \lambda_{q}(\beta) \) for any values of the design variables \( l_{m}, \delta \) and \( \alpha \) and therefore the corresponding flux weakening characteristics like the peak torque \( T_{\text{max}} \) or the \( CPSR \).

We therefore have an analytical model of the IPMSM relating its performances to the design variables (Fig. 4.8). The ability of this model to predict accurately

![Diagram](image_url)

Figure 4.8: Combination of RSM and non-linear modelling of the IPMSM
the performance of the IPMSM will be investigated by testing it using different sets of values of $l_m$, $\delta$ and $\alpha$. For each set of values, the performance predictions are compared with the “measured performances” obtained from FEM flux linkages computation at the $n$ interpolation points. One also has to make sure that the accuracy is good throughout the space of the design variables, and not only at the point were the RSM models have been fitted (CCD points). From the results shown in Table 4.8, the following remarks can be made:

The torque predictions are very good: most of the prediction errors are below 1% and the maximum prediction error is less than 5% (obtained for the extreme value of $l_m = 0.66mm$).

The CPSR predictions are very good for practical values attainable in practice, usually values below 5; for these cases, the prediction error is less than 4%. For very high values, the accuracy of the model becomes poor with prediction errors up to 65% of the measured CPSR. The prediction error is actually made on $\Omega_2$ the upper limit of the CPSR. This can be explained from the equation

$$\Omega = \frac{V_{\text{max}}}{\lambda} \quad (4.27)$$

At high speeds, the speed prediction is extremely sensitive to the flux; for values of design parameters that allow the d-axis flux to be close to zero, a small prediction error made on $\lambda_d$ results in a very high error on $\Omega_2$. This is typically the case of the set of values ($l_m = 0.66mm, \delta = 67.5\%, \alpha = 67.5^\circ$) corresponding to the point(-1.68,0,0) of the CCD and whose flux profiles can be seen in Fig. 4.10. Both d- and q-axis fluxes are nearly 0 when $\beta$ is close to 180°, which results in a prediction error on the CPSR equal to 63.9%.

To conclude, one can say that the RSM models are satisfactory in predicting
<table>
<thead>
<tr>
<th>Design variables values</th>
<th>Performances</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_m$ γ α (mm) (°)</td>
<td>$T_{max}$ measured (Nm)</td>
</tr>
<tr>
<td>2 67.5% 75</td>
<td>4.01</td>
</tr>
<tr>
<td>1.75 71.25% 71.25</td>
<td>3.97</td>
</tr>
<tr>
<td>1.75 71.25% 63.75</td>
<td>3.54</td>
</tr>
<tr>
<td>1.75 63.75% 71.25</td>
<td>3.48</td>
</tr>
<tr>
<td>1.75 63.75% 63.75</td>
<td>3.02</td>
</tr>
<tr>
<td>1.25 71.25% 71.25</td>
<td>3.65</td>
</tr>
<tr>
<td>1.25 71.25% 63.75</td>
<td>3.24</td>
</tr>
<tr>
<td>1.25 63.75% 71.25</td>
<td>3.19</td>
</tr>
<tr>
<td>1.25 63.75% 63.75</td>
<td>2.77</td>
</tr>
<tr>
<td>2 75.00% 75</td>
<td>4.48</td>
</tr>
<tr>
<td>2 75.00% 60</td>
<td>3.63</td>
</tr>
<tr>
<td>2 60.00% 75</td>
<td>3.53</td>
</tr>
<tr>
<td>2 60.00% 60</td>
<td>2.57</td>
</tr>
<tr>
<td>1 75.00% 75</td>
<td>3.76</td>
</tr>
<tr>
<td>1 75.00% 60</td>
<td>3.03</td>
</tr>
<tr>
<td>1 60.00% 75</td>
<td>2.94</td>
</tr>
<tr>
<td>1 60.00% 60</td>
<td>2.12</td>
</tr>
<tr>
<td>2.341 67.50% 67.5</td>
<td>3.74</td>
</tr>
<tr>
<td>0.659 67.50% 67.5</td>
<td>2.51</td>
</tr>
<tr>
<td>1.5 80.10% 67.5</td>
<td>4.24</td>
</tr>
<tr>
<td>1.5 80.10% 67.5</td>
<td>2.56</td>
</tr>
<tr>
<td>1.5 67.50% 80.115</td>
<td>3.97</td>
</tr>
<tr>
<td>1.5 67.50% 80.115</td>
<td>2.46</td>
</tr>
<tr>
<td>1.5 67.50% 67.5</td>
<td>3.39</td>
</tr>
</tbody>
</table>

the field-weakening characteristics of the IPMSM, and the prediction error is no more than a few percent in most cases. Larger errors may arise for special motor
designs where the flux can nearly be cancelled. The next chapter will explain this case in detail.

### 4.5 Conclusion

This chapter has proved that simple second order models can be built from FEM computations to relate the characteristics of the motors to the design variables. These characteristics can be the motor parameters, if a linear model of the IPMSM is considered, or the interpolation points if the non-linear model is preferred. The second case requires much more FEM computations to be carried out for the model building. This is the price to be paid if one wants a more sophisticated model including saturation effects. In both cases, the RSM models give satisfactory predictions, even if one must admit that their accuracy decreases with the distance from the center of the design on which they have been built. The torque and speed predictions obtained from these RSM models for the non-linear model of the IPMSM are globally good; however, the high sensitivity of the $CPSR$ to low flux values can result in large prediction errors. This is, however, unavoidable, no matter which method (RSM or other analytical methods) is used to calculate the flux from the design variables. RSM is therefore not a cause in this problem. The next chapter will show that it is in practice not a real problem, because the assumption of no-loss has stronger effects on the prediction.
Figure 4.9: Comparison of \( \lambda_d(\beta) \) and \( \lambda_q(\beta) \) observed and predicted
Figure 4.10: Comparison of $\lambda_d(\beta)$ and $\lambda_q(\beta)$ observed and predicted
Figure 4.11: Comparison of $\lambda_d(\beta)$ and $\lambda_q(\beta)$ observed and predicted
Chapter 5

Optimization of the IPMSM

5.1 Introduction

5.1.1 Problem formulation

This chapter will focus on the optimization of the IPMSM; the objective is to achieve the widest constant power speed range for given voltage and current limitations, the optimization variables being $l_m$, $\delta$ and $\alpha$. This optimization will be obviously subject to physical constraints that limit the range of the variables. In addition, a performance constraint on the peak torque will be added to ensure the IPMSM have enough torque at its disposal below base speed. In mathematical terms, the problem can be formulated as:

Find the set $l_{m_{opt}}, \delta_{opt}, \alpha_{opt}$ such that

$$CPSR(l_{m_{opt}}, \delta_{opt}, \alpha_{opt}) = \max\{CPSR(l_m, \delta, \alpha)\}$$

(5.1)
and

\[ T_{\text{max}}(l_{\text{opt}}, \delta_{\text{opt}}, \alpha_{\text{opt}}) \geq T_{\text{inf}} \]

\[ l_{\text{inf}} \leq l_{\text{opt}} \leq l_{\text{sup}} \]

\[ \delta_{\text{inf}} \leq \delta_{\text{opt}} \leq \delta_{\text{sup}} \]

\[ \alpha_{\text{inf}} \leq \alpha_{\text{opt}} \leq \alpha_{\text{sup}} \]  \hspace{1cm} (5.2)

### 5.1.2 Review of optimization tools

The traditional step by step procedure to design an electromagnetic device, by prototyping or CAD, is in practice a trial and error process. The final result may just be a suboptimal solution as the procedure strongly relies on the experience of the design engineer. In recent years, the concept of automated optimal design has emerged as device-optimization has turned out to be of increasing significance in industry; such an algorithm replaces the designer to carry out the task in a faster and more efficient manner \[61\]. Numerical optimization provides such tools. The algorithms can be classified into two main categories: the traditional deterministic algorithms where the search of the optimum is based on a predefined scheme, and the more recent stochastic algorithms where the nature of randomness plays an important role in the search.

Deterministic methods are more suitable for local optimization and often based on derivatives of the function to be optimized (cost function). The principle behind is to follow the gradient of the function to find the optimum (Conjugate gradient, Newton or Quasi-Newton, BFGS). These methods converge in a small number of steps (low computational cost) to the nearest optimum. However, they are strongly dependent on the starting point and may fail in finding the optimum of a function having several local optima. Another drawback is the necessity to calculate the
derivative of the cost function which is not an easy task when the differentiation
has to be conducted numerically; it may require an important computational ef-
fort and is also inherently inaccurate. This is the reason why these methods are
not recommended in our problem. Indeed, it has been said that the CPSR, the
quantity to be optimized, cannot be expressed analytically in terms of the design
variables.

Stochastic methods, like simulated annealing and genetic algorithms, on the con-
trary, are global search methods that do not require any information about the
derivatives; random search is instead introduced in the process of searching the
solution space. However, the convergence is slower as many evaluations of the cost
function are needed. This can be considered as a significant drawback if these eval-
uations are time consuming, typically when using FEM. For example, the genetic
algorithms used by Bianchi [35] to optimize the rotor shape of a IPMSM required
4000 FEM computations, which meant several days of computation. Direct search
methods, where such algorithms work directly on the function to be optimized seem
not acceptable when the evaluation is slow. Indirect search methods are alternative
methods where the optimization algorithm is applied to an approximation of the
cost function; this approximation has the significant advantage that its evaluation
is much faster than the evaluation of the original function, since it is expressed as
an analytical function of the optimization variables. This is the approach that has
been adopted in this thesis using the Response Surface Method. Response Sur-
face Method turns out to be the perfect tool to interface Genetic Algorithms with
Finite element, sacrificing a little accuracy in FEM simulation for a much faster
optimization process. Indeed, the FEM is not used to evaluate the cost function
which would result in a high computational cost; instead it is used to build RSM
models which are very convenient to evaluate quickly this function. The high num-
ber of evaluations of the cost function necessary in the GA procedure is therefore
no longer a problem. In fact, the RSM models cancel the main drawbacks of the
GA and the optimization time is drastically reduced.

5.2 Mechanism of Genetic Algorithms

This method was developed by John Holland in the 60’s and 70’s, and popularized
by one of his student, David Goldberg in 1989, who applied it to solve a difficult
problem in his dissertation [62]. GAs work on populations of candidate solutions,
or individuals, according to the mechanism of natural selection [63]. Each individ-
ual is actually a sampling point in the search space. The fittest individuals are
selected and stochastic genetic operators are applied to them in order to gener-
ate new sample points for the next generation. Successive generations yield fitter
solutions which approach the optimal solution of the problem. This optimization
tool is between the total random search and deterministic search; the selection part
introduces a bias in the random search to make it converge to the region of the
optimum solution.

5.2.1 Encoding

In practice, the individuals are encoded into a string of bits that could be considered
as a chromosome. This binary encoding means that each variable, a gene in the
chromosome, is discretized; the search space is therefore reduced to a finite number
of possible solutions. One must thus make sure that the discretization provides enough resolution to make it possible to adjust the cost function with the desired level of precision. If not, it is possible that the optimum solution cannot be found.

In our case, the ranges of three design variables \( l_m, \delta \) and \( \alpha \) are decided as being \([0.63 mm, 2 mm]\), \([60\%, 80\%]\) and \([60^\circ, 80^\circ]\). Each of the variable is coded into 10 bits so that the resolution is slightly higher than the manufacturing tolerances; a much higher resolution would be useless since yielding dimensions not practically realizable at a reasonable cost. Hence,

\[
\begin{align*}
  l_m &= 0.63 + \frac{l_{m_{\text{binary}}}}{2^{10} - 1} \times 1.37 \\
  \delta &= 60\% + \frac{\delta_{\text{binary}}}{2^{10} - 1} \times 20 \\
  \alpha &= 60^\circ + \frac{\alpha_{\text{binary}}}{2^{10} - 1} \times 20
\end{align*}
\] (5.3)

It is interesting to note that this encoding takes into account implicitly the geometrical constraints on the variables expressed in (5.2): any individual will automatically respect these constraints. An example of an individual, representing \( l_m = 1.63 \text{ mm}, \delta = 65.88\% \alpha = 63.68^\circ \) is shown in Fig. 5.1. The population is made of \( N_{\text{pop}} \) individuals. The initial generation is created as follows: \( N_{\text{init}} \) bit strings are generated randomly and the \( N_{\text{pop}} \) best ones are selected to constitute the initial generation.

The creation of the next generation is accomplished by a selection-recombination process that can be described by the following pseudo-code:
For $i = 1, \ldots, N_{\text{pop}}$, do

Select an individual among the population

Select a genetic operator and apply on this individual

Place the offspring in the next generation

Endfor.

5.2.2 Selection

The selection is thus the process of choosing the individuals who will be the parents of the next generation; it has to be done in such a way that high quality individuals have a better chance to mate than lower quality ones. The selection process is thus closely related to the way the individuals are evaluated.

A fitness function has to be defined for this purpose. The function should reflect the quality of the individual through its performance, the function to optimize ($CPSR$), and through its compliance with the optimization constraints. Indeed, the individuals in the population do not necessarily meet the constraint on $T_{\text{max}}$; any violation must be translated into a penalty term that affects the fitness. It is decided here that an individual who fails to meet the $T_{\text{max}}$ constraint has no chance to be selected. The fitness is thus written as:

$$fitness = \begin{cases} 
CPSR & \text{if } T_{\text{max}} \geq T_{\text{inf}} \\
0 & \text{if } T_{\text{max}} \leq T_{\text{inf}} 
\end{cases} \quad (5.4)$$

The efficiency of the GA is strongly affected by the way individuals are selected. Should the selection pressure be too weak and the algorithm will converge very slowly, like a totally random search. On the other hand, if the selection is too
strong, there is a risk of premature convergence; the algorithm may remain stuck in a local optimum.

There are several ways of selecting the individuals that will produce offsprings. The traditional wheel selection where the probability of being chosen is proportional to the fitness is not suitable in our problem as the fitness range is very large (the CPSR can be theoretically infinite). This method would result in the best individuals being systematically chosen leading to premature convergence of the algorithm. For this reason the ranking selection is preferred where all the individuals of the population are evaluated and ranked according to their fitness. The probability of being chosen is then a function of the rank and not of the fitness directly. This form of selection is also a good way to maintain constant selection pressure \[64\].

The probability of being selected for the individual ranked \(k\) is decided as

\[
P(k) = \frac{2(N_{\text{pop}} - k)}{N_{\text{pop}}(N_{\text{pop}} - 1)}
\]

so that \(P(1) + P(2) + \ldots + P(N_{\text{pop}}) = 1\)

The pseudo-code for the evaluation of the population is then

\begin{verbatim}
evaluate(population)
for i=1:popSize
    decode individual(i) to get \(l_m\), \(\alpha\) and \(\delta\)
    calculate the CPSR and \(T_{\text{max}}\) as in Chapter 4
    calculate the fitness of individual(i) according to equation (5.4)
end for
\end{verbatim}
rank *population* according to the fitness

return *ranked population*

and for the selection of an individual among the population

select(*population*)

draw a number *p* randomly between 0 and 1

return *individual*(*k*) such that *P(k−1) < p < P(k)*

Finally, an elitism policy is applied. The best individual in the population is carried forward unchanged in the next generation. This ensures that good solution are not “discarded” and also accelerates the convergence of the algorithm.

### 5.2.3 Recombination

The recombination is the process of generating new individuals from existing ones to explore further the search space and obtain better solutions. Genetic operators inspired from biology, like crossover and mutation, are applied on the individuals that have been selected.

The bulk of the search is carried out by the crossover. This operator consists of selecting a mate for the individual, choosing two random bit positions (crossover points) and making the two individuals swap the bits segment that falls between those positions. An example is given in Fig. 5.2. One of the two offsprings is placed in the next generation while the other one is discarded. As a matter of fact, all the individuals tend to become more and more similar, generation after
Figure 5.2: 2 points crossover operator

generation. This is a sign that the algorithm is converging. However as a result, some part of the search space cannot be explored any more. For example, if none of the chromosomes in the population has the $k^{th}$ bit equal to 1, then none of the offspring obtained from crossover will have it.

The mutation operator is then necessary to reintroduce this genetic material that has been lost or has never been present in the older generations. It simply involves complementing some bits of the chromosome (Fig. 5.3) according to a probability law also called mutation rate. This rate is usually problem dependant; it is difficult

Figure 5.3: Mutation operator

to say a priori which rate would yield best results. However according to literatures [63], an average of one mutation per chromosome is a good starting point.
The probability of each bit to mutate is chosen here to be twice higher as

\[
\text{mutRate} = \frac{2}{\text{chromLength}}
\]  

(5.6)

In pseudo-code, these 2 operators can be described as:

\begin{verbatim}
xover(individual1, individual2)
    draw randomly 2 integers i and j between 1 and chromLength
    swap the strings delimited by i and j between both chromosomes
    choose one of the 2 offspring obtained
    return chosenoffspring

mutation(individual)
    for i=1:chromLength
        draw randomly a number p between 0 and 1
        if p < mutRate, bit(i) = \overline{bit(i)}
    end if
    end for
    return individual
\end{verbatim}

Finally, an operator selection rule has to be decided. This rule is chosen such that the two operators step in at different stages. During the first generations, where the individuals are very different, the mutation operator is less useful than the crossover which uses the best of each individual to build better solutions. After some time, mutation becomes necessary to avoid premature convergence for the
reasons explained above. If we denote the maximum number of generations as $\text{genMax}$ (after which we consider the algorithm has converged to the solution) and the current generation as $\text{gen}$, the probability that the crossover is applied on the selected individual is:

$$P_{\text{xover}} = \begin{cases} 
0.8 & \text{if } \text{gen} < 0.75 \times \text{genMax} \\
0.5 & \text{if } \text{gen} > 0.75 \times \text{genMax} 
\end{cases}$$

(5.7)

The pseudo-code for the whole GA algorithm is then:

$\text{gen}=1$

$\text{pop(gen)}=\text{initialisation}$

$\text{pop(gen)}=\text{evaluate(}\text{pop(gen)}\text{)}$

while $\text{gen} < \text{genMax}$ do

  add($\text{best individual}, \text{pop(gen} + 1)$)

  for $i=2:\text{popSize}$

    $\text{parent1}=\text{select(}\text{pop(gen)}\text{)}$

    if $\text{rand} < P_{\text{xover}}$

      $\text{parent2}=\text{select(}\text{pop(gen)}\text{)}$

      $\text{son}=\text{xover(}\text{parent1, parent2}\text{)}$

    else $\text{son}=\text{mutation(}\text{parent1}\text{)}$

    end if

    add($\text{son}, \text{pop(gen} + 1)$)

  end for
\[ gen = gen + 1 \]

\[ pop(gen) = evaluate(pop(gen)) \]

update \( P_{xover} \) according to \( (5.7) \)

end while

\( (l_{opt}, \delta_{opt}, \alpha_{opt}) = decode(\text{best individual of } pop(genMax)) \)

We choose the values \( popSize = 40 \) and \( genMax = 40 \). It is important to note that the choice of the values of the parameters such as \( popSize, genMax, mutRate, P_{xover} \) and even the selection method cannot be made a priori since they are totally problem dependant; a set of parameters that yield good results for a particular problem can be inefficient in another problem. The selection of the values of the parameters has to be made empirically and requires many trials. This is how the values chosen have been obtained.

Finally, it is important to note that the random part introduced in the GAs means that running the algorithm twice will likely not give exactly the same results. In other words, the solution returned by the GA is not necessarily the optimum solution to the problem; but it should be very close if the GA is well designed. The GA programm must therefore be run several times to enhance the chance of hitting the optimum.

### 5.3 Optimization results

The stator of our IPMSM has been borrowed from a 400 Watts induction machine with a current and voltage respectively rated and limited at \( I_r = 1.92\sqrt{2}A \) and
The optimization of the IPMSM has to be done within the frame defined by this voltage and current values.

It consists of determining the magnet position and dimensions that will maximize the constant power speed range while keeping a high enough peak torque. Indeed, it has been reported that these two quantities are antagonists. The peak torque is representative of the performance in the constant torque region, at low speeds. The \(CPSR\) is representative of the performance in the flux weakening region, at high speeds. If the \(CPSR\) is to be privileged, like in our case, it will be at the expense of \(T_{\text{max}}\). This point will be elaborated in the following section.

5.3.1 Influence of the torque constraint upon the maximum \(CPSR\) achievable

The GA program described in the previous section has been used to optimize the \(CPSR\) of the IPMSM for various values of the torque constraint \(T_{\text{inf}}\) ranging from 2.5 \(Nm\) (the value of the torque that would give rated power of 400 Watts at a rated speed of 1500rpm) to 4 \(Nm\). The optimum designs obtained are shown in Table 5.1 with their \(CPSR\) and peak torque; their power capability is shown on Fig. 5.4.

It is interesting to note that each of these optimum designs present a very low magnet thickness \(l_m\), near or equal to the minimum allowed (0.63 \(mm\)). This offers large flux weakening capabilities to the armature reaction as the \(d\)-axis path reluctance is minimized; in linear terms, \(L_d\) is maximized by the GA.

From these results, the antagonism of \(T_{\text{max}}\) and the \(CPSR\) becomes obvious;
Table 5.1: Optimum designs obtained for different peak torque constraints

<table>
<thead>
<tr>
<th>$T_{inf}$ (Nm)</th>
<th>$l_m$ (mm)</th>
<th>$\delta$ (%)</th>
<th>$\alpha$ (°)</th>
<th>$T_{max}$ (Nm)</th>
<th>CPSR</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>0.89</td>
<td>88.6</td>
<td>87.3</td>
<td>4.0014</td>
<td>2.78</td>
</tr>
<tr>
<td>3.75</td>
<td>0.77</td>
<td>89.1</td>
<td>80.2</td>
<td>3.7500</td>
<td>3.19</td>
</tr>
<tr>
<td>3.5</td>
<td>0.63</td>
<td>88.3</td>
<td>78.1</td>
<td>3.500</td>
<td>3.82</td>
</tr>
<tr>
<td>3.25</td>
<td>0.63</td>
<td>80.2</td>
<td>78.5</td>
<td>3.2503</td>
<td>4.76</td>
</tr>
<tr>
<td>3</td>
<td>0.69</td>
<td>79.6</td>
<td>65.7</td>
<td>3.0002</td>
<td>6.54</td>
</tr>
<tr>
<td>2.85</td>
<td>0.63</td>
<td>77.7</td>
<td>65.2</td>
<td>2.8502</td>
<td>9.34</td>
</tr>
<tr>
<td>2.75</td>
<td>0.64</td>
<td>83.8</td>
<td>55.3</td>
<td>2.7520</td>
<td>12.96</td>
</tr>
<tr>
<td>2.55</td>
<td>0.65</td>
<td>67.2</td>
<td>67.5</td>
<td>2.5500</td>
<td>90</td>
</tr>
</tbody>
</table>

It is clear that each of these optimum designs exhibits a peak torque just equal to the torque constraint $T_{max} = T_{inf}$. In other words, the peak torque is reduced to the minimum allowed to obtain the maximum CPSR. Furthermore, by comparing these optimum designs we observe the trend that the lower the constraint $T_{inf}$, the higher the maximized CPSR obtained. This trend is illustrated by the plot of the maximum CPSR achievable against the peak torque constraint in Fig. 5.5 (using the previous results).

The maximized CPSR seems to take extremely high values when $T_{max}$ is allowed to be as low as 2.55 Nm. This seems to confirm the well known fact that an IPMSM with an infinite CPSR is obtainable (under the unreasonable assumption of neglecting losses and practical mechanical constraints) [18] [20]. The value $CPSR = 90$ obtained under the constraint $T_{inf} = 2.55 \text{Nm}$ can be considered to be infinite. Indeed, the quantization of the optimization variables results in that the exact solution giving an infinite CPSR cannot necessarily be exactly hit; by
coding each variable into 20 bits instead of 10, we obtain a $CPSR$ of 500 for the same torque constraint, which confirms this statement.

5.3.2 The case of the ideal design

This ideal design able to achieve an infinite maximum speed has been often mentioned and investigated with the motor parameters approach. The most fundamental result was established by Schiferl [18]; when $\lambda_m = L_d I_r$, that is, when all the current applied in the d-axis cancels out the magnet flux, then the $CPSR$ is infinite. In non linear terms, this corresponds to a design such that the d-axis flux reaches $0 \, Wb$ at the point C in Fig. 5.6. This is the case for our design ($l_m = 0.65 \, mm$, $\delta = 67\%$, $\alpha = 67.5^\circ$) which seems to present a very large $CPSR$ of 90. A look at the plot of the predicted $\lambda_d(\beta)$ against the current angle $\beta$ confirms this (Fig. 5.7).

The performance predictions of this optimal design have to be verified. Like in the previous chapter, several FEM computations give the flux at the interpolation points (Fig. 5.7) and the actual performances are derived. The comparisons of actual/real performances shown in Table 5.2 are in close agreement. The error on
the CPSR is not negligible for the same reasons as stated in the previous chapter (a small prediction error in $\lambda_d$ results in a large $CPSR$ prediction error when the flux is nearly zero). It is important to understand that such a high $CPSR$ is, in practice, not obtainable; iron losses become preponderant at high speeds as they are proportional to the square of the frequency. Copper losses also increase because of the skin effect that increases the wire resistance with the frequency. These losses, not accounted for in the model, will result in the practical $CPSR$ being far from infinite. And obviously, the rotor, shaft and bearings have a limited mechanical resistance and therefore will be able to withstand the centrifugal force only up to

Figure 5.5: Maximum CPSR achievable versus peak torque constraint

<table>
<thead>
<tr>
<th></th>
<th>$\Omega_1$</th>
<th>$CPSR$</th>
<th>$T_{max}$</th>
<th>$\Omega_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimation</td>
<td>1500</td>
<td>91</td>
<td>2.55</td>
<td>1840</td>
</tr>
<tr>
<td>Verification</td>
<td>1530</td>
<td>79.6</td>
<td>2.5</td>
<td>1850</td>
</tr>
</tbody>
</table>
Figure 5.6: Current trajectory for the infinite CPSR design

A limited speed. So far, the maximum speed range that has been reported in the literature for an IPMSM prototype is around 7.5 \cite{20}. The prediction error on the CPSR is therefore not so important; the point is that the optimization procedure has given the design variable values to chose in order to ensure that the IPMSM has the ability to achieve nearly zero flux, the optimal condition for field weakening purpose.

The theoretical power-speed, torque-speed and flux-speed characteristics of this optimal design are shown in Fig. 5.8.

5.4 Conclusion

A Genetic Algorithm working on a population of 40 individuals on 40 generations requires 1600 evaluations of the CPSR. Using the RSM models, each evaluation takes 1 sec, which means that optimization requires approximately 30 minutes. Using FEM computation, the evaluation requires 2-3 minutes, resulting in an opti-
Figure 5.7: Flux profiles of the optimum design

Optimization time between 50 and 80 hours. The RSM models have therefore allowed us to estimate quickly the function to be optimized and thus made possible the use of GA as the optimization tool in our problem. GA’s present many advantages over gradient based optimization techniques, the main one being the ability to perform a global search for the optimum.

This chapter has described the main decisions regarding the implementation of the GA; the ranked selection to “scale” the fitness that can have a large range of $1: \infty$, the elitism to improve the efficiency and the two genetic operators which perform the search task. The algorithm has been used to optimize the constant power speed range of the motor under peak torque constraints. It has been shown that, with given current and voltage limitations, the maximum $CPSR$ obtainable can be increased without limits, provided one is ready to compromise on the peak torque available. At $T_{max} = 2.55Nm$, the $CPSR$ is infinite under the no-loss assumption. This study has shown that this ideal design, often described in motor parameters terms ($\lambda_m = L_d I_r$) in researcher’s work, can be practically realized by
Figure 5.8: Theoretical performances of the optimal design
a specific set of the design variables \( l_m, \delta, \alpha \). The key point of this design is that \( l_m \) is reduced to the minimum in order to increase the flux weakening capability of the armature; the magnet flux is totally cancelled by the armature flux for a current vector \( I_r \angle 180^\circ \).

However, in reality we cannot ignore mechanical constraints, losses and other effects of high frequency operation, and hence \( CPSR \) will always be limited to a practical value depending on design skills.
Chapter 6

Conclusion and discussion

This thesis dealt with the analysis, design and optimization of the buried type interior permanent magnet synchronous motors. The main objective was to move beyond the convenient but inadequate motor parameters approach; indeed, most researchers agree on the fact that saturation cannot be neglected for the IPMSM unlike for other kinds of motors. The proposed solution has been a new non-linear circuital representation of the machine: the d- and q-axis flux linkages have been expressed as non-linear functions of the d- and q-axis currents in order to integrate the saturation effect in the circuit modelling of the machine. Comparisons have shown that this non-linear circuit modelling provides more reliable predictions of the field weakening capabilities of the machine as it fully accounts for the variation of the saturation level of the machine.

From this modelling of the IPMSM, a special design-optimization procedure has been developed, based on Finite Element Method, Response Surface Method and Genetic Algorithm. Empirical models relating the flux linkages to the design variables have been built using RSM combined with FEM. The quality of the per-
formance predictions of these models has been checked and can be considered to be very satisfactory. Besides their accuracy, the main advantage of these models is that, being analytical, they allow estimating the performances of the motor nearly instantaneously. For such motors where traditional analytical tools such as magnetic circuit modelling are unable to deal with the complex geometry and saturation, RSM is a good alternative to obtain an analytical model of the motor. A simple genetic algorithm has then been used to optimize the constant power speed range of the motor, using RSM models for a fast fitness evaluation. It has been shown that the optimal design reported in the literature can be achieved for a given set of the design variables; such a design exhibits a theoretical infinite constant power speed range if all the losses are neglected. Its main characteristic is that the armature reaction is strong enough to cancel out the magnet flux when all the current is along the d-axis.

We have said that in practice, an infinite $CPSR$ is obviously not possible because of the losses. For more accurate field weakening predictions, especially at high speeds, a resistance $R_{\text{iron}}$ representing iron losses could be added in the non-linear circuit modelling. As the value of such a resistance would be dependent on the flux density level in the machine (in addition to the supply frequency), one could use the same modelling method (RSM+FEM+cubic spline interpolation) as for the flux linkages to model it as a function of the current angle and the design variables. This would not require additional FEM computation as the same experimental data that have been used to build the flux linkage RSM models could also be used to build this $R_{\text{iron}}$ RSM model.

The circuit model can be further developed by the addition of the stator resistance, independent of the design variables and the flux, but that presumably increases
with the supply frequency because of the skin effect. In addition, the end-winding leakage flux could also be added for more accurate predictions, especially when one wants to obtain the optimal design described in the last chapter. Well established analytical formula of the end-winding inductance could be used for this purpose. Assuming that the current trajectory is unchanged when one considers losses, all the procedure described in chapter 2 to calculate the performance of the machine from the non-linear circuit model would still be valid.

One has to keep in mind that the proposed non-linear circuit modelling of the motor is useful only for field weakening capabilities predictions. Indeed, the flux linkages can be predicted only on the rated current circle. As a result, this model cannot be used for control purposes, unless one is ready to build a look-up table of the d- and q-axis flux throughout the whole second quadrant of the $I_d-I_q$ plane.

It has been noted that the CPSR prediction errors from the RSM models can be significant for designs that can achieve very low flux levels. As a result, the GA may have missed the exact position of the optimum design since it uses these RSM prediction models. If it is required to locate this optimum with a better resolution, one can do another iteration of the whole process; a narrower search region centered on the optimum returned previously has to be defined and a new Central Composite Design is built inside. New RSM models of the flux fitted to this small-sized design will likely have better accuracy as they cover a much smaller region. The GA using these new models will then locate the optimum in this new region more precisely.

The method chosen to estimate the coefficients of the polynomials is the popular least square estimation. This method is extremely popular and is the
one chosen in all the research papers related to motor design applications. The reasons are mainly the ease of use (the estimates are obtained by the simple pseudo-inverse formula) and also because it is very popular in traditional fields using RSM (quality control, biology...). However, one should keep in mind that this method is optimal [58] only for these fields where the variance of the experimental data is “high”, that is in fields where the experimental error is larger than the bias error (induced by the polynomial approximation of the response). When the experimentation is made by FEM, the experimental error is zero and one can wonder whether another estimation method, based on bias minimization [58] would not be more appropriate. Other designs of experiments, with bias minimization properties could also be investigated.

Finally, some thoughts should be given on the limitation of the method. The low number of design variables that can be handled by the design procedure is its main drawback. Indeed, the number of experiments to be performed to fit a RSM model increases exponentially with the number of design variables. 250 FEM computations were needed to fit the non-linear model of the IPMSM, as 15 RSM polynomials of three variables had to be fitted. Adding a 4\textsuperscript{th} design variable would lead to around 500 computations and a 5\textsuperscript{th} one to 1000 computations. Even if the experimentation procedure can be automated, the computational time becomes extremely long and unacceptable. As a result, this design method is rather to be applied at the latest stage of the design of the machine; traditional sizing equations are more useful at the initial stages to choose the main dimensions of the motor. In a second stage, RSM could be applied on the motor parameters based modelling of the motor; less costly in FEM experiments as it requires only three RSM models (\(\lambda_m\), \(L_d\) and \(L_q\)) to be fitted, this model could handle twice or three times more
variables than the non-linear model. Finally, the non-linear model described in the thesis can be used at the end to adjust finely the variables in order to achieve the optimal zero flux design described in the last chapter, with much better accuracy.
Bibliography


[64] D. Whitley, *A Genetic Algorithm tutorial*
List of Publications

Published


To be published


Submitted for Review

Appendix A

Cubic spline interpolation

Let \( \{(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)\} \) be the \( n \) points to be interpolated by the function \( f(x) \) (Fig. A.1). \( f \) is made a piece-wise function, such that on each interval \([x_k; x_{k+1}]\),

\[
f(x) = f_k(x) = a_k + b_k(x - x_k) + c_k(x - x_k)^2 + d_k(x - x_k)^3 \tag{A.1}
\]

The piecewise function must meet the following requirements
1. $f$ should go through all the data points and be continuous on $[x_0, x_n]$

2. $f'$ must be continuous on $[x_0, x_n]$

3. $f''$ must be continuous on $[x_0, x_n]$

From 1, one can write

$$f_0(x_0) = y_0$$

$$f_k(x_{k+1}) = f_{k+1}(x_{k+1}) = y_{k+1} \quad \text{for } k = 1 : n - 2 \quad (A.2)$$

$$f_{n-1}(x_n) = y_n$$

From 2,

$$f'_k(x_{k+1}) = f'_{k+1}(x_{k+1}) \quad (A.3)$$

and from 3

$$f''_k(x_{k+1}) = f''_{k+1}(x_{k+1}) \quad (A.4)$$

Using the notation

$$h_k = x_{k+1} - x_k \quad (A.5)$$

Combining (A.2) and (A.1) give

$$a_k = y_k \quad (A.6)$$

for $k = 0 : n$ (which means that all the coefficients $a_k$ are known) and

$$a_{k+1} = a_k + b_k h_k + c_k h_k^2 + d_k h_k^3 \quad (A.7)$$

for $k = 0 : n - 1$

Differentiating (A.1) and combining with (A.3) give for $k = 0 : n - 1$

$$b_{k+1} = b_k + 2c_k h_k + 3d_k h_k^2 \quad (A.8)$$
Differentiating two times (A.1) and combining with (A.4) give for $k = 0 : n−1$

$$c_{k+1} = c_k h_k + 3d_k h_k \quad (A.9)$$

The coefficients $d_k$ can be thus be expressed in terms of the coefficients $c_k$

$$d_k = \frac{c_{k+1} - c_k}{3h_k} \quad (A.10)$$

The coefficients $b_k$ can be expressed in terms of the coefficients $c_k$ and $a_k$, substituting (A.10) into (A.7)

$$b_k = \frac{a_{k+1} - a_k}{h_k} - \frac{h_k}{3}(2c_k + c_{k+1}) \quad (A.11)$$

Finally, the coefficients $c_k$ can be related to the coefficients $a_k$ (known), substituting (A.10) into (A.8)

$$b_{k+1} = b_k + h_k(c_k + c_{k+1}) \quad (A.12)$$

and A.11 into A.12

$$\frac{3}{h_{k+1}}(a_{k+2} - a_{k+1}) - \frac{3}{h_k}(a_{k+1} - a_k) = h_k c_k + 2(h_k + h_{k+1}) + h_{k+1}c_{k+2} \quad (A.13)$$

In the following, the left hand term of A.13, which only depends of the coefficients $a_k$ is denoted by $\alpha_{k+1}$

The value of the second derivative at $x_0$ and $x_n$ is chosen equal to 0 (free spline). It follows that:

$$c_0 = 0 \quad (A.14)$$

and

$$2c_{n-1} + 6d_{n-1}h_{n-1} = 0 \quad (A.15)$$
To sum up, all the coefficients $d_k$ and $b_k$ can be derived from the coefficient $c_k$. Those $c_k$ are expressed in terms of the coefficients $\alpha_k$ (known) as follows:

$$
\begin{pmatrix}
0 \\
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_{n-1} \\
0
\end{pmatrix}
= \begin{pmatrix}
1 & 0 & 0 & \cdots & \cdots & 0 \\
0 & h_0 & 2(h_0 + h_1) & h_1 & 0 & \vdots \\
0 & h_1 & 2(h_1 + h_2) & h_3 & 0 & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & h_{n-2} & 2(h_{n-2} + h_{n-1}) & h_{n-1}
\end{pmatrix}
\begin{pmatrix}
c_0 \\
c_1 \\
c_2 \\
\vdots \\
c_{n-1} \\
c_n
\end{pmatrix}
$$

Solving this system of equation gives us the $c_k$. The $d_k$ are then obtained by (A.10) and the $b_k$ by A.11.
Appendix B

Motor characteristics

1. Magnetic materials characteristics

![B-H curve of the iron (50H470) used for the rotor and stator](image)

Figure B.1: B-H curve of the iron (50H470) used for the rotor and stator
Figure B.2: B-H curve of the NdFeB magnet used in the rotor

2. Main characteristics of the stator
Table B.1: Main characteristics of the stator

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<th>Value</th>
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<td>Rated speed (rpm)</td>
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<tr>
<td>Max voltage (V&lt;sub&gt;rms&lt;/sub&gt;)</td>
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</tr>
<tr>
<td>Rated current (A&lt;sub&gt;rms&lt;/sub&gt;)</td>
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<td>Number of poles</td>
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<td>Winding pole pitch</td>
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<td>Outer diameter (mm)</td>
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<td>Stack length (mm)</td>
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Appendix C

RSM experimental results

The $\lambda_m$, $L_d$ and $L_q$ observations collected on the three designs are

<table>
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<th>Coded values</th>
<th>Observations</th>
<th>$\lambda_m$</th>
<th>$L_d$</th>
<th>$L_q$</th>
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Table C.2: Observations on the Full Factorial Design

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Table C.3: Observations on the Box-Behnken Design

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